Cold play: Learning across bimatrix games

Lensberg, Terje and Schenk-Hoppé, Klaus R.

Department of Finance, NHH–Norwegian School of Economics,
Department of Economics, University of Manchester

17 March 2021

Cold play: Learning across bimatrix games

Terje Lensberg[a] and Klaus Reiner Schenk-Hoppé[a,b]

March 17, 2021

Abstract

We study one-shot play in the set of all bimatrix games by a large population of agents. The agents never see the same game twice, but they can learn 'across games' by developing solution concepts that tell them how to play new games. Each agent's individual solution concept is represented by a computer program, and natural selection is applied to derive a stochastically stable solution concept. Our aim is to develop a theory predicting how experienced agents would play in one-shot games. To use the theory, visit https://gplab.nhh.no/gamesolver.php.

Keywords: One-shot games, solution concepts, genetic programming, evolutionary stability.
JEL classification: C63, C73, C90.

[a]Department of Finance, NHH–Norwegian School of Economics, Bergen, Norway.
[b]Department of Economics, School of Social Sciences, University of Manchester, United Kingdom.
E-mail: terje.lensberg@nhh.no; klaus.schenk-hoppe@manchester.ac.uk.

By three methods we may learn wisdom: First, by reflection, which is noblest; second, by imitation, which is easiest; and third by experience, which is the bitterest. — Confucius.

1 Introduction

One-shot games put players in unfamiliar situations. Playing well in such situations is a difficult task. Games with multiple Nash equilibria raise the question of which one, if any, of those equilibria will be played, and there is ample evidence that equilibrium solution concepts fail to predict actual behavior in many games. However, by playing many one-shot games, an agent can learn 'across games' to form, and to gradually improve, a theory of games. A theory that can be used to solve all games in a given class is a solution concept.

In this paper we apply the idea of learning across games to develop a new solution concept for one-shot bimatrix games.[1] Like Harsanyi & Selten's (1988) general theory of equilibrium selection, it assigns a unique solution to every game, and it agrees with their risk dominance criterion for 2×2 games. However, our solution concept is not a refinement of Nash equilibrium. In (the extensive form of) some games, the solution is not subgame perfect, and in others, it is not even a Nash equilibrium.

Such digressions from rationality have interesting implications: First, they tend to increase the agents' payoffs, which helps protect the solution concept against invasion by Nash players. Second, they allow us to account for observed behavior in many games where Nash equilibrium, or some refinement, fails to predict actual outcomes. Examples include the Ultimatum game, the Centipede game, and the Traveler's dilemma. And third, they suggest a new explanation why people seem to be motivated by fairness or altruism in some games designed to study social norms.

Our analysis is carried out with a numerical evolutionary model. Because the analysis consists of several steps and elements, we proceed with an outline of the main ideas.

Individual solution concepts. The model contains a large number of agents who use individual solution concepts to play random games in random positions (Row and Col) against random opponents across many periods of time. An *individual solution concept* is a map which to each position in every game assigns an *action* for the agent and a *conjecture* about the opponent's action. Only actions entail consequences, but the agents also need conjectures to reason about how to play a game. By prescribing conjectures, the solution reveals the agent's reasoning and explains her actions.

We are agnostic about conjectures and leave them to be shaped by evolution. In particular, we do not require that an agent's action at one position be consistent with her conjecture at the other. Some agents may then appear to believe that they can outsmart their opponents. Our experience with the model suggests that a side effect of allowing such beliefs is to boost the agents' ability to learn.

Aggregate solution concepts. By taking the mean across all individual solution concepts for each game, one obtains an *aggregate solution concept (ASC)*. It assigns a joint probability distribution over all pairs of actions and conjectures for each position in every

[1] A bimatrix game is a two-person simultaneous-move game where each player has a finite number of pure strategies.

game. For a given game and position, one derives *mixed actions and conjectures* as the marginal distributions of the probability distribution for that position. As discussed above, an ASC may not solve all one-shot games at equilibrium points, but we shall require that the ASC itself be an equilibrium point in an appropriate space. Specifically, we shall look for an ASC whose individual components form a *stochastically stable equilibrium (SSE)* when applied to one-shot bimatrix games.

Good replies and good solutions. To that end, we impose some structure on the set of admissible solution concepts. First, we generalize the Nash equilibrium concept by assuming that the agents reason in terms of good replies and good solutions. Each term is meant to capture one aspect of the agent's preferences, and is defined by a function, specific to each agent, on the relevant domain of information. An action is a *good reply* to a conjecture if the agent likes the corresponding vector of deviation losses. And a pair of action and conjecture is a *good solution* if the agent thinks the two are mutual good replies. By composing the good solution function with the good reply function, we obtain a *numerical representation* of the agent's solution concept: To solve a game at a position, the agent maximizes the composite function.

Separability, symmetry and scale invariance. The Nash equilibrium concept has three properties that will also be imposed here. Payoffs represent von Neumann-Morgenstern utilities, and the agents' solution concepts must therefore be *scale invariant* to reflect that fact. Good reply functions are assumed to be *symmetric* and *separable*. An *iteration function* on a domain of fixed dimension can then be used to process games with different dimensions in the same way. We do not impose symmetry on good solution functions because, as mentioned above, we will allow the agents to believe that they can outsmart their opponents.

Genetic programming. To obtain an SSE, we represent each agent's good reply iteration function and his good solution function by computer programs, and use a *genetic programming (GP) algorithm* (Koza 1992)[2] to evolve these programs until the ASC remains constant. The algorithm works as follows: Begin with a population of randomly generated programs. Let the programs play lots of random games against random opponents and measure their performance. Arrange tournaments and replace some low performing programs with copies of high performing ones, cross and mutate some of the copies and let the programs play another random set of games. By continuing in this manner across thousands of iterations, the programs become increasingly better at one-shot play until possibly, the process converges to an SSE.

Data. The data are generated by doing several runs with the GP algorithm. Table 1 contains an overview of the main model parameters. At the end of each run, we save the population of programs. These programs constitute the *data set* for our analysis. In total our data set contains 200,000 computer programs.

Table 1: Model parameters

Model runs	100	Iterations per run	100,000
Number of games	1,000	Tournaments per iteration	50
Number of agents	2,000	Agents per tournament	4

[2]A technique for 'programming computers by means of natural selection', see also the series of handbooks edited by Koza (1998-2007).

The data analysis consists of three parts: First, we try to gain some intuition for the evolved ASC by solving 11 games from the literature and comparing its predictions to the theoretical and experimental evidence (Section 4.1). Second, we apply the ASC to a set of random games and compute a number of statistical performance measures, e.g., the frequency of deviation from Nash play, and the performance of Nash players against the ASC and vice versa. We also use additional data from the model runs to check that the ASC has converged to an SSE (Section 4.2). In the third (and main) part, we analyze all individual solution concepts to identify any common structure (Section 4.3).

Structure of individual solution concepts. Section 4.3 begins with an example of an individual solution concept. It suggests to look for *additive good reply functions* and *scale invariant good solution functions* (Section 4.3.1), and indeed, we find that almost all agents have additive and almost linear good reply functions (Section 4.3.2). It means that an action is usually a good reply to a conjecture if the corresponding *sum of deviation losses* is large.

Individual good solution functions turn out to be more complex, with substantial heterogeneity across agents (Section 4.3.3). But most of the variation can be explained by fitting, for each agent, a collection of CES functions[3] to a fixed partition of the domain of the good solution functions.

2×2 games. In Section 4.3.4, we use the data to construct a *representative solution concept for 2×2 games*. It agrees with Harsanyi & Selten's (1988) risk dominance criterion for games with strict Nash equilibria, and the representative agent is *rational*, in the sense that her action is always a *best reply* to her conjecture.

Additional details. Model details are presented in Appendix A. That includes an algorithm for computing separable good replies, implementation of solution concepts as executable programs, generation of random games, and the genetic programming algorithm. In Appendix B we test the robustness of our ASC with respect to some changes in the model specification. A user interface to the ASC is provided at https://gplab.nhh.no/gamesolver.php. The web site solves one-shot bimatrix games with up to 10 strategies per player.

2 Related literature

Our paper belongs to the literature on learning across games. Following Selten, Abbink, Buchta & Sadrieh (2003), we consider a population of (artificial) agents who use behavior rules as in Stahl (1996) to decide upon some course of action in unfamiliar situations as described by Gilboa & Schmeidler (1995).

Gilboa & Schmeidler (1995) provide a theoretical basis for learning across games. In their 'Case-based decision theory', the agents do not know all states of the world, but they can make decisions by drawing upon their experience with past cases. This situation is what our model is meant to represent. Gilboa, Schmeidler & Wakker (2002) suggest a set of axioms for rational behavior in such situations and show that it can be represented by a similarity-weighted utility function. LiCalzi (1995), Jehiel (2005) and Steiner & Stewart (2008) model learning across games by agents who use exogenous similarity measures, and in Samuelson (2001) and Mengel (2012) the agents learn to partition games into endogeneous

[3]A CES function is a product of power functions.

analogy classes. An empirical test of Mengel's (2012) partition model is provided by Grimm & Mengel (2012).

Stahl (1996, 1999, 2000) introduced a rule–based approach to model learning by boundedly rational agents. The agents have behavior rules, which are maps from information sets to sets of feasible actions, and the reinforcement principle defines a learning dynamic on the space of behavior rules. In our paper, we use a different learning dynamic, but our solution concepts represent the same idea as Stahl's behavior rules. Stahl's rule based learning model covers a number of special cases, including fictitious play (Brown 1951), replicator dynamics (Taylor & Jonker 1978), belief updating (Mookherjee & Sopher 1994) and reinforcement learning (Roth & Erev 1995). Models of these types have been used by LiCalzi (1995), Germano (2007) and Mengel (2012) to represent learning in theoretical analogy-based models, and by Gale, Binmore & Samuelson (1995), Cooper & Kagel (2003, 2008) and Haruvy & Stahl (2012) to study transfer of learning across games. The latter three papers find that human subjects learn to reason across dissimilar games, and with increasing sophistication as they become more experienced.

Stahl's rule based learning model builds on Nagel (1995) and Stahl & Wilson (1994), who introduced level-k reasoning as a model of initial play. In experiments with initial play, one finds that the subjects often deviate in systematic ways from equilibrium play, and that level-k reasoning and other structural non-equilibrium models (Stahl 2001, Costa-Gomes, Crawford & Broseta 2001) do a better job at predicting actual outcomes. A survey of this literature is provided by Crawford, Costa-Gomes & Iriberri (2013), and a recent contribution is Fudenberg & Liang (2019), who use machine learning to re-examine the empirical evidence. Our paper is related to this literature by considering only one-shot games, but it differs in one important respect: In experiments with initial play, the subjects usually play a sequence of one-shot games without intermediate feedback. The purpose is to suppress learning and preserve an impression of initial play throughout the experiment. As a result, inexperienced subjects remain so during the whole experiment. This contrasts with our paper, and with Selten et al. (2003), where the agents receive systematic feedback to become experienced at one-shot play over time.

Selten et al. (2003) is closely related to our paper. They provide a detailed account of an experiment aimed at studying one-shot play in 3×3 games by means of Selten's (1967) *strategy method*. As part of an economics course, students were asked to write computer programs that would determine their choice of actions in randomly chosen 3×3 games. Several contests were held during the teaching term. In each contest the programs played 500,000 random games, with the results of each contest being used by the students to further refine their programs. They quickly introduced a distinction between games with and without pure Nash equilibria. In the former, they ended up coordinating on equilibria with maximal joint payoff. In the latter, their behavior was a more diverse mix of best-reply cascades, as in level-k reasoning.

Also closely related to our paper is a small literature on learning across games by artificial agents. Sgroi & Zizzo (2009) train neural networks (NNs) to play Nash in 3×3 games with one pure equilibrium. They find that the NNs behave as if they try to identify pure Nash equilibria by means of level-k reasoning. When the NNs are applied to unfamiliar games, this 'shortcut' yields a prediction accuracy which is comparable to that of human subjects. Spiliopoulos (2015) considers a population of NNs who learn to play ex post best reply against the field in seven strategically different classes of 2×2 games. He finds strong evidence of cross–game learning, e.g., training on games with more incentives to cooperate yields more cooperation in unfamilar games. Spiliopoulos (2011) uses a population of NNs to play general 3×3 games. He finds that the NNs develop similarity measures which they use to classify games by their strategic properties, consistent with the case-based decision theory of Gilboa & Schmeidler (1995). The same phenomenon occurred in Selten et al.'s (2003) experiment, as mentioned above, and we show that it also occurs in our model.[4]

Many authors have used genetic algorithms to model learning by heterogeneous agents in repeated games and markets. Genetic algorithms suit that purpose because they impose very little structure on the agents' decision rules. Agents are modeled by specifying their information, their feasible actions and a measure of their individual performance. Competition drives behavior, which is commonly found to agree well with that of human subjects, see e.g, Arifovic (1995, 1996) and Chen, Duffy & Yeh (2005). A pioneering contribution to this literature is Arifovic's (1994) analysis of the cobweb market model. Marks (2002) provides a survey, and more recent applications include coordination games (Chen et al. 2005), Traveler's dilemma games (Pace 2009), and financial market microstructure models (Lensberg, Schenk-Hoppé & Ladley 2015).

3 Model

In this section, we introduce a general class of solution concepts. We consider a large population of agents, each one equipped with a solution concept that she uses to solve games. Agents will be randomly assigned to play random bimatrix games in some random position, Row (1) or Col (2), against random opponents.

3.1 Solution concepts

Let Γ denote the set of all bimatrix games. The members of Γ are pairs $G = (S, \pi)$, where $S = S_1 \times S_2$ is a finite set of pure strategy profiles and $\pi : S \to \mathbb{R}^2$ is a payoff function such that $\pi(\mathbf{s}) = (\pi_1(\mathbf{s}), \pi_2(\mathbf{s}))$ are the von Neumann-Morgenstern utilities obtained by the two players when profile $\mathbf{s} \in S$ is played. From now on, the word 'game' will be used to designate the members of Γ.

For any game G, let $\Sigma(G)$ denote the associated set of strategy profiles. A *solution concept* is a map F from games to strategy profiles, such that $F(G) \subset \Sigma(G)$ for all $G \in \Gamma$. $F(G)$ can contain one or more elements, any one of which is a solution to G. Solution concepts allow to solve a game from the perspectives of both players (Row and Col). Let $G = (S, \pi)$ be any game and define its transpose $G^\top$ as $G^\top = (S', \pi')$, where $S'_1 = S_2$, $S'_2 = S_1$, and $(\pi'_1(t, s), \pi'_2(t, s)) = (\pi_2(s, t), \pi_1(s, t))$ for all $(s, t) \in S$. Then:

1. each $(s, t) \in F(G)$ is a solution to G from Row's point of view. s is Row's action and t is her conjecture about Col's action; and

2. each $(t', s') \in F(G^\top)$ is a solution to G from Col's point of view. t' is Col's action and s' is his conjecture about Row's action.

