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Abstract

In this paper, we show that traditional comparisons of Mean Squared Prediction Error (MSPE) between two competing forecasts may be highly controversial. This is so because when some specific conditions of efficiency are not met, the forecast displaying the lowest MSPE will also display the lowest correlation with the target variable. Given that violations of efficiency are usual in the forecasting literature, this opposite behavior in terms of accuracy and correlation with the target variable may be a fairly common empirical finding that we label here as "the MSPE Paradox." We characterize "Paradox zones" in terms of differences in correlation with the target variable and conduct some simple simulations to show that these zones may be non-empty sets. Finally, we illustrate the relevance of the Paradox with two empirical applications.

JEL Codes: C52, C53, G17, E270, E370, F370, L740, O180, R310

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1. Introduction

"*How wonderful that we have met with a paradox. Now we have some hope of making progress*." Niels Bohr.

In this paper, we show that traditional comparisons of Mean Squared Prediction Error (MSPE) between two competing forecasts may be highly controversial. This is so because when some specific conditions of efficiency are not met, the forecast displaying the lowest MSPE will also display the lowest correlation with the target variable. Given that violations of efficiency are usual in the forecasting literature, this opposite behavior in terms of accuracy and correlation with the target variable may be a fairly common empirical finding that we label here as the MSPE paradox[2].

It is safe to say that MSPE is one of the most popular measures in the forecast evaluation literature, with a long tradition in both empirical and theoretical works. Just as an anecdotal illustration of its relevance, the acronym "MSPE" is mentioned 77 times in West's (2006) survey. The rationale for using MSPE as a loss function is as follows: MSPE is a statistical measure of accuracy, then, a forecast displaying a low MSPE is an accurate forecast that, on average, will be close to the target variable. Some of the most iconic empirical contributions in economic forecasting (such as those of Meese and Rogoff (1983, 1988), Goyal and Welch (2008), Stock and Watson (2003), and Timmermann (2008)) rely partially or completely on MSPE comparisons. Due to its importance and tractability, it is not surprising that many theoretical works in this literature treat MSPE as a leading case (e.g., Diebold and Mariano (1995), West (1996), Giacommini and White (2006)).

An alternative avenue to evaluate predictive ability could consider the association between the forecast and the target variable: the tighter the association is, the better the forecast is. Probably the simplest association measure between two random variables, X and Y, is the correlation between them. According to this intuition, a forecast more closely related to Y would be superior to another forecast not as closely related to Y. In other words, a forecast with a higher correlation with Y should be preferable to another forecast displaying a lower correlation.

Interestingly, in this paper, we show analytically and empirically that the forecast with the lowest MSPE does not necessarily display the highest correlation (what we call the MSPE Paradox). We show that both approaches are equivalent when forecasts meet some

[2] Violations of efficiency in the forecast literature are found across multiple variables in a number of articles. See for instance, Ince and Molodtsova (2017); Joutz and Stekler (2000); Ang, Bekaert and Wei (2007); Bentancor and Pincheira (2010); Patton and Timmermann (2012); Nordhaus (1987); Pincheira and Álvarez (2009); Pincheira and Fernández (2011) and Pincheira (2012, 2010) just to mention a few.

conditions of efficiency (Mincer and Zarnowitz (1969)). Given that violations of efficiency are usual in the forecasting literature (see footnote 2), this opposite behavior in terms of accuracy and correlation with the target variable may be a fairly common empirical finding.

We offer a characterization of "Paradox zones" in terms of the differences in correlation with the target variable. Moreover, we carry out simple simulations to show that these Paradox zones are, in general, non-empty sets. As a matter of fact, our analysis shows that we could have an extreme case in which a totally uncorrelated forecast with the target variable could be superior in terms of MSPE to an alternative forecast displaying a positive correlation with the same target variable. Our empirical illustration supports this idea.

Finally, we show the relevance of the MSPE Paradox with two empirical applications in which some of the most accurate forecasts are, in fact, the worst in terms of correlations with the target variable. Both illustrations are related to the commodity-currencies literature. In the first exercise, we predict the returns of eleven commodities with the exchange rates of five commonly studied commodity-exporting economies. In the second exercise, we evaluate several exchange rates forecasts of the same five commodity-exporting economies. In this case, we compare the predictions of the FX4cast survey with some forecasts constructed with commodity returns and some usual benchmarks as well.

The rest of this paper is organized as follows. In section 2 we show with simple examples what we call the MSPE Paradox. We warn the reader that in subsection 2.1 we will be making very restrictive assumptions for the sake of clarity. Nevertheless, in subsection 2.2 we relax these assumptions to analyze the Paradox with more generality. In Section 3 we offer a characterization of "Paradox zones". In section 4 we illustrate the Paradox with simple simulations whereas in section 5 we present two empirical illustrations. Finally, section 6 concludes.

2. The MSPE Paradox

2.1 Simple examples

In this section we illustrate with simple examples what we call "The MSPE Paradox." We use this name to label the fact that when comparing two competing forecasts for the same target variable, it might be the case that the forecast displaying the lowest MSPE will also display the lowest correlation with the target variable.

Let us consider $\{Y_t\}$ to be a mean zero target variable with variance equal 1. At time t, we have two competing forecasts $\{X_{t-1}\}$ and $\{Z_{t-1}\}$ for $\{Y_t\}$. It is important to notice that both $\{X_{t-1}\}$ and $\{Z_{t-1}\}$ are forecasts constructed with information previous to time t and that they are taken as primitives (hence, we are not concerned here about parameter uncertainty). For clarity of exposition, we drop the sub-indexes t in what follows. Let us assume that the vector $(Y, X, Z)'$ is weakly stationary (so here we assume the existence of second moments).

Example 1:

For example 1 we will also assume that both forecasts have the same non-negligible variance: Var(X)=Var(Z)>0, that X is a mean zero forecast and that $E(X^2) > 0$. Many of these assumptions are very restrictive, but they are useful to illustrate the Paradox.

Consider now the MSPE of both forecasts:

$$MSPE_X = E(Y - X)^2 \,;\; MSPE_Z = E(Y - Z)^2$$

and let us also define the corresponding Mean Squared Forecasts (MSF) as follows:

$$MSF_X = E(X^2);\; MSF_Z = E(Z^2)$$

Suppose now that we are interested in a traditional comparison of MSPE, then:

$$\Delta MSPE \equiv MSPE_X - MSPE_Z = E(Y - X)^2 - E(Y - Z)^2$$

$$= (EX^2 - EZ^2) - 2(EYX - EYZ)$$

$$= (MSF_X - MSF_Z) - 2\{Cov(Y, X) - Cov(Y, Z)\}$$

$$= (MSF_X - MSF_Z) - 2\sqrt{Var(Y)}\left\{Corr(Y, X)\sqrt{Var(X)} - Corr(Y, Z)\sqrt{Var(Z)}\right\}$$

$$= (MSF_X - MSF_Z) - 2\sqrt{Var(X)}\{Corr(Y, X) - Corr(Y, Z)\}$$

$$= (MSF_X - MSF_Z) - 2\sqrt{MSF_X - (EX)^2}\{Corr(Y, X) - Corr(Y, Z)\}$$

$$= (MSF_X - MSF_Z) - 2\sqrt{MSF_X}\{Corr(Y, X) - Corr(Y, Z)\} \quad (1)$$

Eq.(1) illustrates an important result: the difference in MSPE depends not only on the correlation between the forecasts with the target variable, but also on MSF that are not directly linked to properties of the target variable. The problem in this illustration relies on a "magnitude" effect associated to the term $(MSF_X - MSF_Z)$: A high MSF of a forecast could more than offset its high correlation with the target variable and therefore the forecast itself could be outperformed by another less informational forecast with a lower MSF. In other words, in this example, traditional MSPE comparisons give a natural advantage to "small forecasts", that is to say, forecasts with small MSF.

