# Uncovering seeds

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**Abstract**

We provide the theoretical foundations for a new estimation algorithm that non-parametrically infers level-$k$ beliefs from laboratory choices in generalized guessing games with heterogeneous interactions. The algorithm takes the strategic dependencies of the game and subjects' choices as an input and returns a detailed histogram (a "pseudo-spectrogram" of seeds) that represents population beliefs about the behavior of level-0 players. As a by-product, the algorithm also returns the estimated population composition of reasoning levels.

The main contributions are as follows. First, we study the equilibrium properties of generalized guessing games and provide an ordinal (visual) characterization for uniqueness. Second, within the level-$k$ model, our key theoretical results establish conditions on the subjective beliefs or the game structure so that the population distributions of level-$k$ choices and the population distribution of beliefs are alike. These results are obtained without any distributional assumptions. We also present a central limit result that supports the use of parametric gaussian approaches often used in the literature. Third, on the basis of the theoretical results, we construct a new a non-parametric maximum likelihood estimation algorithm that fully identifies the belief pattern. Fourth, we apply the algorithm to experimental data. It is found that beliefs cluster around a few focal points and that a few seeds are able to explain a high percentage of observed behavior. Finally, our theoretical results can also be useful in the design of laboratory guessing games with good estimation properties.

**Keywords:** beliefs, estimation, level-$k$, network, mixture model.

# 1 Introduction

Using a level-$k$ approach,[1] this paper analyzes the microfoundation of a non-parametric method for estimating beliefs from laboratory choices in generalized guessing games.[2] Consider a large population of experimental subjects anonymously arranged into groups of three who face the following simultaneous move game. Each subject in a group is assigned a player role $i \in \{1, 2, 3\}$ and has to choose a real number $x_i \in [0, 100]$. The utility of player $i$ for a given profile of choices $\mathbf{x} = (x_1, x_2, x_3)$ within her group is $u_i(\mathbf{x}) = -(x_i - t_i)^2$, where

$$t_i = p \cdot \frac{x_j + x_k}{2}$$

is said to be the target value of player $i$, and $x_j$ and $x_k$ denote the choices of the other two players in the same group as $i$. We set $p = 2/3$ in the example, which implies that the unique Nash equilibrium is $\mathbf{x}^* = \mathbf{0}$. Our analysis allows for more general games in the sense that targets may be asymmetric (*i.e.*, defined by a network) and may include exogenously given anchors (*i.e.*, constants like 50 or 80). We call this class of games *anchored guessing games*. Proposition 1 shows that an anchored guessing game has a unique Nash equilibrium if and only if the dominant eigenvalue of the matrix that represents the strategic dependencies is less than 1.[3] In our motivating example without anchors this condition reduces to $p < 1$, which is well-known in the literature.[4]

Next, we derive the optimal choices in the level-$k$ framework when a player has subjective beliefs about how all players in the group would behave if they were level-0 players and when these beliefs might not match the actual behavior of level-0 players (also see, Ho, Camerer and Weigelt 1998 and Burchardi and Penczynski 2014). In our particular example:

- A level-0 subject chooses a number from $[0, 100]$ (not necessarily uniformly at random).

[1]See, Nagel (1995) and Stahl and Wilson (1994, 1995) for the seminal contributions and Crawford, Costa-Gomes, and Iriberri (2013) for a more recent literature overview. Other well-known models of non-equilibrium behavior include the quantal response equilibrium suggested in McKelvey and Palfrey (1995) and the cognitive hierarchy model of Camerer, Ho and Chong (2004).

[2]A working implementation is available at `https://observablehq.com/@coballester`, in the Javascript notebook "Uncovering seeds". The reader may run the algorithm using her own datasets on any computer or mobile device.

[3]This type of result is standard in the literature on network economics. See, for instance, Ballester, Calvó-Armengol and Zenou (2006), Bramoullé and Kranton (2007), Acemoğlu, Carvalho, Ozdaglar and Tahbaz-Salehi (2012), Elliot and Golub (2020), and Galeotti, Golub and Goyal (2020).

[4]An intermediate value of $p = 2/3$ can facilitate good estimation results from a design perspective. For details, we refer the reader to the concluding discussion.

- A level-1 subject maximizes utility given her beliefs about level-0 behavior. Consider the level-1 subject Alice ($a$) who plays the game in role 2. Her optimal choice is

$$x_2^{1(a)} = \frac{2}{3}\left(0.5\, e_1^{(a)} + 0.5\, e_3^{(a)}\right),$$

where $e_j^{(a)} \in [0, 100]$ is Alice's belief about the choice of a level-0 player in role $j$ and $x_i^{k(s)}$ indicates the choice of subject $s$ in role $i$ who is level-$k$. Even though we refer throughout to $e_j^{(a)}$ as a "belief", it is in fact that the mathematical expectation of any probability distribution on the strategy space of player $j$. For example, Alice could think that the level-0 player $j$ behaves uniformly at random on [0,100]. Then, $e_j^{(a)} = 50$.

- A level-2 subject maximizes utility viewing all other players as level-1. Suppose that Bob ($b$) who also plays the game in role 2, but in a different group than Alice, is a level-2 subject. The *reduced form* for Bob's optimal choice is then

$$\begin{aligned} x_2^{2(b)} &= \frac{2}{3}\left(0.5 \cdot \mathbb{E}^{(b)}[x_1^1] + 0.5 \cdot \mathbb{E}^{(b)}[x_3^1]\right) \\ &= \frac{2}{3}\left(0.5 \cdot \frac{2}{3}\left(0.5\, e_2^{(b)} + 0.5\, e_3^{(b)}\right) + 0.5 \cdot \frac{2}{3}\left(0.5\, e_1^{(b)} + 0.5\, e_2^{(b)}\right)\right) \\ &= \frac{4}{9}\left(0.25\, e_1^{(b)} + 0.5\, e_2^{(b)} + 0.25\, e_3^{(b)}\right), \end{aligned}$$

where $\mathbb{E}^{(b)}[x_j^k]$ is Bob's belief about the choice of the level-$k$ player in role $j$. Thus, the belief vector $\mathbf{e}^{(b)} = (e_1^{(b)}, e_2^{(b)}, e_3^{(b)})$ is interpreted as a starting seed over which Bob iterates twice to obtain his optimal choice.[5] Also note that beliefs are allowed to be heterogeneous both between-subjects (*i.e*, $e_1^{(a)}$ and $e_1^{(b)}$ may differ) and within-subjects (*i.e.*, $e_1^{(b)}$, $e_2^{(b)}$ and $e_3^{(b)}$ may differ). Within-subjects heterogeneity is important in our setting because the strategic dependencies and anchors that define the targets are possibly asymmetric.[6]

- And so forth for all level-$k$ players. The optimal choice $x_i^{k(s)}$ of a level-$k$ subject $s$ in this example falls into the interval $[0, (2/3)^k 100]$ because all other choices are serially dominated.

[5] Seeds may be originated from at least three different sources. First, a seed might be natural or intrinsic to the subject or group (like "popular" numbers such as 50 or 100). Second, it might be part of the description of the game. In the anchored guessing games introduced in this paper, constants may naturally arise as focal points. Third, focal points also emerge through examples in experimental instructions.

[6] Bob achieves deep strategic reasoning (level 2) about others, but his eventual beliefs about a level-0 subject in role 1 only depend on the game position of that player and not on Bob's reasoning path leading to player 1. This simplifying assumption on the highly complex belief system allows us to have a tractable model with belief heterogeneity within-subjects.

The main purpose of our paper is the estimation of the unknown population beliefs about level-0 players from laboratory choices in anchored guessing games. As an illustration, we show in Figure 1 our estimation outcome for the pooled data (over six treatments of $p$) of Brañas-Garza, García-Muñoz and Hernán González (2012).[7]

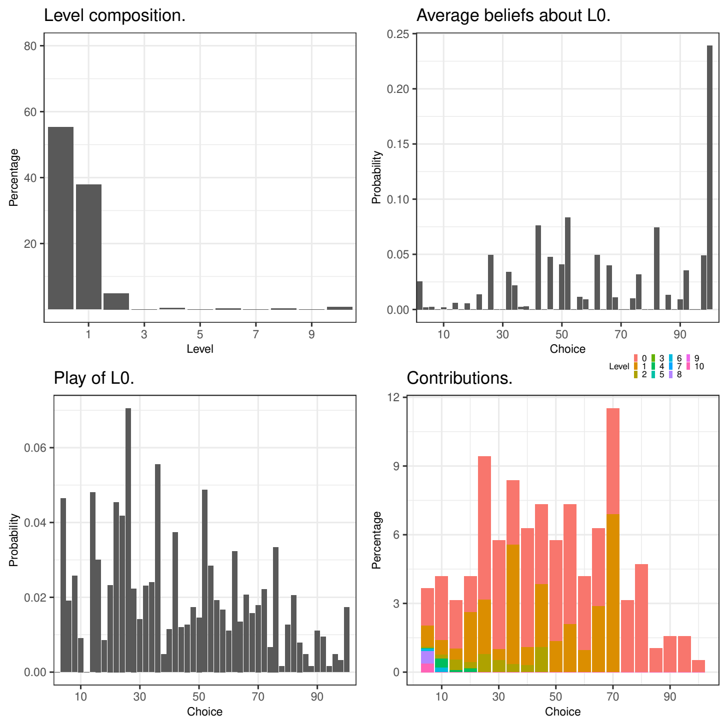



**Figure 1:** Estimation outcome for the pooled data of Brañas-Garza, García-Muñoz and Hernán González (2012). Top-left: estimated distribution of reasoning levels. Top-right: estimated distribution of average beliefs about level-0 choices. Bottom-left: estimated play of level-0 players. Bottom-right: stacked histogram of actual choices ($p = 2/3$ only) by estimated reasoning level.

The top-right panel highlights that the estimated free-form distribution of beliefs about level-0 choices consists of various peaks. The main peak is at 100. Further peaks can be identified at 50, 40, and 80. The estimated belief distribution turns out to be clustered around these peaks. Note that a peak represents a focal point, that is, a starting seed for a subject's reasoning process. For example, 24% of the choices associated with a strictly positive reasoning level have a seed at 100. The level-2 choice for this seed when $p = 2/3$ is $(2/3)^2 \cdot 100 = 44.44$. Associated with this belief distribution is the estimated distribution of reasoning levels in the top-left panel. We can

[7] Brañas-Garza, García-Muñoz and Hernán González (2012) analyze how personal characteristics such as attention, proxied by performance in the cognitive reflection test, and visual reasoning, measured by Raven's progressive matrices, affect behavior in beauty contest games. In total, there are 1146 choices from 191 subjects who play all six treatments.

see that 55% of the guesses are considered level-0 and 38% level-1 play, but a few choices have a rather high reasoning level as well. Since the four above-mentioned seeds of 100, 50, 40, and 80 alone account for 47% of all beliefs of subjects with a non-zero reasoning level and since the estimated fraction of level-0 subjects is 55%, these seeds explain 47% · 45% = 21% of all choices in the data set. The bottom-left panel shows the estimated level-0 choices. These are the choices that are not generated from the estimated beliefs. Finally, the bottom-right panel analyzes the degree to which each reasoning level explains the data. It can be observed that the algorithm does not assign each choice to its maximum compatible reasoning level. For instance, in the first bar of this histogram, which contains all choices in the interval [0,5), the two lowest reasoning levels explain about 70% of these choices, while levels of 8 and beyond explain about 30%. With this brief illustration, the reader should have a clear idea about the objectives and the output of the estimation procedure. In Section 5, we apply our approach to the data of the newspaper experiments of Bosch-Domènech, Montalvo, Nagel and Satorra (2002) and to one of the anchored guessing games of Ballester, Rodriguez-Moral and Vorsatz (2021).

We next detail our estimation technique. For that assume that choices from a large player-2 population have been collected in the introductory example. The corresponding beliefs to be estimated are summarized by the random vector $\tilde{\mathbf{e}}^{(\mathbf{2})} = (\tilde{e}_1^{(2)}, \tilde{e}_2^{(2)}, \tilde{e}_3^{(2)})$. This vector produces, through the reduced-form equations, for all non-zero reasoning levels the random variables of level-$k$ optimal choices $\tilde{x}_2^k = x_2^k(\tilde{e}_1^{(2)}, \tilde{e}_2^{(2)}, \tilde{e}_3^{(2)})$. For simplicity, we remove the (player role) index 2 and write $\tilde{\mathbf{e}}$ and $\tilde{x}^k$ instead of $\tilde{\mathbf{e}}^{(\mathbf{2})}$ and $\tilde{x}_2^k$. The central building block is a *mixture model* that consists of

1. **Level composition**: $p_0, p_1, \ldots, p_K$ are the unknown fractions (to be estimated) of each reasoning level in the player-2 population. The maximum reasoning level allowed for in the estimation is $K$.

2. **Density functions**: $f^0, f^1, \ldots, f^K$ are the unknown density functions of $\tilde{x}^0, \tilde{x}^1, \ldots, \tilde{x}^K$. The only condition imposed is that $f^k$ has domain $[0, (2/3)^k \cdot 100]$. For each reasoning level $k \geq 0$, $f^k$ is approximated by $B$ rectangular buckets with areas $a_1^k, a_2^k, \ldots, a_B^k$ that add up to 1 (to be estimated).

The idea underlying the mixture model is that the density function of the observed data is a convex combination of the densities $f^0, f^1, \ldots, f^K$ with weights $p_0, p_1, \ldots, p_K$. The mixture model can then be solved by using maximum likelihood estimation techniques. The key technical challenge that arises is the introduction of the game-theoretical restrictions into the mixture model. One possible approach is to consider the full set of reduced-form equations for all non-zero levels. Since the mixture model then becomes hard to implement and computationally intractable for general anchored guessing games, we instead impose well-founded game-theoretical conditions that approximate the reduced-form equations. This leads to an easy implementation and a quick and consistent estimation. One further benefit is that a severe identification problem of beliefs is avoided. A priori, if Bob chooses 32, there are many possible values of $e_1^{(b)}$, $e_2^{(b)}$, and $e_3^{(b)}$ that could explain this choice. One possibility is $e_1^{(b)} = e_2^{(b)} = e_3^{(b)} = 72$, but another is $e_1^{(b)} = 88$, $e_2^{(b)} = 100$, and $e_3^{(b)} = 0$.

