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Two-Person Fair Division of Indivisible Items when Envy-Freeness is Impossible

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Abstract

Assume two players, A and B, must divide a set of indivisible items that each strictly ranks from best to worst. If the number of items is even, assume that the players desire that the allocations be balanced (each player gets half the items), item-wise envy-free (EF), and Pareto-optimal (PO).

Meeting this ideal is frequently impossible. If so, we find a balanced maximal partial allocation of items to the players that is EF, though it may not be PO. Then we show how to augment it in a way that makes it a complete allocation that is EF for one player (say, A) and almost-EF for the other player (B) in the sense that it is EF for B except for one item – it would be EF for B if a specific item assigned to A were removed. Moreover, we show how low-ranked that exceptional item can be for B, thereby finding an almost-EF allocation that is as close as possible to EF – as well as complete, balanced, and PO. We provide algorithms to find such almost-EF allocations, adapted from algorithms that apply when complete balanced EF-PO allocations are possible.

Keywords: 2-person fair division, indivisible items, envy-freeness up to one item, Pareto-optimality

1. Introduction

Assume two players, A and B, must divide a set of indivisible items that each strictly ranks from best to worst. It is desirable, if possible, that each player's allocation satisfy two properties:

> *Envy-freeness* (EF): An allocation is *(item-wise) EF* for a player, say A, if there is a one-to-one matching of B's items to A's such that A prefers each of its items to the item of B that is mapped to it. The allocation is EF if it is EF for each player.[1]
>
> *Pareto-optimality* (PO): An allocation is PO if it is not Pareto-dominated by any other allocation – that is, there is no other allocation that is at least equally preferred by both players and strictly preferred by at least one.

We generally assume that the number of items is even, say $2n$. Then an allocation is *balanced* if each player receives n items, and *complete* if all items are allocated.

When the total number of items is even, several algorithms have been proposed that yield at least one complete balanced EF-PO allocation, provided such an allocation exists (for surveys and comparisons of these algorithms, see Kilgour and Vetschera (2018) and Klamler (2020)). Unfortunately, no EF allocation may exist – as occurs, for example, when A and B have identical preferences (Brams and Fishburn, 2000). In Brams, Kilgour, and Klamler (2014), we offered a simple test to determine whether a complete balanced PO-EF allocation exists.

Even if there is no complete balanced EF allocation, as will often be the case, there may be a balanced *partial* EF allocation. We begin by searching for balanced partial EF allocations that are *maximal* in the sense that more items cannot be allocated without making at least one player envious.

Surprisingly, a maximal partial EF allocation may be Pareto-dominated by an allocation of the same size that is not EF. If so, the players face a difficult choice between a maximal partial EF allocation that is not PO and a PO allocation of the same size that one player prefers but which makes the other player envious.

Assuming no complete balanced EF-PO allocation exists, we focus on finding a complete balanced PO allocation that is as close as possible to EF. We do so by augmenting a maximal partial EF allocation in a way that makes it EF for one player and envy-possible (EP) for the other, say A, in the sense that A would not envy B's allocation if one specific item assigned to B were removed. We find an allocation such that the item in question is minimally ranked by A—removing any less-preferred (by A) item from B's allocation would leave A envious. Our definition is modelled on the concept of EF1, introduced by Budish (2011), in which some item must be removed from B's allocation to eliminate A's envy.

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[1] The matching need not be unique. Assume A ranks four items 1 > 2 > 3 > 4, and B ranks them in reverse order: 4 > 3 > 2 > 1. Then giving {1, 2} to A and {3, 4} to B yields two matchings in which A pairwise prefers its allocation to B's: (1 > 3, 2 > 4) and (1 > 4 and 2 > 3). The same two matches, but with the preference relations reversed, show that B pairwise prefers its allocation to A's. If an allocation is EF, A's and B's matchings may be inverses—as illustrated by this example—but need not be, as illustrated by Example 1 in section 2.

We provide algorithms to find a complete balanced allocation that is Almost-EF (to be defined) and PO when there is no complete balanced EF-PO allocation. The key step involves a modification of the algorithm AL that we introduced earlier (Brams, Kilgour, and Klamler, 2014). The allocations that we find improve on EF1, in that only one player has an exceptional item; moreover, we show just how low that item can lie in the player's ordering, or how close an allocation can come to the ideal of EF-PO when, as is often the case, the ideal does not exist.

2. Terminology

Assume that the set of items, S, is to be allocated to the two players, A and B. We will generally assume that the number of items in S is $|S| = 2n > 0$. An allocation of S, denoted (L_A, L_B), consists of two subsets, $L_A \subseteq S$, the subset of S allocated to A, and $L_B \subseteq S$, the subset of S allocated to B, provided that $L_A \cap L_B = \emptyset$. The allocation (L_A, L_B) is *balanced* if and only if $|L_A| = |L_B|$ and *complete* if and only if $L_A \cup L_B = S$. In a complete balanced allocation, each player receives n items. (Below, we will briefly consider allocations of S when $|S|$ is odd.)

Throughout, we assume that the players' preference orderings of the items, which together make up their preference profile, are known and strict (i.e., no ties). We assume that a player's preferences on subsets of items satisfy the strict independence of Barbera, Bossert and Pattanaik (2004) – namely that if $X, Y, Z \subseteq S$, $X \cap Y = \emptyset$, $X \cap Z = \emptyset$, and a player prefers subset Y to subset Z, then the player also prefers subset $X \cup Y$ to subset $X \cup Z$. In particular, the player's preferences on subsets are consistent with its original preference ordering, because if y is preferred to z, then $X \cup \{y\}$ must be preferred to $X \cup \{z\}$.

We follow the convention that the items of S are named 1, 2, 3, …, $2n$, in decreasing order of A's preference.

Example 1: A: $\underline{1}$ 2 $\underline{3}$ 4
B: $\underline{2}$ 3 $\underline{4}$ 1

Note that $2n = 4$. The underscored allocation with $L_A = \{1, 3\}$ and $L_B = \{2, 4\}$, which we write as $(L_A, L_B) = (13, 24)$, is complete and balanced. We follow this convention for indicating allocations when no confusion is possible.

In Brams, Kilgour, and Klamler (2014), we defined item-wise EF for an allocation. (Here, it is convenient to reverse the direction of the mappings that define EF.) An allocation (L_A, L_B) is *item-wise envy-free* (*EF*) *for* A if and only if there exists an injection $f_A: L_B \rightarrow L_A$ such that, for each item $y \in L_B$, A prefers $f_A(y) \in L_A$ to y. If so, we say that, for A, L_A is a (*preferential*) *cover* for L_B. Because f_A is an injection, it follows that, if L_A is a cover for L_B, then $|L_A| \geq |L_B|$.

The meaning of (item-wise) envy-freeness is clear: For each item allocated to B, A prefers one item in her own allocation and, moreover, those preferred items of A are all distinct. In particular, strict independence implies that A definitely does not envy B.