Example 2:

As a second example, let us consider a different econometric context, but similarly simplistic, in which Z is a zero-forecast. Consequently, $Var(Z) = Cov(Y,Z) = EZ^2 = 0$. Furthermore, let us also assume that $EX = 0$, Var(X) > 0 and that Var(Y) = 1. We will have then

$$\Delta MSPE \equiv MSPE_X - MSPE_Z = E(Y-X)^2 - E(Y-Z)^2$$

$$= (EX^2 - EZ^2) - 2(EYX - EYZ)$$

$$= (EX^2 - EZ^2) - 2\{Cov(Y,X) - Cov(Y,Z)\}$$

$$=(EX^2) - 2\{Cov(Y,X)\}$$

$$= (EX^2) - 2\sqrt{Var(Y)}\sqrt{Var(X)}\{Corr(Y,X)\}$$

$$= MSF_X - 2\sqrt{MSF_X}\{Corr(Y,X)\}$$

then if $\sqrt{MSF_X} > 2\,Corr(Y,X) > 0$ we will have that $MSPE_X > MSPE_Z$ despite that $Cov(Y,X) > Cov(Y,Z) = 0$. This is, of course, an extreme situation. The use of MSPE in this case, will suggest that the forecast with no association whatsoever with the target variable is preferable to another forecast with a tighter association. The problem in this example is that MSPE comparisons would fail to detect the usefulness of forecast X if its magnitude (MSF_X) overshadows its informational content.

2.2 A general case

Let us now leave behind our simplifying assumptions to show a more general picture of the MSPE Paradox. We will analyze two leading cases: a case in which Y, X and Z have all positive variances and a case in which Y and X have positive variances but Z is just a constant *c* that might or might not be equal to zero. Beyond weak stationarity, we will also assume that Corr(Y,X) <1. In summary, from now on, we will assume that the following simple assumptions hold true:

A1) Existence of first and second moments for the vector (Y,X,Z)′.

A2) Weak stationarity for the vector (Y,X,Z)′.

A3) Positive variance for Y and X.

A4) Corr(Y,X) <1.

As mentioned before, we are interested in two different scenarios: one in which Z is just a constant, and the other in which Z is a forecast with positive variance. We will refer to these conditions as C1 and C2:

C1) Z is just a constant c.

C2) Z is a forecast with positive variance.

In what follows we will use the notation

$$\Delta MSPE \equiv MSPE_X - MSPE_Z$$

With simple algebra it is straightforward to show that:

$$\Delta MSPE = MSF_X - MSF_Z - 2Cov(Y, X - Z) - 2EY(EX - EZ) \quad (2)$$

If Z has positive variance (case C2) then

$$\Delta MSPE = MSF_X - MSF_Z - 2\left\{Corr(Y,X)\sqrt{V(Y)V(X)} - Corr(Y,Z)\sqrt{V(Y)V(Z)} + EY(EX - EZ)\right\}$$

$$= MSF_X - MSF_Z - 2\sqrt{V(Y)}\left\{Corr(Y,X)\sqrt{V(X)} - Corr(Y,Z)\sqrt{V(Z)}\right\} - 2\{EY(EX - EZ)\}$$

$$= MSF_X - MSF_Z - 2\sqrt{V(Y)}\left\{Corr(Y,X)\sqrt{MSF_x - (EX)^2} - Corr(Y,Z)\sqrt{MSF_z - (EZ)^2}\right\} - 2\{EY(EX - EZ)\} \quad (3.1)$$

Differing from our previous simple examples, where the Paradox emerges entirely by the magnitude effect associated to the MSF, in the more general case of eq.(3.1), the Paradox may also emerge as a consequence of a complex interaction of all the terms involved in that expression.

If Z is just a constant c (case C1) then the correlation between Z and Y is not defined. In this case it is simple to show that expression (2) could also be written as follows:

$$\Delta MSPE = MSF_X - MSF_Z - 2Corr(Y,X)\sqrt{V(Y)V(X)} - 2EY(EX - EZ) \quad (3.2)$$

Here we also see that the Paradox may emerge not only as a consequence of the relative magnitude effects ($MSF_X - MSF_Z$) but also for the action of the term associated to the potential bias of the forecasts: $2EY(EX - EZ)$.

Notice that our decomposition relates to Clark and West (2006, 2007) in the following sense: In the context of out-of-sample comparisons of nested models, Clark and West (2006, 2007) notice that under the null of equal population MSPE, the sample MSPE of the nesting model is expected to be higher than that of the nested one. The intuition is that the nesting model introduces noise into its forecasts through the estimation of parameters that, under the null, are equal to zero. This effect inflates the sample MSPE of the model with additional parameters. Our decomposition resembles the findings by Clark and West, and to some extent, it is even more general. Both Clark and West and us similarly argue that a plain look at MSPE comparisons may be misleading in some circumstances, given that they can be affected by several distortions. In the case of Clark and West, those distortions arise from parameter estimation error. In our approach, these distortions arise at the population level by comparing apple and oranges: forecasts with very different magnitude effects or very different biases. In other words, even at the population level, we may observe that some forecasts have a natural advantage in terms of MSPE relative to others, despite of being far less informational relative to its competitors.

3. Some simple theoretical results

In the following we will assume, without loss of generality, that $Corr(Y,X) \geq Corr(Y,Z)$ if Z has positive variance. In case that Z has zero variance, we will assume, without loss of generality, that $Cov(Y,X) \geq Cov(Y,Z) = 0$. In this setup the Paradox will exist whenever $MSPE_X - MSPE_Z > 0$. As we are considering the two leading cases of positive and zero variance for Z, we will denote by Ω_1 the variance-covariance matrix of the (Y,X,Z)′ vector and by Ω_2 the variance-covariance matrix of the (Y,X)′ vector.

<u>Proposition 1:</u> Let Z be a constant-forecast (say, $Z = c\ \forall t$). Let us also assume that

$$\frac{MSF_X - c^2}{2\sqrt{V(Y)V(X)}} - \frac{EY(EX-c)}{\sqrt{V(Y)V(X)}} > 0$$

Then we will find the Paradox iif $Corr(Y,X) \in [0; \frac{MSF_X - c^2}{2\sqrt{V(Y)V(X)}} - \frac{EY(EX-c)}{\sqrt{V(Y)V(X)}})$.

