Balance sheet and seniority constraints on the repayment value of claims

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Abstract

The problem is addressed of how (different types of) funding transactions may affect the repayment value of (credit or equity) claims; to this purpose a novel prove for the existence and uniqueness of the *payment vector*, which does not make (explicit) use of the fixed point theorem and allows for the presence of claims with different seniorities (i.e. credit and equity claims), is proposed. Different components of the overall displacement (the reduction of repayment value), related to *i*) seniority structure, *ii*) network of bilateral exposure and *iii*) imbalances between external loss and external capital, are calculated by sequentially relaxing different constraints in the mixed linear program used for calculating overall displacement. The possibility that more credit may reduce overall displacement (due to *borrowing-from-Peter-to-pay-Paul* effect) and more equity capital may on the contrary increase overall displacement (due to its role in the transmission of financial displacement) is exemplified, along with the possible negative dependence of relative displacement (the ratio between overall displacement and total claims) on total claims.

1 Introduction, notation and references

Let us consider a set $\mathbb{B}$ of balance sheets of financial and non financial institutions and households. On the left side of each balance sheet there are *i)* debt-type or equity-type claims towards other balance sheets [1] and/or *ii) external assets* i.e. assets to which does not correspond any obligation on the right side of other balance sheets. [2] On the right side there are liabilities toward other balance sheets and, only in the case of households, *external capital* i.e. book entries which do not correspond to claims on the asset side of other balance sheets. *External capital* amounts to the difference between total assets and debt-type obligations in households' balance sheets [3] whose set we denote by $\mathbb{H}$ (a

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[1] In the following, unless otherwise specified, the term *claim* will refer indifferently to both credit claims and equity claims; *overall claims* to the sum of all of them.

[2] The concept of *external assets* is analogous to that of "total assets excluding receivables" in [23]; so it is different from that used in [19] which includes liabilities, shares, bonds and bank loans of final users of funds which are not financial operators. As in the present contest also the balance sheets of the final users of funds are included in $\mathbb{B}$, the expression *external assets* refers only to physical assets, legal intangibles and possibly central bank money to the extent that they do not correspond to liabilities on the right side of any other balance sheet [23]. Although central bank money may be registered on the liability side of the central bank balance sheet, it may still be included in this list as - differently from the liabilities on other balance sheets - its value is assumed to be not related to that of the items on the asset side. In the central bank balance sheet, the items on the liability side would be considered as external capital (see [30]), while the matching claims on the asset side of non central bank balance sheets would be considered as external assets.

[3] In a network scheme, *external asset* would correspond to a source node, whereas *external capital* would correspond to a sink node.

subset of $\mathbb{B}$). Claims are valued at their liquidation value [4] i.e. at the value they would be attributed if all of them were to be simultaneously (and possibly before their natural expiration date) refunded: [5] their value depends on that of the asset side of the balance sheet on whose right side they are registered as liabilities. [6] We consider

- an initial (*before-the-shock* [7]) point in time in which for all of the debt-type obligations, the liquidation value equals the amount that is due to the creditor according to the original contractual arrangement [8] i.e. all the balance sheets are assumed to be solvent,
- a second (*after-the-shock*) point in time in which some of the *external assets* have been impaired and as a consequence some of the balance sheets may possibly become insolvent. [9]

We look at how the *after-the-shock* liquidation values would change in response to changes in *before-the-shock* values as a result of funding transactions. So we have

x_{ijk} = before-the-shock value of the claim of balance sheet i against balance sheet j, [10] where k is an index of seniority: [11]

$$k = \begin{cases} 1 & \text{for equity-type claims} \\ 2 & \text{for uncollateralized credit-type claims} \\ 3 & \text{for fully collateralized credit-like claims} \end{cases}$$

y_{ijk} = after-the-shock balance sheet value of the claim of balance sheet i against balance sheet j

u_{ijk} $= y_{ijk} - x_{ijk}$ is the displacement of x_{ijk}

q_{ij} $= \sum_k x_{ijk}$ is the (equity and credit) bilateral exposure of balance sheet i to balance sheet j

v_{ij} $= -\sum_k u_{ijk}$ is the (negative) displacement on the bilateral exposure of balance sheet i to balance sheet j

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[4] Differently from external assets, whose value is assumed to be determined exogenously. In order to focus on the accounting transmission of the displacement of claims, in the present study the problem of how the value of external assets is determined (particularly in expanded reproduction, see [40]) as well as of its dependence in turn on those claims (see e.g. [12, p. 188], [33]) is completely disregarded.

[5] The word *liquidation* does not have here any connection with any fire sale. Following [42],

> an investment system – [defined as] any set of parties who have agreed upon a method of calculating the obligation of each one of them to each of the others – is said to be in equilibrium if none of its parties has an obligation to any of the others and the liquidation value at a given time of any party A of an investment system is defined to be the net amount (with proper algebraic sign) that would be received by A in the process of instantaneously bringing the system into equilibrium at that time.

[6] The absence of any reference to agents' behavior (like in [20]) may help keep the distinction between agents and balance sheets (so avoiding to equate – to put it as in [34, p. 11] – "the living with the non living")

[7] In the present scheme the word *shock* only refers to a reduction in the value of external assets (including a reduction due to an act of consumption), with no implication as to the speed at which it occurs.

[8] For equity-type obligations this is always the case as in a liquidation process the underlying contractual arrangement provides that they would be paid back only once all the debt-type obligations have been paid at their contractual value.

[9] In this case the new liquidation values may be seen as those which if taken as new face values (after a possible debt-restructuring process) would make all the balance sheets pass the "balance sheet test" which requires "that all liabilities which have accrued so far have to be covered by existing assets", see [44, p. 189].

[10] For credit-type links this amounts to the face value of the obligation.

[11] We assume that $x_{ijk} > 0 \Rightarrow x_{jik} = 0$, i.e. opposite financial obligations of the same type between two balance sheets are offset, but between two balance sheets there can be links of opposite sign if of different seniority: $h \neq k \Rightarrow (x_{ijk} > 0 \not\Rightarrow x_{jih} = 0)$; furthermore, we assume that even after a shock there are enough assets in the debtor's balance sheet to guarantee that collateralized debts are fully repaid (put in another way, x_{ij3} is the amount up to which the collateralization is fully effective).

e_i = before-the shock value of the external assets in balance sheet i: claims which are not the obligation of any other element of $\mathbb{B}$ (physical assets are included in this category)[12]

l_i = loss on the external assets in balance sheet i

k_i = before-the-shock value of the external capital in balance sheet i[13]

g_i = loss on the external capital in balance sheet i [14]

$\theta(.)$ = Heaviside step function

$z_i = \sum_r x_{ri1}$ (with $r = 1, \cdots, |\mathbb{B}|$) overall amount of equity-type obligations of balance sheet i

$p_i = \sum_r x_{ri2}$, overall amount of (uncollateralized) obligations of balance sheet i

$c_i = \frac{\sum_{jk} x_{ijk} + \sum_{jk} u_{ijk} + e_i - l_i - \sum_r x_{ri3}}{\sum_r x_{ri2}}$ is the cover ratio (asset/liability) of balance sheet i

$\mathbb{\Sigma} = \left\{ ijk : (i,j,k) \in \{1,...,|\mathbb{B}|\}^2 \times \{1,2,3\} \wedge i \neq j \right\}$ set of the (three-element) indexes of the claims x_{ijk}

$\mathbb{T} = \left\{ ij : (i,j) \in \{1,...,|\mathbb{B}|\}^2 \wedge i \neq j \right\}$ set of the (two-element) indexes of the exposures q_{ij}

$\phi :\ \mathbb{\Sigma} \rightarrow \{1, \cdots, |\mathbb{\Sigma}|\}$ is a function mapping the three-element index ijk into a single-element index; so $\sigma = \phi(ijk)$ [15]

$\psi :\ \mathbb{T} \rightarrow \{1, \cdots, |\mathbb{T}|\}$ is a function mapping the two-element index ij into a single-element index; so $\varrho = \psi(ij)$

$\mathbb{\Pi}_{\phi(ijk)} = \{\phi(jrt) : r \in \{1,...,|\mathbb{B}|\} \wedge r \neq j \wedge t \in \{1,2,3\}\}$ set of the parent indexes of the index $\sigma = \phi(ijk)$

$\mathbb{A}_{\sigma_0} = \left\{ \sigma_k : \exists(\sigma_0, \cdots, \sigma_k) \wedge \sigma_i \in \mathbb{\Pi}_{\sigma_{i-1}} \wedge 0 < i \leq k \right\}$ set of the ancestor indexes of σ_0

$\mathbb{\Gamma}_{\sigma_0} = \{\sigma_k : \exists(\sigma_0, \cdots, \sigma_k) \wedge \sigma_i \in \mathbb{\Pi}_{\sigma i+1} \wedge 0 \leq i < k\}$ set of the descendant indexes of σ_0

$\mathbb{\Psi}_{\sigma_0} = \{\sigma_k : \mathbb{\Pi}_{\sigma_k} = \mathbb{\Pi}_{\sigma_0} \wedge k \neq 0\}$ is the set of siblings indexes of the index σ_0

$\mathbf{x} = \left[x_1 \cdots x_\sigma \cdots x_{|\mathbb{\Sigma}|} \right]^T$ [16]

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[12] In a graph scheme, that would amount to a claim towards a source node.

[13] Looking at the balance sheet of an individual (a physical person), k amounts to the difference between total assets and total liabilities, i.e. to the part of the individual's total assets which is actually its own as indeed it is not matched by any claim on the asset side of other balance sheets. In a graph scheme that would amount to an obligation towards a sink. Of course $\sum_i e_i = \sum_i k_i$ as, summing the balance sheet equation $e_i + \sum_{jh} x_{ijh} = k_i + \sum_{rh} x_{rih}$ (where $i, j, r = 1, \cdots, |\mathbb{B}|$ and $h = 1, 2, 3$) over all the balance sheets, we obtain $\sum_i e_i + \sum_i \sum_{jh} x_{ijh} = \sum_i k_i + \sum_i \sum_{rh} x_{rih}$ and, given that $\sum_i \sum_{jh} x_{ijh} = \sum_i \sum_{rh} x_{rih}$, we have that $\sum_i e_i = \sum_i k_i$.

[14] Similarly to footnote 13, we have that $\sum_i l_i = \sum_i g_i$ as, summing the after-the-shock balance sheet equation $e_i - l_i + \sum_{jh} y_{ijh} = k_i - g_i + \sum_{rh} y_{rih}$ (where $i, j, r = 1, \cdots, |\mathbb{B}|$ and $h = 1, 2, 3$) over all the balance sheets and, given that $\sum_i \sum_{jh} y_{ijh} = \sum_i \sum_{rh} y_{rih}$, we have that $\sum_i l_i = \sum_i g_i$, which amounts, for the whole set of balance sheets, to the "principle of capital structure irrelevance" mentioned in [28] with reference to a group of firm.

[15] In the following, the three-Latin-letter index or the single-Greek-letter index will be used as more convenient; so while e.g. x_{ijk} and x_σ may refer to the same claim, the former notation will be used whenever the information about the two parties and the type of claim is relevant, whereas the latter will be used when the value of the claim only needs to be identified as an element of a tuple (abusing notation, x_{ijk} will be used instead of $x_{\sigma(ijk)}$).

[16] Vector $\mathbf{x}$ is obtained by putting in lexical order the elements of the three-dimensional array $\left[x_{ijk}\right]_{\{1,...,|\mathbb{B}|\}^2 \times \{1,2,3\}}$ whose index $ijk \in \mathbb{\Sigma}$ ($\mathbf{y}$, $\mathbf{u}$ and the following bold letter vectors are obtained in the same way from $\left[y_{ijk}\right]_{\{1,...,\mathbb{B}\}^2 \times \{1,2,3\}}$, $\left[u_{ijk}\right]_{\{1,...,n\}^2 \times \{1,2,3\}}$ and so on).

$$\mathbf{y} = \begin{bmatrix} y_1 \cdots y_\sigma \cdots y_{|\Sigma|} \end{bmatrix}^T$$

$$\mathbf{u} = \begin{bmatrix} u_1 \cdots u_\sigma \cdots u_{|\Sigma|} \end{bmatrix}^T$$

$$\mathbf{q} = \begin{bmatrix} q_1 \cdots q_\varrho \cdots q_{|\mathbb{T}|} \end{bmatrix}^T$$

$$\mathbf{v} = \begin{bmatrix} v_1 \cdots v_\varrho \cdots v_{|\mathbb{T}|} \end{bmatrix}^T$$

$$\mathbf{e} = \begin{bmatrix} e_1 \cdots e_i \cdots e_{|\mathbb{B}|} \end{bmatrix}^T$$

$$\boldsymbol{\ell} = \begin{bmatrix} l_1 \cdots l_i \cdots l_{|\mathbb{B}|} \end{bmatrix}^T$$

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \cdots \lambda_i \cdots \lambda_{|\mathbb{B}|} \end{bmatrix}^T$$

$$\mathbf{k} = \begin{bmatrix} k_1 \cdots k_i \cdots k_{|\mathbb{B}|} \end{bmatrix}^T$$

$$\mathbf{g} = \begin{bmatrix} g_1 \cdots g_i \cdots g_{|\mathbb{B}|} \end{bmatrix}^T$$

Definition 1. *Overall displacement* is defined as the taxicab length ($\sum_{ijk} |u_{ijk}|$) of the displacement vector $\mathbf{u}$.

In Subsection 2.1 a novel prove for the existence and unicity of the displacement vector [17] is proposed, which – while not relying on the single point theorem as it is the case in [22] and several subsequent studies – [18] exploits the equivalence, in the transmission of displacement, between the role of equity-claims in solvent balance sheets and that of credit-claims in insolvent balance sheets.
In Subsection 2.2 the displacement vector is calculated as the solution of a mixed linear program. Use of linear programming as proposed in [23] was also mentioned in [22] where, however, the fictitious default algorithm was instead adopted [19] as more efficient; [20] nevertheless, in the present contest, the use of a *mixed* linear program [21] allows (in Subsection 2.3) to decompose overall displacement into a component due to the seniority structure and a component due to the structure of bilateral exposures (disregarding seniority); this is done by backward relaxing first the seniority and then the bilateral exposures constraints.
In Subsection 2.3 a quantification of the different contributions to the overall displacement [22] is provided along with some examples (in Appendix A.8) showing that larger overall displacement may be associated with a higher as well as with a lower capitalization and with more as well as with less total claim.
In Section 3 the reasons for this non-monotonicity are shown and exemplified looking into the possible

[17] The displacement vector $\mathbf{u}$ is related to the payment vector $\mathbf{y}$, which is dealt with in [22], by the relationship $\mathbf{u} = \mathbf{y} - \mathbf{x}$; furthermore the definition of payment vector is here more general as it refers to all (credit and equity) obligations so including also *equity values* in the terminology of [24].

[18] See [1] and – also accounting for different types of claims (credit and equity) – [47], [28] and [7].

[19] Due to a typo, the original formulation of function $FF_{p'}$ in the description of the algorithm in [22] was
$FF_{p'} \equiv \Lambda(p')(\Pi^T(\Lambda(p')p + (I - \Lambda(p')\bar{p})) + e) + (I - \Lambda(p'))\bar{p}$ instead of
$FF_{p'} \equiv \Lambda(p')(\Pi^T(\Lambda(p')p + (I - \Lambda(p'))\bar{p}) + e) + (I - \Lambda(p'))\bar{p}$; in [24] the typo was corrected but a closing round bracket was missing.

[20] Furthermore, as defaulting nodes (balance sheets) are added one by one in the fictitious default algorithm, it also provides a measure of the health of each node. A similar algorithm is also adopted in a number of studies, among which [47], [28].

[21] We need introducing binary (solvency) variables in order to account for a possible equity channel of contagion, whereas a simple linear programming in [22] and [23] was adequate, given only the possibility of contagion through uniform seniority claims considered therein.

[22] The idea of adding up the displacement of all exposures is taken from [10], where the amount of total available post-bankruptcy deposits is included in a simplified calculation of social welfare.

effects of different types of funding transactions on overall displacement.
In Subsection 3.1 the effects of small transactions on the displacement vector are described in terms of a deformation gradient. This description differs from the one that is made in [26] (in terms of sensitivity of the payment vector to perturbations on the matrix of bilateral),[23] as in the present study changes in the bilateral exposures, being the results of funding transactions, are always accompanied by changes in other balance sheet items (namely external assets). Furthermore, the analysis is not limited to the interbank market: this allows to highlight possible effects, of changes in bilateral exposures, on external capital (which in the current scheme is typically found to the right of non-bank balance sheets) as well as the role of the latter in decoupling displacement from claims growth.

2 Displacement[24]

The values for u_{ijk} are the solution to the system of $|\Sigma|$ equations

$$u_{ijk} - \left\{ \delta_{1\,k} \theta(c_j - 1) \tfrac{p_j}{z_j} (c_j - 1) + \delta_{2\,k} \Big[[1 - \theta(c_j - 1)]\, c_j + \theta(c_j - 1) \Big] + \delta_{3\,k} - 1 \right\} x_{ijk} = 0 \qquad (1)$$

where $ijk \in \Sigma$, $\delta_{i\,k}$ is the Kronecker delta and if $x_{ijk} = 0$ we set $u_{ijk} = 0$.

Mapping the three-letter index ijk into a single-letter index σ, by the one-to-one correspondence $\sigma : \Sigma \to \{1, \cdots, |\Sigma|\}$, the set of equations (1) may also be written as

$$\begin{cases} f_1 \left(x_1, \cdots, x_\sigma, \cdots, x_{|\Sigma|}, u_1, \cdots, u_\sigma, \cdots, u_{|\Sigma|} \right) = 0 \\ \cdots \\ f_\sigma \left(x_1, \cdots, x_\sigma, \cdots, x_{|\Sigma|}, u_1, \cdots, u_\sigma, \cdots, u_{|\Sigma|} \right) = 0 \\ \cdots \\ f_{|\Sigma|} \left(x_1, \cdots, x_\sigma, \cdots, x_{|\Sigma|}, u_1, \cdots, u_\sigma, \cdots, u_{|\Sigma|} \right) = 0 \end{cases} \qquad (2)$$

By definition, collateralized claims are assumed never to be displaced (i.e. u_{ij3} is always zero); that means that only the part of a collateralized claim that is actually recovered after the shock is denoted with x_{ij3}, whereas the complement to the whole original claim is included in x_{ij2}. As a consequence there is a limit to the possible expansion of collateralized claims, i.e. $\sum_r x_{ri3} \leq \sum_{jk} x_{ijk} + \sum_{jk} u_{ijk} + e_i - l_i$.