That the allocation given in Example 1 is EF for A is demonstrated formally by $f_A(2) = 1$ and $f_A(4) = 3$. In other words, for A, $L_A = 13 = \{1, 3\}$ is a cover for $L_B = 24 = \{2, 4\}$. Moreover, the allocation in Example 1 is also EF for B, because, for B, $L_B = 24$ is a cover for $L_A = 31$. (We write $L_A = 31$, rather than $L_A = 13$, because B's mapping must be by $f_B(3) = 2$ and $f_B(1) = 4$.)

A complete balanced allocation like $(L_A, L_B) = (13, 24)$ in Example 1 is fair because neither player envies the other—it prefers its allocation, item by item, to the other player's. Of course, other complete balanced allocations are available in Example 1; in fact, there are six, which are shown in Table 1. If the objective is to reduce or eliminate envy, none of the other five is as good as (13, 24).

For example, the allocation exactly opposite (13, 24) is (24, 13), wherein each player envies the other. (Because both players would be better off if they reversed their assignments, the allocation (24, 13) cannot be PO.) For instance, for A, B's allocation 13 is a cover for A's allocation 24; similarly, for B, A's allocation is a cover for B's. We call an allocation like (24, 13) in Example 1 *envy-certain* (*EC*) for each player, because each player envies the other.

We call an allocation *envy-possible* (*EP*) for a player if it is neither envy-free (EF) nor envy-certain (EC) for that player. Note that an allocation may have a different status for each player. Table 1 shows the envy status of all six possible allocations in Example 1.

Allocation	Status for A	Status for B	Allocation	Status for A	Status for B
1. (12, 34)	EF	EP	4. (34, 12)	EC	EP
2. (13, 24)	EF	EF	5. (24, 13)	EC	EC
3. (14, 23)	EP	EF	6. (23, 14)	EP	EC

Table 1. Possible allocations for Example 1, with rankings A: 1 2 3 4 and B: 2 3 4 1

EP is very common. For two players, there are 4! = 24 preference profiles on $S = \{1, 2, 3, 4\}$ (taking A's ordering as fixed). Because each profile gives rise to 6 complete balanced allocations, there are a total of $6 \times 24 = 144$ complete balanced allocations. Eighty of them are EP for at least one player.

By comparison, preference profiles that allow for complete balanced (EF, EF) allocations are relatively rare. Exactly 12 of the 24 profiles allow one (EF, EF) allocation, and two more allow two each. Thus, of the 144 possible allocations, only 16 are (EF, EF). Because the reverse of an EF allocation is an EC allocation, 16 of the 144 possible allocations are (EC, EC), and such allocations occur in 14 of the orderings. Exactly 22 of the 24 profiles yield at least one allocation that is EF for one player and either EF or EP for the other.

3. Envy-Possible (EP) Allocations

We now present results showing that EP allocations are always more numerous than EF or EC allocations. In addition, if a preference profile does not admit any allocation that is EF for both players, then we ask if there is an allocation that is not EC for either player.

Assuming that there are at least 3 items, any preference profile must admit at least one allocation that is either EF or EP for both players. To construct such an allocation, give A its top-ranked item and B its two most preferred of the remaining items. Assign additional items in any way that makes the allocation balanced. This allocation cannot be EC for A, because given any mapping of B's items to A's, A must prefer its first-choice item. It also cannot be EC for B, because for any mapping of A's items to B's, B must prefer at least one of its first two items to the item of A that is mapped to it.

The question then arises whether, for any preference profile, there are allocations that are EP for both players (there were none for Example 1 in Table 1). In the case of four items, an (EP, EP) allocation occurs in exactly 8 of the 24 possible preference profiles. (One is illustrated for the example in ftn. 1 by the allocation (14, 23), which for each player is neither EF nor EC.) For any even number of items larger than four, every preference profile admits at least one (EP, EP) allocation. To show this, we start with the following theorem:

Theorem 1: *A complete balanced allocation in which a player receives both her top- and bottom-ranked items, or neither her top- nor bottom-ranked item, is EP.*

Proof: Suppose A's allocation is L_A and B's is L_B, and that L_A contains A's bottom-ranked item. Then L_A cannot cover L_B for A, because A's bottom-ranked item cannot be preferred to any item in L_B, so L_A is not EF for A. Similarly, if L_A contains A's top-ranked item, then L_B cannot cover L_A for A, because nothing in L_B can be preferred by A to that top-ranked item, so L_A is not EC for A. Therefore, L_A is EP for A.

Now suppose that L_A does not contain A's top-ranked item. Then L_B must contain that item, which implies that L_A cannot cover L_B for A, so L_A is not EF for A. Similarly, if L_A does not contain A's bottom-ranked item, then L_B must contain that item. Hence, L_B cannot cover L_A for A, so L_A is not EC for A. Again, L_A must be EP for A. ■

Theorem 2: *Assume that* $|S| = 2n > 4$. *Then every preference profile allows for a complete balanced (EP, EP) allocation.*

Proof: Assume an allocation of S in which A receives her top and bottom-ranked items. Then, from Theorem 1, the set of items allocated to A is EP. Depending on A's preference profile, the allocated set of items to B falls into exactly one of the following four categories:

(i) A's and B's top-ranked items are the same, and their bottom-ranked items are the same. Then B receives neither her top nor her bottom-ranked item. By Theorem 1, the allocation is (EP, EP).

(ii) A and B have the same top-ranked item but a different bottom-ranked item. Assign to A, in addition to her top- and bottom-ranked items, the item which is bottom-ranked by B. Then B receives neither her top- nor bottom-ranked item so, from Theorem 1, the final allocation is (EP, EP).

(iii) A and B have the same bottom-ranked item but a different top-ranked item. Assign to A, in addition to her top- and bottom-ranked items, the item which is top-ranked by B. Then B receives neither her top- nor bottom-ranked item, so again the final allocation must be (EP, EP).

(iv) A and B have different top-ranked items and different bottom-ranked items. If all four of these items are different, assign to B her top- and bottom-ranked items. Then, by Theorem 1, the final allocation is (EP, EP). If exactly one of A's two extreme (i.e., top- or bottom-ranked) items is identical to one of B's, give the item that is different (and therefore not yet assigned) to A. Then B receives neither her top- nor bottom-ranked items and, from Theorem 1, the final allocation is (EP, EP). Finally, if each player's top item is the other's bottom item, then B receives neither her top- nor bottom-ranked item. From Theorem 1, the allocation is (EP, EP).

Categories (ii), (iii), and possibly (iv), require that A's allocation include at least 3 items, which is feasible because $2n > 4$. The proof is now complete. ■

In general, EP allocations are common. It follows from the Chung-Feller Theorem (Chung and Feller, 1949) that the probability that a randomly chosen allocation is EF is $1/_{n+1}$, which also equals the probability that it is EC. Thus, the probability that a randomly chosen allocation is EP is $\frac{n-1}{n+1}$.

In a preference profile in which both players' rankings are the same, any complete balanced allocation that is EF for one player is EC for the other. However, Theorem 2 guarantees that an (EP, EP) allocation exists if $2n > 4$. In fact, such an allocation exists if $2n = 4$, namely (14, 23).