[4]See Section 4.3 where the structural properties of solution concepts are analyzed.

One has consistency of actions and conjectures if the solution concept solves any game G at (s,t) if and only if it solves its transpose at (t,s). Nash equilibrium is a solution concept which satisfies this property. It will not be imposed here because we will leave it for evolution to determine the agents' conjectures.

Solution concepts are applied as follows.

Playing games. Let a and b be two agents, with individual solution concepts F^a and F^b. Let G be a game and suppose a and b are assigned as player 1 and 2, respectively. The game G is played as follows: Agent a makes a uniform random draw of (s,t) from $F^a(G)$ and plays s. Agent b makes a uniform random draw of (t',s') from $F^b(G^\top)$ and plays t'. a receives payoff $\pi_1(s,t')$ and b receives payoff $\pi_2(s,t')$.

Aggregate solution concepts. Consider a population A of agents, each of whom is equipped with an individual solution concept F^a. For any finite set X, let $|X|$ denote the number of elements in X. For any game G, define

$$p_1^a(s,t,G) \;:=\; \frac{1}{|F^a(G)|} \text{ if } (s,t)\in F^a(G) \text{ and 0 otherwise} \tag{1}$$

$$p_2^a(s,t,G) \;:=\; p_1^a(t,s,G^\top) = \frac{1}{|F^a(G^\top)|} \text{ if } (t,s)\in F^a(G^\top) \text{ and 0 otherwise.} \tag{2}$$

$p_1^a(s,t,G)$ is the probability by which agent a solves G at (s,t) as player 1 (Row) and $p_2^a(s,t,G)$ is the probability by which he solves the transposed game $G^\top$ at (t,s) as player 2 (Col). By taking the mean of the probability distributions $\{(p_1^a,p_2^a)\}_{a\in A}$ across all agents we obtain

$$P_i(s,t,G) \;:=\; \frac{1}{|A|}\sum_{a\in A} p_i^a(s,t,G) \tag{3}$$

for each position $i\in\{1,2\}$. $P_1(s,t,G)$ is the percentage of Row players who solve G at (s,t), and $P_2(s,t,G)$ is the percentage of Col players who solve the transposed game $G^\top$ at (t,s). Let $P(s,t,G)=(P_1(s,t,G),P_2(s,t,G))$. The bimatrix $P(\cdot,\cdot,G)$ is the *aggregate solution* to game G for population A, and the function $P(\cdot)$ is the *aggregate solution concept*.

Given an aggregate solution concept P and a game G, one obtains mixed actions and conjectures for the row and column players as the marginal distributions of P, as shown in Table 2.

Table 2: Mixed actions (σ) and conjectures (ϕ) in a game G

$\sigma_1(s,G) := \sum_t P_1(s,t,G)$	Percentage of Row players who do s
$\phi_1(t,G) := \sum_s P_1(s,t,G)$	Percentage of Row players who conjecture that Col will do t
$\sigma_2(t,G) := \sum_s P_2(s,t,G)$	Percentage of Col players who do t
$\phi_2(s,G) := \sum_t P_2(s,t,G)$	Percentage of Col players who conjecture that Row will do s

Mixed Nash equilibria. In our model, the agents solve games by choosing a pair of action and conjecture, using uniform randomizations to select one outcome in games with multiple solutions. There is no mechanism to align the actions or conjectures of indifferent agents to sustain mixed Nash equilibria, which may seem to rig the model in disfavor of such equilibria. However, mixing will also occur at the population level because different agents will typically use (slightly) different solution concepts, and this will enable the population to play mixed Nash equilibria without external intervention. In Section 4.1, we shall see that the agents play plausible mixed strategies in many games.

Numerical representations of solution concepts. A solution concept is (numerically) *representable* if there is a family of functions $V(\cdot, G) : \Sigma(G) \to \mathbb{R}$, such that for each game G, $F(G) = \text{argmax}_{\mathbf{s}\in\Sigma(G)} V(\mathbf{s}, G)$.

We consider a class of representable solution concepts that includes Nash equilibrium as a special case. For any game $G = (S, \pi)$, and any strategy profile $\mathbf{s} = (s, t) \in S$, define pairs of vectors $\delta(\mathbf{s}) = (\delta_1(\mathbf{s}), \delta_2(\mathbf{s}))$ as

$$\delta_1(\mathbf{s}) \;:=\; (\pi_1(s,t) - \pi_1(s',t))_{s'\in S_1\setminus s} \tag{4}$$
$$\delta_2(\mathbf{s}) \;:=\; (\pi_2(s,t) - \pi_2(s,t'))_{t'\in S_2\setminus t}. \tag{5}$$

The vectors (4) and (5) contain the *deviation losses* that players 1 and 2 would incur by unilateral deviations from s and t to each one of their alternative strategies. Next, let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \cup_{n\in\mathbb{N}}\mathbb{R}^n \to \mathbb{R}$ be two functions, where, by definition, g takes a variable number of arguments, and define

$$V(\mathbf{s}, G) := f(g(\delta_1(\mathbf{s})), g(\delta_2(\mathbf{s}))). \tag{6}$$

Several key concepts in game theory can be represented in this fashion:

Nash equilibrium. A numerical representation V^N for the (pure strategy) Nash equilibrium concept F^N can be obtained by setting $f(x, y) = \min(x, y)$ and $g(\delta_i(\mathbf{s})) = \min(0, \delta_i(\mathbf{s}))$. This yields

$$V^N(\mathbf{s}, G) := \min\{\min(0, \delta_1(\mathbf{s})), \min(0, \delta_2(\mathbf{s}))\}. \tag{7}$$

Vectors of non-negative deviation losses represent best replies, and a strategy profile $\mathbf{s}$ is a Nash equilibrium in pure strategies if $V^N(\cdot, G)$ attains its maximal value of 0 at $\mathbf{s}$.

Risk dominance. Another special case of (6) is the risk dominance concept of Harsanyi & Selten (1988) for 2×2 games with two strict Nash equilibria. This is a refinement of the Nash equilibrium concept for that class of games, where the vectors of deviation losses $\delta_i(\mathbf{s})$ are singletons, and where a risk dominant equilibrium is one that maximizes the product of the two players' deviation losses. To represent this solution concept by (6), let g be the identity function on $\mathbb{R}$, $f(x, y) = x \cdot y$ if $(x, y) > 0$, and $f(x, y) = -\infty$ (or any negative number) otherwise. Then

$$V^{RD}(\mathbf{s}, G) := \begin{cases} \delta_1(\mathbf{s}) \cdot \delta_2(\mathbf{s}) & \text{if } \delta(\mathbf{s}) > 0 \\ -\infty & \text{otherwise.} \end{cases} \tag{8}$$

Given a 2×2 game $G = (S, \pi)$, a strategy profile $\mathbf{s}$ is a risk dominant solution if $\mathbf{s}$ uniquely maximizes $V^{RD}(\cdot, G)$ on S with $V^{RD}(\mathbf{s}, G) > 0$.[5] An example to illustrate concepts and notation is provided in Table 3.

Interpretation. Any solution concept that is representable by some version of V in (6) has three features that are worth noting. First, it can be used to solve games of any

[5]If $V^{RD}(\mathbf{s}, G) \le 0$, the game has no strict equilibria. If the maximizer is not unique, it has two, but neither risk dominates the other.

Table 3: A game $G = (S, \pi)$ with two strict Nash equilibria (corresponding payoffs marked in bold). The risk dominant solution is (C, C).

$\pi(S)$

	C	A
C	$\mathbf{4}, \mathbf{4}$	$0, 1$
A	$1, 0$	$\mathbf{1}, \mathbf{1}$

$(g(\delta_1(S)), g(\delta_2(S))) = \delta(S)$

	C	A
C	3, 3	-1, -3
A	-3, -1	1, 1

$V^{RD}(S, G) = f(g(\delta_1(S)), g(\delta_2(S)))$

	C	A
C	9	$-\infty$
A	$-\infty$	1

finite dimension because the function g can take any number of arguments. Second, $V(\cdot) = f(g(\cdot), g(\cdot))$ is separable with respect to the two vectors of deviation losses (the arguments to g). This suggests to think of g as a measure of the extent to which a strategy for one player is a *good reply* to that of the other, and of f as a device that aggregates two good replies into a *good solution*. Third, by relaxing the Nash equilibrium concept in this way, one can construct solution concepts which potentially use more information about games. In particular, it allows to talk about strategies being almost best replies, and to consider if one solution to a game might be better than another because the former provides weaker incentives to deviate than the latter.

The Nash equilibrium concept has some additional properties that do not follow from (6). The following three properties will be imposed on (6) as well. The first two are implemented by means of a 'nudge', as explained in Appendix A.2. The idea is to scramble any information about games that could lead to a violation of the property in question, thereby stimulating development of functional forms that are insensitive to the scrambled information.

Scale invariance. We require individual solution concepts F to be invariant with respect to positive affine transformation of payoffs, because payoffs are assumed to be Neumann-Morgenstern utilities. Adding a constant term to some player's payoffs has no effect on F because the functions g in (6) only depend on payoff differences, but the functions f and g must be jointly chosen to eliminate any scale effect as well.

Symmetric good replies. An individual solution concept has symmetric good replies if it is invariant with respect to the ordering of any player's strategies. Nash equilibrium satisfies this property because g^N is symmetric. We impose this requirement because it prevents the agents from conditioning their actions on some irrelevant aspects of the game.

Separable good replies. A good reply function g is separable if there is a function $\gamma : \mathbb{R}^2 \to \mathbb{R}$ and a constant z_0, such that $g(x_1) = \gamma(z_0, x_1)$ and $g(x_1, \ldots, x_k) = \gamma(g(x_1, \ldots, x_{k-1}), x_k)$ for $k \geq 2$. The Nash good reply function, $g^N(\cdot) = \min(0, \cdot)$, is separable with $\gamma^N(z, x) = \min(z, x)$ and $z_0 \geq 0$. We require individual solution concepts to have separable good replies. An implementation of this condition is provided in Appendix A.1.

Solution concepts with separable good replies have two important benefits: First, they allow to represent games of different dimensions within the same structure and (low-dimensional) domain, parametrized by the game dimensions. Second, this fact, in conjunction with symmetric good replies, will ensure that the solution concepts behave in a similar way across game dimensions. The latter is a desirable property of any solution concept, and without the former our evolutionary approach to solving games would simply not work.

A solution concept F is called *admissible* if it is representable by (6) and satisfies scale invariance, symmetric good replies and separable good replies. For any such F the associated pair of functions (f, γ) will be said to represent F.

3.2 Solving the model

To solve the model, we represent each agent's good solution function f and his good reply iteration function γ as computer programs. A genetic programming (GP) algorithm is then used to search for an aggregate solution concept (ASC) whose individual components constitute a stochastically stable equilibrium (SSE). The GP algorithm is described in Appendix A.4. To train the agents, it uses large numbers of randomly generated games, as explained in Appendix A.3. Because such games differ from the kind that people face in practice, they might fail to prepare the agents for the real thing. We therefore check whether our approach is fit for purpose by applying our ASC to 11 well-known games from the theoretical and experimental literature (Section 4.1).

To obtain the ASC, we do 100 independent runs with the model. Each run is carried out with a population of 2,000 agents. At the end of each run, we save the pair of programs (f^a, γ^a) for each agent a. The ASC consists of this collection of 200,000 program pairs. To find the aggregate solution to a given game, we solve it by means of each program pair of the ASC and take the mean of those solutions.

As explained in Section 3.1, the aggregate solution to a given game is a pair $P = (P_1, P_2)$ of probability distributions on the set of strategy profiles for that game, one probability distribution for each of the two players. For a given player i and strategy profile (s, t), $P_i(s, t)$ is the probability that a randomly chosen agent will solve the game at (s, t) when called upon to play it as player i. For each probability distribution P_i one derives the mixed actions and conjectures for player i as the marginal distributions of P_i.

4 Results

In this section, we present results for the aggregate solution concept (ASC) obtained from the model described in Section 3. Section 4.1 illustrates the behavior of the ASC in some familiar games. Section 4.2 tests convergence and analyzes the performance of the ASC against agents who play best reply, i.e., hypothetical, omniscient agents who know the distribution of strategies in the population for each game. We also look at the performance of the ASC against Nash players in games with one pure Nash equilibrium. Section 4.3 analyzes the structure of individual solution concepts by investigating the functional form of the good reply and the good solution functions. The aim is to understand the logic that drives the agents' behavior.

4.1 Behavior in selected games

When assessing the results of this section, it is important to bear in mind that the agents have no prior experience with any of the games to be considered here. Anything the agents do has been learned by experience with other games, and so the situation is literally one-shot play by experienced agents.

4.1.1 Classical games

Rock, Paper, Scissors is the zero-sum game depicted in Table 4. Panel (a) contains the payoff matrix, and Panel (b) shows the aggregate solution $P = (P_1, P_2)$ and its marginal distributions $\sigma = (\sigma_1, \sigma_2)$ and $\phi = (\phi_1, \phi_2)$. (σ_1, σ_2) are the aggregate mixed actions of the Row and Col players, and (ϕ_1, ϕ_2) are their aggregate conjectures about the opponent's actions. The mixed actions and conjectures are also shown along with the payoff matrix in Panel (a).

Table 4: Rock, Paper, Scissors. Numbers in italics are probabilities (%).

(a) Payoffs, actions and conjectures

(s,t)		R	P	S	
	σ	*33*	*33*	*33*	Col
R	*34*	0, 0	-1, 1	1, -1	*33*
P	*33*	1, -1	0, 0	-1, 1	*33*
S	*33*	-1, 1	1, -1	0, 0	*33*
	Row	*33*	*33*	*33*	ϕ

(b) Solution, actions and conjectures

(s,t)		R	P	S	
	σ	*33*	*33*	*33*	Col
R	*34*	*19,19*	*1,13*	*13, 1*	*33*
P	*33*	*13, 1*	*19,19*	*1,13*	*33*
S	*33*	*1,13*	*13, 1*	*19,19*	*33*
	Row	*33*	*33*	*33*	ϕ

The game has a unique (mixed) Nash equilibrium in which both players do each of their three actions with probability 1/3. The ASC yields the same actions, and (correct) conjectures.

Consider next the details of the solution shown in Panel (b). Given the payoff structure of this game, it seems fair to say that $3 \times 19 = 57\%$ of both players believe in a draw, $3 \times 13 = 39\%$ expect to win, and $3 \times 1 = 3\%$ expect to lose. On the other hand, the agents' tendency to solve the game at the diagonal suggests that they may rather be looking for some kind of equitable compromise. With that interpretation in mind, the agents appear to be 57% egalitarian, 39% selfish, and 3% altruistic.

