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31 March 2021

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Abstract

This work analyses the role of asymmetry in beliefs for price dynamics in a cobweb model with heterogeneous expectations and evolutionary selection of predictors. While heterogeneous but symmetric beliefs result in the rational expectations equilibrium price, the effect of asymmetry depends on whether predictors on one side or the other of rationality have a larger support. A support skewed towards predictors that are anchored to past prices can be destabilizing, and the interaction with the evolutionary selection mechanism can lead to complex dynamics in prices; a support skewed towards predictors that overshoot price changes leads instead to price stability, irrespective of the underlying evolutionary dynamics. The design of the set of beliefs allowed to compete on the market is thus crucial for the possible outcomes of the model. One could interpret a skewed support in terms of sentiments, intended as one-sided systematic biases in expectations.

Key words: expectations; heterogeneity; evolutionary dynamics; sentiments.

JEL classification: C62, D83, D84, E32.

1 Introduction

This paper looks at the impact of asymmetry in agents' beliefs distribution on equilibrium outcomes in a cobweb economy with heterogeneous expectations and evolutionary selection of predictors.

In their seminal paper, [4, Brock and Hommes (1997)] - B&H hereafter - show how highly irregular price dynamics can emerge in a simple cobweb market where agents can switch between rational and naive predictors. After defining the concept of an adaptively rational equilibrium (ARE), where agents adapt their beliefs over time by choosing from a finite set of different predictors according to a discrete logit model, they then show how, in a simple cobweb model where agents can choose between rational (at a cost) and naive (free) expectations, agents keep switching between these two predictors, leading to complex and even chaotic price dynamics.

This paper characterizes the shape of the finite set of different predictors among which agents can choose in terms of their forecast errors, and show that such shape is crucial for the outcomes of the model. A symmetric beliefs space, precisely defined below, gives rise to an equilibrium where prices are at their rational expectations equilibrium (REE) level, while the outcome under an asymmetric set of predictors depends crucially on which side of the rational predictor the set is skewed towards. Depending on this, prices can show either smooth convergence to the REE value, or complicated dynamics of the type depicted in B&H.

The fact that a symmetric beliefs space leads to rational aggregate expectations is not surprising. The coexistence of individual heterogeneity and aggregate rationality was contemplated since the onset of the rational expectations paradigm, with [11, Muth (1961)] acknowledging such possibility, with the caveat:: "Allowing for cross-sectional differences in expectations is a simple matter, because their aggregate effect is negligible as long as the deviation from the rational forecast for an individual firm is not strongly correlated with those of the others." It will be shown in this paper that a key feature in heterogeneous expectations leading to aggregate rationality is the distribution of the forecast errors, rather than their independence: if forecast errors are symmetric around zero, in aggregate expectations are rational, and prices are at the REE level. If forecast errors are not symmetric, complex price dynamics of the type reported in B&H can, but do not necessarily, emerge in a cobweb model. The side of the asymmetry becomes then crucial in determining admissible equilibria of the model.

The analysis in this paper shows indeed that, depending on the side of the asymmetry, the reduced form AR(1) coefficient on prices can be constrained in a region that makes prices stable, or can be allowed to move from a stable to an unstable region, opening the door to complex dynamics. As a logit model of evolutionary selection of beliefs based on forecast errors (directly or indirectly, for example through profits) implies that the distribution of beliefs preserves the same asymmetry as the set of available predictors, the characterization of such set allows to understand whether complex dynamics can emerge or not in the model.

The asymmetry in the distribution of beliefs, and thus in forecast errors, interacts with the negative feedback from expectations and determines prices: when there is a prevalence of beliefs that are anchored to the past, such beliefs tend to be destabilizing, and the evolutionary dynamics on predictors can lead to complex price dynamics. Beliefs that overshoot price changes are, instead, stabilizing, so a set of predictors that forces aggregate beliefs to show this property leads to stable prices, irrespective of the specific evolutionary dynamics on beliefs induced by the logit model.

An important message that emerges from this analysis is that care has to be taken in designing the support for predictors when employing evolutionary selection mechanisms on beliefs, since this largely determines the outcomes that are admissible in the model. There might be perfectly sound motivations for choosing a set of predictors that lead to forecast errors more predominant one side or the other of rationality, for example if one aims at capturing the idea of sentiments in beliefs, where aggregate expectations are systematically overestimating or underestimating actual realized outcomes. In such cases, the evolutionary mechanism allows for heterogeneous beliefs dynamics while preserving the aggregate sentiment.

1.1 Review of the literature

After being proposed in B&H, the concept of evolutionary dynamics on predictors has been applied and extended in various directions in economics. For example, [2, Branch (2002)] extends the original B&H analysis allowing for three types of beliefs: rational, naive and adaptive expectations, while [6, Brock et al (2005)] introduce the concept of a large type limit to study an evolutionary heterogeneous market system with many different strategy types.

Within a cobweb model, [10, Hommes and Wagner (2010)] find that the rational expectations steady state is eductively stable but not evolutionary stable when producers can choose between three strategies: optimistic, fundamentalist and pessimistic. Building on these results, [12, Naimzada and Pireddu (2020)] add rational producers in the market and find that while rational agents enlarge the local stability region of the steady state, they also lead to complex dynamics.

Evolutionary selection of beliefs and the ARE concept have also been used to study the interplay of beliefs and outcomes in other economic settings besides the cobweb model. For example, [5, Brock and Hommes (1998)] and ([9, Gaunersdorfer (2000)]) find similar complex dynamics emerging in an asset pricing model, where the feedback from expectations to prices is positive. Using similar mechanisms, [8, De Grauwe and Grimaldi (2006)] find complex dynamics in exchange rates and [7, De Grauwe (2011)] generates endogenous waves of optimism and pessimism in a macroeconomic model. [3, Branch and MCGough (2016)] find cycles and chaotic dynamics in a monetary model when traders switch between (costly) rational and (costless) adaptive predictors and [1, Agliari et al. (2017)] propose an application of the ARE concept to a New Keynesian macroeconomic model of inflation and output, showing the impact of such beliefs dynamics on the stability and uniqueness of equilibrium.