<u>Corollary 1:</u> If in Proposition 1 we set $Z = c = 0$, we will find the Paradox iif $Corr(Y,X) \in [0; \frac{MSF_X}{2\sqrt{V(Y)V(X)}} - \frac{EYEX}{\sqrt{V(Y)V(X)}})$. Moreover, in the particular case in which $EX = 0$, we will find the Paradox iif $Corr(Y,X) \in [0; \frac{\sqrt{V(X)}}{2\sqrt{V(Y)}})$.

Proof of Proposition 1.

Notice that in this case we have $E(Z) = c, MSF_Z = c^2$ and $V(Z) = Cov(Z,Y) = 0$.

Here eq.(3.2) could also be written as

$$\Delta MSPE = MSF_X - c^2 - 2\left\{Corr(Y,X)\sqrt{V(Y)V(X)} + EY(EX-c)\right\}$$

As $Cov(Z,Y)=0$, we will have the Paradox whenever $Corr(Y,X)\geq 0$ and $\Delta MSPE>0$. Imposing this last condition we get

$$\Delta MSPE>0 \Leftrightarrow MSF_X-c^2-2\left\{Corr(Y,X)\sqrt{V(Y)V(X)}+EY(EX-c)\right\}>0 \quad (4)$$

$$Corr(Y,X)<\frac{MSF_X-c^2}{2\sqrt{V(Y)V(X)}}-\frac{EY(EX-c)}{\sqrt{V(Y)V(X)}}$$

Therefore we will have the Paradox iif $Corr(Y,X)\in[0;\frac{MSF_X-c^2}{2\sqrt{V(Y)V(X)}}-\frac{EY(EX-c)}{\sqrt{V(Y)V(X)}})$ ■

The proof of Corollary 1 follows simply by setting c=0.

Proposition 2 next shows an equivalent result for the case in which Z is a forecast with positive variance.

Proposition 2: Let Z be a forecast with positive variance. Let $\Delta=Corr(Y,X)-Corr(Y,Z)$ and suppose that

$$\frac{MSF_X-MSF_Z}{2\sqrt{V(Y)V(X)}}-\frac{Corr(Y,Z)\left(\sqrt{V(X)}-\sqrt{V(Z)}\right)}{\sqrt{V(X)}}-\frac{EY(EX-EZ)}{\sqrt{V(Y)V(X)}}>0$$

Then we will find the Paradox iif

$$\Delta\in[0;\frac{MSF_X-MSF_Z}{2\sqrt{V(Y)V(X)}}-\frac{Corr(Y,Z)\left(\sqrt{V(X)}-\sqrt{V(Z)}\right)}{\sqrt{V(X)}}-\frac{EY(EX-EZ)}{\sqrt{V(Y)V(X)}}).$$

Proof of Proposition 2.

Reorganizing eq.(3.1) we get

$$\Delta MSPE=MSF_X-MSF_Z-2\left\{Corr(Y,X)\sqrt{V(Y)V(X)}-Corr(Y,Z)\sqrt{V(Y)V(Z)}+EY(EX-EZ)\right\}$$

$$=MSF_X-MSF_Z-2\sqrt{V(Y)}Corr(Y,Z)\left(\sqrt{V(X)}-\sqrt{V(Z)}\right)-2\sqrt{V(Y)}\sqrt{V(X)}\Delta-2EY(EX-EZ) \quad (5)$$

We will find the Paradox whenever $\Delta MSPE>0$ and $\Delta\geq 0$, which is equivalent to

$$0\leq\Delta<\frac{MSF_X-MSF_Z}{2\sqrt{V(Y)V(X)}}-\frac{Corr(Y,Z)\left(\sqrt{V(X)}-\sqrt{V(Z)}\right)}{\sqrt{V(X)}}-\frac{EY(EX-EZ)}{\sqrt{V(Y)V(X)}}$$

And the Paradox condition is simply given by $\Delta\in[0;\frac{MSF_X-MSF_Z}{2\sqrt{V(Y)V(X)}}-\frac{Corr(Y,Z)\left(\sqrt{V(X)}-\sqrt{V(Z)}\right)}{\sqrt{V(X)}}-\frac{EY(EX-EZ)}{\sqrt{V(Y)V(X)}})$ ■

It is also interesting to explore if the Paradox is a simple consequence of the traditional variance-bias trade-off so widely explored in the forecasting literature. It is well known that a biased forecast could be superior to an unbiased forecast in terms of MSPE, if the bias in the first forecast is associated to an important decrease in the variance of the forecast error. To explore this possibility, we analyze the Paradox when comparing two equally biased forecasts, so that the potential presence of the Paradox in this scenario cannot be attributed to a variance-bias trade-off. The following corollary to proposition 2 addresses this case:

<u>Corollary 2:</u> If in Proposition 2 we set $EX = EZ$, then we will find the Paradox iif

$$\Delta \in [0; \frac{V(X)-V(Z)}{2\sqrt{V(Y)V(X)}} - \frac{Corr(Y,Z)\left(\sqrt{V(X)}-\sqrt{V(Z)}\right)}{\sqrt{V(X)}}).$$

Notice that if V(X) = V(Z) then the Paradox is impossible as the Paradox zone is the empty set. Put differently, our decomposition becomes

$$MSPE_X - MSPE_Z = -2\sqrt{V(Y)V(X)}\{Corr(Y,X) - Corr(Y,Z)\}$$

which implies that differences in MSPE are equivalent to differences in correlations.

Nevertheless, whenever $V(X) \neq V(Z)$ the Paradox will be possible as long as

$$\frac{[V(X) - V(Z)]}{2\sqrt{V(Y)V(X)}} - \frac{Corr(Y,Z)\left(\sqrt{V(X)} - \sqrt{V(Z)}\right)}{\sqrt{V(X)}} > 0$$

For this condition to hold true we require either

$$Corr(Y,Z) < \frac{\sqrt{V(X)} + \sqrt{V(Z)}}{2\sqrt{V(Y)}} \quad if\ V(X) > V(Z)$$

Or

$$Corr(Y,Z) > \frac{\sqrt{V(X)} + \sqrt{V(Z)}}{2\sqrt{V(Y)}} \quad if\ V(X) < V(Z)$$

In subsection 4.2 we show some simple simulations showing a non-empty Paradox zone when forecasts are equally biased and $V(X) > V(Z)$. Similarly, in the appendix we show simulations for equally biased forecasts when $V(X) < V(Z)$ with the same conclusion: a non-empty Paradox zone. In summary, the MSPE Paradox is not a direct consequence of the traditional variance-bias trade-off as Corollary 2 and our simulations show that the Paradox also emerges in the context of equally biased forecasts.

Next we will see that for the Paradox to exist we require inefficient forecasts. We need some notation first: Let $\mathbf{u_x}$ and $\mathbf{u_z}$ be the forecast errors of X and Z, respectively. In other words

$$\mathrm{u_x} \equiv Y - X$$

$$\mathrm{u_Z} \equiv Y - Z$$

Let us recall that X and Z are efficient forecasts à la Mincer and Zarnowitz (1969) as long as

$$Cov(X, u_x) = Cov(Z, u_z) = 0$$

$$E(u_x) = E(u_z) = 0$$

<u>Proposition 3:</u> If X and Z are both efficient *à la* Mincer and Zarnowitz, then the Paradox is impossible.