Prisoners' dilemma. We next consider a game where the agents' self-interest prevails. In the Prisoners' dilemma game, Table 5, the players get a sentence depending on whether they deny (d) or confess (C) a crime. Deny is strictly dominated[6], (C, C) is the only Nash equilibrium, and this solution is also selected by 100% of the agents, so (C, C) is the ASC outcome.

Table 5: Prisoners' dilemma. Numbers in italics are probabilities (%).

(a) Payoffs, actions and conjectures

(s,t)		d	C	
	σ	*0*	*100*	Col
d	*0*	-1, -1	-3, 0	*0*
C	*100*	0, -3	**-2, -2**	*100*
	Row	*0*	*100*	ϕ

(b) Solution, actions and conjectures

(s,t)		d	C	
	σ	*0*	*100*	Col
d	*0*	*0,0*	*0, 0*	*0*
C	*100*	*0,0*	*100,100*	*100*
	Row	*0*	*100*	ϕ

[6]We use lower case letters to designate strategies that do not survive iterated elimination of strictly dominated strategies.

In games where (almost) all agents agree on one strategy profile, the solution bimatrix in Panel (b) is not informative and will not be shown from now on.

Battle of the sexes is a coordination game with the payoff structure shown in Table 6. In this game, Row and Col would like to attend a Ballet or a Football match. Row prefers Ballet, Col prefers Football, but in any case, they would like to be together. The game has two equilibria in pure strategies, marked in bold in Table 6, and a mixed equilibrium where the players do their preferred action with probability $x/(x+y)$.

Table 6: Coordination games with $x > y > 0$. (Battle of the sexes.)

(s,t)	B	F
B	$\mathbf{x}, \mathbf{y}$	$0,0$
F	$0,0$	$\mathbf{y}, \mathbf{x}$

The original version of Battle of the sexes, Luce & Raiffa (1957), has $(x,y) = (3,2)$. Applying our ASC to that game, we obtain the results in Table 7.

Table 7: Battle of the sexes with $(x,y) = (3,2)$. Numbers in italics are probabilities (%).

(a) Payoffs, actions and conjectures

(s,t)		B	F	
	σ	*41*	*59*	Col
B	*59*	**3, 2**	$0,0$	*41*
F	*41*	$0,0$	**2, 3**	*59*
Row		*59*	*41*	ϕ

(b) Solution, actions and conjectures

(s,t)		B	F	
	σ	*41*	*59*	Col
B	*59*	*59,41*	*0, 0*	*41*
F	*41*	*0, 0*	*41,59*	*59*
Row		*59*	*41*	ϕ

Panel (b) shows that 59% of both players solve the game at their preferred Nash equilibrium, which is (B,B) for Row and (F,F) for Col. There are several things to note about this solution. First, since the agents solve the game at (B,B) and (F,F) with different probabilities, their solutions cannot result from individual uniform randomizations between equally good solutions. To obtain the solution in Panel (b), there must be some mixing at the population level. Second, the agents' conjectures are wrong: The Row players do B and F with probabilities 59 and 41% while the Col players believe they do it with the opposite probabilities. Third, despite that, the mixed actions are almost equal to the mixed Nash equilibrium of the game, in which the players do their preferred action with 60% probability.

The latter finding turns out to be a coincidence, because the ASC solves *all* Battle of the sexes games at the mixed actions $((59,41),(41,59))$ as long as $x > y > 0$. But this result is in line with the empirical evidence. In experimental studies, games with x/y ranging from 1.2 to 5 have been used. The probability of playing the preferred action ranges from 55 to 83% in the corresponding mixed Nash equilibria, but appears to vary randomly around 60 to 65% among the human subjects.[7] Thus, the ASC seems to beat Nash at predicting actual behavior in general Battle of the sexes games.

[7]See Cooper, DeJong, Forsythe & Ross (1989, 1993), Huck & Müller (2005), Crawford, Gneezy & Rottenstreich (2008), Duffy, Lai & Lim (2017) and He & Wu (2020).

4.1.2 Refinements

We continue with some games from the refinement literature, which analyzes strategic stability of Nash equilibria with respect to criteria such as subgame perfectness, weak dominance, and backward and forward induction. The question is whether, or to what extent, the ASC reflects such considerations.

Market entry game. In this game, which is shown in Table 8, Col is an incumbent monopolist. Row can stay out of the market (O) or enter (E), in which case Col can choose to fight (F) or acquiesce (A). The game has two Nash equilibria in pure strategies, indicated by bold type. Backward induction supports (E, A), and so does the ASC, which plays this pair of strategies with 97% probability.

Table 8: Market entry game. Numbers in italics are probabilities (%).

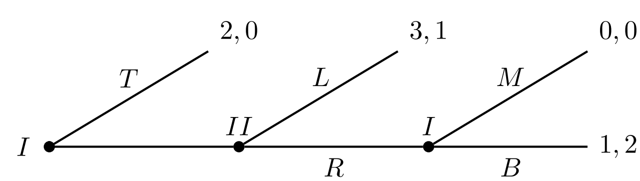
(s,t)		F	A	
	σ	*0*	*100*	Col
O	*3*	**2, 2**	2, 2	*0*
E	*97*	0, 0	**3, 1**	*100*
Row		*3*	*97*	ϕ

The next two games are taken from Kohlberg & Mertens (1986).

Kohlberg and Mertens I. The game in Table 9 has two pure Nash equilibria, (T, R) and (M, L), and a unique strategically stable set, which is the convex hull of (T, R) and $(T, \frac{1}{2}L + \frac{1}{2}R)$. Backward induction selects (T, R) with payoffs $(2, 0)$, but (M, L) is supported by the following (informal) forward induction argument: If Row fails to play T, then Col should understand that Row aims to get 3 by threatening to play M if Col fails to play L. This yields (M, L), which is the solution selected by the ASC.

Table 9: Kohlberg & Mertens (1986, p. 1029). Numbers in italics are probabilities (%).

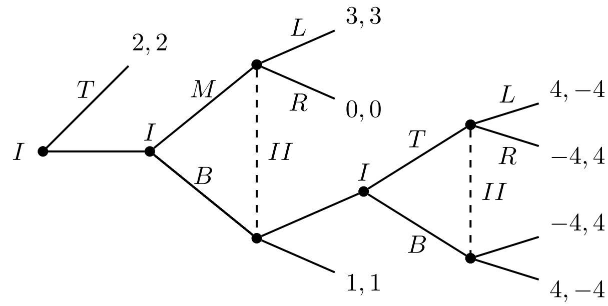
(s,t)		L	R	
	σ	*99*	*1*	Col
T	*4*	2, 0	**2, 0**	*0*
M	*95*	**3, 1**	0, 0	*99*
B	*0*	3, 1	1, 2	*1*
Row		*95*	*5*	ϕ



Kohlberg and Mertens II. The game in Table 10 has one Nash equilibrium in pure strategies (T, R) with payoffs $(2, 2)$, and a mixed equilibrium $(M, \frac{1}{2}LL + \frac{1}{2}LR)$ with superior payoffs $(3, 3)$, which is selected by the ASC. By replacing the subgame with its value (0) and applying iterated dominance, one finds that the mixed equilibrium is also the unique strategically stable set of this game.

Table 10: Kohlberg & Mertens (1986, p. 1016). Numbers in italics are probabilities (%).

(s,t)		LL	LR	R	
	σ	*50*	*50*	*0*	Col
T	*0*	2, 2	2, 2	**2, 2**	*0*
M	*100*	3, 3	3, 3	0, 0	*100*
BT	*0*	4, -4	-4, 4	1, 1	*0*
BB	*0*	-4, 4	4, -4	1, 1	*0*
	Row	*50*	*50*	*0*	ϕ



4.1.3 Equilibrium selection

We next apply the aggregate solution concept to some games in which refinement considerations somehow fail to identify the 'right' outcome with respect to intuition or empirical evidence.

Stag hunt. This game, which is due to Carlson & van Damme (1993), represents the following story: Two hunters can cooperate (C) to catch a stag, or hunt alone (A) to obtain a catch of smaller game amounting to a fraction $x \in (0,1)$ of what each of them would get by cooperating.

Table 11: Stag hunt game. Numbers in italics are probabilities (%).

(a) $x < \frac{1}{2}$

(s,t)		C	A	
	σ	*100*	*0*	Col
C	*100*	**1, 1**	0, x	*100*
A	*0*	x, 0	**x, x**	*0*
	Row	*100*	*0*	ϕ

(b) $x > \frac{1}{2}$

(s,t)		C	A	
	σ	*0*	*100*	Col
C	*0*	**1, 1**	0, x	*0*
A	*100*	x, 0	**x, x**	*100*
	Row	*0*	*100*	ϕ

The game is illustrated in Table 11. It has two strict Nash equilibria: (C,C) and (A,A). When $x < \frac{1}{2}$, the Risk Dominant equilibrium (Harsanyi & Selten 1988) is (C,C), and when $x > \frac{1}{2}$, it is (A,A). Table 11 shows that the ASC always selects the risk dominant equilibrium in the Stag hunt game. When $x = \frac{1}{2}$ (not shown in the table), 50% of the agent population solve the game at (C,C) and 50% solve it at (A,A).

Ultimatum game. Few games have been subject to more empirical analysis than the Ultimatum game of Güth, Schmittberger & Schwarze (1982). In this game, Row and Col get n dollars to share if they can agree how to do it. Row (the proposer) suggests a division by offering an integer amount of x dollars to Col (the responder). Col accepts or rejects. If he accepts, they divide according to Row's suggestion, if Col rejects the offer, both get zero. Any division of the money is the outcome of some Nash equilibrium, but only one is subgame perfect: Row offers zero dollars and Col accepts any offer.

A small version of this game (with 5 dollars to share) is shown in Table 12. Action O_k for Row stands for 'Offer k dollars', and action A_k for Col stands for 'Accept any offer of k or more dollars'. In the ASC, Row offers 2 dollars, and Col accepts all offers of 2 or more. If the total amount is doubled to 10 from 5 dollars, the ASC offers and demands double to 4

Table 12: Ultimatum game. Numbers in italics are probabilities (%).

(s,t)		A_0	A_1	A_2	A_3	A_4	A_5	
	σ	*0*	*0*	*100*	*0*	*0*	*0*	Col
O_0	0	**5, 0**	0, 0	0, 0	0, 0	0, 0	**0, 0**	*0*
O_1	*0*	4, 1	**4, 1**	0, 0	0, 0	0, 0	0, 0	*0*
O_2	*100*	3, 2	3, 2	**3, 2**	0, 0	0, 0	0, 0	*100*
O_3	*0*	2, 3	2, 3	2, 3	**2, 3**	0, 0	0, 0	*0*
O_4	*0*	1, 4	1, 4	1, 4	1, 4	**1, 4**	0, 0	*0*
O_5	*0*	0, 5	0, 5	0, 5	0, 5	0, 5	**0, 5**	*0*
	Row	*0*	*0*	*100*	*0*	*0*	*0*	ϕ

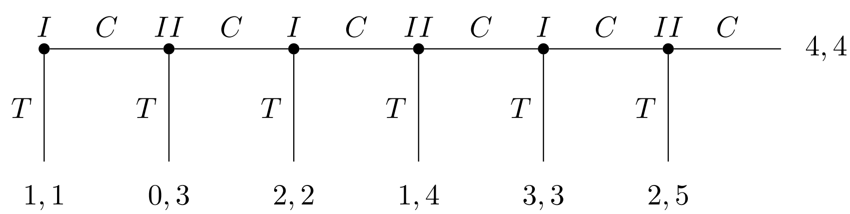
from 2. These results agree well with the experimental evidence, where mean offers amount to some 40% of the stake, and where the responder rejects offers of some 30% or less, see, e.g., Güth & Tietz (1990).

4.1.4 Non-equilibrium behavior

The ultimatum game challenges the idea of backward induction – a basic rationality postulate in game theory. We next consider some games where intuition or experiment suggest that the players will not even play a Nash equilibrium.

The Centipede game by Rosenthal (1981) describes a situation in which two players alternate to decide when to take (T) an increasing pot of money. By continuing (C) for one more round, a player gains if the other player also continues, but loses if the other player then decides to take. A version of this game is shown in Table 13. For each player, C_n denotes the strategy of n C's and then a T if $n < 3$.

Table 13: Centipede game. Numbers in italics are probabilities (%).



(a) Payoffs, actions and conjectures

(s,t)		C_0	C_1	C_2	C_3	
	σ	*8*	*4*	*67*	*21*	Col
C_0	*22*	**1, 1**	1, 1	1, 1	1, 1	*9*
C_1	*1*	0, 3	2, 2	2, 2	2, 2	*3*
C_2	*0*	0, 3	1, 4	3, 3	3, 3	*2*
C_3	*77*	0, 3	1, 4	2, 5	4, 4	*86*
	Row	*20*	*1*	*19*	*60*	ϕ

(b) Solution, actions and conjectures

(s,t)		C_0	C_1	C_2	C_3	
	σ	*8*	*4*	*67*	*21*	Col
C_0	*22*	*20,6*	*1,1*	*1, 1*	*1, 1*	*9*
C_1	*1*	*0,2*	*0,0*	*1, 1*	*0, 0*	*3*
C_2	*0*	*0,0*	*0,2*	*0, 0*	*0, 0*	*2*
C_3	*77*	*0,0*	*0,1*	*17,65*	*60,20*	*86*
	Row	*20*	*1*	*19*	*60*	ϕ

The game has a unique (subgame perfect) Nash equilibrium, in which both players take at the first opportunity. In experiments with human subjects, the game often continues for several moves, but seldom to the end (McKelvey & Palfrey 1992). Under the ASC, 77% of the Row players continue as long as they can, and 86% of the Col players conjecture they will do so. However, Row's willingness to continue seems to be based on the false conjecture that 60% of the Col players will also continue until the end, whereas only 21% of them actually plan to do so. The mixed actions for this game imply that 22% of the player pairs end the game at the first opportunity with payoffs $(1,1)$, $0.77 \times 0.67 = 52\%$ end it at the next to last node with payoffs $(2,5)$, and $0.77 \times 0.21 = 16\%$ go all the way to the end with payoffs $(4,4)$.

Traveler's dilemma. In this game, due to Basu (1994), two travelers have lost their luggage and the airline offers compensation for their loss. They can claim any integer amount in the interval $[\underline{c}, \overline{c}] = [2, 100]$. In any case, the airline will pay both travelers the minimum of the two claims, with the following (slight) modification: If player i claims more than player j, then i pays a penalty of $R = 2$ dollars, and j is rewarded by the same amount. As noted by Basu (1994), intuitively both players should make a high claim and pay little attention to the small penalty/reward. However, the game has a unique Nash equilibrium where both players claim the minimal 2 dollars. In fact, this is the only action pair which survives iterated elimination of strictly dominated strategies.

Capra, Goeree, Gomez & Holt (1999) conduct an experiment with human subjects and find that their behavior is sensitive to the penalty/reward parameter R, with players making large claims for small R and vice versa. The ASC turns out to have the same property. To illustrate, we consider a small version of the Traveler's dilemma game, where $(\underline{c}, \overline{c}) = (4, 11)$ instead of $(2, 100)$. The game is shown in Table 14, where C_n and c_n stand for 'Claim n dollars'.

