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Cost Minimization is Essential for the Sustainable Development of an Industry: A Mathematical Economic Model Approach

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ABSTRACT

The method of Lagrange multiplier is a very useful and powerful technique in multivariable calculus. In this study interpretation of Lagrange multiplier is given with satisfactory mathematical calculations and shows that its value is positive. For the sustainable development of an industry, cost minimization policy is crucial. In any industry the main objective is to minimize production cost for maximizing its profit. By considering Lagrange multiplier technique application an attempt has been taken here in cost minimization problem subject to production function as an output constraint. To predict future performance of an industry, mathematical calculations are necessary and all the procedures are given in this paper with detail mathematical procedures. In this study an attempt has been taken to minimize cost by considering three variables capital, labor, and other inputs of an industry by the application of economic models subject to a budget constraint, using Lagrange multiplier technique, as well as, using necessary and sufficient conditions for minimum value.

Keywords: Lagrange multiplier, cost minimization, mathematical economical models, sustainability

JEL Codes: C02, C61, C62, C65, C67

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Introduction

Mathematical modeling in economics is the application of mathematics to represent theories and analyze problems in economics. Formal economic modeling began in the 19th century with the use of differential calculus to represent and explain economic behavior of optimization [Samuelson, 1947; Herstein, 1953]. At present much of economic theory is presented in terms of mathematical economic models [Carter, 2001]. In this economic model we have used simple mathematical programming, such as differential and integral calculus, matrix algebra, etc. The language of mathematics helps the economists to make specific, positive claims about controversial subjects that would be impossible without mathematics [Chiang, 1984; Weintraub, 2008]. Optimization problems run through modern economics, many with explicit economic constraints. In the society, consumers maximize their utility subject to their budget constraints, and the firms and industries maximize their profits; subject to their production functions, input costs, and market demand [Dixit, 1990].

The method of Lagrange multipliers is a very useful and powerful technique in multivariable calculus that accelerates the determination of necessary conditions. It is a device for transferring a constrained problem to a higher dimensional unconstrained problem [Islam et al. 2010a,b; Mohajan, 2017a].

In this study we have considered cost minimization of a running industry by a Cobb-Douglas production function considering three variables capital, labor, and other inputs. We have also applied necessary and sufficient conditions to make the economic model for the cost minimization problem of an industry for its sustainable development. We have also provided reasonable interpretation of the Lagrange multiplier in the context of cost minimization problem, besides using it as a device for transforming a constrained problem into a higher dimensional unconstrained problem. First, we have examined an example to show the cost minimization problem. Later, we have considered "the implicit function theorem" which is important for solving a system of non-linear equations for the dependent variables and calculating partial derivatives of these variables with respect to the independent variables [Moolio & Islam, 2008; Baxley & Moorhouse 1984; Mohajan 2018b].

In this study the calculations in every step are given more explicitly so that the novice or the economist not sufficiently familiar with advanced mathematical concepts and manipulations can follow the steps relatively easily and interestingly. In the step of detail mathematical procedures, we have included five appendices to understand the mathematical and economical concepts very clearly and precisely.

Literature Review

Mathematical optimization refers to the selection of a best element from some set of available alternatives. An optimization problem includes cost minimization, production maximization, profit or revenue maximization, and utility maximization of a real function by using input values of the function and computing the corresponding values of the function as outputs. Optimization tries to find the best available element of some function and may use a variety of different computational optimization processes [Schmedders, 2008].

John V. Baxley and John C. Moorhouse have considered implicit functions with assumed characteristic qualitative features and have provided illustration of an example by generating meaningful economic behavior. According to them, at the start functions are not explicitly given but these have some assumed characteristic features, which are meaningful for and give insight into economic behavior. Later, explicit functions are considered to clarify the characteristics. For example, a firm wishes to minimize the cost of producing a given output; one may want to know how changes in the input prices will affect the situation. So the problem is not, "find the minimum", but, "assuming the minimum is obtained, what consequences can be deduced" [Baxley & Moorhouse, 1984]. Pahlaj Moolio and Jamal Nazrul Islam have examined the behavior of a competitive firm by the optimization process if the cost of a particular input increases, the firm needs to consider decreasing the level of that particular input; at the same time, there is no effect on the level of other inputs; also that when the demand of product increases, the firm should consider increasing its level of capital, labor and other inputs [Moolio & Islam, 2008].

Haradhan Kumar Mohajan has tried to optimize economic models subject to a budget constraint, using Lagrange multipliers technique, as well as, using necessary and sufficient conditions for optimal value. He has used mathematical economic models to show the economic optimizations in some details [Mohajan, 2017a]. In their book, Haradhan Kumar Mohajan and his coauthors, have analyzed the optimization techniques and social welfare economics. Their work predicts the economic optimization techniques with sufficient mathematical analysis. They have considered theoretically a variation of the problem by assuming that a government agency is allocated an annual budget and required to maximize and make available some sort of services to the community. They have tried to give aspects of economics and sociology with mathematical terms and logics [Mohajan et al., 2013].

In a paper, Haradhan Kumar Mohajan and his coauthors indicate that optimal economic policies increase wealth, change performance of environment policy and enhance sustainable use of natural capital. They are confirmed that present and future production can be maximized if an industry can efficiently minimize its production cost [Mohajan et al., 2012]. In another paper, Haradhan Kumar Mohajan has analyzed that sustainable economic development is an essential issue for the industrial organizations. The main objective of every industry is to follow the cost minimization policy for the profit maximization and sustainability of the industry [Mohajan, 2015].

Methodology of the Study

Word 'Research' is comprised of two syllables: *re-* and *search*. Here *re-* means again, and *search* is a verb means examine very carefully. Together they form a noun describing a careful, systematic, patient study and investigation in some field of knowledge, undertaken to establish facts or principles [Grinnell, 1993]. Research means a systematic investigation or activity to gain new knowledge of the already existing facts. Therefore, research is an essential and powerful tool in leading a researcher towards progress [Pandey & Pandey, 2015]. It emphasizes on creativity, which is carried in a systematic way to improve knowledge that consists of human knowledge, culture, and society. Research is needed to enhance our knowledge of what we already know; to extend our knowledge about aspects of the world of which we know either very little or nothing at all, and to enable us to better understand the world we live in [OECD, 2002; Sekaran, 2000].

'Method' is a word coined of two Greek elements: *meth-* and *odos.* The *meth-* is an element meaning 'after', *odos* means 'way'. A method is, therefore, a following after the way that someone found to be effective in solving a problem, of reaching an objective, in getting a job done. Greek element *ology* means 'the study of' [Leedy & Ormrod, 2001]. Methodology is a system of explicit rules and procedure in which research is based and against which claims of knowledge are evaluated [Ojo, 2003]. Research methodology indicates the logic of development of the process used to generate theory that is procedural framework within which the research is conducted [Remenyi et al., 1998]. Hence, research methodology is the systematic procedure adopted by researchers to solve a research problem that maps out the processes, approaches, techniques, research procedures, and instruments. It may be understood as a science of studying how research is done scientifically [Kothari, 2008]. In this study we have used secondary data to enrich this paper. The secondary data have been collected from both published and unpublished data sources. The published data are collected from books of famous authors, websites, national and international journals, e-journals, various publications of international organizations, handbooks, theses, magazines, newspapers, various statistical reports, historical documents, information on internet, etc. [Mohajan, 2020]

The works of this paper is mathematical modeling in economics where we have discussed simple mathematical techniques of cost minimization. In this study we have discussed the cost minimization policy by the method of Lagrange multiplier by considering three variables capital, labor, and other inputs of an industry. We have used necessary and sufficient conditions to make the economic model for the cost minimization problem of an industry for its sustainable development. We have tried our best to show the mathematical calculations in some details. The reliability and validity are inevitable issues in any research. In this study we have tried our best to maintain the reliability and validity throughout of the study [Mohajan, 2017b, 2018a].