None of these works seem to motivate the choice of the set of predictors made available to agents in terms of the ensuing distribution of forecast errors, nor to discuss how such choice is relevant for their results. It is the aim of this paper to address this issue.

2 A motivating example

As a motivating example for the analysis that follows, I consider the B&H framework with two predictors and show simulations of price dynamics when the set of available predictors is modified in two simple ways, by either adding or substituting one of the available options.

Rational and naive predictors could be represented, parsimoniously, as

$$p^e_{i,t} = \alpha_i p_{t-1} + (1-\alpha_i)\, p_t, i = 1, 2 \tag{1}$$

with $\alpha_1 = 1, \alpha_2 = 0$. The naive predictor is then characterized by $\alpha_1 = 1$ and the rational one by $\alpha_2 = 0$. Consider now a third predictor, derived by setting $\alpha_3 = -1$. Agents using this predictor would make the same forecast errors as those using α_1, the naive agents, but with a reversed sign. In fact, from (1)

$$\varepsilon_{i,t} = p^e_{i,t} - p_t = \alpha_i \left(p_{t-1} - p_t\right),$$

so $\varepsilon_{1,t} = (p_{t-1} - p_t)$ and $\varepsilon_{3,t} = -(p_{t-1} - p_t)$: ex post, once p_t is realized, forecast errors $\varepsilon_{1,t}$ and $\varepsilon_{3,t}$ have the same size, but opposite sign. While naive agents undershoot price changes in their forecasts, agents with a negative value for α_i overshoot those changes (more on this later), forecasting prices that are higher than actual ones when prices increase, and lower than actual ones when they decrease.

Following B&H, I also assume the cost for using a naive predictor is zero, while there is a positive cost for using the rational predictor. As the predictor characterized by $\alpha_3 = -1$ leads to forecast errors of the same size as those from the naive predictor, I also set their cost to zero. It is equally costly to obtain forecasts with the same accuracy.

I then simulate the model under three different settings: i) the original B&H setting, with only predictors characterized by α_1 and α_2 available; ii) a setting where all three predictors, with α_1, α_2 and α_3, are available; iii) a setting where only predictors with α_2 and α_3 are available.

Starting with the B&H setting, I set all parameters as in the original work, in order to replicate their Fig. 8.[1] Results are shown in Figure (1), which in the top panel shows the time series of prices, while in the bottom panel shows the reduced form AR(1) coefficient on prices. Price dynamics replicate exactly those in Fig. 8 of B&H, top panel, where cycles in prices emerge.

Keeping all parameters fixed, I then simulate the model under the two alternative scenarios, first with three available predictors, characterized by $\alpha_i = \{-1, 0, 1\}$ and then with two options, characterized by $\alpha_i = \{-1, 0\}$. Results are reported in Fig 2 and Fig 3, respectively. It can be seen that the behavior of prices is very different in these two cases compared to the first one: with three predictors, prices drop immediately to their REE, remaining there ever after; with two predictors, converge to the REE is instead smooth, but rapid, and once again prices do not move from equilibrium once there.

The set of available predictors, thus, seems to have a major impact on the price dynamics, and the aim of this work is to shed some light on the relationship between the two. In particular, I will characterize the shape of the set of available predictors, how this impacts on the distribution of beliefs and how such distribution determines the law of motion for prices.

[1] Specifically, in terms of the model presented below, parameter values are set as follows: $a = 0$, $z = 0.5$; $b = 1.35$; $C = 1; \beta = 3.8$, where β represents the "intensity of choice" parameter in the discrete logit model for beliefs switching.

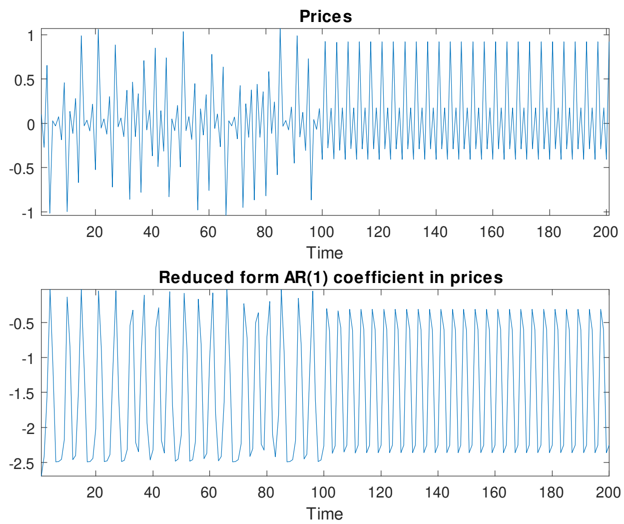
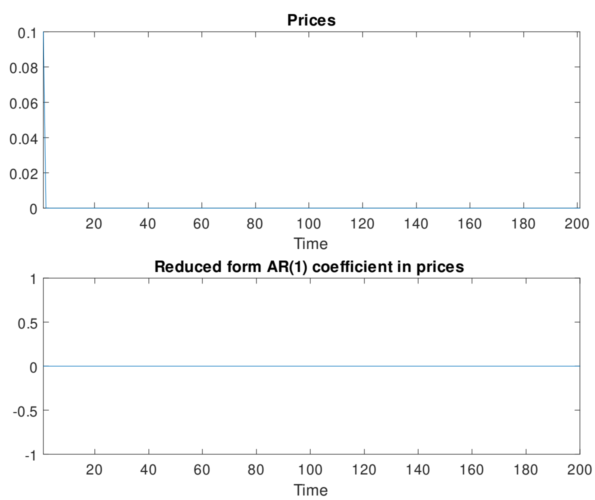


Figure 1: Prices and AR(1) coeficient on prices. B&H setting.

predictors



$_2.pdf$

Figure 2: Prices and AR(1) coeficient on prices with three predictors: naive, rational and contrarian.