Our theoretical results will allow us to assume in the estimation that for all buckets $b = 1, 2, \ldots, B$, we have that $a_b^1 = a_b^2 = \cdots = a_b^K \equiv a_b$. That is, bucket areas are equal across all non-zero levels. This condition is equivalent to what we call a *common shape* of the random variables $\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^K$. As a consequence, the set of bucket areas to be estimated is reduced to $a_1^0, a_2^0, \ldots, a_B^0$ and $a_1, a_2, \ldots, a_B$. The resulting implication in the introductory example is that for $k \geq 1$, the reduced-form equations $\tilde{x}^k = x^k(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ are approximated by

$$\tilde{x}^k \simeq \left(\frac{2}{3}\right)^k \frac{\tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3}{3} = \left(\frac{2}{3}\right)^k \tilde{\mu},$$

where $\tilde{\mu}$ is the random variable of average beliefs of player 2. Observe that the before-mentioned identification problem is now circumvented: the estimated bucket areas $a_1, a_2, \ldots, a_B$ in fact define the density function of the average beliefs $\tilde{\mu}$, which is a one-dimensional proxy of $\tilde{\mathbf{e}}$. For example, a level-2 choice of 32 is now unambiguously explained by the average belief $32 \cdot (2/3)^{-2} = 72$.

We are now ready to outline the theoretical foundations that justify this approximation of $\tilde{x}^k$, $k \geq 1$, through $\tilde{\mu}$. The key assumption is that for all reasoning levels $k \geq 1$, the random vector of beliefs $\tilde{\mathbf{e}}$ is independent of $k$, *i.e.*, Alice and Bob may have different beliefs about level-0 behavior for reasons that are not related with their different reasoning levels. The majority of level-$k$ models implicitly operate under this assumption. Theorem 1 provides conditions under which the approximation is exact. It establishes that if beliefs are homogeneous within subjects (*i.e.*, for each subject $s$, $e_1^{(s)} = e_2^{(s)} = \ldots = e_n^{(s)}$), then the optimal choice distributions $\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^K$ have exactly the same shape as $\tilde{\mu}$, that is, their distribution functions are the same after a suitable affine transformation of the random variables. In our example, this transformation rescales each $\tilde{x}^k$ from the domain $[0, (2/3)^k \cdot 100]$ to $[0, 100]$. Closest to Theorem 1 is the result of Burchardi and Penczynski (2014), who estimate beliefs in the beauty contest game under the assumption that beliefs are homogenous within subjects and follow a normal distribution. Our result is distribution-free and applies to the more general class of anchored guessing games. Second, Theorem 2 obtains a weaker result, similarity of shapes instead of shape equality, by dropping the assumption of homogenous beliefs within subjects. It shows that as the reasoning level $k$ increases, the shape of $\tilde{x}^k$ approximates the shape of $\tilde{\mu}$. Theorem 2 also establishes that the shape of $\tilde{x}^k$ approaches towards the shape of $\tilde{\mu}$ faster in games with a low *eigenvalue ratio*.[8] Consequently, if one cannot rely on belief homogeneity within subjects, the estimation is more consistent for games with a lower eigenvalue ratio by virtue of Theorem 2 because shapes become more similar at earlier reasoning levels. In the introductory example, when allowing for an arbitrary number of players, this ratio is $(n-1)^{-1}$, which is independent of $p$. Hence, our estimation procedure, which assumes equal shapes across non-zero levels, should yield more consistent results for $n = 6$ than for $n = 3$.

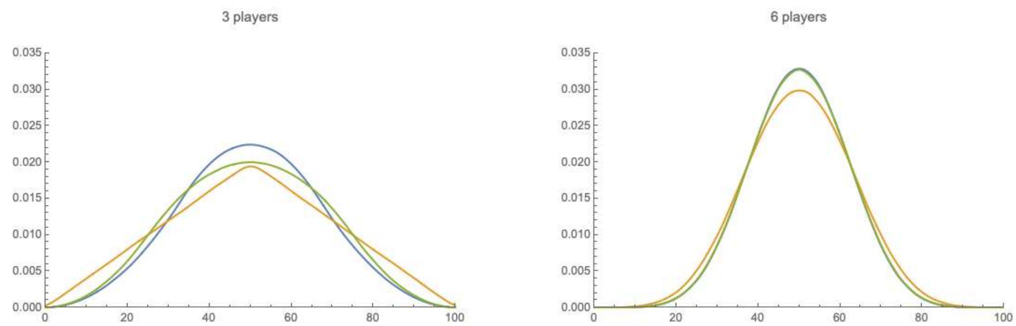



**Figure 2:** Theoretical choice distributions in the introductory example ($\tilde{x}^1$ in orange, $\tilde{x}^2$ in green, and $\tilde{\mu}$ in blue) re-normalized to $[0, 100]$. The beliefs $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n$ about level-0 choices are independently and uniformly distributed on $[0, 100]$. In the panel to the right, it is difficult to visually distinguish $\tilde{x}^2$ and $\tilde{\mu}$ from each other.

Finally, we derive a new central limit result that provides a microfoundation for gaussian mixture models commonly used in the level-$k$ literature. Proposition 2 complements Theorem 2 by showing that the choice distributions for $k \geq 1$ converge towards normal distributions as $n$ grows, as long as the components of the random vector $\tilde{\mathbf{e}}$ are independently (but not necessarily identically) distributed and as long as no player receives too much weight in the targets of other players. Our example satisfies the latter condition because players assign the same weight to all opponents in their targets. The effects of combining Theorem 2 and Proposition 2 can be visually

[8] This is the absolute value of the ratio between the second and the first eigenvalue of the matrix that represents the strategic dependencies of the game. It is a rough measure of the balancedness of the link distribution in the dependency network. For instance, Golub and Jackson (2012) use the second eigenvalue to measure homophily in a network, which affects the speed of convergence to consensus.

appreciated in Figure 2. It can be seen that as $n$ increases, the choice distributions have more similar shapes due to Theorem 2 and are normally distributed due to Proposition 2.

The level-$k$ model has received considerable attention. The main insight from the early experimental studies of guessing games in Nagel (1995), Ho, Camerer and Weigelt (1998), Bosch-Domènech, Montalvo, Nagel and Satorra (2002), and Costa-Gomes and Crawford (2006) as well as from the normal-form game experiments in Stahl and Wilson (1994,1995) and Costa-Gomes, Crawford and Broseta (2001) is that a substantial fraction of the non-equilibrium play is consistent with $k$-rationalizability for low levels of $k$ ($k \leq 3$) if level-0 subjects are assumed to choose uniformly at random over the entire action space.[9] However, according to Crawford, Costa-Gomes and Iriberri (2013) "More work is needed to evaluate the credibility of the models' explanations and to assess their domains of applicability, their portability, and the stability of their parameter estimates across types of games."[10] Not only is the class of anchored guessing games a generalized framework that is ideally suited to compare parameter estimates across different game specifications; more importantly, we develop the microfoundation for a new non-parametric estimation technique that extracts beliefs about the level-0 players directly from the choice data. Our approach therefore complements others in which information about beliefs could be obtained with the help of alternative experimental designs. For instance, Bhatt and Camerer (2005) extract reasoning levels via fMRI; Costa-Gomes and Weizäcker (2008) ask subjects to state their beliefs; Wang, Spezio and Camerer (2010) use eyetracking devices to monitor subjects' decisions; Burchardi and Penczynski (2014) use group chat protocols from team decisions and Fragiadakis, Kovaliukaite and Rojo-Arjona (2019) use an incentive compatible mechanism to elicit beliefs.

We proceed as follows. In Section 2, we introduce the class of all anchored guessing games and study their equilibrium properties. Section 3 introduces the formal level-$k$ framework and derives the theoretical results regarding shape equality/similarity and normality. Section 4 describes the algorithm with which we estimate population beliefs about the behavior of level-0 players. This maximum likelihood estimation technique is applied in Section 5 to data from various experiments. Finally, in the concluding discussion, we illustrate how our results can be helpful for designing laboratory guessing games. All proofs are relegated to the Appendix.

[9] Recently, Agranov, Potamites, Schotter and Tegiman (2012), Georganas Healy and Weber (2015), and Aloui and Penta (2016) introduce the notion of a strategic bound. A player with bound $\bar{k}$ strategically chooses the reasoning level $k \leq \bar{k}$ at which she operates. We abstain from these game-theoretical considerations.

[10] Among others, the level-$k$ model has been applied to strategic information transmission by Cai and Wang (2006), auctions and hide and seek games by Crawford and Iriberri (2007a, 2007b), the 11-20 game by Arad and Rubinstein (2012), and the centipede game by García-Pola, Iriberri and Kovářík (2020).

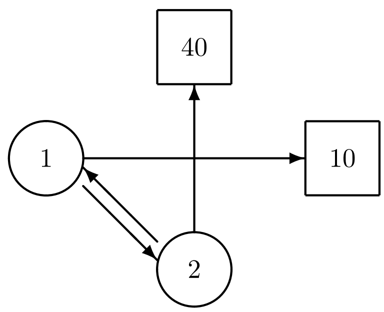
## 2 Anchored guessing games

**Example 1.**
Consider the following simultaneous move game. There are two players $i \in \{1, 2\}$. Each player $i$ has to choose a number $x_i$ from the interval [0,100].

The first player's target guess $t_1$ is the average of 10 and the choice $x_2$ of player 2.

The second player's target guess $t_2$ is the average of 40 and the choice $x_1$ of player 1.

Players have the incentive to minimize the distance between their choices and target guesses. This game has a very simple graphical representation. In the diagram, nodes are players and cells are numbers that we call anchors. Arrows departing from a player point to the information used to determine the target guess. We omit the weights of the arrows, which are 1/2 in all instances.




The unique equilibrium is $\mathbf{x}^* = (20, 30)$, which is the unique solution of the linear system

$$x_1 = t_1 = \frac{x_2 + 10}{2}$$
$$x_2 = t_2 = \frac{x_1 + 40}{2}.$$

This completes the example $\square$

We are ready to introduce the formal model. Let $N = \{1, \ldots, n\}$ be a finite set of $n$ players. Players have to simultaneously and independently choose a number from the interval $X \equiv [\underline{x}, \overline{x}] \subseteq \mathbb{R}$, where $\overline{x} > \underline{x}$.[11] Let $x_i \in X$ be a particular strategy for player $i$. The column vector $\mathbf{x} = (x_1, \ldots, x_n)$ is said to be a strategy vector. The $n \times n$ non-negative matrix $\mathbf{W}$ with generic element $w_{ij}$ describes the network of strategic dependencies between players. For example, if $n = 3$ and the objective of player 1 is to guess the average of $x_2$ and $x_3$, then $w_{12} = w_{13} = 0.5$. The element

[11] The strategy space is assumed to be common exclusively for notational simplicity. The model and results can be adapted in a straightforward way to allow for heterogeneous strategy spaces.

$w_{ii}$ may or may not be equal to zero. If $\mathbf{W}$ has a zero main diagonal, as in Example 1, then we say that the game is represented in its best-reply form. The targets do not only depend on the strategies of other players, but also on constants. Formally, there is a non-empty set of anchors $M = \{1, \ldots, m\}$ with an associated $m \times 1$ vector of real anchor values $\mathbf{v} = (v_1, \ldots, v_m)^\top \in \mathbb{R}^m$. Anchor values enter into the players' objectives through the $n \times m$ non-negative matrix $\mathbf{A}$, which describes how much weight each player assigns to each of the $m$ anchor values. By convention, if there are no anchors, we set $m = 1$ and take $\mathbf{A}$ to be an $n \times 1$ vector of zeroes $\mathbf{0}$ and $\mathbf{v}$ the $1 \times 1$ vector $(0)$. Given a strategy vector $\mathbf{x}$, the target guess of player $i$ is equal to

$$t_i(\mathbf{x}) = \sum_{l=1}^{m} a_{il}\, v_l + \sum_{j=1}^{n} w_{ij}\, x_j.$$

Expressed in matrix form, targets are thus

$$\mathbf{t}(\mathbf{x}) = \mathbf{A}\mathbf{v} + \mathbf{W}\mathbf{x}.$$

The utility function of player $i \in N$ is $u_i(\mathbf{x}) = -\left(x_i - t_i(\mathbf{x})\right)^2$. Let $\mathbf{1}$ be the vector of ones. Targets are said to always be within the strategic bounds if for all $i \in N$ and all $\mathbf{x} \in X^n$, $t_i(\mathbf{x}) \in [\underline{x}, \overline{x}]$. We concentrate throughout on games that satisfy this interiority requirement, which can be succinctly written as follows.

**Assumption 1** (Interiority)**.** $\underline{x}\,(\mathbf{I} - \mathbf{W})\,\mathbf{1} \leq \mathbf{A}\mathbf{v} \leq \overline{x}\,(\mathbf{I} - \mathbf{W})\,\mathbf{1}$.

While Assumption 1 is restrictive, it still leaves a lot of room for designing meaningful laboratory guessing games.[12] The class of all simultaneous move games $(N, X^n, (u_i)_{i \in N})$, parametrized by $\underline{x}$, $\overline{x}$, $\mathbf{W}$, $\mathbf{A}$, and $\mathbf{v}$, that satisfy interiority are called *anchored guessing games.*

Order the (possibly complex) eigenvalues of $\mathbf{W}$ by their absolute value, *i.e.*, $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$. Since $\mathbf{W}$ is non-negative, it is well-known that $\lambda_1 \geq 0$. Also, by Assumption 1, $\lambda_1 \leq 1$ and each row sum of $\mathbf{W}$ is less than or equal to one (see, Lemma 1 and Lemma 2 in the Appendix). Proposition 1 shows that there always is a Nash equilibrium in pure strategies. The equilibrium is unique if and only if $\lambda_1 < 1$. Proposition 1 provides for all anchored guessing games an ordinal characterization of this condition. For this purpose, we say that player $i \in N$ *has a path to an anchor* if there is a sequence of players $(i = j_0, j_1, j_2, \ldots, j_k = j)$ such that $\sum_{l \in N} w_{jl} < 1$ and for all

[12] For instance, given a desired dependency matrix $\mathbf{W}$ and a desired strategy space $X$, one can always find a set of anchors $\mathbf{A}$ with values $\mathbf{v}$ so that the interiority assumption holds. Or, given a desired dependency matrix $\mathbf{W}$ and a set of anchors $\mathbf{A}$ with values $\mathbf{v}$, one can always find a desired strategy space $X$ so that the interiority assumption holds.