Table 14: Traveler's dilemma game with $\underline{c} = 4$, $\overline{c} = 11$ and penalty/reward parameter $R = 2$. Numbers in italics are probabilities (%).

(s,t)		C_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	
	σ	*50*	*0*	*0*	*0*	*0*	*0*	*0*	*50*	Col
C_4	*50*	**4, 4**	6, 2	6, 2	6, 2	6, 2	6, 2	6, 2	6, 2	*50*
c_5	*0*	2, 6	5, 5	7, 3	7, 3	7, 3	7, 3	7, 3	7, 3	*0*
c_6	*0*	2, 6	3, 7	6, 6	8, 4	8, 4	8, 4	8, 4	8, 4	*0*
c_7	*0*	2, 6	3, 7	4, 8	7, 7	9, 5	9, 5	9, 5	9, 5	*0*
c_8	*0*	2, 6	3, 7	4, 8	5, 9	8, 8	10, 6	10, 6	10, 6	*0*
c_9	*0*	2, 6	3, 7	4, 8	5, 9	6, 10	9, 9	11, 7	11, 7	*0*
c_{10}	*0*	2, 6	3, 7	4, 8	5, 9	6, 10	7, 11	10, 10	12, 8	*0*
c_{11}	*50*	2, 6	3, 7	4, 8	5, 9	6, 10	7, 11	8, 12	11, 11	*50*
	Row	*50*	*0*	*0*	*0*	*0*	*0*	*0*	*50*	ϕ

When $R = R^* \equiv 2$, the agents make the minimal and maximal claims with equal probability, as shown in Table 14. When $R > R^*$, all agents claim the minimal 4, and when $R < R^*$ all agents claim the maximal 11. The critical value R^*, relative to the length of the feasible claim interval is $R^*/(\overline{c} - \underline{c}) = 2/(11 - 4) = 0.29$, which is in line with the empirical findings of Capra et al. (1999).

Social norms. There is a large literature on the role of social norms in economic transactions and relationships. In experiments with human subjects on bargaining, public goods, and labor relations, the hypothesis of purely self-interested behavior is often rejected in favor of explanations based on fairness or altruism. We have applied our solution concept to some of the games studied in this literature and found that in many cases, the ASC agrees with the empirical results in the sense of predicting more cooperation than what would be achieved through rational play by self-interested agents.

To illustrate, consider the gift exchange experiment of Van der Heijden, Nelissen, Potters & Verbon (1998). Two players live for two periods. A player who consumes c_1 in period 1 and c_2 in period 2 obtains utility $c_1 \cdot c_2$. In period 1, player 1 is rich and player 2 is poor. In period 2 their situations are reversed. A rich player has income 9 and a poor player has income 1, but the players can smooth consumption by exchanging gifts: Player 1 gives an integer amount $0 \leq s \leq 7$ to player 2 in period 1 and player 2 gives $0 \leq t \leq 7$ to player 1 in period 2. This yields utilities

$$u_1(s,t) = (9-s)\cdot(1+t) \tag{9}$$
$$u_2(s,t) = (9-t)\cdot(1+s) \tag{10}$$

for players 1 and 2, respectively. The simultaneous move version of this game is shown in Table 15, where t_k stands for 'Transfer k dollars to the other player'. Giving zero (T_0) strictly dominates any other action for both players, but the ASC predicts that both players will give one dollar (t_1) to the other player. This agrees with the average gifts of 0.99 and 1.03 observed empirically by Van der Heijden et al. (1998).

Table 15: Gift exchange game. Numbers in italics are probabilities (%).

(s,t)	σ	T_0 *0*	t_1 *100*	t_2 *0*	t_3 *0*	t_4 *0*	t_5 *0*	t_6 *0*	t_7 *0*	Col
T_0	*0*	**9, 9**	18, 8	27, 7	36, 6	45, 5	54, 4	63, 3	72, 2	*0*
t_1	*100*	8, 18	16, 16	24, 14	32, 12	40, 10	48, 8	56, 6	64, 4	*100*
t_2	*0*	7, 27	14, 24	21, 21	28, 18	35, 15	42, 12	49, 9	56, 6	*0*
t_3	*0*	6, 36	12, 32	18, 28	24, 24	30, 20	36, 16	42, 12	48, 8	*0*
t_4	*0*	5, 45	10, 40	15, 35	20, 30	25, 25	30, 20	35, 15	40, 10	*0*
t_5	*0*	4, 54	8, 48	12, 42	16, 36	20, 30	24, 24	28, 18	32, 12	*0*
t_6	*0*	3, 63	6, 56	9, 49	12, 42	15, 35	18, 28	21, 21	24, 14	*0*
t_7	*0*	2, 72	4, 64	6, 56	8, 48	10, 40	12, 32	14, 24	16, 16	*0*
	Row	*0*	*100*	*0*	*0*	*0*	*0*	*0*	*0*	ϕ

To understand why the agents sometimes act as if motivated by social norms, recall that our solution concepts solve games $G = (S, \pi)$ at strategy profiles $\mathbf{s} \in S$ which maximize

$$f(g(\delta_1(\mathbf{s})), g(\delta_2(\mathbf{s}))),$$

where $\delta_1(\mathbf{s})$ and $\delta_2(\mathbf{s})$ are vectors of deviation losses for players 1 and 2. The function $f(g(\cdot), g(\cdot))$ resembles a social welfare function for the two players, except that its arguments are deviation losses instead of payoffs. But in many games, including the Gift exchange game in Table 15 and the Rock, Paper, Scissors game in Table 4, payoffs and deviation losses are positively correlated, so when the ASC solves such games by balancing the players' incentives to deviate, it looks as if it tries to make a fair compromise in terms of payoffs.

4.2 Performance and stability

We have seen that the aggregate solution concept (ASC) sometimes solves games at strategies that do not constitute a Nash equilibrium. In this section we examine how often non-Nash play occurs, how costly it is relative to always playing best reply (if one could), and what non-Nash behavior means in terms of evolutionary stability. We also test whether the 100 model runs have converged to stochastically stable equilibria.

Table 16: Descriptive statistics. The number of observations is 100 for each variable, one observation for each of the 100 independent runs of the GP-algorithm.

Variable		Mean	Std.dev	Min	Max
Panel 1: All games					
Consensus		98.8%	0.2%	98.3%	99.8%
playPureNash		86.2%	0.4%	85.1%	87.2%
playBestReplyToASC		83.8%	0.4%	82.5%	84.8%
meanPayoff		8.90	0.09	8.62	9.15
gainBR		8.0%	0.3%	7.4%	8.9%
Panel 2: Games with one pure Nash equilibrium					
gain2BR		-7.8%	0.6%	-9.4%	-6.1%
pASC_ASC	(a)	9.09	0.06	8.96	9.24
pNash_ASC	(b)	7.76	0.07	7.58	7.91
pNash_Nash	(c)	8.69	0.06	8.57	8.86
pASC_Nash	(d)	7.51	0.07	7.34	7.67
pDiff	(a-b) - (c-d)	0.16	0.03	0.09	0.23
playPureNash		83.8%	0.4%	83.0%	84.6%

Table 16 contains descriptive statistics for a set of variables that measure the performance and stability of the ASC. The performance variables in Panel 1 are computed for each of the 100 model runs from five equally spaced samples taken from the last 2,000 (out of 100,000) iterations. *Consensus* is the percentage of agents who play the modal strategy for a given game and position. With a value close to 100%, it shows that there is very little intra-run heterogeneity among the agents. *playPureNash* is the joint probability of Row and Col playing a pure Nash equilibrium in games with one or more pure equilibria, *playBestReplyToASC* is the percentage of agents whose actions are a best reply to the mixed actions of the ASC, *meanPayoff* is the mean payoff of the ASC against the ASC, and *gainBR* is the percentage net gain in mean payoff from playing best reply, rather than ASC, against the ASC. *Consensus*, *playPureNash*, *playBestReplyToASC* and *meanPayoff* are computed separately for each game and then averaged across all games. *gainBR* is computed at an aggregate level because game payoffs are normally distributed with a zero mean.

The variables in Panel 2 of Table 16 are intended to provide some information about the evolutionary stability of the ASC. Data are obtained by restarting each saved population to solve 10,000 random games with exactly one Nash equilibrium in pure strategies. *gain2BR* is the percentage net gain to player i from deviating to a best reply (if not currently playing a best reply) when that is followed by subsequent best reply by player j, *pASC_ASC* is the mean payoff across all games and positions from playing the ASC against itself (identical to *meanPayoff* in Panel 1 except for considering only games with one pure Nash equilibrium), *pNash_ASC* is the mean payoff from playing the Nash equilibrium actions against the ASC, *pNash_Nash* is the mean payoff from playing the Nash equilibrium against itself, *pASC_Nash* is the mean payoff from playing the ASC against the Nash equilibrium, *pDiff* is the net gain from playing the ASC (rather than Nash) against ASC, minus the net gain from playing Nash (rather than ASC) against Nash, and *playPureNash* is the joint probability of Row and Col playing the pure Nash equilibrium.

The findings in Table 16 can be interpreted as follows. The ASC appears to be well protected against invasion by agents who play Nash because by switching from ASC to Nash they would lose on average $1.33 = 9.09 - 7.76$ (*pASC_ASC* - *pNash_ASC* in Panel 2). The agents play best reply to the ASC 83.8% of the time, which gives an average payoff of 8.90 (*meanPayoff* in Panel 1). An agent could increase her average payoff by 8% if she could play best reply in every game (*gainBR*), but if every deviation to best reply would trigger another best reply from the opponent, the 8% gain would turn into a 7.8% loss (*gain2BR*). Finally, *pDiff* shows that ASC agents outperform Nash agents in an ASC world by a larger margin than Nash agents outperform ASC agents in a Nash world. In other words, ASC is more robust against invasion by Nash agents than vice versa.

We next perform a simple test to check if the 100 model runs have converged to stochastically stable equilibria. This is done by testing for trends in the four variables in Panel 1 of Table 16 towards the end of the model runs. To that end, we use data sampled at every 500th iteration from the last 20,000 iterations of each model run, when mutation and crossover probabilities have reached their common minimum of 1%. We skip the middle part of the data set and test for differences in means between the two intervals 80,000–85,000 and 95,000–100,000 of iterations. The boundary points of each interval are included, which gives 2×11 observations for each run and 2,200 observations in total for each variable in Table 17. The results are consistent with the hypothesis that the 100 model runs have reached stochastically stable equilibria after 80,000 iterations.

Table 17: Convergence tests. Tests of differences in means for the variables *Consensus*, *playBestReplyToASC*, *meanPayoff* and *gainBR* across two intervals of model iterations. The number observations is 2,200 for each variable.

Iterations	*Consensus*	*playBestReplyToASC*	*meanPayoff*	*gainBR*
80,000 – 85,000	98.8%	83.8%	8.91	8.0%
95,000 – 100,000	98.8%	83.8%	8.90	8.0%
p-value	(0.634)	(0.883)	(0.118)	(0.796)

4.3 Structural properties of solution concepts

Recall that the individual solution concept for an agent a is represented by a pair of computer programs (f^a, γ^a), where f^a is a good solution function and γ^a is an iterator which is used to compute the agent's good reply function g^a. In this section, we aim to uncover structural properties of these programs to shed light on the results presented above.

4.3.1 Example

To gain some intuition for the general results, we first look at a typical agent from the model runs. The programs of such an agent are listed in (11) and (12).[8]

$$\gamma^a(z,x) \;=\; z + 0.006 + 4x. \tag{11}$$

$$f^a(x_1,x_2) \;=\; \begin{cases} x_1 \cdot x_2, & \text{if } (x_1,x_2) > 0 \\ x_2, & \text{if } x_1 > 0 \text{ and } x_2 \le 0 \\ -\infty, & \text{if } x_1 < 0 \\ \text{undefined}, & \text{if } x_1 = 0. \end{cases} \tag{12}$$

The good reply score equates to[9]

$$g^a(\mathbf{x}) = \sum_{k=1}^{K} (0.006 + 4x_k). \tag{13}$$

Note that g^a is additive and almost proportional to the sum of deviation losses. Thanks to the constant 0.006 in (11), $g^a(\mathbf{x})$ is positive if $\sum_{k=1}^K x_k \ge 0$ and negative almost always if $\sum_{k=1}^K x_k < 0$. In turn, the function $f^a(g^a(\cdot), g^a(\cdot))$ extends continuously from positive to zero sums of deviation losses and almost never returns undefined values.[10]

We have required that solution concepts be scale invariant, and have taken steps to ensure that this requirement is fulfilled. Consider the solution concept F^a which is represented by $f^a(g^a(\cdot), g^a(\cdot))$. To see if F^a is scale invariant, we shall need the following definition, where '$\circ$' denotes element-wise vector multiplication.

Scale invariant functions on $\mathbb{R}^n$. A function $h : \mathbb{R}^n \to \mathbb{R}$ is *scale invariant* if $h(x) \ge h(x')$ implies $h(\lambda \circ x) \ge h(\lambda \circ x')$ for all x and x' in $\mathbb{R}^n$ and all positive $\lambda \in \mathbb{R}^n$.

If both f^a and g^a are scale invariant functions, then so is $f^a(g^a(\cdot), g^a(\cdot))$, and F^a is a scale invariant solution concept. There are many ways in which f^a can be scale invariant, e.g., if (i) it is constant, or (ii) constant in one argument and monotone in the other one, or (iii)

[8] The programs have been simplified without altering their function, and the constants are truncated.

[9] Given a K-vector $\mathbf{x}$ of deviation losses, initialize z to 0, iterate $z \leftarrow \gamma(z, x_k)$ for $k = 1, \ldots, K$, and finally set $g^a(\mathbf{x}) = z$ to obtain (13).

[10] To see this, observe first that each x_k is a random integer c_k, scaled by some random real $\alpha \in [0.01, 100]$, cf. Appendix A.2 and A.3. Hence $\sum_{k=1}^K x_k = \alpha \sum_{k=1}^K c_k \in (-\infty, -0.01] \cup \{0\} \cup [0.01, \infty)$. Consequently, $g^a(\mathbf{x}) \ne 0$ almost surely, and $g^a(\mathbf{x}) \ge 0.006K > 0$ whenever $\sum_{k=1}^K x_k \ge 0$. But g^a may fail to preserve the sign of $\sum_{k=1}^K x_k$ when $\sum_{k=1}^K x_k$ is negative and close to 0. The maximal negative value of $\sum_{k=1}^K x_k$ is -0.01. Then $g^a(\mathbf{x}) = 0.006K - 0.040$, which is negative if and only if $K < 6.67$. So $g^a(\mathbf{x})$ preserves the sign of $\sum_{k=1}^K x_k$ for $K \le 6$, but may fail to do so for $K \ge 7$. However, such failures occur less than twice per million random games, which explains why this 'bug' escapes removal by the genetic programming algorithm.

depends only on the sign of its arguments, or (iv) is a CES function, i.e., if $f^a(x) = x_1^{\rho_1} \cdot x_2^{\rho_2}$. And indeed, the f^a in (12) is scale invariant by exploiting all those possibilities. In particular, on the positive quadrant, $f^a(x)$ is the CES function $x_1 \cdot x_2$. Since g^a is almost linear, then $f^a(g^a(\cdot), g^a(\cdot))$ is almost scale invariant as well, and so is F^a. The lack of full scale invariance makes evolutionary sense: The small constant term in g^a assigns positive good reply scores to weak best replies, and this ensures that $f^a(g^a(\cdot), g^a(\cdot))$ is well-defined at weak best replies.