Proof of Proposition 3.

Notice that $Y = X + u_x = Z + u_Z$, therefore

$$V(Y) = V(X) + V(u_x) + 2Cov(X, u_x)$$

$$V(Y) = V(Z) + V(u_Z) + 2Cov(Z, u_Z)$$

Under efficiency à la Mincer and Zarnowitz (1969) these expressions reduce to:

$$V(Y) = V(X) + MSPE_X$$

$$V(Y) = V(Z) + MSPE_Z$$

Where $MSPE_X = E(Y - X)^2$; $MSPE_Z = E(Y - Z)^2$.

Therefore

$$\Delta MSPE \equiv MSPE_X - MSPE_Z = V(Z) - V(X)$$

Notice also that under efficiency à la Mincer and Zarnowitz (1969) we will have that

$$Cov(Y,X) = Cov(\mathrm{X} + \mathrm{u_x}, X) = V(X) > 0$$

$$Cov(Y,Z) = Cov(\mathrm{Z} + \mathrm{u_Z}, Z) = V(Z) \geq 0$$

Which means that

$$\Delta\mathrm{MSPE} \equiv MSPE_X - MSPE_Z = \mathrm{V(Z)} - \mathrm{V(X)} = \mathrm{Cov(Y,Z)} - \mathrm{Cov(Y,X)} \quad (6)$$

If Z is just a constant c (case C1) then $\mathrm{Cov(Y,Z)} = \mathrm{V(Z)} = 0$. Therefore

$$\Delta\mathrm{MSPE} \equiv MSPE_X - MSPE_Z = -\mathrm{Cov(Y,X)} < 0$$

And clearly $Cov(Y,X) > Corr(Y,Z) = 0$, so the Paradox is impossible.

If Z has positive variance (case C2) then $\mathrm{Cov(Y,Z)} = \mathrm{V(Z)} > 0$. Let us recall that, without loss of generality, we are assuming that $Corr(Y,X) \geq Corr(Y,Z)$ whenever Z has positive variance. But

$$Corr(Y,X) = \frac{Cov(X + u_x, X)}{\sqrt{V(Y)V(X)}} = \frac{V(X) + Cov(X, u_x)}{\sqrt{V(Y)V(X)}}$$

Then, under efficiency à la Mincer and Zarnowitz (1969) $Corr(Y,X) = \frac{\sqrt{V(X)}}{\sqrt{V(Y)}}$.

Following the same argument, $Corr(Y,Z) = \frac{\sqrt{V(Z)}}{\sqrt{V(Y)}}$.

With these results, the assumption $Corr(Y,X) \geq Corr(Y,Z)$ is equivalent to

$$\sqrt{V(X)} \geq \sqrt{V(Z)}$$

This and eq.(6) implies that

$$\Delta\mathrm{MSPE} \equiv MSPE_X - MSPE_Z \leq 0$$

so the Paradox is, again, impossible ■

4. Simulations

In proposition 1 the existence of the Paradox relies on the following assumption:

$$\frac{MSF_X - c^2}{2\sqrt{V(Y)V(X)}} - \frac{EY(EX - c)}{\sqrt{V(Y)V(X)}} > 0$$

Whereas in proposition 2 it relies on the following more complex assumption:

$$\frac{MSF_X - MSF_Z}{2\sqrt{V(Y)V(X)}} - \frac{Corr(Y,Z)\left(\sqrt{V(X)} - \sqrt{V(Z)}\right)}{\sqrt{V(X)}} - \frac{EY(EX - EZ)}{\sqrt{V(Y)V(X)}} > 0$$

To illustrate that these assumptions may hold true and that the “Paradox zone” may be a non-empty set, we carry out a set of simple simulations. In each simulation, we show that the Paradox zone coincides with the intervals derived in Section 3. Here we show three different cases: i) Z as a zero-forecast with arbitrary parameters, ii) Z is a more general case with arbitrary parameters, and iii) Z is a zero-forecast in the context of a data generating process calibrated with exchange rates forecasts. In addition, in the appendix we consider a case in which both forecasts display equal bias and $\mathrm{V(Z)} > \mathrm{V(X)}$.

4.1 Simulation with a "zero-forecast."

Let us suppose that we want to compare two competing forecasts, X and Z, where Var(X) > 0 and Z is a "zero-forecast." According to corollary 1 in Section 3, the Paradox zone is defined by

$$Corr(X,Y) \in [0; \frac{MSFx}{2\sqrt{V(Y)V(X)}} - \frac{E(Y)E(X)}{\sqrt{V(Y)V(X)}})$$

To show that $[0; \frac{MSFx}{2\sqrt{V(Y)V(X)}} - \frac{E(Y)E(X)}{\sqrt{V(Y)V(X)}})$ may be a non-empty region, we consider the following simulation: We start by setting $EY = 0.1$, $EX = 1$, $E(Y^2) = 2$ and $E(X^2) = 2$, then from expression (4) we have

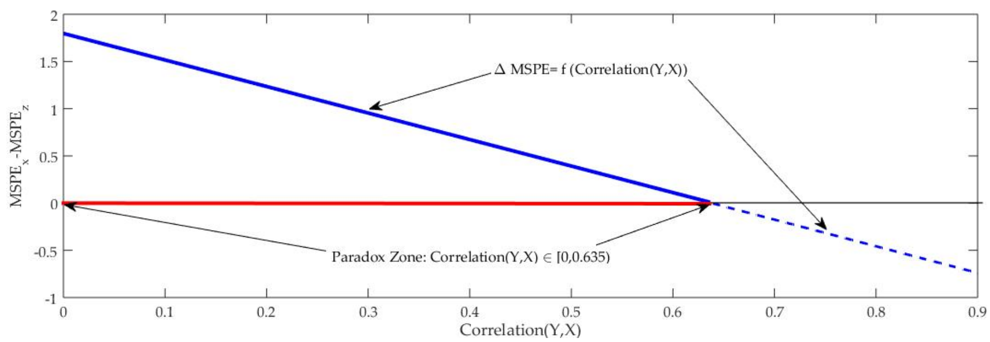
$$\Delta MSPE = \mathrm{MSPE_x} - MSPE_z = 1.8 - 2.8213 * Corr(X,Y)$$

Keeping $EY, EX, E(Y^2)$ and $E(X^2)$ constant, our decomposition is just a linear function between $\Delta MSPE$ and Corr(X,Y), with a slope of -2.8213 and an intercept of 1.8. In order to analyze this linear function without changing the slope nor the intercept, we generate different values of $Corr(X,Y)$ just by changing EYX but keeping in mind that the covariance matrix Ω_2 must be positive definite:

$$\Omega_2 = \begin{pmatrix} E(X^2) - (EX)^2 & EYX - EYEX \\ EYX - EYEX & E(Y^2) - (EY)^2 \end{pmatrix} = \begin{pmatrix} 1 & EYX - 0.1 \\ EYX - 0.1 & 1.99 \end{pmatrix}$$

We parameterize $\mathrm{EYX} = 0.1 + \delta$, where δ is a sequence of small positive incremental changes of 0.001. Notice that the slope and the intercept of our linear function remain unaltered in this simulation. In this case, the Paradox zone is given by $Corr(X,Y) \in \left[0; \frac{E(X^2)}{2\sqrt{V(Y)V(X)}} - \frac{E(Y)E(X)}{\sqrt{V(Y)V(X)}}\right[= [0; 0.638)$. In other words, despite that forecast Z has no covariance with Y, it outperforms the forecast X in terms of MSPE whenever $Corr(X,Y) \in [0; 0.638)$.

