Childcare Support and Public Capital in an Ultra-Declining Birthrate Society

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Abstract This paper analyzes whether public capital investment or childcare support maximizes the growth rate in an ultra-declining birth rate society using a labor-augmented model with public capital. We clarify the global stability of the private–public capital ratio in the steady state. In addition, we analyze the effect of increasing the expenditure share of all tax revenues on economic growth. Furthermore, we are interested in analyzing which policy boosts economic growth considering childcare support as an opportunity cost to raise children or price subsidy if we take having children as consuming nominal rather than capital goods. The results of this analysis show that an increased share of public capital investment leads to higher economic growth. This means that, if all tax revenue is allocated to public capital investment, the growth rate will be maximized. Furthermore, in the second case, the model is reconstructed such that the child is regarded as a nominal consumer good in the first period, and the childcare cost is regarded as a price. In that case, the impact of increased public capital on growth is minor compared to the former case.

Keywords: Public capital investment - Childcare support - Income tax - Economic growth

JEL classification: D91 - E62 - O41

1. Introduction

The number of children born in Japan continues to decrease. The total fertility rate was 1.36[2] in 2019, the lowest level to date, as indicated by the Japanese Ministry of Health, Labor and Welfare (MHLW). The Cabinet Office continues to insist that Japan has been in a state of declining birth rates for many years, resulting in what is referred to as an "ultra-declining birth rate society." The demographic trends are such that, by 2050, one in 2.5 people will be elderly (aged 65 or older).[3] Viewing life in the long term, workers should determine their spending based on their estimated lifetime income. According to the overlapping generations (OLG) model proposed by Diamond (1965), the lifetime income of an individual is assumed to consist of earnings received in two periods: working and later life. Individuals make decisions from a lifetime perspective while adhering to budgetary constraints. Becker (1981)

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2 "Current population survey," MHLW website (https://www.mhlw.go.jp/toukei/list/81-1a.html) (accessed on September 20, 2020)

3 "Situations of Aging" (Japanese), Cabinet Office website (https://www8.cao.go.jp/kourei/whitepaper/w-2012/zenbun/pdf/1s1s_1.pdf) (accessed on June 15, 2020)

and Becker and Lewis (1973) showed that the number of children in developed countries will decline; at first glance, this is seemingly a contradiction considering that children are positive to societies; however, it results from the fact that the cost of childcare is proportional in scale to its quantity multiplied by its quality. In this study, models are established based on a neoclassical theory that suggests that growth in capital boosts gross domestic product (GDP) and leads to a greater growth rate for the whole nation. The main portion of this study utilizes Romer's endogenous growth model (1986) to introduce the public capital models proposed by Barro (1990), Barro and Sala-i-Martin (1992), Futagami et al. (1993), Turnovsky (1997), Yakita (2008), and Maebayashi (2013). These models indicate that public capital stock boosts labor productivity. Investment in public capital is financed by levying income taxes (labor income and capital income). Yakita (2008) used a birth rate internalization model that considers two public expenditures: public capital investment and public capital maintenance. Maebayashi (2013) showed the dynamics of the private–public capital ratio and confirmed the existence of a steady state and global stability. Furthermore, the author analyzed the optimal allocation of tax revenue between expenditure on public capital investment and public pension subsidies under a pay-as-you-go pension system. The study concluded that the best policy for growth is to allocate all financial resources to public capital investment. However, from a social-welfare perspective, the optimal tax revenue allocation rate depends on the magnitude of the social discount rate.

In this study, we analyze the policy trade-off between public capital investment and childcare support and the effects on the growth rate under government budget constraints, where the government sources revenue only from income taxes on labor and capital. Furthermore, as an important point in this study, the child-rearing support policy should be subsidized for the direct opportunity cost to workers (we will call this "Case A."), or the child should be regarded as a normal consumer good rather than a capital good, and a subsidy policy on the price should be implemented. (We call this "Case B.") The point is that comparisons are made and explicitly derived the effect of policy on growth in these cases. In both cases, we first prove the existence of a steady state and confirm that the economy converges to the steady state globally and stably. We show that all variables -public capital, private capital, and GDP- grow at the same rate on the balanced growth path. Second, we analyze the effect of increasing the share of public capital investment on growth under constant tax revenue, and, using a numerical example, we find that this growth is positive. In addition, the elasticity of an increase in the relative share of the public capital investment ratio on private–public capital and the labor share of GDP is considered. In the first case, the sign of this elasticity is positive, suggesting that the additional increase in public capital pushes up private capital more than that increase. Clearly, this is driving economic growth. In the second case, the sign turned out to be negative, the absolute value of elasticity was less than one for a marginal increase in public capital investment, and the sign of the effect on the relative value was negative. This means that the effect of increasing the wage rate due to the increase in public capital does not contribute much to the increase in savings. The reason for this was very clear. First, it depends on the shape of the utility function, as shown in the linear logarithm. The use of this function means that savings depend only on income in the first period and not on the interest rate. In other words, in this change in interest rate, the substitution and income effects cancel each other out, and the effect on savings against changes in the interest rate becomes zero. The second thing to consider is that governmental childcare support measures do not contribute to an increase in the labor force. In this analysis, the public pension system and long-term care insurance system in the social security system are not considered, so there is no externality to the parent generation. Therefore, the incentive for parents to have children is related to them being considered consumer goods rather than capital goods from an economic point of view. We constructed a two-period OLG model using Diamond (1965), a two-period OLG model. We introduce public capital stock to construct a model that incorporates labor-augmented production technology. Therefore, whether or not to have children depends on the preference rate for children as general consumer goods and consumption in the second period—that is, how much deposit is required in the second period because there is no public pension system in this model. Additionally, whether to leave for the second term depends greatly on the preference rate.