Objective of the Study

The main objective of this study is to apply the cost minimization techniques to a running industry for the welfare both of the industry and the society. So that efficient production policy is essential for the sustainability of this industry. The other particular objectives are as follows:

- To provide a mathematical procedure for showing the findings more accurately.
- To discuss mathematical analysis of the model with interpretation of Lagrange multiplier.
- To give detail mathematical calculations with necessary and sufficient conditions for optimization.

Some Preliminary Concepts

Before going to the main study we are interested to discuss some basic ideas related to the paper. So that who are new in this field, will capture the full concept efficiently and interestingly.

Economic Sustainable Development Policy

A development is sustainable if the economic progress is widespread, productions in industries are in tremendous, extreme poverty is eliminated, social trust is encouraged through policies that strengthen the community, and the environment is protected from human-induced degradation [Welfare for the Future, 2002]. According to the direct Government website UK [Government Website UK, 2021] "*Sustainable development means a better quality of life now and for generations to come. It means not using up resources faster than the planet can replenish, or re-stock influences decision making with organizations, and therefore can go towards forming principles and business 'values'*." In the eye of Ecologist [Kates et al., 2005] "*Sustainable development is about maintaining a good quality of life for humanity by developing societal structures that recognize that humans are just one species living within and dependent on the environment. The focus is on preparing for a better future by recognizing humanity's dependence on nature*." Again we can define sustainable development as follows [Arrow et al., 2010; Dasgupta, 2010]: "*Sustainable development is an economic program along which average well-being of present and future generations, taken together, do not decline over time.*"

An economic development is sustainable if [Dasgupta & Mäler, 2001],

$$\frac{dU}{dt} \geq 0, \tag{1}$$

where U is utility function. The inequality (1) offers greater flexibility in ethical reasoning. It permits initial sacrifices in the current standard of living but requires that no future generation should have to experience a decline in their standard of living. If we consider the utility be a function of consumption, S and labor, L; then we can write (1) as;

$$\frac{dU}{dt} = U_S \frac{dC}{dt} + U_L \frac{dL}{dt} \geq 0. \tag{2}$$

An industry cannot be sustainable if it cannot apply efficient modern mathematical models and proper use of it resources. Every industry tries to minimize its all costs to increase profit for the survival in the market, and for the welfare of the workers and the society [Mohajan et al., 2012].

Sustainable Economy

A socially and environmentally sustainable economic system operating with the purpose of facilitating a good life with dignity for all while respecting nature as an integral part of life [Schildberg, 2014]. In a sustainable and caring society, the economy should be perceived as an instrument for assuring the development of human capabilities and the expansion of real freedoms [Dias, 2013]. A sustainable industry can be developed by using cost minimizing techniques for the maximization of its profit by the proper use of its capital, manpower, and other resources. If an industry do not follow the sustainable economic policy, in near future it will be collapsed. In the study we have only stressed on cost minimization strategy of the industry and have encouraged it to follow sustainable techniques for the welfare both of the industry and the society.

Optimization Techniques

Every industry's first aim is to optimize its costs (minimum), products and profits (maximum) in an efficient and satisfied way. Let us consider a function $f(\mathbf{x})$ of one variable $\mathbf{x}$, where $\mathbf{x}=(x_1,x_2,...,x_n)$. For a function $f(x)$ to be optimum (maximum or minimum) $\frac{df}{dx}=f'(x)=0$. If $\frac{d^2f}{dx^2}<0$ at $x=x_0$ the function is maximum at a point $x=x_0$, and if $\frac{d^2f}{dx^2}>0$ at $x=x_0$ the function is minimum at a point $x=x_0$. If $f(x,y)$ be a function of two variables x and y then for optimum, $\frac{\partial f}{\partial x}(i.e. f_x)=0=\frac{\partial f}{\partial y}(\text{i.e., } f_y)$, and $f_{xx}f_{yy}-f_{xy}^2>0$. If $f_{xx}>0(\text{and } f_{yy}>0)$, then the function has a minimum point, if $f_{xx}<0(\text{and } f_{yy}<0)$ then the function has a maximum point. For $f_{xx}f_{yy}-f_{xy}^2<0$, there is neither a maximum nor a minimum, but a saddle point. In all cases, the tangent plane at the extremum (maximum or minimum) or a saddle point to the surface $z=f(x,y)$, is parallel to the z-plane [Mohajan, 2018b] .

Constant Returns to Scale

Constant returns to scale (CRS) explained by Swedish economist Erik Lindahl [Lindahl, 1933]. The word scale refers to the long-run situation where all inputs are changed in the same proportion. If we increase all factors (scale) in a given proportion and the output increases in the same proportion, returns to scale are said to be constant. Hence CRS is a constant ratio between inputs and outputs. It occurs when increasing the number of inputs leads to an equivalent increase in the output. A plant with a CRS is equally efficient in producing small batches as it is in producing large batches. Let us consider a homogeneous production function $f(K,L)$ of degree 1, where K and L are factors of production capital and labor, respectively. Constant returns to scale indicates $f(aK,aL)=af(K,L)$ where constant $a\geq 0$. CRS exists if an industry increases all resources; labor, capital, and other inputs, by 20% (say), and output also increases by 20%. For example, an industry employs 5,000 workers in factory to produce 1 million units of a product each year. CRS exists if the scale of operation expands to 10,000 workers in that factory and production increases to exactly 2 million units each year [Mohajan, 2018b].

Increasing Returns to Scale

Increasing returns to scale (IRS) occurs when a firm increases its inputs, and a more than proportionate increase in production results. Mathematically, we can write, an industry has IRS if $f(aK,aL)>af(K,L)$ where constant $a\geq 0$. For example, in a year an industry employs 1,000 workers, uses 100 machines, and produces 1 million products. In the next year, it employs 2,000 workers, uses 200 machines (inputs doubled), and produces 2.5 million products (output more than doubled) [Mohajan, 2018b].

Decreasing Returns to Scale

Decreasing returns to scale (DRS) happens when the firm's output rises proportionately less than its inputs rise. Mathematically, we can write, a firm has DRS if $f(aK,aL)<af(K,L)$ where constant $a\geq 0$. For example, in year one, an industry employs 2,000 workers, uses 100 machines, and produces 2 million products. In the next year, it employs 4,000 workers, uses 200 machines (inputs

doubled), and produces 1.5 products million (output less than doubled) [Mohajan, 2018b].

The Implicit Function Theorem

Let us suppose $f(x_0, c_0) = 0$ and $\frac{\partial f(x_0, c_0)}{\partial x} \neq 0$. Then there exists a continuous implicit solution *x*(*c*), where *c* is some parameter, with derivative, $\frac{\partial x(c)}{\partial x} = -\frac{f_c(x(c), c)}{f_x(x(c), c)}$ for *c* close to c_0 [Nicholson & Snyder, 2008; Mohajan et al., 2013].

Comparative Static Analysis

In the society the behavior of the buyers and sellers often changes, which causes the shift of demand and supply curves to itself over time. In economics, it is important to analyze how these shifts affect equilibrium. Mathematically, we can write twelve partial derivatives as follows [Mohajan, 2017a]:

$$\begin{bmatrix} X_{P_1} & Y_{P_1} & L_{P_1} & \lambda_{P_1} \\ X_{P_2} & Y_{P_2} & L_{P_2} & \lambda_{P_2} \\ X_w & Y_w & L_w & \lambda_w \end{bmatrix}. \qquad (3)$$

Here *X* and *Y* are two commodities and *L* indicates total labors, and λ is Lagrange multiplier. Moreover, P_1 and P_1 are the prices of commodities *X* and *Y* respectively, and *w* is wage rate. The twelve partial derivatives in matrix (3) are called the *comparative statics* of the problem [Chiang 1984]. For example, if *P* be the price of a commodity *Y*, then $\frac{\partial Y}{\partial P} < 0$ indicates that if the price of commodity *Y* increases, the level of consumption of *Y* will decrease [Islam et al., 2010a,b; Mohajan, 2018b].