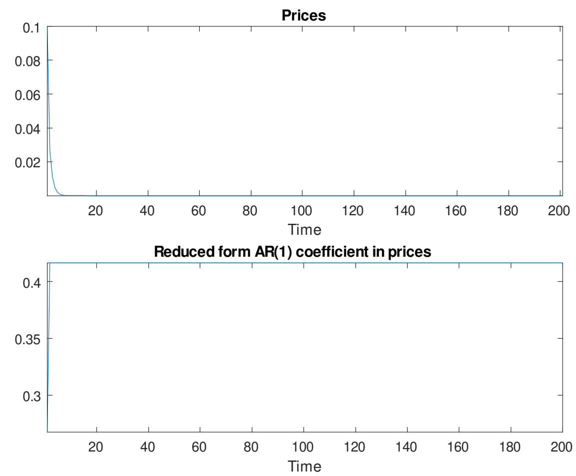


Figure 3: Prices and AR(1) coeficient on prices with two predictors: rational and contrarian.

3 The model

3.1 A cobweb model

In this section, following B&H, I derive a cobweb model with heterogeneous agents, where firms need to decide how much to produce based on the expected selling price.

The profit function for a generic firm of type i (where the type will then represent the predictor the firm uses) is specified as

$$\pi_{i,t} = p_t s_{i,t} - c\left(s_{i,t}\right), \tag{2}$$

where $\pi_{i,t}$ are profits, p_t is the selling price at time t and $s_{i,t}$ is the quantity the firm produces and sells at time t. The cost function $c\left(.\right)$ is quadratic

$$c\left(s_{i,t}\right) = \frac{s_{i,t}^2}{2b},$$

with $b > 0$.

As firms need to produce before knowing the selling price, maximizing expected profits w.r.t. $s_{i,t}$ gives rise to a linear supply curve

$$s_{i,t} = bp_{i,t}^e,$$

where $p_{i,t}^e$ is the expected selling price at time of production for a firm of type i. Aggregating over firms, supply S_t is then given by

$$S_t = \sum_{i=1}^{n} \mu_{i,t} s_{i,t},$$

where $\mu_{i,t}$ represents the relative number of firms of the same type.

Demand D_t is assumed linear in prices

$$D_t = a - zp_t,$$

with $a \geq 0$ and $z > 0$ parameters.

Putting together demand and supply, setting $a = 0$ for simplicity and defining $B = -b/z$, the cobweb model can then be summarized as

$$p_t = Bp_t^e, \tag{3}$$

with

$$p_t^e = \sum_{i=1}^{n} \mu_{i,t} p_{i,t}^e.$$

Since $B < 0$, the cobweb model features negative feedback from expectations to outcomes: higher expected prices mean higher production, which lowers actual prices. Under rational expectations, there is only one equilibrium, $p_t = 0$, for $B \neq 1$. For the non generic case $B = 1$, instead, any p_t is an equilibrium.

3.2 Expectations

In B&H, a large part of the paper is devoted to derive and characterize complex dynamics for the case with two predictors available to firms, $p_{1,t}^e = p_{t-1}$ and $p_{2,t}^e = p_t$, with $p_{1,t}^e$ corresponding to naive expectations and $p_{2,t}^e$ to rational expectations (RE). Naive expectations are free to obtain, while RE entail a cost $C > 0$. Firms switch between the two forecasting rules based on profits, and the switching is modeled using a discrete logit model. In the REE, the two predictors deliver the same (correct) forecast, $p_t = 0$, though one at a cost. B&H show that, under certain conditions (in particular, $B < -1$ and C high enough), cycles and even chaotic dynamics can emerge in this setting.

The purpose of this work is to clarify the relationship between the available predictors and market outcomes. As costs affect the relative selection of different predictors under evolutionary schemes, an assumption needs to be made about their relative cost: I will model such costs as linear in the distance of each predictor from RE, but results are robust to alternative choices.

In order to capture the range of different predictors available for selection, I model them as a linear combination of past and current prices. Agents can thus choose among a finite set of n predictors represented by

$$p^e_{i,t} = \alpha_i p_{t-1} + (1 - \alpha_i)\, p_t,\;\; i = 1, 2, ..., n. \tag{4}$$

Each of the n predictors available is thus characterized by a different α_i, with $\alpha_i \in S \subset \mathbb{R}$. Here S is the support interval for α_i, defined as $S = [-l, r]\,,\; r, l \geq 1$. The n values α_i are chosen to be equally spaced over S, with $\alpha_1 = -l$ and $\alpha_n = r$.[2] Such modelling device allows for a parsimonious parameterization of expectations. Care has to be taken, in designing the space S, in case one wants to make sure the rational predictor is included among the n available predictors.

Restricting $S = [0, 1]$ forces expectations to be a convex combination of past and actual prices, but there is no reason why agents should not be able to use a predictor which forecasts a price outside this range.

A useful way to rewrite (4) is

$$p^e_{i,t} = p_t - \alpha_i \left(p_t - p_{t-1}\right),$$

which shows that positive values of α_i lead to underestimation when prices are increasing and overestimation when they are decreasing. Such beliefs are thus anchored by past values and undershoot price changes, in both directions. A negative value for α_i, instead, has the opposite effect, giving rise to expectations that are "extreme", as they overshoot the direction of movement of prices (up or down).