$s = 1, 2, \ldots, k$, $w_{j_{s-1}j_s} > 0$. That is, player $i$ has a path leading to a player $j \in N$ (not necessarily distinct from $i$) whose row sum $\sum_l w_{jl}$ is strictly smaller than 1. This condition is essentially graph-theoretical (*i.e.*, ordinal) in the sense that only the existence of links in the graph matter, but not the particular weights.

**Proposition 1.**

(a) *Every anchored guessing game has Nash equilibrium (in pure strategies).*

(b) *The equilibrium is unique if and only if every player $i \in N$ has a path to an anchor.*

(c) *If the equilibrium is unique, then $\mathbf{x}^* = (\mathbf{I} - \mathbf{W})^{-1}\mathbf{A}\mathbf{v}$.*

(d) *If the equilibrium is unique, then it is globally stable, dominance solvable, and it can be obtained by applying to any initial vector of guesses $\mathbf{x}^0 = (x_1^0, \ldots, x_n^0)^\top$ the iterative vector function*

$$\mathbf{x}^{\tau+1} = \mathbf{t}(\mathbf{x}^\tau), \ \ \textit{where } \tau = 0, 1, \ldots \, .$$

Proposition 1 follows from standard results. Since the game is smooth supermodular, existence follows from Milgrom and Roberts (1990). The ordinal characterization is useful because uniqueness can be checked by merely inspecting the network of strategic dependencies. The asymptotic convergence factor of the iterative process $\mathbf{x}^{\tau+1} = \mathbf{t}(\mathbf{x}^\tau)$ towards $\mathbf{x}^*$ is $\lambda_1$ and, therefore, a smaller dominant eigenvalue $\lambda_1$ is associated with a higher speed of convergence of this process to the unique equilibrium. Finally, all results in Proposition 1 except for speed of convergence are robust to changes in the targets $\mathbf{t}$ whenever the underlying best-reply mapping remains unaffected. We conclude with another example.

**Example 2 (beauty contest game).**
Let there be $n$ players. The strategy space is $X = [0, \overline{x}]^n$, where $\overline{x} > 0$. Target guesses are equal to a fraction $p \geq 0$ of the average guess, that is, $t_i = (p/n) \sum_{j \in N} x_j$. Thus, $\mathbf{v} = (0)$, $\mathbf{A} = \mathbf{0}$, and $\mathbf{W} = (p/n)\,\mathbf{U}$, where $\mathbf{U} = \mathbf{1}\mathbf{1}^\top$ is an $n \times n$ matrix of ones. It is easy to check that the interiority assumption is satisfied if and only if $p \leq 1$. Since the dominant eigenvalue of $\mathbf{W}$ is $\lambda_1 = p$, the equilibrium is unique if and only if $p < 1$. Observe that the asymptotic speed of convergence to equilibrium is decreasing in $p$ and that it does not depend on the number of players. Section 3 also analyzes this game in its best-reply form. $\square$

# 3 Level-$k$ framework

## 3.1 Decision-making process

In the level-$k$ model, each player is endowed with a reasoning level $k \in \{0, 1, 2, \ldots\}$. A player with reasoning level $k = 0$ chooses any number from $X$. We do not make any assumption on the behavior of level-0 players. A player with reasoning level $k \in \{1, 2, \ldots\}$ is defined recursively as one who best replies to the subjective belief that all other players employ reasoning level $k - 1$. This creates a belief hierarchy that collapses at level 0. We also note that the actual level-0 behavior may differ from the beliefs that players hold about level-0.

In order to introduce our results, suppose that a large number of experimental subjects are randomly assigned into groups of $n$ to play a given anchored guessing game. From an econometric point of view, each experimental subject $s$ is endowed with an observable role $i \in N$, an unobservable reasoning level $k \in \{0, 1, 2, \ldots\}$, and unobservable beliefs $\mathbf{e}^{(s)} = (e_1^{(s)}, \ldots, e_n^{(s)}) \in [\underline{x}, \overline{x}]^n$. Here, $e_j^{(s)} \in [\underline{x}, \overline{x}]$ denotes the belief of subject $s$ about the play of a level-0 subject in role $j$.[13] The random vector $\tilde{\mathbf{e}} = (\tilde{e}_1, \ldots, \tilde{e}_n)$ gathers the beliefs of all subjects in the population. Denote the restriction of $\tilde{\mathbf{e}}$ to any given role $i \in N$ by $\tilde{\mathbf{e}}^{(i)}$. Also, define the *average beliefs* of the subjects in role $i$ as $\tilde{\mu}^{(i)} = \sum_{j=1}^{n} \gamma_j \, \tilde{e}_j^{(i)}$, where $\boldsymbol{\gamma}$ is a vector of non-negative weights that add up to 1. It is assumed throughout that for all non-zero reasoning levels, $\tilde{\mathbf{e}}^{(i)}$ is independent $k$. This assumption, which is implicit in much of the level-$k$ literature, turns out to be crucial for our estimation strategy in Section 4.

**Assumption 2** (Level-independent beliefs)**.** *For all roles, beliefs are independent of non-zero reasoning levels. That is, for all $i \in N$, $\mathbf{e} \in [\underline{x}, \overline{x}]^n$ and all $k, k' \geq 1$,*

$$\Pr(\tilde{\mathbf{e}}^{(i)} \leq \mathbf{e} \,|\, k) = \Pr(\tilde{\mathbf{e}}^{(i)} \leq \mathbf{e} \,|\, k')$$

We next derive the optimal choices under the level-$k$ model. Subject $s$ with reasoning level $k \geq 1$ in role $i$ solves the following vector iterative process:

$$\begin{aligned} &\mathbf{x}^{1(s)} = \mathbf{A}\mathbf{v} + \mathbf{W}\mathbf{e}^{(s)} \text{ and} \\ &\mathbf{x}^{\tau(s)} = \mathbf{t}(\mathbf{x}^{(\tau-1)(s)}) = \mathbf{A}\mathbf{v} + \mathbf{W}\mathbf{x}^{(\tau-1)(s)} \text{ for all } \tau \in \{2, 3, \ldots, k\}. \end{aligned} \tag{1}$$

[13] More concretely, the subjective beliefs over level-0 play are a probability distribution over $X$. Due to the linearity of the targets, an expected utility maximizing subject only considers the mathematical expectation of this probability distribution. Hence, formally $e_j^{(s)}$ is the mathematical expectation of the subjective beliefs.

The optimal choice $x_i^{k(s)}$ of subject $s$ corresponds to the $i$-th entry of the $k$-th iteration vector $\mathbf{x}^{k(s)}$. Let $\tilde{x}_i^k$ be the random variable induced by the optimal choices of all subjects with reasoning level $k$ in role $i$. Since $\tilde{x}_i^k$ depends (linearly) on the beliefs $\tilde{\mathbf{e}}^{(i)}$, we write $\tilde{x}_i^k = x_i^k(\tilde{\mathbf{e}}^{(i)})$ to make this dependence explicit. We refer to $\tilde{x}_i^k = x_i^k(\tilde{\mathbf{e}}^{(i)})$, $k \geq 1$, as *reduced-form equations*.

By Proposition 1, if there is a unique equilibrium $\mathbf{x}^*$, $\tilde{x}_i^k$ degenerates towards $x_i^*$ as $k$ grows. The speed of convergence depends inversely on the dominant eigenvalue $\lambda_1$ of $\mathbf{W}$. We thus highlight here the importance of $\lambda_1$ as a crucial parameter in anchored guessing games under the level-$k$ model because it helps delimiting scenarios where the level-$k$ theory can explain out-of-equilibrium behavior. For instance, the level-$k$ model is more likely to explain out-of-equilibrium behavior of players with moderate levels of reasoning in games with a high $\lambda_1$ (because these games converge slower). Also, note that $\tilde{x}_i^k$ lies necessarily within the set of $k$-iterated undominated strategies $U_i^k \equiv [\underline{x}_i^k, \overline{x}_i^k]$. Due to Assumption 1 the bounds $\underline{x}_i^k$ and $\overline{x}_i^k$ can be computed from the iterative formula in (1) by substituting the initial vector $\mathbf{e}^{(s)}$ by $\underline{x}\mathbf{1}$ and $\overline{x}\mathbf{1}$, respectively. The bounds of $U_i^k$ then correspond to the $i$-th entries of the $k$-th iteration vectors.

## 3.2 Main results

Our main purpose is to estimate, given the choices of the subjects in a given role $i \in N$, the features of the beliefs $\tilde{\mathbf{e}}^{(i)}$. From a theoretical point of view, we are interested in analyzing conditions on the game characteristics and on the beliefs so that the "shapes" of the distributions $\tilde{x}_i^k$, $k \geq 1$, are alike. Formally, consider two real random variables $\tilde{y}$ and $\tilde{z}$, and two domain intervals $[\underline{y}, \overline{y}]$ and $[\underline{z}, \overline{z}]$ that contain the supports of $\tilde{y}$ and $\tilde{z}$, respectively. We say that $\tilde{y}$ and $\tilde{z}$ are *equally shaped*, written $\tilde{y} \overset{\mathcal{S}}{=} \tilde{z}$, if their normalizations within these domain intervals are equally distributed:

$$\frac{\tilde{y} - \underline{y}}{\overline{y} - \underline{y}} \overset{\mathcal{D}}{=} \frac{\tilde{z} - \underline{z}}{\overline{z} - \underline{z}}.$$

The geometric interpretation of the condition is that the probability distribution functions of $\tilde{y}$ and $\tilde{z}$ have exactly the same visual form within their respective domains (they are affine transformations of each other).[14] Let $\{\tilde{y}^k\}_{k=1}^{\infty}$ be a sequence of random variables defined on the domains $\{[\underline{y}^k, \overline{y}^k]\}_{k=1}^{\infty}$ and let $\tilde{y}$ be a random variable with given domain $[\underline{y}, \overline{y}]$. We say that the random variables $\{\tilde{y}^k\}_{k=1}^{\infty}$

[14] In statistics, two random variables $\tilde{y}$ and $\tilde{z}$ are said to be of the same type whenever

$$\frac{\tilde{y} - \mathbb{E}[\tilde{y}]}{\sigma_y} \overset{\mathcal{D}}{=} \frac{\tilde{z} - \mathbb{E}[\tilde{z}]}{\sigma_z}.$$

Our definition is stronger than this notion because it is parametrized by the given domain intervals.

*converge in shape to* $\tilde{y}$ if

$$\frac{\tilde{y}^k - \underline{y}^k}{\overline{y}^k - \underline{y}^k} \xrightarrow{\mathcal{D}} \frac{\tilde{y} - \underline{y}}{\overline{y} - \underline{y}} \quad \text{as } k \to \infty.$$

We write $\tilde{y}^k \xrightarrow{\mathcal{S}} \tilde{y}$ to denote that the shape of $\tilde{y}^k$ approaches towards the shape of $\tilde{y}$ as $k$ grows.[15]

Theorems 1 and 2 establish that under suitable conditions, the population choices $\tilde{x}_i^k$ have for $k \geq 1$ the same or similar shapes as the average beliefs $\tilde{\mu}^{(i)}$. Hence, if one knew the choice distribution of the level-1 subjects in role $i$, then one would only need to rescale and shift this distribution in order obtain the choice distributions for all reasoning levels $k > 1$ for the same role $i$. In Theorem 1, equal shapes are the consequence of imposing restrictions on beliefs. Formally, the beliefs $\tilde{\mathbf{e}}^{(i)}$ are said to be *individually homogeneous* if for each subject $s$ in role $i \in N$, $e_1^{(s)} = e_2^{(s)} = \ldots = e_n^{(s)} \equiv e^{(s)}$. In other words, if beliefs are individually homogeneous, then the random belief vector $\tilde{\mathbf{e}}^{(i)}$ is equal to $\tilde{\mu}^{(i)}\mathbf{1}$ for any weight vector $\boldsymbol{\gamma}$. Note that beliefs remain nevertheless heterogeneous between subjects because two different subjects $s$ and $s'$ may think differently about how level-0 subjects behave. Observe that while $\tilde{x}_i^k$ has domain $U_i^k$, $\tilde{\mu}^{(i)}$ has domain $[\underline{x}, \overline{x}]$.

**Theorem 1.** *Suppose that all subjects in role $i \in N$ have individually homogenous beliefs. Then, the random variables $\{\tilde{x}_i^k\}_{k=1}^{\infty}$ are equal in shape. In particular, for all $k \in \{1, 2, \ldots\}$, $\tilde{x}_i^k \stackrel{\mathcal{S}}{=} \tilde{\mu}^{(i)}$.*

Next, we analyze the case when beliefs are not assumed to be individually homogenous. We assume for simplicity that $\mathbf{W}$ is diagonalizable, which holds generically. To proceed, the following graph-theoretical definitions are needed. An anchored guessing game is said to be connected if the dependency matrix $\mathbf{W}$ is irreducible, that is, if any two players can be connected by some path in the network represented by $\mathbf{W}$. The matrix $\mathbf{W}$ is primitive[16] if there is some integer $k > 0$ such that all entries of $\mathbf{W}^k$ are strictly positive. It is well-known that if $\mathbf{W}$ is primitive, then the corresponding anchored guessing game is connected.

Theorem 2 guarantees shape convergence whenever the underlying dependency matrix $\mathbf{W}$ is primitive without imposing any condition on beliefs. The shape of the choice distribution $\tilde{x}_i^k$

[15] Convergence in distribution ($\xrightarrow{\mathcal{D}}$) to a non-degenerate random variable is a special case of convergence in shape with suitably defined domains. For instance, $\mathcal{U}(0, 1 + 1/k) \xrightarrow{\mathcal{S}} \mathcal{U}(0, 1)$ when the domains are the corresponding supports of the uniform distributions (in fact, the shapes are equal). However, we stress the importance of imposing a non-degenerate limit random variable $\tilde{y}$. For instance, $\mathcal{U}(0, 1/k)$ converges in distribution to the degenerate random variable 0. However, $\mathcal{U}(0, 1/k)$ does not converge in shape to any degenerate random variable. In fact, for each $k$, the shape of $\mathcal{U}(0, 1/k)$ is equal to the shape of $\mathcal{U}(0, 1)$ with corresponding domains $[0, 1/k]$ and $[0, 1]$. In this sense, convergence in shape to $\tilde{y}$ means convergence in shape to the whole class of random variables that have the same shape as $\tilde{y}$.