2.1 Existence and uniqueness of the displacement vector

The existence and uniqueness of the solution (and the relative conditions) are shown in this Section without resorting to the fixed-point theorem.
Existence
As $\theta(c_j - 1) \in \{0, 1\}$, system (1) belongs to the following family of $2^{|\mathbb{B}|}$ systems of linear equations parameterized by the parameter $t = 1, \cdots, 2^{|\mathbb{B}|}$:

$$u_{ijk} - \left\{ \delta_{1\,k} s_j(t) \tfrac{p_j}{z_j} (c_j - 1) + \delta_{2\,k} \Big[(1 - s_j(t))\, c_j + s_j(t) \Big] + \delta_{3\,k} - 1 \right\} x_{ijk} = 0$$

in which if $x_{ijk} = 0$ we set $u_{ijk} = 0$ and where $ijk \in \Sigma$, $\delta_{i\,k}$ is the Kronecker delta and $s_j(t) = \left[1 - \lfloor \frac{t}{2^{j-1}} \rfloor\right] \pmod 2$ is the j^{th} component of the vector valued function $\mathbf{s} : \left\{1, \cdots, 2^{|\mathbb{B}|}\right\} \to \{0, 1\}^{|\mathbb{B}|}$

[23] Sensitivity to various risk factors (external assets, risk-free interest rate, etc.) is instead dealt with in [7].
[24] We only use the word *loss* to denote a reduction of the value of external assets, whereas a decline in the value of a financial claim would be referred to as a *displacement*; this is in line with the *National Wealth Approach* (NWA) as described in [29] according to which "...wealth or net worth is by construction equal to real, nonfinancial assets, since financial assets are conceptually equal to financial liabilities" and "...In the aggregate, national wealth is invariant to declines in the value of loans."

which provides, for every value of the parameter t, a (hypothetical) combination of solvency statuses, one for each balance sheet. [25] The family of systems may also be written [26] as

$$u_{ijk} - \left[\delta_{1\,k} s_j(t) \frac{x_{ijk}}{z_j} + \delta_{2\,k} \left(1 - s_j(t)\right) \frac{x_{ijk}}{p_j}\right] \sum_{rh} u_{jrh} = -\delta_{1\,k} \left[z_j + s_j(t) \left(l_j - z_j\right)\right] \frac{x_{ijk}}{z_j} - \delta_{2\,k} \left(1 - s_j(t)\right) \frac{x_{ijk}}{p_j} \left(l_j - z_j\right)$$

and, after defining a matrix $\mathbf{M}(t)$ such that [27]

$$m_{\phi(ijk),\phi(qrh)}(t) = \begin{cases} 1 & \text{if } qrh = ijk \\ -\delta_{1\,k} s_j(t) \frac{x_{ijk}}{z_j} - \delta_{2\,k} \left(1 - s_j(t)\right) \frac{x_{ijk}}{p_j} & \text{if } q = j \wedge \delta_{1\,k} z_j + \delta_{2\,k} p_j > 0 \\ 0 & \text{all the other cases} \end{cases} \tag{3}$$

(where $q, r, i, j = 1, \cdots, |\mathbb{B}|$ and $h, k = 1, 2, 3$)

and a vector $\mathbf{a}(t)$ such that [28] $a_{\sigma(ijk)}(t) = -\delta_{1\,k} \left[z_j + s_j(t) \left(l_j - z_j\right)\right] \frac{x_{ijk}}{z_j} - \delta_{2\,k} \left(1 - s_j(t)\right) \frac{x_{ijk}}{p_j} \left(l_j - z_j\right)$, as

$$\mathbf{M}(t)\mathbf{u} = \mathbf{a}(t) \tag{4}$$

If system (1) does have a solution, it must be equal to the solution of that single system belonging to the family (4) for which $s_j(t) = \theta(c_j - 1)$ if such a system exists. In order to prove that it does, we consider the system $S : \mathbf{M}(2^{|\mathbb{B}|})\mathbf{u} = \mathbf{b}(2^{|\mathbb{B}|}; \boldsymbol{\lambda})$ where $\mathbf{b}(2^{|\mathbb{B}|}; \boldsymbol{\lambda})$ is obtained by replacing in $\mathbf{a}(t)$ the constant vector $\boldsymbol{\ell}$ with a variable vector $\boldsymbol{\lambda} \in [\mathbf{0}, \boldsymbol{\ell}]$ and setting $t = 2^{|\mathbb{B}|}$; we start moving [29] $\boldsymbol{\lambda}$ from $\mathbf{0}$ (corresponding to the *before-the-shock* situation when all the balance sheets are solvent, i.e. $t = 2^{|\mathbb{B}|}$) towards $\boldsymbol{\ell}$ up to (and included) the value where for one balance sheet (say the j^{th}) the asset/liability ratio becomes one ($\frac{\sum_{ik} x_{jik} + \sum_{ik} u_{jik} + e_j - \lambda_j - \sum_i x_{ij3}}{\sum_i x_{ij2}} = 1$); this happens just before $t = 2^{j-1}$ (if the j^{th} balance sheet is the first to become insolvent along the chosen path) [30] so we call $\boldsymbol{\lambda}_{t=2^{j-1}}$ the correspondent value of $\boldsymbol{\lambda}$; as long as no balance sheet is insolvent, [31] $\mathbf{u}$ is a monotone non

[25] The notation $\lfloor . \rfloor$ stands for the floor function.

[26] In case of claims on the right side of household balance sheets (which may be only credit-claims), the corresponding equations are instead $u_{ij2} - \left(1 - s_j(t)\right) \frac{x_{ij2}}{p_j} \sum_{rk} u_{jrk} = -\left(1 - s_j(t)\right) \frac{x_{ij2}}{p_j} \left(l_j - k_j\right)$.

[27] Matrix $\mathbf{M}(t)$ shares all the properties of matrix $\mathbf{F}_u$ in Section 3.1.3, which in the following will be referred to when needed, including non-singularity (see the paragraph on *Uniqueness* at the end of this Subsection).

[28] Again, it would be $a_{\sigma(ijk)}(t) = -\left(1 - s_j(t)\right) \frac{x_{ijk}}{p_j} \left(l_j - k_j\right)$ in case of claims to the right of household balance sheets.

[29] We make the assumption that the set of balance sheets which are insolvent at a level of external asset losses ($\boldsymbol{\ell}$) element-wise lower or equal to its actual level ($\boldsymbol{\lambda}$) is a subset of the actual set of insolvent balance sheets; as a consequence the actual path is not relevant as long as it traces a continuous curve from $\mathbf{0}$, which is a vector of all zeros to $\boldsymbol{\ell} = \left[l_1 \cdots l_i \cdots l_{|\mathbb{B}|}\right]^T$: different paths may only change the order in which different balance sheets will show an asset-liability ratio equal to one.

[30] In general, if the set of the indexes of the insolvent balance sheets is Υ, the value of t corresponding to the actual solvency situation is $t = \sum_{j \in \Upsilon} 2^{j-1}$. These procedure should not be confused with the *fictitious default algorithm* in [22] where the spread of contagion is divided into several stages and a new default occurs every time the effect of the (full amount of) the original shock is transmitted from one balance sheet to another; in the present proof, instead, the solution is approached by adding further fractions of the original shock step by step.

[31] The value of $\boldsymbol{\lambda}_{t=2^{j-1}}$ sets the limit beyond which the displacement would become irreversible as it would start affecting also credit claims, which – differently from equity claims accounted for to the right of solvent balance sheets – would not recover their original value once the external losses were removed; this limit is analogous to what in plasticity theory is called an *elastic limit*, which in this case would define the *elastic region* $[\mathbf{0}, \boldsymbol{\lambda}_{t=2^{j-1}}]$ (on plasticity theory see [39] p. 60).
As a consequence, while equity claims do transmit displacement not differently from credit claims, they make the system more resilient; as long as the external losses are within the elasticity region, a reduction of the value of equity

positive function [32] of $\boldsymbol{\lambda}$, so at that point [33] a (partial) displacement $\mathbf{u}_{[2^{|\mathbb{B}|},\boldsymbol{\lambda}_{t=2^{j-1}}]}$ may be calculated as the solution of $\mathbf{M}(t)\mathbf{u} = \mathbf{b}(t;\boldsymbol{\lambda})$ after setting $t = 2^{|\mathbb{B}|}$ and $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{t=2^{j-1}}$. We consider then a new (*before-the-shock*) system S' : $\mathbf{M}'(t)\mathbf{u} = \mathbf{b}'(t;\boldsymbol{\lambda})$ where, starting from S, we incorporate this first partial displacement into the *before-the-shock* values of the claims: $\mathbf{x}' = \mathbf{x} + \mathbf{u}_{[2^{|\mathbb{B}|},\boldsymbol{\lambda}_{t=2^{j-1}}]}$, the first partial loss $\boldsymbol{\lambda}_{[t=2^{j-1}]}$ into the external assets $\mathbf{e}' = \mathbf{e} - \boldsymbol{\lambda}_{[t=2^{j-1}]}$ (resetting variable $\boldsymbol{\lambda}$ lambda to zero: $\boldsymbol{\lambda} = \mathbf{0}$) and rename [34] each credit claim to the right of the j^{th} balance sheet x'_{ij2} ($i = 1, ..., |\mathbb{B}|$) as $x'_{(ij2/ij1)}$. [35]
At this point in S' we set $t = 2^{|\mathbb{B}|}$ and start again moving $\boldsymbol{\lambda}$ from $\mathbf{0}$ towards $\boldsymbol{\ell}' = \boldsymbol{\ell} - \boldsymbol{\lambda}_{[t=2^{j-1}]}$ up to (and included) the point $\boldsymbol{\lambda}_{[t=2^{r-1}]}$ where for a second balance sheet (say the r^{th}) the asset/liability ratio becomes one and here we can calculate the displacement $\mathbf{u}'_{[2^{|\mathbb{B}|},\boldsymbol{\lambda}_{t=2^{r-1}}]}$ as the solution to S' in which again we have set [36] $t = 2^{|\mathbb{B}|}$ and $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{[t=2^{r-1}]}$; we repeat the process until in the $S^{[n]}$ system we have that $\boldsymbol{\lambda}_{[n]} = \boldsymbol{\ell} - \sum_{i\in\Upsilon} \boldsymbol{\lambda}_{[t=2^{i-1}]}$ where Υ is the set of the indexes of the balance sheets whose asset/liability ratios happened to become one in the process of increasing $\boldsymbol{\lambda}$; so in the end the total displacement vector will amount to $\mathbf{u} = \mathbf{u}_{[2^{|\mathbb{B}|},\boldsymbol{\lambda}_{t=2^{j-1}}]} + \mathbf{u}'_{[2^{|\mathbb{B}|},\boldsymbol{\lambda}_{t=2^{r-1}}]} + \cdots + \mathbf{u}^{[n-1]}_{[2^{|\mathbb{B}|},\boldsymbol{\lambda}_{t=2^{q-1}}]} + \mathbf{u}^{[n]}_{[2^{|\mathbb{B}|},\boldsymbol{\lambda}_{[n]}]}$ (if q is the index of the balance sheet whose asset/liability ratio was the last to become one).
Uniqueness
In order to prove that the system defined by Eq. (1) has a unique solution it would suffice to show that this is the case for every system belonging to family of systems defined by Eq. (4); for each value of t we get a particular system $\bar{\mathbf{M}}\mathbf{u} = \bar{\mathbf{a}}$ in which matrix $\bar{\mathbf{M}}$ shares the same properties of matrix $\mathbf{F}_u$ in Section 3.1.3 (but for the fact that it does not need Assumption 2) including non-singularity (see Proposition 3), provided that the assumption (analogous to Assumption 4) is made that at least one of the (credit-type or equity-type) obligations [37] included in each strongly connected component of $G\left(\bar{\mathbf{M}}\right)$ is recorded on the liability side of a balance sheet which has also some other obligation not included in the same strongly connected component.

claims – the only claims affected in this region – does not imply insolvency (i.e. the extinction of the balance sheet on whose right side the claims are accounted for): the higher their amount the wider the possibility that total claims return to their original value, once external losses have been recovered.
These results are consistent with [35], which (in the case of banks) finds that there is "no association between more capital and less risk of banking crisis" but "...economies with better capitalized banking systems recover more quickly from financial crises".

[32] Given that $\frac{\partial b_j(t;\lambda_j)}{\partial \lambda_j} \leq 0$ and $\mathbf{M}^{-1}(t)$ is nonnegative (see the proof of Proposition 3 in Section 3.1.3).

[33] As well as at all the previous points $\boldsymbol{\lambda} < \boldsymbol{\lambda}_{t=2^{j-1}}$

[34] This is justified by the fact that for the newly insolvent balance sheet j in absence of equity claims on the right side, the credit claims *de facto* may be considered as equity. The notation, which follows the convention introduced in [49] for a variable substitution (according to which α/β means substitution of the variable β for the variable α), allows to keep track of the former seniority of a claim now downgraded to equity; in this case it stands for substitution of the index $ij1$ for the index $ij2$; correspondingly we also substitute the new index σ' for the synthetic index σ (introduced on page 5), where σ' is defined by $x'_{(\sigma/\sigma')} = x'_{\sigma(ij2/ij1)}$.

[35] It should be noted that as a consequence $\mathbf{M}'(t) \neq \mathbf{M}(t)$ and $\mathbf{b}'(t;\boldsymbol{\lambda}) \neq \mathbf{b}(t;\boldsymbol{\lambda})$ even if they are all calculated at $t = 2^{|\mathbb{B}|}$; more specifically $\mathbf{M}'(2^{|\mathbb{B}|}) = \mathbf{M}(2^{j-1})$: while balance sheet solvency status does not change over the different systems, conventional before-the-shock amounts and seniority of claims do.

[36] As a matter of fact we always deal with a set of solvent balance sheets, as switching from solvency to insolvency is avoided by renaming as equity the credit claims towards the insolvent balance sheet.

[37] As in Section 3.1.3, we say that an obligation is included in a subgraph of G if its relative share ($\bar{\mathbf{M}}[ijk, jmv] = \delta_{1\,k}\frac{x_{ijk}}{z_j} + \delta_{2\,k}\frac{x_{ijk}}{p_j}$ (with $i, j, m = 1, \cdots, |\mathbb{B}|$ and $k, v = 1, 2, 3$)) is the weight of an edge of the subgraph. We refer to Section 3.1.3 also for the notions of $G\left(\bar{\mathbf{M}}\right)$, the graph of a matrix and of strongly connected component of a graph.

2.2 Solving Eq. (1) as a mixed linear programming problem

In order to have positive decision variables in the linear programming problem we introduce the set of variables $\{w_{ijk}\}_{ijk\in\mathbb{I}}$, defined as follows

$$w_{ijk} = -u_{ijk}$$

We minimize the absolute value of the overall displacement subject to constraints stemming from the balance sheet identity, seniority rules and the need to guarantee the level playing field among same seniority siblings: [38]

$$\begin{aligned}
&\text{minimize} \sum_{ijh} w_{ijh} \\
&\text{subject to} \\
&\text{(balance sheet)} \sum_{ih} w_{ijh} - \sum_{rh} w_{jrh} + g_j = l_j \\
&\text{(seniority)} \begin{cases} w_{ij2} + x_{ij2}s_j & \leq x_{ij2} \\ w_{ij1} + x_{ij1}s_j & \geq x_{ij1} \\ w_{ij1} & \leq x_{ij1} \\ w_{ij3} & = 0 \\ g_j + k_j s_j & \geq k_j \\ g_j & \leq k_j \end{cases} \\
&\text{(siblings)} w_{ijh} - \frac{x_{ijh}}{\sum_r x_{rjh}} \sum_r w_{rjh} = 0 \\
&w_{ijh} \geq 0 \\
&s_j \in \{0,1\} \\
&i,j,r = 1, ..., |\mathbb{B}| \\
&h = 1,2,3
\end{aligned} \tag{5}$$

Given that Eq. (1) has a unique solution, there would be no need to minimize an objective function (the solutions would be found by maximizing the objective as well), however (beside allowing to make the calculation by means of a mixed linear programming application [39]) this underlines the role that seniority rules may have in increasing the overall displacement. As in a kind of benchmarking exercise, the next section shows how a better solution to the optimization problem (i.e. smaller overall absolute displacement) may be found by lifting the seniority constraint (i.e. by *ex-post* adopting those seniority rules which minimize the overall absolute displacement).

[38] Of course, in what follows, $j \in \mathbb{H} \implies g_j \geq 0 \wedge w_{ih1} = 0$ and $j \notin \mathbb{H} \implies g_j = 0 \wedge w_{ik1} \geq 0$ (in subscripts, the letter k is replaced by the letter h when needed to avoid confusion with the notation for external capital).

[39] As in the examples in Appendix A.8.

2.3 Total displacement decomposition

In order to look into a possible best seniority structure, we find out what the overall displacement would be when lifting the seniority and siblings constraints in the linear programming problem described in the previous section; to do that we define the set of variables $\{v_{ij}\}$ - representing the (negative) displacement of bilateral exposure - as

$$v_{ij} = -\sum_k u_{ijk}$$

and the "new" (mixed) linear programming problem as [40]

$$
\begin{aligned}
&\text{minimize} \sum_{ij} v_{ij} \\
&\text{subject to} \\
&\text{(balance sheet)} \sum_i v_{ij} - \sum_r v_{jr} + g_j = l_j \\
&\text{(lower seniority of external capital)} \begin{cases} v_{ij} + s_j \sum_h x_{ijh} & \leq \sum_h x_{ijh} \\ g_j + s_j k_j & \geq k_j \\ g_j & \leq k_j \end{cases} \\
&v_{ij} \geq 0 \\
&s_j \in \{0, 1\} \\
&i, j, r = 0, 1, ..., |\mathbb{B}| \\
&h = 1, 2, 3
\end{aligned}
\tag{6}
$$

which, if solved starting from the same data of the first example in Appendix A.8.1 (though disregarding seniority), would produce the following solution:

$$
\mathbf{v} = \begin{pmatrix} 15 \\ 56.7 \\ 11.2 \\ 8 \\ 0 \\ 0 \\ 53 \\ 0 \end{pmatrix} \begin{matrix} v_{14} \\ v_{15} \\ v_{21} \\ v_{31} \\ v_{43} \\ v_{56} \\ v_{71} \\ v_{73} \end{matrix} \quad \mathbf{g} = \begin{pmatrix} 0 \\ 11.2 \\ 8 \\ 0 \\ 0 \\ 2.25 \\ 53 \end{pmatrix} \begin{matrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{matrix}
$$

[40]In this formulation the seniority constraint has no bearing on the displacement of bilateral exposures but only accounts for the necessarily lower seniority of the external capital.

from which we can define an optimal claims vector e.g. by setting [41]

$$\begin{aligned} x^*_{ij1} &= \bar{v}_{ij} \\ x^*_{ij2} &= \sum_k x_{ijk} - \bar{v}_{ij} \text{ for } j \notin \{1, \cdots, \mathbb{H}\} \end{aligned}$$

and

$$\begin{aligned} x^*_{ij2} &= \bar{v}_{ij} \\ x^*_{ij3} &= \sum_k x_{ijk} - \bar{v}_{ij} \text{ for } j \in \{1, \cdots, \mathbb{H}\} \end{aligned}$$

$$\mathbf{x}^* = \begin{pmatrix} 15 \\ 15 \\ 56.7 \\ 26.3 \\ 11.2 \\ 18.8 \\ 8 \\ 32 \\ 15 \\ 20 \\ 53 \\ 20 \end{pmatrix} \begin{array}{l} x^*_{141} \\ x^*_{142} \\ x^*_{151} \\ x^*_{152} \\ x^*_{211} \\ x^*_{212} \\ x^*_{311} \\ x^*_{312} \\ x^*_{432} \\ x^*_{562} \\ x^*_{711} \\ x^*_{732} \end{array}$$

The new claims vector $\mathbf{x}^*$ differs from the one of A.8.1 only for the seniority attributed to the claims while keeping unchanged the size of the total flows from the i^{th} to the j^{th} balance sheet; the associated overall displacement $\sum_{ijk} |u^*_{ijk}| = 143.9$ is 5% lower than that of the original seniority structure: the percentage reduction in the overall displacement which can result from an optimized seniority structure may be expressed as[42]

$$1 - \frac{\sum_{ijk} |u^*_{ijk}|}{\sum_{ijk} |u_{ijk}|}$$

[41]The value for $\bar{v}_{ij}$ is the result of the mixed linear programming problem in 6. The optimal claims vector $\mathbf{x}^*_{ijk}$ of course is not unique; e.g. $\mathbf{x}^*$ could be defined instead by minimizing its (euclidean or taxicab) distance from $\mathbf{x}$:

$$\begin{aligned} &\text{minimize} \sum_{ijk \in \mathbb{Z}} \left| x^*_{ijk} - x_{ijk} \right| \\ &\text{subject to} \\ &\sum_k x^*_{ijk} = \sum_k x_{ijk} \\ &\sum_k u^*_{ijk} = -\bar{v}_{ij} \\ &u^*_{ij1} = -\min\left(\bar{v}_{ij}, x^*_{ij1}\right) \\ &u^*_{ijk} = \frac{x^*_{ijk}}{\sum_r x^*_{rjk}} \sum_r u^*_{rjk} \\ &i, j, r = 1, ..., |\mathbb{B}| \\ &k = 1, 2, 3 \end{aligned} \tag{7}$$

[42]If for a given loss vector $\boldsymbol{\ell}$ this seniority-related overall displacement is small, any reduction of the overall displacement potentially obtainable by changing the seniority structure (e.g. through regulatory interventions) would also be small; on the other hand, the effect of adding further constraints to the "new" mixed linear program in 6 (such as those brought about by regulatory interventions) would be, if any, to make its result worse (possibly even worse than the result of the original program 5).