Shadow Price

The shadow price of a commodity is defined as its social opportunity cost, i.e., the net loss (gain) associated with having 1 unit less (more) of it. For example, if $\frac{\partial C}{\partial Q} = \lambda$, then if the firm wants to increase (decrease) 1 unit of its production, it would cause total cost to increase (decrease) by approximately λ units [Mohajan, 2018b].

Price Vector and Budget Constraint

Let P_1 be the cost of per kg of rice, and P_2 be the cost of per kg of wheat in dollar. We call $\mathbf{P} = (P_1, P_2)$ the price vector of possible bundles of rice and wheat. The total cost of the bundle x_1 and x_2 is, $P_1 x_1 + P_2 x_2 = \mathbf{P}.\mathbf{x}$. For bundle **x** with a price vector **P** let us consider one has maximum *c* amount of dollars to spend, then we can write, $\mathbf{P}.\mathbf{x} \le c$; ($\mathbf{P}.\mathbf{x}$ is the price of the bundle **x**) which is referred to as budget constraint [Mohajan, 2017b].

Mathematical Discussion of the Model

We consider that for the fixed price, an industry is under contract to produce and deliver quantity *Q* units of a commodity during a specified time, with the use of *K* quantity of capital, *L* quantity of labor, and *R* quantity of other inputs, into its production process. If the industry seeks to maximize its profit while meeting the terms of the contract, it's production policy can be characterized as a constrained cost minimization problem in which the firm chooses the least cost combination of three factors *K*, *L*, and *R* to produce *Q* quantity of products. To reach its target the industry must minimize its cost function [Moolio & Islam, 2008; Mohajan et al., 2013];

$$C(K, L, R) = rK + wL + \rho R, \tag{4}$$

subject to the constraint of production function;

$$Q = f(K, L, R), \tag{5}$$

where *r* is rate of interest or services of capital per unit of capital *K*; *w* is the wage rate per unit of labor *L*; and *ρ* is the cost per unit of other inputs *R*; while *f* is a suitable production function. We assume that second order partial derivatives of the function *f* with respect to the independent variables (factors) *K*, *L*, and *R* exist. Now we apply Lagrange multiplier *λ* in (4) and (5) with the Lagrangian function *U*, in a four-dimensional unconstrained problem as follows [Moolio & Islam, 2008]:

$$U(K, L, R, \lambda) = C(K, L, R) + \lambda(Q - f(K, L, R)). \tag{6}$$

We assume that the industry minimizes its cost, the optimal quantities K^*, L^*, R^*, λ^* of *K*, *L*, *R*, and λ that necessarily satisfy the first order conditions; which we obtained by partially differentiation of the Lagrangian function (6) with respect to four variables *K*, *L*, *R*, and λ; and setting them equal to zero we get,

$$U_\lambda = Q - f(K, L, R) = 0, \quad (7a)$$

$$U_K = C_K - \lambda f_K = 0, \quad (7b)$$

$$U_L = C_L - \lambda f_L = 0, \quad (7c)$$

$$U_R = C_R - \lambda f_R = 0, \quad (7d)$$

where $C_K = \frac{\partial C}{\partial K}$, etc. are partial derivatives. From (7b-d) we get the Lagrange multiplier as,

$$\lambda = \frac{C_K}{f_K} = \frac{C_L}{f_L} = \frac{C_R}{f_R}. \quad (8)$$

Considering the infinitesimal changes *dK*, *dL*, and *dR* in *K*, *L*, and *R* respectively; and the corresponding changes *dQ* and *dC* we yield,

$$dC = C_K dK + C_L dL + C_R dR, \quad (9)$$

$$dQ = f_K dK + f_L dL + f_R dR. \quad (10)$$

Dividing (9) by (10) and using (8) we get,

$$\frac{dC}{dQ} = \frac{C_K dK + C_L dL + C_R dR}{f_K dK + f_L dL + f_R dR} = \lambda. \quad (11)$$

Hence, the Lagrange multiplier can be interpreted as the marginal cost of production. It indicates that total cost will be increased from the production of an additional unit *Q* [Mohajan et al., 2013].

Cobb-Douglas Production Function

Let us consider the Cobb-Douglas production function *f* is given by [Humphery, 1997],

$$Q = f(K, L, R) = AK^a L^b R^c, \quad (12)$$

where *A* is the efficiency parameter reflecting the level of technology. Here *a*, *b*, and *c* are constants; *a* indicates the output of elasticity of capital measures the percentage change in *Q* for 1% change in *K* while *R* and *L* are held constants; *b* indicates the output of elasticity of labor, and *c* indicates the output of elasticity of other inputs in the production process, are exactly parallel to *a*. Now these three constants *a*, *b*, and *c* must satisfy the following three inequalities:

$$0 < a < 1, 0 < b < 1, \text{ and } 0 < c < 1. \quad (13)$$

A strict Cobb-Douglas production function, in which $a+b+c=1$, indicates constant returns to scale, $a+b+c<1$ indicates decreasing returns to scale, and $a+b+c>1$ indicates increasing returns to scale. A Cobb-Douglas production function is optimized subject to a budget constraint [Mohajan, 2018b]. Now using (4), (5), and (12) in (6) we get [Moolio & Islam, 2008],

$$U(K,L,R,\lambda)=rK+wL+\rho R+\lambda\left(Q-AK^{a}L^{b}R^{c}\right). \tag{14}$$

Taking the partial differentiations in (14) for minimization we get;

$$U_{\lambda}=Q-AK^{a}L^{b}R^{c}=0, \tag{15a}$$

$$U_{K}=r-a\lambda AK^{a-1}L^{b}R^{c}=0, \tag{15b}$$

$$U_{L}=w-b\lambda AK^{a}L^{b-1}R^{c}=0, \tag{15c}$$

$$U_{R}=\rho-c\lambda AK^{a}L^{b}R^{c-1}=0. \tag{15d}$$

From (15) we get,

$$K^{a}L^{b}R^{c}=\frac{Q}{A} \text{ and } \lambda=\frac{rK}{aAK^{a}L^{b}R^{c}}=\frac{wL}{bAK^{a}L^{b}R^{c}}=\frac{\rho R}{cAK^{a}L^{b}R^{c}}. \tag{16}$$

$$\Rightarrow \frac{rK}{a}=\frac{wL}{b}=\frac{\rho R}{c}. \tag{17}$$

Now we can write minimum cost as [See Appendix-I, equation (AI-8)],

$$C^{*}=\frac{\Phi r^{\left(\frac{a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}. \tag{18}$$

Equation (18) is the optimal cost in terms of r, w, A, a, b, c, Q, ρ, and $\Phi=a+b+c$. Now putting the known values in right side of (18) we can easily calculate the value of minimum cost $C=C^{*}$. From (4) and (15b-d) we get,

$$\frac{\partial C}{\partial Q}=C_{K}\frac{\partial K}{\partial Q}+C_{L}\frac{\partial L}{\partial Q}+C_{R}\frac{\partial R}{\partial Q}$$

$$=r\frac{\partial K}{\partial Q}+w\frac{\partial L}{\partial Q}+\rho\frac{\partial R}{\partial Q}$$

$$=\lambda\left[aAK^{a-1}L^{b}R^{c}\frac{\partial K}{\partial Q}+bAK^{a}L^{b-1}R^{c}\frac{\partial L}{\partial Q}+\lambda AK^{a}L^{b}R^{c-1}\frac{\partial R}{\partial Q}\right], \text{ by (15b-d)}. \tag{19}$$

Differentiating (15a) with respect to Q we get,

$$1=aAK^{a-1}L^{b}R^{c}\frac{\partial K}{\partial Q}+bAK^{a}L^{b-1}R^{c}\frac{\partial L}{\partial Q}+cAK^{a}L^{b}R^{c-1}\frac{\partial R}{\partial Q}. \tag{20}$$

From (19) and (20) we get,

$$\frac{\partial C}{\partial Q}=\lambda, \text{ i.e., } \frac{\partial C^{*}}{\partial Q}=\lambda^{*}. \tag{21}$$

We have observed that (21) verifies (11). So that, Lagrange multiplier λ^{*} indicates that if the industry wants to increase (decrease) one unit of its production, it would cause total cost to increase (decrease) by approximately λ^* units, i.e., the Lagrange multiplier is a shadow price. Hence, we have observed that Lagrange multiplier has some sort of reasonable interpretation [Mohajan, 2018b].