[16] Primitive matrices are generic. Even if $\mathbf{W}$ has a zero diagonal, primitivity remains generic for $n \geq 3$.

converges to the shape of the average beliefs $\tilde{\mu}^{(i)} = \sum_{j=1}^{n} \gamma_j \, \tilde{e}_j^{(i)}$, where the weight $\gamma_j$ assigned to each role $j \in N$ is precisely the centrality of player $j$ in the network induced by $\mathbf{W}$. The centrality vector $\boldsymbol{\gamma}$ is computed as the left eigenvector (with $\sum_j \gamma_j = 1$) associated with the dominant eigenvalue $\lambda_1$ of $\mathbf{W}$. That is, $\boldsymbol{\gamma}^\top \mathbf{W} = \lambda_1 \boldsymbol{\gamma}^\top$. The centrality $\gamma_j$ summarizes the influence of role $j$ on the choices of all players.

**Theorem 2.** *Suppose that* $\mathbf{W}$ *is primitive. Then, for each role* $i \in N$*, the random variables* $\{\tilde{x}_i^k\}_{k=1}^{\infty}$ *converge in shape. In particular,* $\tilde{x}_i^k \xrightarrow{\mathcal{S}} \tilde{\mu}^{(i)}$ *as* $k \to \infty$*.*

The asymptotic convergence factor of this result is given by the eigenvalue ratio $|\lambda_2|/\lambda_1 < 1$. An extreme case of Theorem 2 occurs when $\mathbf{W}$ has rank one and can thus be written as $\mathbf{W} = \mathbf{r} \cdot \boldsymbol{\gamma}^\top$. The shape of the choice distributions of non-zero levels then turn out to be equal to $\tilde{\mu}$ because the eigenvalue ratio is 0. The beauty contest game of Example 2 has an associated rank-one matrix $\mathbf{W}$ with $\mathbf{r} = p\,\mathbf{1}$ and $\boldsymbol{\gamma} = (1/n)\,\mathbf{1}$.

We also highlight that Theorem 2 provides insights about the proper design of anchored guessing games for the laboratory because games with a low eigenvalue ratio $|\lambda_2|/\lambda_1$ are those for which shape convergence may be observed at the earliest non-zero reasoning levels.[17] Our simulations show that an eigenvalue ratio of 0.65 seems generally low enough to obtain similar shapes starting at levels 1 or 2.[18] Under these circumstances it is therefore more justified to impose in the econometric analysis developed in Section 4 that the choice distributions of non-zero levels are equally shaped. As an illustration, consider the beauty contest game with at least three players in its best-reply form (zero main diagonal) and targets $t_i(\mathbf{x}) = p/(n-p) \sum_{j \neq i} x_j$. Also, suppose that beliefs are not assumed individually homogenous. Then, $\mathbf{W}$ is primitive and Theorem 2 applies (but not Theorem 1). It follows from the anonymity of the interactions that $\boldsymbol{\gamma}$ of $\mathbf{W}$ is the uniform vector $n^{-1}\mathbf{1}$, which implies that $\tilde{x}_i^k$ converges in shape to $\tilde{\mu}^{(i)} = \sum_{j=1}^{n} \tilde{e}_j^{(i)}/n$. Moreover, the eigenvalue ratio $|\lambda_2|/\lambda_1 = (n-1)^{-1}$ is bounded above by 0.5 when there are at least three players, which facilitates similar shapes at early reasoning levels.

[17] In the discussion section, we address this design problem by comparing a variety of games with $n \leq 4$.

[18] Convergence in shape is not only determined by the eigenvalue ratio, which should be interpreted as an optimistic measure of convergence, it is also affected by the random vector of beliefs $\tilde{\mathbf{e}}^{(i)}$ as well as by other graph-theoretical features like the diameter of the network associated with the dependency matrix $\mathbf{W}$.

### 3.3 Large games

In this part, we study games with a growing number of players. Proposition 2 below identifies conditions under which each $\tilde{x}_i^k$ approximates a truncated normal distribution. The proposition thus rationalizes the application of parametric gaussian estimation techniques in certain types of anchored guessing games. For example, Ho, Camerer and Weigelt (1998) and Burchardi and Penczynski (2014) assume that a normal distribution for level-0 choices or beliefs is transmitted to all non-zero reasoning levels. In our setting, normality arises naturally without making this kind of assumption. Two types of convergences take place under our result. First, the random variables $\tilde{x}_i^k$ and $\tilde{\mu}$ have a similar shape by Theorem 2. Second, $\tilde{\mu}$ is distributed normally as $n$ grows by a central limit theorem.

We introduce the necessary notation and definitions. Given role $i \in N$, beliefs are said to be *role-independent* if the random variables $\tilde{e}_1^{(i)}, \tilde{e}_2^{(i)}, \ldots, \tilde{e}_n^{(i)}$ are independently (but not necessarily identically) distributed. Since role-independence implies that beliefs are not individually homogenous, this condition is incompatible with Theorem 1. We thus have to build the central limit result on top of Theorem 2. Remember that in Theorem 2, $\tilde{x}_i^k$ converges in shape towards the average beliefs $\tilde{\mu}^{(i)} = \sum_{j=1}^n \gamma_j \, \tilde{e}_j^{(i)}$. Let $\overline{\gamma} = \max_j\{\gamma_j\}$ be the maximum network centrality, $\mathrm{Var}(\tilde{e}_j^{(i)})$ be the variance of the beliefs subjects in role $i$ have about the play of level-0 subjects in role $j$, and $\bar{\gamma}_j = \gamma_j/\overline{\gamma} \in [0, 1]$ be the centrality of role $j$ relative to the maximum centrality. In the context of a growing network all parameters and variables associated with the game, except for the strategy space, like $\mathbf{W}$, $\tilde{\mathbf{e}}^{(i)}$, $\overline{\gamma}$, $\bar{\gamma}_j$, and $\tilde{x}_i^k$, may vary with the size of the network $n$. However, in order to keep the notation as simple as possible, we abstain from making theses dependencies explicit.

**Proposition 2.** *Consider any role $i \in N$ and suppose that $\mathbf{W}$ is primitive for all $n \geq 1$. If beliefs are role-independent and if $\sum_{j=1}^n \overline{\gamma}_j^2 \,\mathrm{Var}(\tilde{e}_j^{(i)}) \to \infty$ as $n \to \infty$, then $\tilde{x}_i^k$ is asymptotically normal. In particular,*

$$\tilde{x}_i^k \xrightarrow{\mathcal{D}} \mathcal{N}_{[\underline{x}_i^k, \overline{x}_i^k]} \left( \underline{x}_i^k + \frac{\overline{x}_i^k - \underline{x}_i^k}{\overline{x} - \underline{x}} \left( \mathbb{E}\left[\tilde{\mu}^{(i)}\right] - \underline{x} \right), \left( \frac{\overline{x}_i^k - \underline{x}_i^k}{\overline{x} - \underline{x}} \right)^2 \mathrm{Var}\left(\tilde{\mu}^{(i)}\right) \right) \quad \textit{as } k, n \to \infty,$$

*where $\mathcal{N}_{[\underline{x}_i^k, \overline{x}_i^k]}$ is the truncation of a normal distribution to the interval $[\underline{x}_i^k, \overline{x}_i^k]$.*

To see why role independence is a necessary condition imagine an extreme case where beliefs are individually homogeneous with $\tilde{e}^{(i)} \sim \mathcal{U}(\underline{x}, \overline{x})$. Then, by Theorem 1, choices are uniformly (*i.e.*, not normally) distributed. Next, we discuss the condition on the variance of the beliefs. First, it is required that $\mathrm{Var}(\tilde{e}_j^{(i)})$ does not vanish too fast (if it vanishes) as $n$ grows. In fact, without enough belief variability, it would be difficult to obtain normally distributed choices when these are mixed in the average belief $\tilde{\mu}^{(i)}$. Second, no player should have an excessive impact on the targets of other players, that is, as $n$ grows, there is no role $j$ whose centrality $\gamma_j$ becomes large compared to that of the other roles. Corollary 1 below illustrates this point when the belief variance is bounded away from zero: normality arises when the maximum centrality $\overline{\gamma}$ vanishes faster than $n^{-1/2}$.

**Corollary 1.** *Consider any role $i \in N$ and suppose that $\mathbf{W}$ is primitive for all $n \geq 1$. If beliefs are role-independent, if there is $\varepsilon > 0$ independent of $n$ such that for all $j \in N$, $\mathrm{Var}(\tilde{e}_j^{(i)}) \geq \varepsilon$ as $n \to \infty$, and if $\lim_{n\to\infty} \overline{\gamma}\sqrt{n} = 0$, then $\tilde{x}_i^k$ is asymptotically normal as $k, n \to \infty$.*

A simple scenario of a moderate maximum influence, which includes the beauty contest game with at least three players, occurs when $\mathbf{W}$ approaches towards a column-regular matrix, *i.e.*, there is $c > 0$ such that $\mathbf{1}^\top \mathbf{W} \to c\,\mathbf{1}^\top$ as $n \to \infty$. Then, for all $j \in N$, we have that $\gamma_j \simeq \overline{\gamma} \simeq 1/n$ for large $n$, which vanishes sufficiently fast. More generally, anonymous games (*i.e.*, games where roles cannot be distinguished from each other in the network $\mathbf{W}$) satisfy the conditions of Corollary 1.

# 4 Maximum likelihood estimation

We propose a mixture model, whose components are the different reasoning levels, in order to non-parametrically infer the beliefs about the behavior of level-0 subjects. By Theorem 1 or by Theorem 2, the choice distributions $\tilde{x}_i^k$ of all non-zero levels are assumed to have the same shape as the average beliefs $\tilde{\mu}$. Since these choice distributions only differ in known scale and shift parameters from each other, we therefore implicitly accomplish our goal of estimating the distribution of average beliefs. Formally, let $x_l$, $l = 1, ..., L$, be a typical observation or choice. For notational simplicity, it is assumed that all choices belong to the same role $i \in N$ in a given anchored guessing game. Since the role is fixed, we suppress the index $i$ throughout this section. In the log-likelihood function

$$\mathcal{L} = \sum_{l=1}^{L} \log\left(\sum_{k=0}^{K} p_k\, f^k(x_l)\right)$$

$K$ denotes the maximum reasoning level considered in the estimation, $p_k \in [0, 1]$ corresponds to the (unknown and to be estimated) fraction of observations that are generated by subjects with reasoning level $k$, and $f^k$ is the (unknown and to be estimated) conditional density function that describes the choices of the level-$k$ subjects. Regarding $f^0$, we consider first what we call an $L0$-*unrestricted* estimation according to which the density function $f^0$ forms part of the estimation. The domain of the density function $f^k$ is the $k$-iterated undominated set $U^k = [\underline{x}^k, \overline{x}^k]$.

The interest of Theorems 1 and 2 lies in that they produce shape restrictions that reduce the number of unknown density functions from $K+1$ to 2 ($f^0$ and $f$). In particular, by applying either of these results, we obtain that for each $k \geq 1$ and each $x \in \mathbb{R}$,

$$f^k(x) = \begin{cases} \frac{\overline{x}-\underline{x}}{\overline{x}^k-\underline{x}^k}\, f\left(\underline{x} + \frac{\overline{x}-\underline{x}}{\overline{x}^k-\underline{x}^k}(x-\underline{x}^k)\right) & \text{if } x \in U^k \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The unknown density $f$ with domain $[\underline{x}, \overline{x}]$ corresponds to the average beliefs $\tilde{\mu}$. Given an observation $x$, let $k(x)$ be the highest $k$ such that $x \in U^k$. So, $k(x)$ is the maximum reasoning level that is compatible with choice $x$. Substituting (2) into the original log-likelihood function yields

$$\mathcal{L} = \sum_{l=1}^{L} \log\left(p_0\, f^0(x_l) + \sum_{k=1}^{k(x_l)} p_k\, \frac{\overline{x}-\underline{x}}{\overline{x}^k-\underline{x}^k}\, f\left(\underline{x} + \frac{\overline{x}-\underline{x}}{\overline{x}^k-\underline{x}^k}(x_l-\underline{x}^k)\right)\right).$$

We approximate the density functions by partitioning their domains in a precise way that takes advantage of our theoretical results. In particular, each set of undominated strategies $U^k = [\underline{x}^k, \overline{x}^k]$ is divided into a total of $B$ buckets of equal width $(\overline{x}^k - \underline{x}^k)/B$. These buckets form a histogram that approximates the density function $f^k$. The non-negative bucket areas (to be estimated) for reasoning level $k$ are denoted by $a_1^k, a_2^k, \ldots, a_B^k$. The total bucket area for reasoning level $k$ is $\sum_{b=1}^{B} a_b^k = 1$. For each observation $x$ and each reasoning level $k$ such that $x \in U^k$, let $b^k(x)$ be the index $1, 2, \ldots, B$ of the bucket that contains $x$ in the histogram of $f^k$. The discrete version of the original log-likelihood function (without shape restrictions) is then

$$\mathcal{L}_D = \sum_{l=1}^{L} \log\left(\sum_{k=0}^{k(x_l)} p_k\, \frac{a^k_{b^k(x_l)}\, B}{\overline{x}^k-\underline{x}^k}\right),$$

where $a^k_{b^k(x_l)}\, B/(\overline{x}^k - \underline{x}^k)$ is the height of the $b^k(x_l)$-th bucket that indicates the approximated value of $f^k(x_l)$. Theorems 1 and 2 allow us to assume equal bucket areas across non-zero levels. That is, for each bucket $b = 1, 2 \ldots, B$, we have that $a_b^1 = a_b^2 = \cdots = a_b^K \equiv a_b$. The log-likelihood function to be maximized is then

$$\max_{(p_k)_{k=0}^K \in \Delta^K, (a_b^0)_{b=1}^B, (a_b)_{b=1}^B \in \Delta^{B-1}} \sum_{l=1}^{L} \log\left(p_0\, \frac{a^0_{b^0(x_l)}\, B}{\overline{x}-\underline{x}} + \sum_{k=1}^{k(x_l)} p_k\, \frac{a_{b^k(x_l)}\, B}{\overline{x}^k-\underline{x}^k}\right), \tag{3}$$

where $\Delta^r$ is the $r$-dimensional simplex. The total number of parameters to be estimated is $K + 2(B-1)$, which is possibly a large number.