Furthering the reasoning, we may find what the minimum level of overall displacement would be - given only $\mathbf{e}$ (external assets), $\boldsymbol{\ell}$ (external asset losses), $\mathbf{k}$ (external capital) and total claims ($\sum_{ijk} x_{ijk} = 291$); in order to do that, we redefine the (mixed) linear programming problem described in 6 introducing, as a new variable, the overall (debt and equity) bilateral exposure q_{ij}:

$$
\begin{gathered}
\text{minimize} \sum_{i,j \in \{1,\cdots,|\mathbb{B}|\}} v_{ij} \\
\text{subject to} \\
\text{(balance sheet)} \sum_i v_{ij} - \sum_s v_{js} + g_j = l_j \\
\sum_i q_{ij} - \sum_s q_{js} = e_j - k_j \\
\text{(total claims)} \sum_{ij} q_{ij} = 291 \\
\text{(total external loss)} \sum_j g_j = \sum_j l_j \\
\text{(lower seniority of external capital)}\ g_j = \min\left(\sum_s v_{js} + l_j, k_j \right) \\
0 \leq v_{ij} \leq q_{ij} \\
i, j, r, s = 1, ..., |\mathbb{B}|
\end{gathered}
\tag{8}
$$

Solving problem 8, starting from the same data of A.8.1 (i.e. given $\mathbf{e}$, $\mathbf{k}$, $\mathbf{l}$, $\mathbb{B}$ and total claims), would result in an overall displacement of the bilateral exposures of $\sum_{ij} v_{ij} = 72.4$:

$$
\mathbf{q} = \begin{pmatrix} 22.3 \\ 1.9 \\ 8.3 \\ 2.7 \\ 40.5 \\ 11.2 \\ 21.2 \\ 3 \\ 35.6 \\ 6.5 \\ 6.6 \\ 4.4 \\ 2.3 \\ 27.9 \\ 23.7 \\ 4.1 \\ 48.6 \\ 7.6 \\ 5.2 \\ 7.4 \end{pmatrix}
\begin{matrix} q_{14} \\ q_{15} \\ q_{16} \\ q_{21} \\ q_{23} \\ q_{24} \\ q_{25} \\ q_{26} \\ q_{31} \\ q_{34} \\ q_{35} \\ q_{36} \\ q_{45} \\ q_{46} \\ q_{65} \\ q_{71} \\ q_{72} \\ q_{73} \\ q_{74} \\ q_{75} \end{matrix}
\quad
\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.2 \\ 0 \\ 9.4 \\ 20.9 \\ 0 \\ 0.2 \\ 1.8 \\ 6.1 \\ 0 \\ 0 \\ 0 \\ 22.8 \\ 0.2 \\ 0 \\ 0 \\ 3.8 \\ 7 \end{pmatrix}
\begin{matrix} v_{14} \\ v_{15} \\ v_{16} \\ v_{21} \\ v_{23} \\ v_{24} \\ v_{25} \\ v_{26} \\ v_{31} \\ v_{34} \\ v_{35} \\ v_{36} \\ v_{45} \\ v_{46} \\ v_{65} \\ v_{71} \\ v_{72} \\ v_{73} \\ v_{74} \\ v_{75} \end{matrix}
$$

To sum up, overall displacement ($\sum_{ijk} |u_{ijk}| = 151.1$ in example A.8.1) may be decomposed into

- a component related to the **seniority** structure of the system, which amounts to the difference between the optimal value of the objective function in the *fully constrained* mixed linear programming problem (5) and that of problem (6), in which the seniority constraints have been relaxed (this component amounts to 5% of overall displacement in example A.8.1);
- a component related to the structure of **bilateral exposure**, amounting to the difference between the optimal value of the objective function in the *seniority relaxed* problem (6) and that of problem (8), in which also the bilateral exposure constraints have been relaxed (this component amounts to 47% of overall displacement in example A.8.1);
- a component representing the **minimum level** of overall displacement for a given $\mathbf{k}$, $\mathbf{l}$ and $\mathbf{e}$, which can be calculated as the optimal value of the objective function in problem (8) or more simply as the sum of that part of the external losses on the left side of each balance sheet which is not absorbed by the external capital on the right side of the same balance sheet, i.e. $\sum_i \max(l_i - k_i, 0), i = 1, ..., |\mathbb{B}|$; (48% of overall displacement in example A.8.1). [43]

2.4 Relative displacement

In problem (8) a total exposure constraint has been added in order to make the results as close as possible to those of example A.8.1 after removing bilateral exposures from the list of parameters. However the same result in terms of overall displacement (i.e. $\sum_{i,j\in\{1,\cdots,|\mathbb{B}|\}} v_{ij} = 72.4$) could be associated with a very different vector of bilateral exposures $\mathbf{q}$. [44] In the same way, overall displacement in example A.8.1 ($\sum_{ijk} |u_{ijk}| = 151.1$) may be associated with a much higher total exposure, as in the following example where, taking the same values for $\mathbf{k}$, $\mathbf{l}$ and $\mathbf{e}$ of example A.8.1, a different exposure vector (whose taxicab length is $\sum_{ij} q_{ij} = 1000$) would still be associated with a overall displacement [45] of $\sum_{ij} v_{ij} = 151.1$:

[43]Of course these imbalances include the ones in household balance sheets, which are referred to in [6], [9] and [38], but also those in all the other balance sheets, in which by definition there is no external capital (and as a consequence any external asset loss would necessarily be greater than external capital).

[44]In System (8) bilateral exposures only set upper bounds to displacements.

[45]It should be noted that this is not trivially due to an increase of the exposures which suffer no displacement.

$$
\mathbf{q} = \begin{array}{c} \begin{pmatrix} 150.7 \\ 69 \\ 53.5 \\ 2.7 \\ 4 \\ 4.1 \\ 9.6 \\ 9.5 \\ 2.9 \\ 112.1 \\ 56.4 \\ 64.7 \\ 4.4 \\ 152.8 \\ 67.9 \\ 36.8 \\ 0.1 \\ 125.6 \\ 0.1 \\ 17 \\ 10 \\ 12.1 \\ 24.1 \\ 9.8 \end{pmatrix} \end{array} \begin{array}{l} q_{13} \\ q_{14} \\ q_{15} \\ q_{16} \\ q_{21} \\ q_{23} \\ q_{24} \\ q_{25} \\ q_{26} \\ q_{31} \\ q_{34} \\ q_{35} \\ q_{36} \\ q_{41} \\ q_{43} \\ q_{45} \\ q_{46} \\ q_{54} \\ q_{56} \\ q_{71} \\ q_{73} \\ q_{74} \\ q_{75} \\ q_{76} \end{array} \quad \mathbf{v} = \begin{array}{c} \begin{pmatrix} 7.2 \\ 6.7 \\ 18.5 \\ 0 \\ 2.4 \\ 2.5 \\ 7.1 \\ 7 \\ 0 \\ 9.4 \\ 9.1 \\ 13 \\ 0 \\ 8.6 \\ 7.7 \\ 2.8 \\ 0 \\ 3.8 \\ 0 \\ 12.5 \\ 6.1 \\ 7.5 \\ 19.2 \\ 0 \end{pmatrix} \end{array} \begin{array}{l} v_{13} \\ v_{14} \\ v_{15} \\ v_{16} \\ v_{21} \\ v_{23} \\ v_{24} \\ v_{25} \\ v_{26} \\ v_{31} \\ v_{34} \\ v_{35} \\ v_{36} \\ v_{41} \\ v_{43} \\ v_{45} \\ v_{46} \\ v_{54} \\ v_{56} \\ v_{71} \\ v_{73} \\ v_{74} \\ v_{75} \\ v_{76} \end{array}
$$

Of course more claims may be the result of more intermediation [46] and, given the mechanism of claims generation through financial transactions - as described in Section 3 -, the presence of cycles makes it always possible to expand total claims without limits; [47] however, even though the exposure network sets the channels through which the displacement may spread, a structure of bilateral exposures with more cycles does not necessarily imply higher overall displacement. [48] Possible cycles in the exposure network, involving household balance sheets, [49] may not necessarily provide displacement amplifier channels, if the external capital of those household balance sheets is large enough to absorb the incoming displacement (along with their own external losses). [50] As a result the ratio between overall displacement and total claims $\frac{\sum_{ij} v_{ij}}{\sum_{ij} q_{ij}}$ would be very different in the two systems (0.52 in example A.8.1 and 0.15 in the example above). [51]

* * *

[46]See [2] on the building of long intermediation chains.

[47]This is shown in Appendix A.1; in general the distinction may be done between claims expansion through new links and claims expansion due to more financial flow through already existing links between balance sheets - see [17, p. 11].

[48]See Appendix A.3.

[49]Intermediation may also take place *de facto* through household balance sheets, see e.g. [14], [15], [16, pp. 300-301], [45, p. 10], [43, p. 71], [46], [48, p. 44].

[50]A possible reduction of relative displacement is one of the accounting effects of household balance sheet involvement in intermediation (and cycles), the other being a reallocation of external capital losses between household balance sheets; both effects are exemplified in Appendix A.2.

[51]In this case the assertion that more finance may reverse its function of facilitating the management of risk (as in [5]) does not apply as long as overall displacement is not increased.

Given the amount of total external assets [52] $\sum_j e_j = \sum_j k_j$, moving from one element of the set of possible (before-the-shock) claims vectors to another is the result of funding transactions. The effect of the external asset losses on this set may be described as its deformation, in analogy with the deformation of a set of material points (body) due to an external shock in continuous mechanics. [53] In the following, the deformation will be dealt with, first with reference to small transactions and then to finite transactions.

3 Funding transactions

Funding transactions may be seen as those sets of balancing changes in the involved balance sheets which include the (*synchronic* [54]) change of at least one claim (Δx_{ijk}). Their effects on the involved balance sheets depend on the nature of the claim x_{ijk} (uncollateralized credit, equity, fully collateralized credit), and on its accounting counterparties (external assets, other claims). However the analysis may be restricted to the effects of those funding transactions in which a claim changes in the context of the transfer of external assets. [55] Funding transactions in which the counterparty to the claim change (increase) is a change (decrease) of another claim (credit-type or equity-type) may always be expressed as the result of a sequence of transactions, funding the transfer of external assets. E.g., a funding transaction resulting in the transfer to the j^{th} balance sheet of a credit that the i^{th} balance sheet originally held toward another balance sheet (say the m^{th}) – expressed by the set $\{-\Delta x_{im2}, +\Delta x_{ijk}, +\Delta x_{jm2}\}$ – amounts to the sequence $\{-\Delta x_{im2}, +\Delta e_i, -\Delta e_m\}, \{+\Delta x_{ijk}, -\Delta e_i, +\Delta e_j\}, \{+\Delta x_{jm2}, -\Delta e_j, +\Delta e_m\}$.

3.1 Small transactions

In describing a small funding transaction between balance sheets i and j, we may express the external assets e_i and e_j on left side of the involved balance sheets as a function of the claim x_{ijk} against which they play the role of the counterparty (where $\frac{\partial e_i}{\partial x_{ijk}} = -1$ and $\frac{\partial e_j}{\partial x_{ijk}} = 1$). In order to avoid that a reduced overall displacement – e.g., if this happens to be the result of more credit – [56] might be attributed to a post-transaction reduction of the external losses, it is assumed that the loss on external assets is a non-negative function of the amount of those assets [57] (furthermore it is assumed that it may not grow more than that amount): [58]

Assumption 1. $0 \le \frac{\partial l_j}{\partial e_j} \le 1$.

[52] See footnote 13.

[53] See [39], in particular the first pages of Chapter 8, from which the setting followed, by analogy, in the next section is taken: we have here *before-the-shock* ($\mathbf{x}$) and *after-the-shock* ($\mathbf{y} = \mathbf{x} + \mathbf{u}$) claims vectors, instead of the "material" and "displaced configuration" points dealt with in [39].

[54] A financial claim x_{ijk} may undergo two types of changes: a change due to a financial transaction ($\mathrm{d}\, x_{ijk}$) and a change due to the displacement (u_{ijk}) which occurs as we go from a *before-the-shock* to an *after-the-shock* point in time. Borrowing terminology from [18], we may call the first one a *synchronic* change and the second a *diachronic* change.

[55] They may be physical assets or legal intangibles as in a sale (trade credit) or money as in a loan.

[56] See Eq. (10).

[57] In example A.8.1 the simplifying assumption is also made that the relation between e_i and l_i is linear and equal for both the balance sheets involved in the transaction.

[58] If the increase of external loss related to an increase of external assets were to be greater than the latter, the sign of the impact of a claim on the value of its siblings (See Subsection 3.1.2 - Eq.(10)) could never be positive: the second part of the assumption ($\frac{\partial l_j}{\partial e_j} \le 1$) allows us to focus attention on cases where the sign of this impact could instead be positive.

Furthermore, in order to differentiate the system of equations (1) [59], we assume that wherever it has a solution the asset/debt ratio c_j is not one and that p_j is positive and – unless on the right side of a household balance sheet – also z_j is always positive:

Assumption 2. *For every solution $\bar{u}_{ijk\in\Sigma}$ of the system of equations (1) we have that $\sum_{jk} x_{ijk} + \sum_{jk} \bar{u}_{ijk} + e_i - l_i - \sum_r x_{ri3} \neq \sum_r x_{ri2}$*

Assumption 3. *$p_j > 0$ for $j \in \mathbb{B}$ and $z_j > 0$ for $j \in \mathbb{B} \setminus \mathbb{H}$*

We may now define $\mathbf{F}_x = \left[\frac{\partial f_\rho}{\partial x_\sigma}\right]_{|\Sigma|\times|\Sigma|}$ and $\mathbf{F}_u = \left[\frac{\partial f_\rho}{\partial u_\sigma}\right]_{|\Sigma|\times|\Sigma|}$ so that, if $\det \mathbf{F}_u \neq 0$, [60] the displacement gradient $\mathbf{H} = \left[\frac{\partial u_\rho}{\partial x_\sigma}\right]_{|\Sigma|\times|\Sigma|}$ may be defined as

$$\mathbf{H} = -\mathbf{F}_u^{-1}\mathbf{F}_x \tag{9}$$

3.1.1 Deformation and displacement gradients

Given a combination of (before-the-shock) financial claims $\mathbf{x}$, corresponding to $\mathbf{y}$ in terms of (after-the-shock) balance sheet value, we may wonder how $\mathbf{y}$ changes as we move from $\mathbf{x}$ to $\mathbf{x} + \mathrm{d}\,\mathbf{x}$. For a small move $\mathrm{d}\,\mathbf{x}$ we have that $\mathrm{d}\,\mathbf{y} = (\mathbf{H} + \mathbf{I})\,\mathrm{d}\,\mathbf{x}$, where $(\mathbf{H} + \mathbf{I})$ is the deformation gradient. If $\mathrm{d}\,\mathbf{y} = \mathrm{d}\,\mathbf{x}$ (i.e. $\mathbf{H} = \mathbf{0}_{|\Sigma|\times|\Sigma|}$ where $\mathbf{0}_{|\Sigma|\times|\Sigma|}$ is the null matrix) this means that the two points ($\mathbf{x}$ and $\mathbf{x} + \mathrm{d}\,\mathbf{x}$) are equivalent in terms of displacement due to a negative shock.[61] If for some σ it happens that $\mathrm{d}\,y_\sigma \leq \mathrm{d}\,x_\sigma$, that means that, following a negative shock, the new point would face a greater absolute displacement in his σ-th component than the one from which we started. We move from $\mathbf{x}$ to $\mathbf{x} + \mathrm{d}\,\mathbf{x}$ through a set of funding transactions, so the deformation gradient $(\mathbf{H} + \mathbf{I})$ describes the effect produced on after-the-shock balance sheet values by changes in $\mathbf{x}$ due to funding transactions. This effect depends on the displacement gradient $\mathbf{H}$, that in turn may be decomposed into two components: $-\mathbf{F}_x$ that accounts for the initial impact of these transactions on the displacement of the liabilities in the balance sheets directly involved in the transaction [62] and $\mathbf{F}_u^{-1}$ that accounts for network effect (the transmission of the displacement from one claim to another).

3.1.2 Impact of funding transactions on the allocation of external asset losses (-$\mathbf{F}_x$)

The entries of the displacement gradient $\mathbf{H}$ describe the impact of a funding transaction on the displacement of a given claim. The opposite of the second factor of the right side of Eq. (9), i.e. $-\mathbf{F}_x$, refers to the initial impact of a funding transaction ($\mathrm{d}\,x_{ijk}$) on the displacements ($\mathrm{d}\,u_{*j*}$ and $\mathrm{d}\,u_{*i*}$) of the items to the right of the two balance sheets (the i^{th} and the j^{th} ones) directly involved in the transaction; they may be classified according to the kinship relation that links the impacted displacement ($\mathrm{d}\,u_{rst}$) to the claim change brought about by the funding transaction ($\mathrm{d}\,x_{ijk}$). These relations may be self relations $-\frac{\partial f_{ijk}}{\partial x_{ijk}}$, sibling relations $-\frac{\partial f_{rjt}}{\partial x_{ijk}}$ and parent-children relations $-\frac{\partial f_{rit}}{\partial x_{ijk}}$.[63]

`Self-relation`

[59] As formalized in Eq. (2).
[60] The conditions for that are specified in 3.1.3.
[61] This situation (corresponding to one case of rigid body in continuum mechanics) would occur (e.g.) if the original loss or its reallocation, which occurs as we move from $\mathbf{x}$ to $\mathbf{x} + \mathrm{d}\,\mathbf{x}$, only affected the external capital with no effect on the displacement of claims.
[62] We call *initial* the impact on the displacement of a given claim which is not the (second round) consequence of the impact of the funding transaction on a different claim.
[63] Children may have more than two parents here.

$$-\frac{\partial f_{ij1}}{\partial x_{ij1}} = \theta\,(c_j-1)\left[\left(1-\frac{x_{ij1}}{z_j}\right)\frac{p_j}{z_j}\,(c_j-1)+\frac{x_{ij1}}{z_j}\left(1-\frac{\partial l_j}{\partial e_j}\right)\right]-1\leq 0$$

$$-\frac{\partial f_{ij2}}{\partial x_{ij2}} = \left[1-\theta\,(c_j-1)\right]\left[\left(1-\frac{x_{ij2}}{p_j}\right)c_j+\frac{x_{ij2}}{p_j}\left(1-\frac{\partial l_j}{\partial e_j}\right)\right]+\theta\,(c_j-1)-1\leq 0$$

`Sibling-relations` (where $m \neq i$)

$$\begin{aligned}
-\frac{\partial f_{ij1}}{\partial x_{mj1}} &= \theta\,(c_j-1)\,\frac{x_{ij1}}{z_j}\left[1-\frac{p_j}{z_j}\,(c_j-1)-\frac{\partial l_j}{\partial e_j}\right]\gtreqless 0\\
-\frac{\partial f_{ij2}}{\partial x_{mj1}} &= \left[1-\theta\,(c_j-1)\right]\frac{x_{ij2}}{p_j}\left(1-\frac{\partial l_j}{\partial e_j}\right)\geq 0\\
-\frac{\partial f_{ij1}}{\partial x_{mj2}} &= -\theta\,(c_j-1)\,\frac{x_{ij1}}{z_j}\frac{\partial l_j}{\partial e_j}\leq 0\\
-\frac{\partial f_{ij2}}{\partial x_{mj2}} &= \left[1-\theta\,(c_j-1)\right]\frac{x_{ij2}}{p_j}\left(1-c_j-\frac{\partial l_j}{\partial e_j}\right)\gtreqless 0\\
-\frac{\partial f_{ij1}}{\partial x_{mj3}} &= -\theta\,(c_j-1)\,\frac{x_{ij1}}{z_j}\frac{\partial l_j}{\partial e_j}\leq 0\\
-\frac{\partial f_{ij2}}{\partial x_{mj3}} &= -\left[1-\theta\,(c_j-1)\right]\frac{x_{ij2}}{p_j}\frac{\partial l_j}{\partial e_j}\leq 0
\end{aligned}\tag{10}$$

`Child-relations` ($k \in \{1,2,3\}$)

$$-\frac{\partial f_{ij1}}{\partial x_{jmk}} = \theta\,(c_j-1)\,\frac{x_{ij1}}{z_j}\frac{\partial l_j}{\partial e_j}\geq 0$$

$$-\frac{\partial f_{ij2}}{\partial x_{jmk}} = \left[1-\theta\,(c_j-1)\right]\frac{x_{ij2}}{p_j}\frac{\partial l_j}{\partial e_j}\geq 0$$

By definition collateralizated claims may not be displaced so $\frac{\partial f_{ij3}}{\partial x_{mrk}} = 0$ for $i,j,m,r \in \{1,2,...|\mathbb{B}|\}$ and $k \in \{1,2,3\}$, whereas on uncollateralized credit claims and equity-type claims it all depends on kinship and seniority.[64]

Proposition 1. *In terms of the initial reallocation of the external asset loss in the balance sheets in which it is recorded, a (small) increase in an uncollateralised claim has an impact which is*

- ***non positive*** *on* ***its own*** *displacement and on that of its* ***lower seniority siblings,*** [65]
- ***non negative*** *on* ***its children and its higher seniority siblings*** *displacement,*
- ***mixed*** *on* ***same seniority siblings*** *displacement.*

The impact on same seniority siblings is positive if the increase in the external-assets loss due the higher external assets ($\frac{\partial l_j}{\partial e_j}$) is lower than the fraction of same seniority claims not covered by total assets – i.e. $(1-c_j)$ for credit claims to the right of insolvent balance sheets and $1-\frac{p_j}{z_j}\,(c_j-1)$ for equity claims to the right of solvent balance sheets – a condition which may be verified e.g. if

[64] Even in the original field of linguistic anthropology, in which they have been firstly developed (with reference to inheritance rights), the concepts of kinship and seniority tend to complement each other. So e.g. in Fanti language the distinction is made between *nua panyin* (senior sibling) and *nua kakraba* (junior sibling); see [37, p. 305].