One can then write the forecast errors $\varepsilon_{i,t}$ as

$$\varepsilon_{i,t} \equiv p^e_{i,t} - p_t = -\alpha_i \left(p_t - p_{t-1}\right).$$

An interesting empirical question that could help design the beliefs space of agents is whether the sign of the forecast errors tends to be the same as that of $(p_{t-1} - p_t)$: that is, whether or not people's expectations tend to be anchored by current values and forecast errors tend to fall predominantly on one side. In the absence of any such evidence, it would be perhaps advisable to allow forecast errors to fall equally on either side of zero.

[2]Restricting $l = 0$, $r = 1$ and $n = 2$ leads to the B&H setting, where $\alpha_i = \{0, 1\}$.

I then denote as $\mu_{i,t}$ the relative fraction of firms at time t using a specific predictor, characterized by α_i, with $\sum_{i=1}^{n}\mu_{i,t}=1$. Then, aggregating

$$\begin{aligned} p_t^e &= \sum_{i=1}^{n}\mu_{i,t}\left(\alpha_i p_{t-1}+(1-\alpha_i)\,p_t\right) \\ & p_{t-1}\sum_{i=1}^{n}\mu_{i,t}\alpha_i + p_t\sum_{i=1}^{n}\mu_{i,t}\left(1-\alpha_i\right) \\ &= p_t + \sum_{i=1}^{n}\mu_{i,t}\alpha_i\left(p_{t-1}-p_t\right) \end{aligned} \tag{5}$$

and substituting into (3)

$$\begin{aligned} p_t &= B\left[p_{t-1}\sum_{i=1}^{n}\mu_{i,t}\alpha_i + p_t\sum_{i=1}^{n}\mu_{i,t}\left(1-\alpha_i\right)\right] \\ &= B\left(\sum_{i=1}^{n}\mu_{i,t}\alpha_i\right)p_{t-1} + B\left(\sum_{i=1}^{n}\mu_{i,t}\left(1-\alpha_i\right)\right)p_t \\ &= \frac{B\left(\sum_{i=1}^{n}\mu_{i,t}\alpha_i\right)}{1-B\left(\sum_{i=1}^{n}\mu_{i,t}\left(1-\alpha_i\right)\right)}p_{t-1}. \end{aligned} \tag{6}$$

It will be convenient to denote the reduced form AR(1) coefficient on prices as Ω_t; that is:

$$\Omega_t \equiv \frac{B\left(\sum_{i=1}^{n}\mu_{i,t}\alpha_i\right)}{1-B\left(\sum_{i=1}^{n}\mu_{i,t}\left(1-\alpha_i\right)\right)} = \frac{B\left(\sum_{i=1}^{n}\mu_{i,t}\alpha_i\right)}{(1-B)+B\left(\sum_{i=1}^{n}\mu_{i,t}\alpha_i\right)}. \tag{7}$$

Equation (6) shows the law of motion for prices, given a certain distribution of beliefs, characterized by $\sum_{i=1}^{n}\mu_{i,t}\alpha_i$. While parameters α_i, $i=1,..,n$, depend on the set of predictors made available to agents, the fractions $\mu_{i,t}$ will depend on the evolutionary mechanism adopted. Before looking at it, though, I will discuss briefly in the next Section the choice of modelling predictors according to (4).

3.3 Representing expectations

In this section I motivate the choice of representing different predictors, and thus expectations, according to (4) by looking at two different alternative representations of heterogeneous beliefs that might seem more intuitive, but that would not be suitable for the analysis of this paper. These two examples help clarifying the desirable properties that an expectations representation should feature in studies where the interplay of beliefs and outcome is to be analysed.

The main reason for modelling expectations according to (4) is that such representation offers a parsimonious way to model a class of beliefs where the degree of rationality is characterized by one parameter only and prices depend on the aggregation of heterogeneous beliefs in the population.

One could, for example, think of modelling different predictors as

$$p_{i,t}^e = p_t + \alpha_i,$$

where α_i would represent the forecast error for predictor i. Rational firms would then be characterized by $\alpha_i = 0$. The problem with this representation is that individual forecast errors (and the ensuing profits for a firm) would be exogenous and independent of price movements, so there would not be any endogenous dynamics in the evolutionary selection of beliefs. Moreover, prices would always be equal to the constant $\alpha/(1 - B)$, where α is the aggregation of the (exogenous) individual forecast errors.

Another possibility would be to represent expectations according to

$$p_{i,t}^e = \alpha_i p_t.$$

Rational firms would now be characterized by $\alpha_i = 1$, and forecast errors would be equal to $(\alpha_i - 1)\, p_t$. Now individual forecast errors do depend on price movements but, with this specification, $p_t = 0$, independently of agents' beliefs. Price dynamics are thus trivial, no matter what beliefs agents hold.

Neither of these representations, thus, would be suitable for a setting where one wants to analyse the interplay of beliefs and outcomes. Of course, one can come up with an infinite number of alternative representations, some of which might well offer useful insights, but I believe that equation (4) offers a sensible and parsimonious way to think about the issues at the core of this paper.

4 Evolutionary dynamics on beliefs

Having defined the set of predictors available to firms, it is now necessary to specify how such predictors are selected. Following B&H, I assume that the fraction of firms using each predictor i at time t, $\mu_{i,t}$, is determined endogenously through a simple discrete logit model based on relative profits, with no memory:

$$\mu_{i,t+1} = \frac{\exp\left(\beta \tilde{\pi}_{i,t}\right)}{\sum_j \exp\left(\beta \tilde{\pi}_{j,t}\right)}, \tag{8}$$

where

$$\tilde{\pi}_{i,t} = \pi_{i,t} - C\left(\alpha_i\right) \tag{9}$$

represents overall profits once the cost of using a certain predictor is accounted for. Next period predictors are thus chosen according to their relative performance this period.