If the best one can do is an EP allocation, how can items be allocated fairly? We consider two general strategies: Either allocate only some of the items, subject to the condition that this partial allocation is EF, or allocate all the items and try to minimize the "distance" from EF. The latter strategy may allow a player who does not receive an EF allocation to be compensated in some way, though our model does not specify how.

Although our main objective is to study EP allocations when there is no complete EF allocation, we begin with what is known about EF allocations.

4. Envy-Free (EF) Allocations

The following material, adapted from Brams, Kilgour, and Klamler (2014), shows how to determine whether a particular pair of preference rankings admits a complete balanced EF allocation. We begin by defining some conditions that may or may not be satisfied by a preference profile.

Assume $1 \leq k \leq |S|$. Denote by $T_A(S, k) \subseteq S$ the set containing A's k most-preferred items in S, which of course are the first k items in A's preference ranking. Because by convention A's preference ranking is

A: 1 2 3 … $|S|$

it follows that $T_A(S, k) = \{1, 2, \ldots, k\}$. $T_A(S, k)$ is called A's *top k set* in S. Similarly, let $T_B(S, k)$ denote B's *top k set* in S. In Example 1, $T_A(S, 3) = \{1, 2, 3\}$ and $T_B(S, 3) = \{2, 3, 4\}$.

We now consider some conditions that top k sets may satisfy.

Condition $C(S, k)$: $T_A(S, k) = T_B(S, k)$.

Condition $C(S, k)$ holds if and only if A's and B's top k items in S—considered as sets—are identical. Note that, although the items are the same, A's and B's rankings may be different. If Condition $C(S, k)$ holds, then the common set of top k items is denoted $T(S, k)$.

Condition $D(S)$: *Condition $C(S, k)$ fails for all odd values of k satisfying* $1 \leq k \leq |S|$.

When $|S| = 2n$, Condition $D(S)$ fails if and only if there is some odd value of k satisfying $1 \leq k < 2n$ and $T_A(S, k) = T_B(S, k)$. Note that it is always true that $S = T_A(S, |S|) = T_B(S, |S|)$. Therefore, Condition $D(S)$ fails when $|S| = 2n + 1$, because Condition $C(S, k)$ is true when $k = |S|$.

Theorem 3 (Brams, Kilgour, and Klamler, 2014). *A preference profile on a finite set of items, S, admits a complete balanced EF allocation if and only if it satisfies Condition* $D(S)$.

In a balanced allocation, the number of items to be allocated must be even. Therefore, there cannot be a complete balanced allocation of S if $|S| = 2n + 1$. This case is included in Theorem 3 because, as noted above, Condition $D(S)$ fails whenever $|S|$ is odd.

If $|S|$ is even and Condition $D(S)$ holds, then we know from Theorem 3 that a complete balanced EF allocation must exist. Several algorithms have been proposed to find such allocations, which may also satisfy other properties (see Kilgour and Vetschera (2018) and Klamler (2020)). But in many real-life allocation settings, there is no complete balanced EF allocation. What then can be said?

In this paper, we show how to find both maximal partial EF allocations and complete allocations that are as close as possible to being EF. We begin by studying the consequences of the failure of Condition $D(S)$ when, by Theorem 3, no complete balanced EF allocation exists.

5. Utility Representation

We obtain a different understanding of the consequences of the success or failure of Condition $D(S)$ by expressing A's and B's preferences using utilities for items that are additive on subsets.

Note that our utilities for a player will always be consistent with that player's preference ranking of the items. Of course, a player's preferences on subsets of items can be consistent with a given ordering even if they cannot be expressed using utilities, so our assumption of a preference ranking for items does not imply the existence of a utility.

Assume that A's *utility* for item i is x_i, where $1 \geq x_1 \geq x_2 \geq \ldots \geq x_{|S|} \geq 0$; we say that A's *utility vector* is $x = (x_1, x_2, \ldots, x_{|S|})$. Let X denote the set of all possible utility vectors. We also assume that B has a utility vector, but in B's case the ordering of the utilities must match B's ordering of the items of S.

We assume that utilities are additive, so that A's *utility* for subset $L_A \subseteq S$ is

$$U_x(L_A) = U(L_A) = \sum_{i \in L_A} x_i$$

To be explicit, A prefers $L \subseteq S$ to $L' \subseteq S$, or is indifferent, if and only if $U_x(L) \geq U_x(L')$.

For any allocation (L_A, L_B), A can evaluate A's own subset as well as the subset assigned to B. This motivates us to define A's *envy* to be

$$E_A = E_A(x) = U_x(L_B) - U_x(L_A)$$

Thus, E_A measures how much net utility A would gain if the allocations L_A and L_B were reversed. Note that E_A depends on (L_A, L_B), as well as on A's utility vector, x. The allocations that we consider may be incomplete (which occurs when $L_A \cup L_B \neq S$). But they must be non-trivial in the sense that there exists some utility vector x such that $U_x(L_A) \neq U_x(L_B)$.

For now, fix the allocation (L_A, L_B). For $1 \leq h \leq 2n$, define

$$c^h = (\underbrace{1, 1, \ldots, 1}_{h}, \underbrace{0, 0, \ldots, 0}_{|S| - h})$$

Thus, c^h is the utility vector in which the first h items have utility 1 and the last $|S| - h$ items have utility 0. It follows that

$$E_A(c^h) = |\{i \in L_B : i \leq h\}| - |\{i \in L_A : i \leq h\}|$$

Thus, $E_A(c^h)$ equals the number of the first h items that lie in L_B minus the number of those items that lie in L_A.

We now define A's *index vector* for each item in S. The h^{th} component of this $2n$-vector is $I_A(h) = -E_A(c^h)$, which of course equals the number of the first h items (including possibly item h itself) that are in L_A minus the number of those items that are in L_B. For instance, consider again

Example 1, $|S| = 2n = 4$.

A: $\underline{1}$ 2 3 $\underline{4}$
B: $\underline{2}$ $\underline{3}$ 4 1

A's index vector for the sets of underscored items, i.e., $L_A = \{1, 4\}$ and $L_B = \{2, 3\}$—usually written $(L_A, L_B) = (14, 23)$—is $I_A = (1, 0, -1, 0)$. Similarly, B's index vector is (1, 2, 1, 0).

Theorem 4: Let $|S| = 2n$ *and consider a balanced allocation* (L_A, L_B). *There exists* $x \in X$ *such that* $E_A(x) > 0$ *if and only if, for some h,* $I_A(h)$ *is negative.*

Comment: Thus, for $n = 4$ and $(L_A, L_B) = (14, 23)$, the fact that $I_A(3) = -1$ signals that it is possible for A to envy B.

