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Abstract

This paper examines the conditions for the existence of a maximal element of a relation on every nonempty compact subset of its ground set. A preliminary analysis establishes some connections between the maximals of a relation and those of its right trace. Via this analysis, various results of the literature are unified by identifying a common property of their assumptions that concerns the right trace of the transitive closure of the objective relation. Next, a generalization is provided so as to accommodate some relations of interest to economics. Finally, a necessary and sufficient condition is presented for the existence of a maximal on every nonempty compact subset of the ground set of a relation.

Keywords: Maximal element; Existence; Right trace; Transitive closure; Suzumura-consistency.

JEL Classification: C60, C61, D11.

1 Introduction

The maximal elements of a preference relation on a set of feasible alternatives are often interpreted as the optimal choices of a rational agent. Given this interpretation, any set of conditions that guarantees the existence of a maximal of a relation supplies us with information about the circumstances that allow for an optimal choice. Some authors, however, are inclined to evaluate the importance of these sets of conditions according to their capability to be applied to parametric optimization problems of economic interest. In this regard, Walker (1977) observed:

> *We are generally interested not in a maximal element of just a single set X, but rather in a whole family $\mathcal{F}$ of subsets of some underlying set X, and in whether each member of $\mathcal{F}$ has a maximal element.*

Walker's observation was motivated by the structure of fundamental problems of economic theory like, for instance, the nonemptiness of the consumer's demand correspondence. In the basic version of that problem, the consumer is endowed with a nonnegative level of wealth w and with a preference relation B defined on a commodity space X identified with the nonnegative orthant of some n-dimensional Euclidean space. As long as all components of the market price vector p are positive and the consumption set Y is a nonempty closed subset of the commodity space X, the image of the demand correspondence is the set of maximals of B on a budget set $\{y \in Y : p \cdot y \leq w\}$ that is nonempty and compact. Without additional assumptions, the budget set could be any nonempty compact subset of the ground set of B and the issue of the non-vacuity of consumer's choice for every admissible configuration of the triple (Y, p, w) boils down, simply, to the abstract problem of the existence of a maximal of B on every nonempty compact subset of the ground set of B. The present work deals, precisely, with such abstract problem.

Alexander Doniphan Wallace proved, in a 1945 article on fixed points (see Wallace (1945)), that any transitive and reflexive relation with closed upper sections possesses a maximal on every nonempty compact subset of its ground set. That result—which, in fact, is asserted under additional assumptions of topological nature—is presented as one of many auxiliary lemmas of Wallace's article. Only some years later Lewis Edes Ward, Jr.—a student of Wallace—reformulated it as a formal theorem on the existence of a maximal element in his 1954 article on partially ordered topological spaces (see Ward (1954)). In the literature, the result obtained in the 1945 article by Wallace is sometimes reckoned "a folk theorem in optimization theory" (see, e.g., Evren and Ok (2011)) and, indeed, the very Wallace (see Wallace (1962)) seemed to share that opinion when claiming that his result "was certainly to be obtained by any mathematician who was interested in these matters". It is a fact, however, that it was first asserted and proved in Wallace (1945).

About a quarter of a century later, a series of works—among them, the article by Walker (1977) mentioned at the beginning of this Introduction—contributed to the formulation of a result according to which any acyclic relation with open lower sections has a maximal element on every nonempty compact subset of its ground set. Subsequently, at the end of the nineties, that result has been generalized in Subiza and Peris (1997) by relaxing the openness of lower sections. The natural question is: What connection is there between the old existence result that assumed the closedness of the upper sections of a preorder relation and the relatively new results assuming the openness of the lower sections of an acyclic relation (or its weakening introduced in Subiza and Peris (1997))? This work answers to the previous question by proving that:

> *The existence of a maximal element of the right trace of the transitive closure of a Suzumura-consistent relation B implies the existence of a maximal element of B.*

As right traces are transitive and reflexive by nature, the closedness of their upper sections implies the existence of a maximal element of the right trace by virtue of Wallace's result: observing that a relation B that satisfies the assumptions of any existence result mentioned sofar is Suzumura-consistent and has a transitive closure whose right trace possesses closed upper sections, the desired connection is readily established. In the light of this fact, the Suzumura-consistent relations whose transitive closure possesses a right trace with closed upper sections will be called relations with the W-property. It must be remarked that the previous observation about the closedness of the upper sections of the right trace of the transitive closure of a relation is far from being obvious and that—for the important case of a relation with open lower sections—it has been first proved in Banks et al. (2006, Proposition A.4): the application of such observation in Duggan (2011) testifies to its value to the economic literature on the existence of maximal elements. Further, it must be noted that the idea of using one-sided traces to prove the existence of undominated maximals—a selection of unconstrained maximals due to Peris and Subiza (2002)—is pursued in Alcantud et al. (2010); however, the use made in that article and the purposes thereof differ substantially from those of the present paper.

This work considers also a weakening of the W-property that is satisfied by some relations of interest to economics and vector optimization (like, e.g., transitive—but possibly not reflexive—relations with closed upper sections, lexicographic order relations on product spaces endowed with their natural topology, relations induced by strictly supported cones of real topological vector spaces and a class of justifiable preference relations) and proves the sufficiency of that condition for a relation to possess a maximal on every nonempty compact subset of its ground set. The mentioned weakening of the W-property is, in fact, a generalization of the notion of a quasi upper semicontinuous preorder recently introduced in Bosi and Zuanon (2017): the generalization concerns both the order-theoretic and the topological conditions imposed in the definition of a quasi upper semicontinuous preorder. Interestingly, a relativized version of the W-property—that further weakens the definition of the W-property—turns out to be necessary and sufficient for a relation to possess a maximal on every nonempty compact subset of its ground set.

The paper is structured as follows. Sect. 2 recalls some definitions and notation. Sect. 3 investigates the connections between the set of maximals of a relation and those of its right trace. Sect. 4 examines the conditions for the closedness of the upper section of the right trace of a relation. Sect. 5 introduces the W-property and shows an existence result that unifies those in Wallace (1945) and Walker (1977), as well as others. Sect. 6 contains some mathematical facts of topological nature. Sect. 7 introduces a weakening of the W-property and proves a generalization of the aforementioned unifying existence result that accommodates some relations of interest to economics and vector optimization; notably, Sect. 7 proves that the reflexivity assumption in Wallace's result on the existence of a maximal element can be simply dropped. Sect. 8 provides a necessary and sufficient condition for the existence of a maximal on every nonempty compact subset of the ground set of a relation.

2 Preliminaries

A **relation on a set** X is a subset of $X \times X$, where the second factor of the Cartesian product $X \times X$ is here understood as the domain of the relation and the first as its codomain. Let B be a relation on a set X. When $(y, x) \in B$, we say that y is **related through** B **to** x. The set of all elements related through B to $x \in X$ is denoted by

$$B(x) = \{y \in X : (y, x) \in B\}$$

and is called the **upper section of** B **at** x.

2.1 Operations

In this Sect. 2.1, the letter X denotes a set, the letter Y denotes a subset of X while the letters B and R denote relations on X. The relation B is also alternatively denoted by B^1. The **restriction of** B **to** Y is the relation $B|_Y$ on Y defined by

$$B|_Y = B \cap (Y \times Y).$$

The **composition of** B **with** R is the relation $R \circ B$ on X defined by

$$R \circ B = \{(z, x) \in X \times X : (z, y) \in R \text{ and } (y, x) \in B \text{ for some } y \in X\}.$$

The n**-power of** B is the relation B^n on X recursively defined by

$$B^n = B \circ B^{n-1} \text{ for every integer } n > 1.$$

The **converse of** B is the relation $B^{\mathfrak{c}}$ on X defined by[1]

$$B^{\mathfrak{c}} = \{(x, y) \in X \times X : (y, x) \in B\}.$$

The **transitive closure of** B is the relation $B^{\mathfrak{t}}$ on X defined by

$$B^{\mathfrak{t}} = \bigcup_{n=1}^{+\infty} B^n.$$

The **reflexive closure of** B is the relation $B^{\mathfrak{r}}$ on X defined by

$$B^{\mathfrak{r}} = B \cup \{(x, x) : x \in X\}.$$

The **irreflexive part of** B is the relation $B^{\mathfrak{i}}$ on X defined by

$$B^{\mathfrak{i}} = B \backslash \{(x, x) : x \in X\}.$$

The **asymmetric part of** B is the relation $B^{\mathfrak{a}}$ on X defined by

$$B^{\mathfrak{a}} = B \backslash B^{\mathfrak{c}}.$$

The **right trace of** B is the relation $\mathcal{T}_B$ on X defined by

$$\mathcal{T}_B = \{(y, x) \in X \times X : B(y) \subseteq B(x)\}.$$

So, $y \in \mathcal{T}_B(x) \Leftrightarrow B(y) \subseteq B(x)$. The notion of a trace is often ascribed to Luce (1956) and Luce (1958). One-sided decompositions of a trace into the left and right trace are formulated in Doignon et al. (1986): the definition of a right trace adopted here is, exactly, that of the last mentioned article[2] as well as the one provided in Aleskerov et al. (2007, p. 69).

Notation 1 *Given a relation* B *and* $(\mathfrak{p}, \mathfrak{q}) \in \{\mathfrak{a}, \mathfrak{c}, \mathfrak{i}, \mathfrak{r}, \mathfrak{t}\} \times \{\mathfrak{a}, \mathfrak{c}, \mathfrak{i}, \mathfrak{r}, \mathfrak{t}\}$*, we henceforth write* $B^{\mathfrak{p}\mathfrak{q}}$ *instead of* $(B^{\mathfrak{p}})^{\mathfrak{q}}$*.*

[1] The upper section of $B^{\mathfrak{c}}$ at x is called the **lower section of** B **at** x.

[2] Some authors (see, e.g., Bouyssou and Marchant (2011) or Bouyssou and Doignon (2020)) have used an inverted nomenclature for one-sided traces. Further, in the economic literature, one-sided traces and traces appear also under alternative names, like *umbra* or *transitive core*.

2.2 Topological definitions

Let (X, τ) be a topological space and B be a relation on X. Denote by σ the product topology of $X \times X$. We say that: B is **open-valued for** τ iff $B(x)$ is τ-open for every $x \in X$; B is **closed-valued for** τ iff $B(x)$ is τ-closed for every $x \in X$; B is **graph-open for** τ iff B is σ-open; B is **graph-closed for** τ iff B is σ-closed. The graph-closedness (graph-openness) for τ of B implies the closed-valuedness (open-valuedness) for τ of both B and $B^{\mathfrak{c}}$.