Figure 1: Illustration of the Paradox zone when Z is a zero-forecast.



Source: Author's elaboration

4.2 Simulation with a general forecast Z

We consider a more general case now in which $V(Z) \neq 0$. According to Proposition 2, the Paradox zone is given by

$$\Delta \equiv Corr(X,Y) - Corr(Z,Y) \in [0; \frac{MSF_X - MSF_Z}{2\sqrt{V(Y)V(X)}} - \frac{Corr(Y,Z)\left(\sqrt{V(X)} - \sqrt{V(Z)}\right)}{\sqrt{V(X)}} - \frac{EY(EX - EZ)}{\sqrt{V(Y)V(X)}}).$$

In order to show that this interval may be a non-empty region, we carry out the following simulation: We start by setting $EZ = 1, EYZ = 0.7, EXZ = 1.1, EY = 0.6$, $EX = 1$, $E(Y^2) = 3$, $E(Z^2) = 1.5$ and $E(X^2) = 2.5$. This implies the following values:

$V(Y) = 2.64$; $\sqrt{V(Y)} = 1.6248$; $V(X) = 1.5$; $\sqrt{V(X)} = 1.2247$; $V(Z) = 0.5$; $\sqrt{V(Z)} = 0.7071$; Corr(Y,Z)=0.0870 and $EY(EX - EZ) = 0$.

Then using expression (5) we have

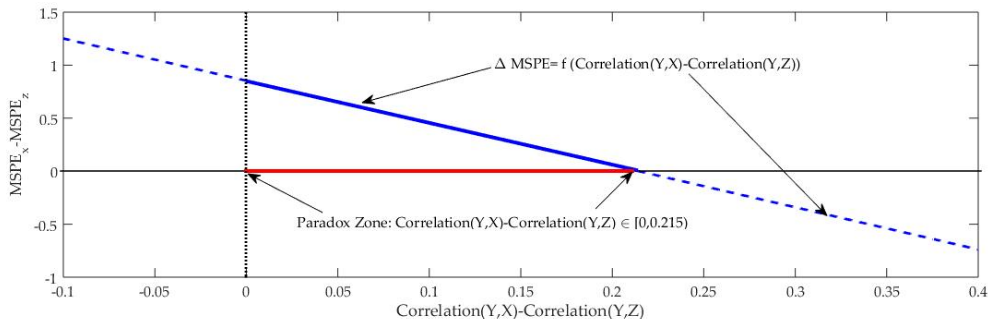
$$\Delta MSPE = \text{MSPE}_\text{x} - MSPE_z = 0.8536 - 3.9799 * \Delta$$

Keeping $EZ, EYZ, EXZ, EY, EX, E(Y^2), E(Z^2)$ and $E(X^2)$ constant, our decomposition is just a linear function between $\Delta MSPE$ and Δ with an approximate slope of -3.98 and an approximate intercept of 0.85. In order to draw this linear function without affecting its slope and intercept, we generate different values of Δ just by changing EYX but keeping in mind that the following covariance matrix Ω_1 must be positive definite:

$$\Omega_1 = \begin{pmatrix} 3 - 0.6^2 & EYX - 0.6 * 1 & 0.7 - 0.6 * 1 \\ EYX - 0.6 * 1 & 2.5 - 1 & 1.1 - 1 * 1 \\ 0.7 - 0.6 * 1 & 1.1 - 1 * 1 & 1.5 - 1 \end{pmatrix}$$

We consider EYX = 0.15 + δ, where δ is a sequence of small positive increments of 0.001. Notice that the slope and the intercept of our linear function remain unaltered in this simulation. In this case, the Paradox zone is given by $Corr(X,Y) - Corr(Z,Y) \in [0; 0.215)$.

Figure 2: Illustration of the Paradox zone in a general framework.



Source: Author's elaboration

4.3 Simulation calibrated to exchange rates.

Here we show results when we compare a generic forecast X against a zero-forecast Z. So, again, we are in the framework of Subsection 4.1. Differing from our first simulation in which we consider arbitrary population parameters for our forecasts, here we pick our parameters so to match sample moments of forecasts and exchange rates at the monthly frequency. We use data of the Australian dollar obtained from Thomson Reuters Datastream (our target variable Y). Our forecast X is simply a survey-based expectation for the Australian exchange rate: we use data on professional exchange rate forecasts from FX4Casts (previously known as The Financial Times Currency Forecaster and Currency Forecasters' Digest). Our database goes from October 2001 through May 2019.

Let us define our target variable as follows:

$$Y_t = \ln(ER_t) - \ln(ER_{t-3})$$

Where ER_t is the Australian Dollar (spot) at time t, and Y_t is simply the three-month cumulative log-return. Let

$$X_t = \ln(FX_{t-3}) - \ln(ER_{t-3})$$

where FX_{t-3} is the FX4CASTS three-month ahead forecast for the Australian Dollar (e.g., the forecast for ER_t, and X_t is simply the forecast of the three-month ahead cumulative return. We set the parameters of our simulations using the following sample moments:

$$EY = 0.0045,\ EX = -0.0010,\ E(Y^2) = 0.0042 \text{ and } E(X^2) = 0.0002$$

This implies the following values (rounding to the fourth decimal):

$V(Y)$ = 0.0042; $\sqrt{V(Y)} = 0.0647$; $V(X)$ =0.0002; $\sqrt{V(X)} = 0.0141$; $EYEX = -0.00005$.