Proof: For $1 \le h \le 2n$, define $a_h = \begin{cases} -1 \text{ if } h \in L_A \\ 1 \text{ if } h \in L_B \end{cases}$. Then $E_A(x) = \sum_{h=1}^{2n} a_h x_h$. Now define $y_{2n} = x_{2n}$ and, for $h = 2n - 1, 2n - 2, \ldots, 1$, set $y_h = x_h - x_{h+1}$. Note that $y_h \ge 0$ and that $x_h = y_h + y_{h+1} + \ldots + y_{2n}$. Therefore, by reversing the order of summation,

$$E_A(x) = \sum_{h=1}^{2n} a_h x_h = \sum_{h=1}^{2n} a_h \sum_{k=h}^{2n} y_k = \sum_{k=1}^{2n} y_k \sum_{h=1}^{k} a_h = -\sum_{k=1}^{2n} I_A(k) y_k$$

because $\sum_{h=1}^{k} a_h = -I_A(k)$. Now since $E_A(x)$ is positive, it must be the case that $\sum_{k=1}^{2n} I_A(k) y_k$ is negative. But $y_1, y_2, \ldots, y_{2n}$ are all non-negative. Therefore, for some value of k, $I_A(k)$ must be negative. ■

Corollary 1: *If* (L_A, L_B) *is EF for A, then* $I_A(h) \ge 0$ *for all* $h = 1, 2, \ldots, 2n$.

Proof: If (L_A, L_B) is EF for A, then L_A is a cover for L_B, which means that, for any $y \in L_B$ there exists $f_A(y) \in L_A$ such that A prefers $f_A(y)$ to y. It follows that, for every h,

$$I_A(h) = |\{x \in L_A: x \le h\}| - |\{y \in L_B: y \le h\}| \ge 0. \quad ■$$

Theorem 5: *There exists* $x \in X$ *such that* $E_A(x) < 0$ *if and only if, for some h,* $I_A(h)$ *is positive.*

Proof: Similar to Theorem 4. ■

For an allocation (L_A, L_B), recall that L_A is EC (envy-certain) for A if L_B is EF for A.

Corollary 2: *If* (L_A, L_B) *is EC for A, then* $I_A(h) \le 0$ *for all* $h = 1, 2, \ldots, 2n$.

Proof: Similar to Corollary 1.

We have shown that, if a non-trivial allocation (L_A, L_B) is EF for A, and A's preferences are given by a utility, then $\overline{E_A} = \max_{x \in X} E_A(x) \le 0$. Similarly, if a non-trivial allocation (L_A, L_B) is EC for A, then $\underline{E_A} = \min_{x \in X} E_A(x) \ge 0$. Making use of the fact that $\underline{E_A} < \overline{E_A}$, we can also see that an allocation is EP iff $\underline{E_A} < 0 < \overline{E_A}$. It follows from Theorems 4 and 5 that, if an allocation is EP for a player whose preferences are given by a utility, then that player's index vector must contain both negative and positive values. Table 2 shows the index vectors for both A and B for all six possible allocations in Example 1.

Allocation	Index for A	Index for B	Allocation	Index for A	Index for B
1. (12, 34)	(1,2,1,0)	(–1, 0, 1, 0)	4. (34, 12)	(–1, –2, –1, 0)	(1, 0, –1, 0)
2. (13, 24)	(1, 0, 1, 0)	(1, 0, 1, 0)	5. (24, 13)	(–1, 0, –1, 0)	(–1, 0, –1, 0)
3. (14, 23)	(1, 0, –1, 0)	(1, 2, 1, 0)	6. (23, 14)	(–1, 0, 1, 0)	(–1, –2, –1, 0)

Table 2. Index values for all possible allocations for Example 1, where A: 1 2 3 4 and B: 2 3 4 1

Next, we ask how many items must be unallocated if there is to be a balanced allocation of the remaining items that is EF for both players. The answer defines a maximal balanced partial EF allocation.

6. Maximal Balanced Partial EF Allocations

A *partial* allocation of S is an allocation (L_A, L_B) satisfying $S - (L_A \cup L_B) \neq \emptyset$; thus, some items in S are not allocated in (L_A, L_B). Our aim is to identify balanced partial allocations that are EF and maximal in the sense that no balanced partial EF allocation allocates more items. For now, we make no assumption about the parity of $|S|$; it may be even or odd.

First we prove a result that parallels Theorem 3.

Theorem 6: *Suppose that $|S|$ is odd and that Condition C(S, k) fails for all odd values of k satisfying $k < |S|$. Then a maximal balanced partial EF allocation of S allocates all except one item of S.*

Proof: Because $|S|$ is odd, a balanced allocation of S can allocate at most $|S| - 1$ items. We show that a balanced partial EF allocation exists that allocates $|S| - 1$ items.

In the proof of Theorem 3 in Brams, Kilgour, and Klamler (2014), consider the situation in stage t, when A and B have already received t items. If the algorithm AL cannot allocate any more items, then A's and B's top ($2t + 1$) items must be identical. By assumption, the smallest odd number for which Condition $C(S, k)$ holds is $|S|$. It follows that AL can find a balanced allocation of $|S| - 1$ items that is EF. ■

Now we allow $|S|$ to be even or odd, and suppose that, for a given preference profile on S, Condition $D(S)$ fails. Then there must exist an odd value of k such that $1 \leq k \leq |S|$ and $T_A(S, k) = T_B(S, k)$. Let $k = k_1$ be the smallest odd solution of the equation $T_A(S, k) = T_B(S, k)$.

We now define a sequence of values of k, denoted $(k_1, k_2, \ldots, k_h)$. There are $h \geq 1$ entries in this sequence; the last entry is k_h. We call this sequence the *k-sequence* of the preference profile.

For convenience, we define $S_0 = S$ and $k_0 = 0$. Consider first the set $S_1 = S_0 - T(S_0, k_1)$, the subset of S containing the last $|S_1| = |S| - k_1$ items in both A's and B's original preference rankings. (Note that A's and B's preference rankings on S_1 are simply the last $|S| - k_1$ items in their respective original rankings.)

If $k_1 = |S|$, or if Condition $D(S_1)$ is satisfied, set $h = 1$; the k-sequence is (k_1), i.e., it contains only one entry. Otherwise, there is some odd value of k satisfying $1 \leq k \leq |S_1|$ such that $T_A(S_1, k) = T_B(S_1, k)$. Let k_2 be the smallest odd value of k such that $T_A(S_1, k_2) = T_B(S_1, k_2)$, i.e., $C(S_1, k_2)$ holds. Now define $S_2 = S_1 - T(S_1, k_2)$. Note that $|S_2| = |S_1| - k_2 = |S| - k_1 - k_2$ items.