Notation 2 *Let (X, τ) be a topological space. The **collection of all non-empty τ-compact subsets of** X is denoted by $\mathcal{K}(X, \tau)$.*

2.3 Quasi-extensions

Let B be a relation on a set X. A relation R is a **quasi-extension** of B iff R is a relation on X such that $B^{\mathfrak{a}} \subseteq R^{\mathfrak{a}}$. When R is a quasi-extension of B, we also say that B admits the quasi-extension R and that R quasi-extends B. The idempotence of the operation of asymmetrization implies the equivalence of the inclusions $B^{\mathfrak{a}} \subseteq R^{\mathfrak{a}}$ and $B^{\mathfrak{aa}} \subseteq R^{\mathfrak{a}}$: we thus infer the conclusion in Remark 1.

Remark 1 *Let B and R be relations. Then, R is a quasi-extension of B if and only if R is a quasi-extension of $B^{\mathfrak{a}}$.*

2.4 Relations

A relation B on a set X is: **asymmetric** iff $B = B^{\mathfrak{a}}$; **irreflexive** iff $B = B^{\mathfrak{i}}$; **reflexive** iff $B = B^{\mathfrak{r}}$ (iff $\mathcal{I}_B \subseteq B$); **transitive** iff $B = B^{\mathfrak{t}}$ (iff $B \subseteq \mathcal{T}_B$, iff $B \circ B \subseteq B$); **acyclic** iff $B^{\mathfrak{t}}$ is asymmetric (iff $B^{\mathfrak{t}}$ is irreflexive, iff $B^{\mathfrak{t}} = B^{\mathfrak{ta}}$, iff $B^{\mathfrak{t}} = B^{\mathfrak{ti}}$); a **strict partial order** iff B is transitive and asymmetric (iff B is transitive and irreflexive, iff $B = B^{\mathfrak{ta}}$, iff $B = B^{\mathfrak{ti}}$); a **preorder** iff B is transitive and reflexive (iff $B = \mathcal{T}_B$); **total** iff $X \times X = B \cup B^{\mathfrak{c}}$; **connex** iff $B^{\mathfrak{r}}$ is total; a **strict total order** iff B is a connex strict partial order.

2.5 Maximals

Let B be a relation on a set X and $Y \subseteq X$. The **set of B-maximals on** Y is the set $\mathcal{M}(B, Y)$ defined by

$$\mathcal{M}(B, Y) = \{y \in Y : B^{\mathfrak{a}}(y) \cap Y = \emptyset\}.$$

A member of $\mathcal{M}(B, Y)$ is called a **B-maximal on** Y: when the specification of the constraint set Y is immaterial, it is called a B-maximal; when even the specification of the relation is immaterial, it is called a maximal. The **set of unconstrained B-maximals** is the set $\mathcal{M}(B)$ defined by

$$\mathcal{M}(B) = \{x \in X : B^{\mathfrak{a}}(x) = \emptyset\}.$$

A member of $\mathcal{M}(B)$ is called an **unconstrained B-maximal**: when the specification of the relation B is immaterial, it is called an unconstrained maximal. Proposition 1 points out that unconstrained maximals are special types of maximals and that any maximal can be expressed as an unconstrained maximal. Proposition 2 recalls that, when the constraint set Y is an upper section, the set of maximals of a transitive relation is just the intersection of the set of its unconstrained maximals with the constraint set.

Proposition 1 *Let B be a relation on a set X and $Y \subseteq X$.*

1. $\mathcal{M}(B) = \mathcal{M}(B, X)$.
2. $\mathcal{M}(B, Y) = \mathcal{M}(B|_Y)$.

Proof. Part 1 of Proposition 1 is obvious. The proof of part 2 of Proposition 1 is as follows. The operations of restriction and asymmetrization commute with each other. So, $B^{\mathfrak{a}}|_Y = B|_Y^{\mathfrak{a}}$. As the equality $B^{\mathfrak{a}}(y) \cap Y = B^{\mathfrak{a}}|_Y(y)$ holds true for every $y \in Y$, we have that $B^{\mathfrak{a}}(y) \cap Y = B|_Y^{\mathfrak{a}}(y)$ for every $y \in Y$. Consequently, $\mathcal{M}(B, Y) = \mathcal{M}(B|_Y)$. ■

Proposition 2 *Let B be a transitive relation on a set X and $Y \subseteq X$. If $Y = B(x)$ for some $x \in X$, then $\mathcal{M}(B, Y) = \mathcal{M}(B) \cap Y$.*

Proof. Assume the existence of $x \in X$ such that $Y = B(x)$. The inclusion $\mathcal{M}(B) \cap Y \subseteq \mathcal{M}(B, Y)$ is obvious. We prove only the inclusion $\mathcal{M}(B, Y) \subseteq \mathcal{M}(B) \cap Y$, as follows. Suppose $y \in \mathcal{M}(B, Y)$. The membership $y \in \mathcal{M}(B, Y)$ and the equality $Y = B(x)$ entail that $y \in B(x)$ and $B^{\mathfrak{a}}(y) \cap B(x) = \emptyset$. As $y \in B(x)$, the transitivity of B implies $B(y) \subseteq B(x)$: so $B^{\mathfrak{a}}(y) \subseteq B(x)$ in that $B^{\mathfrak{a}}(y) \subseteq B(y)$. As $B^{\mathfrak{a}}(y) \cap B(x) = \emptyset$, the inclusion $B^{\mathfrak{a}}(y) \subseteq B(x)$ implies $B^{\mathfrak{a}}(y) = \emptyset$. We are in a position to conclude that $y \in \mathcal{M}(B) \cap Y$. ■

2.6 Cones

Let X be a real vector space. A **cone** of X is a subset C of X such that $\lambda c \in C$ for every $c \in C$ and every scalar $\lambda > 0$. So, a cone can be empty and need not contain the zero vector. The union and intersection of an arbitrary—possibly empty—family of cones of a real vector space are cones. Let C be a cone of X. The **asymmetric part of** C is the cone $C^\bullet$ defined by

$$C^\bullet = C \setminus -C.$$

The **relation induced by** C is the relation B on X defined by

$$B(x) = x + C \text{ for all } x \in X,$$

with $x + C$ denoting the Minkowski sum of $\{x\}$ and C. Henceforth in this Sect. 2.6, let (X, τ) be a real topological vector space with a topology τ. The topological closure of a subset S of X for the topology τ is denote by $\mathrm{cl}_\tau S$. A homogeneous τ-open (τ-closed) half-space of X is a cone H of X satisfying the equality $H = \{x \in X : f(x) > 0\}$ ($H = \{x \in X : f(x) \geq 0\}$) for some non-zero continuous linear functional $f : X \to \mathbb{R}$, where $\mathbb{R}$ is endowed with the natural topology. A cone C of a real topological vector space X is **strictly τ-supported** iff $C^\bullet$ is included in a homogeneous τ-open half-space of X.

3 Connections between B- and $\mathcal{T}_B$-maximality

In general, the existence of a B-maximal neither implies nor is implied by the existence of a $\mathcal{T}_B$-maximal. Remark 2 clarifies the point. The rest of this Sect. 3 investigates the connections between B- and $\mathcal{T}_B$-maximality in the case of a transitive relation B and derives a property of Suzumura-consistent relations.

Remark 2 *It is well possible that a relation B possesses an empty set of unconstrained B-maximals and a nonempty set of unconstrained $\mathcal{T}_B$-maximals: consider, for instance, the relation B on $X = \mathbb{Z}$ specified by*

$$B(x) = \{x + 1\} \text{ for all } x \in X$$

and observe that

$$\emptyset = \mathcal{M}(B) \subset \mathcal{M}(\mathcal{T}_B) = X.$$

Conversely, it is well possible that a relation B possesses a nonempty set of unconstrained B-maximals and an empty set of unconstrained $\mathcal{T}_B$-maximals: consider, for instance, the relation B on $X = \mathbb{Z}$ specified by

$$B(x) = \{0\} \cup \{y \in X : |y| > |x|\} \text{ for all } x \in X$$

and observe that

$$\emptyset = \mathcal{M}(\mathcal{T}_B) \subset \mathcal{M}(B) = \{0\}.$$

Remark 3 *The sets of B- and $\mathcal{T}_B$-maximals illustrated in Remark 2 are nested. In general, however, they need not be so. Indeed, it is well possible that a relation B possesses* nonempty and disjoint *sets of unconstrained B- and $\mathcal{T}_B$-maximals. Consider, for instance, the relation B on $X = \{0, 1, 2\}$ specified by*

$$B(x) = \begin{cases} \{2\} & \text{if } x = 0 \\ \{1, 2\} & \text{if } x \neq 0 \end{cases}$$

and observe that the nonempty set $\mathcal{M}(B) = \{1, 2\}$ is disjoint from the nonempty set $\mathcal{M}(\mathcal{T}_B) = \{0\}$. Indeed, $\{\mathcal{M}(B), \mathcal{M}(\mathcal{T}_B)\}$ is a partition of X.

3.1 Transitive relations and maximality

This Sect. 3.1 proves that the existence of a $\mathcal{T}_B$-maximal for a transitive relation B implies the existence of a member of the constraint set that is both a $\mathcal{T}_B$-maximal and a B-maximal.[3] Remark 4 refutes the conjecture that the existence of a B-maximal for a transitive relation B implies the existence of a $\mathcal{T}_B$-maximal.

Theorem 1 *Let B be a transitive relation on a set X and $Y \subseteq X$. Assertions I and II are equivalent.*

I. $\mathcal{M}(\mathcal{T}_B, Y) \neq \emptyset$

II. $\mathcal{M}(\mathcal{T}_B, Y) \cap \mathcal{M}(B, Y) \neq \emptyset$.

Proof. The implication $II \Rightarrow I$ is obvious and hence we prove only the implication $I \Rightarrow II$, as follows. Suppose $\mathcal{M}(\mathcal{T}_B, Y) \neq \emptyset$ and pick

$$y \in \mathcal{M}(\mathcal{T}_B, Y). \tag{1}$$

Then $y \in Y$ and

$$B(v) \subset B(y) \text{ for no } v \in Y. \tag{2}$$

[3]Theorem 1 does not assert that every $\mathcal{T}_B$-maximal of a transitive relation B is a B-maximal: Remark 5 of Sect. 3.2 tacitly shows that such an assertion is false.