We adapt the expectation maximization (EM) algorithm to solve the maximization problem (3) iteratively. The maximization problem is generally difficult to solve, yet since we can rely in the M-step on closed-form formulae rather than on solving an interim maximization program, it turns out that the algorithm is very fast in practice. The algorithm typically yields a local maximum for any arbitrary starting point and, due to the efficient implementation, it can be executed many times producing in this way a set of local maxima that can be further analyzed. In the following, superscript $[t]$ denotes the current parameter value at time $t$ during the execution of the algorithm. Start at time 0 with arbitrary parameter values $(p_k^{[0]})_{k=0}^K \in \Delta^K$ and $(a_b^{0[0]})_{b=1}^B, (a_b^{[0]})_{b=1}^B \in \Delta^{B-1}$. Then, at time $t = 0, 1, \ldots$, repeat the following two steps until the distance between two consecutive solutions reaches a desired tolerance level:

- **E-step (expectation).**

  Given the iteration $t$ density functions $f^{0[t]}(x_l) = a^{0[t]}_{b^0(x_l)}\, B/(\overline{x} - \underline{x})$ and $f^{k[t]}(x_l) = a^{[t]}_{b^k(x_l)}\, B/(\overline{x}^k - \underline{x}^k)$ for all $k \geq 1$, compute for each reasoning level $k$ and each observation $x_l$,

  $$q_k^{[t]}(x_l) = \frac{p_k^{[t]}\, f^{k[t]}(x_l)}{\sum_{k'=0}^K p_{k'}^{[t]}\, f^{k'[t]}(x_l)}.$$

- **M-step (maximization).**

  The parameters are updated.[19,20]

  1. Update the level composition. For each $k \geq 0$,

     $$p_k^{[t+1]} = \frac{1}{L} \sum_{l=1}^{L} q_k^{[t]}(x_l).$$

  2. Update bucket areas. For each bucket $b = 1, \ldots, B$,

     $$a_b^{0[t+1]} = \frac{1}{L\, p_0^{[t+1]}} \sum_{\substack{l=1,\ldots,L \\ b^0(x_l)=b}} q_0^{[t]}(x_l) \text{ and } a_b^{[t+1]} = \frac{1}{L\left(1-p_0^{[t+1]}\right)} \sum_{k=1}^{K} \sum_{\substack{l=1,\ldots,L \\ b^k(x_l)=b}} q_k^{[t]}(x_l).$$

[19] It can be shown that the the closed-form formulae for level composition and bucket areas given in the M-step are the solution to the interim maximization program

$$\max_{(p_k^{[t+1]})_{k=0}^K \in \Delta^{\bar{K}}, (a_b^{0[t+1]})_{b=1}^B, (a_b^{[t+1]})_{b=1}^B \in \Delta^{B-1}} \sum_{l=1}^{L} \sum_{k=0}^{K} q_k^{[t]}(x_l) \log\left(p_k^{[t+1]}\, f^{k[t+1]}(x_l)\right).$$

[20] When a solution is reached after a high number of iterations $T$, the probability that the choice $x \in [\underline{x}, \overline{x}]$ is generated by the reasoning level $k$ is approximated by $q_k^{[T]}(x)$ from the $E$-step.

Instead of the $L0$-unrestricted estimation procedure shown above, one could also make assumptions about the actual behavior of the level-0 subjects. Two types of restricted estimation procedures seem particularly appealing. In an $L0$-*fixed* estimation, a concrete $f^0$ is imposed. For example, if level-0 choices are assumed to be uniformly distributed, then we set for all $x \in [\underline{x}, \overline{x}]$ and all iterations $t$, $f^{0[t]}(x) = 1/(\overline{x} - \underline{x})$. Alternatively, in an $L0$-*consistent* estimation, the density $f^0$ of level-0 play is assumed to be equal to the density $f$ of average beliefs. In this case, the average beliefs are correct and we substitute $a_b^0$ by $a_b$ for all buckets $b = 1, \ldots, B$ and update the bucket areas in the $M$-step as

$$a_b^{[t+1]} = \frac{1}{L} \sum_{k=0}^{K} \sum_{\substack{l=1,\ldots,L \\ b^k(x_l)=b}} q_k^{[t]}(x_l).$$

Finally, if Proposition 2 applies, we may adopt a gaussian approach in which the algorithm has to estimate the parameters $\mu$ and $\sigma$ of the normal density $f$ of average beliefs.[21] For the sake of simplicity in the estimation, we do not truncate the normal distributions. It is important to observe that gaussian $L0$-unrestricted estimations generally lead to trivial solutions with $p_0 = 1$ because $B$ buckets of $f^0$ tend to explain the data better alone than when mixed with gaussian shapes of choices with level $k \geq 1$. Therefore, we only consider an L0-restricted estimation framework. We can still consider $L0$-consistent and $L0$-fixed estimations. In the equations below, the $L0$-consistent estimation corresponds to the model parameter $\mathcal{C} = 0$, while an $L0$-fixed estimation is obtained if $\mathcal{C} = 1$. Let

$$z^k(x) = \underline{x} + \frac{\overline{x} - \underline{x}}{\overline{x}^k - \underline{x}^k}(x - \underline{x}^k)$$

be the normalization of $x \in U^k$ from $[\underline{x}^k, \overline{x}^k]$ to $[\underline{x}, \overline{x}]$. Start at time $t = 0$ with arbitrary parameter values $(p_k^{[0]})_{k=0}^K \in \Delta^K$, $\mu^{[0]} \in [\underline{x}, \overline{x}]$, and $\sigma^{[0]} > 0$. The E-step and M-step for the gaussian estimation are as follows:

- **E-step (expectation)**.

  From current parameter values available at time $t$, compute for each reasoning level $k$ and each observation $x_l$

$$q_k^{[t]}(x_l) = \frac{p_k^{[t]} f^{k[t]}(x_l)}{\sum_{k'=0}^{K} p_{k'}^{[t]} f^{k'[t]}(x_l)},$$

  where for all $k \geq \mathcal{C}$,

[21] In typical gaussian mixture models, gaussian level-0 beliefs imply gaussian choices of higher reasoning levels. We allow for any distribution of beliefs, and gaussian choices arise by the central limit result of Proposition 2.

$$f^k(x) = \begin{cases} \frac{\overline{x}-\underline{x}}{\overline{x}^k-\underline{x}^k} \frac{1}{\sqrt{2\pi\sigma^{[t]^2}}} \exp\left\{-\frac{\left(z^k(x)-\mu^{[t]}\right)^2}{2\sigma^{[t]^2}}\right\} & \text{if } x \in U^k \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{C} = 1$, we have $f^{0[t]} = f^0$ (fixed ex-ante).

- **M-step (maximization)**.

  The parameters are updated.

  1. Update the level composition. For each $k \geq 0$,

  $$p_k^{[t+1]} = \frac{1}{L} \sum_{l=1}^{L} q_k^{[t]}(x_l).$$

  2. Update the gaussian parameters:

  $$\mu^{[t+1]} = \frac{1}{L\left(1-\mathcal{C}\, p_0^{[t+1]}\right)} \sum_{l=1}^{L} \sum_{k=\mathcal{C}}^{k(x_l)} q_k^{[t]}(x_l)\, z^k(x_l)$$

  and

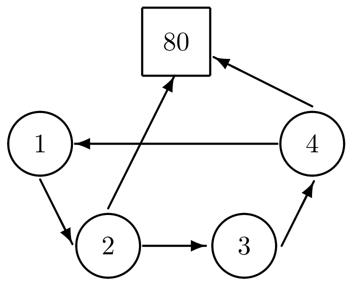
  $$\sigma^{[t+1]^2} = \frac{1}{L\left(1-\mathcal{C}\, p_0^{[t+1]}\right)} \sum_{l=1}^{L} \sum_{k=\mathcal{C}}^{k(x_l)} q_k^{[t]}(x_l) \left(z^k(x_l) - \mu^{[t+1]}\right)^2.$$

# 5 Experimental analysis

This section estimates the mixture model for the data from the newspaper experiments of Bosch-Domènech, Montalvo, Nagel and Satorra (2002), who implement the beauty contest game with $p = 2/3$ in the Financial Times (England), Spektrum (Germany), and Expansión (Spain), and for the first of the three anchored guessing games detailed in Ballester, Rodriguez-Moral and Vorsatz (2021).[22] The latter study is tailored as a more direct test of the equilibrium behavior in Proposition 1 and of whether the estimated beliefs depend on exogenous characteristics such as gender. In Ballester, Rodriguez-Moral and Vorsatz (2021) subjects are asked, using an incentive compatible mechanism, to register during 8 periods as many choices as they wish to without receiving feedback regarding the actions or the payoffs of any player.[23]

[22] We also performed estimations with the beauty contest dataset of Matthew O. Jackson from his Game Theory course in Coursera. The results are similar to those obtained in the newspaper experiments, but they are noisier and the estimation yields, maybe due to the absence of incentives, lower reasoning levels.

[23] There are also treatments with feedback between periods, which affects subject behavior. We therefore concentrate here on the treatments without feedback.



$$[\mathbf{W}|\mathbf{A}] = \left(\begin{array}{cccc|c} 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0 & 0.25 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0.25 & 0 & 0 & 0 & 0.25 \end{array}\right)$$

$$\mathbf{x}^* = (11, 23, 11, 23)$$

$$\lambda_1 = 0.35$$

**Figure 3:** Anchored guessing game of Ballester, Rodriguez-Moral and Vorsatz (2021).

The targets, induced by means of the matrix in Figure 3, have simple verbal equivalents that are used in the instructions in order to keep the experiment as simple as possible. Subjects are also provided with the graphical representation of the strategic dependencies (network structure) to further simplify the experiment. Players 1 and 3 as well as players 2 and 4 are isomorphic in the sense that their respective positions in the game are identical. We pool all registered choices because not much additional insight is gained from separating player roles or periods.

Theorem 1 reasonably applies to Bosch-Domènech, Montalvo, Nagel and Satorra (2002) because their game is anonymous which supports Assumption 2. But one could also make use of Theorem 2 since the eigenvalue ratio $(n-1)^{-1}$ is negligible when subjects use best-replies and $n$ is large. For Ballester, Rodriguez-Moral and Vorsatz (2021) we have to impose Assumption 2 in order to apply Theorem 1. Theorem 2 cannot be used because $\mathbf{W}$ is not primitive. We set the maximal reasoning level to $K = 10$, approximate the density functions via $B = 50$ buckets, and fix an error tolerance of $10^{-8}$. The results are quite stable with respect to these choices. Finally, since the algorithm identifies local maxima for each starting point, we run for each data set a total of 1,000 estimations with random starting points. In most cases, we get one or a few local maximum with small differences in the value of the log-likelihood function. The global maximum is always the most common outcome. The reported outcome is the average of these 1,000 estimations.

The results of the $L0$-unrestricted estimations are presented in Figure 4. The top-left panel shows the percentages of the subjects that are assigned to the different reasoning levels. We find that the percentage of level-0 subjects is between 35% and 55%. Some subjects are assigned to the highest reasoning level in the newspaper experiments, which is different for the anchored guessing game of Ballester, Rodriguez-Moral and Vorsatz (2021), where the highest reasoning level with positive estimated mass is $k = 3$. Also, the percentage of level-1 subjects is rather low and smaller

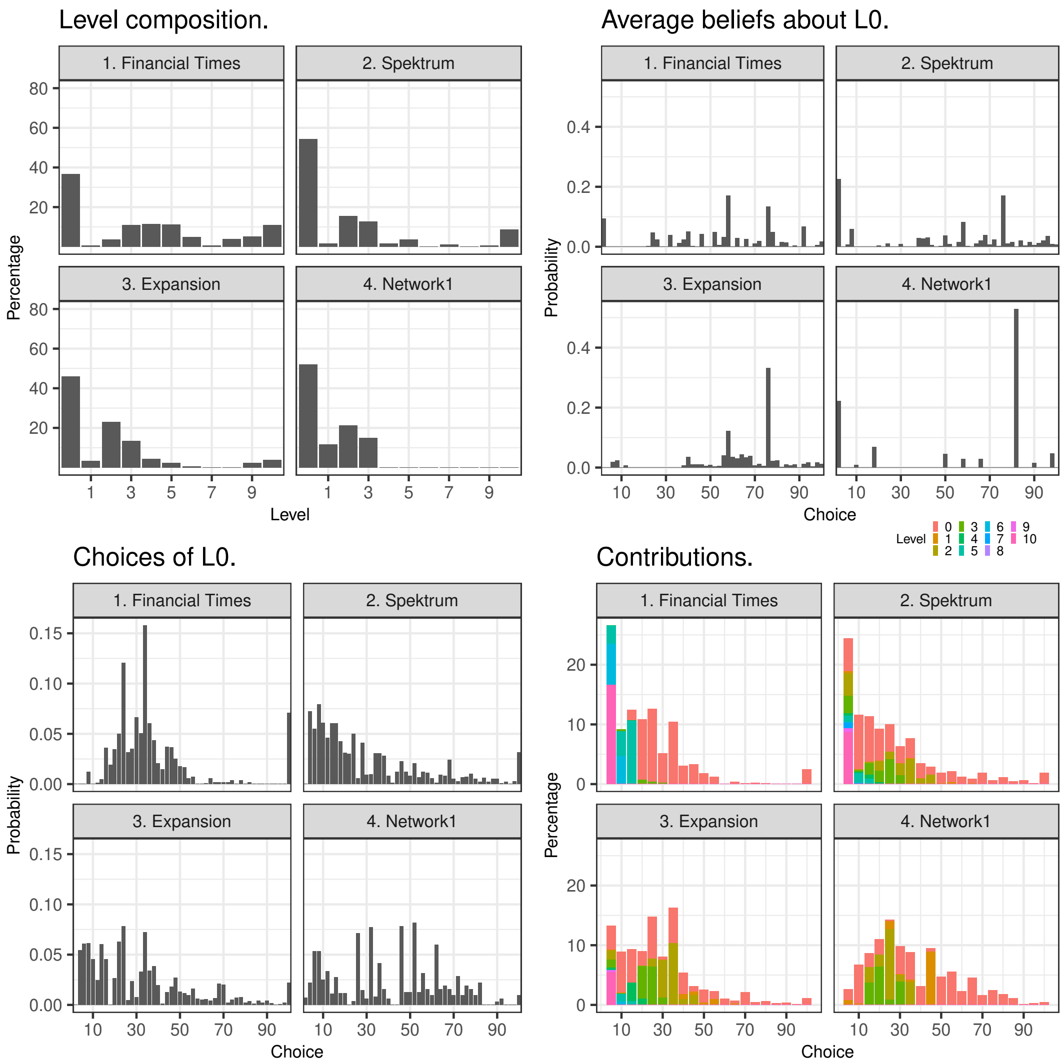



**Figure 4:** Unrestricted estimation results.

than the percentage of level-2 subjects. However, this has a straightforward explanation that directly brings us to the top-right panel.