[65] Of course this is nothing new, e.g. in [4] "...the use of collateral in repos withdraws securities from the pool of assets that would be available to unsecured creditors in the event of a bankruptcy".

the external asset which is the counterpart in the funding transaction is money. [66] In general due to network effects $\mathbf{H}[\rho,\sigma] \neq -\mathbf{F}_x[\rho,\sigma]$, however in absence of feedback effects $-\mathbf{F}_x$ also provide a measure of the final impact of a claim change on its own displacement as well as on that of its siblings, [67] i.e.,

$$(\det \mathbf{F}_u = 1) \implies \mathbf{H}[ijk, rjt] = -\mathbf{F}_x[ijk, rjt] \tag{11}$$

where the $[ijk, rjt]$ entries are those related to self and siblings relations (with $i, j, r = 1, \cdots, |\mathbb{B}|$ and $k, t = 1, 2, 3$)). [68]

3.1.3 Network amplification ($\mathbf{F}_u^{-1}$)

The only entries of $\mathbf{F}_u$ that are not zero are those referring to self-relations (which are equal to one) and to parent-children relations, i.e.

$$\frac{\partial f_{ijk}}{\partial u_{rst}} = \begin{cases} 1 & \text{if } rst = ijk \\ -\delta_{1\,k}\theta(c_j-1)\frac{x_{ijk}}{z_j} - \delta_{2\,k}\left[1-\theta(c_j-1)\right]\frac{x_{ijk}}{p_j} & \text{if } r = j \wedge \delta_{1\,k}z_j + \delta_{2\,k}p_j > 0 \\ 0 & \text{all the other cases} \end{cases} \tag{12}$$

(where $i, j, r, s = 1, \cdots, |\mathbb{B}|$ and $k, t = 1, 2, 3$)

Following [13], in order to calculate the inverse of $\mathbf{F}_u$, we consider $G(\mathbf{F}_u)$ the flow graph of $\mathbf{F}_u$ which is a $|\Sigma|$-node, weighted, labeled, directed graph such that if $\mathbf{F}_u[\sigma,\rho] \neq 0$, there is an edge (ρ,σ) directed from nodes ρ to σ with associated weight $\mathbf{F}_u[\sigma,\rho]$. If $\mathbf{F}_u[\sigma,\rho] = 0$, there is no edge directed from nodes ρ to σ. A directed path from node ρ to node σ, $P_{\rho\sigma} = (\rho,\gamma_1)(\gamma_1,\gamma_2)\cdots(\gamma_\mu,\sigma)$, where ρ, σ, γ_τ (with $\tau = 1, 2, \cdots, \mu$) are (all distinct) nodes in $G(\mathbf{F}_u)$, has weight

$$w(P_{\rho,\sigma}) = \mathbf{F}_u[\sigma,\gamma_\mu]\,\mathbf{F}_u[\gamma_\mu,\gamma_{\mu-1}], \cdots, \mathbf{F}_u[\gamma_2,\gamma_1]\,\mathbf{F}_u[\gamma_1,\rho] \tag{13}$$

(In general, if T is a subgraph of G, then $w(T)$ amounts to the product of the weights of the edges of T). If ρ and σ coincide, we have a directed circuit. A directed circuit consisting only of one edge is called a self-loop. A 1-factorial connection from ρ to σ is a subgraph which includes all the nodes of $G(\mathbf{F}_u)$ and contains *(a)* a directed path P from ρ to σ and *(b)* a set of node-disjoint directed circuits that include all the nodes of $G(\mathbf{F}_u)$ except those contained in P. A 1-factor is a set of directed disjoint circuits which include all nodes of $G(\mathbf{F}_u)$. A directed graph is called strongly connected if there is a path in each direction between each pair of vertices of the graph; a strongly connected component of a directed graph G is a subgraph that is strongly connected and is maximal with the property that no additional edges or vertices from G can be included in the subgraph without breaking its property of being strongly connected.
From equation (12) we have that there is one 1-factor which consists only of self-loops. Its weight is

[66] As pointed out by Graeber (see [31]), the case where credit transactions are beneficial to siblings credit claims is already mentioned by Rabelais in Gargantua and Pantagruel (see Book 3, Chapter III): "...he [the creditor] will always speak good of you in every company, ever and anon purchase new creditors unto you; to the end, that through their means you may make a shift by borrowing from Peter to pay Paul...".

[67] The reason for that will become clearer in Subsection 3.2, after describing network effects in Subsection 3.1.3, which will also show (see Eq. (17)) that, in absence of feedback, $\det \mathbf{F}_u = 1$.

[68] A sufficient, but not necessary, condition would be that $\det(\mathbf{I} - \mathbf{P}) = 1$ where $\mathbf{I}$ is the identity matrix, $\mathbf{P}$ is the $|\Sigma| \times |\Sigma|$ matrix of the contractual links between claims whose only non-zero entries are $\mathbf{P}[ijk, juv] = \delta_{1\,k}\frac{x_{ijk}}{z_j} + \delta_{2\,k}\frac{x_{ijk}}{p_j}$ (with $i, j, u = 1, \cdots, |\mathbb{B}|$ and $k, v = 1, 2, 3$).

1 as it amounts to the product of the diagonal terms. We call a cycle a directed circuit which is not a self-loop. In the case of $G(\mathbf{F}_u)$, if we consider a cycle C of ν elements, we have

$$C = (i_1 j_1 k_1, i_2 j_2 k_2)(i_2 j_2 k_2, i_3 j_3 k_3) \cdots (i_{\nu-1} j_{\nu-1} k_{\nu-1}, i_\nu j_\nu k_\nu)(i_\nu j_\nu k_\nu, i_1 j_1 k_1) \tag{14}$$

so,

the weight of a cycle is

$$w(C) = (-1)^\nu \left| \frac{\partial f_{i_1 j_1 k_1}}{\partial u_{i_2 j_2 k_2}} \frac{\partial f_{i_2 j_2 k_2}}{\partial u_{i_3 j_3 k_3}} \cdots \frac{\partial f_{i_{\mu-1} j_{\mu-1} k_{\mu-1}}}{\partial u_{i_\mu j_\mu k_\mu}} \cdots \frac{\partial f_{i_{\nu-1} j_{\nu-1} k_{\nu-1}}}{\partial u_{i_\nu j_\nu k_\nu}} \frac{\partial f_{i_\nu j_\nu k_\nu}}{\partial u_{i_1 j_1 k_1}} \right|$$

or, given that if the weight is not zero it must be that $j_{\mu-1} = i_\mu$,

$$w(C) = (-1)^\nu \left| \frac{\partial f_{i_1 i_2 k_1}}{\partial u_{i_2 j_2 k_2}} \frac{\partial f_{i_2 i_3 k_2}}{\partial u_{i_3 j_3 k_3}} \cdots \frac{\partial f_{i_{\mu-1} i_\mu k_{\mu-1}}}{\partial u_{i_\mu j_\mu k_\mu}} \cdots \frac{\partial f_{i_{\nu-1} i_\nu k_{\nu-1}}}{\partial u_{i_\nu j_\nu k_\nu}} \frac{\partial f_{i_\nu i_1 k_\nu}}{\partial u_{i_1 j_1 k_1}} \right|$$

and, reflecting the feedback character of a cycle,

$$w(C) = (-1)^\nu \left| \hat{x}_{i_1 i_2 k_1} \hat{x}_{i_2 i_3 k_2} \cdots \cdots \hat{x}_{i_{\mu-1} i_\mu k_{\mu-1}} \hat{x}_{i_\mu i_{\mu+1} k_\mu} \cdots \cdots \hat{x}_{i_{\nu-1} i_\nu k_{\nu-1}} \hat{x}_{i_\nu j_1 k_1} \right| \tag{15}$$

where

$$\hat{x}_{ijk} = \begin{cases} -\delta_{1\,k} \theta(c_j - 1) \frac{x_{ijk}}{z_j} - \delta_{2\,k} \left[1 - \theta(c_j - 1)\right] \frac{x_{ijk}}{p_j} & (\text{if } \delta_{1\,k} z_j + \delta_{2\,k} p_j > 0) \\ 0 & (\text{otherwise}) \end{cases} .$$

From [13], we have that [69]

$$\det \mathbf{F}_u = (-1)^{|\Sigma|} \sum_h (-1)^{L_h} w(h) \tag{16}$$

where

h is a 1-factor which includes all the nodes in $G(\mathbf{F}_u)$

L_h is the number of directed circuits in h

If h' is a 1-factor which includes at least one cycle and $L_{h'}$ is the number of directed circuits in h' the previous equation may be rewritten as

$$\det \mathbf{F}_u = 1 + (-1)^{|\Sigma|} \sum_{h'} (-1)^{L_{h'}} w(h')$$

as for the 1-factor which includes only self-loops (h_0) we have that $|\Sigma| = L_{h_0}$ and $(-1)^{2|\Sigma|} = 1$.

Given that the weight of all the self-loops included in h' is 1, if $C_{h'}$ is a cycle included in h', $\nu_{C_{h'}}$ is its number of edges and $\nu_{h'} = \sum_{C_{h'}} \nu_{C_{h'}}$, we have that

$$\begin{aligned} w(h') &= \prod_{C_{h'}} \left[(-1)^{\nu_{C_{h'}}} \left|w(C_{h'})\right|\right] \\ &= (-1)^{\nu_{h'}} \prod_{C_{h'}} \left|w(C_{h'})\right| \end{aligned}$$

Given that $L_{h'}$ is the sum of *(a)* the number of cycles $L_{C_{h'}}$ and *(b)* the number of self loops $L_{S_{h'}}$ and that $|\Sigma| = \nu_{h'} + L_{S_{h'}}$, we have that

$$(-1)^{|\Sigma| + L_{h'} + \nu_{h'}} = (-1)^{2\nu_{h'} + 2L_{S_{h'}} + L_{C_{h'}}} = (-1)^{L_{C_{h'}}}$$

so

$$\begin{aligned} \det \mathbf{F}_u &= 1 + \sum_{h'} (-1)^{|\Sigma| + L_{h'} + \nu_{h'}} \prod_{C_{h'}} \left|w(C_{h'})\right| \\ &= 1 + \sum_{h'} (-1)^{L_{C_{h'}}} \prod_{C_{h'}} \left|w(C_{h'})\right| \end{aligned} \tag{17}$$

[69] See Theorem 1 in [13].

Definition 2. An uncollateralized-credit (equity) obligation is actual if it is accounted for to the right of an insolvent (solvent) balance sheet.

Definition 3. A path (cycle) is actual if all of the obligations out of which it is built are actual. A path (cycle) is potential if it is not actual.

Assumption 4. *At least one of the (credit-type or equity-type) actual obligations* [70] *included in each strongly connected component of* $G(\boldsymbol{F}_u)$ *is recorded on the liability side of a balance sheet which has also some other actual obligations not included in the same strongly connected component.* [71]

Proposition 2. *Under Assumption 4,* $0 < \det \boldsymbol{F}_u \leq 1$.

Proof. If $\mathbf{F}_u$ is reducible – after row and column permutations – it may be partitioned into a triangular block matrix with the diagonal blocks corresponding to the strongly connected components of $G(\mathbf{F}_u)$; [72] so, given that $\det \mathbf{F}_u$ amounts to the product of the determinants of the diagonal blocks (of the permutated matrix), we only need proving the above mentioned bounds for the determinant of any of these blocks. Given Assumption 4, each of these blocks is an irreducibly diagonally dominant matrix with positive diagonal entries, so the real part of its eigenvalues is positive[73], furthermore, as all of its off-diagonal entries are non-positive, all its leading principal minors are positive [74], so also $\det \mathbf{F}_u$ is positive. If instead $\mathbf{F}_u$ may not be partitioned into a triangular block matrix, it is itself an irreducible diagonally dominant matrix to which all the properties mentioned above for the diagonal block also apply, including $\det \mathbf{F}_u > 0$. Given that, from Eq. (17), if $G(\mathbf{F}_u)$ has no cycles $\det \mathbf{F}_u = 1$, in order to prove that $\det \mathbf{F}_u \leq 1$ it suffices to show that $\det \mathbf{F}_u$ decreases monotonically as the absolute value of the weight of any of the cycles included in $G(\mathbf{F}_u)$ increases i.e. $\frac{\partial \det \mathbf{F}_u}{\partial |w(C)|} < 0$: let $\mathbb{K}$ be the set of all the cycles included in $G(\mathbf{F}_u)$, $\mathbb{K} \setminus \{C\}$ the set of the cycles included in $\mathbb{K}$ but C, $\mathbb{K}_{\not\cap C}$ the set of the cycles included in $\mathbb{K}$ which are pairwise disjoint from (i.e share no node with) C, [75] then, making explicit the contribution of C from Eq. (17), $\det \mathbf{F}_u$ may be rewritten as[76]

$$\det \mathbf{F}_u = \det \mathbf{F}_u \restriction_{\mathbb{K}\setminus\{C\}} - |w(C)| \det \mathbf{F}_u \restriction_{\mathbb{K}_{\not\cap C}} \tag{18}$$

where $\mathbf{F}_u \restriction_{\mathbb{K}\setminus\{C\}}$ (resp. $\mathbf{F}_u \restriction_{\mathbb{K}_{\not\cap C}}$) is the matrix obtained from $\mathbf{F}_u$, by setting to zero all the off-diagonal entries whose corresponding edges are not included in any of the cycles in $\mathbb{K} \setminus \{C\}$ (resp. $\mathbb{K}_{\not\cap C}$). Given that $\mathbf{F}_u \restriction_{\mathbb{K}_{\not\cap C}}$ has the same structure as $\mathbf{F}_u$, i.e. it is a square matrix with ones on the main diagonal, non-positive off-diagonal entries and non-negative (but for at least one column positive) column sums, $\det \mathbf{F}_u \restriction_{\mathbb{K}_{\not\cap C}} > 0$ and $\frac{\partial \det \mathbf{F}_u}{\partial |w(C)|} < 0$. □

Proposition 3. $\boldsymbol{F}_u^{-1}[\rho, \sigma] \geq 0$ *with* $\rho, \sigma = 1, 2, \cdots, |\Sigma|$.

Proof. This is another consequence of the fact that the real part of the eigenvalues of $\mathbf{F}_u$ is positive [77] □

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[70] Abusing language, we say that an obligation is included in a subgraph of G if its relative share ($\mathbf{P}[ijk, juv] = \delta_{1\,k} \frac{x_{ijk}}{z_j} + \delta_{2\,k} \frac{x_{ijk}}{p_j}$ (with $i, j, u = 1, \cdots, |\mathbb{B}|$ and $k, v = 1, 2, 3$)) is the weight of an edge of a subgraph of G.

[71] This condition amounts to Assumption 2 in [19] where it is made to ensure the uniqueness of a historical cost propagation.

[72] See Proposition 1 and Theorem 2 in [11]; the original matrix – after possible rows and column permutations – may be partitioned by simultaneous permutation of rows and columns leaving unchanged diagonal entries - see also [27, p. 176].

[73] See Theorem 1.21. in [52]

[74] See Theorem 2.1 in [41]

[75] Of course as C is not disjoint from itself, $\mathbb{K}_{\not\cap C}$ does not include C.

[76] See Appendix A.4

[77] See Theorem 2.1 [41]

Proposition 4. *a) Any ρ,σ entry of $\boldsymbol{F}_u^{-1}$ amounts to the sum of the absolute value of weights of all the paths from node ρ to node σ each one possibly amplified by a (greater than one) multiplier if the path shares a node with some cycle, i.e.*

$$\boldsymbol{F}_u^{-1}[\rho,\sigma]=\sum_{P_{\rho\sigma}}\left[|w(P_{\rho\sigma})|\frac{\det \boldsymbol{F}_u\restriction_{\mathbb{K}_{\not\cap P_{\rho\sigma}}}}{\det \boldsymbol{F}_u}\right] \tag{19}$$

where $\mathbb{K}_{\not\cap P_{\rho\sigma}}$ is the set of all the cycles which are disjoint from (i.e. share no node with) $P_{\rho\sigma}$ and $\boldsymbol{F}_u\restriction_{\mathbb{K}_{\not\cap P_{\rho\sigma}}}$ is the matrix obtained from $\boldsymbol{F}_u$, by setting to zero all the off-diagonal entries not corresponding to edges of cycles in $\mathbb{K}_{\not\cap P_{\rho\sigma}}$ b) Each multiplier is positively related to the absolute value of the weights of the cycles which are joint to the given path.