If $y \in \mathcal{M}(B, Y)$, then we are done. So, henceforth suppose $y \notin \mathcal{M}(B, Y)$. Then $B^{\mathfrak{a}}(y) \cap Y$ is nonempty and hence there exists an element

$$z \in B^{\mathfrak{a}}(y) \tag{3}$$

such that

$$z \in Y. \tag{4}$$

As $B^{\mathfrak{a}} \subseteq B$, the membership in (3) implies $z \in B(y)$: thus $B(z) \subseteq B(y)$ by the transitivity of B. The last inclusion and the memberships in (1) and (4) imply $z \in \mathcal{M}(\mathcal{T}_B, Y)$: to conclude the proof, it then suffices to show that $z \in \mathcal{M}(B, Y)$. By way of contradiction, suppose $z \notin \mathcal{M}(B, Y)$. Then $B^{\mathfrak{a}}(z) \cap Y$ is nonempty and hence there exists an element

$$x \in B^{\mathfrak{a}}(z) \tag{5}$$

such that

$$x \in Y. \tag{6}$$

The membership in (5) implies $z \notin B(x)$: thus $B(x) \neq B(y)$ in that $z \in B(y)$. As $B^{\mathfrak{a}} \subseteq B$, the membership in (5) implies $x \in B(z)$: thus $B(x) \subseteq B(z)$ by the transitivity of B. The inequality $B(x) \neq B(y)$ and the inclusions $B(x) \subseteq B(z)$ and $B(z) \subseteq B(y)$ imply $B(x) \subset B(y)$: a contradiction with (2) and (6). ■

Corollary 1 *Let B be a transitive relation on a set X and $Y \subseteq X$. Then,*

$$\mathcal{M}(\mathcal{T}_B, Y) \neq \emptyset \Leftrightarrow \mathcal{M}(\mathcal{T}_B, Y) \cap \mathcal{M}(B, Y) \neq \emptyset \Rightarrow \mathcal{M}(B, Y) \neq \emptyset.$$

Proof. A consequence of Theorem 1 and of the obvious implication $\mathcal{M}(\mathcal{T}_B, Y) \cap \mathcal{M}(B, Y) \neq \emptyset \Rightarrow \mathcal{M}(B, Y) \neq \emptyset$. ■

Remark 4 *The converse of the one-way implication in Corollary 1 is generally false. Consider, for instance, the transitive relation B on $X = \mathbb{R}$ specified by*

$$B(x) = \begin{cases} [1, +\infty) & \text{if } x = 1 \\ [x+1, +\infty) & \text{if } x \neq 1 \end{cases}$$

and observe that

$$\mathcal{M}(\mathcal{T}_B, Y) = \emptyset \neq (0, 1] = \mathcal{M}(B, Y) \text{ when } Y = [0, 1].$$

3.2 Transitive relations and unconstrained maximality

This Sect. 3.2 proves that, in the case of a transitive relation B, the existence of an unconstrained $\mathcal{T}_B$-maximal is equivalent to the existence of an unconstrained B-maximal. Remark 5 refutes the conjecture that nonempty sets of unconstrained B- and $\mathcal{T}_B$-maximals of a transitive relation B coincide.

Theorem 2 *Let B be a transitive relation on a set X. The following assertions are equivalent.*

I. $\mathcal{M}(B) \neq \emptyset$.

II. $\mathcal{M}(B) \cap \mathcal{M}(\mathcal{T}_B) \neq \emptyset$.

Proof. The implication $II \Rightarrow I$ is obvious and hence we prove only the implication $I \Rightarrow II$, as follows. Suppose $\mathcal{M}(B) \neq \emptyset$ and pick

$$x \in \mathcal{M}(B). \tag{7}$$

The last membership implies $x \in X$. If $x \in \mathcal{M}(\mathcal{T}_B)$, then we are done. So, suppose $x \notin \mathcal{M}(\mathcal{T}_B)$. As $x \in X \backslash \mathcal{M}(\mathcal{T}_B)$, there exists $y \in X$ such that

$$B(y) \subset B(x). \tag{8}$$

It is readily seen that the validity of the equality $B(y) = \emptyset$ implies the validity of the membership $y \in \mathcal{M}(B) \cap \mathcal{M}(\mathcal{T}_B)$ and hence the validity of the desired inequality $\mathcal{M}(B) \cap \mathcal{M}(\mathcal{T}_B) \neq \emptyset$. To conclude the proof it then suffices to show that $B(y) = \emptyset$. By way of contradiction, suppose $B(y) \neq \emptyset$. Pick $z \in B(y)$. The membership $z \in B(y)$ and the inclusion in (8) imply

$$z \in B(x). \tag{9}$$

The membership $z \in B(y)$ and the transitivity of B imply $B(z) \subseteq B(y)$. As $B(z) \subseteq B(y)$, from the inclusion in (8) we infer that $B(z) \subset B(x)$. The transitivity of B and the last inclusion imply

$$x \notin B(z). \tag{10}$$

But (9) and (10) are in contradiction with (7). ■

Corollary 2 *Let B be a transitive relation on a set X. Then*

$$\mathcal{M}(\mathcal{T}_B) \neq \emptyset \Leftrightarrow \mathcal{M}(\mathcal{T}_B) \cap \mathcal{M}(B) \neq \emptyset \Leftrightarrow \mathcal{M}(B) \neq \emptyset.$$

Proof. A consequence of Corollary 1 and of Theorem 2. ■

Remark 5 *It is well possible that a transitive relation possesses nonempty sets of unconstrained B- and $\mathcal{T}_B$-maximals and that an unconstrained $\mathcal{T}_B$-maximal is not an unconstrained B-maximal: for instance, consider the transitive relation B on $X = \{0, 1\}$ specified by*

$$B(0) = \{0\} \text{ and } B(1) = \{0\}$$

and observe that

$$\{0\} = \mathcal{M}(B) \subset \mathcal{M}(\mathcal{T}_B) = \{0, 1\}.$$

Likewise, it is well possible that a transitive relation possesses nonempty sets of unconstrained B- and $\mathcal{T}_B$-maximals and that an unconstrained B-maximal is not an unconstrained $\mathcal{T}_B$-maximal: for instance, consider the transitive relation B on $X = \{0, 1\}$ specified by

$$B(0) = \emptyset \text{ and } B(1) = \{1\}$$

and observe that

$$\{0\} = \mathcal{M}(\mathcal{T}_B) \subset \mathcal{M}(B) = \{0, 1\}.$$

3.3 Basic properties of S-consistent relations

The notion of consistency considered in the sequel has been introduced in Suzumura (1976, p. 387) and has been subject to a variety of applications: see also Bossert (2008) and Bossert (2018).[4] The definition adopted here is equivalent to the original definition in Suzumura (1976): on this point, see also Duggan (1999, p. 5), where Suzumura-consistency is called *transitive-consistency*. Proposition 3 recalls known sufficient conditions for a relation to be Suzumura-consistent and is stated here for completeness and future reference. Proposition 4 recalls that every $B^{\mathfrak{t}}$-maximal of an S-consistent relation is a B-maximal.

Definition 1 *A relation B is* ***Suzumura-consistent*** *iff $B^{\mathfrak{a}} \subseteq B^{\mathfrak{ta}}$. (So, a relation is Suzumura-consistent iff it is quasi-extended by its transitive closure.) Suzumura-consistent relations will be referred to as* ***S-consistent*** *relations.*

Proposition 3 *Let B be a relation. Each of the following conditions is sufficient for B to be S-consistent.*

1. *B is transitive.*
2. *B is acyclic.*

Proof. The transitivity of B is equivalent to $B = B^{\mathfrak{t}}$ and implies $B^{\mathfrak{a}} = B^{\mathfrak{ta}}$. The acyclicity of B is equivalent to $B^{\mathfrak{t}} = B^{\mathfrak{ta}}$ and from the obvious inclusions $B^{\mathfrak{a}} \subseteq B \subseteq B^{\mathfrak{t}}$ we infer that $B^{\mathfrak{a}} \subseteq B^{\mathfrak{ta}}$. ■

Proposition 4 *Let B be an S-consistent relation on a set X and $Y \subseteq X$. Then*

$$\mathcal{M}(B^{\mathfrak{t}}, Y) \subseteq \mathcal{M}(B, Y).$$

Proof. Suppose $y \in \mathcal{M}(B^{\mathfrak{t}}, Y)$. Then $y \in Y$ and $B^{\mathfrak{ta}}(y) \cap Y = \emptyset$. So, $B^{\mathfrak{a}}(y) \cap Y = \emptyset$ by the S-consistency of B and hence $y \in \mathcal{M}(B, Y)$. ■

3.4 A further property of S-consistent relations

Theorem 3 asserts that the existence of a $\mathcal{T}_{B^{\mathfrak{t}}}$-maximal of an S-consistent relation B implies the existence of a B-maximal.

Theorem 3 *Let B be an S-consistent relation on a set X and $Y \subseteq X$. Then*

$$\mathcal{M}(\mathcal{T}_{B^{\mathfrak{t}}}, Y) \neq \emptyset \Rightarrow \mathcal{M}(B, Y) \neq \emptyset.$$

Proof. Suppose $\mathcal{M}(\mathcal{T}_{B^{\mathfrak{t}}}, Y) \neq \emptyset$. As $B^{\mathfrak{t}}$ is transitive, the last inequality and Theorem 1 imply $\mathcal{M}(B^{\mathfrak{t}}, Y) \neq \emptyset$. As B is S-consistent, the last inequality and Proposition 4 imply $\mathcal{M}(B, Y) \neq \emptyset$. ■

Remark 6 *As is clear from the relation presented in Remark 4, the converse of the one-way implication of Theorem 3 is generally false. As is clear from first (second) relation presented in Remark 5, the inclusion $\mathcal{M}(\mathcal{T}_{B^{\mathfrak{t}}}) \subseteq \mathcal{M}(B)$ ($\mathcal{M}(B) \subseteq \mathcal{M}(\mathcal{T}_{B^{\mathfrak{t}}})$) need not hold true for an S-consistent relation B.*

[4] It is worth underlining that the cited contribution define as *acyclic* those relations whose asymmetric part is acyclic according to the current nomenclature.

4 Closed-valuedness of the right trace

This Sect. 4 illustrates sufficient conditions for the right trace of (the transitive closure of) a relation to possess closed upper sections.

Definition 2 *Let (X, τ) be a topological space and B be a relation on X:*

- *B is $\mathcal{T}$***-closed-valued for** *τ iff $\mathcal{T}_B$ is closed-valued for τ;*
- *B is $\mathcal{T}_{\mathfrak{t}}$***-closed-valued for** *τ iff $\mathcal{T}_{B^{\mathfrak{t}}}$ is closed-valued for τ.*

Remark 7 *The notions of $\mathcal{T}$- and $\mathcal{T}_{\mathfrak{t}}$-closed-valuedness are independent of each other. The following examples illustrate the point. Put $X = [0, 1]$ and endow it with the subspace topology τ induced from the natural topology of $\mathbb{R}$; the S-consistent relation B on X defined by*

$$B(x) = \{1/2 + x/2\} \text{ for all } x \in X$$

is $\mathcal{T}$-closed-valued for τ, but not $\mathcal{T}_{\mathfrak{t}}$-closed-valued for τ. Put $X = \mathbb{R}$ and endow it with the natural topology; the S-consistent relation B on X defined by

$$B(x) = \begin{cases} (x, 0] & \text{if } x < 0 \\ (x, +\infty) & \text{if } x \geq 0 \end{cases}$$

is $\mathcal{T}_{\mathfrak{t}}$-closed-valued for τ, but not $\mathcal{T}$-closed-valued for τ. The last example is taken from Subiza and Peris (1997, Example 2).