To solve a game, one selects a strategy profile which maximizes $f^a(g^a(\cdot), g^a(\cdot))$. Since $g^a(\mathbf{x}) > 0$ if $\sum_{k=1}^{K} x_k \geq 0$, then, for each conjecture t about player 2, there is an action s for player 1 (e.g., a best reply to t) which yields a positive g^a-score to player 1 at (s, t). This implies that $f^a(g^a(\cdot), g^a(\cdot))$ is maximized at case 1 or 2 of (12). *Case 1:* Games with strategy profiles that yield two positive g^a-scores (e.g., pure Nash equilibria) are solved at some strategy profile (not necessarily a Nash equilibrium) which maximizes their product. *Case 2:* All other games are solved at some strategy profile that maximizes the (non-positive) g^a-score to player 2 among those that yield positive g^a-scores to player 1. In other words, the action is a good reply to the conjecture, which is a least bad reply to any such action.

The ASC, which consists of the programs of 200,000 agents, is a more complex object than an individual agent's solution concept. To study the ASC, we proceed in three steps. First, we show that the additive structure of the good reply function in (13) is shared by almost all agents. Second, we show that most good solution functions share a common structure, of which (12) is a representative example. Finally, we construct a representative solution concept for 2×2 games and provide a graphical illustration of its numerical representation.

4.3.2 Good reply functions

A good reply function g is said to be *additive* if

$$g(\mathbf{x}) = \sum_{k=1}^{K} (\alpha + \beta x_k), \tag{14}$$

where $\beta \neq 0$. Negative β's may occur because the signed effect of the arguments to g is determined by the composite function $f(g(\cdot), g(\cdot))$.

To test if the g-functions of the ASC are additive, we proceed as follows: For each agent a, generate a data set with 100 observations (y^a, x_1, x_2), where (x_1, x_2) is a vector of two random deviation losses and $y^a = g^a(x_1, x_2)$. Then estimate the linear model

$$y^a = \alpha + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2, \tag{15}$$

and conclude that agent a has an additive good reply function if $\beta_1 = \beta_2$, $\beta_{12} = 0$, and the R^2 of the regression exceeds 0.99. For each run, compute the mean R^2 and the median values of the parameter estimates across all agents.[11] This yields a data set of 100 observations which is described in Table 18.

The table shows that β_1 is not significantly different from β_2, that β_{12} does not differ significantly from zero, and that $R^2 > 0.99$ for all runs. We can therefore conclude that additive good replies is a characteristic feature of the aggregate solution concept. Furthermore,

[11] We use medians to aggregate the parameter estimates, because they can potentially vary widely across agents. But most agents have R^2's close to 1, so we use means to obtain conservative averages of R^2.

Table 18: Test of additive good reply functions by means of (15). The number of observations is 100, one observation for each run with the model. P-values (from left to right) refer to Wilcoxon tests against the null hypotheses that $\alpha = 0$, $\beta_1 = \beta_2$, and $\beta_{12} = 0$.

Parameter	α	β_1	β_2	β_{12}	R^2
Min	-0.047	1	1	0	0.999
Max	0.015	15	15	8.2e−05	1
Median	0	5	5	0	1
Mean	0.000	5.760	5.760	8.4e−07	1.000
P-value	0.394	0.995		0.371	

since α is insignificant and generally small relative to $\beta_1 = \beta_2$, the good reply function g is almost proportional to the sum of deviation losses. Loosely speaking, if the sum of Row's deviation losses is large at some strategy profile (s, t), then s is a good reply to t.

4.3.3 Good solution functions

In this section, we analyze all agents' good solution functions and show that they share many structural properties with the example good solution function in (12).

The following notation and terminology will be used: $\mathbb{R}_{++}$, $\mathbb{R}_{+}$, $\mathbb{R}_{-}$ and $\mathbb{R}_{--}$ denote, respectively, the positive, non-negative, non-positive and negative real numbers. The positive quadrant in $\mathbb{R}^2$ is $Q_1 := \mathbb{R}^2_{++}$. Similarly, $Q_2 := \mathbb{R}_{--} \times \mathbb{R}_{++}$, $Q_3 := \mathbb{R}_{--} \times \mathbb{R}_{--}$, and $Q_4 := \mathbb{R}_{++} \times \mathbb{R}_{--}$. For each Q_k, $\varsigma_k = (\varsigma_{k1}, \varsigma_{k2})$ denotes the signs of the coordinates in Q_k. Given a subset X of $\mathbb{R}^2$, ∂X and $\bar{X}$ denote the boundary and the closure of X, e.g., $\bar{Q}_1$ is the non-negative quadrant. Consider a game G which is solved at a strategy profile $\mathbf{s}$ by some agent a. If the corresponding pair of good reply scores $(g^a(\delta_1(\mathbf{s})), g^a(\delta_2(\mathbf{s})))$ belongs to $X \subset \mathbb{R}^2$, the game is said to be solved in X for that agent. Two good solution functions are *equivalent* if they yield the same ranking, i.e., if they are related by a positive monotone transformation.

Table 19: Some properties of the individual solution concepts. The data consist of program pairs for all agents and 1,000 random games for each model run, a total of 100,000 games.

Subdomain	Q_1	Q_2	Q_3	Q_4	$\partial Q_1 \cup \partial Q_4$
% of games solved in	86	2	2	9	1
% of good solution functions constant in	0	22	22	11	–

We begin with some basic observations about individual solution concepts in Table 19. The table shows that the agents solve 95% of all randomly generated games in Q_1 or Q_4, i.e., at strategy profiles that yield positive good reply scores for player 1 (the decision maker). Another 1% are solved on the boundaries of these two quadrants, and only 4% are solved in $Q_2 \cup Q_3$, i.e., at points that yield a negative good reply score to player 1. The table also shows that the agents' good solution functions tend be constant on $Q_2 \cup Q_3$, less so on Q_4, and not at all on Q_1. Both properties are matched by the example good solution function f^a in (12), which is constant on $Q_2 \cup Q_3$ and solves all games in $\bar{Q}_1 \cup \bar{Q}_4$.

Recall that f^a in (12) of Section 4.3.1 is scale invariant with $f^a(x) = x_1 \cdot x_2$ on Q_1 and $f^a(x) = x_2$ on Q_4. Therefore, on Q_1, f^a is a CES function, and on Q_4 it is equivalent to the CES function $x_1^0 \cdot (-x_2)^{-1}$. We next investigate whether this observation carries over to the whole population of agents, i.e., are most good solution functions scale invariant, and, in particular, are they equivalent to collections of CES functions fitted on separate quadrants.

To that end we consider the two-dimensional grid $D_0 := \{-10, -9.5, \ldots, 10\}^2$ along with scaled versions λD_0 for $\lambda \in \{1, 10, 100\}$. For each agent a, each quadrant Q_k and each scale λ, compute $f^{ak\lambda} := f^a(\lambda D_0 \cap Q_k)$. This gives a data set with $20 \times 20 = 400$ observations. If $f^{ak\lambda}$ is not constant, a CES function $\hat{f}^{ak\lambda}(x, \rho) = (\varsigma_{k1} x_1)^{\varsigma_{k1}\rho_1} \cdot (\varsigma_{k2} x_2)^{\varsigma_{k2}\rho_2}$ is fitted to $f^{a\lambda k}$ by searching for parameter values ρ which maximize the rank correlation between $\hat{f}^{ak\lambda}(\cdot, \rho)$ and the data $f^{a\lambda k}$. For that purpose, only relative values of ρ matter, so we can do a one–dimensional search for angles θ and let ρ be proportional to the corresponding point on the unit circle. With θ measured in degrees we have $(\rho_1, \rho_2) \propto (\cos(\frac{\pi}{180}\theta), \sin(\frac{\pi}{180}\theta))$.

Table 20 displays summary statistics across agents by quadrant and scale for the estimated θ's. In each case, the number of observations is 200,000 minus the number of agents whose good solution functions are constant at that combination of quadrant and scale. The mean rank correlations between actual and estimated good solution functions show that the CES functions fit the data well, especially on Q_1. On that quadrant, θ is close to 45° for all agents, as can be seen from the mean, median and mode, the low standard deviations, and the large concentrations of observations at the mode. $\theta = 45°$ corresponds to $\rho \propto (1, 1)$, which yields the good solution function $f(x) = x_1 \cdot x_2$, as in (12).

On Q_4, the mean θ is about 79°. But standard deviations are large, and rank correlations are relatively low. Thus there is a variety of good solution functions, some of which are not scale invariant because they are neither constant nor CES. There is, however, a large concentration of agents with $\theta = 90°$, which is the common mode and median in Q_4. It corresponds to $\rho \propto (0, 1)$ and $f(x) = (-x_2)^{-1}$, or equivalently, $f(x) = x_2$, as in (12).

On $Q_2 \cup Q_3$, the good solution functions tend to prefer higher x_1 and x_2 closer to 0. But there is substantial heterogeneity, which is well captured by the CES functions, as evidenced by the high standard deviations and the high rank correlations. The number of observations is relatively small because 22% of the agents have constant good solution functions on $Q_2 \cup Q_3$, cf. Table 19. Since only 4% of all games are solved at points in that subdomain, it makes evolutionary sense that many agents do not care to distinguish between them.

In order for the good solution functions to be scale invariant, the estimated θ's should be the same on different scales within the same quadrant. In Table 20, the evidence against this hypothesis is somewhat mixed: We find significant differences in means, but the differences are small, and with one exception, modes and medians coincide and are constant across scales.

We conclude this subsection with a rule of thumb to find approximate solutions to many games.[12] Let Γ_1 denote the set of games which contain some strategy profile that yields a pair of positive good reply scores, i.e., a point in Q_1. For instance, Γ_1 includes all games with one or more strict Nash equilibria. Our data show that 99.6% of all individual good solution

[12]Exact solutions to all games (up to 10×10) can be found at https://gplab.nhh.no/gamesolver.php.

Table 20: Estimated CES good solution functions $\hat{f}(x, \rho)$. $\rho = (\rho_1, \rho_2)$ is proportional to $(\cos(\frac{\pi}{180}\theta), \sin(\frac{\pi}{180}\theta))$, where θ is an angle measured in degrees. The table contains summary statistics for θ by quadrant (Q_k), and scale (λ). Numbers in parentheses are p-values from Watson-Wheeler tests of equal distributions for θ across different scales within each quadrant. All statistical measures involving θ are circular, see e.g., Pewsey et al. (2013).

Quadrant	Q_2			Q_1		
$\hat{f}(x, \rho)$	$(-x_1)^{-\rho_1} \cdot x_2^{\rho_2}$			$x_1^{\rho_1} \cdot x_2^{\rho_2}$		
Scale (λ)	1	10	100	1	10	100
Mean	324.4	323.2	323.1	44.8	45.0	45.0
St.dev.	42.9	43.1	43.2	1.9	1.6	1.5
P(H_0: Equal means)		(0.000)			(0.000)	
Median	316.2	315.0	315.0	45.0	45.0	45.0
Mode	315.0	315.0	315.0	45.0	45.0	45.0
ρ(mode)	(1, -1)	(1, -1)	(1, -1)	(1, 1)	(1, 1)	(1, 1)
Obs. at mode	22%	36%	46%	87%	99%	98%
Obs. (1,000)	157	156	156	200	200	200
Mean rank corr.	0.986	0.989	0.989	0.999	0.999	0.999

Quadrant	Q_3			Q_4		
$f(x, \rho)$	$(-x_1)^{-\rho_1} \cdot (-x_2)^{-\rho_2}$			$x_1^{\rho_1} \cdot (-x_2)^{-\rho_2}$		
Scale (λ)	1	10	100	1	10	100
Mean	18.2	18.8	19.5	79.9	78.9	78.9
St.dev.	53.3	52.2	54.4	53.3	55.6	54.2
P(H_0: Equal means)		(0.000)			(0.000)	
Median	0.0	0.0	0.0	90.0	90.0	90.0
Mode	0.0	0.0	0.0	90.0	90.0	90.0
ρ(mode)	(1, 0)	(1, 0)	(1, 0)	(0, 1)	(0, 1)	(0, 1)
Obs. at mode	50%	56%	55%	28%	33%	34%
Obs. (1,000)	155	154	155	180	178	175
Mean rank corr.	0.971	0.977	0.973	0.932	0.931	0.952

functions rank all points in Q_1 above all points in $\mathbb{R}^2 \setminus Q_1$. This implies that all games in Γ_1 are solved in Q_1. Since individual good reply scores are almost scaled sums of deviation losses (Table 18), and since on Q_1 the good solution functions are almost equivalent to the product function (Table 20), it follows that almost all games in Γ_1 are solved at a pair of action and conjecture which maximizes the product of the two players' sums of deviation losses. Table 19 shows that 86% of all randomly generated games can be solved in this way. Note, however, that when the maximizer is not unique, as would be the case in the Battle of the sexes game of Section 4.1.1, one must consult the ASC to get the correct randomization. In that sense, the rule of thumb yields only approximate solutions.

4.3.4 2×2 games

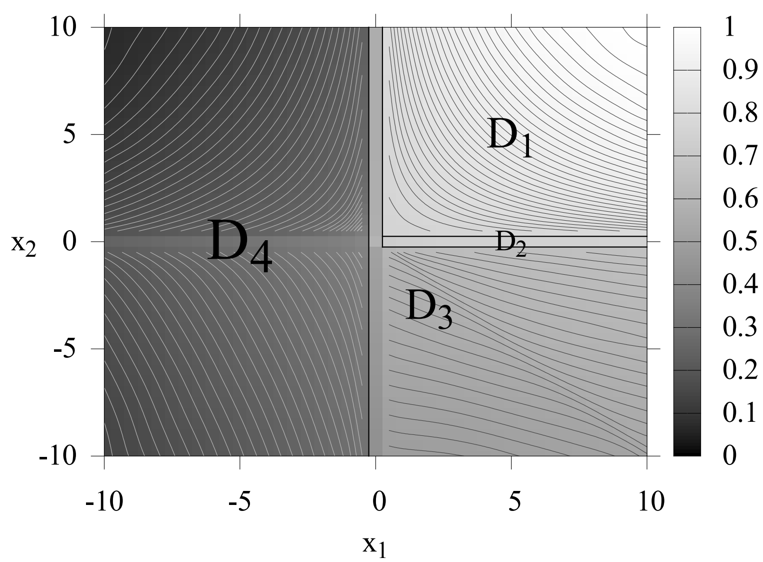
In this subsection, we construct a representative solution concept for 2×2 games. We do this because 2×2 games constitute an important special case, and because it allows to visualize its numerical representation in two dimensions.