Assume now that $k_1, k_2, \ldots, k_r$ and $S_1, S_2, \ldots, S_r$ have been defined, and note that $|S_r| = |S| - k_1 - k_2 - \ldots - k_r$. Set $h = r$ if either $\sum_{j=1}^{r} k_j = |S|$ or $S_r \neq \emptyset$ but $D(S_r)$ is satisfied. Otherwise, we must have $\sum_{j=1}^{r} k_j < |S|$, $S_r \neq \emptyset$ and $D(S_r)$ fails. Then define $k = k_{r+1}$ to be the smallest odd value of k satisfying $1 \leq k \leq |S_r|$ and $T_A(S_r, k) = T_B(S_r, k)$. Set $S_{r+1} = S_r - T(S_r, k_{r+1})$. This process must end after a finite number of steps since $k_r \geq 1$ for all r. When it ends at, say, $r = h$, exactly one of the following statements is true for the k-sequence $(k_1, k_2, \ldots, k_h)$:

$$\sum_{r=1}^{h} k_r = |S| \text{ and } S_h = \emptyset, \text{ or}$$

$$\sum_{r=1}^{h} k_r < |S|,\ S_h \neq \emptyset, \text{ and } D(S_h) \text{ holds. } \blacksquare$$

Example 2, $|S| = 4$.

A: 1 2 3 4
B: 2 3 1 4

Note that $|S|$ is even, but $D(S)$ fails because $T_A(S, 3) = T_B(S, 3)$. Therefore, $k_1 = 3$ and $S_1 = \{4\}$. Obviously, A's and B's preferences on S_1 are the same, so $D(S_1)$ fails and $T_A(S_1, 1) = T_B(S_1, 1)$. We therefore define $k_2 = 1$. Because $k_1 + k_2 = |S|$, the process is complete, $h = 2$, and the k-sequence of Example 2 is (3, 1).

Essentially, the k-sequence divides the pair of preference orderings into odd-length blocks containing common items. In Example 2, the first block is {1, 2, 3}, of length $k_1 = 3$, and the second block is {4}, of length $k_2 = 1$.

For any preference profile on S such that $D(S)$ fails, identification of the k-sequence makes it relatively easy to find all maximal EF allocations, which must of course be partial.

Lemma 1: *Let W_1 and W_2 be non-empty disjoint sets of items such that both players, A and B, prefer any item in W_1 to every item in W_2. In any balanced EF allocation of $W_1 \cup W_2$, each of A and B must receive equal numbers of items from W_1 and from W_2.*

Proof: The Lemma follows because, otherwise, the player who receives more items from W_2 must envy the opponent, who receives more items from W_1. ■

In particular, the Lemma applies to $W_1 = S - S_1$, $W_2 = S_1$. Extending it further leads to the following theorem, which applies whether $|S|$ is even or odd.

Theorem 7: *If Condition D(S) fails, then a maximal balanced partial EF allocation of S is the union of balanced EF allocations of* $T(S_{r-1}, k_r)$ *for* $r = 1, 2, \ldots, h$, *and, if* $S_h \neq \emptyset$, *a complete balanced EF allocation of* S_h.

Proof: It follows from Theorem 6 that, for each $r = 1, 2, \ldots, h$, a maximal balanced partial EF allocation of $T(S_{r-1}, k_r)$ is an allocation of $k_r - 1$ items. It follows from Theorem 3 that there is a complete balanced EF allocation of S_h whenever $S_h \neq \emptyset$. The theorem follows from Lemma 1, because each player prefers every item in $W_1 = T(S_{j-1}, k_j)$ to any item in $W_2 = T(S_j, k_{j+1})$ whenever $1 \leq j < r \leq h$, and every item in $W_1 = T(S_{h-1}, k_h)$ to any item in $W_2 = S_h$. ■

Example 3, $|S| = 8$.

A: 1 2 3 4 5 6 7 8
B: 2 3 1 4 7 5 6 8

The rankings can be divided into odd-length blocks as follows:

A: 1 2 3 | 4 | 5 6 7 | 8
B: 2 3 1 | 4 | 7 5 6 | 8

Thus, $k_1 = 3$, $S_1 = \{4, 5, 6, 7, 8\}$, $k_2 = 1$, $S_2 = \{5, 6, 7, 8\}$, $k_3 = 3$, $S_3 = \{8\}$, $k_4 = 1$, $S_4 = \emptyset$, so $h = 4$ and the k-sequence is (3, 1, 3, 1). Therefore, a maximal balanced EF partial allocation allocates $k_1 - 1 = 2$ items from $T(S_0, k_1) = \{1, 2, 3\}$, plus $k_2 - 1 = 0$ items from $T(S_1, k_2) = \{4\}$, plus $k_3 - 1 = 2$ items from $T(S_2, k_3) = \{5, 6, 7\}$, plus $k_4 - 1 = 0$ items from $T(S_3, k_4) = \{8\}$. Note that $k_1 + k_2 + k_3 + k_4 = |S|$, and that $T(S_0, k_1) \cup T(S_1, k_2) \cup T(S_2, k_3) \cup T(S_3, k_4) = S$. All fair-division algorithms that we know of give (1, 2) as a PO-EF allocation from $T(S_0, k_1) = \{1, 2, 3\}$ and (5, 7) as a PO-EF allocation from $T(S_2, k_3) = \{5, 6, 7\}$; therefore, the maximal balanced partial EF allocation found by Theorem 3 is (15, 27).

The index vectors for (15, 27) are $I_A = (1, 0, 0, 0, 1, 1, 0, 0)$ and $I_B = (1, 0, 0, 0, 1, 0, 0, 0)$. As expected for EF allocations, every index component is non-negative.

Example 4, $|S| = 7$.

A: 1 2 3 4 5 6 7
B: 2 3 1 7 4 5 6

In this case, the first odd-length block contains {1, 2, 3}, and there are no subsequent odd-length blocks. Thus, $k_1 = 3$, $T(S_0, k_1) = \{1, 2, 3\}$, and $S_1 = \{4, 5, 6, 7\}$. Because $D(S_1, n - k_1) = D(S_1, 4)$ is true, it follows that $h = 1$. Therefore, the k-sequence of Example 4 is (3). Theorem 7 shows that a maximal partial EF allocation consists of an EF allocation of $k_1 - 1 = 2$ items from $T(S_0, k_1) = \{1,$

2, 3}, plus an EF allocation of all $n - k_1 = 4$ items of S_1. All fair-division algorithms that we know of give (1, 2) as the PO-EF allocation from $T(S_0, k_1) = \{1, 2, 3\}$, and (46, 75) as the PO-EF allocation from S_1. Thus, the balanced partial EF allocation found by Theorem 3 is (146, 275). Note that the index vectors are $I_A = (1, 0, 0, 1, 0, 1, 0)$ and $I_B = (1, 1, 0, 1, 0, 1, 0)$; the fact that these allocations are EF is confirmed, because every index component is non-negative.

Corollary 3: *In a maximal balanced EF partial allocation,* $\sum_{r=1}^{h}(k_r - 1) + |S_h| = |S| - h$ *items are allocated.*

Proof: From an odd-length block of length k_r, there is an EF allocation of $k_r - 1$ items. There is always an EF allocation of all items in S_h. Thus, the total number of items not allocated in an EF partial allocation equals h, the number of odd-length blocks. ■

Example 3, again: $|S| = 8$. A maximal balanced EF partial allocation is (15, 27). Note that $h = 4$, and that $|S| - h = 4$ items are allocated.