The remainder of this paper is organized as follows. The next section presents the model and its dynamics in terms of private and public capital. The global stability of the dynamics in the steady state was confirmed. The effects of governmental increases in income tax and public capital investment share in the steady state are analyzed. The final section concludes the paper.

2. Model

Case A: The childcare cost is regarded as an opportunity cost.

2.1 Individuals

The two-period OLG model presented by Diamond (1965) with fully competitive markets is considered. A homogeneous individual is assumed, who obtains utility from consumption in the working and later periods and selects the number of children that they have. We consider a child to be a consumer good rather than a capital good, and there is no public pension. Individuals supply labor inelastic in only the first period, and it is assumed that every individual has one unit of labor to supply to the labor market. Individuals allocate income for consumption, saving, and childcare costs in the first period. The individual consumes all income, including savings and interest, in the first period, with no bequests in the second period. A logarithmic linear utility function and lifetime budget constraint, which must hold for the economy to be sustainable in the long term, are specified as follows:

(1)

$$max.\ u_t = logc_t + \rho logd_{t+1} + \varepsilon logn_t$$

Where c_t, d_{t+1} denote family consumption in the first and second periods, respectively, and n_t indicates the number of children and where the time preference factor for consumption in the second period and the utility of having children are denoted as $\rho \in (0,1)$, $\varepsilon > 0$. Budget constraint in the first period is as follows:

$$w_t\,(1-\tau)[1-n_t(z-h_t)] = c_t + s_t \tag{2}$$

Family supply their labor time obtained by subtracting the opportunity cost of child rearing from the one unit of labor that they hold, and the second period's constraint is shown as follows:

$$d_{t+1} = s_t r_{t+1}(1-\tau) \tag{3}$$

Where it indicates that they spend all capital income only on consumption in the second period. Therefore, from these equations, we obtain the lifetime budget constraint:

$$s.t\, w_t\,(1-\tau)[1-n_t(z-h_t)] = c_t + \frac{d_{t+1}}{r_{t+1}(1-\tau)} \tag{4}$$

Where childcare cost, childcare support, and income tax rate are denoted by $z \in (0,1)$, $h_t \in (0,1)$, $\tau \in [0,1]$, respectively. We assume here that childcare support is less than or equal to childcare cost $z > h_t$. The income tax rate indicates that both labor income tax and capital income tax are included. By maximizing the utility of equation (1) subject to the budget constraint in equation (4), we can obtain the first-order conditions as follows:

$$\frac{1}{c_t} = \frac{\rho r_{t+1}}{d_{t+1}} \tag{5}$$

This is a relation that indicates a trade-off between the consumption of the first and second periods. The next condition indicates whether individuals consume or have children in the first period.

$$\frac{1}{c_t} = \frac{\varepsilon}{w_t n_t(z-h_t)} \tag{6}$$

The last condition is whether the family decides to have children in the first period or make a consumption in second period as follows:

$$\frac{\rho r_{t+1}}{d_t} = \frac{\varepsilon}{w_t n_t (z - h_t)} \tag{7}$$

Therefore, we substitute these conditions for lifetime budget constraints and derive the optimal values as follows:

$$c_t^* = \frac{(1-\tau)w_t}{(1+\varepsilon+\rho)} \tag{8}$$

Where consumption in the first period depends only on consumable labor income, not capital income, because we use a logarithmic utility function—that is, since the effects of both substitution and income are offset, there is no effect of interest rate, which is indicated as the price.

$$d_{t+1}^* = \frac{\rho r_{t+1} w_t (1-\tau)^2}{(1+\varepsilon+\rho)} \tag{9}$$

The optimal consumption in the second period depends on both the consumable wage rate and interest rate. The optimal solution for the number of children is as follows:

$$n_t^* = \frac{\varepsilon}{(1+\varepsilon+\rho)(z-h_t)} \tag{10}$$

The point to note here is that it has nothing to do with the wage rate and depends only on the time used to raise children and parameters. The important optimal value of savings is as follows:

$$s_t^* = \frac{(\varepsilon+\rho)}{(1+\varepsilon+\rho)}(1-\tau)w_t \tag{11}$$

This value is significantly related to the derivation of the growth rate and is increasing function for consumable labor income, which depends on preferences regarding both having children and consumption in first period only.