4.1 Sufficient conditions for $\mathcal{T}$-closed-valuedness

Proposition 5 provides sufficient conditions for the $\mathcal{T}$-closed-valuedness of B. Proposition 5 is actually well-known: its part 1 is a consequence of the definition of a preorder while its part 2 is, exactly, the statement of Lemma 2 in Duggan (2011).[5] The proof of Lemma 2 in Duggan (2011) makes use of the theory of nets; the proof of part 2 of Proposition 5 proposed here is more elementary.

Proposition 5 *Let (X, τ) be a topological space and B be a relation on X. Then B is $\mathcal{T}$-closed-valued for τ if at least one of the following conditions holds.*

1. *B is closed-valued for τ, transitive and reflexive.*
2. *$B^{\mathfrak{c}}$ is open-valued for τ.*

Proof. 1. A consequence of the fact that $B = \mathcal{T}_B$ when B is a preorder.

2. The case $X = \emptyset$ is obvious. Assume that $X \neq \emptyset$ and $x \in X$. It is shown that $X \backslash \mathcal{T}_B(x)$ is τ-open. Suppose $y \in X \backslash \mathcal{T}_B(x)$. Then $B(y) \nsubseteq B(x)$ and there exists $z \in B(y)$ such that $z \notin B(x)$. As $z \in B(y)$, the open-valuedness for τ of $B^{\mathfrak{c}}$ implies the existence of a τ-neighborhood N_y of y such that $z \in B(t)$ for all $t \in N_y$. As $z \notin B(x)$, $B(t) \nsubseteq B(x)$ for any $t \in N_y$. So, $N_y \subseteq X \backslash \mathcal{T}_B(x)$. ■

[5] An earlier version thereof is proved in Banks et al. (2006, Proposition A.4).

4.2 Sufficient conditions for $\mathcal{T}_{\mathrm{t}}$-closed-valuedness

Proposition 6 provides sufficient conditions for the $\mathcal{T}_{\mathrm{t}}$-closed-valuedness of a relation. Definition 3 recalls the notion of lower quasi-continuity adopted in Subiza and Peris (1997) by enunciating an equivalent[6] reformulation thereof.

Definition 3 *Let (X, τ) be a topological space. A relation B on X is* ***lower quasi-continuous*** *iff the membership $(y, x) \in B^{\mathrm{c}}$ implies the existence of a τ-neighborhood N_y of y included in $B^{\mathrm{ct}}(x)$.*

Proposition 6 *Let (X, τ) be a topological space and B be a relation on X. Then B is $\mathcal{T}_{\mathrm{t}}$-closed-valued for τ if at least one of the following conditions holds.*

1. *B^{t} is closed-valued for τ and reflexive.*
2. *B is closed-valued for τ, transitive and reflexive.*
3. *B is lower quasi-continuous.*
4. *B^{ct} is open-valued for τ.*
5. *B^{c} is open-valued for τ.*

Proof. 1. The transitive closure of any relation is transitive. So, part 1 of Proposition 6 is an immediate consequence of part 1 of Proposition 5.

2. Suppose B is closed-valued for τ, transitive and reflexive. A transitive relation is equal to its own transitive closure. So, B^{t} is closed-valued for τ and reflexive. Said this, part 1 of Proposition 6 delivers the desired result.

3. Suppose B is lower quasi-continuous. The case $X = \emptyset$ is obvious. So, assume that $X \neq \emptyset$ and $x \in X$. It is shown that $X \backslash \mathcal{T}_{B^{\mathrm{t}}}(x)$ is τ-open. Suppose

$$y \in X \backslash \mathcal{T}_{B^{\mathrm{t}}}(x).$$

Then, $B^{\mathrm{t}}(y) \nsubseteq B^{\mathrm{t}}(x)$. By basic properties of the transitive closure operator, the inclusion $B(y) \subseteq B^{\mathrm{t}}(x)$ implies the inclusion $B^{\mathrm{t}}(y) \subseteq B^{\mathrm{t}}(x)$. Therefore, $B(y) \nsubseteq B^{\mathrm{t}}(x)$ and there exists $z \in B(y)$ such that $z \notin B^{\mathrm{t}}(x)$. The last membership and the lower quasi-continuity of B imply the existence of a τ-neighborhood N_y of y included in $B^{\mathrm{ct}}(z)$. The operations of transitive closure and conversion commute with each other. So, $N_y \subseteq B^{\mathrm{tc}}(z)$ and hence $z \in B^{\mathrm{t}}(t)$ for all $t \in N_y$. Since $z \notin B^{\mathrm{t}}(x)$, we infer that $B^{\mathrm{t}}(t) \nsubseteq B^{\mathrm{t}}(x)$ for any $t \in N_y$. Consequently,

$$N_y \subseteq X \backslash \mathcal{T}_{B^{\mathrm{t}}}(x).$$

We are in position to conclude that $X \backslash \mathcal{T}_{B^{\mathrm{t}}}(x)$ is τ-open.

4. Suppose B^{ct} is open-valued for τ. If $(y, x) \in B^{\mathrm{c}}$, then $(y, x) \in B^{\mathrm{ct}}$ and there exists a τ-neighborhood N_y of y included in $B^{\mathrm{ct}}(x)$ by virtue of the open-valuedness of B^{ct} for τ. So, B is lower quasi-continuous. Said this, part 3 of Proposition 6 delivers the desired result.

5. Suppose B^{c} is open-valued for τ. Then, B^{ct} is open-valued for τ in that the transitive closure of any relation that is open-valued for τ is open-valued for τ. Said this, part 4 of Proposition 6 delivers the desired result. ■

[6] The operations of transitive closure and conversion commute with each other. So, the replacement of "$B^{\mathrm{ct}}(x)$" with "$B^{\mathrm{tc}}(x)$" leaves essentially unchanged Definition 3.

Remark 8 *It is known—and it is clear from the proof of Proposition 6—that the condition in part 2 (part 4, part 5) of Proposition 6 implies the condition in part 1 (part 3, part 4) of Proposition 6; however, the converse implications do not generally hold true.*

5 Existence of maximals

A classic theorem on the existence of maximals was first asserted in Wallace (1945, pp. 414-415), recast in Birkhoff (1948, Theorem 16 at p. 63) and taken back to a formulation similar to the original one by Ward (1954, Theorem 1). Ward's version of Wallace's result essentially reads as follows.

Theorem 4 (Ward (1954, Theorem 1)) *Let (X, τ) be a nonempty compact topological space and B be a relation on X. If B is closed-valued for τ, transitive and reflexive, then $\mathcal{M}(B) \neq \emptyset$.*

Ward's version subsumes Birkhoff's one but, in point of fact, it is not comparable to the original result by Wallace. For instance, Wallace (1945) assumes that the ground set of the objective relation is a compact Hausdorff space but proves the existence of a maximal on every closed—equivalently, compact—subset thereof. For expositional convenience, we here provide a straightforward generalization of Theorem 4 that subsumes the results by Wallace, Birkhoff and Ward. In the main, however, Theorem 5 must be credited to Wallace.

Theorem 5 (Wallace, Birkhoff, Ward) *Let (X, τ) be a topological space and B be a relation on X. If B is closed-valued for τ, transitive and reflexive, then $\mathcal{M}(B, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$.*

Proof. Suppose B is closed-valued for τ, transitive and reflexive and $Y \in \mathcal{K}(X, \tau)$. Endow Y with the subspace topology $\hat{\tau}$. Then, $(Y, \hat{\tau})$ is nonempty compact topological space and $B|_Y$ is a preorder on Y that is closed-valued for $\hat{\tau}$. So, $\mathcal{M}(B|_Y) \neq \emptyset$ by Theorem 4. Part 2 of Proposition 1 ensures that $\mathcal{M}(B, Y) = \mathcal{M}(B|_Y)$. Consequently, $\mathcal{M}(B, Y) \neq \emptyset$. ■

Another theorem on the existence of maximals is proved in Walker (1977, Theorem at p. 472) and antecedents thereof can be found in Bergstrom (1975, Theorem at p. 403), Brown (1973, Theorem 7) and Sloss (1971). Indeed, there are also some earlier versions that should be mentioned: Rader (1972, Theorem 4 of Ch. 5), Sonnenschein (1971, Theorem 3) and, in particular, Schmeidler (1969, Lemma 2). Walker's version reads as follows.

Theorem 6 (Walker (1977, Theorem)) *Let (X, τ) be a topological space and B be a relation on X. If $B^{\mathfrak{c}}$ is open-valued for τ and B is acyclic, then $\mathcal{M}(B, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$.*

Theorem 6 has been generalized in Subiza and Peris (1997, Theorem 3) by replacing the open-valuedness of the converse of the objective relation with its lower quasi-continuity (on the connection between these two conditions, see Remark 8). Subiza and Peris' result reads as follows.

Theorem 7 (Subiza and Peris (1997, Theorem 3)) *Let (X, τ) be a topological space and B be a relation on X. If B is lower quasi-continuous and acyclic, then $\mathcal{M}(B, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$.*

5.1 A unifying theorem

This Sect. 5.1 introduces the W-property. Theorem 8 shows a result on the existence of maximals that, by consequence of Propositions 3 and 6, subsumes and unifies Theorems 5 and 7. The unification is brought about by making use of Theorem 5 and, as a matter of fact, the key-argument about the existence of a maximal that upholds Theorem 8 is still that first employed in Wallace (1945): in a sense, Theorem 7—and its special case, Theorem 6—can be viewed as a consequence of Theorem 5 and, in particular, of Wallace's 1945 result.

Definition 4 *Let (X, τ) be a topological space. A relation B on X has the* ***W-property for*** *τ iff B is $\mathcal{T}_{\mathfrak{t}}$-closed-valued for τ and S-consistent.*

Theorem 8 *Let (X, τ) be a topological space and B be a relation on X. If B has the W-property for τ, then $\mathcal{M}(B, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$.*

Proof. Suppose B has the W-property for τ and $Y \in \mathcal{K}(X, \tau)$. As B is $\mathcal{T}_{\mathfrak{t}}$-closed-valued for τ, the relation $\mathcal{T}_{B^{\mathfrak{t}}}$ is closed-valued for τ. The relation $\mathcal{T}_{B^{\mathfrak{t}}}$ is a preorder in that so is any right-trace. So, $\mathcal{M}(\mathcal{T}_{B^{\mathfrak{t}}}, Y) \neq \emptyset$ by Theorem 5. As B is S-consistent, Theorem 3 and the last inequality imply $\mathcal{M}(B, Y) \neq \emptyset$. ■

5.2 On the intersection of B- and $\mathcal{T}_B$-maximals

This Sect. 5.2 introduces a strong version of the W-property to examine the nonemptiness of the intersection of the sets of B- and $\mathcal{T}_B$-maximals.