Let Γ^2 denote the set of 2×2 games. Consider an agent a and the composite function $v^a : \mathbb{R}^2 \to \mathbb{R}$, defined as

$$v^a(x) := f^a(g^a(x_1), g^a(x_2)), \tag{16}$$

where $x = (x_1, x_2) = \delta(\mathbf{s})$ is a pair of deviation losses corresponding to some strategy profile $\mathbf{s}$ for some $G \in \Gamma^2$. The function v^a in (16) is a numerical representation of agent a's solution concept on Γ^2. We construct a representative solution concept by aggregating those numerical representations across all agents. For each agent a, rank all the points in $D_0 = \{-10, -9.5, \ldots, 10\}^2$ according to v^a. Define $v : D_0 \to [0,1]$ as the Borda count of all the 200,000 rankings obtained in this way, normalized to yield rank scores in $[0,1]$. Let D denote the convex hull of D_0 and extend v to D by interpolation. For each $G = (S, \pi) \in \Gamma^2$ such that $\delta(S) \subset D$, let $F^2(G) := \mathrm{argmax}_{\mathbf{s}\in S} v(\delta(\mathbf{s}))$. Then F^2 is a representative solution concept for the ASC on Γ^2.

Figure 1: Contour plot of the numerical representation v for the solution concept F^2 on the set $D = [-10, 10]^2$ of pairs of deviation losses. The subsets $\{D_k\}_{k=1}^4$ constitute a partition of D such that $v(x) > v(y) > v(z) > v(w)$ for any $(x, y, z, w) \in D_1 \times D_2 \times D_3 \times D_4$. Straight black lines represent boundaries between those partition elements, and gray curves represent indifference with respect to v.



A contour plot of v is provided in Figure 1. Each quadrant illustrates the information in the corresponding quadrant in Table 20.[13] In Q_1, v has a rank correlation of 0.9999 with the

[13]Figure 1 and Table 20 deal with different functions, but the comparison is still meaningful because $v^a = f^a(g^a(), g^a())$ of the former only differs from f^a of the latter by the affine transformations g^a.

CES function $x_1 \cdot x_2$, which shows that F^2 agrees with the Harsanyi-Selten risk dominance concept (8) for 2×2 games. Furthermore, the wriggly contour lines in Q_4 reflect our finding in Table 20 that some good solution functions fail to be scale invariant on Q_4.

The four subsets $\{D_k\}_{k=1}^4$ are defined as

$$\begin{aligned} D_1 &= D \cap Q_1, & D_3 &= D \cap (Q_4 \cup (\{0\} \times \mathbb{R})), \\ D_2 &= D \cap (\mathbb{R}_{++} \times \{0\}), & D_4 &= D \cap (\mathbb{R}_{--} \times \mathbb{R}). \end{aligned} \tag{17}$$

They constitute a partition of D such that

$$v(x) > v(y) > v(z) > v(w) \text{ for any } (x, y, z, w) \in D_1 \times D_2 \times D_3 \times D_4. \tag{18}$$

i.e., the elements of $\cup_{k=1}^4 D_k$ are coarse equivalence classes for v. The straight black lines in Figure 1 represent borders between those equivalence classes, and the gray curves represent indifference with respect to v. Plotting of indifference curves is suppressed along the zero bins of D_0 because v is discontinuous at such points. But the data show that v increases in x_1 for $x_2 = 0$, whereas for $x_1 = 0$, v increases as $x_2 \to 0$.

A 2×2 game G is solved with the representative solution concept F^2 by using the contour lines in Figure 1 to rank the strategy profiles $\mathbf{s} \in \Sigma(G)$ by their deviation losses $\delta(\mathbf{s})$.

For many games, this ranking can be obtained directly from the partition $\{D_k\}_{k=1}^4$. Since F^2 solves games by maximizing $v(\delta(\cdot))$, the definition of $\{D_k\}_{k=1}^4$ and the relation (18) imply the following: First, games with strict Nash equilibria are solved in D_1. Second, any other game with one or more weak Nash equilibria is solved in D_2 if player 1 has a strict best reply in any such equilibrium. Otherwise the game is solved in D_3. Finally, since any game can be solved in $D \setminus D_4$ by choosing a best reply for player 1 to any conjecture about player 2, it follows that no game is solved in D_4 and that all games without pure Nash equilibria are solved in D_3. This implies that the solution concept F^2 is rational in the sense that its action is always a best reply to its conjecture.

5 Conclusion

The paper uses an evolutionary model to develop a new solution concept for one-shot bimatrix games. It is constructed by aggregating the individual solution concepts of 200,000 artificial agents who have learned to play one-shot games through natural selection. The agents solve games by reasoning in terms of good replies and good solutions. This can be thought of as a soft, non-equilibrium generalization of the Nash equilibrium concept. By taking the mean of all individual solutions to each game we obtain an aggregate solution concept (ASC), whose individual components constitute a stochastically stable equilibrium.

Almost all agents have additive and almost linear good reply functions. The good solution functions are heterogeneous, but most of the variation can be accounted for by fitting a collection of CES functions to each agent's good solution function. In particular, on the positive quadrant, most good solution functions are the product of their two arguments.

This means that games with strict Nash equilibria, and other games which contain some strategy profile that yields a pair of positive good reply scores, can be solved by finding strategy profiles which maximize the product of the two players' sums of deviation losses. This yields risk dominance for 2×2 games and an extension of that criterion to games with higher dimensions. 86% of all games can be solved by this rule of thumb. There is no simple recipe to solve the remaining 14% because the solutions to those games depend on the full spectrum of individual heterogeneity. We therefore provide a web page that can be used to solve all bimatrix games with up to 10 strategies per player (https://gplab.nhh.no/gamesolver.php).

Applying the ASC to a number of well-known games, we find that it agrees well with intuition and empirical evidence. Examples include the Ultimatum game, the Traveler's dilemma, the Centipede game and a collection of games from the refinement literature. The ASC also behaves as if the agents were motivated by fairness or altruism in some games that were designed to study social norms.

Our approach to modeling one-shot play can be extended in several directions. (1) We have assumed that payoffs represent von Neumann-Morgenstern utility. A model with monetary payoffs might be built by evolving utility functions along with the good reply and good solution functions. (2) Our agents are boundedly rational due to computational constraints on program length and memory. These parameters can be varied to study behavioral effects of variations in bounded rationality. (3) By representing games in terms of vectors of deviation losses, our model forces the agents to focus on strategic stability, i.e., variations in player i's payoffs for a given action by player j, with no focus on risk, i.e., variations in i's payoffs for a given action by player i. In experiments with human subjects, such risk considerations seem to play a role, and it would be of interest to see if our artificial agents would make the same considerations if they were provided with the relevant information.

Appendix A Model details

A.1 Implementation of separable good replies

A good reply function g is separable if there exists a function $\gamma : \mathbb{R}^2 \to \mathbb{R}$ and a constant z_0, such that $g(x_1) = \gamma(z_0, x_1)$ and $g(x_1, \ldots, x_k) = \gamma(g(x_1, \ldots, x_{k-1}), x_k)$ for $k \geq 2$. An implementation of this condition is provided in Table 21. $\mathbf{x}$ is a vector of deviation losses, and $\mathbf{z}$ is a real vector of scratch memory for the algorithm, whose first element (z_1) is taken to be its return value. K is the number of deviation losses, one less than the number of pure strategies available to player i. For instance the value of K can be used by solution concepts that rely on some kind of average. $d(k)$ is a dummy variable to indicate whether the current iteration k is the first one. This information will allow γ to re-initialize one or more of the memory slots $\mathbf{z}$ at the beginning of the first iteration for solution concepts that need some initial value other than 0.

A.2 Implementation of solution concepts

Let F be an admissible solution concept, let (f, γ) be a numerical representation for F, and let g be the good reply function generated by γ by means of the algorithm in Table 21. To solve games, the functions f and γ, which are specific to each agent, must be implemented as computer programs. Because computing time is going to be an issue, we implement f

Table 21: Algorithm to compute the function g for a player i at strategy profile $\mathbf{s}$ in a game G by means of an iteration function γ. $\mathbf{x} = (x_1, ..., x_K)$ is a vector of length K containing the deviation losses in $\delta_i(\mathbf{s})$ for G at $\mathbf{s} \in \Sigma(G)$ and $d(k)$ is a dummy variable which is 1 if $k = 1$ and 0 otherwise. $\mathbf{z}$ is a real vector of scratch memory for the algorithm, whose first element (z_1) is taken to be its return value.

Pseudo-code	Comment
$\mathbf{z} = \mathbf{0}$	Initialize memory
For $k = 1$ to K	Loop over deviation losses
$\quad\mathbf{z} \leftarrow \gamma(\mathbf{z}, x_k, d(k), K)$	Update memory
End For	End of loop
$g(\mathbf{x}) = z_1$	Return value

and γ in machine code,[14] following Nordin (1997). Each program consists of at most 32 machine instructions for the x86-64 processor. The processor has 16 floating point registers, and we use four of those as scratch memory for the programs. For the iteration program γ, the contents of the memory slots (denoted $\mathbf{z}$ in Table 21) are preserved across iterations.

Program instructions specify one or more operators and one or more operands. Operators consist of +, -, /, ×, *maximum, minimum, change sign, absolute value*, variable manipulations *copy*, program-flow instructions, *if, goto*, and relational operators $<$, $>$, $\leq$, $\geq$, $=$, $\neq$. This set of operators allows for conditional arithmetic operations and assignments, as well as conditional jumps.[15] Operands consist of the relevant input variables, the four memory slots, and randomly chosen constants. When a program executes, the memory slots are initialized to 0 and the instructions are performed in order. The output from a program is taken to be the value of the first memory slot after the program has executed.[16]

We next describe how scale invariance and symmetric good replies can be imposed on F. Because solution concepts are subject to random modifications, strict compliance is difficult to achieve. Instead, we provide strong incentives by means of a 'nudge', which scrambles any information that could lead to a violation of the property in question. To see how, consider a game $G = (S, \pi)$, and a player position $i \in \{1, 2\}$.

First, we impose symmetric good replies by randomly shuffling the deviation losses in $\delta_j(\mathbf{s})$ before computing $g(\delta_j(\mathbf{s}))$ for each player $j \in \{1, 2\}$ and each strategy profile $\mathbf{s}$. This scrambles the ordering of strategies and removes any possibilities for the agents to coordinate, or otherwise condition, their actions on the ordering of strategies.

Second, to enforce scale invariance, we introduce a distinction between the payoffs that

[14]The machine code representation is used for fast execution of programs. In addition, we use a byte code representation to simplify program generation and manipulation, a small compiler to translate byte code to binary machine code, and a byte code disassembler to produce program representations that can be read by humans and analyzed by computer algebra applications.

[15]All jumps are forward jumps to avoid infinite loops.

[16]The agents' programs will sometimes produce $\pm\infty$ or NaN (not a number). The function g will be restricted to return only real numbers to ensure that the arguments to f are real, while f will be allowed to return $\pm\infty$ as well. To this end, any NaN or $\pm\infty$ from g and any NaN from f will be replaced by a random draw from a normal distribution with large standard deviation.

will be used as arguments to the solution concept F and the payoffs that will be used to measure its performance. To measure performance, we use the original payoffs π_i, whereas the arguments to F are obtained by multiplying both players' payoffs by two separate real random numbers from the interval $[0.01, 100]$. This scrambles the agents' information about the stakes of the game, which provides them with an incentive to develop scale invariant solution concepts.[17]

A.3 Games

Agents develop solution concepts by playing lots of random games. To generate the dimensions and payoffs of those games, a probability distribution on the space of games is needed.

Game payoffs are generated by independent draws from a normal distribution with mean 0 and standard deviation 10. Each payoff is rounded to the nearest integer to produce some games with weak best replies, weakly dominated strategies, and connected components of Nash equilibria. Games with these features are the subject matter of the large literature on equilibrium refinements.

To generate game dimensions, we need a probability distribution with finite support to ensure that the computing time to solve a random game is bounded, and it should select larger games with lower probability in order to save computing time. Moreover, because we shall compare results with alternative experiments where the agents are not allowed to play strictly dominated strategies, we want the game dimensions to be identically distributed across those experiments.

Table 22: Auxiliary probability distribution to select a number n of strategies for one player.

n	2	3	4	5	6	7	8	9	10
$p(n)$	0.222	0.243	0.152	0.117	0.088	0.065	0.050	0.039	0.024

To meet those ends, we consider games where the number of strategies per player is a number between 2 and 10, inclusive. To produce a game G, we first generate a pair of dimensions (n'_1, n'_2) by means of two independent draws from the probability distribution p in Table 22, and then randomly generate payoffs for a game G^1 with those dimensions. Second, we iteratively eliminate all strictly dominated strategies from G^1 to obtain a game G^2 of dimension $(n_1, n_2) \leq (n'_1, n'_2)$. If $n_i < 2$ for any $i \in \{1, 2\}$, we discard G^1 and G^2 and repeat the first two steps until both players in G^2 have at least two undominated strategies. Third, set $G = G^2$ if we want a game without strictly dominated strategies, otherwise, randomly generate a new game G^3 with the same dimensions (n_1, n_2) as G^2, and set $G = G^3$.

The first step of this procedure produces a large proportion of asymmetrically shaped games. The actual or simulated elimination of dominated strategies in steps 2 and 3 improves the distribution by removing some asymmetric games, while at the same time shifting it towards smaller games. The numbers in Table 22 are somewhat arbitrary, but the shape of

[17] As noted earlier, F is already immune against the constant term in such transformations because it only depends on the players' deviation losses. So there is no need to also add a random number.

the distribution has been chosen in order for the whole procedure to produce a reasonable number of medium-sized and large games. For example, it yields 2×2 games with probability 0.21, 4×5 games with probability 0.05, and 10×10 games with probability 0.003.

A.4 Evolution

We apply a genetic programming algorithm (Koza 1992) to model the evolution of solution concepts. The algorithm starts by creating 1,000 random games and 2,000 agents, each equipped with a random pair of programs (f^a, γ^a). These programs are then applied to solve each game for each agent from the point of view of each player, as described in Section 3.1.

The genetic programming algorithm is run for 100,000 iterations, each of which consists of the following three stages:

1. *Performance measurement:* Each agent a plays each game in a random position against a random opponent $b \neq a$ in the opposite position. The payoffs for agent a are summed up across all games to obtain a measure of a's performance.[18]

2. *Tournament selection:* Using these performance measures, the algorithm arranges 50 tournaments, each involving four randomly selected agents. In each tournament, the algorithm replaces the programs of the two losers by recombining the programs of the two winners. Equipped with new programs, both losers then solve all 1,000 games.