2.2 Production

A Cobb-Douglas production technology, in which labor increases with public capital investment, as in Romer (1986), is used. It is assumed that there are many firms in the goods market, and these firms have access to the same technology. The inputs were private capital, public capital stock, and labor.

We consider that the depletion rates of both capitals are 1—that is, all capital investments are exhausted in one period. Taking company ii as an example, the production function is specified as follows:

$$Y_{it} = K_{it}^{\alpha}(A_t L_{it})^{1-\alpha} \tag{12}$$

This Cobb-Douglas-type production function indicates labor augmented with technology progress A. Here, the contents of A are explained as follows. Technological progress is shown as per capita public capital.

$$A_t = \frac{G_t}{L_t} \tag{13}$$

Next, I substitute this technology for equation (12) and obtain the following equation:

$$Y_t = K_t^{\alpha} {G_t}^{1-\alpha} = \left(\frac{K_t}{G_t}\right)^{\alpha} G_t = x_t^{\alpha} G_t \tag{14}$$

Since the labor market is perfectly competitive, it is always in equilibrium and shows its conditional equation. The labor force in period t is determined as follows:

$$L_t = N_t[1 - n_t(z - h_t)] \tag{15}$$

where N_t is the number of households in period t. We assume a perfectly competitive market and solve the profit maximization problem as follows:

$$(1-\alpha)\left(\frac{K_{it}}{L_{it}}\right)^{\alpha-1} {A_t}^{1-\alpha} = w_t \tag{16}$$

We define w_t as the wage rate in period t, and α is the share of private capital on gross domestic product (GDP), so $(1-\alpha)$ indicates the share of effective labor on GDP, and A_t is shown as technology progress G_t/L_t. Concretely, the government extends the road, the transformation industry will be streamlined, and it is clear that the development of information and communication networks increases labor productivity in all industries. Then, an interest rate in period t is shown as r_t.

$$\alpha\left(\frac{K_{it}}{L_{it}}\right)^{\alpha-1} {A_t}^{1-\alpha} = r_t \tag{17}$$

In equations (16) and (17), the private capital-labor ratio will become the same value as in $K_{ti}/L_{ti} = K_t/L_t$ because the market is perfectly competitive and there is an infinite number of homogeneous firms in the market. That is, we can say $\sum_{i=1}^{\infty} L_{it} = L_t$ and $\sum_{i=1}^{\infty} K_{it} = K_t$ where L_t and K_t denote the total labor supply and total private capital, respectively. By defining a new variable, $x = K/G$, to be the ratio of private and public capital, (16) can be rewritten as the following equation:

$$(1-\alpha)\left(\frac{K_t}{G_t}\right)^{\alpha}\frac{G_t}{L_t} = (1-\alpha)x_t^{\alpha}\frac{G_t}{L_t} = w_t \tag{18}$$

In the above equation, the wage rate equals the productivity of effective labor, which is shown as an increasing function of both the relative value of capital and per capita public capital. The interest rate shown in equation (17) can be written as follows:

$$\alpha\left(\frac{K_t}{G_t}\right)^{\alpha-1} = \alpha x_t^{\alpha-1} = r_t \tag{19}$$

Contrary to the wage rate, interest rate is indicated as a decreasing function of the relative value of capital. There are no profits for all firms because the market is competitive. Therefore, corporate taxes do not appear in the next section.