Then from expression (4), we have

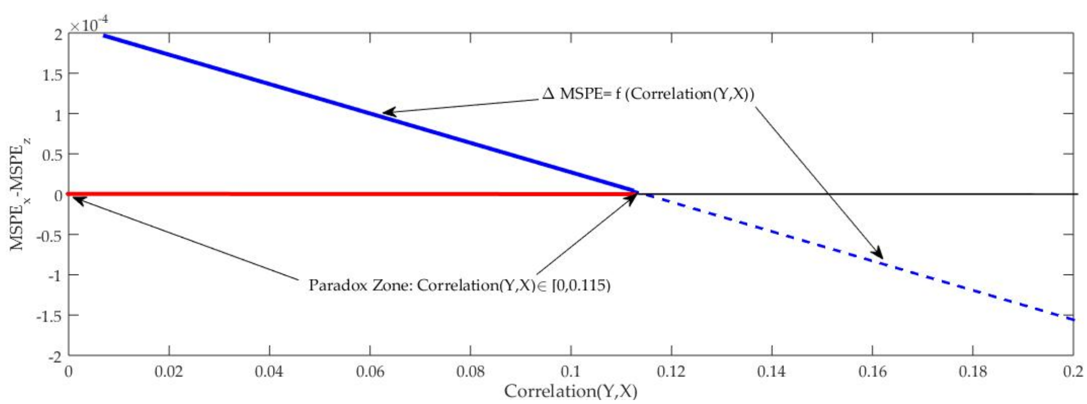
$$\Delta MSPE = \mathrm{MSPE_x} - MSPE_z = 0.0002 - 0.0018 * Corr(X, Y)$$

Keeping $EY, EX, E(Y^2)$ and $E(X^2)$ constant, our decomposition is just a linear function between $\Delta MSPE$ and Corr(X,Y), with a slope of -0.0018 and an intercept of 0.0002 (both values rounded at the fourth decimal). In order to analyze this linear function without changing the slope nor the intercept, we generate different values of $Corr(X, Y)$ just by changing EYX but keeping in mind that the following covariance matrix Ω_2 must be positive definite:

$$\Omega_2 = \begin{pmatrix} E(X^2) - (EX)^2 & EYX - EYEX \\ EYX - EYEX & E(Y^2) - (EY)^2 \end{pmatrix}$$

We consider EYX = 0 + δ, where δ is a sequence of small positive increments of 0.00001. Notice that the slope and the intercept of our linear function remain unaltered in this simulation. In this case, the Paradox zone is given by $Corr(X, Y) \in \left[0; \frac{E(X^2)}{2\sqrt{V(Y)V(X)}} - \frac{E(Y)E(X)}{\sqrt{V(Y)V(X)}}\right[= [0; 115)$. In other words, despite that forecast Z has no covariance with Y, it outperforms the forecast X in terms of MSPE whenever $Corr(X, Y) \in [0; 115)$.

Figure 3: Paradox zone with parameters calibrated to exchange rates



Source: Author's elaboration

5. Empirical illustrations of the MSPE Paradox

In this section we illustrate the Paradox with two empirical applications using commodities and commodity currencies. In both cases, we will assume for simplicity that population moments are well approximated by their sample counterparts.

5.1 The Paradox in commodity forecasts

Our first empirical illustration is inspired by the commodity-currencies literature. Chen, Rogoff and Rossi (2010, 2011) seminal papers report strong predictive ability from the exchange rates of some exporting countries such as Australia, Canada, Chile, New Zealand and South Africa to some country-specific commodity indices. Additionally, Pincheira and Hardy (2019, 2021) find strong predictive ability from the same currencies to some base-metal prices.

In this context, we construct and compare different forecasts for 11 series of commodities (aluminum, copper, lead, nickel, zinc, tin, LMEX, gold, silver, S&P GSCI, and platinum) using the exchange rates of Australia, Canada, Chile, New Zealand and South Africa (relative to the U.S dollar)[3]. The econometric specifications for our forecasts closely follow Pincheira and Hardy (2019, 2021):

$$\Delta CP_t = \beta \Delta E\mathrm{R}_{\mathrm{t-1}} + \varepsilon_{\mathrm{t}} \ \ (M1)$$

Where ΔCP_t stands for the log-difference of a commodity price, ΔER is the log-difference of a generic exchange rate, β is a regressor coefficient and ε_{t} is the error term. Note that we are only evaluating one-step-ahead forecasts. The database is collected from Thomson Reuters Datastream, considering monthly closing prices on commodity prices and exchange rates (relative to the U.S dollar). In this analysis, we consider exclusively a period in which all the economies pursue a pure flotation exchange rate regime; hence our database goes from October 1999 through May 2019 (a total of T=236 observations for each series).

In addition to the five forecasts generated by each exchange rate using (M1), we also consider the forecast of a Driftless Random Walk (a zero-forecast, DRW), a Random Walk (a forecast with the historical mean, RW), and an AR(1). Finally, the parameter β in (M1), and the parameters of the AR(1) and the RW are estimated by OLS and updated with rolling windows of R=48 monthly observations. Notice that all our forecasts are evaluated out-of-sample, with a total of P=T-R=188 predictions.

[3] The S&P GSCI was formerly known as the Goldman Sachs Commodity Index.

Table 1: Evaluation of commodity forecasts with correlations with the target variable and RMSPE.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	Australia	Canada	Chile	NewZealand	SouthAfrica	AR(1)	RW	DRW
S&P GSCI								
Correlation	0.070	-0.023	0.147	-0.118	0.031	0.151	-0.065	-
RMSPE	0.067	0.067	0.066	0.069	0.067	0.066	0.067	0.066
LMEX								
Correlation	0.049	-0.025	0.146	-0.178	-0.046	0.161	0.009	-
RMSPE	0.068	0.067	0.067	0.069	0.068	0.067	0.067	0.066
Aluminum								
Correlation	0.029	-0.068	0.139	0.007	-0.087	0.137	-0.043	-
RMSPE	0.063	0.063	0.062	0.063	0.063	0.063	0.063	0.062

Notes: RMSPE stands for Root MSPE. Source: Author's elaboration

Table 1 reports our results when predicting GSCI, LMEX and aluminum[4]. First, 12 out of 21 non-zero forecasts have a positive correlation with the target variable, suggesting some useful information in those forecasts. Notably, the DRW is the forecast with the lowest RMSPE in the three commodities, despite having zero covariance with the target variable.

Second, note that the forecasts for each commodity show very similar RMSPE, but very different correlations. For instance, the RMSPE for the LMEX goes between 0.066 and 0.069, but the correlations vary between -0.178 and 0.161. In other words, relative to the maximums we find changes of around 4% in RMSPE and changes of around 210% in correlations.