3. *Game replacement:* 10 games are randomly selected and replaced with another 10 randomly generated games. The 10 new games are solved by all 2,000 agents.

By replacing only 10 out of 1,000 games in stage 3 of each iteration, most games will be played several times by most agents across subsequent iterations. By keeping records of each agent's solutions to each game, it can be solved once and then played repeatedly without having to execute the agent's programs. This allows to complete a run with the genetic programming algorithm in a couple of days, as compared to months if one were to replace all games in every iteration.

With all this repeated play, the reader may wonder what became of our story of one-shot games, in which the agents are supposed to never play the same game twice. Fortunately, it is still intact, because the agents have no memory of previously played games, except for whatever is contained in their programs. From the agents' perspective, the situation looks like a one-shot game, provided the set of games exhibits enough variation over time to prevent overfitting (knowing the solutions to specific games) and induce learning (knowing how to play games). To that end, it will suffice to replace 10 out of 1,000 games in each iteration.

Tournament selection uses the standard genetic operators copy, crossover and mutation to produce programs that perform increasingly better over time. We implement this mechanism as follows:

1. *Tournament:* Randomly select four agents from the population, and rank them by decreasing performance to get an ordered set $\{a_1, a_2, a_3, a_4\}$ of agents.

[18] The performance of a's opponents is computed separately but in the same way, i.e., by randomly selecting an opponent and a position for each game, and accumulating payoffs across all games.

2. *Copy:* Replace the programs of agents 3 and 4 with copies of the programs of agents 1 and 2. Denote the copied programs by (f^3, γ^3) and (f^4, γ^4).

3. *Crossover:* With probability χ_1, cross f^3 with f^4 by swapping randomly selected sublists of instructions among them, and cross γ^3 with γ^4 in the same way.

4. *Mutation:* Each of the four new programs undergoes a mutation with probability χ_2: A single instruction in the program is randomly selected, and replaced with a randomly generated instruction.

The crossover and mutation rates, χ_1 and χ_2, are initially set to 0.5 and 0.8. Between iteration 40,000 and 80,000 both rates decay to 0.01 and stay there until the last iteration. To begin with, this produces a noisy environment with lots of experimentation, and then a period with increasing imitation as the system cools down to possibly settle in a stable state. We collect data from the last 20,000 iterations, and examine whether the distribution of solution concepts has then reached a *stochastically stable equilibrium* in the sense of Young (1994).

Appendix B Robustness checks

The ASC is tested for robustness with respect to two changes to the model specification. First, we consider the algorithm which computes good replies, and ask if initialization by zero values could have introduced a bias towards additive good replies. Second, we analyze to what extent individual solution concepts are affected by the presence of strictly dominated strategies. This yields the four experiments shown in Table 23. The case D0 is the one considered so far.

Table 23: Robustness checks

Strictly dominated strategies	Initial memory	
	Zero	Random
Allowed	D0	DR
Not allowed	N0	NR

B.1 Memory initialization

The algorithm in Table 21, computing good reply scores for strategy profiles, initializes its memory slots $\mathbf{z}$ to zero. On exit from the algorithm, the first memory slot z_1 contains its return value, which is taken to be the good reply score for the given strategy profile. In this setting, additive good reply function can be obtained as a single instruction which simply adds the next deviation loss to z_1. To gauge the extent to which the existence of this 'shortcut' may have influenced the results, we re-run the model with the memory slots initialized to random values. This experiment, called DR in Table 23, turns out to sometimes produce a new type of agent with *multiplicative* good reply functions of the following form:

$$g(\mathbf{x}) = \begin{cases} \prod_{k=1}^{K}(\alpha + \beta x_k), & \text{if } \mathbf{x} \geq 0 \\ \xi(\mathbf{x}) < \min\limits_{\mathbf{x}' \geq 0} \prod_{k=1}^{K}(\alpha + \beta x'_k), & \text{otherwise.} \end{cases} \tag{19}$$

The first case assigns high scores to vectors of deviation losses $\mathbf{x}$ that correspond to best replies, and the second case assigns low scores ($\xi(\mathbf{x})$) to all other $\mathbf{x}$. This suggests that solution concepts with multiplicative good replies are geared towards solving games at pure Nash equilibria whenever they exist.

The functions γ^a and f^a for a typical agent a of this type are listed in (20)–(21).

$$\gamma^a(z, x_k, k) \;=\; \begin{cases} 3.7 \cdot 10^{13} \cdot (1.7 \cdot 10^{-4} + x_k) \cdot \max(0, 1), & \text{if } k = 1 \\ 3.7 \cdot 10^{13} \cdot (1.7 \cdot 10^{-4} + x_k) \cdot \max(0, z), & \text{if } k > 1. \end{cases} \tag{20}$$

$$f^a(x_1, x_2) \;=\; \begin{cases} x_1 \cdot x_2, & \text{if } (x_1, x_2) > 0 \\ x_1, & \text{if } x_1 > 0 \text{ and } x_2 \leq 0 \\ 0, & \text{if } x_1 \leq 0. \end{cases} \tag{21}$$

At the first iteration ($k = 1$) of γ^a, it deals with the initial random z by replacing it with 1. After K iterations, one obtains the good reply function g^a in (22), which has the structure (19).

$$g^a(\mathbf{x}) = \begin{cases} \prod_{k=1}^{K}(6.3 \cdot 10^9 + 3.7 \cdot 10^{13} x_k) \geq (6.3 \cdot 10^9)^K, & \text{if } \mathbf{x} \geq 0 \\ 0, & \text{if } x_k < 0 \text{ for some } k < K \\ \prod_{k=1}^{K}(6.3 \cdot 10^9 + 3.7 \cdot 10^{13} x_k) < 0, & \text{if } \mathbf{x}_{-K} \geq 0 \text{ and } x_K < 0. \end{cases} \tag{22}$$

Pure Nash equilibria are represented by Case 1 of (21). Again we see that the good solution function scores such strategy profiles by the product of the good reply scores. Parallel to (11) of Section 4.3, the small constant in (20) guarantees that the function $f^a(g^a(\cdot), g^a(\cdot))$ extends continuously from strict to weak best replies because $(1.7 \cdot 10^{-4} + x_k)$ is positive for $x_k \geq 0$ and negative for all other deviation payoffs. This is shown below along with a proof of (22).[19]

The large constant factor in (20) is typical for multiplicative good reply functions. Larger constants cause more games to be solved at pure Nash equilibria, by producing larger g–scores which increase the likelihood that f^a reaches a maximum at case 1 of (21).[20] As an

[19] We show that (22) holds. The first case follows directly from (20). Consider next the second and third cases of (22). As shown in footnote 10, $\text{abs}(x_k) \geq 10^{-2}$ if $x_k \neq 0$. Consequently, the term $(1.7 \cdot 10^{-4} + x_k)$ in (20) is positive if $x_k \geq 0$ and negative otherwise, and $\gamma^a(z, x_k, \ldots) \leq 0$ if $x_k < 0$. If $x_k < 0$ for $k < K$, the term $\max(0, z)$ in (20) ensures that $\gamma^a(x_{k'}, \ldots) = 0$ for all $k' > k$, hence $g^a(\mathbf{x}) = 0$, which proves case 2 of (22). But if $x_k \geq 0$ for $k < K$ and $x_K < 0$ then $\gamma^a(z, x_k, \ldots) > 0$ for all $k < K$ and $g^a(\mathbf{x}) = \gamma^a(z, x_K, \ldots) < 0$, which proves case 3 of (22). Thus the random order in which deviation losses are presented to γ^a can lead to a negative or a zero score if one or more deviation losses are strictly negative. But the agent still behaves in a consistent manner because the good solution function f^a in (21) does not distinguish between zero and negative arguments.

[20] To see this, let (x_1, x_2) and (x'_1, x'_2) satisfy the conditions of case 1 and 2 of (21), respectively. Then $\lambda x_1 \cdot \lambda x_2 > \lambda x'_1$, for sufficiently large λ.

upshot, games without pure Nash equilibria are solved somewhat arbitrarily: The g–score to player 1 is maximized without regard for that of player 2 (case 2 of (21)). By comparison, the additive solution concept in (12)–(13) of Section 4.3 solves games without pure Nash equilibria by maximizing the g–score to player 2 on the set of strategy profiles that yield a positive g–score for player 1.

We do 100 runs with experiment DR and use the same procedure as in Section 4.3.2 to test for additive and multiplicative good replies: For each agent a, we generate a data set with 100 observations (y^a, x_1, x_2), where (x_1, x_2) is a vector of two random deviation losses and $y^a = g^a(x_1, x_2)$. We restrict the deviation losses to be positive and bounded away from 0 because we cannot exclude the possibility that some good reply functions have discontinuities close to zero values of the arguments, cf. (19) and (22). We then estimate the linear model (15) for each agent and aggregate the parameter estimates by runs.

A summary of the results is contained in Table 24, where $P(\beta_{12})$ is the P-value associated with the multiplicative term β_{12} in (15). We sort the sample by $P(\beta_{12})$, split it at $P(\beta_{12}) = 0.1$, and examine the estimated parameters to find that the 48 runs in Panel A (with parameters $\alpha = 0$, $\beta_1 = \beta_2$, $\beta_{12} = 0$ and $R^2 > 0.99$) have additive good reply functions, while the 52 runs in Panel B (with $\beta_{12} \neq 0$) are consistent with the multiplicative good reply structure in (19).

Table 24: Test of good reply functions by means of (15) for experiment DR. The number of observations is 100, one observation for each run with the model. Parameters α, β_1, β_2, β_{12} and R^2 are defined as in Table 18. $P(\beta_{12})$ is the median P-value by run associated with β_{12}, and *playNash* is the frequency of Nash equilibrium play in games with one Nash equilibrium. P-values (from left to right) refer to Wilcoxon tests against the null hypotheses that $\alpha = 0$, $\beta_1 = \beta_2$, and $\beta_{12} = 0$.

Panel A: $P(\beta_{12}) > 0.1$, 48 runs, none with *playNash* ≥ 0.99.							
Parameter	α	β_1	β_2	β_{12}	R^2	$P(\beta_{12})$	*playNash*
Min	-0.013	-6	-6	$-6.4e{-}09$	0.997	0.606	0.828
Max	1.000	18	18	$3.4e{-}08$	1.000	1.000	0.846
Median	0	2	2	0	1.000	1.000	0.838
Mean	0.043	3.667	3.667	$8.8e{-}10$	0.999	0.927	0.837
P-value	0.363	1.000		0.236			
Panel B: $P(\beta_{12}) \leq 0.1$, 52 runs, 46 with *playNash* ≥ 0.99.							
Parameter	α	β_1	β_2	β_{12}	R^2	$P(\beta_{12})$	*playNash*
Min	$-2.1e{+}27$	$-1.7e{+}26$	$1.1e{-}01$	$-7.9e{-}03$	0.161	0.000	0.838
Max	$4.1e{+}31$	$5.3e{+}30$	$3.7e{+}30$	$4.5e{+}29$	0.929	0.051	1.000
Median	$2.8e{+}17$	$8.3e{+}19$	$4.9e{+}21$	$2.7e{+}20$	0.659	0.000	0.997
Mean	$1.0e{+}30$	$1.6e{+}29$	$1.5e{+}29$	$1.5e{+}28$	0.593	0.001	0.990
P-value	0.000	0.449		0.000			

Table 24 supports our conjecture that multiplicative good replies are closely associated with Nash equilibrium play. Variable *playNash* is the frequency of Nash equilibrium play in games with one Nash equilibrium, as explained in Section 4.2. On average, agents with multiplicative good replies play the Nash equilibrium 99% of the time, against 83.7% for the agents with additive good replies. By looking at individual runs, we find that 46 of the 52 runs in Panel B have *playNash* ≥ 0.99 and that all 48 runs in Panel A have *playNash* < 0.85. The remaining 6 runs seem to represent a mix of agents with additive and multiplicative and good reply functions.

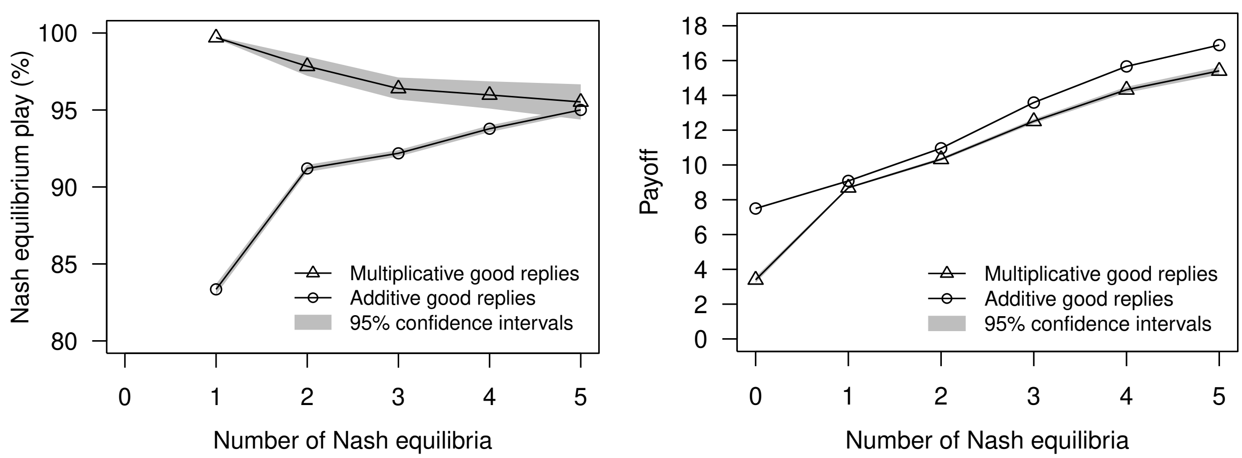
In what follows, we will disregard those 6 runs and reserve the term *multiplicative* for agents and model runs with *playNash* ≥ 0.99. Analogously, an *additive agent* is one whose good reply function yields $\beta_1 = \beta_2$, $\beta_{12} = 0$ and $R^2 > 0.99$ when fitted to (15), and an additive model run is one for which the median β's and the mean R^2 satisfy these conditions.

In the remainder of this subsection, we compare the 48 additive runs of Table 24, Panel A with the 46 multiplicative runs from Panel B. The 48 additive runs constitute an additive ASC, and the 46 multiplicative runs form a multiplicative ASC.

Applying the multiplicative ASC to the games in Section 4.1, we find less cooperation and lower aggregate payoffs as compared to the additive one: The 'refinement' games in Tables 9 and 10 are both solved at the inferior equilibrium (T, R), and in the Centipede game the agents take the money at the first opportunity. In ultimatum games with 5 or 10 dollars to share, the players offer and demand one dollar, and with 50 or 100 dollars to share, offers and demands amount to only 8% of the total.