Sufficient Conditions for Cost Minimization

To minimize cost we consider the determinant of the Hessian matrix,

$$|H| = \begin{vmatrix} 0 & -Q_K & -Q_L & -Q_R \\ -Q_K & U_{KK} & U_{KL} & U_{KR} \\ -Q_L & U_{LK} & U_{LL} & U_{LR} \\ -Q_R & U_{RK} & U_{RL} & U_{RR} \end{vmatrix}. \tag{22}$$

We can simplify (22) as [See Appendix-II, equation (AII-4)],

$$|H| = -\Phi\left(\frac{r^{2\left(\frac{b+c}{\Phi}\right)} w^{2\left(\frac{a+c}{\Phi}\right)} \rho^{2\left(\frac{a+b}{\Phi}\right)} a^{\left(\frac{3a}{\Phi}\right)} b^{\left(\frac{3b}{\Phi}\right)} c^{\left(\frac{3c}{\Phi}\right)} A^{\left(\frac{4}{\Phi}\right)}}{r^{\left(\frac{2a}{\Phi}\right)} w^{\left(\frac{2b}{\Phi}\right)} \rho^{\left(\frac{2c}{\Phi}\right)} a^{\left(\frac{b+c}{\Phi}\right)} b^{\left(\frac{a+c}{\Phi}\right)} c^{\left(\frac{a+b}{\Phi}\right)} Q^2} \right) \tag{23}$$

Since $A, a, b, c, \Phi > 0$ and *r*, *w*, *ρ* are the rate of inputs and hence are positive, while *Q* is production, which will never be negative. Hence, $|H| < 0$, as required, consequently cost is minimum.

Comparative Statics Analysis

We consider necessary conditions and examine the sufficiency conditions for a solution K^*, L^*, R^* and λ^* to be least for a minimum cost of the industry. Now we solve the four equations (15a-d) for *K*, *L*, *R*, and *λ* in terms of *r*, *w*, *ρ* and *Q*, and compute sixteen partial derivatives (comparative statics). We have assumed that the left side of each equation in (15) is continuously differentiable and that the solution exists, then by the implicit function theorem *K*, *L*, *R*, and *λ* will each be continuously differentiable function of *r*, *w*, *ρ* and *Q*, if the following Jacobian matrix [Moolio & Islam, 2008; Mohajan et al., 2013]:

$$J = \begin{bmatrix} 0 & -Q_K & -Q_L & -Q_R \\ -Q_K & U_{KK} & U_{KL} & U_{KR} \\ -Q_L & U_{LK} & U_{LL} & U_{LR} \\ -Q_R & U_{RK} & U_{RL} & U_{RR} \end{bmatrix}, \tag{24}$$

is non-singular at the optimum point K^*, L^*, R^* and λ^*; and the determinant of Jacobian matrix $|J|$, is negative. In each case given above, the determinant of Jacobian matrix $|J|$ is equal to the determinant of Hessian matrix $|H|$, supporting the 'widespread economic folklore' and the 'economist's deep wish'! [Baxley & Moorhouse, 1984].

Let F be the vector-valued function which may be regarded as points in a 8-dimensional Euclidean space as $(\lambda^*, K^*, L^*, R^*, r, w, \rho, Q) \in R^8$. By the implicit functions theorem we get,

$\mathbf{F} = (F_1, F_2, F_3, F_4)$, $F_i = F_i(\lambda^*, K^*, L^*, R^*, r, w, \rho, Q) = 0$; $i = 1, 2, 3, 4$, (25)

may be solved in the form as,

$$\begin{bmatrix} \lambda^* \\ K^* \\ L^* \\ R^* \end{bmatrix} = \mathbf{G}(r, w, \rho, Q). \tag{26}$$

where $\mathbf{G} = (G_1, G_2, G_3, G_4)$, being a four vector valued function of r, w, ρ, and Q. Moreover, the Jacobian matrix for G, J_G is given by;

$$J_G = \begin{bmatrix} \frac{\partial \lambda^*}{\partial r} & \frac{\partial \lambda^*}{\partial w} & \frac{\partial \lambda^*}{\partial \rho} & \frac{\partial \lambda^*}{\partial Q} \\ \frac{\partial K^*}{\partial r} & \frac{\partial K^*}{\partial w} & \frac{\partial K^*}{\partial \rho} & \frac{\partial K^*}{\partial Q} \\ \frac{\partial L^*}{\partial r} & \frac{\partial L^*}{\partial w} & \frac{\partial L^*}{\partial \rho} & \frac{\partial L^*}{\partial Q} \\ \frac{\partial R^*}{\partial r} & \frac{\partial R^*}{\partial w} & \frac{\partial R^*}{\partial \rho} & \frac{\partial R^*}{\partial Q} \end{bmatrix} = -J^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{27}$$

where the i^{th} row in the last matrix on the right is obtained by differentiating the i^{th} left side in (27) with respect to r then w, then ρ, and then Q. Let C_{ij} be the cofactor of the element in the i^{th} row and j^{th} column of J, and then inverting J using the method of cofactor gives:

$J^{-1} = \frac{1}{|J|} C^T$, where $C = (C_{ij})$, the matrix of cofactors of J, and T for transpose.

Thus, equation (27) can further be expressed in the following form:

$$\begin{bmatrix} \frac{\partial \lambda^*}{\partial r} & \frac{\partial \lambda^*}{\partial w} & \frac{\partial \lambda^*}{\partial \rho} & \frac{\partial \lambda^*}{\partial Q} \\ \frac{\partial K^*}{\partial r} & \frac{\partial K^*}{\partial w} & \frac{\partial K^*}{\partial \rho} & \frac{\partial K^*}{\partial Q} \\ \frac{\partial L^*}{\partial r} & \frac{\partial L^*}{\partial w} & \frac{\partial L^*}{\partial \rho} & \frac{\partial L^*}{\partial Q} \\ \frac{\partial R^*}{\partial r} & \frac{\partial R^*}{\partial w} & \frac{\partial R^*}{\partial \rho} & \frac{\partial R^*}{\partial Q} \end{bmatrix} = -\frac{1}{|J|} C^T \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} \frac{\partial \lambda^*}{\partial r} & \frac{\partial \lambda^*}{\partial w} & \frac{\partial \lambda^*}{\partial \rho} & \frac{\partial \lambda^*}{\partial Q} \\ \frac{\partial K^*}{\partial r} & \frac{\partial K^*}{\partial w} & \frac{\partial K^*}{\partial \rho} & \frac{\partial K^*}{\partial Q} \\ \frac{\partial L^*}{\partial r} & \frac{\partial L^*}{\partial w} & \frac{\partial L^*}{\partial \rho} & \frac{\partial L^*}{\partial Q} \\ \frac{\partial R^*}{\partial r} & \frac{\partial R^*}{\partial w} & \frac{\partial R^*}{\partial \rho} & \frac{\partial R^*}{\partial Q} \end{bmatrix} = -\frac{1}{|J|} \begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} \\ C_{12} & C_{22} & C_{32} & C_{42} \\ C_{13} & C_{23} & C_{33} & C_{43} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} \frac{\partial \lambda^*}{\partial r} & \frac{\partial \lambda^*}{\partial w} & \frac{\partial \lambda^*}{\partial \rho} & \frac{\partial \lambda^*}{\partial Q} \\ \frac{\partial K^*}{\partial r} & \frac{\partial K^*}{\partial w} & \frac{\partial K^*}{\partial \rho} & \frac{\partial K^*}{\partial Q} \\ \frac{\partial L^*}{\partial r} & \frac{\partial L^*}{\partial w} & \frac{\partial L^*}{\partial \rho} & \frac{\partial L^*}{\partial Q} \\ \frac{\partial R^*}{\partial r} & \frac{\partial R^*}{\partial w} & \frac{\partial R^*}{\partial \rho} & \frac{\partial R^*}{\partial Q} \end{bmatrix} = -\frac{1}{|J|} \begin{bmatrix} C_{21} & C_{31} & C_{41} & C_{11} \\ C_{22} & C_{32} & C_{42} & C_{12} \\ C_{23} & C_{33} & C_{43} & C_{13} \\ C_{24} & C_{34} & C_{44} & C_{14} \end{bmatrix}. \qquad (28)$$