The last term in (9) represents the cost of using the predictor characterized by α_i. As said before, I model such costs as a linear decreasing function of the distance between α_i and 0, which characterizes the RE predictor: the cost of a predictor, thus, increases (linearly) with its degree of rationality, as, e.g., agents need to acquire and process more information for it. In particular, the cost of the least rational predictor(s) available, characterized by the largest α_i in absolute value, is normalized to zero, and the cost of the rational predictor ($\alpha_i = 0$) is fixed to the level $\bar{C}$, that is $C\left(\max(l, r)\right) = 0$ and $C\left(0\right) = \bar{C}$:[3]

$$C\left(\alpha_i\right) = \bar{C} - \frac{\left|\alpha_i\right| \bar{C}}{\max(l, r)}.$$

Overall profits are thus given by

$$\tilde{\pi}_{i,t} = p_t b p^e_{i,t} - \frac{\left(b p^e_{i,t}\right)^2}{2b} - \bar{C} + \frac{\left|\alpha_i\right| \bar{C}}{\max(l, r)}.$$

Using (4) to substitute out expectations and (6)-(7) to represent prices, and denoting $\bar{\alpha} =$

[3] Alternative cost functions, such as the quadratic one

$$C\left(\alpha_i\right) = \bar{C} - \frac{\alpha_i^2 \bar{C}}{\bar{\alpha}},$$

could be assumed, and all results in this paper would hold. In general, any cost function that implies the same cost for predictors with the same accuracy would lead to equivalent results.

$\max(l, r)$, one obtains

$$\tilde{\pi}_{i,t} = \frac{b}{2}\Omega_t^2 p_{t-1}^2 - \alpha_i^2 \left(\frac{b}{2} (1 - \Omega_t)^2 p_{t-1}^2 \right) + \bar{C} \left(\frac{|\alpha_i|}{\bar{\alpha}} - 1 \right), \tag{10}$$

which shows that predictors characterized by the same $|\alpha_i|$ lead to the same profits.

If one looks at mean squared errors, then, it can be seen that also the accuracy of forecasts is the same for equal $|\alpha_i|$:

$$MSE_{i,t} \equiv \left(p_{i,t}^e - p_t \right)^2 = \alpha_i^2 \left(p_{t-1} - p_t \right)^2 \tag{11}$$

$$= \alpha_i^2 \left(\Omega_t - 1 \right)^2 p_{t-1}^2. \tag{12}$$

Moreover, the accuracy of forecasts is monotonic and decreasing in $|\alpha_i|$, with a maximum at the rational predictor ($\alpha_i = 0$).

The fact that two predictors with equal $|\alpha_i|$ have the same performance, both in terms of MSE and in terms of profits, justifies the idea of allowing predictors with both positive and negative values for α_i to coexist in the set of available options. For example, allowing the existence of a predictor characterized by $\alpha_i = -1$ in addition to one with $\alpha_i = 1$ simply means to allow agents to use a predictor with the same forecast error as the naive expectations, only "on the other side" of rationality.

Equation (8) then implies that predictors generating the same profits (or the same MSE) are adopted by the same number of agents, for any finite value of β. Given that, as seen, profits depend on the absolute value of the forecast error, two predictors characterized by the same $|\alpha_i|$, if both available, will be adopted by the same fraction of firms at each time. In other words, if the support of predictors is symmetric, so is the distribution of firms adopting those predictors, and an asymmetric set of predictors translates into an asymmetric distribution of firms, for any finite value of β.

While the size of the forecast errors depends on $|\alpha_i|$, their sign depends on the sign of α_i: a positive α_i generates a negative forecast error if prices increase (i.e., prices are underestimated) and a positive error if prices decrease (prices are overestimated), while a negative α_i leads to a positive error if prices increase (prices are overestimated) and a negative error of prices decrease (prices are underestimated). A positive α_i, thus, means that expectations are anchored to past values (undershooting any price change), while a negative one means they are overshooting price changes (in both direction).

For example, consider the set of predictors represented by $\alpha_i = \{-1, 0, 1\}$, which generates expectations $p_{i,t}^e = \{-p_{t-1} + 2p_t, p_t, p_{t-1}\}$, with forecast errors equal to $\left(p_{i,t}^e - p_t \right) =$

$\{-\Delta_{p,t}, 0, \Delta_{p,t}\}$, with $\Delta_{p,t} = p_t - p_{t-1}$. In the first case $p^e_{1,t} = -p_{t-1} + 2p_t = p_t + \Delta_{p,t} = p_{t-1} + 2\Delta_{p,t}$: agents expect twice the change in prices that actually takes place. In the third case, instead, $p^e_{1,t} = p_{t-1} = p_t - \Delta_{p,t}$: agents' forecasts miss out completely the change in prices that is going to happen.

To sum up, a set of predictors $\{\alpha_i,\ i = 1...n\}$, chosen to be equally spaced over the closed interval $S \subset \mathbb{R}$, gives rise to a distribution of beliefs, through (8). As (8) gives equal weight to predictors with the same $|\alpha_i|$, whether the distribution of beliefs is symmetric or not will be determined by the shape of the set S.

5 Beliefs distribution and market outcomes

5.1 Symmetry and equilibrium

I now define precisely what is meant here by symmetry (and asymmetry) of beliefs or predictors and derive implications for price dynamics and equilibrium. As said before, the set of available predictors gives rise to a distribution of beliefs in the economy through (8). As predictors with the same $|\alpha_i|$ imply forecast errors of the same magnitude, the distribution of beliefs will preserve the same (a)symmetry as that of the set of available predictors. The notion of (a)symmetry can thus be applied equivalently to the set of predictors, to the distribution of beliefs or to that of forecast errors.