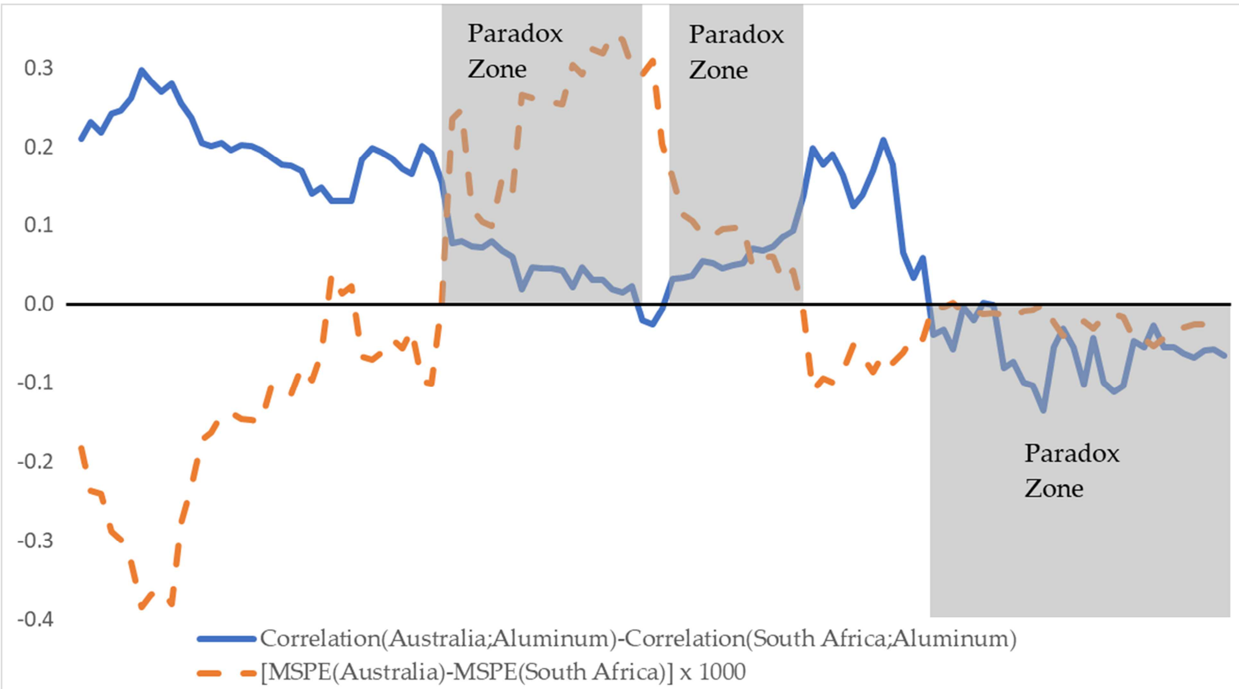
Third, there are some cases of paradoxes worth to be mentioned. For instance, for the LMEX, the forecast of the AR(1) has a particularly high correlation of 0.161; nevertheless, the DRW has a lower RMSPE. Moreover, the forecast constructed with the Australian dollar has a correlation of 0.049, but notably, it has greater RMSPE than the forecast constructed with the Canadian dollar, even though the latter has a negative correlation of -0.025.

Figures 4 and 5 display the differences in MSPE and correlations between two forecasts using rolling windows of 48 observations. Figure 4 compares two different forecasts for aluminum: one constructed with the Australian Dollar and the other with the South African Rand. Figure 5 reports our results when forecasting the LMEX with the Australian

[4] Results for the rest of commodities are presented in Table A1 in the Appendix. With subtleties, some entries of Table A1 also illustrate the Paradox. For instance, in the case of Zinc, the RMSPE ranges from 0.087 to 0.090. Notably, with this commodity, the forecast constructed with the Chilean peso is the only one with a positive correlation with the target variable, and at the same time, it exhibits a RMSPE of 0.090. In other words, the forecast with the Chilean peso is the best in term of correlations, and one of the worsts in terms of RMSPE.

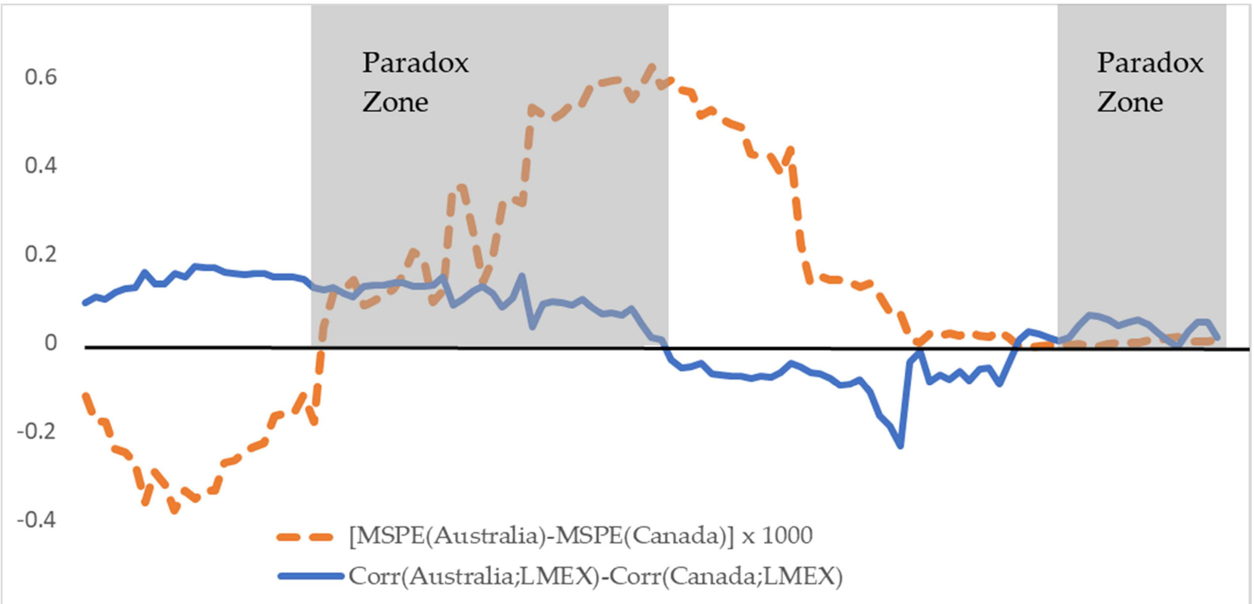
and the Canadian Dollar. In both figures, whenever the differences in MSPE and correlations have the same sign, we have the MSPE Paradox. Notably, both figures suggest that the Paradox may appear quite often.

Figure 4: Differences in MSPE and Correlations using rolling windows. Forecasting aluminum returns with the Australian and South African exchange rates.



Notes: Figure 4 displays the differences in MSPE and correlations between two competing forecasts, using rolling windows of 48 observations. In this illustration the target variable is aluminum one-month returns. We compare a forecast using the Australian Dollar with another using the South African Rand. Whenever both series have the same sign, we have the MSPE Paradox. The differences in MSPE have been scaled so the left axis represents both differences in MSPE and differences in correlations. Source: Author's elaboration

Figure 5: Differences in MSPE and Correlations using rolling windows. Forecasting LMEX with the Australian and Canadian exchange rates.



Notes: Figure 5 displays the differences in MSPE and correlations between two competing forecasts using rolling windows of 48 observations. In this illustration the target variable is LMEX one-month returns. We compare a forecast using the Australian Dollar with another using the Canadian Dollar. Whenever both series have the same sign, we have the MSPE Paradox. The differences in MSPE have been scaled so the left axis represents both differences in MSPE and differences in correlations. Source: Author's elaboration

5.2 The Paradox in exchange rates forecasts

In our second empirical illustration, we evaluate the predictive relationship between commodities and commodity-currencies, but in the opposite direction (i.e., the ability of fundamentals to predict exchange rates). Previous studies like Chen et al. (2010), Engel and West (2005), and Ferraro et al. (2015) have evaluated this predictive performance with rather weak results. For instance, Ferraro et al. (2015) conclude that the ability of commodity prices to predict exchange rates is unstable and appears only for some commodities at some frequencies. Chen et al. (2010) find evidence that exchange rates can predict commodity prices at the quarterly frequency; nevertheless, there is little evidence in the opposite direction. Moreover, Engel and West (2005) conclude that there is very weak evidence of Granger-causality from fundamentals to exchange rates, yet results in the opposite direction are more encouraging.