The top-right panel, the main output of the algorithm, shows the estimated average beliefs about the behavior of level-0 players. The estimated average beliefs are much more concentrated than the original choice histograms in bottom-right panel, which suggests that subjects usually start their reasoning process from a few common focal points. To understand the lack of level-1 subjects in the level compositions of the newspapers' beauty contests, note that many subjects choose the number 33 in this game. If subjects assume that level-0 play is uniformly distributed on [0,100], then the average belief is 50 and the choice of 33 is best associated with a level-1 behavior.

Since the unrestricted estimations do not make specific assumptions (other than shape similarity) about the distribution of average beliefs, a guess of 33 is now not only consistent with a level-1 play under the average belief of 50, but also with a level-2 play under the average belief of about 75. In fact, under shape similarity, a single observation of 33 is more likely to be generated by level 2 than by level 1, which illustrates the algorithm's tendency to assign subjects to higher levels. To see this, suppose that the average beliefs 50 and 75 are equally likely, that is, $f(50) = f(75)$. Then, by shape similarity,

$$f^2(33) = \left(\frac{3}{2}\right)^2 f(75) > \left(\frac{3}{2}\right) f(50) = f^1(33).$$

The tendency to assign a guess of 33 in the newspaper experiments to level 2 can be observed in the top-right panel: instead of a spike at 50, a lot of mass is assigned to the 70-80 range. With respect to Ballester, Rodriguez-Moral and Vorsatz (2021), about 50% of the subjects with a non-zero reasoning level have an average belief of 80. This suggests that the unique anchor of the game description is a strong focal point in the belief formation process. Moreover, about 20% of these subjects have an average belief of 0, which may be caused by the fact that this number involves simpler calculations. Since 50% of all choices are level-0 play in this game (see, the top-left panel), the seeds of 80 and 0 explain about 35% of all choices.

The estimated behavior of the level-0 players in the bottom-left panel is interpreted as the residual of observations that the algorithm cannot explain by higher reasoning levels. Notably, the distributions are not uniformly distributed. Hence, for the data at hand, the outcome of the $L0$-restricted estimations, which we detail below and which assume that level-0 subjects play uniformly at random, might differ from what we find for the $L0$-unrestricted estimations.

The stacked histograms in the bottom-right panel show for every range of guesses, the percentage of choices in that range that correspond to each level. These percentages are obtained from the E-step after the execution of the algorithm. For example, in case of the newspaper experiment in the Financial Times, it turns out that choices below 15 come predominantly from subjects with a reasoning level $k \geq 6$, whereas for the Spektrum and Expansión treatments, choices in this range are partly attributed to level-0 subjects. For Ballester, Rodriguez-Moral and Vorsatz (2021), the interior equilibrium causes that the choices between 10 and 30 are mainly assigned to subjects with a relatively higher reasoning level and that choices close to the extremes of the action space correspond almost exclusively to level-0 subjects.

Figure 5 presents the results of the $L0$-fixed estimations when level-0 play is distributed uni-

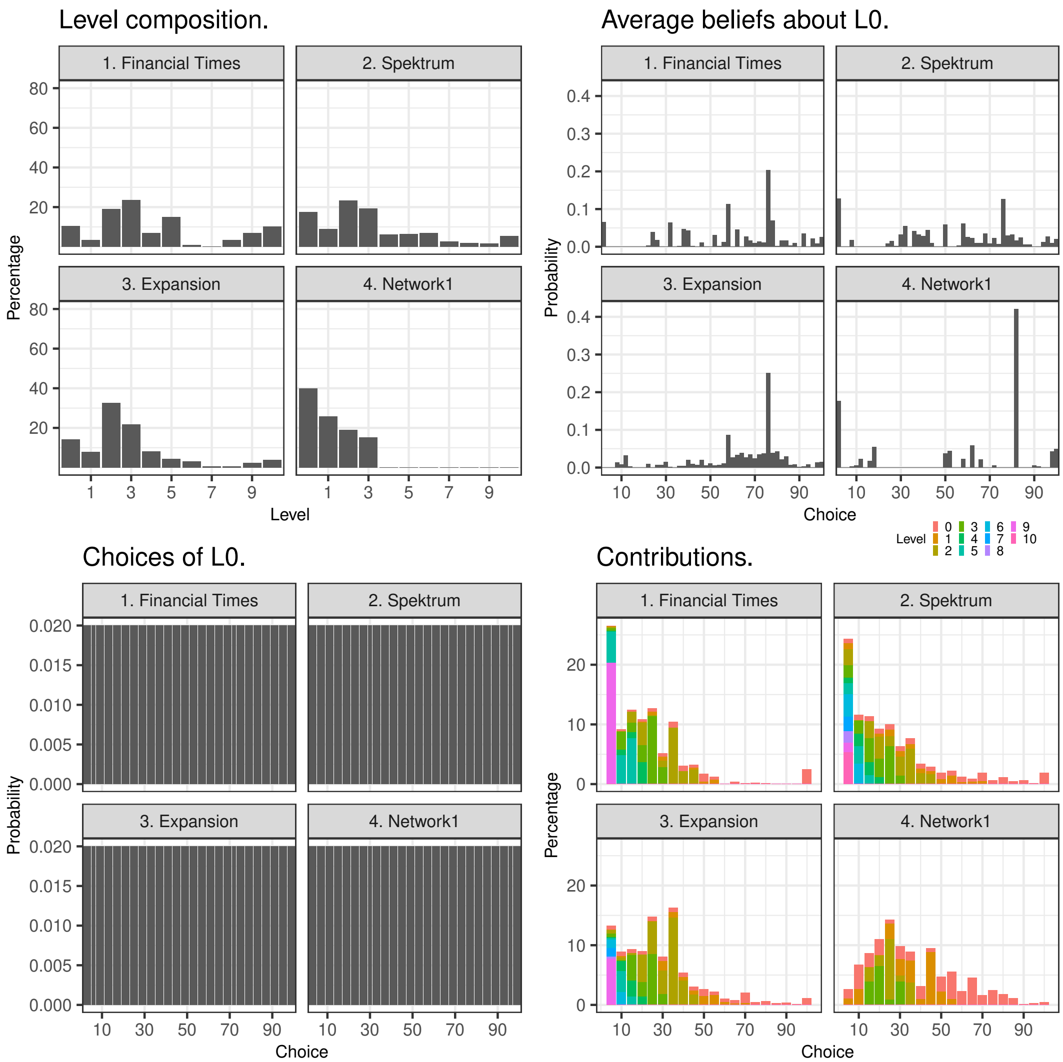



**Figure 5:** Restricted estimation results.

formly on the interval $[0, 100]$ (see, bottom-left panel). If we compare Figures 4 and 5, then it is evident that the level composition in the top-left panel changes substantially. Most strikingly, the percentage of level-0 subjects in the population is now much lower. The level composition has consequently a tendency to shift upwards for the $L0$-fixed estimation. An important question to be answered is whether the change in the level composition between the estimation procedures is mainly due to the different behavior of the level-0 subjects or whether this model assumption causes the estimated behavior of subjects with a higher reasoning level to be different. A comparison of the corresponding top-right panel is helpful in providing an answer to this question. One observes that the estimated average beliefs is rather robust between estimation procedures. The estimated histograms display peaks at similar locations under both methodologies, with a slightly less clustered distribution in the $L0$-fixed estimation. The explanation for the reduced clustering of the $L0$-fixed estimation is that this procedure is generally forced to classify more observations above level 0, which implies that the algorithm needs to figure out new beliefs that can generate the new observations that are not level 0. Hence, the shape of the non-zero choice distributions does not vary too much between the two approaches and the differences in the level composition seems to be mainly driven by the level-0 subjects themselves.

Finally, Bosch-Domènech, Montalvo, Nagel and Satorra (2010) estimate a mixture model for the data from the newspaper experiments using generalized beta distributions when level-0 play is uniformly distributed on the interval [0,100], the estimated distributions are anchored at the theoretical play, the width of the support of the distributions is exogenous and equal to 20, and beliefs about level-0 play are common and set to 50 or 100, respectively. They find for the pooled data that about 30% of the subjects are level-0, about 10% are level-1, about 20% are level-2, and about 40% are level-$\infty$. Our $L0$-fixed estimations assign between 15% and 20% of the choices to level-0 play, but finds similar percentages of level-1 and level-2 guesses as in Bosch-Domènech, Montalvo, Nagel and Satorra (2010). A direct comparison of higher levels play is not possible because Bosch-Domènech, Montalvo, Nagel and Satorra (2010) pool all remaining levels at infinity. Even though our methodology also relies on a mixture model, the results between the two studies are still hard to compare because the density functions of the mixture model are defined on completely different domains.

# Concluding discussion

This paper contributes to the literature on strategic models of bounded rationality and, in particular, on the level-$k$ model, by proposing for a general class of guessing games a new maximum likelihood estimation procedure that uncovers the belief structure that underlies actual choices and by deriving the microeconomic/statistical foundations of this econometric approach. Section 2 generalizes beauty contest games, which have been widely applied in laboratory experiments, by allowing for asymmetries and constants in the players' best reply functions. These anchored guessing games have a natural visualization in terms of networks. It is worth mentioning in this respect that the characterization of equilibrium uniqueness in Proposition 1 is ordinal (*i.e.* purely graph-theoretical). Section 3 studies the optimal choices under the level-$k$ framework and derives the foundations of our estimation technique. Throughout, the central assumption is that beliefs are independent of reasoning levels. We then find that if each subject assigns the same belief to all level-0 players in all roles, then the random variables that collect the optimal level-$k$ choices must have exactly the same shape as the unknown distribution of average beliefs (Theorem 1). Said differently, the distribution of optimal level-$k$ choices can be remapped to obtain the distribution of average beliefs by simply applying scale and shift parameters to its associated domain of serially undominated strategies. If beliefs are not individually homogenous, the shape of the level-$k$ choice distribution approaches towards the shape of average beliefs (Theorem 2). We also provide extra conditions on Theorem 2 so that the distribution is normal as the number of players grows (Proposition 2). Section 4 presents the maximum likelihood estimation procedure of a mixture model with a free-form belief distribution when, due to Theorem 1 or Theorem 2, the choice distributions of all non-zero levels are assumed to have the same shape as the average beliefs. We solve the maximization problem iteratively with the help of the expectation maximization algorithm, which turns out to be fast since we can rely in the maximization step on a closed-form formulae. Finally, we apply the estimation technique to the data of Bosch-Domènech, Montalvo, Nagel, and Satorra (2002) and Ballester, Rodriguez-Moral and Vorsatz (2021). We identify clustered beliefs in both data sets. Two main focal points appear for the anchored guessing game of Ballester, Rodriguez-Moral and Vorsatz (2021). The first focal point is 80, which is precisely the unique anchor value of the game. The second focal point is 0, which is the lower bound of the strategy space.

In the remainder of this section, we adapt a mechanism design perspective. Apart from personal research interests and other factors like simplicity in terms of the number of player and the number of connections in the network, the following factors affect the quality of the estimation:

1. **Connectedness**. If a game is connected (in the associated network, there is a path between any two players), the choice of a player will potentially be determined by her level-0 beliefs about all players. Theorem 2 requires that the dependency matrix $\mathbf{W}$ is primitive, which implies connectedness.

2. **Eigenvalue ratio** $|\lambda_2|/\lambda_1$. The estimation algorithm assumes common shapes for all reasoning levels $k \geq 1$, but Theorem 2 applies only in the limit. As we have seen in Section 3, games with a lower eigenvalue ratio theoretically provide better estimations because convergence emerges at earlier reasoning levels.

3. **Heterogeneity**. As the game becomes more heterogeneous in terms of dependencies or

anchors, it is richer in terms of variability across different roles. However, the assumption of individually homogeneous beliefs necessary for Theorem 1 is probably harder to sustain as heterogeneity increases. In those cases, one must rely on Theorem 2 and make sure that the eigenvalue ratio is moderate. In a similar fashion, if Theorem 2 does not apply (because the matrix is not primitive or, more generally, the eigenvalue ratio is close to one), one has to rely entirely on Theorem 1. Anchor value variability should then probably be avoided.

4. **Dominant eigenvalue** $\lambda_1$. Recall that lower values of $\lambda_1$ imply faster convergence towards the equilibrium as the reasoning level increases. Next, we provide a new result, which shows that intermediate values of the dominant eigenvalue $\lambda_1$ may improve the quality of the estimation. The intuition is simple in the beauty contest game when it is taken into account that $\lambda_1 = p$. Suppose that $p = 0.1$. This has two effects. First, it allows us to classify all observations in the interval $[10, 90]$ as level 0. Second, it reduces the choice range for all $k \geq 1$ to $[0, 10]$, which can make it difficult to correctly discriminate between these levels when players make mistakes. The same idea applies when $p$ is too high. If $p = 0.9$, it not so easy to classify some choices as low-level. As we show below, $p = 2/3$ is a compromise eigenvalue when it is the objective to estimate 4 reasoning levels ($K = 3$).

To formalize the last point, let the *discriminating sets* of player $i$ be the disjoint sets

$$D_i^k = \begin{cases} U_i^k \setminus U_i^{k+1} & \text{if} \quad k < K \\ U_i^K & \text{if} \quad k = K. \end{cases}$$

For $k < K$, the $k$-th discrimination set $D_i^k$ for a player in role $i$ contains a particular choice $x \in X$ if and only if the maximum reasoning level $x$ can be assigned to is exactly $k$. And $D_i^K$ is the set of choices with a maximum reasoning level of at least $K$. Intuitively, we would like to jointly maximize the size of all discriminating sets of all players subject to the interiority condition. For that, we say that the dependency matrix $\mathbf{W}$ *admits interiority* if $\underline{x}\ (\mathbf{I} - \mathbf{W})\,\mathbf{1} \leq \overline{x}\ (\mathbf{I} - \mathbf{W})\,\mathbf{1}$, that is, if $\mathbf{W}$ is substochastic. We fix a baseline non-negative matrix $\mathbf{H}$ in the sense that the maximum of its row sums is one. Then, we solve the maximization problem for games with dependency matrix $\mathbf{W} = q\mathbf{H}$, where $q$ is a scaling factor such that $\mathbf{W}$ admits interiority, that is, $q \in [0, 1]$. The dominant eigenvalue of the dependency matrix $\mathbf{W}$ is simply $\lambda_1(\mathbf{W}) = q\lambda_1(\mathbf{H})$. The following result assumes that $\mathbf{H}$ (and hence $\mathbf{W}$) is regular[24], which includes all anonymous games such as beauty contest games.