Example 4, again: $|S| = 7$. A maximal balanced EF partial allocation is (146, 275). Note that $h =$ 1, and $|S| - h = 6$ items are allocated.

We now proceed to discuss Pareto-optimality. In this context, an allocation is PO if there is no other allocation, in which each player receives the same number of items, that is preferred by both players. A maximal balanced partial EF allocation determined by Theorem 7 is rarely PO. In Example 3, the maximal balanced EF partial allocation, (15, 27), is Pareto-inferior to the balanced partial allocations (13, 27) and (15, 23). In Example 4, the maximal balanced EF partial allocation (146, 275) is Pareto-inferior to the balanced partial allocations (134, 275) and (146, 237).

Theorem 8: *If Condition D(S) fails, then a maximal balanced partial EF allocation of S is Pareto-optimal if and only if* $k_2 = k_3 = \ldots = k_h = 1$ *and* $\sum_{r=1}^{h} k_r = |S|$.

Proof: If $k_r > 1$ for some $r > 1$, or if $S_h \neq \emptyset$, then a maximal balanced partial EF allocation contains at least one item not in $T(S, k_1)$. Because one item in $T(S, k_1)$ does not appear in the maximal balanced EF allocation, that item can be substituted for any item not in $T(S_0, k_1)$, creating a Pareto-improvement. ■

Thus, a maximal balanced partial EF allocation, as found in Theorem 7, cannot be PO unless it contains only items from $T(S, k_1)$, the first top k set. This can occur if and only if all k-values after the first equal 1 and the set S_h is empty.

From a societal point of view, it is not clear that a Pareto-inferior EF maximal partial allocation is preferable to a partial allocation (with the same number of items) that weakly Pareto-dominates it. Take the EF partial allocation (15, 27) in Example 3. A prefers the PO allocation (13, 27), and B prefers the PO allocation (15, 23), where the player who does not benefit from a PO allocation receives exactly the same items as in the EF allocation. Is it better to choose the PO or the EF allocation? And if the choice is in favor of PO, which player should benefit?

Curiously enough, such questions do not arise for the eight *complete* allocations that the algorithm SA (Brams, Kilgour, and Klamler, 2015) gives, one of which, for example, is (1356, 2478). All eight of these allocations are supersets of the maximal partial EF allocation (15, 27), and all are PO, according to the necessary and sufficient condition given in Brams and King (2005). To be sure, these complete allocations are not EF; however, if complete EF allocations were to exist, at least one would be PO (Brams, Kilgour, and Klamler, 2015). Note that the eight SA allocations all include the maximal partial EF allocation, so they are *as EF as possible.* But is there a way of distinguishing them further?

7. Almost-EF Allocations

We now assume that adding an extra item to an allocation makes it preferable, for example because all items are goods (as opposed to "bads," like chores). In this way, we consider allocations that are not necessarily balanced. First, we observe that our definition of EF cannot be applied directly, even when it is appropriate, because it assumes balanced allocations.

Example 5: $|S| = 3$.

A: $\underline{1}$ 2 $\underline{3}$
B: $\underline{2}$ 3 1

Consider allocation (13, 2). This allocation is EF for A, because $L_A = \{1, 3\}$ provides A with a preferential cover for $L_B = \{2\}$ in that A prefers 1 to 2. But the allocation is not EF for B, because B has no preferential cover for $L_A = \{1, 3\}$ since there are no injections from L_A to L_B. Note that the index vectors for (13, 2) are $I_A = (1, 0, 1)$ and $I_B = (1, 0, -1)$, showing that this allocation is EF for A and EP for B.

If (L_A, L_B) is EF for A, then A has a cover for L_B; in Example 5, $\{1\}$ covers $\{2\}$. In general, if $|L_B| = k$, then A's cover for L_B can always be taken to be $T_A(L_A, k)$, A's k most preferred items in A's allocation.

Although (13, 2) is not EF for B in Example 5, it is almost-EF in the sense that the only reason it is not EF is that, while B can cover $\{1\}$, A's best item, B has no cover for $\{3\}$, A's "extra" item. Our new definition of almost-EF allows for such extra items that cannot be covered.[2]

Definition: An allocation (L_A, L_B) is *item-wise EF for* A *up to* $Z_B \subseteq L_B$ if and only if some top k set of L_A covers $L_B - Z_B$, i.e., for some k, there exists an injection f_A: $(L_B - Z_B) \rightarrow T_A(L_A, k)$ such that, for each item $y \in L_B - Z_B$, A prefers $f_A(y)$ to y. (Note that $k = |L_B - Z_B|$.)

In Example 5, we have that (13, 2) is EF for B up to $\{3\}$, because $T_B(L_B, 1) = \{2\}$ covers $L_A - \{3\} = \{1\}$. Also, we can say that (13, 2) is EF for B up to $\{1\}$, because $\{2\}$ covers $L_A - \{1\} =$

[2] The definition to follow can be compared to the definition of "EF up to 1 item, or EF1" in the literature (Budish, 2011). In our context, where the only available information about preferences over subsets is what can be inferred from the rankings of items, it is a natural analog and extension. For a different approach to almost-EF, see Bilò *et al.* (2018).

{3}. Noting that B prefers item 3 to item 1, our convention is to use the second description, which indicates how close this almost-EF allocation is to EF for the player in question, B.

Example 1, again: $|S| = 4$.

A: 1 2 3 4
B: 2 3 4 1

In Table 1, the third allocation listed is (14, 23), which is EP for A and EF for B. In terms of our new definition, it is EF up to {3} for A and, of course, EF for B. Note that the index vectors are $I_A = (1, 0, -1, 0)$ and $I_B = (1, 2, 1, 0)$. In particular, if {3} is removed from B, then A's index vector becomes (1, 0, 0, 1).

Example 4, again: $|S| = 7$.

A: 1 2 3 4 5 6 7
B: 2 3 1 7 4 5 6

The allocation (1346, 275) is EF for A and EF up to {1} for B. Note that the index vectors are $I_A = (1, 0, 1, 2, 1, 2, 1)$ and $I_B = (1, 0, -1, 0, -1, 0, -1)$. Removing from A item {1}, B's third-most-preferred item, changes B's index vector to (1, 0, 0, 1, 0, 1, 0).

An alternative allocation in this example is (1267, 345), which is EF up to {5} for A and EF up to {2, 7} for B, with index vectors $I_A = (1, 2, 1, 0, -1, 0, 1)$ and $I_B = (-1, 0, -1, -2, -1, 0, -1)$. As discussed below, the index vectors show this allocation is EF for A up to A's fifth-most preferred item and EF for B up to B's first- and fourth-most preferred items.

Example 3, again: $|S| = 8$.

A: 1 2 3 4 5 6 7 8
B: 2 3 1 4 7 5 6 8

The allocation (1356, 2478) is EF for A; for B, it is EF up to {1}, B's third-most preferred item. Index vectors are $I_A = (1, 0, 1, 0, 1, 2, 1, 0)$ and $I_B = (1, 0, -1, 0, 1, 0, -1, 0)$.