2.3 Government

The government taxes income and divides tax revenues between public capital investment, $E > 0$, and gross childcare support, $H > 0$. The share of spending on public capital investment and the income tax rate is denoted as $\varphi \in [0,1], \tau \in [0,1]$. The depreciation rates of public and private capital were both 1. Government budget constraint is given by the following equations:

$$E_t + H_t = \tau Y_t = \tau x_t^{\alpha} G_t \tag{20}$$

As in the formula above, all tax revenue on income is allocated to public capital investment and childcare support. Public capital investment is represented by the constant difference equation of public capital, as follows:

$$E_t = G_{t+1} - G_t = \varphi \tau Y_t = \varphi \tau x_t^{\alpha} G_t \tag{21}$$

An allocation for childcare capita on all tax revenue is as follows:

(22)

$$w_t h_t n_t N = (1-\varphi)\tau Y_t = (1-\varphi)\tau x_t^{\alpha} G_t$$

The per-capita childcare support is determined using (22) and is indicated by the following equation (the value of which will be constant):

$$h_t = \frac{(1-\varphi)(1-\varepsilon z)\tau}{\varepsilon[1-(1-\varphi)\tau]} \tag{23}$$

The equations (10) and (15), which indicate the labor force in period t, can be rewritten as follows:

$$L_t = N_t\left(\frac{1+\rho}{1+\varepsilon+\rho}\right) \tag{24}$$

This equation implies that the labor force in period t does not depend on the value of childcare support or on the share of government tax revenue allocated to childcare support expenditure. This clearly means that the government cannot intervene in childcare support as a policy. If late marriage is resolved, the preference for having children increases.

The labor force will continue to decline in the future. Next, we use equation (24) to derive an expression for labor growth:

$$g_L = \frac{L_{t+1}}{L_t} = \frac{N_{t+1}}{N_t} \tag{25}$$

where the number of households in period t+1 is denoted as $N_{t+1} = N_t n_t$, and the number of children is constant. Therefore, equation (25) can be rewritten in the following form:

$$g_L = \frac{L_{t+1}}{L_t} = \frac{n_t N_t}{N_t} = n_t \tag{26}$$

Which relates the growth in the labor force to the number of children.

3. Equilibrium

There are three markets, and we consider only the capital market using Walras' law. The equilibrium conditions were as follows:

$$s_t N_t = K_{t+1} \tag{27}$$

We substitute the optimal savings (6) for the equilibrium condition (27) and substitute for the wage rate (16). These allow us to rewrite condition (27) as follows:

$$K_{t+1} = \frac{(\varepsilon+\rho)}{(1+\varepsilon+\rho)}(1-\tau)w_t = \frac{(\varepsilon+\rho)}{(1+\varepsilon+\rho)}(1-\tau)(1-\alpha)x_t^{\alpha}N_t\frac{G_t}{L_t} \tag{28}$$

Equation (29) can be obtained by dividing both sides of equation (28) by K_t:

$$g_K = \frac{K_{t+1}}{K_t} = \frac{(\varepsilon+\rho)(1-\tau)(1-\alpha)}{(1+\rho)}x_t^{\alpha-1} \tag{29}$$

The dynamics of private–public capitals are obtained in the following section.

4. Dynamics

The dynamics of public capital are indicated by equation (30):

$$g_G = \frac{G_{t+1}}{G_t} = \varphi\alpha\tau x_t^{\alpha} + 1 \tag{30}$$

The growth of x is indicated by the following equation, which combines the capital dynamic equations (29) and (30).

$$g_x = \frac{x_{t+1}}{x_t} = \frac{\frac{K_{t+1}}{K_t}}{\frac{G_{t+1}}{G_t}} = \frac{(\varepsilon+\rho)(1-\tau)(1-\alpha)x_t^{\alpha-1}}{(\varphi\alpha\tau x_t^{\alpha}+1)(1+\rho)} \tag{31}$$

We will attempt to prove this stabilization using the total derivative of the above equation with respect to x_t and x_{t+1}.

$$\frac{dx_{t+1}}{dx_t} = \frac{A(x_t, x_{t+1})}{B(x_t\,)} = f(x_t, x_{t+1}) \tag{32}$$

$$A(x_t, x_{t+1}) = \alpha(\varepsilon+\rho)(1-\tau)(1-\alpha)x_t^{\alpha-1} - (1+\rho)\varphi\alpha^2\tau x_t^{\alpha-1}x_{t+1} \tag{33}$$

$$B(x_t\,) = (\varphi\alpha\tau x_t^{\alpha}+1)(1+\rho) > 0 \tag{34}$$

For the signs of “A” in equation (33) to be positive, the sign of equation (33) is not clear. Parameters in equation (33) are quantified concretely as $(\alpha, \varepsilon, \rho, \tau, x, z, \varphi) = (0.4, 0.7, 0.7, 0.3, 3, 0.06, 0.83)$. As a result, the sign becomes positive. When x_t approaches 0, the growth of x is zero in equation (31) $\left(\lim_{x_t \to 0} \frac{x_{t+1}}{x_t} = 0\right)$. In other words, the curve passed through the origin. Next, we derive the second derivative of equation (31):

$$\frac{d^2 x_{t+1}}{(dx_t)^2} = \frac{\partial f(x_t, x_{t+1})}{\partial x_t} = \frac{A' B - AB'}{B^2} \tag{35}$$