[24] A non-negative matrix $\mathbf{S}$ is regular if $\mathbf{S}\mathbf{1} = s\mathbf{1}$ for some constant $s > 0$.

**Proposition 3.** *Fix a regular baseline matrix* $\mathbf{H} \geq \mathbf{0}$. *The optimal scale* $\hat{q}$ *that jointly maximizes the size of discriminating sets is*

$$\hat{q} = \underset{q \in [0,1]}{\operatorname{argmax}} \prod_{i=1}^{n} \prod_{k=0}^{K} |D_i^k| = \frac{K+1}{K+3}.$$

The interpretation is that given a baseline $\mathbf{H}$, the associated optimal dependency matrix $\mathbf{W} = \hat{q}\mathbf{H}$ has a dominant eigenvalue of $(K+1)/(K+3)$ (because $\lambda_1(\mathbf{H}) = 1$ by the regularity of $\mathbf{H}$). In order to illustrate how the before-mentioned factors may affect the quality of the estimation, we consider a family of anchored guessing games with two types of players. Players of type A have one out-link in the network represented by $\mathbf{W}$ and are also directly connected to an anchor. Players of type B have two neighbors in the network $\mathbf{W}$. We assume that $\underline{x} \leq 0 \leq \overline{x}$ and that the dependency matrix $\mathbf{W}$ has a zero main diagonal. The targets are

$$t_i = \begin{cases} q\,\frac{x_{i_1}+v_i}{2} & \text{if } i \text{ is of type A} \\ q\,\frac{x_{i_1}+x_{i_2}}{2} & \text{if } i \text{ is of type B,} \end{cases}$$

where $q > 0$ is a parameter, $v_i$ is the anchor value of player $i$ of type A, and $i_j$ denotes a neighbor of player $i$ in the dependency network $\mathbf{W}$. It is not difficult to see that the interiority assumption translates into $q \leq 1$ and for any type A player $i$, $v_i \in [(2-q)\underline{x}/q, (2-q)\overline{x}/q]$.

Figure 6 shows all connected three-player and four-player games in this class. Type A players are indicated by black dots and type B players by white dots. The games are ordered by the network structure and the eigenvalue ratio $r = |\lambda_2|/\lambda_1$. The number of links in the network is $e$ and the dominant eigenvalue $\lambda_1$ is shown as a function of the parameter $q$. For each game in the figure, $\hat{q}$ is the value of $q$ that maximizes the objective function of Proposition 3 with $K = 3$.[25] By Proposition 1($b$), all games with at least one type A player must have a unique equilibrium (every player has a path to an anchor). And all games with only type B players have a unique equilibrium ($\mathbf{x}^* = \mathbf{0}$) if and only if $q < 1$. Finally, Theorem 2 only applies if $r < 1$ (*i.e.*, $\mathbf{W}$ is primitive).[26] Interestingly, Theorem 2 cannot be applied to any of the "square" games.

Consider the first game in the second row of all "diamond" games, which consists of two players of each type. In this game, the eigenvalue ratio is $r = 0.62$, which is substantially bounded away from 1 so that shape similarity should ceterus paribus be observed at relatively early non-zero reasoning levels by Theorem 2. And it should be rather easy for subjects to understand the

[25] We have numerically solved the maximization problem for each game.

[26] The second game in the first row of all "diamond" games is an exception because $\mathbf{W}$ is not diagonalizable.

game. The number of links is relatively small ($e = 6$) and $\hat{q} \approx 3/4$. Finally, setting $v_1 = 0$ and $v_2 = 100$ does not impose too complex calculations on experimental subjects. This example shows how Figure 6 can be used to design anchored guessing games for laboratory experiments with reasonable estimation properties.

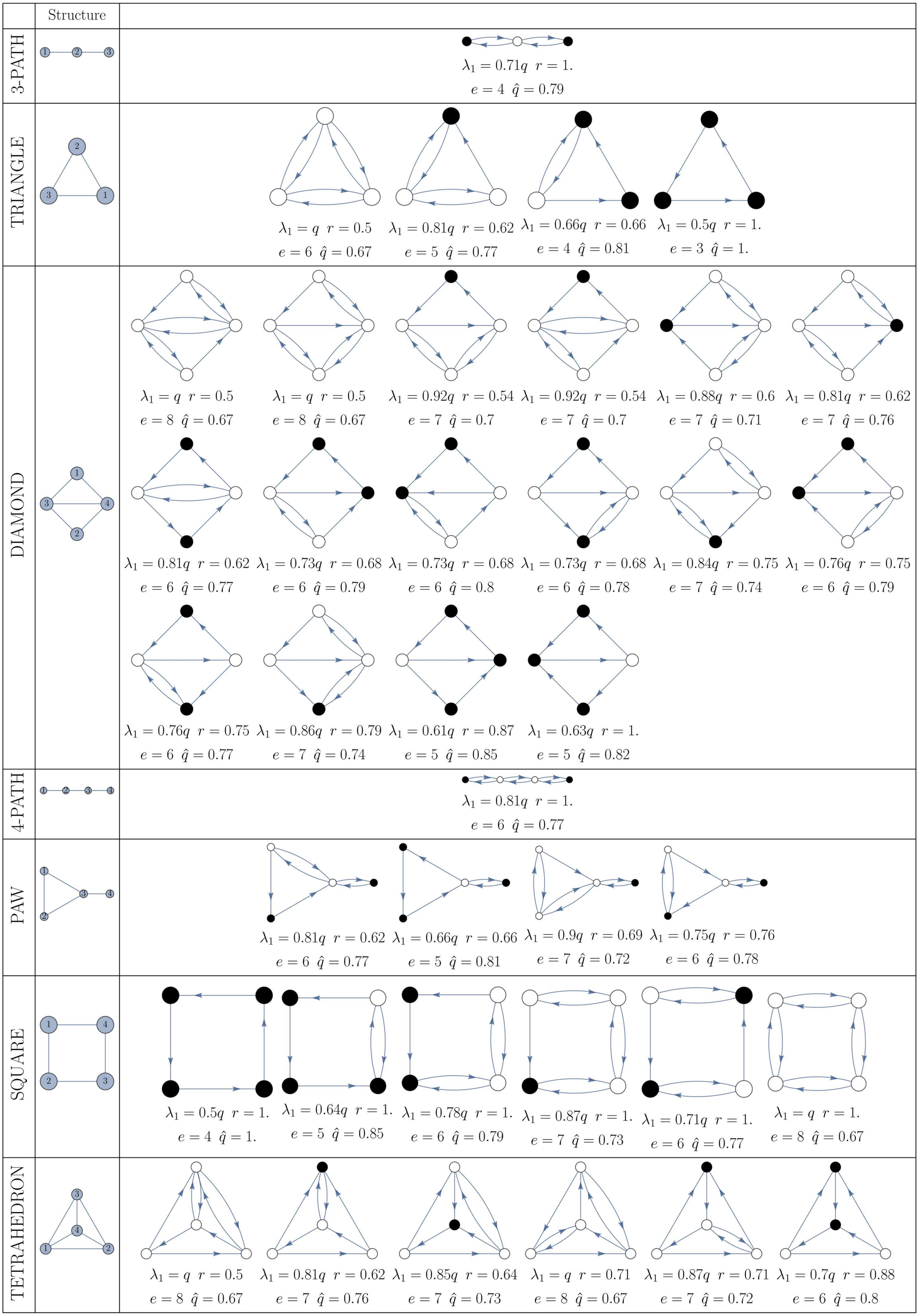



**Figure 6:** Three-player and four-player anchored guessing games.

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# Appendix: Proofs

We establish a couple of auxiliary results before proving Proposition 1. A matrix is (row)-substochastic if all its entries are non-negative and all row sums are smaller than or equal to 1. If all row sums are exactly equal to 1, the matrix is called (row)-stochastic.

**Lemma 1.** *For all anchored guessing games,* $\mathbf{W}$ *is row-substochastic.*

**Proof of Lemma 1.** Suppose by contradiction that $\mathbf{W}$ is not row-substochastic. Then, there is a player $i \in N$ for whom $\sum_j w_{ij} \equiv r_i > 1$. By Assumption 1, $\underline{x}\,(\mathbf{I} - \mathbf{W})\,\mathbf{1} \leq \mathbf{Av} \leq \overline{x}\,(\mathbf{I} - \mathbf{W})\,\mathbf{1}$. Considering the $i$-th row of this system of equations, we can see that $\underline{x}(1 - r_i) \leq \overline{x}(1 - r_i)$. Since $r_i > 1$, this contradicts the assumption that $\underline{x} < \overline{x}$. $\square$

**Lemma 2.** *For all anchored guessing games, the dominant eigenvalue* $\lambda_1$ *of* $\mathbf{W}$ *satisfies* $\lambda_1 \in [0, 1]$.

**Proof of Lemma 2.** First, $\lambda_1 \geq 0$ because $\mathbf{W} \geq \mathbf{0}$. Also, $\lambda_1 \leq 1$ because (a) $\lambda_1$ is monotone in $\mathbf{W}$ due to Perron-Frobenius theory, (b) $\lambda_1 = 1$ whenever $\mathbf{W}$ is row-stochastic, and (c) since $\mathbf{W}$ is row-substochastic by Lemma 1, it can be obtained by decreasing a stochastic matrix. $\square$

**Proof of Proposition 1.**

(a) An anchored guessing game is smooth supermodular whenever for all $i \in N$, the strategy space $X_i$ is a closed interval, the utility function $u_i$ is twice continuously differentiable on $X_i$, and for all $j \in N \setminus \{i\}$, $\frac{\partial^2 u_i(x)}{\partial x_i \partial x_j} \geq 0$. Since $X_i = [\underline{x}, \overline{x}]$, and $\frac{\partial^2 u_i(x)}{\partial x_i \partial x_j} = 2w_{ij} \geq 0$, the game is smooth supermodular. Equilibrium existence in pure strategies follows from Theorem 5 in Milgrom and Roberts (1990).

(b) Suppose that every player $i \in N$ has a path to an anchor. This, combined with the fact that $\mathbf{W}$ is row-substochastic by Lemma 1, implies that there is an integer $k$ such that all row sums of $\mathbf{W}^k$ are strictly less than 1. The dominant eigenvalue of $\mathbf{W}^k$ is $\lambda_1^k$ and it is bounded by the maximum of its row sums. Hence, $\lambda_1^k < 1$. This implies that $\lambda_1 < 1$. Consequently, $\mathbf{I} - \mathbf{W}$ has an inverse. By interiority, every equilibrium $\mathbf{x}^*$ must satisfy $\mathbf{x}^* = \mathbf{Av} + \mathbf{Wx}^*$. Then, $\mathbf{x}^* = (\mathbf{I} - \mathbf{W})^{-1}\mathbf{Av}$ must be the unique equilibrium.

Suppose next that some player $i \in N$ does not have a path to an anchor. Consider the set $S \subseteq N$ of all such players. By construction, the square matrix $\mathbf{W}_S$, the restriction of $\mathbf{W}$ to the set $S$, is stochastic. Multiplicity of equilibria is attained as follows. For each $c \in [\underline{x}, \overline{x}]$, we construct an equilibrium $\mathbf{x}^*(c)$ such that $x_i^*(c) = c$ for all $i \in S$. To see this, note that the interiority assumption restricted to the set $S$ becomes $\mathbf{0} \leq (\mathbf{Av})_S \leq \mathbf{0}$, which implies that $(\mathbf{Av})_S = \mathbf{0}$, that is, the target of each member of $S$ is a weighted average of choices made within $S$. This means that all members of $S$ are best-replying by choosing the number $c$. Now, taking this action $c$ as a (possibly) new anchor for players in $N \setminus S$, these players choose their unique equilibrium strategies (since they all have a path to some anchor). Hence, each choice $c$ by all the members of $S$ determines an equilibrium of the entire game.

(c) Since there is a unique equilibrium by assumption, each player has a path to anchor by part (b). We have already seen in the proof of part (b) that a player has a path to an anchor if and only if the dominant eigenvalue $\lambda_1$ of $\mathbf{W}$ is such that $\lambda_1 < 1$. This implies that the matrix $\mathbf{I} - \mathbf{W}$ is invertible. Since best replies are interior by assumption, we have that the unique solution must be $\mathbf{x}^* = (\mathbf{I} - \mathbf{W})^{-1}\mathbf{Av}$.

(d) Iterative convergence and dominance solvability follows from smooth supermodularity and equilibrium uniqueness. Global stability follows from iterative convergence and equilibrium uniqueness.