Taking the rational predictor as the reference point, with a zero forecast error, symmetric forecast errors in available predictors require that for any predictor characterized by $\alpha_i = a$, $a \in \mathbb{R}^+$, a "mirror" predictor characterized by $\alpha_j = -a$ is also available for selection. In terms of the set S, this translates into the requirement $l = r$.

For any finite intensity of choice, β, a symmetric set $S = [-l, r]$, $l = r$, translates into a symmetric beliefs distribution, since any two predictors characterized by $\alpha_i = a$ and $\alpha_j = -a$ are adopted by the same proportion of agents. It also gives rise to a symmetric distribution of forecast errors, centered at zero.

A useful way to summarize whether a distribution of beliefs is symmetric or not is through the following measure:

$$\lambda_t \equiv \sum_{i=1}^{n} \mu_{i,t}\alpha_i. \tag{13}$$

It is clear that a symmetric distribution is characterized by $\lambda_t = 0$. It is also clear from (7) that $\lambda_t = 0$ implies $\Omega_t = 0$ and thus, from 6, $p_t = 0$. A symmetric distribution of beliefs thus leads to an aggregate expected price equal to its REE value, which is a stable fixed point of the system.

Proposition 1 *An heterogeneous expectations cobweb model with evolutionary dynamics over predictors symmetrically distributed around the rational one is equivalent to an homogeneous, rational expectations cobweb model. The market is stable at the REE equilibrium* $p_t = 0$.

From (8) and (13), it is clear that symmetry in beliefs is both a necessary and sufficient condition for $\lambda_t = 0$, for any finite β. Since any predictor will be adopted by a strictly positive fraction of agents, and this fraction is the same for any two predictors characterized by $\alpha_i = a$ and $\alpha_j = -a$, all pairs of terms where both $\alpha_i = a$ and $\alpha_j = -a$ are available cancel out: the sign of λ_t is then characterized by those predictors (α_i) for which a "mirror" counterpart is not available. In other words, (8) imposes the same mass μ_i on beliefs with the same absolute forecast error: their values will cancel out, and only forecast errors of predictors that don't have a counterpart will contribute to characterize aggregate expectations, and thus price dynamics.

5.2 Asymmetry and dynamics

Having defined symmetry in beliefs, it is straightforward now to present the concept of an asymmetric distribution of beliefs, arising from an asymmetric support set S with $l \neq r$. If $|l - r|$ is large enough compared to n, the largest interval between $[-l, 0)$ and $(0, r]$ will include more predictors. In particular, this requires $|l - r| > \frac{l+r}{n-1}$, which I will assume to be always the case whenever $l \neq r$.[4] In this case, then, $\lambda_t \neq 0$, with the sign of λ_t depending on which side of rationality includes more predictors: beliefs are not symmetrically distributed around the rationality and, in aggregate, expectations differ from the rational expectations price.

According to (8), predictors characterized by the same $|\alpha_i|$ will get chosen by the same fraction of agents: if S is not symmetric around zero, there will be predictors on one side of rationality that don't have the corresponding predictor on the other side: since all predictors are chosen by a positive fraction of agents for finite β, and predictors with the same absolute value for α_i get the same relative selection, the largest interval between $[-l, 0)$ and $(0, r]$,

[4] This assumption simplifies the exposition as it ensures that the largest interval includes at least one more predictor, thus allowing to characterize the distribution of predictors in terms of the boundaries of set S.

including more predictors, will determine the overall sign of λ_t. In other words, forecast errors are not equally distributed around zero, and so they don't aggregate up to zero.

It is essential at this point to characterise the sign of λ_t and, through it, pin down possible values for Ω_t. Clearly, a positive λ_t means that there are more predictors available with positive α_i than with negative α_i. The following Proposition lays out the ensuing restrictions on Ω_t and the induced properties of prices.

Proposition 2 *If the set $S = [-l, r]$, $l, r \geqslant 0$, is asymmetric around zero, with $|l - r| > \frac{l+r}{n-1}$, two cases arise:*
i. For $l > r$, $\lambda_t < 0$ and $0 < \Omega_t < 1$: prices evolving according to (6) converge to their REE value of zero over time.
ii. For $l < r, \lambda_t > 0$ and $\Omega_t \in (-\infty, +\infty)$. Prices evolving according to (6) can display complex dynamics under an evolutionary selection mechanism for beliefs like (8).

The proof is straightforward and goes as follows.

Noting that (8) determines the same $\mu_{i,t}$ for predictors characterized by the same absolute value for α_i, it is clear that all the terms in λ_t for which both $\alpha_i = a$ and $\alpha_j = -a$ exist cancel out, and the sign of λ_t will depend on the relative size of l and r: if $l > r$, there will be more terms with negative than with positive α_i and $\lambda_t < 0$, while if $l < r$, positive terms will instead dominate and $\lambda_t > 0$. We thus have two possible scenarios:

- Case 1: $l > r \Longrightarrow \lambda_t < 0$.
  If $\lambda_t < 0$ then $0 < \Omega_t < 1$ (since $B < 0$, $B\lambda_t > 0$, $B\lambda_t < (1 - B(1 - \lambda_t))$) and prices converge monotonically to zero, the REE equilibrium.
- Case 2: $l < r \Longrightarrow \lambda_t > 0$.
  When $\lambda_t > 0$, the characterization of Ω_t is more complicated. First note that in this case $B\lambda_t < 0$. It it also trivially true that $B\lambda_t < (1 - B(1 - \lambda_t))$, as the inequality reduces to $0 < 1 - B$, which is always satisfied in a cobweb model ($B < 0$). The sign of the denominator in Ω_t, though, cannot be characterized uniquely and two cases are possible:
  - $B\lambda_t < (1 - B(1 - \lambda_t)) < 0$: then $\Omega_t > 1$ and prices are unstable.
  - $B\lambda_t < 0 < (1 - B(1 - \lambda_t))$: then $\Omega_t < 0$, and prices could be stable or unstable, depending on whether $-1 < \Omega_t < 0$ or not.