We have for minimum cost, $|J| = |H|$, therefore, by substituting the value of $|\overline{H}|$ from equation (AII-3) we can write,

$$|J| = |H| = -\Phi abc\lambda^2 A^4 K^{4a-2} L^{4b-2} R^{4c-2}. \qquad (28a)$$

Now, we study the effects of changes in r, w, ρ, and Q, on K, L, and R. We can find the effect on capital K when the output Q of the industry increases [See Appendix-III, equation (AIII-4)],

$$\frac{\partial K^*}{\partial Q} > 0, \qquad (29)$$

which indicates that if the capital K increases, the firm has to consider increasing the level of output Q. Similarly, we can obtain,

$$\frac{\partial L^*}{\partial Q} > 0, \qquad \text{and} \qquad (30)$$

$$\frac{\partial R^*}{\partial Q} > 0. \tag{31}$$

Hence, the demand of the product increases the firm may consider increasing its level of inputs: capital, labor, and other inputs.

We can find the effect on capital K when its interest rate r increases [See Appendix-IV, equation (AIV-4)],

$$\frac{\partial K^*}{\partial r} < 0, \tag{32}$$

which indicates that if the interest rate of the capital K increases, the firm has to consider decreasing the level of input K.

We can examine the effects on labor L when the interest rate of capital K increases [See Appendix-V, equation (AV-3)],

$$\frac{\partial L^*}{\partial r} > 0 \tag{33}$$

which indicates that when the interest rate of the capital increases the firm can increase the level of labor L. In our common sense both inputs K and L are not related to each other, i.e., $C_{KL} = 0$. Hence, both inputs K and L are neither complement nor supplement, but they are unrelated.

Finally, we comment from our assumption and common sense that when the demand of the product of an industry increases, it must try to increase the production, then the industry has to consider increasing its level of inputs, such as capital, labor and other inputs. But in any situation the industry tries to minimize the cost for the maximization of its profit. As a result, in the long-run the industry will be sustainable.

Conclusion and Recommendation

In this study we have used Lagrange multiplier method to obtain cost minimization problem of an industry subject to Cobb-Douglas production function as an output constraint. We have shown that the value of the Lagrange multiplier is positive, and in our study it indicates shadow price. We have also used necessary and sufficient conditions to obtain minimum cost of the industry. In the beginning we have used sustainable economic policies for the sustainable development of the industry. We have included five appendices to make the paper interesting to the readers. In the study we have applied implicit function theorem and comparative static analysis. Throughout the paper we have tried to present mathematical calculations in some details.

Appendix-I

From (16) we get,

$$K=\frac{Q^{1/a}}{A^{1/a}L^{b/a}R^{c/a}},\quad L=\frac{Q^{1/b}}{A^{1/b}K^{a/b}R^{c/b}},\text{ and}\quad R=\frac{Q^{1/c}}{A^{1/c}K^{a/c}L^{b/c}}. \qquad \text{(AI-1)}$$

Again from (16) we get,

$$K^{\left(\frac{a+c}{c}\right)}=\frac{a\,\rho\,Q^{1/c}}{c\,rA^{1/c}L^{b/c}}$$

$$\Rightarrow K=\left[\frac{a\,\rho\,Q^{1/c}}{c\,r\,A^{1/c}L^{b/c}}\right]^{\left(\frac{c}{a+c}\right)}$$

$$\Rightarrow K=\frac{a^{\left(\frac{c}{a+c}\right)}\rho^{\left(\frac{c}{a+c}\right)}Q^{\left(\frac{1}{a+c}\right)}}{c^{\left(\frac{c}{a+c}\right)}r^{\left(\frac{c}{a+c}\right)}A^{\left(\frac{1}{a+c}\right)}L^{\left(\frac{b}{a+c}\right)}}. \qquad \text{(AI-2)}$$

Similarly from (16) we get,

$$L=\frac{b^{\left(\frac{c}{b+c}\right)}\rho^{\left(\frac{c}{b+c}\right)}Q^{\left(\frac{1}{b+c}\right)}}{c^{\left(\frac{c}{b+c}\right)}w^{\left(\frac{c}{b+c}\right)}A^{\left(\frac{1}{b+c}\right)}K^{\left(\frac{a}{b+c}\right)}}. \qquad \text{(AI-3)}$$

Now using the values of *K* and *L* from equation (AI-2) and (AI-3) respectively in (17), we get,

$$K=\frac{awL}{br}=\frac{awb^{\left(\frac{c}{b+c}\right)}\rho^{\left(\frac{c}{b+c}\right)}Q^{\left(\frac{1}{b+c}\right)}}{brc^{\left(\frac{c}{b+c}\right)}w^{\left(\frac{c}{b+c}\right)}A^{\left(\frac{1}{b+c}\right)}K^{\left(\frac{a}{b+c}\right)}}$$

$$\Rightarrow K^{\left(\frac{a+b+c}{b+c}\right)}=\frac{awb^{\left(\frac{c}{b+c}\right)}\rho^{\left(\frac{c}{b+c}\right)}Q^{\left(\frac{1}{b+c}\right)}}{brc^{\left(\frac{c}{b+c}\right)}w^{\left(\frac{c}{b+c}\right)}A^{\left(\frac{1}{b+c}\right)}}$$

$$\Rightarrow K=\left[\frac{awb^{\left(\frac{c}{b+c}\right)}\rho^{\left(\frac{c}{b+c}\right)}Q^{\left(\frac{1}{b+c}\right)}}{brc^{\left(\frac{c}{b+c}\right)}w^{\left(\frac{c}{b+c}\right)}A^{\left(\frac{1}{b+c}\right)}}\right]^{\left(\frac{b+c}{a+b+c}\right)}$$

$$=\frac{a^{\left(\frac{b+c}{a+b+c}\right)}w^{\left(\frac{b}{a+b+c}\right)}\rho^{\left(\frac{c}{a+b+c}\right)}Q^{\left(\frac{1}{a+b+c}\right)}}{b^{\left(\frac{b}{a+b+c}\right)}c^{\left(\frac{c}{a+b+c}\right)}r^{\left(\frac{b+c}{a+b+c}\right)}A^{\left(\frac{1}{a+b+c}\right)}}$$

$$K=K^{*}=\frac{a^{\left(\frac{b+c}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}r^{\left(\frac{b+c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}} \qquad \text{(AI-4)}$$

where $\Phi=a+b+c.$ Similarly we get,

$$L=L^{*}=\frac{b^{\left(\frac{a+c}{\Phi}\right)}r^{\left(\frac{a}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}w^{\left(\frac{a+c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}, \text{ and} \qquad \text{(AI-5)}$$

$$R=R^{*}=\frac{c^{\left(\frac{a+b}{\Phi}\right)}r^{\left(\frac{a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}b^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{a+b}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}. \qquad \text{(AI-6)}$$

Using (AI-4-6) in (16) we get the optimal value of Lagrange multiplier λ^{*} as;

$$\lambda=\frac{rK}{aAK^{a}L^{b}R^{c}}=\frac{r\dfrac{a^{\left(\frac{b+c}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}r^{\left(\frac{b+c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}}{aA\left[\dfrac{a^{\left(\frac{b+c}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}r^{\left(\frac{b+c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]^{a}\left[\dfrac{b^{\left(\frac{a+c}{\Phi}\right)}r^{\left(\frac{a}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}w^{\left(\frac{a+c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]^{b}\left[\dfrac{c^{\left(\frac{a+b}{\Phi}\right)}r^{\left(\frac{a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}b^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{a+b}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]^{c}}$$

.