Definition 5 *Let (X, τ) be a topological space. A relation B on X has the* ***strong W-property for*** *τ iff B is $\mathcal{T}$-closed-valued for τ and transitive.*

Proposition 7 *Let (X, τ) be a topological space and B be a relation on X. If B has the strong W-property for τ, then B has the W-property for τ.*

Proof. A consequence of part 1 of Propositions 3 and of the fact that the transitivity of B implies $\mathcal{T}_B = \mathcal{T}_{B^{\mathfrak{t}}}$. ■

Theorem 9 *Let (X, τ) be a topological space and B be a relation on X. If B has the strong W-property for τ, then $\mathcal{M}(B, Y) \cap \mathcal{M}(\mathcal{T}_B, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$.*

Proof. Suppose B has the strong W-property for τ and $Y \in \mathcal{K}(X, \tau)$. The relation $\mathcal{T}_B$ is a preorder in that so is any right-trace. As $\mathcal{T}_B$ is closed-valued for τ, $\mathcal{M}(\mathcal{T}_B, Y) \neq \emptyset$ by Theorem 5. As B is transitive, Theorem 1 and the last inequality imply $\mathcal{M}(B, Y) \cap \mathcal{M}(\mathcal{T}_B, Y) \neq \emptyset$. ■

5.3 An example

Consider the topological space (X, τ), where X is a cone of $\mathbb{R}^n$ endowed with the subspace topology τ induced on X by the natural topology of $\mathbb{R}^n$. Pick a real $\lambda \geq 1$ and let B be the strict partial order relation on X specified by

$$B(x) = \begin{cases} \{\mu x : \mu > \lambda\} & \text{if } x \neq 0 \\ X \backslash \{0\} & \text{if } x = 0 \end{cases} .$$

Clearly, B is transitive as well as acyclic: a fortiori, B is S-consistent by Proposition 3. The relation B possesses a right-trace $\mathcal{T}_B$ specified by

$$\mathcal{T}_B(x) = \begin{cases} \{\mu x : \mu \geq 1\} & \text{if } x \neq 0 \\ X & \text{if } x = 0 \end{cases}$$

that is independent of the choice of λ and that is closed-valued for τ. Consequently, B has the strong W-property for τ: a fortiori, B has the W-property for τ by Proposition 7. So, B satisfies all assumptions of Theorems 8 and 9. Let us now restrict to the particular case in which $n > 1$ and X is the nonnegative orthant $\mathbb{R}^n_+$. Fix an arbitrary $x \in X \backslash \{0\}$. It is readily observed that

$$B^{\mathfrak{ct}}(x) = B^{\mathfrak{c}}(x) = \{\mu x : 0 \leq \mu < 1/\lambda\}.$$

With respect to the topology τ, the topological interior of $B^{\mathfrak{ct}}(x)$ is empty: as $B^{\mathfrak{c}}(x) \neq \emptyset$, we are in a position to conclude that B is not lower quasi-continuous. Clearly, B is not even closed-valued for τ. Therefore, B satisfies neither the conditions of Theorem 5 nor those of Theorem 7. In particular, B is an instance of a relation that is $\mathcal{T}$- and $\mathcal{T}_{\mathfrak{t}}$-closed-valued for τ but that does not satisfy any of the seven conditions listed in Propositions 5 and 6. Note that, when $\lambda > 1$, not even the reflexive closure $B^{\mathfrak{r}}$ of B is closed-valued for τ.

5.4 An observation and a corollary

Let B be an arbitrary relation on an arbitrary set X. If $x \in \mathcal{M}(B)$, then $B^{\mathfrak{a}}(x) = \emptyset$ and hence $x \in \mathcal{M}(\mathcal{T}_{B^{\mathfrak{a}}})$. We conclude that

$$\mathcal{M}(B) \subseteq \mathcal{M}(\mathcal{T}_{B^{\mathfrak{a}}}).$$

So, every *unconstrained* B-maximal is an *unconstrained* $\mathcal{T}_{B^{\mathfrak{a}}}$-maximal.[7] This conclusion is contained in the second paragraph of Duggan (2013, Sect. 3), where the author states that, "[i]nterestingly, the core is always a subset of the maximal set of Fishburn shading" (in Duggan (2013), the sets $\mathcal{M}(B)$ and $\mathcal{M}(\mathcal{T}_{B^{\mathfrak{a}}})$ are called, respectively, the *core* and the *maximal set of Fishburn shading*). However, a B-maximal need not be a $\mathcal{T}_{B^{\mathfrak{a}}}$-maximal and Duggan's observation does not extend, in general, to constrained optimization problems. Remark 9 clarifies the point employing the strong Pareto dominance relation on $\mathbb{R}^2$. Next, Corollary 3 provides sufficient conditions for the existence of a B-maximal that is also a $\mathcal{T}_{B^{\mathfrak{a}}}$-maximal.

Remark 9 *Let B be the strict partial order relation on $X = \mathbb{R}^2$ induced by the positive orthant $\mathbb{R}^2_{++}$. Endow X with the natural topology τ of $\mathbb{R}^2$. Observing that $B = B^{\mathfrak{a}}$ and that $B^{\mathfrak{a}}$ has the strong W-property for τ, put*

$$Y = \{(0,0), (1,0)\}.$$

Noting that $\mathcal{M}(B, Y) = Y$ and $\mathcal{M}(\mathcal{T}_B, Y) = \mathcal{M}(\mathcal{T}_{B^{\mathfrak{a}}}, Y) = \{(1,0)\}$, we are in a position to conclude that

$$\mathcal{M}(\mathcal{T}_{B^{\mathfrak{a}}}, Y) \subset \mathcal{M}(B, Y).$$

So, in general, a B-maximal need not be a $\mathcal{T}_{B^{\mathfrak{a}}}$-maximal.

[7] Considering the first (asymmetric) relation illustrated in Remark 2, it is readily checked that an unconstrained $\mathcal{T}_{B^{\mathfrak{a}}}$-maximal need not be an an unconstrained B-maximal.

Corollary 3 *Let (X, τ) be a topological space and B be a relation on X. If $B^{\mathfrak{a}}$ has the strong W-property for τ, then $\mathcal{M}(B, Y) \cap \mathcal{M}(\mathcal{T}_{B^{\mathfrak{a}}}, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$.*

Proof. Suppose $B^{\mathfrak{a}}$ has the strong W-property for τ and $Y \in \mathcal{K}(X, \tau)$. Then $\mathcal{M}(B^{\mathfrak{a}}, Y) \cap \mathcal{M}(\mathcal{T}_{B^{\mathfrak{a}}}, Y) \neq \emptyset$ by Theorem 9. The idempotence of asymmetrization implies $\mathcal{M}(B, Y) = \mathcal{M}(B^{\mathfrak{a}}, Y)$ and thus we have desired result. ■

5.5 Final observations

Part of the literature has tackled the problem of the existence of an unconstrained maximal by making use of some notion of "transfer continuity".[8] One of the strongest versions of "transfer continuity" is the following: given a topological space (X, τ), a relation B on X is **transfer lower continuous** iff the membership $(y, x) \in B^{\mathfrak{c}}$ implies the existence of a τ-neighborhood N_y of y included in $B^{\mathfrak{c}}(\bar{x})$ for some $\bar{x} \in X$. It is readily seen that transfer lower continuity is a weakening of the condition that $B^{\mathfrak{c}}$ is open-valued for τ. Example 4 in Subiza and Peris (1997)—which is an adaptation of Example 1 in Tian and Zhou (1995)—shows that a condition weaker than transfer lower continuity of a relation is not sufficient to guarantee the existence of a maximal on every nonempty compact subset of its ground set. Indeed, also transfer lower continuity does not suffice[9] to this end, even in the case of a strict total order relation. Example 1—which is, again, an adaptation of Example 1 in Tian and Zhou (1995)—illustrates the point.

Example 1 *Endow $X = [0, 1]$ with the topology τ induced by the natural topology of $\mathbb{R}$. Let $f : X \to \mathbb{R}$ be the function defined by*

$$f(x) = \begin{cases} x + 1 & \text{if } x \text{ is rational} \\ x & \text{otherwise} \end{cases}$$

and let B be the strict total order relation on X defined by

$$B(x) = \{z \in X : f(z) > f(x)\}.$$

Put $Y = [0, e/3]$ and note that Y is a τ-compact subset of X and that

$$\mathcal{M}(B) \neq \emptyset \text{ and } \mathcal{M}(B, Y) = \emptyset.$$

The relation B is an instance of a strict total order relation—defined on a compact ground set—that is transfer lower continuous and that fails to possess a B-maximal on some nonempty compact subset of its ground set.

Part of the literature has tackled the problem of the existence of an unconstrained maximal by making use of the notion of "convex-valuedness" or of

[8]See, e.g., Border (1985, Theorem 7.2), Mehta (1989), Zhou and Tian (1992), Tian and Zhou (1995), Subiza and Peris (1997), Alcantud (2002) as well as Rodríguez-Palmero and Garcìa-Lapresta (2002), Andrikopoulos and Zacharias (2012).

[9]The second relation in Remark 7—which is taken from Subiza and Peris (1997, Example 2)—is not transfer lower continuous but satisfies all assumptions of Theorem 7. Therefore, transfer lower continuity is not even a necessary condition for a relation to possess a maximal on every nonempty compact subset of its ground set.

some weakening thereof.[10] For clarity, the definition of convex-valuedness is the following: given a subset X of a real vector space V, a relation B on X is **convex-valued** iff $B(x)$ is convex for all $x \in X$. Example 2 shows that the acyclicity assumption in Theorem 6 cannot be replaced by convex-valuedness (or by some weakening thereof).

Example 2 *Endow $X = [0, 6]$ with the topology τ induced by the natural topology of $\mathbb{R}$. Let B be the asymmetric—not acyclic—relation B on X defined by*

$$B(x) = \begin{cases} (2,4) & \text{if } x \in (0,2) \\ (4,6) & \text{if } x \in (2,4) \\ (0,2) & \text{if } x \in (4,6) \\ \emptyset & \text{otherwise.} \end{cases}$$

It is readily seen that the relation B is graph-open for τ. Put $Y = \{1, 3, 5\}$ and note that Y is a τ-compact subset of X and that

$$\mathcal{M}(B) \neq \emptyset \text{ and } \mathcal{M}(B, Y) = \emptyset.$$

The relation B is an instance of an asymmetric and convex-valued relation—defined on a compact ground set—that is graph-open for τ and that fails to possess a B-maximal on some nonempty compact subset of its ground set. It is worth to remark that, in particular, the relation B satisfies all assumptions of Corollary 7.5 in Border (1985).

6 A digression on compactness

Let τ_1 and τ_2 be topologies on a set X. The topology τ_2 is **finer than** the topology τ_1 iff $\tau_1 \subseteq \tau_2$. The "finer than" is a well-known relation that enables comparison of topologies. Definition 6 introduces two others. Note that—as to these two relations—it is immaterial whether the definition of $\mathcal{K}(X, \tau)$ allows or not for the empty set.