Proof. See Appendix A.5 □

Proposition 5. *The weight of a cycle C which shares no node with $P_{\rho\sigma}$ has a bearing on the multiplier $\mu=\frac{\det \boldsymbol{F}_u\restriction_{\mathbb{K}_{\not\cap P_{\rho\sigma}}}}{\det \boldsymbol{F}_u}$ in Eq. (19), which is a) zero, if the cycle is not strongly connected with any cycle $\tilde{C}$ which in turn shares some node with $P_{\rho\sigma}$, b) non-negative if it is strongly connected with a cycle $\tilde{C}$*

Proof. $\det \mathbf{F}_u\restriction_{\mathbb{K}_{\not\cap P_{\rho\sigma}}}$ (resp. $\det \mathbf{F}_u$) may be expressed as the product of the determinants of the irreducible components of its matrix $\mathbf{F}_u\restriction_{\mathbb{K}_{\not\cap P_{\rho\sigma}}}$ (resp. $\mathbf{F}_u$). If C and $\tilde{C}$ belong to different strongly connected components of $G\left(\mathbf{F}_u\restriction_{\mathbb{K}_{\not\cap P_{\rho\sigma}}}\right)$ (resp. $G(\mathbf{F}_u)$), the determinant of the corresponding irreducible component in $\mathbf{F}_u\restriction_{\mathbb{K}_{\not\cap P_{\rho\sigma}}}$ (resp. $\mathbf{F}_u$) which C belongs to is a factor of both the numerator and the denominator, so it has no bearing on their ratio. If this is not the case, making explicit the contribution of a cycle C as in Eq. (18), we have that [78]

$$\mu=\frac{\det \mathbf{F}_u\restriction_{\mathbb{K}_{\not\cap P_{\rho\sigma}}\setminus\{C\}}-|w(C)|\det \mathbf{F}_u\restriction_{\mathbb{K}_{\not\cap P_{\rho\sigma},C}}}{\det \mathbf{F}_u\restriction_{\mathbb{K}\setminus\{C\}}-|w(C)|\det \mathbf{F}_u\restriction_{\mathbb{K}_{\not\cap C}}}$$

and

$$\frac{\partial\mu}{\partial|w(C)|}=\frac{-\det \mathbf{F}_u\restriction_{\mathbb{K}_{\not\cap P_{\rho\sigma},C}}\det \mathbf{F}_u+\det \mathbf{F}_u\restriction_{\mathbb{K}_{\not\cap P_{\rho\sigma}}}\det \mathbf{F}_u\restriction_{\mathbb{K}_{\not\cap C}}}{(\det \mathbf{F}_u)^2}$$

which is non-negative if

$$\det \mathbf{F}_u\restriction_{\mathbb{K}_{\not\cap P_{\rho\sigma},C}}\det \mathbf{F}_u\leq\det \mathbf{F}_u\restriction_{\mathbb{K}_{\not\cap P_{\rho\sigma}}}\det \mathbf{F}_u\restriction_{\mathbb{K}_{\not\cap C}} \tag{20}$$

To show that this is always true, let for any subset $\mathbb{A}$ of $\{1,\cdots,|\Sigma|\}$ the principal minor on the rows and columns of $\mathbf{F}_u$ indexed by $\mathbb{A}$ be denoted by $\mathbf{F}_u(\mathbb{A})$. The rows and columns of $\mathbf{F}_u$ corresponding to the nodes of $G(\mathbf{F}_u)$ which are included in the cycles in $\mathbb{K}_{\not\cap P_{\rho\sigma}}$ are indexed by $\mathbb{A}_1$, those included in $\mathbb{K}_{\not\cap C}$ are indexed by $\mathbb{A}_2$ and those included in $\mathbb{K}\setminus\left(\mathbb{K}_{\not\cap P_{\rho\sigma}}\cup\mathbb{K}_{\not\cap C}\right)$ are indexed by $\mathbb{A}_3$. As $\mathbf{F}_u$ is an M-matrix,[79] the following inequality holds: [80]

$$\det \mathbf{F}_u(\mathbb{A}_1\cap\mathbb{A}_2)\det \mathbf{F}_u(\mathbb{A}_1\cup\mathbb{A}_2)\leq\det \mathbf{F}_u(\mathbb{A}_1)\det \mathbf{F}_u(\mathbb{A}_2) \tag{21}$$

and, given that
$\det \mathbf{F}_u(\mathbb{A}_1\cap\mathbb{A}_2)=\det \mathbf{F}_u\restriction_{\mathbb{K}_{\not\cap P_{\rho\sigma},C}}$
$\det \mathbf{F}_u(\mathbb{A}_1)=\det \mathbf{F}_u\restriction_{\mathbb{K}_{\not\cap P_{\rho\sigma}}}$

[78] $\mathbb{K}_{\not\cap P_{\rho\sigma},C}$ denotes the set of the set of all the cycles which are disjoint from both $P_{\rho\sigma}$ and C.
[79] See Theorem 2.1 in [41]
[80] The result follows from Ineq. (5) and Theorem 2 in [25] where we put the unit matrix in place of matrix $\mathbf{B}$.

$\det \mathbf{F}_u (\mathbb{A}_2) = \det \mathbf{F}_u \restriction_{\mathbb{K}_{\not\subset C}}$ and
$\det \mathbf{F}_u (\mathbb{A}_1 \cup \mathbb{A}_2) \geq \det \mathbf{F}_u (\mathbb{A}_1 \cup \mathbb{A}_2 \cup \mathbb{A}_3) = \det \mathbf{F}_u$ (as $\det \mathbf{F}_u$ decreases monotonically as the absolute value of the weight of any of the cycles included in $G(\mathbf{F}_u)$ increases), so inequality (20) is always true. □

3.1.4 Combining external loss allocation and network effect

The generic term $\frac{\partial u_{ijk}}{\partial x_{rst}}$ is a weighted sum of the initial impacts of a claim change $\mathrm{d}\, x_{rst}$ on the balance sheets directly involved in a funding transaction; each weight in particular equals the sum of the weights of the paths from x_{rst} to x_{ijk} each one possibly amplified by (the weights of) the cycles which share some element with the path [81] . Given the initial impact $-\mathbf{F}_x$, the overall effect of a claim increase due to a funding transaction will depend on the structure of paths and cycles linking the different claims (i.e. on $\mathbf{F}_u$). In terms of the initial impact, a funding transaction may always have a negative impact on the displacement of some claims, along with a possible positive impact on some others; this means that, e.g., more credit claims (higher leverage) may result in a reduced overall absolute displacement ($\Sigma_{ijk} |u_{ijk}|$) after the shock, whereas more equity capital (lower leverage) may have the opposite effect. As pointed out in the previous section, the relevant paths and cycles are those built out of actual obligations depending on the solvency status of the balance sheet which, by definition, small funding transaction may not alter. Although the existence of a contractual obligation is a necessary condition for that of a actual obligation, it may be the case that more numerous contractual obligations give rise to a less connected network (i.e. with less numerous actual paths and/or cycles) as the example in A.8.1 shows; this may happen when we move from small transactions to finite transactions which is the argument of the following section.

3.2 Finite transactions

The analysis of small transactions has shown that the displacement of a given financial claim u_{ijk} may increase or decrease as a result of a move from point $\mathbf{x}$ to point $\mathbf{x} + \mathrm{d}\,\mathbf{x}$, depending on initial impact $-\mathbf{F}_x$ and network effects $\mathbf{F}_u$; in finite transactions the additional possibility of changing balance sheet solvency status further widens the range of possible results, which may also include a reduction of overall displacement [82] as a consequence of increased credit transactions and higher leverage or, conversely, more acute displacement following an increase in capital financing. An attempt to reduce overall displacement by changing the seniority structure – e.g. allocating most of the displacement to less connected balance sheets – [83] might be frustrated by the fact that the relevant links are not the contractual links, but the actual ones and the latter may be even reduced by an increase in contractual connections. [84]

In order to look more closely into the finite transaction case we define the *over*-allocation [85] of

[81] And indirectly by the cycles to which those cycles are in turn strongly connected.
[82] And also of credit claims displacement.
[83] E.g. in [21, p. 129] "...a wide distribution of bail-inable instruments outside the banking sector is preferable".
[84] See Appendix A.8.1
[85] While in a given balance sheet, e.g. the j^{th}, the sum $\sum_{ik} \bar{a}_{ijk}$ (to which we add, for household balance sheets, the external capital absorption $h_j = -k_j - \theta(c_j - 1)(l_j - k_j)$) amounts to the (negative) external loss ($\sum_{ik} \bar{a}_{ijk} + h_j = -l_j$), the reallocation for a specific claim may be higher/lower (in absolute value) than the external loss of the balance sheet on whose liability side it is registered, due to possible transfers to/from its siblings of higher/lower seniority (or from external capital); so it is an *over*-reallocation. The term $\bar{a}_{ijk}$ may be seen as the sum of two components, the allocation of the external asset loss and a transfer between siblings of different seniority:

$$\bar{a}_{ijk} = \frac{x_{ijk}}{\delta_{1\,k} z_j + \delta_{2\,k} p_j} \left[\underbrace{-\delta_{1\,k} \min\{l_j, z_j\} + \delta_{2\,k} \min\{0, (z_j - l_j)\}}_{\text{external asset loss allocation}} + \underbrace{(\delta_{2\,k} - \delta_{1\,k})\,[1 - \theta(c_j - 1)] \max\{0, (z_j - l_j)\}}_{\text{transfer between siblings of different seniority}} \right]$$

the external assets losses $\bar{a}_{ijk}$ on the original claim x_{ijk} as

$$\bar{a}_{ijk} = -\delta_{1\,k}\left[z_j + \theta(c_j - 1)\,(l_j - z_j)\right]\frac{x_{ijk}}{z_j} - \delta_{2\,k}\left[1 - \theta(c_j - 1)\right]\frac{x_{ijk}}{p_j}\,(l_j - z_j)$$

or, for household balance sheets,

$$\bar{a}_{ij2} = -\left[1 - \theta(c_j - 1)\right]\frac{x_{ij2}}{p_j}\,(l_j - k_j)$$

and putting the terms $\bar{a}_{ijk}$ in lexical order $\bar{\mathbf{a}} = \left[\bar{a}_1 \cdots \bar{a}_\sigma \cdots \bar{a}_{|\Sigma|}\right]^T$. So from Eq. (1) we obtain

$$\mathbf{u} = \bar{\mathbf{M}}^{-1}\bar{\mathbf{a}} \tag{22}$$

where $\bar{\mathbf{M}}$ and $\bar{\mathbf{a}}$ are those particular matrix [86] and vector belonging respectively to $\mathbf{M}(t)$ and $\mathbf{a}(t)$ of Section 2.1 for which $s_j(t) = \theta(c_j - 1)$. Under Assumption 4, Eq. (22) has a solution which may be found by solving a linear programming problem as shown in Section 2.2. [87]

As in the case of small transactions also the displacement of finite transactions may be seen as the the product of two components: one describing the effect of the transaction on the allocation of both external assets losses and (pre-transaction) displacement, i.e. $(\Delta\bar{\mathbf{a}} - \Delta\bar{\mathbf{M}}\mathbf{u}_0)$, the other accounting for the network effect, i.e. $\bar{\mathbf{M}}_1^{-1}$. So from Eq. (22) we have (see Appendix A.6)

$$\Delta\mathbf{u} = \bar{\mathbf{M}}_1^{-1}(\Delta\bar{\mathbf{a}} - \Delta\bar{\mathbf{M}}\mathbf{u}_0) \tag{23}$$

where $\Delta\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_0$, $\Delta\bar{\mathbf{a}} = \bar{\mathbf{a}}_1 - \bar{\mathbf{a}}_0$, $\Delta\bar{\mathbf{M}} = \bar{\mathbf{M}}_1 - \bar{\mathbf{M}}_0$, and subscripts 0 and 1 refer respectively to point $\mathbf{x}$ and $\mathbf{x} + \mathrm{d}\,\mathbf{x}$. Eq. (23) may be seen as the finite transaction counterpart of Eq. (9) (times $\mathrm{d}\,\mathbf{x}$), [88] but differently from (what happens with a column of) $-\mathbf{F}_x$, in $(\Delta\bar{\mathbf{a}} - \Delta\bar{\mathbf{M}}\mathbf{u}_0)$ the impact of the transaction is not limited to the two involved balance sheets as it also includes the effects of changes in solvency status which may be induced also in other balance sheets. For self and siblings, the boundaries to the sign of the entries of $-\mathbf{F}_x$ set in Proposition 1 also apply to the correspondent entries of $(\Delta\bar{\mathbf{a}} - \Delta\bar{\mathbf{M}}\mathbf{u}_0)$ [89] and – in analogy with the case of small transaction – in absence of feedback

$$\left(\Delta\mathbf{x} = [0 \cdots \Delta x_{rjt} \cdots 0]^T \wedge (\mathbf{I} - \mathbf{P})^{-1}[jsl, pjh] = 0\right) \implies \Delta\mathbf{u}\,[ijk] = (\Delta\bar{\mathbf{a}} - \Delta\bar{\mathbf{M}}\mathbf{u}_0)\,[ijk] \tag{24}$$

(where $i, j, p, r, s = 1, \cdots, |\mathbb{B}|$ and $h, k, l, t = 1, 2, 3$)
I.e., if there are no feedback effects from any of the claims to the right of j^{th} balance sheet – a funding transaction $\{\Delta x_{rjt}, +\Delta e_j, -\Delta e_r\}$ generating (or increasing) a claim to the right side of the j^{th} balance sheet may not have any effect on an entry of $(\Delta\bar{\mathbf{a}} - \Delta\bar{\mathbf{M}}\mathbf{u}_0)$ corresponding to a node

---

or, for entries on the liability side of households balance sheets:

$$\bar{a}_{ij2} = \frac{x_{ij2}}{p_j}\left[\underbrace{\min\{0, (k_j - l_j)\}}_{\text{external asset loss}} + \underbrace{[1 - \theta(c_j - 1)]\max\{0, (k_j - l_j)\}}_{\text{transfer from external capital}}\right]$$

[86] Matrix $\bar{\mathbf{M}}$ amounts to $\mathbf{F}_u$ (but for the fact that it does not need Assumption 2 to be defined) so it shares all the properties of the latter as described in Section 3.1.3.

[87] Of course Eq. (22) does not provide an explicit solution for $\mathbf{u}$, which besides appearing explicitly to the left of the equal sign is also an argument of $\bar{\mathbf{M}}$ and $\bar{\mathbf{a}}$.

[88] Again differently from the case of small transactions, Eq. (23) does not provide an explicit solution for $\Delta\mathbf{u}$, which given $\mathbf{u}_0$ is an argument of $\bar{\mathbf{M}}$ and $\Delta\bar{\mathbf{M}}$; while the solution may be found by solving a (mixed) linear programming problem, Eq. (22) and Eq. (23) only describe the intertwining of external asset impact and network effects.

[89] See Appendix A.7.

which is also an ancestor of ijk [90] other than $(\Delta\bar{\mathbf{a}} - \Delta\bar{\mathbf{M}}\mathbf{u}_0)[ijk]$. In general, for claims not to the right of the balance sheets involved in the transaction (due to possible changes in solvency status) and also for those involved (if there are feedback effects), $\Delta\mathbf{u}[ijk] \neq (\Delta\bar{\mathbf{a}} - \Delta\bar{\mathbf{M}}\mathbf{u}_0)[ijk]$, as it is shown in example A.8.1 (see footnote 115). However, consistently with Proposition 1, also in the case of finite transactions,

- higher leverage ratios for some balance sheets may result in lower overall absolute displacement (as measured by $\sum_{ijk}|u_{ijk}|$) and
- higher collateralized claims may result in higher overall displacement,

as the examples in Appendix A.8 indicate.

4 Conclusions

Assuming no self-generating claims – claims generated through simultaneous (direct or indirect) reciprocal financing – the presence of balance sheet deficits (external assets higher than external capital) is a necessary condition for the existence of financial claims. Beyond the minimum level of claims that would suffice to make up for balance sheets imbalances, further claims are generated in the activity of intermediation, giving rise to a network of bilateral exposures with different senioritiy. Total displacement of claims may be (backward) decomposed into a component related to the seniority structure of claims, one related to the structure of bilateral links (which exposure a balance sheet has vis-à-vis which other balance sheet – irrespective of claim seniority) and one related to the imbalances between external losses and external capital.
In doing so it may be shown that – limiting the analysis to direct balance sheet contagion (as in [22]) for a given set of external loss-capital imbalances and structure of bilateral exposures – the sign of the relation between leverage and overall displacement may not be necessarily positive: [91] equity claims may transmit displacement not differently from credit claims. Nevertheless – differently from credit claims – until the first balance sheet in the system gets insolvent, before-the-shock values of equity claims may be restored if external losses are reversed: so if equity claim may not necessarily reduce contagion, it can make the system more resilient.
Furthermore, [92] more total claims does not imply more overall displacement: on the one hand, the presence of cycles in the network may account for a possibly infinite growth of total claims; on the other hand the involvement of household balance sheets (the only ones with external capital) in cycles may prevent a corresponding growth of overall displacement.
So – given external assets, losses and capital – a set of balance sheets may be thought for which

- greater leverage may reduce the overall financial claims displacement,
- the same overall displacement may be associated to very different levels of total claims (even at external loss-capital imbalances unchanged),
- increasing the number of contractual links may reduce that of the actual ones ...

No attempt has been made here at identifying the conditions under which such results may occur or how realistic they may be. Still they are possible according to A.V. Smirnov's definition:

[90] A sufficient, but not necessary condition for that, is $\det(\mathbf{I} - \mathbf{P}) = 1$; a node uvz is an ancestor of ijk if $\bar{\mathbf{M}}_1^{-1}[ijk, uvz] \neq 0$ (see [32, p. 1176]).
[91] Not even at the level of a single balance sheet, as the example in Appendix A.8.1, footnote 116 indicates.
[92] Again, given the set of external losses.

If something is possible, then it is not required that it has been the case some time, takes place now, or will be some time later. It is not excluded that it has never been, does not take place now, and will never be. [93]

[93] See [36].

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A Appendix

A.1 Financial claims generation

Assuming no self-generating claims (claims generated through simultaneous reciprocal financing) the presence of balance sheet deficits is a necessary condition for the existence of financial claims. [94] Total balance sheet deficit ($\sum_i \max(e_i - k_i, 0), i = 1, ..., |\mathbb{B}|$) sets the minimum level of total claims in order for surplus balance sheets to make up deficit ones; [95] financial intermediation accounts for total claims above this level. [96] Given the mechanism of claims generation through financial transactions – as described in Section 3 – the presence of cycles make it always possible to expand total claims without limits. The claims generation process is illustrated below in terms of bilateral exposures:

`A) Minimum level of total claims`

1				2				3				4			
e_1	10	k_1	0	e_2	0	k_2	0	e_3	1	k_3	10	e_4	0	k_4	1
		q_{31}	9					q_{31}	9			q_{41}	1		
		q_{41}	1												

`B) Intermediation`

1				2				3				4			
e_1	10	k_1	0	e_2	0	k_2	0	e_3	1	k_3	10	e_4	0	k_4	1
		q_{21}	9	q_{21}	9	q_{32}	9	q_{32}	9			q_{41}	1		
		q_{41}	1												

`C) Cycles`

1				2				3				4			
e_1	10	k_1	0	e_2	0	k_2	0	e_3	1	k_3	10	e_4	0	k_4	1
q_{13}	5	q_{21}	14	q_{21}	14	q_{32}	14	q_{32}	14	q_{13}	5	q_{41}	1		
		q_{41}	1												

The cycle generating sequence of transactions [97] that bring the system from `B)` to `C)` could be repeated with the result of inflating the balance sheets while leaving the original distribution of external assets $\mathbf{e}$ and capital $\mathbf{k}$ unchanged.

[94] Which here include both credit and equity claims.

[95] This level grows as the distance between $\mathbf{e}$ and $\mathbf{k}$ increases.

[96] This distinction follows the description of the different roles of finance as described in [51, p. 11] where the distinction is made between the "translation of the savings of households into corporate business investment" – a process which "occurs mainly outside the market, as retention of earnings gradually and irregularly augments the value of equity shares" – and the activity of "Capital markets and financial intermediaries [which] assist this process by facilitating transfer from surplus companies to deficit companies".

[97] i.e.: $\Delta q_{13} = -\Delta e_1 = \Delta e_3 = 5$,
$\Delta q_{32} = -\Delta e_3 = \Delta e_2 = 5$,
$\Delta q_{21} = -\Delta e_2 = \Delta e_1 = 5$

A.2 Accounting effects of household involvement in financial intermediation: 1) on relative displacement and 2) on external capital.

The involvement of household balance sheets in financial intermediation cycles may result *ceteris paribus* in a reduction of the relative displacement, which is merely due to inflated financial claims: [98]

`A) No household involvement`

before the shock:

1				2				3			
e_1	10	x_{212}	10	x_{212}	10	x_{322}	10	x_{322}	10	k_3	10

after the shock: (e_1 is reduced of a half)

1				2				3			
e_1	5	y_{212}	5	y_{212}	5	y_{322}	5	y_{322}	5	k_3	5

Relative displacement = 0.50

`B) Household balance sheet involved in financial intermediation`

before the shock:

1				2				3			
e_1	10	x_{212}	20	x_{212}	20	x_{322}	20	x_{322}	20	k_3	10
x_{132}	10									x_{132}	10

after the shock: (e_1 is reduced of a half)

1				2				3			
e_1	5	y_{212}	15	y_{212}	15	y_{322}	15	y_{322}	15	k_3	5
y_{132}	10									y_{132}	10

Relative displacement = 0.25

and distributional effects [99] – in the following example, the involvement of balance sheet 3 in an intermediation path (actually it is a cycle) results in an improvement of after the shock external capital of balance sheet 4 at expense of balance sheet 3:

`A) No household involvement`

before the shock:

1				2				3				4			
e_1	10	x_{212}	5	x_{212}	5	x_{322}	5	x_{322}	5	k_3	5	x_{412}	5	k_3	5
		x_{412}	5												

after the shock: (e_1 is reduced of a half)

1				2				3				4			
e_1	5	y_{212}	2.5	y_{212}	2.5	y_{322}	2.5	y_{322}	2.5	k_3	2.5	y_{412}	2.5	k_3	2.5
		y_{412}	2.5												

Relative displacement = 0.50

`B) Household balance sheet involved in financial intermediation`

before the shock:

1				2				3				4			
e_1	10	x_{212}	15	x_{212}	15	x_{322}	15	x_{322}	15	k_3	5	x_{412}	5	k_3	5
x_{132}	10	x_{412}	5							x_{132}	10				

[98]For the assumption of a constant ratio of bad debts to lending, see instead e.g. [3, p. 29].