After some straightforward calculations we obtain,

$$\lambda=\lambda^{*}=\frac{r^{\left(\frac{a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1-\Phi}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}. \qquad \text{(AI-7)}$$

Now using the values of *K*, *L*, and *R* from (AI-4-6) in equation (4) we can write,

$$C=r\frac{a^{\left(\frac{b+c}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}r^{\left(\frac{b+c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}+w\frac{b^{\left(\frac{a+c}{\Phi}\right)}r^{\left(\frac{a}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}w^{\left(\frac{a+c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}+\rho\frac{c^{\left(\frac{a+b}{\Phi}\right)}r^{\left(\frac{a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}b^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{a+b}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}$$

$$=\frac{ar^{\left(\frac{a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}+br^{\left(\frac{a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}+cr^{\left(\frac{a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}$$

$$C=C^{*}=\frac{ar^{\left(\frac{a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}(a+b+c)}{a^{\left(\frac{a}{\Phi}\right)}b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}$$

$$C^{*}=\frac{ar^{\left(\frac{a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}\Phi}{a^{\left(\frac{a}{\Phi}\right)}b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}. \quad \text{(AI-8)}$$

Appendix-II

From (22) we have the determinant of the Hessian matrix,

$$|H|=\begin{vmatrix}0 & -Q_K & -Q_L & -Q_R\\ -Q_K & U_{KK} & U_{KL} & U_{KR}\\ -Q_L & U_{LK} & U_{LL} & U_{LR}\\ -Q_R & U_{RK} & U_{RL} & U_{RR}\end{vmatrix}$$

$$=Q_K\begin{vmatrix}-Q_K & U_{KL} & U_{KR}\\ -Q_L & U_{LL} & U_{LR}\\ -Q_R & U_{RL} & U_{RR}\end{vmatrix}-Q_L\begin{vmatrix}-Q_K & U_{KK} & U_{KR}\\ -Q_L & U_{LK} & U_{LR}\\ -Q_R & U_{RK} & U_{RR}\end{vmatrix}+Q_R\begin{vmatrix}-Q_K & U_{KK} & U_{KL}\\ -Q_L & U_{LK} & U_{LL}\\ -Q_R & U_{RK} & U_{RL}\end{vmatrix}$$

$$=Q_K\left\{-Q_K\begin{vmatrix}U_{LL} & U_{LR}\\ U_{RL} & U_{RR}\end{vmatrix}-U_{KL}\begin{vmatrix}-Q_L & U_{LR}\\ -Q_R & U_{RR}\end{vmatrix}+U_{KR}\begin{vmatrix}-Q_L & U_{LL}\\ -Q_R & U_{RL}\end{vmatrix}\right\}$$

$$-Q_L\left\{-Q_K\begin{vmatrix}U_{LK} & U_{LR}\\ U_{RK} & U_{RR}\end{vmatrix}-U_{KK}\begin{vmatrix}-Q_L & U_{LR}\\ -Q_R & U_{RR}\end{vmatrix}+U_{KR}\begin{vmatrix}-Q_L & U_{LK}\\ -Q_R & U_{RK}\end{vmatrix}\right\}$$

$$+Q_R\left\{-Q_K\begin{vmatrix}U_{LK} & U_{LL}\\ U_{RK} & U_{RL}\end{vmatrix}-U_{KK}\begin{vmatrix}-Q_L & U_{LL}\\ -Q_R & U_{RL}\end{vmatrix}+U_{KL}\begin{vmatrix}-Q_L & U_{LK}\\ -Q_R & U_{RK}\end{vmatrix}\right\}$$

$$=-Q_KQ_KU_{LL}U_{RR}+Q_KQ_KU_{LR}U_{LR}+2Q_KQ_LU_{KL}U_{RR}-2Q_KQ_RU_{KL}U_{LR}-2Q_KQ_LU_{KR}U_{LR}+2Q_KQ_RU_{KR}U_{LL}$$
$$-Q_LQ_LU_{KK}U_{RR}+2Q_LQ_RU_{KK}U_{LR}+Q_LQ_LU_{KR}U_{KR}-2Q_LQ_RU_{KL}U_{KR}-Q_RQ_RU_{KK}U_{LL}+Q_RQ_RU_{KL}U_{KL}.$$

First order derivatives of (12) are,

$$Q_K=aAK^{a-1}L^bR^c,\ Q_L=bAK^aL^{b-1}R^c,\ Q_R=cAK^aL^bR^{c-1}. \quad \text{(AII-1)}$$

From (15) we get,

$$U_{KK}=-a(a-1)\lambda AK^{a-2}L^bR^c,\ U_{LL}=-b(b-1)\lambda AK^aL^{b-2}R^c,$$
$$U_{RR}=-c(c-1)\lambda AK^aL^bR^{c-2},\ U_{KL}=U_{LK}=-ab\lambda AK^{a-1}L^{b-1}R^c, \quad \text{(AII-2)}$$
$$U_{KR}=U_{RK}=-ac\lambda AK^{a-1}L^bR^{c-1},\ U_{LR}=U_{RL}=-bc\lambda AK^aL^{b-1}R^{c-1}.$$

Using (AII-1) and (AII-2) we can calculate the determinant of Hessian matrix as,

$$
\begin{aligned}
|H| =& -a^2bc(b-1)(c-1)\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2} + a^2b^2c^2\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2} \\
&+ 2a^2b^2c(c-1)\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2} - 2a^2b^2c^2\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2} \\
&- 2a^2b^2c^2\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2} + 2a^2bc^2(b-1)\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-1} \\
&- ab^2c(a-1)(c-1)\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2} + 2ab^2c^2(a-1)\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2} \\
&+ a^2b^2c^2\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2} - 2a^2b^2c^2\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2} \\
&- abc^2(a-1)(b-1)\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2} + a^2b^2c^2\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2} \\
=& abc\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2}\begin{Bmatrix} -a(b-1)(c-1)+abc+2ab(c-1)-2abc-2abc+2ac(b-1) \\ -b(a-1)(c-1)+2bc(a-1)+abc-2abc-c(a-1)(b-1)+abc \end{Bmatrix} \\
=& -(a+b+c)abc\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2} \\
=& -\Phi abc\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2}. \qquad \text{(AII-3)}
\end{aligned}
$$

By substituting the critical values of *K*, *L*, *R,* and *λ* from (AI-4-7) into the equation (AII-3) we can write the determinant of Hessian matrix as,

$$
|H| = -\frac{\Phi abcA^4\left[\dfrac{r^{\left(\frac{a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1-\Phi}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]^2\left[\dfrac{a^{\left(\frac{b+c}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}r^{\left(\frac{b+c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]^{4a-2}}{\left[\dfrac{b^{\left(\frac{a+c}{\Phi}\right)}r^{\left(\frac{a}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}w^{\left(\frac{a+c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]^{2-4b}\left[\dfrac{c^{\left(\frac{a+b}{\Phi}\right)}r^{\left(\frac{a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}b^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{a+b}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]^{2-4c}}.
$$