Definition 6 *Let τ_1 and τ_2 be topologies on a set X.*

- *τ_2 is **compactly finer than** τ_1 iff $\mathcal{K}(X, \tau_1) \subseteq \mathcal{K}(X, \tau_2)$.*
- *τ_2 is **compactly equivalent to** τ_1 iff $\mathcal{K}(X, \tau_1) = \mathcal{K}(X, \tau_2)$.*

6.1 On the cofinite topology

Theorem 10 asserts that any topology τ_1 admits a T_1 topology τ_2 finer than τ_1 but compactly equivalent to τ_1. Before presenting Theorem 10, it is recalled that the **cofinite topology on a set** X is the topology on X whose members are—exactly—the empty set and every subset of X with a finite complement. For clarity, it is recalled that the **topology on a set X generated by a family $\mathcal{F}$ of subsets of** X is the smallest topology on X that includes $\mathcal{F}$.

[10] See, e.g., Sonnenschein (1971, Theorem 4), Aliprantis and Brown (1983, Theorem 3.5) (as well as the works of Fan and of Borglin and Keiding mentioned therein), Yannelis and Prabhakar (1983), Mehta (1984), Yannelis (1985) Border (1985, Theorem 7.2), Mehta (1987, Sect. 3), Mehta (1989), Bergstrom (1992) (as well as his 1975 paper mentioned therein) and the more recent Duggan (2011).

Theorem 10 *Let X be a set and τ_1 be a topology on X. Let τ_0 be the cofinite topology on X and τ_2 be the topology on X generated by $\tau_0 \cup \tau_1$.*

1. *The topology τ_2 is a T_1 topology finer than the topology τ_1.*
2. *The topology τ_2 is compactly equivalent to the topology τ_1.*

Proof. 1. An immediate consequence of the definition of the topology τ_2 and of the fact that any cofinite topology is T_1.

2. The topology τ_1 is compactly finer than the topology τ_2 in that τ_2 is finer than τ_1. To prove part 2 of Theorem 10 it then suffices to show that the topology τ_2 is compactly finer than the topology τ_1. So, suppose

$$Y \in \mathcal{K}(X, \tau_1).$$

Let $\hat{\tau}_0$, $\hat{\tau}_1$ and $\hat{\tau}_2$ be the subspace topologies on Y induced by τ_0, τ_1 and τ_2, respectively. As $\tau_0 \cup \tau_1$ is a subbase of open sets for τ_2, the union $\hat{\tau}_0 \cup \hat{\tau}_1$ is a subbase of open sets for $\hat{\tau}_2$. Let γ be an arbitrary cover of Y by members of $\hat{\tau}_0 \cup \hat{\tau}_1$. Assume for a moment that the cover γ contains a nonempty member S of the topology $\hat{\tau}_0$: since every element of the—then necessarily finite—complement $Y \backslash S$ is contained in some member of γ and since S is a member of γ, there exists a finite subcover of γ. Assume now that the cover γ contains no nonempty member of $\hat{\tau}_0$: then $\gamma \subseteq \hat{\tau}_1$ and the compactness of the topological space $(Y, \hat{\tau}_1)$ implies the existence of a finite subcover of γ. We thus infer the existence of a finite subcover of γ. By virtue of Alexander's subbase theorem, we conclude that $(Y, \hat{\tau}_2)$ is a compact topological space and hence that

$$Y \in \mathcal{K}(X, \tau_2).$$

We are now in a position to assert that $\mathcal{K}(X, \tau_1) \subseteq \mathcal{K}(X, \tau_2)$ and hence that τ_2 is compactly finer than τ_1. ■

6.2 On the lexicographic lower topology

Theorem 11 asserts the existence of a topology on $\mathbb{R}^\alpha$ that is compactly finer than the natural topology and for which the lexicographic order is closed-valued. Before enunciating Theorem 11—whose proof makes use of Lemma 2.1 in Schouten (2018)—it is worth recalling the definitions of the lexicographic order relation and of the associated lower topology. Let α be a non-zero ordinal. The product of α copies of $\mathbb{R}$ is denoted by $\mathbb{R}^\alpha$. The **lexicographic order on** $\mathbb{R}^\alpha$ is the preorder relation Λ_α on $\mathbb{R}^\alpha$ defined by

$$\Lambda_\alpha(x) = \left\{ \begin{array}{c} y \in \mathbb{R}^\alpha : \text{either } y = x \text{ or there exists a non-zero} \\ \text{ordinal } \beta \leq \alpha \text{ such that } x_\beta < y_\beta \text{ and } x_\gamma = y_\gamma \\ \text{for every non-zero ordinal } \gamma < \beta. \end{array} \right\}.$$

The **lexicographic lower topology on** $\mathbb{R}^\alpha$ is the topology that arises by taking the family $\{\Lambda_\alpha(x) : x \in \mathbb{R}^\alpha\} \cup \{\mathbb{R}^\alpha\}$ as a subbase of closed sets (equivalently, by taking the family $\{\Lambda_\alpha^{\mathfrak{ca}}(x) : x \in \mathbb{R}^\alpha\} \cup \{\emptyset\}$ as a subbase of open sets). The **natural topology of** $\mathbb{R}^\alpha$ is the product topology that arises by endowing each copy of $\mathbb{R}$ with its natural topology.

Theorem 11 *Let α be a non-zero ordinal. Let τ_1 be the natural topology on $\mathbb{R}^\alpha$ and τ_2 be the lexicographic lower topology on $\mathbb{R}^\alpha$.*

1. *The lexicographic order Λ_α is closed-valued for the topology τ_2.*

2. *The topology τ_2 is compactly finer than the topology τ_1.*

Proof. 1. An immediate consequence of the definition of the lexicographic lower topology on $\mathbb{R}^\alpha$.

2. Note that $\Lambda_\alpha^{\mathfrak{ca}}$ is transitive and connex. Noted this, suppose

$$Y \in \mathcal{K}(X, \tau_1).$$

Lemma 2.1 in Schouten (2018) ensures the existence of $y \in Y$ such that $z \in \Lambda_\alpha^{\mathfrak{c}}(y)$ for all $z \in Y$. The connexity of $\Lambda_\alpha^{\mathfrak{ca}}$ then implies

$$z \in \Lambda_\alpha^{\mathfrak{ca}}(y) \text{ for all } z \in Y \backslash \{y\}. \tag{11}$$

Put $\sigma_2 = \{\Lambda_\alpha^{\mathfrak{ca}}(x) : x \in \mathbb{R}^\alpha\} \cup \{\emptyset\}$. The set σ_2 is a subbase of open sets for τ_2. Let $\hat{\tau}_2$ be the subspace topology on Y induced by τ_2. Put $\hat{\sigma}_2 = \{S \cap Y : S \in \sigma_2\}$ and note that $\hat{\sigma}_2$ is a subbase of open sets for $\hat{\tau}_2$. Assume that γ is an arbitrary cover of Y by members of $\hat{\sigma}_2$. Then there exists $S \in \gamma$ such that $y \in S$. By virtue of the last two memberships, the inequality $S \neq Y$ implies the existence of $\bar{x} \in \mathbb{R}^\alpha$ such that $S = \Lambda_\alpha^{\mathfrak{ca}}(\bar{x}) \cap Y$, that $y \in \Lambda_\alpha^{\mathfrak{ca}}(\bar{x}) \cap Y$ and that $z \notin \Lambda_\alpha^{\mathfrak{ca}}(\bar{x}) \cap Y$ for some $z \in Y \backslash \{y\}$: a contradiction with (11) and the transitivity of $\Lambda_\alpha^{\mathfrak{ca}}$. Therefore $S = Y$ and $\{S\}$ is a finite subcover of γ. Said this, Alexander's subbase theorem implies the compactness of the topological space $(Y, \hat{\tau}_2)$ and hence the validity of the membership

$$Y \in \mathcal{K}(X, \tau_2).$$

We are now in a position to assert that $\mathcal{K}(X, \tau_1) \subseteq \mathcal{K}(X, \tau_2)$ and hence that τ_2 is compactly finer than τ_1. ■

Corollary 4 *Let α be a non-zero ordinal. There exists a T_1 topology on $\mathbb{R}^\alpha$ that is compactly finer than the natural topology on $\mathbb{R}^\alpha$ and for which the lexicographic order on $\mathbb{R}^\alpha$ is closed-valued.*

Proof. Apply Theorem 11 and 10, in this order. ■

7 A generalization

Definition 7 introduces the weak W-property. Sect. 7.1 establishes a connection with the W-property. Sect. 7.2 provides a generalization of Theorem 8 and an extension of Theorem 9. Sect. 7.3 concludes showing sufficient conditions for a relation to possess the weak W-property.

Definition 7 *Let (X, τ) be a topological space. A relation B on X has the **weak W-property for** τ iff B admits a quasi-extension R possessing the W-property for at least one topology on X compactly finer than τ.*

7.1 Connection with the W-property

Proposition 8 asserts that every relation with the W-property has the weak W-property and Proposition 9 points out that—unlike the W-property—the weak W-property is in fact a property of the asymmetric part of a relation.

Proposition 8 *Let (X, τ) be a topological space and B be a relation on X. If B has the W-property for τ, then B has the weak W-property for τ.*

Proof. A consequence of the fact that every topology is compactly finer than itself and that every relation is a quasi-extension of itself. ■

Proposition 9 *Let (X, τ) be a topological space and B be a relation on X. The relation B has the weak W-property for τ if and only if the relation $B^{\mathfrak{a}}$ has the weak W-property for τ.*

Proof. A consequence of Remark 1. ■

7.2 The generalized theorem

Theorem 12 generalizes Theorem 8 and its Corollary 5 extends Theorem 9 restricting attention to the unconstrained maximals of a transitive relation.

Theorem 12 *Let (X, τ) be a topological space and B be a relation on X. If B has the weak W-property for τ, then $\mathcal{M}(B, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$.*

Proof. Suppose B has the weak W-property for τ and $Y \in \mathcal{K}(X, \tau)$. Then there exist a quasi-extension R of B and a topology $\bar{\tau}$ on X such that R has the W-property for $\bar{\tau}$ and $Y \in \mathcal{K}(X, \bar{\tau})$. So, $\mathcal{M}(R, Y) \neq \emptyset$ by Theorem 8. As $\mathcal{M}(R, Y) \neq \emptyset$ and $B^{\mathfrak{a}} \subseteq R^{\mathfrak{a}}$, we conclude that $\mathcal{M}(B, Y) \neq \emptyset$. ■

Corollary 5 *Let B be a transitive relation on a nonempty compact topological space (X, τ). If B has the weak W-property for τ, then $\mathcal{M}(B) \cap \mathcal{M}(\mathcal{T}_B) \neq \emptyset$.*

Proof. Suppose B has the weak W-property for τ. As $X \in \mathcal{K}(X, \tau)$, Theorem 12 implies $\mathcal{M}(B, X) \neq \emptyset$. Consequently, $\mathcal{M}(B) \neq \emptyset$ by part 1 of Proposition 1 and $\mathcal{M}(B) \cap \mathcal{M}(\mathcal{T}_B) \neq \emptyset$ by Theorem 2. ■

7.3 Sufficient conditions for the weak W-property

This Sect. 7.3 shows sufficient conditions for a relation to possess the weak W-property and the connection with Bosi and Zuanon (2017)'s quasi upper semicontinuity is clarified.