[99]Redistribution effects of changes in cross-holding are mentioned in [7], whereas in [43, p. 73] other distributional consequences of a growing involvement of household in finance are described, which – differently from the simple accounting effects of higher claims *volume* described in this Appendix – are related to the *composition* of claims.

after the shock: (e_1 is reduced of a half)

1				2				3				4			
e_1	5	y_{212}	11.25	y_{212}	11.25	y_{322}	11.25	y_{322}	11.25	k_3	1.25	y_{412}	3.75	k_3	3.75
y_{132}	10	y_{412}	3.75							y_{132}	10				

Relative displacement $= 0.25$

A.3 A representation of bilateral exposure and bilateral displacement in terms of imbalances and network amplification.

Bilateral exposure $\mathbf{q}$ and (negative) bilateral displacement $\mathbf{v}$ [100] can be expressed – in analogy with the procedure followed in Section 2.1 for claims displacement – as the product of

- the inverse of a diagonally dominant matrix with unitary diagonal and non-positive off-diagonal entries (which accounts for network effects) and
- a vector accounting for imbalances between external capital and
  - external assets (as for exposure $\mathbf{q}$) or
  - external losses (as for negative displacement $\mathbf{v}$).

As for exposures, $\mathbf{q}$ may be seen as the solution of the system of equations

$$q_{ij} - \alpha_{ij} \sum_i q_{ji} = \alpha_{ij}(e_j - k_j)$$

where $i, j = 1, \cdots, |\mathbb{B}|$ and α_{ij} is the incidence of the exposure of balance sheet i to balance sheet j on total assets of balance sheet j (net of possible external capital); we may write it as $\mathbf{q} = \mathbf{R}^{-1}\mathbf{o}$, where
$\mathbf{R}$ is defined by [101]

$$r_{\psi(ij),\psi(ps)} = \begin{cases} 1 & \text{if } ps = ij \\ -\alpha_{ij} & \text{if } p = j \\ 0 & \text{all the other cases} \end{cases}$$
$$\text{where } i, j, p, s = 1, \cdots, |\mathbb{B}|$$

$\mathbf{o}$ is defined by $o_{\psi(ij)} = \alpha_{ij}(e_j - k_j)$, and
$\psi = \psi(ij)$ is a function which maps the two-letter index into a single-letter index.

As for (negative) displacement of bilateral exposure, $\mathbf{v}$ may be seen as the solution of the system of equations

$$v_{ij} - \beta_{ij} \sum_i v_{ji} = \beta_{ij}(l_j - k_j)$$

where $i, j = 1, \cdots, |\mathbb{B}|$ and β_{ij} is the incidence of the displacement of the claims that balance sheet i holds toward balance sheet j on total displacement and loss of the j^{th} balance sheet's assets (net of possible external capital); we may write it as $\mathbf{v} = \mathbf{N}^{-1}\mathbf{d}$, where
$\mathbf{N}$ is defined by

$$n_{\psi(ij),\psi(ps)} = \begin{cases} 1 & \text{if } ps = ij \\ -\beta_{ij} & \text{if } p = j \\ 0 & \text{all the other cases} \end{cases}$$
$$\text{where } i, j, p, s = 1, \cdots, |\mathbb{B}| \text{ and}$$

$0 \leq \beta_{ij} \leq 1$ – depending on the seniority structure actually associated with the structure of bilateral exposure – and $\beta_{ij} = 0$ for solvent household balance sheets

[100] Whether calculated as a solution to problem (6) or given as data.

[101] Both $\mathbf{R}$ and $\mathbf{N}$ are diagonally dominant matrices with positive diagonal entries and non positive off-diagonal entries so sharing all the properties of $\mathbf{F}_u$; furthermore, in order for them to be invertible, as it was the case for $\mathbf{F}_u$ – we assume that the relative graphs ($G(\mathbf{R})$ and $G(\mathbf{N})$) entail no closed strongly connected components.

$\mathbf{d}$ is defined by $d_{\psi(ij)} = \beta_{ij}(l_j - k_j)$.

As shown in Appendix A.1, given a set of balance sheet deficits, total exposure may be expanded quite indefinitely by the presence of cycles. [102]

By the same token, given the loss-capital imbalances, overall displacement may be expanded by the presence of cycles. However, more cycles in $\mathbf{R}$ (which is relevant for exposure) do not necessary mean more cycles in $\mathbf{N}$ (which is relevant for displacement); cycles involving household balance sheets in $\mathbf{R}$, might not be matched by cycles in $\mathbf{N}$, e.g., if the external capital of those household balance sheet is large enough to absorb the incoming displacement (along with the external loss), [103] so preventing its further propagation.

Comparing the relative displacement of Example A.8.1 with that of Example in Section 2.4, the higher denominator in the second displacement/exposure ratio is mainly due to the presence of more cycles in the matrix of bilateral relationship $\mathbf{R}$ whose determinant [104] is 0.86 in the first case and 0.20 in the second one.

The values for $\mathbf{R}$ and $\mathbf{o}$ in the two systems are

$$
\mathbf{R} = \begin{array}{c} \begin{array}{cccccccc} q_{14} & q_{15} & q_{21} & q_{31} & q_{43} & q_{56} & q_{71} & q_{73} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -0.244 & -0.244 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.325 & -0.325 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.429 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -0.431 & -0.431 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -0.571 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad \mathbf{o} = \begin{array}{cc} \begin{pmatrix} 15.000 \\ 63.000 \\ 2.439 \\ 3.232 \\ -2.143 \\ 20.000 \\ 4.309 \\ -2.857 \end{pmatrix} & \begin{array}{c} q_{14} \\ q_{15} \\ q_{21} \\ q_{31} \\ q_{43} \\ q_{56} \\ q_{71} \\ q_{73} \end{array} \end{array}
$$

in the first (Example A.8.1) system and

$$
\mathbf{R} = \begin{array}{c} \begin{array}{cccccccccccccccccccccccc} w_{13} & w_{14} & w_{15} & w_{16} & w_{21} & w_{23} & w_{24} & w_{25} & w_{26} & w_{31} & w_{34} & w_{35} & w_{36} & w_{41} & w_{43} & w_{45} & w_{46} & w_{54} & w_{56} & w_{71} & w_{73} & w_{74} & w_{75} & w_{76} \end{array} \\ \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.648 & -0.648 & -0.648 & -0.648 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.253 & -0.253 & -0.253 & -0.253 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.284 & -0.284 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.014 & -0.014 & -0.014 & -0.014 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -0.017 & -0.017 & -0.017 & -0.017 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -0.035 & -0.035 & -0.035 & -0.035 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.05 & -0.05 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.392 & -0.392 & -0.392 & -0.392 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -0.207 & -0.207 & -0.207 & -0.207 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -0.343 & -0.343 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.535 & -0.535 & -0.535 & -0.535 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.292 & -0.292 & -0.292 & -0.292 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -0.195 & -0.195 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.46 & -0.46 & -0.46 & -0.46 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-0.059 & -0.059 & -0.059 & -0.059 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.043 & -0.043 & -0.043 & -0.043 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.045 & -0.045 & -0.045 & -0.045 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.128 & -0.128 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \end{array}
$$

$$
\mathbf{o} = (-3.238 \quad 3.797 \quad 17.88 \quad 2.72 \quad 0.139 \quad -0.087 \quad 0.528 \quad 3.172 \quad 2.86 \quad 3.922 \quad 3.102 \quad 21.603 \quad 4.41 \quad 5.345 \quad -1.46 \quad 12.301 \quad 0.1 \quad 6.906 \quad 0.11 \quad 0.594 \quad -0.214 \quad 0.668 \quad 8.044 \quad 9.8)^T
$$

[102] Cycles in $\mathbf{R}$ increase the entries of $\mathbf{R}^{-1}$ (see Proposition 5 in Section 3.1.3) and – despite $\mathbf{o}$ may have some negative entries (if there are claims toward surplus balance sheets) – also total claims: given the mechanism of claims generation, the effect of negative entries in $\mathbf{o}$ is always offset by that of positive ones.

[103] Of course, the (in a way) opposite case may also occur that the matching cycle in $\mathbf{N}$ has a greater weight than the correspondent cycle in $\mathbf{R}$ e.g., if it stems from equity claims on the right side of a solvent balance sheet to which there are other higher seniority claims.

[104] As shown in Eq. (18), the determinant is negatively related to the weight of cycles.

in the second (Example in Section 2.4).

In order to assess to what extent the different ratio is due to the presence of cycles, for each of the two systems (the one with $\sum_i q_{ij} = 291$ and the other with $\sum_i q_{ij} = 1000$) the acyclic system S^* closest to each one may be calculated as the solution $\mathbf{q}^*$ to the following problem [105]

$$\begin{gathered}
\text{minimize} \sum_{i,j\in\{1,\cdots,|\mathbb{B}|\}} \left(q^*_{ij} - \bar{q}_{ij}\right)^2 \\
\text{subject to} \\
\text{(balance sheet)} \sum_i v^*_{ij} - \sum_s v^*_{js} + g^*_j = l_j \\
\sum_i q^*_{ij} - \sum_s q^*_{js} = e_j - k_j \\
\text{(total external loss)} \sum_j g^*_j = \sum_j l_j \\
\text{(lower seniority of external capital)}\ g^*_j = \min\left(\sum_s v^*_{js} + l_j, k_j\right) \\
\text{(overall displacement)} \sum_{ij} v^*_{ij} = 151.1 \\
\text{(acyclicality)} \det(\mathbf{R}) = 1 \\
0 \leq v^*_{ij} \leq q^*_{ij} \\
i, j, s = 1, ..., |\mathbb{B}|
\end{gathered} \tag{25}$$

which if setting $\bar{\mathbf{q}}$ at the values of the first system (where $\sum_{ij} q_{ij} = 291$) would produce

$$\mathbf{q}^* = \begin{pmatrix} 20.617 \\ 77.383 \\ 26.245 \\ 3.739 \\ 32.507 \\ 3.746 \\ 1.871 \\ 18.129 \\ 49.248 \\ 23.768 \end{pmatrix} \begin{array}{l} q^*_{14} \\ q^*_{15} \\ q^*_{21} \\ q^*_{23} \\ q^*_{31} \\ q^*_{45} \\ q^*_{46} \\ q^*_{56} \\ q^*_{71} \\ q^*_{73} \end{array} \text{ and } \mathbf{v}* = \begin{pmatrix} 16.390 \\ 55.310 \\ 20.649 \\ 1.424 \\ 13.791 \\ 1.390 \\ 0.000 \\ 0.000 \\ 37.688 \\ 4.367 \end{pmatrix} \begin{array}{l} v^*_{14} \\ v^*_{15} \\ v^*_{21} \\ v^*_{23} \\ v^*_{31} \\ v^*_{45} \\ v^*_{46} \\ v^*_{56} \\ v^*_{71} \\ v^*_{73} \end{array}$$

[105] Where $\bar{\mathbf{q}}$ is the bilateral exposure vector of the correspondent cyclic system.

and if set at the values of the second system (where $\sum_{ij} q_{ij} = 1000$) would produce

$$\mathbf{q}^* = \begin{array}{cc} \begin{pmatrix} 61,249 \\ 27,215 \\ 6,655 \\ 66,249 \\ 20,000 \\ 35,000 \\ 44,034 \\ 3,870 \\ 25,096 \end{pmatrix} & \begin{array}{c} q^*_{13} \\ q^*_{21} \\ q^*_{25} \\ q^*_{35} \\ q^*_{46} \\ q^*_{54} \\ q^*_{71} \\ q^*_{72} \\ q^*_{75} \end{array} \end{array} \text{ and } \mathbf{v}{*} = \begin{array}{cc} \begin{pmatrix} 31.945 \\ 14.958 \\ 6.654 \\ 39.945 \\ 0.000 \\ 15.000 \\ 17.487 \\ 0.000 \\ 25.100 \end{pmatrix} & \begin{array}{c} v^*_{13} \\ v^*_{21} \\ v^*_{25} \\ v^*_{35} \\ v^*_{46} \\ v^*_{54} \\ v^*_{71} \\ v^*_{72} \\ v^*_{75} \end{array} \end{array}$$

As a consequence, the displacement/claim ratios for the two acyclic sistems would be much closer (respectively -0.59 and -0.57).

A.4 From Eq. (17) to Eq. (18)

We consider a network with n cycles $(C_1, C_2, \cdots, C_n)$ and, in order to calculate the determinant of its weighted adjacency matrix, we start from the determinant of a network without cycles (which equals 1) and then we add one cycle at a time, allowing for all possible pairwise combinations of joint $(C_i \cap C_j \neq \emptyset)$ or disjoint $(C_i \cap C_j = \emptyset)$ cycles; starting with the first 3 cycles we have

`one cycle` `two cycles` `three cycles`

$$
1-|w(C_1)|
\begin{cases}
\begin{array}{l} 1-|w(C_1)|-|w(C_2)| \\ \text{if } C_1\cap C_2\neq\emptyset \end{array}
&
\begin{cases}
\begin{array}{l} 1-|w(C_1)|-|w(C_2)|-|w(C_3)| \\ \text{if } C_1\cap C_2\neq\emptyset \wedge C_1\cap C_3\neq\emptyset \wedge C_2\cap C_3\neq\emptyset \end{array} \\[1ex]
\begin{array}{l} 1-|w(C_1)|-|w(C_2)|-|w(C_3)|\,(1-|w(C_1)|) \\ \text{if } C_1\cap C_2\neq\emptyset \wedge C_1\cap C_3=\emptyset \wedge C_2\cap C_3\neq\emptyset \end{array} \\[1ex]
\begin{array}{l} 1-|w(C_1)|-|w(C_2)|-|w(C_3)|\,(1-|w(C_2)|) \\ \text{if } C_1\cap C_2\neq\emptyset \wedge C_1\cap C_3\neq\emptyset \wedge C_2\cap C_3=\emptyset \end{array} \\[1ex]
\begin{array}{l} 1-|w(C_1)|-|w(C_2)|-|w(C_3)|\,(1-|w(C_1)|-|w(C_2)|) \\ \text{if } C_1\cap C_2\neq\emptyset \wedge C_1\cap C_3=\emptyset \wedge C_2\cap C_3=\emptyset \end{array}
\end{cases}
\\[4ex]
\begin{array}{l} 1-|w(C_1)|-|w(C_2)|\,(1-|w(C_1)|) \\ \text{if } C_1\cap C_2=\emptyset \end{array}
&
\begin{cases}
\begin{array}{l} 1-|w(C_1)|-|w(C_2)|\,(1-|w(C_1)|)-|w(C_3)| \\ \text{if } C_1\cap C_2=\emptyset \wedge C_1\cap C_3\neq\emptyset \wedge C_2\cap C_3\neq\emptyset \end{array} \\[1ex]
\begin{array}{l} 1-|w(C_1)|-|w(C_2)|\,(1-|w(C_1)|)-|w(C_3)|\,(1-|w(C_1)|) \\ \text{if } C_1\cap C_2=\emptyset \wedge C_1\cap C_3=\emptyset \wedge C_2\cap C_3\neq\emptyset \end{array} \\[1ex]
\begin{array}{l} 1-|w(C_1)|-|w(C_2)|\,(1-|w(C_1)|)-|w(C_3)|\,(1-|w(C_2)|) \\ \text{if } C_1\cap C_2=\emptyset \wedge C_1\cap C_3\neq\emptyset \wedge C_2\cap C_3=\emptyset \end{array} \\[1ex]
\begin{array}{l} 1-|w(C_1)|-|w(C_2)|\,(1-|w(C_1)|)- \\ |w(C_3)|\,[1-|w(C_1)|-|w(C_2)|\,(1-|w(C_1)|)] \\ \text{if } C_1\cap C_2=\emptyset \wedge C_1\cap C_3=\emptyset \wedge C_2\cap C_3=\emptyset \end{array}
\end{cases}
\end{cases}
$$

I.e. as we add one more cycle C_i, the new determinat will equal that associated with a network lacking the new cycle, less the absolute value of the weight of the new cycle times the determinant associated with a network lacking all the cycles joint to C_i.

A.5 Proof of Proposition 4

a) From Theorem 1 in [13]

$$\mathbf{F}_u^{-1}[\rho,\sigma] = \frac{\sum_{H_{\rho\sigma}}(-1)^{L_{H_{\rho\sigma}}-1}w(H_{\rho\sigma})}{\sum_h(-1)^{L_h}w(h)} \tag{26}$$

where [106]

$L_h, L_{H_{\rho\sigma}}$	is the number of directed circuits in h and $H_{\rho\sigma}$ respectively
h	is a 1-factor in $G(\mathbf{F}_u)$
$H_{\rho\sigma}$	is a 1-factorial connection from node ρ to node σ in $G(\mathbf{F}_u)$

Furthermore, if

$[\mathbf{F}_u]_{\rho,\sigma}$	is a matrix obtained from $\mathbf{F}_u$ by removing the ρ-th and the σ-th columns
h'	is a 1-factor including at least one cycle and covering all the nodes in $G(\mathbf{F}_u)$
$P_{\rho\sigma}$	is a path from node ρ to node σ in $G(\mathbf{F}_u)$
$h_{\not\ni P_{\rho\sigma}}$	is a 1-factor covering all the nodes in $G(\mathbf{F}_u)$, but the ones already included in $P_{\rho\sigma}$
$h'_{\not\ni P_{\rho\sigma}}$	is a 1-factor which includes at least one cycle and covering all the nodes in $G(\mathbf{F}_u)$, but the ones already included in $P_{\rho\sigma}$
$C_{h'_{\not\ni P_{\rho\sigma}}}$	is a cycle in $h'_{\not\ni P_{\rho\sigma}}$
$L_{C_{h_{\not\ni P_{\rho\sigma}}}}$	is the number of cycles included in $h_{\not\ni P_{\rho\sigma}}$
$L_{C_{h'_{\not\ni P_{\rho\sigma}}}}$	is the number of cycles included in $h'_{\not\ni P_{\rho\sigma}}$
$L_{C_{h'}}$	is the number of cycles included in h'
$V_{P_{\rho\sigma}}$	is the number of nodes included in $P_{\rho\sigma}$
$V_{C_{h_{\not\ni P_{\rho\sigma}}}}$	is the number of nodes (as well as of edges) belonging to the cycles included in $h_{\not\ni P_{\rho\sigma}}$
$L_{S_{h_{\not\ni P_{\rho\sigma}}}}$	is the number of self-loops included in $h_{\not\ni P_{\rho\sigma}}$
$L_{h_{\not\ni P_{\rho\sigma}}}$	is the number of directed circuits included in $h_{\not\ni P_{\rho\sigma}}$

from [50] and [13][107] we have that

$$(-1)^{\rho+\sigma}\det[\mathbf{F}_u]_{\rho,\sigma} = (-1)^{|\Sigma|}\sum_{H_{\rho\sigma}}(-1)^{L_{H_{\rho\sigma}}-1}w(H_{\rho\sigma}) \tag{27}$$

and given that
$L_{h_{\not\ni P_{\rho\sigma}}} = L_{S_{h_{\not\ni P_{\rho\sigma}}}} + L_{C_{h_{\not\ni P_{\rho\sigma}}}}$,
$|\Sigma| = V_{P_{\rho\sigma}} + V_{C_{h_{\not\ni P_{\rho\sigma}}}} + L_{S_{h_{\not\ni P_{\rho\sigma}}}}$ and
$\left(|\Sigma| + L_{h_{\not\ni P_{\rho\sigma}}} + V_{P_{\rho\sigma}} + V_{C_{h_{\not\ni P_{\rho\sigma}}}}\right) \pmod 2 \equiv L_{C_{h_{\not\ni P_{\rho\sigma}}}} \pmod 2$,

[106] The original formula in [13] only applies if $\rho \neq \sigma$. If $\rho = \sigma$, it should be replaced (Theorem 6.28 in [50]) by

$$\mathbf{F}_u^{-1}[\rho,\rho] = \frac{\sum_H(-1)^{L_H-1}w(H)}{\sum_h(-1)^{L_h}w(h)}$$

where H is a 1-factor in the graph obtained by removing node ρ from $G(\mathbf{F}_u)$. Nevertheless, given that all the diagonal entries of $\mathbf{F}_u$ equal 1, the two expressions coincide if $\rho = \sigma$.