After a straightforward mathematical calculations we finally obtain,

$$
|H| = -\Phi\left(\frac{r^{2\left(\frac{b+c}{\Phi}\right)}w^{2\left(\frac{a+c}{\Phi}\right)}\rho^{2\left(\frac{a+b}{\Phi}\right)}a^{\left(\frac{3a}{\Phi}\right)}b^{\left(\frac{3b}{\Phi}\right)}c^{\left(\frac{3c}{\Phi}\right)}A^{\left(\frac{4}{\Phi}\right)}}{r^{\left(\frac{2a}{\Phi}\right)}w^{\left(\frac{2b}{\Phi}\right)}\rho^{\left(\frac{2c}{\Phi}\right)}a^{\left(\frac{b+c}{\Phi}\right)}b^{\left(\frac{a+c}{\Phi}\right)}c^{\left(\frac{a+b}{\Phi}\right)}Q^2}\right). \qquad \text{(AII-4)}
$$

Appendix-III

Let us suppose, the industry gets an additional order of its products. So, it wants to increase its production to produce and supply highest quantity of its products to yield maximum profit. Then naturally, we can expect that it will take attempts to increase its inputs (capital, labor, and other inputs). From equation (28) we get,

$$
\frac{\partial K}{\partial Q} = -\frac{1}{|J|}\times \text{cofactor of } [C_{12}]
$$

$$=\frac{1}{|J|}\begin{vmatrix} -Q_K & U_{KL} & U_{KR} \\ -Q_L & U_{LL} & U_{LR} \\ -Q_R & U_{RL} & U_{RR} \end{vmatrix}$$

$$=\frac{1}{|J|}\left\{-Q_K\begin{vmatrix} U_{LL} & U_{LR} \\ U_{RL} & U_{RR} \end{vmatrix}-U_{KL}\begin{vmatrix} -Q_L & U_{LR} \\ -Q_R & U_{RR} \end{vmatrix}+U_{KR}\begin{vmatrix} -Q_L & U_{LL} \\ -Q_R & U_{RL} \end{vmatrix}\right\}$$

$$=\frac{1}{|J|}\left\{-Q_K\left(U_{LL}U_{RR}-U_{LR}U_{RL}\right)-U_{KL}\left(-Q_LU_{RR}+Q_RU_{LR}\right)+U_{KR}\left(-Q_LU_{RL}+Q_RU_{LL}\right)\right\}$$

$$=\frac{1}{|J|}\left\{-Q_KU_{LL}U_{RR}+Q_KU_{LR}U_{RL}+Q_LU_{KL}U_{RR}-Q_RU_{KL}U_{LR}-Q_LU_{KR}U_{RL}+Q_RU_{LL}U_{KR}\right\}. \quad \text{(AIII-1)}$$

By using (AII-1) and (AII-2) in (AIII-1) we can write,

$$\frac{\partial K}{\partial Q}=\frac{1}{|J|}\left\{\begin{array}{l} -ab(b-1)c(c-1)\lambda^2A^3K^{3a-1}L^{3b-2}R^{3c-2}+ab^2\gamma^2\lambda^2A^3K^{3a-1}L^{3b-2}R^{3c-2} \\ +ab^2c(c-1)\lambda^2A^3K^{3a-1}L^{3b-2}R^{3c-2}-ab^2c^2\lambda^2A^3K^{3a-1}L^{3b-2}R^{3c-2} \\ -ab^2c^2\lambda^2A^3K^{3a-1}L^{3b-2}R^{3c-2}+ab(b-1)c^2\lambda^2A^3K^{3a-1}L^{3b-2}R^{3c-2} \end{array}\right\}.$$

$$=\frac{1}{|J|}abc\lambda^2A^3K^{3a-1}L^{3b-2}R^{3c-2}\left\{-(b-1)(c-1)+bc+b(c-1)-bc-bc+c(b-1)\right\}.$$

$$=-\frac{1}{|J|}abc\lambda^2A^3K^{3a-1}L^{3b-2}R^{3c-2}$$

$$=\frac{abc\lambda^2A^3K^{3a-1}L^{3b-2}R^{3c-2}}{\Phi abc\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2}}, \qquad \text{by (28a)},$$

$$=\frac{1}{\Phi AK^{a-1}L^bR^c}. \quad \text{(AIII-2)}$$

Now using the values of *K*, *L*, and *R* from (AI-4-6) in equation (AIII-2) we get,

$$\frac{\partial K}{\partial Q}=\frac{\left[\dfrac{c^{\left(\frac{a+b}{\Phi}\right)}r^{\left(\frac{a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}b^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{a+b}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]^{-c}}{\Phi A\left[\dfrac{a^{\left(\frac{b+c}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}r^{\left(\frac{b+c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]^{a-1}\left[\dfrac{b^{\left(\frac{a+c}{\Phi}\right)}r^{\left(\frac{a}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}w^{\left(\frac{a+c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]^{b}}.$$

After some mathematical calculations we get,

$$\frac{\partial K}{\partial Q}=\frac{1}{\Phi}\left[\frac{a^{\left(\frac{\Phi-a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1-\Phi}{\Phi}\right)}}{r^{\left(\frac{\Phi-a}{\Phi}\right)}b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]. \quad \text{(AIII-3)}$$

Since $A,b,c,\Phi>0$ and $r,w,\rho>0$, and also Q is the output of the firm that can never be negative, obviously;

$$\frac{\partial K}{\partial Q}=\frac{\partial K^{*}}{\partial Q}>0. \qquad \text{(AIII-4)}$$

Appendix-IV

Now we want to study the effects on capital *K* when its interest rate increases. From the equation (28) we find that,

$$\frac{\partial K}{\partial r}=-\frac{1}{|J|}\times \text{Cofactor of } [C_{22}]$$

$$=-\frac{1}{|J|}\begin{vmatrix} 0 & -Q_L & -Q_R \\ -Q_L & U_{LL} & U_{LR} \\ -Q_R & U_{RL} & U_{RR} \end{vmatrix}$$

$$=-\frac{1}{|J|}\left\{Q_L\begin{vmatrix} -Q_L & U_{LR} \\ -Q_R & U_{RR}\end{vmatrix}-Q_R\begin{vmatrix} -Q_L & U_{LL} \\ -Q_R & U_{RL}\end{vmatrix}\right\}$$

$$=-\frac{1}{|J|}\{Q_L(-Q_LU_{RR}+Q_RU_{LR})-Q_R(-Q_LU_{RL}+Q_RU_{LL})\}$$

$$=-\frac{1}{|J|}\{-Q_LQ_LU_{RR}+Q_LQ_RU_{LR}+Q_LQ_RU_{RL}-Q_RQ_RU_{LL}\}.$$

$$=-\frac{1}{|J|}\{-Q_LQ_LU_{RR}+2\,Q_LQ_RU_{LR}-Q_RQ_RU_{LL}\}.$$

(AIV-1)