Convention: If an allocation (L_A, L_B) is EF for A up to $Z_B \subseteq L_B$, we say that the allocation is EF for A up to $|Z_B|$ items. When this statement is true for several different choices of Z_B, we choose Z_B to contain items that are as low-ranked as possible for A. This convention was introduced earlier in the discussion of Example 5. For a more complex example, consider the allocation (1267, 345) in Example 4. For B, it is EF up to two items, which could be {2, 1} or {2, 7}. We choose the latter, because B prefers item 1 to item 7. Thus, the allocation is EF for B up to B's first- and fourth-most preferred items. The convention serves to highlight the "shortest distance," in ordinal terms, between the actual allocation and EF.

To summarize, in an allocation (L_A, L_B) which is EF for, say, A up to one item, we say that the allocation is EF for A up to position q when q is the rank of the least-preferred item for player A that, if not assigned, would make the allocation EF for A. The same ideas can, of course, be extended to cases in which more than one item must be removed to make the allocation EF.

It is clear that a player prefers an allocation which is EF up to position q to an allocation which is EF up to position p if and only if $q > p$, i.e., the player prefers the allocation in which the items that must change status to guarantee EF are as low-ranked as possible. Almost-EF, therefore, provides a way to evaluate EP allocations in terms of their ordinal distance from EF.

In general, EF up to position $|S|$ is very close to EF, because only the lowest-ranked item cannot be covered, whereas EF up to position 1 is much less preferable, because the player's highest-ranked item is assigned to the other player. This descent from position 1 to position $|S|$ can be seen as a "continuous" process in which allocations move closer to EF.

Similarly, the almost-EF allocation that is closest to an EC allocation would be one in which only the lowest-ranked item can be covered. For example, if A's ranking is 1 2 3 4 5 6, then the allocation 345, which is EF for A up to positions 1 and 2, is the closest EP allocation to the allocation 346, which is EC.

The index vector provides both information about the number of items to be removed and about their positions, as can be seen in the following example:

Example 3, again: $|S| = 8$.   A: 1 2 3 4 5 6 7 8
B: 2 3 1 4 7 5 6 8

The allocation (1356, 2478) is EF for A, and EF up to position 3 (or EF up to {1}) for B. Index vectors are $I_A = (1, 0, 1, 0, 1, 2, 1, 0)$ and $I_B = (1, 0, -1, 0, 1, 0, -1, 0)$, with the first negative number in B's index vector appearing at position 3. Consider now the allocation (1345, 2768). The index vectors are $I_A = (1, 0, 1, 2, 3, 2, 1, 0)$ and $I_B = (1, 0, -1, -2, -1, -2, -1, 0)$. Hence, this allocation is EF for A, and EF for B up to {1, 4}, i.e., up to positions 3 and 4. The fact that the minimum of B's index vector is –2 indicates that two items must change status for B's allocation to be EF. Moreover, the entries at the 3rd and 4th positions are negative, and strictly less than any previous entry, indicating that the allocation is EF for B up to the 3rd and 4th items in B's ranking, or 1 and 4.

Surprisingly, for any preference profile there is an allocation that is EF for one player and either EF or EF up to one item for the other. As well, the allocation is PO.

Theorem 9: *Any preference profile on a set of items, S, with* $|S|$ even, *admits a complete allocation that is PO, EF for one player, and either EF or EF up to one item for the other.*

Proof: To construct a complete PO allocation that is EF for A and EF up to 1 item for B, we apply a greedy algorithm which assigns A's most preferred item to A, then B's most preferred available item to B, then A's most preferred available item to A, etc. The resulting allocation must be PO, because it results from a sincere sequence of choices by the players (Brams and King, 2005).

This allocation must be EF for A, because for each item, *y*, assigned to B, A has already received an item it prefers to *y,* guaranteeing that a mapping from L_B to L_A can be constructed so that A prefers its item to the item of B that is mapped to it.

If the first item assigned to A is removed from A's allocation, then B can construct a similar cover, because each item received by B is preferred by B to the item assigned to A in the subsequent step. Thus, B's allocation is EF up to the first item assigned to A. ■

The algorithm of Theorem 9 can sometimes produce an EF allocation—that is, the EF-up-to-1 item exception may not be needed.

Example 1, again: $|S| = 4$.

A: 1 2 3 4
B: 2 3 4 1

The allocation constructed according to Theorem 9, with A to start, is (13, 24), which is EF. The algorithm with B to start is (14, 23), which is EF for B and EF for A up to the third position,{3}.

But the greedy-algorithm procedure of Theorem 9 does not always give the best allocation, as demonstrated by

Example 6: $|S| = 8$.

A: 1 2 3 4 5 6 7 8
B: 4 1 2 3 6 7 8 5

If A starts, the Theorem 9 allocation is (1257, 4368) and B envies A. If B starts, the Theorem 9 allocation is (1358, 4267) and A envies B. But this example satisfies Condition $D(S)$, so there is an EF allocation. The EF allocation (1357, 4268) will be found by most of the usual allocation algorithms described in Kilgour and Vetschera (2018) and Klamler (2020), including SA.

Example 7: $|S| = 10$.

A: 1 2 3 4 5 6 7 8 9 a
B: 4 1 2 3 6 7 8 5 9 a

(Instead of 10, we write "a.") Here, C(S, 9) is true, so there is no EF allocation. If A starts, the greedy allocation is (12579, 4368a), which is EF for A and EF for B up to {2}, in the third position. If B starts, the greedy allocation is (1358a, 42679), which is EF for B and EF for A up to {7}, in the seventh position. But there are better allocations: SA and other algorithms give (13579, 4268a), which is EF for A and EF up to {9}, i.e., position 9, for B; and (1347a, 42689), which is EF for B and EF up to {9}, i.e., position 9, for A.

In Example 6, the greedy allocations are not EF, although an EF allocation exists. In Example 7, no EF allocation is available, but nonetheless the greedy allocations create more envy than is necessary.

We can now link maximal partial EF allocations with almost-EF allocations. Consider again

Example 3, again: $|S| = 8$.

A: 1 2 3 4 5 6 7 8

B: 2 3 1 4 7 5 6 8

First, $D(S)$ fails, so there is no EF allocation. In fact, the k-sequence is (3, 1, 3, 1), and the maximal partial EF allocation is (15, 27). Notice that the unallocated item in the first (i.e., k_1) block is item 3. In any almost-EF allocation that starts with (15, 27), one player will receive item 3, and the other will be envious, but the envy can be eliminated by the removal of item 3 or of some item in the first block less preferred by the player who does not receive item 3.

The greedy allocations provide good examples. Starting with B, the allocation is (1458, 2376), which is EF for B and EF for A up to item {3}. Starting with A, it is (1356, 2478), which is EF for A and EF for B up to item {1}, i.e., up to position 3. The fact that the exceptional item lies in the first block is the key to constructing an almost-EF allocation with minimum envy.