[107] See Theorem 6.28 in [50] and Theorem 1 in [13].

then

$$
\begin{aligned}
& (-1)^{|\Sigma|} \textstyle\sum_{H_{\rho\sigma}} (-1)^{L_{H_{\rho\sigma}}-1} w(H_{\rho\sigma}) \\
= \; & (-1)^{|\Sigma|} \textstyle\sum_{P_{\rho\sigma}} w\left(P_{\rho\sigma}\right) \sum_{h_{\not\pitchfork P_{\rho\sigma}}} (-1)^{L_{h_{\not\pitchfork P_{\rho\sigma}}}-1} w\left(h_{\not\pitchfork P_{\rho\sigma}}\right) \\
= \; & (-1)^{|\Sigma|} \textstyle\sum_{P_{\rho\sigma}} (-1)^{V_{P_{\rho\sigma}}-1} \left|w\left(P_{\rho\sigma}\right)\right| \sum_{h_{\not\pitchfork P_{\rho\sigma}}} (-1)^{L_{h_{\not\pitchfork P_{\rho\sigma}}}-1+V_{C_{h_{\not\pitchfork P_{\rho\sigma}}}}} \left|w\left(h_{\not\pitchfork P_{\rho\sigma}}\right)\right| \\
= \; & \textstyle\sum_{P_{\rho\sigma}} \left|w\left(P_{\rho\sigma}\right)\right| \sum_{h_{\not\pitchfork P_{\rho\sigma}}} (-1)^{|\Sigma|+L_{h_{\not\pitchfork P_{\rho\sigma}}}+V_{P_{\rho\sigma}}-2+V_{C_{h_{\not\pitchfork P_{\rho\sigma}}}}} \left|w\left(h_{\not\pitchfork P_{\rho\sigma}}\right)\right| \\
= \; & \textstyle\sum_{P_{\rho\sigma}} \left|w\left(P_{\rho\sigma}\right)\right| \sum_{h_{\not\pitchfork P_{\rho\sigma}}} (-1)^{L_{C_{h_{\not\pitchfork P_{\rho\sigma}}}}} \left|w\left(h_{\not\pitchfork P_{\rho\sigma}}\right)\right| \\
= \; & \textstyle\sum_{P_{\rho\sigma}} \left|w\left(P_{\rho\sigma}\right)\right| \left[1+\sum_{h'_{\not\pitchfork P_{\rho\sigma}}} (-1)^{L_{C_{h'_{\not\pitchfork P_{\rho\sigma}}}}} \prod_{C_{h'_{\not\pitchfork P_{\rho\sigma}}}} \left|w\left(C_{h'_{\not\pitchfork P_{\rho\sigma}}}\right)\right|\right]
\end{aligned}
$$

So, given Eq. (16), Eq. (17) and Eq. (26),

$$
\mathbf{F}_u^{-1}\left[\rho,\sigma\right] = \sum_{P_{\rho\sigma}} \left[|w(P_{\rho\sigma})| \frac{1+\sum_{h'_{\not\pitchfork P_{\rho\sigma}}} (-1)^{L_{C_{h'_{\not\pitchfork P_{\rho\sigma}}}}} \prod_{C_{h'_{\not\pitchfork P_{\rho\sigma}}}} \left|w\left(C_{h'_{\not\pitchfork P_{\rho\sigma}}}\right)\right|}{1+\sum_{h'} (-1)^{L_{C_{h'}}} \prod_{C_{h'}} \left|w\left(C_{h'}\right)\right|} \right] \tag{28}
$$

and, if $\mathbb{K}_{\not\pitchfork P_{\rho\sigma}}$ is the set of all the cycles which are disjoint from (i.e. share no node with) $P_{\rho\sigma}$ and $\mathbf{F}_u \restriction_{\mathbb{K}_{\not\pitchfork P_{\rho\sigma}}}$ is the matrix obtained from $\mathbf{F}_u$, by setting to zero all the off-diagonal entries which are not included in any of the cycles included in any $h_{\not\pitchfork P_{\rho\sigma}}$,

$$
\mathbf{F}_u^{-1}\left[\rho,\sigma\right] = \sum_{P_{\rho\sigma}} \left[|w(P_{\rho\sigma})| \frac{\det \mathbf{F}_u \restriction_{\mathbb{K}_{\not\pitchfork P_{\rho\sigma}}}}{\det \mathbf{F}_u} \right] \tag{29}
$$

which amounts to Proposition 4, given that $\det \mathbf{F}_u \restriction_{\mathbb{K}_{\not\pitchfork P_{\rho\sigma}}} \geq \det \mathbf{F}_u$ as $\frac{\partial \det \mathbf{F}_u}{\partial |w(C_i)|} < 0$. [108]
b) This is a consequence of the fact that the cycles joint to a given path are negatively related to the determinant at the denominator of Eq. (19) and have no bearing on the denominator.

[108] See proof of Proposition 2.

A.6 Decomposing $\Delta\mathbf{u}$

From Eq. (22)

$$
\begin{aligned}
\mathbf{u}_1 - \mathbf{u}_0 &= \bar{\mathbf{M}}_1^{-1}\bar{\mathbf{a}}_1 - \bar{\mathbf{M}}_0^{-1}\bar{\mathbf{a}}_0 \\
&= \bar{\mathbf{M}}_1^{-1}\bar{\mathbf{a}}_1 - \bar{\mathbf{M}}_0^{-1}\bar{\mathbf{a}}_0 + \bar{\mathbf{M}}_1^{-1}\bar{\mathbf{a}}_0 - \bar{\mathbf{M}}_1^{-1}\bar{\mathbf{a}}_0 \\
&= \bar{\mathbf{M}}_1^{-1}\left(\bar{\mathbf{a}}_1 - \bar{\mathbf{a}}_0\right) + \left(\bar{\mathbf{M}}_1^{-1} - \bar{\mathbf{M}}_0^{-1}\right)\bar{\mathbf{a}}_0 \\
&= \bar{\mathbf{M}}_1^{-1}\left(\bar{\mathbf{a}}_1 - \bar{\mathbf{a}}_0\right) - \bar{\mathbf{M}}_1^{-1}\left(\bar{\mathbf{M}}_1 - \bar{\mathbf{M}}_0\right)\bar{\mathbf{M}}_0^{-1}\bar{\mathbf{a}}_0 \\
&= \bar{\mathbf{M}}_1^{-1}\left(\Delta\bar{\mathbf{a}} - \Delta\bar{\mathbf{M}}\mathbf{u}_0\right)
\end{aligned}
$$

(In the same way it may be shown that $\mathbf{u}_1 - \mathbf{u}_0 = \bar{\mathbf{M}}_0^{-1}\left(\Delta\bar{\mathbf{a}} - \Delta\bar{\mathbf{M}}\mathbf{u}_1\right)$)

A.7 Sign of $(\Delta\bar{\mathbf{a}} - \Delta\bar{\mathbf{M}}\mathbf{u}_0)\,[ijk]$

Making explicit the dependence of c_j on $\mathbf{x}$, we denote by ${}_0c_j$ and ${}_1c_j$ the asset/liability ratio of the j^{th} balance sheet at points $\mathbf{x}$ and $\mathbf{x}+\Delta\mathbf{x}$ respectively and by ${}_0u_{ijk}$ the displacement u_{ijk} at point $\mathbf{x}$. We calculate $b_{ijk} = (\Delta\bar{a}_{ijk} - \Delta\hat{x}_{ijk}\sum_{st} {}_0u_{jst})$ for self and siblings relations in different possible cases; [109]

`Effect of an equity funding on itself`
If $\Delta\mathbf{x} = [0\cdots\Delta x_{ij1}\cdots 0]^T$ (with $\Delta x_{ij1} \geq 0$), we have

$$
\begin{array}{lcrcll}
{}_0c_j < 1 \wedge {}_1c_j < 1 & \Longrightarrow & b_{ij1} & = & -\Delta x_{ij1} & \leq\ 0 \\
{}_0c_j < 1 \wedge {}_1c_j \geq 1 & \Longrightarrow & b_{ij1} & = & x_{ij1} - \frac{x_{ij1}+\Delta x_{ij1}}{z_j+\Delta x_{ij1}}\left(l_j + \Delta l_j - \sum\limits_{st} {}_0u_{jst}\right) & \leq\ 0
\end{array}
$$

(as at point $\mathbf{x}$, $x_{ij1} - \frac{x_{ij1}}{z_j}(l_j - \sum\limits_{st} {}_0u_{jst}) \leq 0$)

$$
{}_0c_j \geq 1 \wedge {}_1c_j \geq 1 \Longrightarrow b_{ij1} = -\Delta l_j \frac{x_{ij1}+\Delta x_{ij1}}{z_j+\Delta x_{ij1}} - (l_j - \sum_{st} {}_0u_{jst})\left(\frac{x_{ij1}+\Delta x_{ij1}}{z_j+\Delta x_{ij1}} - \frac{x_{ij1}}{z_j}\right) \leq 0
$$

`Effect of a credit funding on itself` [110]
If $\Delta\mathbf{x} = [0\cdots\Delta x_{ij2}\cdots 0]^T$ (with $\Delta x_{ij2} \geq 0$), we have

$$
\begin{array}{lcrcll}
{}_0c_j < 1 \wedge {}_1c_j < 1 & \Longrightarrow & b_{ij2} & = & -\left(\frac{x_{ij2}+\Delta x_{ij2}}{p_j+\Delta x_{ij2}} - \frac{x_{ij2}}{p_j}\right)\left(l_j - z_j - \sum\limits_{st} {}_0u_{jst}\right) - \frac{x_{ij2}+\Delta x_{ij2}}{p_j+\Delta x_{ij2}}\Delta l_j & \leq\ 0 \\
{}_0c_j \geq 1 \wedge {}_1c_j \geq 1 & \Longrightarrow & b_{ij2} & = & 0 & \\
{}_0c_j \geq 1 \wedge {}_1c_j < 1 & \Longrightarrow & b_{ij2} & = & -\frac{x_{ij2}+\Delta x_{ij2}}{p_j+\Delta x_{ij2}}\left(l_j + \Delta l_j - z_j - \sum\limits_{st} {}_0u_{jst}\right) & \leq\ 0
\end{array}
$$

`Effect of an equity funding on a higher seniority (credit) sibling`
If $\Delta\mathbf{x} = [0\cdots\Delta x_{rj1}\cdots 0]^T$ (with $\Delta x_{rj1} \geq 0$), we have

$$
\begin{array}{lcrcll}
{}_0c_j < 1 \wedge {}_1c_j < 1 & \Longrightarrow & b_{ij2} & = & -\frac{x_{ij2}}{p_j}(\Delta l_j - \Delta z_j) & \geq\ 0 \\
{}_0c_j < 1 \wedge {}_1c_j \geq 1 & \Longrightarrow & b_{ij2} & = & \frac{x_{ij2}}{p_j}\left(l_j - z_j - \sum\limits_{st} {}_0u_{jst}\right) & \geq\ 0 \\
{}_0c_j \geq 1 \wedge {}_1c_j \geq 1 & \Longrightarrow & b_{ij2} & = & 0 &
\end{array}
$$

[109] In order to limit the analysis to the sole effect of the accounting mechanisms due to balance sheet and seniority constraints, we assume that the relation between e_i and l_i is linear and equal for both the balance sheets involved in the transaction, as a consequence the two cases in which $\Delta\mathbf{x} = [0\cdots\Delta x_{ij1}\cdots 0]^T \wedge \Delta x_{ij1} \geq 0 \wedge ({}_0c_j \geq 1 \wedge {}_1c_j < 1)$ and $\Delta\mathbf{x} = [0\cdots\Delta x_{ij2}\cdots 0]^T \wedge \Delta x_{ij2} \geq 0 \wedge ({}_0c_j < 1 \wedge {}_1c_j \geq 1)$ are not considered. The increase of an equity claim on the right side of the balance sheet may not bring about a change of the solvency status from solvent to insolvent: if ${}_0c_j \geq 1$, even in the extreme case of a full loss of the counterpart of the new equity claim ($\Delta l_j = \Delta e_j$) the loss would be fully covered by the increase in equity claim Δx_{ij1}; as a consequence there would be no effect on the solvency of balance sheet j. By the same reasoning, the increase of a credit claim may not bring about a change from insolvent to solvent: if ${}_1c_j \geq 1$, even in the extreme case of zero loss ($\Delta l_j = 0$) on the external asset counterpart of the new credit claim the latter should be still fully covered by the original amount of its counterpart $\Delta x_{ij2} = \Delta e_j$ and the two amounts would compensate each other leaving unchanged the solvency status of the original balance sheet j, i.e.
$\Delta\mathbf{x} = [0\cdots\Delta x_{ij1}\cdots 0]^T \wedge \Delta x_{ij1} \geq 0 \implies \neg({}_0c_j \geq 1 \wedge {}_1c_j < 1)$ and
$\Delta\mathbf{x} = [0\cdots\Delta x_{ij2}\cdots 0]^T \wedge \Delta x_{ij2} \geq 0 \implies \neg({}_0c_j < 1 \wedge {}_1c_j \geq 1)$.

[110] In order to describe the effect of a credit claim to the right of a household balance sheet (there cannot be equity claims to the right of a household balance sheet), just replace z_j with k_j in the relative expressions.

`Effect of a credit funding on a lower seniority (equity) sibling`
If $\Delta\mathbf{x} = [0 \cdots \Delta x_{rj2} \cdots 0]^T$ (with $\Delta x_{rj2} \geq 0$), we have

$$\begin{array}{lcrcll} {}_0c_j < 1 \wedge {}_1c_j < 1 & \Longrightarrow & b_{ij1} & = & 0 & \\ {}_0c_j \geq 1 \wedge {}_1c_j \geq 1 & \Longrightarrow & b_{ij1} & = & -\frac{x_{ij1}}{z_j}\Delta l_j & \leq \; 0 \\ {}_0c_j \geq 1 \wedge {}_1c_j < 1 & \Longrightarrow & b_{ij1} & = & \frac{x_{ij1}}{z_j}\left(l_j - z_j - \sum\limits_{st} {}_0u_{jst}\right) & \leq \; 0 \end{array}$$

`Effect of an equity funding on its equity sibling` [111]
If $\Delta\mathbf{x} = [0 \cdots \Delta x_{rj1} \cdots 0]^T$ (with $\Delta x_{rj1} \geq 0$), we have

$$\begin{array}{lcrcll} {}_0c_j < 1 \wedge {}_1c_j < 1 & \Longrightarrow & b_{ij1} & = & 0 & \\ {}_0c_j < 1 \wedge {}_1c_j \geq 1 & \Longrightarrow & b_{ij1} & = & x_{ij1} - \frac{x_{ij1}}{z_j+\Delta z_j}\left(l_j + \Delta l_j - \sum\limits_{st} {}_0u_{jst}\right) & \gtreqless \; 0 \\ {}_0c_j \geq 1 \wedge {}_1c_j \geq 1 & \Longrightarrow & b_{ij1} & = & \left(\frac{x_{ij1}}{z_j} - \frac{x_{ij1}}{z_j+\Delta z_j}\right)\left(l_j - \sum\limits_{st} {}_0u_{jst}\right) - \frac{x_{ij1}}{z_j+\Delta z_j}\Delta l_j & \gtreqless \; 0 \end{array}$$

`Effect of a credit funding on its credit sibling`
If $\Delta\mathbf{x} = [0 \cdots \Delta x_{rj2} \cdots 0]^T$ (with $\Delta x_{rj2} \geq 0$), we have

$$\begin{array}{lcrcll} {}_0c_j < 1 \wedge {}_1c_j < 1 & \Longrightarrow & b_{ij2} & = & \left(\frac{x_{ij2}}{p_j} - \frac{x_{ij2}}{p_j+\Delta p_j}\right)\left(l_j - z_j - \sum\limits_{st} {}_0u_{jst}\right) - \frac{x_{ij2}}{p_j+\Delta p_j}\Delta l_j & \gtreqless \; 0 \\ {}_0c_j \geq 1 \wedge {}_1c_j \geq 1 & \Longrightarrow & b_{ij2} & = & 0 & \\ {}_0c_j \geq 1 \wedge {}_1c_j < 1 & \Longrightarrow & b_{ij2} & = & -\frac{x_{ij2}}{p_j+\Delta p_j}\left(l_j + \Delta l_j - z_j - \sum\limits_{st} {}_0u_{jst}\right) & \leq \; 0 \end{array}$$

[111] In analogy with the case of small transactions, also for finite transactions the impact on same seniority siblings is non-negative if the increase in the external-assets loss due to greater external assets is lower than the fraction of same seniority claims not covered by total assets:
if $\theta({}_0c_j - 1) = \theta({}_1c_j - 1) = 1$, we have $b_{ij1} = \frac{x_{ij1}}{z_j+\Delta x_{ij1}}\Delta x_{ij1}\left[1 - \frac{p_j}{z_j}\left({}_0c_j - 1\right) - \frac{\Delta l_j}{\Delta x_{ij1}}\right]$ for equity funding, and
if $\theta({}_0c_j - 1) = \theta({}_1c_j - 1) = 0$, we have $b_{ij2} = \frac{x_{ij2}}{p_j+\Delta x_{ij2}}\Delta x_{ij2}\left[1 - {}_0c_j - \frac{\Delta l_j}{\Delta x_{ij2}}\right]$ for credit funding,
in line with the corresponding equations for small transactions in Subsection 3.1.2.