We next compare the additive and the multiplicative ASC across a large number of games with a varying number of pure Nash equilibria. We create six sets of 1,000 games with the number of pure Nash equilibria ranging from 0 to 5 and solve each one of those 6,000 games with the two ASC's. The results are presented in Figure 2.

Figure 2: Behavior of the additive and the multiplicative aggregate solution concepts from experiment DR in games with a varying number of pure Nash equilibria. The number of observations is 94.



The left panel of Figure 2 shows the frequency of Nash equilibrium play.[21] In games with one pure Nash equilibrium, the multiplicative agents play that strategy profile in 99.7% of

[21]If agents would independently randomize between the n row and n column strategies that support n pure Nash equilibria, the generic probability of playing some Nash equilibrium is $1/n$.

those games. As the number of pure Nash equilibria increases, the frequency of Nash play declines, but remains above 95%. The additive agents are not equipped to identify Nash equilibria. Instead they look for strategy profiles with positive sums of deviation losses, which become more prevalent as the number of pure Nash equilibria increases. In games with one pure Nash equilibrium, these agents play Nash only 84% of the time, but this frequency is increasing in the number of equilibria. For games with 5 pure Nash equilibria there is no significant difference between the two ASC's with respect to the frequency of Nash equilibrium play.

The right panel of Figure 2 plots payoffs against the number of pure Nash equilibria for the two ASC's. Payoffs increase as the number of Nash equilibria increases, with additive agents doing better throughout. The difference is small for games with one pure Nash equilibrium, but widens as the number of equilibria increases. The multiplicative agents fare particularly badly in games with no pure Nash equilibrium, obtaining less than half the payoff of the additive agents.

B.2 Rationalizability

We have seen in Section 4.1 that the additive solution concept sometimes produces solutions that are not subgame perfect, or not Nash, or include strictly dominated strategies. While strictly dominated solutions agree with intuition or experiments for some games, it raises the issue to what extent the solution concept is robust with respect to addition of dominated strategies.[22] To illustrate the issue, we consider the game in Table 25.

Table 25: A game with strictly dominated strategies. Numbers in italics are probabilities (%).

(s,t)		A	b	c	
	σ	*0*	*100*	*0*	Col
A	*0*	**1, 1**	11, 0	-1, -2	*0*
b	*100*	0, 11	10, 10	-2, 0	*100*
c	*0*	-2, -1	0, -2	-3, -3	*0*
	Row	*0*	*100*	*0*	ϕ

The game is symmetric and has one Nash equilibrium in pure strategies at (A, A), with payoffs $(1, 1)$. The additive ASC solves the game at (b, b), which yields payoffs $(10, 10)$. Human players might also be able to solve the game at (b, b) because it yields high, identical payoffs and only weak incentives to deviate to A. But this is not quite how the ASC arrives at its solution: When the good solution function takes sums of deviation losses as inputs, (b, b) is selected because it has a high g-score of 9 since $(10-11)+(10-0) = -1+10 = 9$. The smaller negative term -1 is associated with the weak incentives to deviate. But the larger positive term 10 is due to the presence of the dominated action c. Although the ASC seems to have found the 'right' solution to this game, it may have done so for the wrong reason. If

[22] Kohlberg & Mertens (1986) dismiss the idea of robustness with respect to addition of strictly dominated strategies in relation to strategic stability, but in our case, there are additional considerations to be made.

the dominated action c is eliminated from the game, we obtain a Prisoner's dilemma game which is solved at (A, A) by the ASC, cf. Section 4.1.

It is easy to construct this type of examples by adding strictly dominated strategies to an existing game. An obvious remedy would be to iteratively eliminate strictly dominated strategies (IESDS) before presenting the game to the ASC for solution. The modified ASC would then solve the game in Table 25 at (A, A) and any other game at some rationalizable[23] pair of strategies. However, the ASC may no longer be stochastically stable if IESDS is imposed on it ex post. We will therefore impose IESDS ex ante and see if, and how, this affects the aggregate solution concept.

To that end, we carry out two additional experiments, N0 and NR, each one consisting of 100 runs with the model. N0 and NR are identical to D0 and DR, respectively, except that the agents are not allowed to play strictly dominated strategies, see Table 23. This restriction is imposed by iteratively removing all strictly dominated strategies from any game before applying some individual solution concept.

Experiment NR (IESDS and random initial memory) yields 81 additive runs and one multiplicative one out of 100 runs in total. Further, 100 runs of experiment N0 (IESDS and zero initial memory) yields 93 additive runs and no multiplicative ones. Thus IESDS strengthens the additive solution concept by removing some potentially irrelevant information which the additive solution concept is unable to detect. Apparently, this effect is strong enough, or the competition from multiplicative Nash players is weak enough, for the additive solution concept to prevail when IESDS is imposed.

References

Arifovic, J. (1994). Genetic algorithm learning and the cobweb model, *Journal of Economic Dynamics and Control* **18**(1): 3–28.

Arifovic, J. (1995). Genetic algorithms and inflationary economies, *Journal of Monetary Economics* **36**(1): 219–243.

Arifovic, J. (1996). The behavior of the exchange rate in the genetic algorithm and experimental economies, *Journal of Political Economy* **104**(3): 510–541.

Basu, K. (1994). The Traveler's dilemma: Paradoxes of rationality in game theory, *American Economic Review: Papers and Proceedings* **84**(2): 391–395.

Bernheim, B. D. (1984). Rationalizable strategic behavior, *Econometrica* **52**(4): 1007–1028.

Brown, G. W. (1951). Iterative solution of games by fictitious play, *Activity Analysis of Production and Allocation* **13**(1): 374–376.

Capra, C. M., Goeree, J. K., Gomez, R. & Holt, C. A. (1999). Anomalous behavior in a traveler's dilemma?, *American Economic Review* **89**(3): 678–690.

[23] Bernheim (1984) and Pearce (1984).

Carlson, H. & van Damme, E. (1993). Equilibrium selection in stag hunt games, *in* K. Binmore, A. Kirman & P. Tani (eds), *Frontiers of Game Theory*, Addison-Wesley, Reading MA, pp. 237–254.

Chen, S.-H., Duffy, J. & Yeh, C.-H. (2005). Equilibrium selection via adaptation: Using genetic programming to model learning in a coordination game, *in* A. S. Nowak & K. Szajowski (eds), *Advances in Dynamic Games*, Springer, New York NY, pp. 571–598.

Cooper, D. J. & Kagel, J. H. (2003). Lessons learned: Generalizing learning across games, *American Economic Review* **93**(2): 202–207.

Cooper, D. J. & Kagel, J. H. (2008). Learning and transfer in signaling games, *Economic Theory* **34**(3): 415–439.

Cooper, R., DeJong, D. V., Forsythe, R. & Ross, T. W. (1989). Communication in the battle of the sexes game: Some experimental results, *RAND Journal of Economics* **20**(4): 568–587.

Cooper, R., DeJong, D. V., Forsythe, R. & Ross, T. W. (1993). Forward induction in the battle-of-the-sexes games, *American Economic Review* **83**(5): 1303–1316.

Costa-Gomes, M., Crawford, V. P. & Broseta, B. (2001). Cognition and behavior in normal-form games: An experimental study, *Econometrica* **69**(5): 1193–1235.

Crawford, V. P., Costa-Gomes, M. A. & Iriberri, N. (2013). Structural models of nonequilibrium strategic thinking: Theory, evidence, and applications, *Journal of Economic Literature* **51**(1): 5–62.

Crawford, V. P., Gneezy, U. & Rottenstreich, Y. (2008). The power of focal points is limited: Even minute payoff asymmetry may yield large coordination failures, *American Economic Review* **98**(4): 1443–1458.

Duffy, J., Lai, E. K. & Lim, W. (2017). Coordination via correlation: An experimental study, *Economic Theory* **64**(2): 265–304.

Fudenberg, D. & Liang, A. (2019). Predicting and Understanding Initial Play, *American Economic Review* **109**(12): 4112–4141.

Gale, J., Binmore, K. G. & Samuelson, L. (1995). Learning to be imperfect: The ultimatum game, *Games and Economic Behavior* **8**(1): 56–90.

Germano, F. (2007). Stochastic evolution of rules for playing finite normal form games, *Theory and Decision* **62**(4): 311–333.

Gilboa, I. & Schmeidler, D. (1995). Case-based decision theory, *Quarterly Journal of Economics* **110**(3): 605–639.

Gilboa, I., Schmeidler, D. & Wakker, P. P. (2002). Utility in case-based decision theory, *Journal of Economic Theory* **105**(2): 483–502.

Grimm, V. & Mengel, F. (2012). An experiment on learning in a multiple games environment, *Journal of Economic Theory* **147**(6): 2220–2259.

Güth, W. R., Schmittberger, R. & Schwarze, B. (1982). An experimental analysis of ultimatum bargaining, *Journal of Economic Behaviour & Organization* **3**(4): 367–388.

Güth, W. R. & Tietz, R. (1990). Ultimatum bargaining behavior: A survey and comparison of experimental results, *Journal of Economic Psychology* **11**(3): 417–449.

Harsanyi, J. & Selten, R. (1988). *A General Theory of Equilibrium Selection in Games*, MIT Press, Cambridge MA.

Haruvy, E. & Stahl, D. O. (2012). Between-game rule learning in dissimilar symmetric normal-form games, *Games and Economic Behavior* **74**(1): 208–221.

He, S. & Wu, J. (2020). Compromise and coordination: An experimental study, *Games and Economic Behavior* **119**: 216–233.

Huck, S. & Müller, W. (2005). Burning money and (pseudo) first-mover advantages: an experimental study on forward induction, *Games and Economic Behavior* **51**(1): 109–127.

Jehiel, P. (2005). Analogy-based expectation equilibrium, *Journal of Economic Theory* **123**(2): 81–104.

Kohlberg, E. & Mertens, J.-F. (1986). On the strategic stability of equilibria, *Econometrica* **54**(5): 1003–1037.

Koza, J. R. (1992). *Genetic Programming: On the Programming of Computers by Means of Natural Selection*, MIT Press, Cambridge MA.

Koza, J. R. (ed.) (1998-2007). *Genetic Programming Book Series*, Springer, New York.

Lensberg, T., Schenk-Hoppé, K. R. & Ladley, D. (2015). Costs and benefits of financial regulation: Short-selling bans and transaction taxes, *Journal of Banking & Finance* **51**: 103–118.

LiCalzi, M. (1995). Fictitious play by cases, *Games and Economic Behavior* **11**(1): 64–89.

Luce, R. D. & Raiffa, H. (1957). *Games and decisions: Introduction and critical survey*, Wiley.

Marks, R. E. (2002). Playing games with genetic algorithms, *in* S. H. Chen (ed.), *Evolutionary Computation in Economics and Finance. Studies in Fuzziness and Soft Computing*, Vol. 100, Physica, Heidelberg, pp. 31–44.

McKelvey, R. D. & Palfrey, T. R. (1992). An experimental study of the centipede game, *Econometrica* **60**(4): 803–836.

Mengel, F. (2012). Learning across games, *Games and Economic Behavior* **74**(2): 601–619.

Mookherjee, D. & Sopher, B. (1994). Learning behavior in an experimental matching pennies game, *Games and Economic Behavior* **7**(1): 62–91.

Nagel, R. (1995). Unraveling in guessing games: An experimental study, *American Economic Review* **85**(5): 1313–1326.

Nordin, P. (1997). *Evolutionary Program Induction of Binary Machine Code and its Applications*, Krehl Verlag, Münster.

Pace, M. (2009). How a genetic algorithm learns to play traveler's dilemma by choosing dominated strategies to achieve greater payoffs, *2009 IEEE Symposium on Computational Intelligence and Games*, IEEE, pp. 194–200.

Pearce, D. G. (1984). Rationalizable strategic behavior and the problem of perfection, *Econometrica* **52**(4): 1029–1050.

Pewsey, A., Neuhäuser, M. & Ruxton, G. D. (2013). *Circular Statistics in R*, Oxford University Press, Oxford.

Rosenthal, R. W. (1981). Games of perfect information, predatory pricing and the chain-store paradox, *Journal of Economic Theory* **25**(1): 92–100.

Roth, A. E. & Erev, I. (1995). Learning in extensive-form games: Experimental data and simple dynamic models in the intermediate term, *Games and Economic Behavior* **8**(1): 164–212.

Samuelson, L. (2001). Analogies, adaptation, and anomalies, *Journal of Economic Theory* **97**(2): 320–366.

Selten, R. (1967). Die Strategiemethode zur Erforschung des eingeschränkt rationalen Verhaltens im Rahmen eines Oligopolexperiments, *in* H. Sauermann (ed.), *Beiträge zur experimentellen Wirtschaftsforschung*, Mohr, Tübingen, pp. 136–168.

Selten, R., Abbink, K., Buchta, J. & Sadrieh, A. (2003). How to play (3×3)-games. A strategy method experiment, *Games and Economic Behavior* **45**(1): 19–37.

Sgroi, D. & Zizzo, D. J. (2009). Learning to play 3×3 games: Neural networks as bounded-rational players, *Journal of Economic Behavior & Organization* **69**(1): 27–38.

Spiliopoulos, L. (2011). Neural networks as a unifying learning model for random normal form games, *Adaptive Behavior* **19**(6): 383–408.

Spiliopoulos, L. (2015). Transfer of conflict and cooperation from experienced games to new games: A connectionist model of learning, *Frontiers in Neuroscience* **9**(102): 1–18.

Stahl, D. O. (1996). Boundedly rational rule learning in a guessing game, *Games and Economic Behavior* **16**(2): 303–330.

Stahl, D. O. (1999). Evidence based rules and learning in symmetric normal-form games, *International Journal of Game Theory* **28**(1): 111–130.

Stahl, D. O. (2000). Rule learning in symmetric normal-form games: Theory and evidence, *Games and Economic Behavior* **32**(1): 105–138.

Stahl, D. O. (2001). Population rule learning in symmetric normal-form games: Theory and evidence, *Journal of Economic Behavior & Organization* **45**(1): 19–35.

Stahl, D. O. & Wilson, P. W. (1994). Experimental evidence on players' models of other players, *Journal of Economic Behavior & Organization* **25**(3): 309–327.

Steiner, J. & Stewart, C. (2008). Contagion through learning, *Theoretical Economics* **3**(4): 431–458.

Taylor, P. D. & Jonker, L. B. (1978). Evolutionary stable strategies and game dynamics, *Mathematical Biosciences* **40**(1-2): 145–156.

Van der Heijden, E. C., Nelissen, J. H., Potters, J. J. & Verbon, H. A. (1998). The poverty game and the pension game: The role of reciprocity, *Journal of Economic Psychology* **19**(1): 5–41.

Young, H. P. (1994). The evolution of conventions, *Econometrica* **61**(1): 57–84.