$$A' = \frac{\partial A(x_t, x_{t+1})}{\partial x_t} = -\alpha(\varepsilon + \rho)(1-\tau)(1-\alpha)^2 x_t^{\alpha-2} + (1-\alpha)(1+\rho)\varphi\alpha^2\tau x_t^{\alpha-2} x_{t+1} \tag{36}$$

$$B' = \frac{dB(x_t)}{dx_t} = \emptyset\varphi\alpha^2\tau x_t^{\alpha-1} \tag{37}$$

$$\lim_{x_t \to 0} \frac{dx_{t+1}}{dx_t} = \infty \text{ , } \lim_{x_t \to \infty} \frac{dx_{t+1}}{dx_t} = 0 \tag{38}$$

Equation (38) indicates that the curve in figure 1 intersects the 45° line. The private-public capital ratio increases, and the steady state of x is denoted by x^*. If equation (39) is satisfied with x^*, the growth rate of GDP, private capital, and public capital will be the same:

$$(\varepsilon + \rho)(1-\tau)(1-\alpha)x_t^{\alpha-1} = (\varphi\alpha\tau x_t^{\alpha} + 1)(1+\rho) \tag{39}$$

Proposition 1. There is a unique value that shows the public–private capital ratio in a steady state. If equation (39) is satisfied, public capital, private capital, and GDP will grow at the same rate—that is, the growth path is balanced and globally stable.

Next, we will try to analyze the effect on growth when the government increases the share of public capital investment. We differentiate equation (30) by share.

$$\frac{\partial g}{\partial \varphi} = \tau\alpha^2 (x^*)^{\alpha} \left[\frac{1}{\alpha} + \frac{\varphi}{x^*}\frac{dx^*}{d\varphi}\right] \tag{40}$$

$$\frac{dx^*}{d\varphi} = \frac{A}{B} < 0 \tag{41}$$

(42)

$$A = \varphi^2 \alpha \tau (x^*)^{\alpha} > 0$$

$$B = -[\varphi\, \alpha^2 \tau (x^*)^{\alpha-1} + (\varepsilon + \rho)(1-\tau)(1-\alpha)^2 (x^*)^{\alpha}] < 0 \tag{43}$$

where the second term in brackets indicates the elasticity of the share for the relative capital value, and the sign is negative- that is, an increase in the share of public capital investment reduces the magnitude of the private-public capital ratio. This suggests that the economy will grow regardless of the private capital share of GDP or the size of the share's elasticity.

Proposition 2. The economy will grow independently of the private capital share of GDP or the elasticity of the allocation rate to private and public capital.

$$\frac{\partial g}{\partial \varphi} = \tau \alpha^2 (x^*)^{\alpha} \times$$
$$\left\{ \frac{[\varphi\, \alpha^2 \tau (x^*)^{\alpha-1} + (\varepsilon+\rho)(1-\tau)(1-\alpha)^2 (x^*)^{\alpha}] - \varphi^2 \alpha^2 \tau (x^*)^{\alpha}}{[\varphi\, \alpha^2 \tau (x^*)^{\alpha-1} + (\varepsilon+\rho)(1-\tau)(1-\alpha)^2 (x^*)^{\alpha}]} \right\} > 0 \tag{44}$$

An increase in the share of public capital investment increases the wage rate. A policy in which all tax revenue is spent on public capital investment yields the best results in terms of growth.

Proposition 3. A policy in which all tax revenue is spent on public capital investment is the optimal policy in terms of growth.

5. Case B: The childcare cost is regarded as the price.

5.1 Households

Next, we consider the case where the children are goods and pay the childcare cost as a price, such as consumer goods. The function of utility is the same as in Case A, which shows the log-linear type, and equation (2), which shows that the budget constraint is rewritten as follows:

$$w_t\,(1-\tau) = c_t + n_t(z - h_t) + \frac{d_{t+1}}{r_{t+1}(1-\tau)} \tag{45}$$

We solve the problem and can find the optimal solution as follows.

(46)

$$c_t^* = \frac{1}{(1+\varepsilon+\rho)} w_t(1-\tau)$$

$$d_{t+1}^* = \frac{\rho}{(1+\varepsilon+\rho)} w_t r_{t+1}(1-\tau)^2 \tag{47}$$

$$n_t^* = \frac{\varepsilon}{(1-h_t)(1+\varepsilon+\rho)}(1-\tau)w_t \tag{48}$$

$$s_t^* = \frac{\rho}{(1+\varepsilon+\rho)}(1-\tau)w_t \tag{49}$$

5.2 Firms and government

The technology of firms can be drawn in the same way as in Case A, and the government budget constraint in Case A, indicated by equation (22), is rewritten as the following equation:

$$h_t n_t L_t = (1-\varphi)\tau Y_t = (1-\varphi)\tau x_t^{\alpha} G_t \tag{50}$$