Theorem 10: *Consider a preference profile on a set, S, with $|S|$ even, such that D(S) fails. Then any complete balanced PO allocation is EF for some player up to an item in the first odd-length block, and possibly other items.*

Proof: Let $k = k_1$ be the smallest odd solution of the equation $T_A(S, k) = T_B(S, k)$ and define $S_1 = S - T(S, k_1)$. Then both players prefer every item in $W_1 = S - S_1$ to any item $W_2 = S_1$. Because both $|W_1|$ and $|W_2|$ are odd, in any complete balanced allocation one player, say A, must receive fewer items from W_1 than B. Items from the first block can be covered only by other items from the first block, so the items that A receives are not sufficient to cover what B receives from the first block. Therefore, for A the allocation cannot be better than EF up to some item in the first block. ■

With the restriction imposed by Theorem 10, we can determine the best available complete balanced almost-EF allocation – that is, the one closest to EF.

Theorem 11: *Consider a preference profile on a set, S, with $|S|$ even, such that D(S) fails, and let k_1 be the length of the first odd-length block. Then there is a complete balanced PO allocation that is EF for A and EF for B up to the k_1 item in B's ranking.*

Proof: Find a maximal balanced EF partial allocation, as in Theorem 7. Note that this partial allocation assigns all but one item from each odd-length block. To construct a complete PO allocation that is EF for A, assign the missing item from the first block to A, then the missing item from the second block to B, then alternate in assigning all remaining missing items. Note that the number of blocks, h, must be even because $\sum_{r=1}^{h} k_r + |S_h| = |S|$ is even, $|S_h|$ is even, and all values of k_r are odd. Therefore, the constructed allocation must be balanced.

This allocation must be EF for A, because the original items of A covered the original items of B, and every new item added for A covers the next new item added for B. Similarly, the constructed allocation must be EF for B except for the first item added for A, because every item added for B (except the last) covers the next item added for A.

To show that this allocation is EF for B up to the item in B's k_1 position, we first show that B cannot have been assigned this item in the maximal partial EF allocation, in which each player receives $(k_1 - 1)/2$ of the items in $T(S, k_1)$. If B were to receive its least preferred item in $T(S, k_1)$ in this allocation, then B would hold only $(k_1 - 1)/2 - 1 = (k_1 - 3)/2$ items that it could possibly rank higher than A's $(k_1 - 1)/2$ items. This is impossible, since the partial allocation is EF, so B's items must cover A's for B. By contradiction, B does not receive the item in its k_1 position in the maximal partial EF allocation.

When the final item from the first block is assigned to A, this coverage still applies. If the originally unassigned item is last in B's ordering, then B's items cover all except that last item assigned to A, as before. If the originally unassigned item is not last in B's ordering, then B's coverage of A's items can be adjusted so that only A's last item is uncovered because, otherwise, the item could have been assigned to B in the maximal partial EF allocation, which would have been a Pareto-improvement for B. Since the maximal partial EF allocation is known to be PO, this contradiction shows that the constructed complete balanced allocation is EF for B up to the k_1 item in B's ordering. ■

Whenever a complete balanced EF-PO allocation is impossible, Theorem 11 shows how close the best possible allocations – there are two of them – come to EF. They are EF except for the k_1 item of one player or the other. We illustrate with two examples.

Example 3, again: $|S| = 8$.

A: 1 2 3 4 5 6 7 8
B: 2 3 1 4 7 5 6 8

There are four blocks, and the k-sequence is (3, 1, 3, 1). The maximal partial EF allocation is (15. 27), so the unassigned items are 3, 4, 6, and 8. The two almost-EF allocations are (1356, 2478) which is EF for A and EF for B up to {1}, B's third item, and (1458, 2376), which is EF for B and EF for A up to {3}, A's third item.

In the previous example, both almost-EF allocations could have been found using greedy algorithms as in Theorem 9. But this is not always the case. In the following example, the allocations of Theorem 11 are strictly better than the greedy allocations.

Example 8: $|S| = 20$

A: 1 2 3 4 5 6 7 8 9 a b c d e f g h i j k
B: 4 1 2 3 5 7 8 9 6 a c d e b f j g h i k

It can be easily checked that there is no complete EF allocation; actually, the k-sequence is (5, 5, 5, 5), so that only 16 items are allocated in the unique maximal balanced EF partial allocation, (1368bdgi, 4279cejh). The allocation constructed in Theorem 11, with A to start, is (13568bdfgi, 4279acejhk). The greedy allocation of Theorem 9, with A to start, is (12569bdfgi, 4378acejhk); note that it does not contain the maximal partial EF allocation. Moreover, the Theorem 11 allocation is EF for A and EF for B up to {5}, B's fifth-ranked item, while the Theorem 9 allocation is EF for A and EF for B up to {2}, B's third-ranked item. The situation is similar for the allocations obtained when the algorithms start with B rather than A.

8. Conclusions

We assumed that (i) two players can strictly rank a set of items from best to worst and (ii) their preferences for subsets of items—and any additive utilities they may have for the items—are consistent with these rankings. If the number of items is even, we asked how to equally divide them between the players so that the allocation, insofar as possible, is envy-free (EF) and Pareto-optimal (PO).

Our notion of EF is item-wise: There is a one-to-one matching of B's items to A's such that A prefers each of its items to the item of B that is matched to it, and likewise for the matching of A's items to B's. It is known that there always exists an allocation that for both players is EF except for one item (Caragiannis *et al.*, 2019).

We also know that if there is a complete EF allocation that is balanced (each player receives half the items), at least one such allocation will be PO as well as EF (Brams, Kilgour, and Klamler, 2015). But in the usual situation, where there is no complete balanced EF allocation, there will be a maximal EF partial allocation, which may not be PO (in fact, this case will be common).

The items not assigned in the maximal partial allocation will be ranked the same by both players. In Brams, Kilgour, and Klamler (2012), we proposed the Undercut Procedure to divide identically-ranked items into two subsets such that different players prefer different subsets. But this notion of EF depends on players' subset preferences rather than the item-wise matchings that define EF, which constitute the basis for this paper. (For a discussion of fair division of indivisible items without balance, see Bouveret, Chevelayre, and Maudet, 2016.)

EF based on item-wise preferences produced three classes of complete balanced allocations: EF, envy-certain (EC), and envy-possible (EP), for each player. If there is no complete EF allocation, we showed that there is always a complete allocation that is EF for one player and EP for the other, for whom the allocation is EF except for one item. We showed how to measure the distance between the EP player's allocation and EF for that player. We called the resulting allocation Almost-EF and showed that it is PO, even though it may include a maximal partial allocation that is not PO among allocations with the same number of items.

The algorithm for building a complete balanced PO allocation that is EF for one player and gives the closest possible approximation to EF for the EP player is given by Theorem 11. Theorem 10 shows that no allocation can be better than the one so constructed. Thereby we have specified the gap between a complete balanced EF-PO allocation and, when there is no such allocation, its closest approximation. This approximation yields a complete balanced PO allocation that is as close to EF as possible.

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