This concludes the proof of the proposition. □

**Proof of Theorem 1.** Let $s$ be a level-$k$ subject playing role $i$. By the interiority assumption, for all $t \geq 1$,

$$\overline{\mathbf{x}}^t = \mathbf{Av} + \mathbf{W}\overline{\mathbf{x}}^{t-1}$$

and

$$\underline{\mathbf{x}}^t = \mathbf{Av} + \mathbf{W}\underline{\mathbf{x}}^{t-1}.$$

By Equation (1),

$$\mathbf{x}^{k(s)} - \underline{\mathbf{x}}^k = \mathbf{W}^k \left(\mathbf{e}^{(s)} - \underline{\mathbf{x}}^0\right) = (e^{(s)} - \underline{x})\mathbf{W}^k\mathbf{1}$$

and

$$\overline{\mathbf{x}}^k - \underline{\mathbf{x}}^k = \mathbf{W}^k \left(\overline{\mathbf{x}}^0 - \underline{\mathbf{x}}^0\right) = (\overline{x} - \underline{x})\mathbf{W}^k\mathbf{1},$$

where $\underline{\mathbf{x}}^0 = \underline{x}\,\mathbf{1}$ and $\overline{\mathbf{x}}^{\mathbf{0}} = \overline{x}\,\mathbf{1}$ due to the fact that the strategy space is common, and $\mathbf{e}^{(s)} = e^{(s)}\,\mathbf{1}$ by individually homogeneous beliefs. Then,

$$\begin{aligned}
\mathbf{x}^{k(s)} - \underline{\mathbf{x}}^k &= \mathbf{W}^{k-1}\left(\mathbf{x}^{1(s)} - \underline{\mathbf{x}}^1\right) \\
&= \mathbf{W}^{k-1}\left(\mathbf{A}\mathbf{v} + e^{(s)}\mathbf{W}\mathbf{1} - \mathbf{A}\mathbf{v} - \mathbf{W}\underline{\mathbf{x}}^0\right) \\
&= \mathbf{W}^{k-1}\left(e^{(s)}\mathbf{W}\mathbf{1} - \underline{x}\mathbf{W}\mathbf{1}\right) \\
&= \left(e^{(s)} - \underline{x}\right)\mathbf{W}^k\mathbf{1} \\
&= \frac{e^{(s)} - \underline{x}}{\overline{x} - \underline{x}}\left(\overline{x} - \underline{x}\right)\mathbf{W}^k\mathbf{1} \\
&= \frac{e^{(s)} - \underline{x}}{\overline{x} - \underline{x}}\mathbf{W}^k\left(\overline{\mathbf{x}}^0 - \underline{\mathbf{x}}^0\right) \\
&= \frac{e^{(s)} - \underline{x}}{\overline{x} - \underline{x}}\left(\overline{\mathbf{x}}^k - \underline{\mathbf{x}}^k\right).
\end{aligned}$$

The optimal choice $x_i^{k(s)}$ of subject $s$ is the $i$-th entry of $\mathbf{x}^{k(s)}$. It follows then from the former equation that

$$\frac{x_i^{k(s)} - \underline{x}_i^k}{\overline{x}_i^k - \underline{x}_i^k} = \frac{e^{(s)} - \underline{x}}{\overline{x} - \underline{x}}.$$

Since $\tilde{e}^{(i)}$ is independent of the reasoning level $k$ by Assumption 2,

$$\frac{\tilde{x}_i^k - \underline{x}_i^k}{\overline{x}_i^k - \underline{x}_i^k} \overset{\mathcal{D}}{=} \frac{\tilde{e}^{(i)} - \underline{x}}{\overline{x} - \underline{x}}.$$

Thus, $\tilde{x}_i^k \overset{\mathcal{S}}{=} \tilde{e}^{(i)}$ for each role $i \in N$. $\square$

**Proof of Theorem 2.** We have for each level-$k$ subject $s$ playing role $i$ that

$$\begin{aligned}
\frac{x_i^{k(s)} - \underline{x}_i^k}{\overline{x}_i^k - \underline{x}_i^k} &= \frac{\left(\mathbf{W}^k\right)_i\left(\mathbf{e}^{(s)} - \underline{\mathbf{x}}^0\right)}{\left(\mathbf{W}^k\right)_i\left(\overline{\mathbf{x}}^0 - \underline{\mathbf{x}}^0\right)} \\
&= \frac{\sum_{j=1}^n (\mathbf{W}^k)_{ij}\left(e_j^{(s)} - \underline{x}\right)}{\sum_{j=1}^n (\mathbf{W}^k)_{ij}\left(\overline{x} - \underline{x}\right)} \\
&\underset{k\to\infty}{\to} \frac{\sum_{j=1}^n \gamma_j\left(e_j^{(s)} - \underline{x}\right)}{\sum_{j=1}^n \gamma_j\left(\overline{x} - \underline{x}\right)} \\
&= \frac{\sum_{j=1}^n \gamma_j e_j^{(s)} - \underline{x}}{\overline{x} - \underline{x}},
\end{aligned}$$

where $\gamma_j > 0$ for all $j \in N$ and $\sum_{j=1}^{n} \gamma_j = 1$. The limit follows because $\mathbf{W}$ is primitive and, by the Perron-Frobenius theorem, it has a unique dominant eigenvalue $\lambda_1 > 0$ ($\lambda_1 > |\lambda_j|$ for $j = 2, 3, \ldots, n$) whose associated right and left eigenvectors, $\mathbf{r}$ and $\boldsymbol{\gamma}$, with $\sum_{j=1}^{n} \gamma_j = 1$, are strictly positive. Thus, for high $k$, $(\mathbf{W}^k)_{ij} \sim C\, \lambda_1^k r_i \gamma_j$ for some constant $C > 0$ independent of $k$. By definition, $\tilde{\mu}^{(i)} = \sum_{j=1}^{n} \gamma_j\, \tilde{e}_j^{(i)}$ is the random variable of average beliefs on the domain $[\underline{x}, \overline{x}]$. Since $\tilde{\mu}^{(i)}$ is independent of the reasoning level $k$ by Assumption 2, it follows that

$$\frac{\tilde{x}_i^k - \underline{x}_i^k}{\overline{x}_i^k - \underline{x}_i^k} \xrightarrow{\mathcal{D}} \frac{\tilde{\mu}^{(i)} - \underline{x}}{\overline{x} - \underline{x}}.$$

This means that $\tilde{x}_i^k \xrightarrow{\mathcal{S}} \tilde{\mu}^{(i)}$ for each role $i \in N$. □

**Proof of Proposition 2.** Apply Theorem 2 to see that for all $i \in N$, $\tilde{x}_i^k \xrightarrow{\mathcal{S}} \tilde{\mu}^{(i)}$, where $\gamma_j \geq 0$ for all $j \in N$ and $\sum_j \gamma_j = 1$. Let $m_j^{(i)} = \mathbb{E}[\tilde{e}_j^{(i)}]$, ${\sigma_j^{(i)}}^2 = \mathrm{Var}(\tilde{e}_j^{(i)})$ and $\mathbf{m}^{(i)} = (m_1^{(i)}, \ldots, m_n^{(i)})$. Also, define

$$m^{(i)} \equiv \mathbb{E}[\tilde{\mu}^{(i)}] = \sum_{j=1}^{n} \gamma_j\, m_j^{(i)}$$

and

$${s^{(i)}}^2 \equiv \mathrm{Var}(\tilde{\mu}^{(i)}) = \sum_j \gamma_j^2\, {\sigma_j^{(i)}}^2,$$

where the last equality follows because the random variables $\tilde{e}_1^{(i)}, \ldots, \tilde{e}_n^{(i)}$ are independent by assumption. We first show that $(\tilde{\mu}^{(i)} - m^{(i)})/s^{(i)} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$ by applying Lyapunov's Central Limit Theorem to the independent (but not identically distributed) random variables $\gamma_1 \tilde{e}_1^{(i)}, \ldots, \gamma_n \tilde{e}_n^{(i)}$. To shorten notation, let $\tilde{\mathbf{z}}^{(i)} = \tilde{\mathbf{e}}^{(i)} - \mathbf{m}^{(i)}$. Lyapunov's CLT requires the existence of $\delta > 0$ (independent of $n$) such that

$$\lim_{n \to \infty} \frac{1}{{s^{(i)}}^{2+\delta}} \sum_{j=1}^{n} \mathbb{E}\left[\left|\gamma_j\, \tilde{e}_j^{(i)} - \gamma_j\, m_j^{(i)}\right|^{2+\delta}\right] = 0.$$

We find an upper bound that tends to zero. Note that $\tilde{\mathbf{z}}^{(i)}$ is bounded because both $\tilde{\mathbf{e}}^{(i)}$ and $\mathbf{m}^{(i)}$ are. So let $D \geq |\tilde{z}_j^{(i)}|$ for all $i, j \in N$, where $D$ is fixed, independent of $n$. Then,

$$\begin{aligned}
\frac{1}{{s^{(i)}}^{2+\delta}} \sum_{j=1}^{n} \mathbb{E}\left[\left|\gamma_j\, \tilde{e}_j^{(i)} - \gamma_j\, m_j^{(i)}\right|^{2+\delta}\right] &= \frac{1}{{s^{(i)}}^{2+\delta}} \sum_{j=1}^{n} \gamma_j^{2+\delta}\, \mathbb{E}\left[\left|\tilde{z}_j^{(i)}\right|^{2+\delta}\right] \\
\leq \frac{1}{{s^{(i)}}^{2+\delta}} \sum_{j=1}^{n} (D\, \gamma_j)^{\delta}\, \gamma_j^2\, \mathbb{E}\left[\left(\tilde{z}_j^{(i)}\right)^2\right] &\leq \frac{1}{{s^{(i)}}^{2+\delta}} (D\, \bar{\gamma})^{\delta} \sum_{j=1}^{n} \gamma_j^2\, \mathbb{E}\left[\left(\tilde{z}_j^{(i)}\right)^2\right] \\
= \frac{1}{{s^{(i)}}^{2+\delta}} (D\, \bar{\gamma})^{\delta}\, {s^{(i)}}^2 &= \left(\frac{D\, \bar{\gamma}}{s^{(i)}}\right)^{\delta}
\end{aligned}$$

Since $D$ and $\delta$ are independent of $n$, Lyapunov's condition is satisfied if $\bar{\gamma}/s^{(i)} \to 0$ as $n \to \infty$. As $n \to \infty$, we have that

$$\frac{\bar{\gamma}}{s^{(i)}} = \frac{\bar{\gamma}}{\sqrt{\sum_j \gamma_j^2 {\sigma_j^{(i)}}^2}} = \frac{1}{\sqrt{\sum_j \overline{\gamma}_j^2 {\sigma_j^{(i)}}^2}} \to 0,$$

where the limit follows from $\sum_j \overline{\gamma}_j^2 {\sigma_j^{(i)}}^2 \to \infty$. Thus, $(\tilde{\mu}^{(i)} - m^{(i)})/s^{(i)} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$, which is equivalent to $\tilde{\mu}^{(i)} \xrightarrow{\mathcal{S}} \mathcal{N}(0,1)$. Since $\tilde{x}_i^k \xrightarrow{\mathcal{S}} \tilde{\mu}^{(i)}$ by Theorem 2, we obtain that $\tilde{x}_i^k \xrightarrow{\mathcal{S}} \mathcal{N}(0,1)$. Then, since the normal distribution is a stable distribution (*i.e.*, a normal random variable remains normal after affine transformations), we conclude that $\tilde{x}_i^k \xrightarrow{\mathcal{D}} \mathcal{N}$. In particular,

$$\tilde{x}_i^k \xrightarrow{\mathcal{D}} \mathcal{N}\left(\underline{x}_i^k + \frac{\overline{x}_i^k - \underline{x}_i^k}{\overline{x} - \underline{x}}(m^{(i)} - \underline{x}), \left(\frac{\overline{x}_i^k - \underline{x}_i^k}{\overline{x} - \underline{x}}\right)^2 {s^{(i)}}^2\right).$$

Finally, given that the support of $\tilde{x}_i^k$ is $U_i^k = [\underline{x}_i^k, \overline{x}_i^k]$, the limiting distribution of $\tilde{x}_i^k$ is the corresponding truncated normal in $U_i^k$. □

**Proof of Corollary 1.** We have that

$$\lim_{n\to\infty} \sum_{j=1}^{n} \overline{\gamma}_j(n)^2 \, {\sigma_j^{(i)}}^2(n) \geq \varepsilon \lim_{n\to\infty} \sum_{j=1}^{n} \overline{\gamma}_j(n)^2 = \varepsilon \lim_{n\to\infty} \frac{1}{\overline{\gamma}(n)^2} \sum_{j=1}^{n} \gamma_j(n)^2 \geq \varepsilon \lim_{n\to\infty} \frac{1}{\overline{\gamma}(n)^2} \sum_{j=1}^{n} \frac{1}{n^2} =$$
$$= \varepsilon \lim_{n\to\infty} \frac{1}{\overline{\gamma}(n)^2 \, n} = \infty.$$

This concludes the proof. □

**Proof of Proposition 3.** The matrix $\mathbf{W}$ is regular and, therefore, $\mathbf{W1} = \lambda_1 \mathbf{1}$ by the Perron-Frobenius theorem, where $\lambda_1$ is the dominant eigenvalue of $\mathbf{W}$. Also $\lambda_1 = q$ because $\lambda_1(\mathbf{H}) = 1$ by the regularity of the baseline matrix $\mathbf{H}$. Since for all $i \in N$ and all $k \in \mathbb{N}_+$, $U_i^{k+1}$ is a subinterval of $U_i^k$ , the vector $\mathbf{D}^k \equiv (D_i^k)_{i=1}^n$ that contains the $k$-th discrimination sets of all players is equal to

$$\mathbf{D}^k = \begin{cases} [\underline{\mathbf{x}}^k, \underline{\mathbf{x}}^{k+1}] \cup [\overline{\mathbf{x}}^{k+1}, \overline{\mathbf{x}}^k] & \text{if} \quad k < K \\ [\underline{\mathbf{x}}^K, \overline{\mathbf{x}}^K] & \text{if} \quad k = K. \end{cases}$$

Given a reasoning level $k < K$, the sizes of the $k$'th discrimination sets $|\mathbf{D}^k| \equiv (|D_i^k|)_{i=1}^n$ are given by the expression

$$\begin{aligned}|\mathbf{D}^k| &= (\underline{\mathbf{x}}^{k+1} - \underline{\mathbf{x}}^k) + (\overline{\mathbf{x}}^k - \overline{\mathbf{x}}^{k+1}) \\ &= (\overline{\mathbf{x}}^k - \underline{\mathbf{x}}^k) - (\overline{\mathbf{x}}^{k+1} - \underline{\mathbf{x}}^{k+1}) \\ &= (\overline{x} - \underline{x})\left(\mathbf{W}^k\mathbf{1} - \mathbf{W}^{k+1}\mathbf{1}\right) \\ &= (\overline{x} - \underline{x})\lambda_1^k(1-\lambda_1)\mathbf{1}.\end{aligned}$$

Similarly, $|\mathbf{D}^K| = (\overline{x} - \underline{x})\lambda_1^K\mathbf{1}$. Hence,$|D_i^k|$ is independent of the player role $i$, which allows us to write the objective function to be maximized as follows:

$$L(\lambda_1; K) \equiv \prod_{i=1}^{n}\prod_{k=0}^{K}|D_i^k| = \left((\overline{x} - \underline{x})^{K+1}\lambda_1^{\frac{K(K+1)}{2}}(1-\lambda_1)^K\right)^n.$$

We finally show that the global maximum is attained at $\hat{\lambda}_1 = (K+1)/(K+3)$. Since $\lambda_1 \in [0,1]$, the base of the exponentiation in $L(\lambda_1; K)$ is non-negative on the whole domain of $\lambda_1$, which implies that the we can maximize instead the function $\tilde{L}(\lambda_1; K) = \lambda_1^{\frac{K(K+1)}{2}}(1-\lambda_1)^K$ in $\lambda_1$. It follows from straightforward calculus that the unique critical point of $\tilde{L}(\lambda_1; K)$ is $\hat{\lambda}_1 = (K+1)/(K+3)$. Finally, since $L(0; K) = L_2(1; K) = 0$ and since $\tilde{L}$ is continuous and non-negative on $\lambda_1 \in [0,1]$, it must be the case that $\hat{\lambda}_1 = (K+1)/(K+3)$ is a global maximum. $\square$