The alternating between stable and unstable regions for Ω_t as μ_i evolves gives rise to the irregular behavior described by B&H. In particular, when a large proportion of firms are using the more rational predictors, the distribution of beliefs shifts closer to zero and λ_t takes on small positive values, determining small negative values for Ω_t: price dynamics become stable, inducing agents to adopt less rational predictors, thus increasing λ_t and driving Ω_t into the unstable region, with either $\Omega_t < -1$ or $\Omega_t > 1$, which in turn leads then firms to switch to more rational predictors, and so on.

5.3 Discussion

Having presented Proposition 2, it is useful now to discuss some of its implications and delve some more into the properties of the relationship between beliefs and prices, that is, of the function $\Omega_t(\lambda_t)$.

If $\lambda_t < 0$ then $0 < \Omega_t < 1$ and prices converge monotonically to zero, the REE equilibrium. In this case the evolutionary dynamics on predictors do not matter, as they preserve $\lambda_t < 0$ and thus price stability. The intuition is straightforward: when more firms use a predictor characterized by a negative α_i than by a positive α_i, aggregate expectations overshoot any price change. This means that when prices are decreasing, on average firms expect prices to be lower than they turn out to be, which induces them to produce less, thus keeping prices up. Similarly if prices are increasing, firms expect a larger rise than what actually happens, thus increasing production and keeping prices low: again, prices are stabilized. Due to the negative feedback on prices, expectations that overshoot price changes have a dampening effect in a cobweb model, generating a contraction mapping on prices. The evolutionary dynamics on predictors in this case do not matter, since λ_t remains negative and thus $0 < \Omega_t < 1$, no matter what the distribution of μ_i is.

One might wonder why this stable behavior is not found in B&H: the answer is that it is explicitly ruled out by their assumptions A2 and A2', which make the equilibrium price $p = 0$ not stable and thus imply that Ω_t must be greater than one in absolute value at $p = 0$. Assumption A2 ensures such instability for the case where all costs $C(\alpha_i) = 0$, $\forall \alpha_i$, and agents are uniformly distributed over the predictor space, with all fractions of agents fixed at $1/n$, while assumption A2' ensures the instability when there are different costs and all agents use the cheapest predictor. In both cases, a support of beliefs with $l < r$, which would generate $\lambda_t < 0$ and price stability, is ruled out.

The second case, with $\lambda_t > 0$, is more complicated, and thus more interesting. For a given cobweb model (that is, for a given $B < 0$), as λ_t increases (more firms adopting less rational predictors) Ω_t turns from a stable to an unstable region, first crossing the threshold of -1, and then becoming positive and larger than 1.

As prices become unstable, more people will be driven to use a more rational predictor, so

λ_t decreases towards zero and the system returns temporarily to stability with $-1 < \Omega_t < 0$. As λ_t approaches zero (more and more firms use the more rational predictors, characterized by values of α_i close to zero, both positive and negative) and the price converges towards its REE, the advantage of using the more rational, and more costly, predictors vanishes and more and more firms start using the cheaper and less accurate predictors, leading λ_t to increase once again. This is the B&H story of complex dynamics, and the intensity of choice parameter β, determining the rate of switch among predictors, acquires now relevance in determining these dynamics.

At this point, it is useful to investigate further the relationship between λ_t and Ω_t, since λ_t is characterized by the shape of the beliefs distribution and Ω_t determines the dynamics of prices. From (7) and (13)

$$\Omega_t = \frac{B\lambda_t}{1 - B + B\lambda_t},$$

and it can be seen that there is a vertical asymptote at $\lambda_t = 1 - \frac{1}{B}$, where Ω_t tends to $-\infty$ or $+\infty$ infinity, depending on whether convergence is from the right or the left. Specifically

$$\begin{aligned} \lim_{\lambda_t \to 1 - \frac{1}{B}^+} \frac{B\lambda_t}{1 - B + B\lambda_t} &= \infty, \\ \lim_{\lambda_t \to 1 - \frac{1}{B}^-} \frac{B\lambda_t}{1 - B + B\lambda_t} &= -\infty. \end{aligned}$$

Moreover, there is a horizontal asymptote at $\Omega_t = 1$: for $\lambda_t \to \pm\infty$, $\Omega_t \to 1$ from below or above, that is

$$\begin{aligned} \lim_{\lambda_t \to \infty} \frac{B\lambda_t}{1 - B + B\lambda_t} &= 1^+, \\ \lim_{\lambda_t \to -\infty} \frac{B\lambda_t}{1 - B + B\lambda_t} &= 1^-. \end{aligned}$$

Fig. (4) depicts the relationship between and λ_t and Ω_t.

It is also possible to determine the point where the system changes from stable to unstable: for $0 < \lambda_t < \frac{1}{2} - \frac{1}{2B}$, $-1 < \Omega_t < 0$ and prices are stable, while for $\frac{1}{2} - \frac{1}{2B} < \lambda_t < 1 - \frac{1}{B}$, $\Omega_t < -1$ and prices are unstable. The point $(\lambda_t, \Omega_t) = \left(\frac{1}{2} - \frac{1}{2B}, -1\right)$ is thus the crucial point where prices change from stable to unstable, and the switching back and forth across this