A.8 Examples

A.8.1 Example 1

From the input data [112]

$$
\mathbf{x} = \begin{array}{c} \begin{pmatrix} 30 \\ 83 \\ 30 \\ 40 \\ 15 \\ 20 \\ 0 \\ 53 \\ 20 \end{pmatrix} \end{array} \begin{array}{l} x_{141} \\ x_{151} \\ x_{212} \\ x_{312} \\ x_{432} \\ x_{562} \\ x_{612} \\ x_{711} \\ x_{732} \end{array}
\quad \mathbf{k} = \begin{pmatrix} 90 \\ 8 \\ 25 \\ 73 \end{pmatrix} \begin{array}{l} k_2 \\ k_3 \\ k_6 \\ k_7 \end{array}
\quad \mathbf{e} = \begin{pmatrix} 10 \\ 60 \\ 3 \\ 15 \\ 63 \\ 45 \\ 0 \end{pmatrix} \begin{array}{l} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{array}
\quad \mathbf{l} = \begin{pmatrix} 0.50 \\ 0.00 \\ 0.00 \\ 15.00 \\ 56.70 \\ 2.25 \\ 0.00 \end{pmatrix} \begin{array}{l} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \\ l_6 \\ l_7 \end{array}
$$

solving Eq. (1), we obtain

$$
\mathbf{u} = \begin{pmatrix} -16.7 \\ -56.7 \\ -8.9 \\ -11.9 \\ -1.7 \\ 0.0 \\ 0.0 \\ -53.0 \\ -2.2 \end{pmatrix} \begin{array}{l} u_{141} \\ u_{151} \\ u_{212} \\ u_{312} \\ u_{432} \\ u_{562} \\ u_{612} \\ u_{711} \\ u_{732} \end{array}
\quad \mathbf{g} = \begin{pmatrix} -8.9 \\ -8.0 \\ -2.2 \\ -55.2 \end{pmatrix} \begin{array}{l} g_2 \\ g_3 \\ g_6 \\ g_7 \end{array}
$$

By Eq. (22) the displacement $\mathbf{u}$ may be decomposed as the product of $\bar{\mathbf{M}}^{-1}$ and $\bar{\mathbf{a}}$, where

$$
\bar{\mathbf{a}} = \begin{pmatrix} -15.00 \\ -56.70 \\ 22.50 \\ 30.00 \\ 3.43 \\ 0.00 \\ 0.00 \\ -53.00 \\ 4.57 \end{pmatrix} \begin{array}{l} \bar{a}_{141} \\ \bar{a}_{151} \\ \bar{a}_{212} \\ \bar{a}_{312} \\ \bar{a}_{432} \\ \bar{a}_{562} \\ \bar{a}_{612} \\ \bar{a}_{711} \\ \bar{a}_{732} \end{array} \quad \text{and}
$$

[112] In order to limit the analysis to the sole effect of the accounting mechanisms due to balance sheet and seniority constraints, we assume that the relation between e_i and l_i is linear and equal for both the balance sheets involved in the transaction.

$$\bar{\mathbf{M}} = \begin{array}{cc} \begin{array}{ccccccccc} u_{141} & u_{151} & u_{212} & u_{312} & u_{432} & u_{562} & u_{612} & u_{711} & u_{732} \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 & \underline{-1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -0.43 & -0.43 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \underline{-0.57} & -0.57 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{-.43} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -0.57 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} f_{141} \\ f_{151} \\ f_{212} \\ f_{312} \\ f_{432} \\ f_{562} \\ f_{612} \\ f_{711} \\ f_{732} \end{array} \end{array}$$

in which there is one (potential) cycle (whose entries are underlined in red) which is also an actual cycle.
If, after a credit-type funding transaction $\{\Delta x_{612} = 35, \Delta e_6 = -35, \Delta e_1 = 35\}$, we move from the original point $\mathbf{x}$ to the new point $\mathbf{x}'$, we have

$$\mathbf{x}' = \begin{pmatrix} 30 \\ 83 \\ 30 \\ 40 \\ 15 \\ 20 \\ 35 \\ 53 \\ 20 \end{pmatrix} \begin{matrix} x_{141} \\ x_{151} \\ x_{212} \\ x_{312} \\ x_{432} \\ x_{562} \\ x_{612} \\ x_{711} \\ x_{732} \end{matrix} \quad \mathbf{k}' = \begin{pmatrix} 90 \\ 8 \\ 25 \\ 73 \end{pmatrix} \begin{matrix} k_2 \\ k_3 \\ k_6 \\ k_7 \end{matrix} \quad \mathbf{e}' = \begin{pmatrix} 45 \\ 60 \\ 3 \\ 15 \\ 63 \\ 10 \\ 0 \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{matrix} \quad \mathbf{l}' = \begin{pmatrix} 2.25 \\ 0.00 \\ 0.00 \\ 15.00 \\ 56.70 \\ 0.50 \\ 0.00 \end{pmatrix} \begin{matrix} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \\ l_6 \\ l_7 \end{matrix}$$

solving Eq. (1), we obtain

$$\mathbf{u}' = \begin{pmatrix} -15.0 \\ -56.7 \\ -6.0 \\ -8.0 \\ 0.0 \\ 0.0 \\ -7.0 \\ -53.0 \\ 0.0 \end{pmatrix} \begin{matrix} u_{141} \\ u_{151} \\ u_{212} \\ u_{312} \\ u_{432} \\ u_{562} \\ u_{612} \\ u_{711} \\ u_{732} \end{matrix} \quad \mathbf{g}' = \begin{pmatrix} -6.0 \\ -8.0 \\ -7.5 \\ -53.0 \end{pmatrix} \begin{matrix} g_2 \\ g_3 \\ g_6 \\ g_7 \end{matrix}$$

By Eq. (22) the displacement $\mathbf{u}$ may be decomposed as the product of $\bar{\mathbf{M}}^{-1}$ and $\bar{\mathbf{a}}$, here we have

$$\bar{\mathbf{a}}' = \begin{pmatrix} -15.00 \\ -56.70 \\ 14.50 \\ 19.33 \\ 0.00 \\ 0.00 \\ 16.92 \\ -53.00 \\ 0.00 \end{pmatrix} \begin{matrix} \bar{a}_{141} \\ \bar{a}_{151} \\ \bar{a}_{212} \\ \bar{a}_{312} \\ \bar{a}_{432} \\ \bar{a}_{562} \\ \bar{a}_{612} \\ \bar{a}_{711} \\ \bar{a}_{732} \end{matrix} \quad \text{and}$$

$$\bar{\mathbf{M}}' = \begin{array}{cc} \begin{matrix} u_{141} & u_{151} & u_{212} & u_{312} & u_{432} & u_{562} & u_{612} & u_{711} & u_{732} \end{matrix} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 & \underline{-1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \underline{-1} & 0 & 0 & 0 \\ -0.29 & -0.29 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \underline{-0.38} & -0.38 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{0} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \underline{0} & 0 & 0 \\ -0.33 & \underline{-0.33} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \begin{matrix} f_{141} \\ f_{151} \\ f_{212} \\ f_{312} \\ f_{432} \\ f_{562} \\ f_{612} \\ f_{711} \\ f_{732} \end{matrix} \end{array}$$

in which there are two potential cycles (whose entries are underlined respectively with red and blue colours)[113] none of which is an actual cycle. So, moving from $\mathbf{x}$ to $\mathbf{x}'$ despite increasing

- the overall debt $\sum_{ijk} x_{ijk}$,
- the before-the-shock leverage ratio for the 1st balance sheet (all the others unchanged) and
- the number of before-the-shock cycles (from one to two),[114]

would result in

- a lower overall displacement [115] as measured by the taxicab length of the displacement vector $\sum_{ijk} |u_{ijk}|$ (from 151.2 to 145.6) and [116]
- a reduction of after-the-shock actual cycles (from one to zero).

[113] Despite both entries $\bar{\mathbf{M}}[5,4]$ and $\bar{\mathbf{M}}[6,7]$ are zero, the correspondent potential links are not zero given that both x_{432} and x_{562} are not zero.

[114] Already in 1913 A.A.Bogdanov showed that the results of making a system more connected may be ambiguous in terms of its vulnerability; see [8, pp. 144-146].

[115] The reduction in the overall displacement stems from an initial *borrowing-from-Peter-to-pay-Paul* effect whereby the granting of a new credit (Δx_{612}) from balance sheet *6* to balance sheet *1* results in an initial reallocation of both external assets losses and (pre-transaction) displacement to the advantage of his siblings ($(\Delta\bar{\mathbf{a}} - \Delta\bar{\mathbf{M}}\mathbf{u}_0)_{212} = 2.3$ and $(\Delta\bar{\mathbf{a}} - \Delta\bar{\mathbf{M}}\mathbf{u}_0)_{312} = 3.3$ respectively), which *network* effect amplifies (to $\Delta u_{212} = 2.9$ and $\Delta u_{312} = 3.9$) and (differently from the case of small transactions) extends to other claims ($\Delta u_{432} = 1.7$) not involved in the transaction..

[116] It may also occur that the overall displacement of the claims to a given balance sheet decreases due to an increase of its liabilities as in the following example (which shows the effects of a funding transaction $\{\Delta x_{432} = 6, \Delta e_3 = 6, \Delta e_4 = -6\}$)

$$\mathbf{x} = \begin{pmatrix} 9.0 \\ 10.0 \\ 10.0 \\ 5.0 \end{pmatrix} \begin{matrix} x_{132} \\ x_{212} \\ x_{321} \\ x_{432} \end{matrix} \quad \Delta\mathbf{x} = \begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 6.0 \end{pmatrix} \quad \mathbf{e} = \begin{pmatrix} 4.0 \\ 0.0 \\ 4.5 \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix} \quad \mathbf{l} = \begin{pmatrix} 4.00 \\ 0.00 \\ 0.225 \end{pmatrix} \begin{matrix} l_1 \\ l_2 \\ l_3 \end{matrix} \quad \Delta\mathbf{l} = \begin{pmatrix} 0.00 \\ 0.00 \\ 0.30 \end{pmatrix} \quad \mathbf{k} = \begin{pmatrix} 3.0 \\ 0.0 \\ 0.5 \end{pmatrix} \begin{matrix} k_1 \\ k_2 \\ k_3 \end{matrix} \quad \Delta\mathbf{u} = \begin{pmatrix} 0.47 \\ 0.47 \\ 0.47 \\ -0.30 \end{pmatrix} \begin{matrix} \Delta u_{132} \\ \Delta u_{212} \\ \Delta u_{321} \\ \Delta u_{432} \end{matrix}$$

Here the overall displacement on the right side of balance sheet 3 decreases, despite the absolute value of the negative direct impact of the funding transaction on u_{432} is higher that the positive direct impact on u_{132}, as the latter is amplified by the presence of a cycle; in Eq. (22) we have

$$\bar{\mathbf{M}}_1 = \begin{array}{cc} \begin{matrix} u_{132} & u_{212} & u_{321} & u_{432} \end{matrix} & \\ \begin{pmatrix} 1.00 & 0.00 & \underline{-0.45} & 0.00 \\ \underline{-1.00} & 1.00 & 0.00 & 0.00 \\ 0.00 & \underline{-1.00} & 1.00 & 0.00 \\ 0.00 & 0.00 & -0.55 & 1.00 \end{pmatrix} & \begin{matrix} f_{132} \\ f_{212} \\ f_{321} \\ f_{432} \end{matrix} \end{array} \quad \bar{\mathbf{M}}_1^{-1} = \begin{array}{c} \begin{matrix} u_{132} & u_{212} & u_{321} & u_{432} \end{matrix} \\ \begin{pmatrix} 1.82 & 0.82 & 0.82 & 0.00 \\ 1.82 & 1.82 & 0.82 & 0.00 \\ 1.82 & 1.82 & 1.82 & 0.00 \\ 1.00 & 1.00 & 1.00 & 1.00 \end{pmatrix} \end{array} \Delta\bar{\mathbf{a}} = \begin{pmatrix} -0.19 \\ 0.00 \\ 0.00 \\ -0.11 \end{pmatrix} \quad \Delta\bar{\mathbf{M}}\mathbf{u}_0 = \begin{pmatrix} -0.45 \\ 0.00 \\ 0.00 \\ 0.45 \end{pmatrix}$$

A.8.2 Example 2

If an equity-type transaction $\{\Delta x_{611} = 35, \Delta e_6 = -35, \Delta e_1 = 35\}$ were put in place, instead of a credit-type transaction, at the new point

$$\mathbf{x}'' = \begin{array}{c}\begin{pmatrix}30\\83\\30\\40\\15\\20\\35\\53\\20\end{pmatrix}\end{array}\begin{array}{l}x_{141}\\x_{151}\\x_{212}\\x_{312}\\x_{432}\\x_{562}\\x_{611}\\x_{711}\\x_{732}\end{array} \text{ given } \mathbf{k}'' = \begin{pmatrix}90\\8\\25\\73\end{pmatrix}\begin{array}{l}k_2\\k_3\\k_6\\k_7\end{array} \; \mathbf{e}'' = \begin{pmatrix}45\\60\\3\\15\\63\\10\\0\end{pmatrix}\begin{array}{l}e_1\\e_2\\e_3\\e_4\\e_5\\e_6\\e_7\end{array} \; \mathbf{l}'' = \begin{pmatrix}2.25\\0.00\\0.00\\15.00\\56.70\\0.50\\0.00\end{pmatrix}\begin{array}{l}l_1\\l_2\\l_3\\l_4\\l_5\\l_6\\l_7\end{array}$$

solving Eq. (1), we obtain

$$\mathbf{u}'' = \begin{pmatrix}-15.0\\-64.9\\0.0\\0.0\\0.0\\-8.2\\-32.7\\-49.4\\0.0\end{pmatrix}\begin{array}{l}u_{141}\\u_{151}\\u_{212}\\u_{312}\\u_{432}\\u_{562}\\u_{611}\\u_{711}\\u_{732}\end{array} \; \mathbf{g}'' = \begin{pmatrix}0.0\\0.0\\-25.0\\-49.4\end{pmatrix}\begin{array}{l}g_2\\g_3\\g_6\\g_7\end{array}$$

By Eq. (22) the displacement $\mathbf{u}$ may be decomposed as the product of $\bar{\mathbf{M}}^{-1}$ and $\bar{\mathbf{a}}$, where

$$\bar{\mathbf{a}}'' = \begin{pmatrix}-15.00\\-56.70\\0.00\\0.00\\0.00\\24.50\\-0.89\\-1.36\\0.00\end{pmatrix}\begin{array}{l}\bar{a}_{141}\\\bar{a}_{151}\\\bar{a}_{212}\\\bar{a}_{312}\\\bar{a}_{432}\\\bar{a}_{562}\\\bar{a}_{611}\\\bar{a}_{711}\\\bar{a}_{732}\end{array} \text{ and}$$

$$\bar{\mathbf{M}}'' = \begin{array}{c} \begin{array}{ccccccccc} u_{141} & u_{151} & u_{212} & u_{312} & u_{432} & u_{562} & u_{611} & u_{711} & u_{732} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & \underline{-1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \underline{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \underline{0} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{0} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \underline{-1} & 0 & 0 \\ -0.40 & \underline{-0.40} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -0.60 & -0.60 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ f_{141}\\f_{151}\\f_{212}\\f_{312}\\f_{432}\\f_{562}\\f_{611}\\f_{711}\\f_{732}\end{array}$$

Here again there are two potential cycles, but this time one of them (the blue one) is an actual cycle In this case the equity-type transaction, despite bringing about a reduction of the leverage ratio of the 1st balance sheet (all the other unchanged), would result in a higher overall displacement as measured by the taxicab length of the displacement vector $\sum_{ijk} |u_{ijk}|$ (from 151.2 to 170.1). [117]

A.8.3 Example 3

Finally if the funding transaction between the 6$^{\text{th}}$ and the 1$^{\text{st}}$ balance sheet were to be a collateralized one (i.e. $\{\Delta x_{613} = 35, \Delta e_6 = -35, \Delta e_1 = 35\}$), at the new point

$$\mathbf{x}'^{v} = \begin{array}{c} \begin{pmatrix} 30 \\ 83 \\ 30 \\ 40 \\ 15 \\ 20 \\ 35 \\ 53 \\ 20 \end{pmatrix} \end{array} \begin{array}{l} x_{141} \\ x_{151} \\ x_{212} \\ x_{312} \\ x_{432} \\ x_{562} \\ x_{613} \\ x_{711} \\ x_{732} \end{array} \quad \mathbf{k}'^{v} = \begin{pmatrix} 90 \\ 8 \\ 25 \\ 73 \end{pmatrix} \begin{array}{l} k_2 \\ k_3 \\ k_6 \\ k_7 \end{array} \quad \mathbf{e}'^{v} = \begin{pmatrix} 45 \\ 60 \\ 3 \\ 15 \\ 63 \\ 10 \\ 0 \end{pmatrix} \begin{array}{l} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{array} \quad \mathbf{l}'^{v} = \begin{pmatrix} 2.25 \\ 0.00 \\ 0.00 \\ 15.00 \\ 56.70 \\ 0.50 \\ 0.00 \end{pmatrix} \begin{array}{l} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \\ l_6 \\ l_7 \end{array}$$

[117] Here capitalization transactions show a non-monotonic effect on the overall displacement: a smaller capitalization transaction (e.g. $\{\Delta x_{611} = 20, \Delta e_6 = -20, \Delta e_1 = 20\}$) would produce a smaller displacement ($\sum_{ijk} |u_{ijk}| = 144.9$) even with respect to the original example (where $\Delta x_{611} = 0$). Doing the calculation would produce:

$$\mathbf{u}''' = \begin{pmatrix} -15.0 \\ -56.7 \\ -0.1 \\ -0.1 \\ 0.0 \\ 0.0 \\ -20.0 \\ -53.0 \\ 0.0 \end{pmatrix} \begin{array}{l} u_{141} \\ u_{151} \\ u_{212} \\ u_{312} \\ u_{432} \\ u_{562} \\ u_{611} \\ u_{711} \\ u_{732} \end{array} \quad \mathbf{g}''' = \begin{pmatrix} -0.1 \\ -0.1 \\ -21.2 \\ -53.0 \end{pmatrix} \begin{array}{l} g_2 \\ g_3 \\ g_6 \\ g_7 \end{array} \quad \bar{\mathbf{a}}''' = \begin{pmatrix} -15.00 \\ -56.70 \\ 30.64 \\ 40.85 \\ 0.00 \\ 0.00 \\ -20.00 \\ -53.00 \\ 0.00 \end{pmatrix} \begin{array}{l} \bar{a}_{141} \\ \bar{a}_{151} \\ \bar{a}_{212} \\ \bar{a}_{312} \\ \bar{a}_{432} \\ \bar{a}_{562} \\ \bar{a}_{611} \\ \bar{a}_{711} \\ \bar{a}_{732} \end{array} \quad \text{and}$$

$$\bar{\mathbf{M}}''' = \begin{array}{c} \begin{array}{ccccccccc} u_{141} & u_{151} & u_{212} & u_{312} & u_{432} & u_{562} & u_{611} & u_{711} & u_{732} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -0.43 & -0.43 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.57 & -0.57 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ f_{141} \\ f_{151} \\ f_{212} \\ f_{312} \\ f_{432} \\ f_{562} \\ f_{611} \\ f_{711} \\ f_{732} \end{array}$$

In this case there would be no after-the-shock cycle ($\det \bar{\mathbf{M}}''' = 1$).

solving Eq. (1), we obtain

$$\mathbf{u}'^{v} = \begin{pmatrix} -17.2 \\ -56.7 \\ -9.9 \\ -13.3 \\ -2.2 \\ 0.0 \\ 0.0 \\ -53.0 \\ -3.0 \end{pmatrix} \begin{array}{l} u_{141} \\ u_{151} \\ u_{212} \\ u_{312} \\ u_{432} \\ u_{562} \\ u_{613} \\ u_{711} \\ u_{732} \end{array} \quad \mathbf{g}'^{v} = \begin{pmatrix} -9.9 \\ -8.0 \\ -0.5 \\ -56.0 \end{pmatrix} \begin{array}{l} g_2 \\ g_3 \\ g_6 \\ g_7 \end{array}$$

By Eq. (22) the displacement $\mathbf{u}$ may be decomposed as the product of $\bar{\mathbf{M}}^{-1}$ and $\bar{\mathbf{a}}$, where

$$\bar{\mathbf{a}}'^{v} = \begin{pmatrix} -15.00 \\ -56.70 \\ 21.75 \\ 29.00 \\ 3.43 \\ 0.00 \\ 0.00 \\ -53.00 \\ 4.57 \end{pmatrix} \begin{array}{l} \bar{a}_{141} \\ \bar{a}_{151} \\ \bar{a}_{212} \\ \bar{a}_{312} \\ \bar{a}_{432} \\ \bar{a}_{562} \\ \bar{a}_{613} \\ \bar{a}_{711} \\ \bar{a}_{732} \end{array} \quad \text{and}$$

$$\bar{\mathbf{M}}'^{v} = \begin{array}{c} \begin{array}{ccccccccc} u_{141} & u_{151} & u_{212} & u_{312} & u_{432} & u_{562} & u_{612} & u_{711} & u_{732} \end{array} \\ \left(\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & \underline{-1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -0.43 & -0.43 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \underline{-0.57} & -0.57 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{-.43} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -0.57 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{l} \\ f_{141} \\ f_{151} \\ f_{212} \\ f_{312} \\ f_{432} \\ f_{562} \\ f_{612} \\ f_{711} \\ f_{732} \end{array}$$

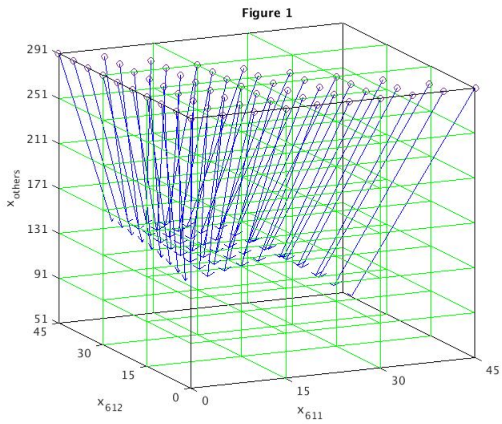
Here again there are two potential cycles, one of them (the red one) is an actual cycle; overall displacement as measured by the taxicab length of the displacement vector $\sum_{ijk} |u_{ijk}| = 155.4$ is higher than that in the case of the uncollateralized funding transaction of Example 1 (145.6).

* * *

Figure 1 shows the displacement field on the portion of plane defined by[118]

$$\begin{cases} 0 \le x_{611} \le 45 \\ 0 \le x_{612} \le 45 \\ 0 \le x_{611} + x_{612} \le 45 \\ x_{other} = 291 \end{cases}$$

(to which the points of the previous examples belong); the overall displacement increases (after initially slightly decreasing) as we move toward higher capital points and decreases as we move toward higher leverage points.



[118]With $x_{other} = \sum_{ijk \notin \{611,612\}} x_{ijk}$.