This can lead to the childcare support scale in Case B, and we will compare these characteristics of the two equations. Let $\emptyset > 0$ be $(1+\varepsilon+\rho)$ to simplify the formula. Here, $\partial h_t/\partial\varphi < 0$, and childcare support does not depend on the relative scale of capital.

$$h_t(\varphi) = \frac{\tau(1-\varphi)\emptyset}{\tau(1-\varphi)\emptyset + \varepsilon(1-\tau)(1-\alpha)} \tag{51}$$

5.3 Equilibrium

We substitute the optimal number of children (48) and wage rate (18) for savings (49).

$$s_t^* = \frac{\rho}{(1+\varepsilon+\rho)}(1-\tau)(1-\alpha)x_t^{\alpha}\frac{G_t}{L_t} \tag{52}$$

Furthermore, we derive the dynamic equation of private capital using the capital market equilibrium condition.

$$g_K = \frac{K_{t+1}}{K_t} = \frac{\rho(1-\tau)(1-\alpha)}{(1+\varepsilon+\rho)} x_t^{\alpha-1} \tag{53}$$

5.4 Dynamics

The above equation and equation (30) can be combined to obtain the dynamic equation of capital, which is the relative capital value between private and public.

$$g_x = \frac{x_{t+1}}{x_t} = \frac{\frac{K_{t+1}}{K_t}}{\frac{G_{t+1}}{G_t}} = \frac{\rho(1-\tau)(1-\alpha)}{(\varphi\alpha\tau x_t^{\alpha}+1)(1+\varepsilon+\rho)} x_t^{\alpha-1} \tag{54}$$

Here, we will try $(1+\varepsilon+\rho)$ as $\emptyset > 0$ for simplicity, which allows me to rewrite the above equation as follows:

$$g_x = \frac{x_{t+1}}{x_t} = \frac{\frac{K_{t+1}}{K_t}}{\frac{G_{t+1}}{G_t}} = \frac{\rho(1-\tau)(1-\alpha)}{(\varphi\alpha\tau x_t^{\alpha}+1)\emptyset} x_t^{\alpha-1} \tag{55}$$

Next, we will see if this economy converges to a steady state globally. For that purpose, when illustrating the abovementioned dynamic equation of capital, we consider that the curve must trough the origin, then rise to the right, and finally have a concave shape with respect to the origin. We do the total derivative with respect to x_{t+1} and x_t in equation (55).

$$\frac{dx_{t+1}}{dx_t} = \frac{A(x_t, x_{t+1})}{B(x_t)} = f(x_t, x_{t+1}) = C \tag{56}$$

$$A(x_t, x_{t+1}) = [\alpha\rho(1-\tau)(1-\alpha)x_t^{\alpha-1} - \emptyset x_{t+1}\varphi\alpha^2\tau x_t^{\alpha-1}] \tag{57}$$

$$B(x_t) = (\varphi\alpha\tau x_t^{\alpha}+1)\emptyset \tag{58}$$

Where the sign of equation (58), which is the denominator of equation (56), is clearly positive. Unfortunately, the sign of the numerator "A" is ambiguous. Therefore, we would like to derive this sign using numerical approach. (Table.1) The table.1 indicates that an absolute value of C is smaller than 1 in all cases and the first derivative will be negative; in other words, it becomes clear that the curve goes down to the right. Next, to investigate the stability of the economy, we further differentiate equation (56) with respect to x_t.

$$\frac{d^2x_{t+1}}{(dx_t)^2} = \frac{\partial f(x_t, x_{t+1})}{\partial x_t} = \frac{A'B - AB'}{B^2} = D \tag{59}$$

$$A' = \frac{dA(x_t, x_{t+1})}{dx_t} = -\alpha\rho(1-\tau)(1-\alpha)^2 x_t^{\alpha-2} + \emptyset(1-\alpha)x_{t+1}\varphi\alpha^2\tau x_t^{\alpha-2} \tag{60}$$

$$B' = \frac{dB(x_t)}{dx_t} = \emptyset\varphi\alpha^2\tau x_t^{\alpha-1} \tag{61}$$

As with the above, since the sign cannot be explicitly determined, we try to derive it using numerical values (Table.2); the sign of the second derivative of the dynamic equation of capital will be positive, and it turns out that the decreasing rate of x_{t+1} gradually increases as x_t increases. Finally, we consider whether this curve intersects the 45° line at one point. It has the same meaning as the so-called "Inada condition."

$$\lim_{x_t \to 0} \frac{dx_{t+1}}{dx_t} = indefinite, \quad \lim_{x_t \to \infty} \frac{dx_{t+1}}{dx_t} = 0 \tag{62}$$

As shown in equation (62), if x_t approaches 0 as much as possible, the value of the first derivative indefinite; conversely, if x_t expands close to infinity, then its value is approximately 0.