7.3.1 Quasi upper semicontinuous preorders

Recently, Bosi and Zuanon (2017) have introduced a notion of continuity that allows for a unification of some results on the existence of a maximal element; applications can be found also in Bosi and Zuanon (2019, Sect. 3). Definition 8 recalls the notion of quasi upper semicontinuity. Proposition 10 shows that every quasi upper semicontinuous relation has the weak W-property.

Definition 8 *Let (X, τ) be a topological space. A relation B on X is* ***quasi upper semicontinuous*** *iff B admits a quasi-extension R that is a preorder and that is closed-valued for τ.*

Proposition 10 *Let (X, τ) be a topological space and B be a relation on X. If B is quasi upper semicontinuous, then B has the weak W-property for τ.*

Proof. Suppose B is quasi upper semicontinuous. Then B admits a quasi-extension R that is a preorder and that is closed-valued for τ. Part 1 of Proposition 3 ensures that R is S-consistent and part 2 of Proposition 6 ensures that R is $\mathcal{T}_\mathfrak{t}$-closed-valued for τ. So, R has the W-property for τ. As every topology is compactly finer than itself, B has the weak W-property for τ. ∎

Remark 10 *Let (X, τ) be a topological space and B be a preorder on X. The closed-valuedness of B for τ and the open-valuedness of $B^{\mathfrak{ca}}$ for τ are sufficient conditions for B to be quasi upper semicontinuous: see Bosi and Zuanon (2017, Remark 2.5). Other sufficient conditions are shown therein. In particular—and related to a result in Nosratabadi (2014)—Theorem 2.11 in Bosi and Zuanon (2017) asserts that, when τ is a second countable topology, the quasi upper semicontinuity of B is equivalent to the representability of $B^{\mathfrak{a}}$ by means of an upper semicontinuous weak utility function (i.e., to the existence of an upper semicontinuous function $u : X \to \mathbb{R}$ such that $(y, x) \in B^{\mathfrak{a}} \Rightarrow u(y) > u(x)$).*[11]

7.3.2 Lexicographic orders

Proposition 11 shows that the lexicographic order on $\mathbb{R}^\alpha$ is an instance of a relation with the weak W-property for the natural topology of $\mathbb{R}^\alpha$. Proposition 12 clarifies that, when $\mathbb{R}^\alpha$ is endowed with the natural topology, the lexicographic order on $\mathbb{R}^\alpha$ is not, in general, a quasi upper semicontinuous preorder. Indeed, Proposition 12 proves the usefulness of the generalization of notion of a quasi upper semicontinuous preorder brought about by the weak W-property.

Proposition 11 *Let α be a non-zero ordinal—possibly, a positive integer—and endow $\mathbb{R}^\alpha$ with the natural topology. The lexicographic order on $\mathbb{R}^\alpha$ has the weak W-property for the natural topology of $\mathbb{R}^\alpha$.*

Proof. Let τ_1 denote the natural topology of $\mathbb{R}^\alpha$ and τ_2 the lexicographic lower topology on $\mathbb{R}^\alpha$. Part 1 of Theorem 11 ensures that Λ_α is closed-valued for τ_2: as Λ_α is a preorder, from part 1 of Proposition 3 and part 2 of Proposition 6 we infer that Λ_α has the W-property for τ_2. Part 2 of Theorem 11 ensures that τ_2 is compactly finer than τ_1: as every relation is a quasi-extension of itself, Λ_α has the weak W-property for τ_1. ∎

Proposition 12 *Endow $\mathbb{R}^2$ with the natural topology. The lexicographic order on $\mathbb{R}^2$ is a preorder but not a quasi upper semicontinuous preorder.*

[11] For instance, endowing $X = [0, 2]$ with the subspace topology inherited from the natural topology of $\mathbb{R}$, the preorder relation B on X defined by $B(x) = [x, 2]$ if $x \geq 1$ and by $B(x) = [x, 1)$ if $x < 1$ is quasi upper semicontinuous (in that id_X is a continuous weak utility for $B^{\mathfrak{a}}$). It is worth to point out that, when B is understood as a constant "variable preference relation" in the sense of Luc and Soubeyran (2013), B is not "upper closed" in the sense of Luc and Soubeyran (2013, Definition 10). In fact, Luc and Soubeyran (2013)'s upper closedness does not subsume Bosi and Zuanon (2017)'s quasi upper semicontinuity.

Proof. Let τ denote the natural topology of $\mathbb{R}^2$. It is known—and readily verified—that Λ_2 is a preorder and that Λ_2 can be equivalently defined as the relation induced by the lexicographic cone

$$C = (\mathbb{R}_{++} \times \mathbb{R}) \cup (\{0\} \times \mathbb{R}_+).$$

Note that $C^\bullet = C\backslash\{(0,0)\}$ and that $\Lambda_2^{\mathfrak{a}}(x) = x + C^\bullet$ for all $x \in \mathbb{R}^2$. By way of contradiction, assume the existence of a preorder relation R that quasi-extends Λ_2 and that is closed-valued for τ. As R quasi-extends Λ_2, $\Lambda_2^{\mathfrak{a}}(x) \subseteq R^{\mathfrak{a}}(x)$ for all $x \in \mathbb{R}^2$; as $R^{\mathfrak{a}} \subseteq R$, we have that $R^{\mathfrak{a}}(x) \subseteq R(x)$ for all $x \in \mathbb{R}^2$. So, $x + C^\bullet \subseteq R(x)$ for all $x \in \mathbb{R}^2$ and basic properties of the topological closure operation entail that $\mathrm{cl}_\tau\,(x + C^\bullet) \subseteq \mathrm{cl}_\tau\, R(x)$ for all $x \in \mathbb{R}^2$. Since the topology τ is translation invariant and since R is closed-valued for τ, we infer that $x + \mathrm{cl}_\tau\, C^\bullet \subseteq R(x)$ for all $x \in \mathbb{R}^2$. Noting that $\mathrm{cl}_\tau\, C^\bullet = \mathbb{R}_+ \times \mathbb{R}$, we conclude that

$$x + (\mathbb{R}_+ \times \mathbb{R}) \subseteq R(x) \text{ for all } x \in \mathbb{R}^2. \tag{12}$$

Put $p = (0,0)$ and $q = (0,1)$ and note that $q \in \Lambda_2^{\mathfrak{a}}(p)$. From (12) we conclude that $q \in R(p)$ and $p \in R(q)$: a contradiction with the membership $q \in \Lambda_2^{\mathfrak{a}}(p)$ and the assumption that R quasi-extends Λ_2. ■

7.3.3 Transitivity and closed-valuedness

A transitive and closed-valued relation need not have the W-property: this claim is readily verified by considering the relation illustrated in Remark 4. Proposition 13 proves that any transitive and closed-valued relation has, however, the weak W-property (even when the topology is not T_1). Recalling that the graph-closedness of a relation implies its closed-valuedness, it is thus clear that Theorem 12 subsumes Theorem I in Wallace (1962). Also, in the light of Proposition 2, it is thus clear that Theorem 12 subsumes Proposition A.1 in Banks et al. (2006) (equivalently, Lemma 1 in Duggan (2011)).

Proposition 13 *Let (X,τ) be a topological space and B be a relation on X. If B is transitive and closed-valued for τ, then B has the weak W-property for τ.*

Proof. Put $R = B^{\mathrm{r}}$ and $\tau_1 = \tau$. Let τ_0 denote the cofinite topology on X and let τ_2 be the topology generated by $\tau_0 \cup \tau_1$. Suppose B is transitive and closed-valued for τ_1: we show that B has the weak W-property for τ_1. As R is the reflexive closure of the transitive relation B, the relation R is a preorder that quasi-extends B. By Theorem 10, τ_2 is finer than τ_1 and compactly equivalent to τ_1; a fortiori, τ_2 is compactly finer than τ_1. As B is closed-valued for τ_1 and τ_2 is finer than τ_1, the relation B is closed-valued also for τ_2. By Theorem 10, the topology τ_2 is T_1. As τ_2 is a T_1 topology and B is closed-valued for τ_2, its reflexive closure R is closed-valued also for τ_2. Said this, part 1 of Proposition 3 and part 2 of Proposition 6 ensure that R has the W-property for τ_2. We are now in a position to conclude that B has the weak W-property for τ_1. ■

An obvious—yet interesting—consequence of Proposition 13 and Theorem 12 is that the assumption of reflexivity of B in statements of Theorems 4 and 5 can be simply dropped. Corollary 6 provides a restatement of Theorems 5 dispensing with the unnecessary reflexivity assumption imposed therein.

Corollary 6 *Let (X,τ) be a topological space and B be a relation on X. If B is closed-valued for τ and transitive, then $\mathcal{M}(B,Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X,\tau)$.*

7.3.4 Cones

Parts 1 and 2 of Proposition 14 show that the strong W-property is possessed by any relation induced by a convex cone of a real topological vector space that is either closed or open: the weak and the strong Pareto dominance relations on $\mathbb{R}^n$ are instances of economic interest.[12] Part 3 of Proposition 14 shows that the—possibly neither transitive nor acyclic—relation induced by a strictly supported cone of a real topological vector space has the weak W-property. The fact that the relation induced by a strictly supported cone possesses a maximal element on every nonempty compact constraint set is a classic result of vector optimization: see, e.g., Luc (1989, Corollary 3.6).

Proposition 14 *Let (X, τ) be a real topological vector space.*

1. *The relation induced by a τ-closed convex cone of X has the strong W-property for τ.*
2. *The relation induced by a τ-open convex cone of X has the strong W-property for τ.*
3. *The relation induced by a strictly τ-supported cone of X has the weak W-property for τ.*

Proof. 1. Suppose C is a τ-closed convex cone of X and let B be the relation induced by C. If C is empty, then B is the empty relation on X and the proof is obvious. Suppose C is nonempty. In a real topological vector space, any nonempty τ-closed cone contains the zero vector; also, the relation induced on a real vector space by a convex cone that contains the zero vector is a preorder. So, B is a preorder. As the topology of a real topological vector space is invariant under translation, the preorder B is closed-valued for τ. Said this, part 1 of Proposition 5 ensures that R has the strong W-property for τ.

2. Suppose C is a nonempty τ-open convex cone of X and let B be the relation induced by C. Recalling that the relation induced on a real vector space by a convex cone is transitive, we infer that B is transitive. As the topology of a real topological vector space is invariant under translation and scalar multiplication by a non-zero scalar, $B^{\mathfrak{c}}$ is open-valued. Said this, part 2 of Proposition 5 ensures that R has the strong W-property for τ.