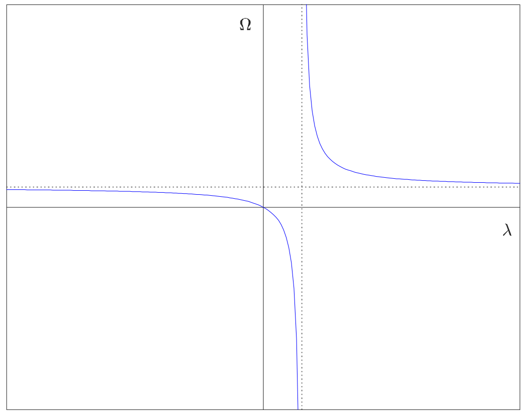


Figure 4: Relationship between λ_t and Ω_t.

point is what generates the complex dynamics in prices described in B&H. Since

$$\begin{aligned} \lim_{B\to-\infty} \frac{1}{2} - \frac{1}{2B} &= \frac{1}{2} \\ \lim_{B\to0^-} \frac{1}{2} - \frac{1}{2B} &= +\infty, \end{aligned}$$

this means that as B takes on larger and larger negative values the system can be destabilized for milder asymmetries in beliefs, while values of B closer to zero require stronger beliefs asymmetries in order for the market to be destabilized.

Before concluding this discussion, a few words should be spent over the role played by the intensity of choice parameter β. Such parameter is key for the complex dynamics depicted in B&H, and it is clear from the above analysis that its key role is limited to the case where $\lambda_t > 0$, as it governs the switch from a region of stability to one of instability for prices, as λ_t jumps from one side to the other of the threshold $\frac{1}{2} - \frac{1}{2B}$. For a given set of available predictors $\{\alpha_1, ..., \alpha_n\}$, in fact, the condition $l < r$ ensures that $\lambda_t > 0$, but the precise value of λ_t depends on the distribution of μ_i, which represent the weights on α_i in λ_t, and such distribution depends on β. The intensity of choice is instead irrelevant for the case where $\lambda_t \leqslant 0$, as the system is in this case stable for any dynamics on beliefs, i.e., for any distribution of μ_i. In other words, changes in μ_i do not affect the sign of λ_t when $l \geq r$ and thus Ω_t remains constrained in the stable region $0 \leqslant \Omega_t < 1$ (with equality holding for symmetric beliefs, i.e., $\lambda_t = 0$).

5.4 Asymmetric support of predictors and sentiments

While a symmetric support for predictors allows agents to make forecast errors that fall equally on both sides of zero, there might be valid reasons to assume that, in certain settings, predictors falling on one side of rationality are more prevalent than those falling on the other side. In particular, one could interpret an asymmetric support for beliefs as a way to capture a prevailing sentiment in an economy, intended as one-sided systematic biases in expectations. Evolutionary dynamics over an asymmetric support ensure that agents can switch predictors over time, but the aggregate sentiment remains unchanged as it is determined by the shape of the support set. Designing a skewed support S, thus, allows for a parsimonious way to model sentiments in an economy, while allowing evolutionary forces to operate on the distribution of predictors in the population.

Under this interpretation, in a cobweb model, bullish beliefs when prices are increasing and bearish beliefs when prices are decreasing are stabilizing (this happens when $\lambda_t < 0$, that is, $l > r$). The negative feedback from expectations to actual prices, in fact, ensures that the overshooting in expected price changes has a dampening effect on price dynamics. The evolutionary dynamics on predictors, then, capture the adjustment of individual beliefs, within the aggregate sentiment defined by S.

On the other hand, when $l < r$, that is, when aggregate expectations are bullish at times of decreasing prices and bearish when prices are increasing, sentiments are destabilizing if strong enough. As seen above, for mild sentiments (when $0 < \lambda_t < \frac{1}{2} - \frac{1}{2B}$) the AR(1) coefficient on prices (Ω_t) is between -1 and 0 and there is oscillatory convergence of prices to the REE value, while for stronger sentiments ($\lambda_t > \frac{1}{2} - \frac{1}{2B}$) the AR(1) coefficient is larger than 1 in absolute value and prices become unstable. In this case, the evolutionary selection of predictors, through (8), generates dynamics in λ_t that, interacting with actual prices, give rise to the complex dynamics characterized in B&H, while preserving the general nature of the sentiments.

6 Conclusions

Looking back at the motivating example in Section (2), one can now see that if there were, for example, three predictors available, with say $\alpha_i = \left\{0, \frac{1}{2}, 1\right\}$, representing beliefs that are rational, naive and half-way through the two, complex dynamics would still emerge. Indeed, no matter how many predictors one includes with $\alpha_i \in [0, 1]$, irregular dynamics in prices would still emerge. On the other hand, if one allows enough predictors with forecast errors that fall on the other side of zero (that is, if one allows enough predictors with $\alpha_i < 0$), then stable prices can obtain. The shape of the set of beliefs allowed to compete under evolutionary dynamics is thus crucial for the outcomes of the model.

An essential element required in order to have cycles and complex dynamics in prices in the original B&H work is the difference in cost among predictors: if the cost of using different predictors is the same, in fact, the homogeneous equilibrium where all agents use the rational predictor would prevail. In order to have irregular price dynamics, an unstable homogeneous equilibrium is needed: since the fundamental equilibrium with rational agents is always stable (see B&H, Theorem 3.1), costs need to be introduced in order to destabilize it. But, as shown here, costs alone do not suffice to destabilize the market: one also needs a set of available predictors skewed in one particular direction.

These results highlight the importance of the set of predictors, or heuristics, allowed to compete under evolutionary schemes, as this determines the distribution of beliefs, which in turns restricts possible market outcomes. A skewed support for beliefs could be introduced, for example, with the purpose of capturing the idea of sentiments, where aggregate expectations are tilted to one side or the other of the rational predictor, due to optimism or pessimism of economic agents. Evolutionary forces then work on the skewed support, but cannot change it, and thus the general behavior of the system is, to some extent, predetermined. Being aware of this link between available strategies and admissible outcomes is important when employing evolutionary dynamics as a way to study the interplay of beliefs and market outcomes.

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