5.5 An analysis in the steady state

Now that the global stability of the economy has been proven, we focus our analysis on the steady-state economy. Using equation (55) of the dynamic equation of capital and setting the left-hand side of equation to 1, it is possible to derive an equation that satisfies in the steady state.

$$[\varphi\alpha\tau(x^*)^\alpha + 1]\emptyset = \rho(1-\tau)(1-\alpha)(x^*)^{\alpha-1} \tag{63}$$

where x^* indicates the relative value of capital in the steady state. Next, we try to differentiate equation (30), which shows the dynamics of public capital by the share of expenditure on public capital investment to see the effect of increasing this share on growth.

$$\frac{\partial g}{\partial \varphi} = \tau\alpha^2(x^*)^\alpha \left[\frac{1}{\alpha} + \frac{\varphi}{x^*}\frac{dx^*}{d\varphi}\right] \tag{64}$$

Here, the second item in parentheses indicates the elasticity of the share for the relative value of capital. To measure the magnitude of this elasticity, we completely differentiate equation (59) with respect to φ and x^*. The results are as follows.

(65)

$$\frac{dx^*}{d\varphi} = \frac{A}{B} < 0$$

$$A = -\varphi\alpha\tau(x^*)^{\alpha}\emptyset < 0 \tag{66}$$

$$B = [\varphi\alpha^2\tau(x^*)^{\alpha-1}\emptyset + \rho(1-\tau)(1-\alpha)(x^*)^{\alpha-2}] > 0 \tag{67}$$

As can be seen, the sign of (66) is negative. This means that the share rise pulled down the relative value of capitals. This is because the expansion of public capital overtakes the rise in private capital. Here, this elasticity is the rate of change of x^* with respect to the rate of change of 1% φ. It indicates that this is synonymous with the well-known price elasticity of price. Therefore, the value was higher than 0 and lower than 1. Since the value of the share of private capital on GDP is between 0 and 1, the sign of parentheses will be positive. Then, our question is why the sign of elasticity will be negative? This is because the effect of the rising price exceeds the effect of booting income. That is, the birth rate will decline with price support for child rearing, although an increase in the share of public investment pushes up the wage rate.

Proposition 4. The sign of elasticity of the share on public capital investment for the relative value between private and public capital will be negative in both Cases, case A and B.

Therefore, we derive the effect of rising share on growth, which is explicitly shown below.

$$\frac{\partial g}{\partial \varphi} = \tau\alpha^2 \left\{ \frac{[\varphi\alpha^2\tau(x^*)^{\alpha}\emptyset(1-\varphi) + \rho(1-\tau)(1-\alpha)(x^*)^{\alpha-1}]}{[\varphi\alpha^2\tau\emptyset + \rho(1-\tau)(1-\alpha)(x^*)^{-1}]} \right\} > 0 \tag{68}$$

It can be seen that the sign in the above equation is explicitly positive.

6. Comparison of the impact of public capital on growth.

The next thing of interest is whether Case A or B has a higher growth rate. Equations (44) and (68) show the effect of rising share on growth. Substituting numerical examples of parameters explicitly indicate these.

$$\frac{\partial g}{\partial \varphi} = \tau\alpha^2(x^*)^{\alpha} \times \tag{69}$$

$$\left\{ \frac{[\varphi\,\alpha^2\tau(x^*)^{\alpha-1} + (\varepsilon+\rho)(1-\tau)(1-\alpha)^2(x^*)^{\alpha}] - \varphi^2\alpha^2\tau(x^*)^{\alpha}}{[\varphi\,\alpha^2\tau(x^*)^{\alpha-1} + (\varepsilon+\rho)(1-\tau)(1-\alpha)^2(x^*)^{\alpha}]} \right\} = 0.911$$

$$\frac{\partial g}{\partial \varphi}=\tau\alpha^2\left\{\frac{[\varphi\alpha^2\tau(x^*)^{\alpha}\emptyset(1-\varphi)+\rho(1-\tau)(1-\alpha)(x^*)^{\alpha-1}]}{[\varphi\alpha^2\tau\emptyset+\rho(1-\tau)(1-\alpha)(x^*)^{-1}]}\right\}=0.078 \qquad (70)$$

The above equations show that Case B has a greater effect on growth. What is the main cause of this difference? Intuitively, first, it is considered that the effect of the increase in wages (income effect) due to the increase in public capital is very large in Case B. In both cases, wages increase. Furthermore, in Case A, the interest rate also increased. At first glance, Case B is likely to have a higher growth rate, but the numbers point to the opposite. Second, in Case A, the increase in public capital should reduce the allocation for childcare support in government tax revenues, resulting in a decrease in the number of births and an increase in the opportunity cost of workers. Considering this and considering that the impact on the growth of Case A is small, the government's policy for reducing the opportunity cost of workers—for example, in Japan, free childcare fees for nursery schools for children over 3 years old and a second or more children and introduction of subsidies for the establishment of daycare centers in companies—can be considered insignificant.