3. Suppose C is a cone of X and H is a τ-open half-space including $C^{\bullet}$. Let B be the relation induced by C and R be the relation induced by H. As H is a τ-open convex cone of X, R has the W-property for τ by part 2 of Proposition 14 and Proposition 7. Note that $B^{\mathfrak{a}}$ and $R^{\mathfrak{a}}$ are, respectively, the relations induced by $C^{\bullet}$ and $H^{\bullet}$. As $C^{\bullet} \subseteq H = H^{\bullet}$, we infer that R quasi-extends B. As every topology is compactly finer than itself, we are in a position to conclude that B has the weak W-property for τ. ■

Corollary 7 shows two consequences of Proposition 14. Remark 11 points out that a certain class of Bewley (justifiable) preferences lies within the class of relations with the strong (weak) W-property.

[12] Agreeing that $\mathbb{R}^n$ is the set of all conceivable utility levels of an economy with n agents, the weak (strong) Pareto dominance relation is the relation on $\mathbb{R}^n$ induced by the nonnegative (positive) orthant of $\mathbb{R}^n$, which is a closed (open) cone of $\mathbb{R}^n$.

Corollary 7 *Let (X, τ) be a real topological vector space.*

1. *The relation induced by the intersection of a family of τ-closed half-spaces of X has the strong W-property for τ.*
2. *The relation induced by the union of a family of τ-closed half-spaces of X has the weak W-property for τ.*

Proof. 1. Let $\{H_i\}_{i \in I}$ be a family of τ-closed half-spaces of X. Put

$$C = \bigcap_{i \in I} H_i$$

and let B be the relation induced by C. Note that C is a τ-closed convex cone of X. As C is a τ-closed convex cone of X, B has the strong W-property for τ by part 1 of Proposition 14.

2. Let $\{H_i\}_{i \in I}$ be a family of τ-closed half-spaces of X. Put

$$C = \bigcup_{i \in I} H_i$$

and let B be the relation induced by C. If I is the empty set, then B is the empty relation and the proof is obvious. Suppose I is nonempty and pick an arbitrary $H \in \{H_i\}_{i \in I}$. Let R be the relation induced by H. From part 1 of Proposition 14 and Proposition 7 we infer that R has the W-property for τ. A moment's reflection shows that $C^\bullet \subseteq H^\bullet$. Consequently, $B^\mathfrak{a} \subseteq R^\mathfrak{a}$ and hence R is a quasi-extension of B. Therefore, B has the weak W-property for τ. ∎

Remark 11 *We here adopt the definitions of a Bewley and of a justifiable preference set forth and discussed at more length in Cerreia-Vioglio and Ok (2018, Sect. 6). Let $\mathbb{R}^n$ be endowed with the natural topology and Π denote a nonempty, closed and convex subset of the simplex $\Delta^{n-1} \subseteq \mathbb{R}^n$. A* ***Bewley (a justifiable) preference with a prior set*** *Π is a relation B on $\mathbb{R}^n$ defined by*

$$B(x) = \{y \in \mathbb{R}^n : \pi \cdot y \geq \pi \cdot x \textit{ for all (for some) } \pi \in \Pi\}.$$

Putting $H_\pi = \{z \in \mathbb{R}^n : \pi \cdot z \geq 0\}$ for all $\pi \in \Pi$ and $C = \bigcap_{\pi \in \Pi} H_\pi$ (and $C = \bigcup_{\pi \in \Pi} H_\pi$), it is readily checked that a relation B on $\mathbb{R}^n$ is a Bewley (a justifiable) preference with a prior set Π if and only if

$$B(x) = x + C \textit{ for all } x \in \mathbb{R}^n.$$

Therefore, a relation B on $\mathbb{R}^n$ is a Bewley (a justifiable) preference with prior set Π only if it coincides with the relation induced by the intersection (the union) of a family of half-spaces of $\mathbb{R}^n$ that are closed for the natural topology of $\mathbb{R}^n$. Said this, part 1 (part 2) of Corollary 7 implies that a Bewley (a justifiable) preference is a relation with the strong (the weak) W-property.[13]

[13] Bewley (justifiable) preferences can be defined in the more general setting specified in f.n. 27 of Cerreia-Vioglio and Ok (2018). Part 1 (part 2) of Corollary 7 implies that a Bewley (a justifiable) preference is a relation with the strong (the weak) W-property also in that setting: the argument that leads to such a conclusion is essentially the same as that just exposed.

8 A necessary and sufficient condition

This Sect. 8 introduces another weakening of the W-property.

Definition 9 *Let (X, τ) be a topological space. A relation B on X has the* ***relativized W-property for*** *τ iff for every $Y \in \mathcal{K}(X, \tau)$ there exists a compact topology τ_Y on Y such that the restriction $B|_Y$ admits a quasi-extension R_Y possessing the W-property for τ_Y.*

Proposition 15 points out that—like the weak W-property—the relativized W-property is in fact a property of the asymmetric part of a relation. By making use of Theorem 4 in Alcantud (2002) and of Theorem 8 of this paper, Theorem 13 proves that the relativized W-property is necessary and sufficient for a relation to possess a maximal on every nonempty compact subset of its ground set. Corollary 8 concludes asserting that every relation with the weak W-property has the relativized W-property and hence that the latter is a (possibly non-proper) generalization of the former.

Proposition 15 *Let (X, τ) be a topological space and B be a relation on X. The relation B has the relativized W-property for τ if and only if the relation $B^{\mathfrak{a}}$ has the relativized W-property for τ.*

Proof. Let Y be an arbitrary subset of X and τ_Y be a (possibly not compact) topology on Y. By virtue of Remark 1, $B|_Y$ admits a quasi-extension R_Y possessing the W-property for τ_Y if and only if $B|_Y^{\mathfrak{a}}$ admits a quasi-extension R_Y possessing the W-property for τ_Y. The operations of restriction and asymmetrization commute with each other and hence $B|_Y^{\mathfrak{a}} = B^{\mathfrak{a}}|_Y$. We are in a position to conclude that B has the relativized W-property for τ if and only if the relation $B^{\mathfrak{a}}$ has the relativized W-property for τ. ∎

Theorem 13 *Let (X, τ) be a topological space and B be a relation on X. Assertions I and II are equivalent.*

I. $\mathcal{M}(B, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$.

II. B has the relativized W-property for τ.

Proof. *$I \Rightarrow II$.* Suppose $\mathcal{M}(B, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$. By virtue of Proposition 15, we are done if we show that $B^{\mathfrak{a}}$ has the relativized W-property for τ. Since $\mathcal{M}(B, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$ and since every nonempty finite subset of X is an element of $\mathcal{K}(X, \tau)$, the relation $B^{\mathfrak{a}}$ is acyclic. Now, fix an arbitrary $Y \in \mathcal{K}(X, \tau)$. The idempotence of asymmetrization implies $\mathcal{M}(B, Y) = \mathcal{M}(B^{\mathfrak{a}}, Y)$ and part 2 of Proposition 1 implies $\mathcal{M}(B^{\mathfrak{a}}, Y) = \mathcal{M}(B^{\mathfrak{a}}|_Y)$: the inequality $\mathcal{M}(B, Y) \neq \emptyset$ then entails that $\mathcal{M}(B^{\mathfrak{a}}|_Y) \neq \emptyset$. Any restriction of an acyclic relation is acyclic. So, $B^{\mathfrak{a}}|_Y$ is acyclic and hence S-consistent by part 2 of Proposition 3. As $B^{\mathfrak{a}}|_Y$ is acyclic and $\mathcal{M}(B^{\mathfrak{a}}|_Y) \neq \emptyset$, Theorem 4 in Alcantud (2002) ensures the existence of a compact topology τ_Y on Y for which the relation $B^{\mathfrak{a}}|_Y^{\mathfrak{c}}$ is open-valued. Consequently, $B^{\mathfrak{a}}|_Y$ is $\mathcal{T}_{\mathfrak{t}}$-closed-valued for τ_Y by part 5 of Proposition 6 and we conclude that the S-consistent relation $B^{\mathfrak{a}}|_Y$ has the W-property for τ_Y. As $B^{\mathfrak{a}}|_Y$ is a quasi-extension of itself and Y is an arbitrary element of $\mathcal{K}(X, \tau)$, $B^{\mathfrak{a}}$ has the relativized W-property for τ.

II ⇒ *I*. Suppose B has the relativized W-property for τ and $Y \in \mathcal{K}(X,\tau)$. Then there exists a compact topology τ_Y on Y such that $B|_Y$ admits a quasi-extension R_Y possessing the W-property for τ_Y. As $Y \in \mathcal{K}(Y,\tau_Y)$, Theorem 8 implies $\mathcal{M}(R_Y, Y) \neq \emptyset$: so, $\mathcal{M}(R_Y) \neq \emptyset$ by part 1 of Proposition 1. As R_Y is a quasi-extension of $B|_Y$, we have that $B|_Y^{\mathfrak{a}} \subseteq R_Y^{\mathfrak{a}}$ and hence that $\mathcal{M}(R_Y) \subseteq \mathcal{M}(B|_Y)$. The last inequality and the last inclusion imply $\mathcal{M}(B|_Y) \neq \emptyset$: so, $\mathcal{M}(B,Y) \neq \emptyset$ by part 2 of Proposition 1. ■

Corollary 8 *Let (X,τ) be a topological space and B be a relation on X. If B has the weak W-property for τ, then B has the relativized W-property for τ.*

Proof. A consequence of Theorems 12 and 13. ■

It is natural to question whether every relation with the relativized W-property has the weak W-property. An affirmative answer would imply—by consequence of Theorem 13 and Corollary 8—that the weak W-property is a necessary and sufficient condition for a relation to possess a maximal on every nonempty compact subset of its ground set; a negative answer would imply—by consequence of Corollary 8—that the relativized W-property is a proper generalization of the weak W-property. Presently, it is an open question. Part of the literature on the existence of a maximal has restricted attention to a special class of strict partial orders called interval orders. Within this literature, Campbell and Walker (1990, Theorem 2) have provided sufficient conditions for an interval order on a general topological space to possess a maximal on every nonempty compact subset of its ground set[14] and Kukushkin (2008, Theorem 3) has provided necessary and sufficient conditions for an interval order on a metric space to possess a maximal on every nonempty compact subset of its ground set (the last result subsumes, essentially, Theorem 4.1 in Smith (1974)). Clearly, Theorem 13 implies that every interval order that satisfies the sufficient condition in Campbell and Walker (1990, Theorem 2)—called, by the authors, weak lower continuity—has the relativized W-property and that every interval order on a metric space that satisfies the necessary and sufficient condition in Kukushkin (2008, Theorem 3)—called, by the author, ω-acyclicity—has the relativized W-property. It is an open question, however, whether such conditions imply the weak W-property. An example of a relation with the relativized W-property that does have the weak W-property, if any exists, might be found within the classes of relations considered in the two aforementioned articles.

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[14] Strictly speaking, Theorem 2 in Campbell and Walker (1990) concerns only the unconstrained maximals of an interval order with a compact ground set. However, all conditions imposed in that result—other than the compactness of the ground set—are inherited by any subset of the ground set and so they suffice to guarantee the existence of a maximal on every nonempty compact subset of the ground set of the objective relation.

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