By using (AII-1) and (AII-2) in (AIV-1) we can write,

$$\frac{\partial K}{\partial r}=-\frac{1}{|J|}\left\{b^2c(c-1)\lambda A^3K^{3a}L^{3b-2}R^{3c-2}-2b^2c^2\lambda A^3K^{3a}L^{3b-2}R^{3c-2}+b(b-1)c^2\lambda A^3K^{3a}L^{3b-2}R^{3c-2}\right\}$$

$$=-\frac{1}{|J|}\left(bc\lambda A^3K^{3a}L^{3b-2}R^{3c-2}\right)\{b(c-1)-2bc+c(b-1)\}$$

$$=-\frac{1}{|J|}\left(bc\lambda A^3K^{3a}L^{3b-2}R^{3c-2}\right)\{bc-b-2bc+bc-c\}$$

$$=\frac{1}{|J|}(b+c)bc\lambda A^3K^{3a}L^{3b-2}R^{3c-2}$$

$$=-\frac{(b+c)bc\lambda A^3K^{3a}L^{3b-2}R^{3c-2}}{(a+b+c)abc\lambda^2A^4K^{4a-2}L^{4b-2}R^{4c-2}}, \qquad \text{by (28a),}$$

$$=-\frac{(b+c)}{a(a+b+c)\lambda AK^{a-2}L^bR^c}. \qquad \text{(AIV-2)}$$

Now using the values of *K*, *L*, *R,* and *λ* from (AI-4-7) in equation (AIV-2) we get,

$$\frac{\partial K}{\partial r}=-\frac{(\Phi-a)\left[\frac{b^{\left(\frac{a+c}{\Phi}\right)}r^{\left(\frac{a}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}w^{\left(\frac{a+c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]^{-b}\left[\frac{c^{\left(\frac{a+b}{\Phi}\right)}r^{\left(\frac{a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}b^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{a+b}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]^{-c}}{a\Phi A\left[\frac{r^{\left(\frac{a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1-\Phi}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)}b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]\left[\frac{a^{\left(\frac{b+c}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}r^{\left(\frac{b+c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]^{a-2}}.$$

After some mathematical calculations we find,

$$\frac{\partial K}{\partial r}=-\frac{(\Phi-a)}{\Phi}\left[\frac{a^{\left(\frac{\Phi-a}{\Phi}\right)}w^{\left(\frac{b}{\Phi}\right)}\rho^{\left(\frac{c}{\Phi}\right)}Q^{\left(\frac{1}{\Phi}\right)}}{r^{\left(\frac{2\Phi-a}{\Phi}\right)}b^{\left(\frac{b}{\Phi}\right)}c^{\left(\frac{c}{\Phi}\right)}A^{\left(\frac{1}{\Phi}\right)}}\right]. \qquad \text{(AIV-3)}$$

Since $A,b,c>0$, $r,w,\rho>0$, and *Q* is the output of the firm that can never be negative. Hence, (AIV-3) offers,

$$\frac{\partial K^*}{\partial r}=\frac{\partial K^*}{\partial r}<0, \qquad \text{(AIV-4)}$$

which indicates that if the interest rate of the capital *K* increases, the firm has to consider decreasing the level of input *K*.

Appendix-V

Now we examine the effects on labor *L* when the interest rate of capital *K* increases. From the equation (22), we find that,

$$\frac{\partial L}{\partial r}=-\frac{1}{|J|}\times \text{Cofactor of } [C_{23}]$$

$$=\frac{1}{|J|}\begin{vmatrix} 0 & -Q_K & -Q_R \\ -Q_L & U_{LK} & U_{LR} \\ -Q_R & U_{RK} & U_{RR} \end{vmatrix}$$

$$=\frac{1}{|J|}\left\{Q_K\begin{vmatrix} -Q_L & U_{LR} \\ -Q_R & U_{RR}\end{vmatrix}-Q_R\begin{vmatrix} -Q_L & U_{LK} \\ -Q_R & U_{RK}\end{vmatrix}\right\}$$

$$=\frac{1}{|J|}\{Q_K(-Q_LU_{RR}+Q_RU_{LR})-Q_R(-Q_LU_{RK}+Q_RU_{LK})\}$$

$$= \frac{1}{|J|}\left\{-Q_K Q_L U_{RR} + Q_K Q_R U_{LR} + Q_L Q_R U_{RK} - Q_R Q_R U_{LK}\right\}. \qquad \text{(AV-1)}$$

By using (AII-1) and (AII-2) in (AV-1) we can write,

$$\frac{\partial L}{\partial r} = \frac{1}{|J|}\left\{\begin{matrix} abc(c-1)\lambda A^3 K^{3a-1} L^{3b-1} R^{3c-2} - abc^2 \lambda A^3 K^{3a-1} L^{3b-1} R^{3c-2} \\ - abc^2 \lambda A^3 K^{3a-1} L^{3b-1} R^{3c-2} + abc^2 \lambda A^3 K^{3a-1} L^{3b-1} R^{3c-2} \end{matrix}\right\}$$

$$= \frac{1}{|J|}\left(abc\lambda A^3 K^{3a-1} L^{3b-1} R^{3c-2}\right)(c-1-c-c+c)$$

$$= -\frac{1}{|J|} abc\lambda A^3 K^{3a-1} L^{3b-1} R^{3c-2}$$

$$= \frac{abc\lambda A^3 K^{3a-1} L^{3b-1} R^{3c-2}}{\Phi abc\lambda^2 A^4 K^{4a-2} L^{4b-2} R^{4c-2}}, \qquad \text{by (28a),}$$

$$= \frac{1}{\Phi \lambda A K^{a-1} L^{b-1} R^{c}}. \qquad \text{(AV-2)}$$

Now using the values of *K*, *L*, *R,* and *λ* from (AI-4-7) in (AV-2) we get,

$$\frac{\partial L}{\partial r} = \frac{\left[\dfrac{b^{\left(\frac{a+c}{\Phi}\right)} r^{\left(\frac{a}{\Phi}\right)} \rho^{\left(\frac{c}{\Phi}\right)} Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)} c^{\left(\frac{c}{\Phi}\right)} w^{\left(\frac{a+c}{\Phi}\right)} A^{\left(\frac{1}{\Phi}\right)}}\right]^{1-b} \left[\dfrac{c^{\left(\frac{a+b}{\Phi}\right)} r^{\left(\frac{a}{\Phi}\right)} w^{\left(\frac{b}{\Phi}\right)} Q^{\left(\frac{1}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)} b^{\left(\frac{b}{\Phi}\right)} \rho^{\left(\frac{a+b}{\Phi}\right)} A^{\left(\frac{1}{\Phi}\right)}}\right]^{-c}}{\Phi A \left[\dfrac{r^{\left(\frac{a}{\Phi}\right)} w^{\left(\frac{b}{\Phi}\right)} \rho^{\left(\frac{c}{\Phi}\right)} Q^{\left(\frac{1-\Phi}{\Phi}\right)}}{a^{\left(\frac{a}{\Phi}\right)} b^{\left(\frac{b}{\Phi}\right)} c^{\left(\frac{c}{\Phi}\right)} A^{\left(\frac{1}{\Phi}\right)}}\right] \left[\dfrac{a^{\left(\frac{b+c}{\Phi}\right)} w^{\left(\frac{b}{\Phi}\right)} \rho^{\left(\frac{c}{\Phi}\right)} Q^{\left(\frac{1}{\Phi}\right)}}{b^{\left(\frac{b}{\Phi}\right)} c^{\left(\frac{c}{\Phi}\right)} r^{\left(\frac{b+c}{\Phi}\right)} A^{\left(\frac{1}{\Phi}\right)}}\right]^{a-1}}.$$

After some mathematical calculations we obtain the simplified form as,

$$\frac{\partial L}{\partial r} = \frac{1}{\Phi}\left\{\frac{a^{\left(\frac{\Phi-a}{\Phi}\right)} b^{\left(\frac{\Phi-b}{\Phi}\right)} \rho^{\left(\frac{c}{\Phi}\right)} Q^{\left(\frac{1}{\Phi}\right)}}{r^{\left(\frac{\Phi-a}{\Phi}\right)} w^{\left(\frac{\Phi-b}{\Phi}\right)} c^{\left(\frac{c}{\Phi}\right)} A^{\left(\frac{1}{\Phi}\right)}}\right\}.$$

Since $a, b, c, A > 0$, and $r, w, \rho > 0$, *Q* is the output of the firm that can never be negative. Consequently, we confirm,

$$\frac{\partial L}{\partial r} = \frac{\partial L^*}{\partial r} > 0, \qquad \text{(AV-3)}$$

which indicates that when the interest rate of the capital increases the firm can increase the level of labor *L*.

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