7. Concluding remarks

This study focused on the relative value of private–public capital in the presence of a childcare support policy. First, the global stability of economic growth and the unique steady state to which economic converges is clarified. In the steady state, the economy is on a balanced growth path in which private capital, public capital, and GDP grow at the same rate. Second, the effect of increasing the share of public capital investment on the steady-state growth rate was analyzed and was found to depend on the absolute value of the elasticity of increasing the share of public capital investment relative to capital value or the labor share of GDP. More specifically, a smaller absolute elasticity value and larger labor share of GDP were found to be more likely to result in a positive growth rate. This is because the magnitude of the effect that increases public capital exceeds the effect of rising private capital; thus, a larger increase in income is needed to increase savings.

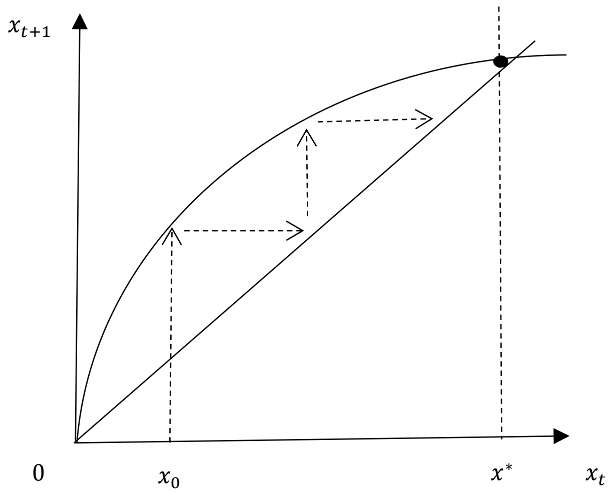

Fig. 1. Dynamics of x (Case A)

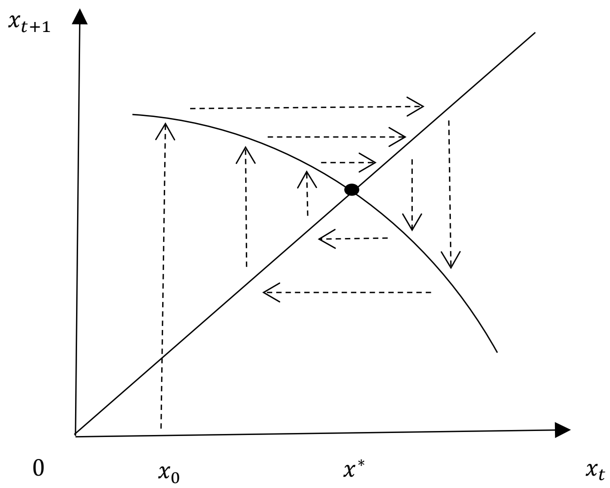

Fig. 2. Dynamics of x (Case B)

x	1	2	3	4	5
α	0.4	0.4	0.4	0.4	0.4
ε	0.7	0.7	0.7	0.7	0.7
ρ	0.7	0.7	0.7	0.7	0.7
φ	0.83	0.83	0.83	0.83	0.83
τ	0.3	0.3	0.3	0.3	0.3
A	-0.0266	-0.0574	-0.097	-0.115	-0.097
B	1.075	3.53	2.81	2.82	2.81
C	**-0.0083**	**-0.0179**	**-0.035**	**-0.041**	**-0.0626**

Table.1. Stability of x (Case A)

x	1	2	3	4	5
α	0.4	0.4	0.4	0.4	0.4
ε	0.7	0.7	0.7	0.7	0.7
ρ	0.7	0.7	0.7	0.7	0.7
φ	0.83	0.83	0.83	0.83	0.83
τ	0.3	0.3	0.3	0.3	0.3
A'	0.052	0.015	0.018	0.017	0.017
B'	0.096	0.063	0.049	0.042	0.38
A	-0.0266	-0.0574	-0.097	-0.115	-0.097
B	1.075	3.53	2.81	2.82	2.81
D	**0.05845**	**0.05657**	**0.05533**	**0.05277**	**0.08463**

Table.1. Stability of x (Case B)