Due to the evidence reported in Ferraro et al. (2015), Chen et al. (2010) and Pincheira and Hardy (2019, 2021) we consider the following commodities as predictors for exchange rates: Aluminum, LMEX, Oil and a commodity index (MSCI GSCI). The commodity-currencies considered here are the same than in Chen et al. (2010): Australia, Canada, Chile, New Zealand and South Africa.

In addition to our forecasts constructed with commodities, we evaluate the predictive performance of the exchange rates FX4cast survey considered in previous studies such as Ince and Molodtsova (2017)[5]. The main sources for our data are Datastream (for the case of commodities) and FX4cast (for the exchange rates and their respective forecasts). Due to data availability in FX4cast, we consider the following sample periods at the monthly frequency: Australia (August 1986 through May 2019), Canada (August 1986 through May 2019), Chile (October 2001 through May 2019), New Zealand (December 1993 through May 2019) and South Africa (October 2001 through May 2019).

Our econometric specifications are very simple, mainly inspired by Pincheira and Hardy (2019, 2021):

$$\Delta E\mathrm{R}_{\mathrm{t+h}} = \sum_{k=1}^{p} \gamma \Delta CP_{t-k+1} + \varepsilon_{t+h}$$

Where ΔER_{t+h} stands for the log-difference of a generic exchange rate at time t+h. ΔCP_{t-k+1} stands for the log-difference of a generic commodity price. The number of lags "p" is determined with AIC in each case, using the first 48 monthly observations. We update estimates of the parameters for our forecasts using rolling windows of R=48 observations. Additionally, as in Subsection 5.1, we consider forecasts from a RW (a forecast using the historical mean), a DRW (a zero-forecast) and an AR(p) (where "p" in each case is determined again using AIC with the first 48 observations). We report out-of-sample results for h=3 and h=6. Consequently, for h=3 we report results of the 3-months ahead FX4cast survey and, for h=6, we report the 6-months ahead FX4cast survey[6]. Finally, γ is a regressor coefficient and ε_{t+h} is an error term.

[5]FX4cast considers five different horizons: 1, 3, 6, 12 and 24-months ahead. Nevertheless, the 1-month and the 24 months-ahead surveys are only available since May 2008 for all the exchange rates; for this reason, we only use in this exercise the 3-, 6- and 12-months ahead forecasts.

[6] One-step-ahead results (i.e., h=1) are available upon request.

Table 2: Evaluation of exchange rate forecasts with correlations with the target variable and RMSPE.

(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*
	FX4cast 6	*Rolling*	*AR(p)*	*LMEX*	*GSCI*	*Oil WTI*
			Australia h=6			
Correlation	*0.083*	*-0.051*	*-0.036*	*0.000*	*-0.039*	*-0.016*
RMSPE	*0.043*	*0.033*	*0.033*	*0.033*	*0.034*	*0.034*
			Canada h=6			
Correlation	*0.166*	*0.008*	*0.077*	*-0.076*	*-0.093*	*-0.088*
RMSPE	*0.026*	*0.023*	*0.022*	*0.024*	*0.024*	*0.024*
			Chile h=6			
Correlation	*0.088*	*-0.060*	*-0.027*	*-0.068*	*-0.062*	*-0.035*
RMSPE	*0.036*	*0.034*	*0.033*	*0.035*	*0.036*	*0.036*
			NewZealand h=6			
Correlation	*0.069*	*-0.053*	*-0.028*	*0.000*	*-0.049*	*0.048*
RMSPE	*0.047*	*0.038*	*0.038*	*0.038*	*0.038*	*0.037*
			SouthAfrica h=6			
Correlation	*-0.004*	*-0.043*	*-0.052*	*0.007*	*0.083*	*0.056*
RMSPE	*0.052*	*0.046*	*0.048*	*0.048*	*0.047*	*0.047*

Notes: RMSPE stands for Root MSPE, *h* is the forecasting horizon and FX4cast 6 is the 6 months-ahead forecast. "Rolling" simply represents the forecast using the historical mean. Source: Author's elaboration

Table 2 shows some striking results[7]. First of all, with one exception (the South African Rand), FX4CAST is always the forecast displaying the highest correlation with the target variable, and notably, at the same time, it is always the forecast displaying the worst RMSPE. For instance, for the Australian Dollar, FX4CAST has a RMSPE about 30% higher than the other forecasts, despite the fact that it is the only forecast with a positive correlation with the target variable.

Second, in three out of our five exchange rates, the forecast displaying the lowest RMSPE has also a zero or negative correlation with the target variable; in other words, the "most accurate" forecast is frequently one of the worsts in terms of correlations. Notably, in Table 2 we do not find any cases in which the forecast displaying the highest correlation exhibits simultaneously the smallest RMSPE.

[7] Results for h=3 are presented in Table A2 in the Appendix. With subtleties, some entries of Table A2 also illustrate the Paradox. For instance, for the case of South Africa, every forecast displays a RMSPE of 0.048. This result suggests that the six forecasts are similarly accurate. However, the correlation of the forecasts with the South African rand ranges from -0.099 to 0.087; in other words, relative to the maximum, we observe differences of more than 213%.

All in all, Tables 1 and 2 support the main message of our paper: Sometimes, a set of competing forecasts may exhibit a similar RMSPE (i.e., they are “similarly accurate”), and, at the same time, they may have important differences in terms of correlations with the target variable. In this scenario, no one would be surprised if no differences whatsoever were found between our competing forecasts using traditional tests of equality in MSPE (e.g., Diebold and Mariano (1995) and West (1996)), despite the fact that our forecasts contains fairly different information about the future evolution of the target variable.

6. Concluding remarks

In this paper we show that traditional comparisons of Mean Squared Prediction Error (MSPE) between two competing forecasts may be highly controversial. This is so because when some specific conditions of efficiency are not met, the forecast displaying the lowest MSPE will also display the lowest correlation with the target variable. Given that violations of efficiency are usual in the forecasting literature, this opposite behavior in terms of accuracy and correlation with the target variable may be a fairly common empirical finding that we label here as "the MSPE Paradox."

We characterize "Paradox zones" in terms of differences in correlation with the target variable and conduct some simple simulations to show that these zones may be non-empty sets. Moreover, our analysis shows that we could have an extreme case in which a forecast and a target variable are independent random variables, which speaks of a useless forecast, yet, in terms of MSPE, this useless forecast might outperform a useful forecast displaying a positive correlation with the target variable.

Finally, we illustrate the relevance of the MSPE Paradox with two empirical applications in which some of the most accurate forecasts in terms of MSPE are, in fact, some of the worst in terms of correlations with the target variable.

Our paper emphasizes the need to look beyond MSPE when evaluating two or more competing forecasts, as a blind search for the minimum out-of-sample MSPE forecast may lead to an incorrect evaluation of the information contained within those predictions. In light of these results, an interesting avenue for future research is the elaboration of a simple asymptotically normal test to evaluate two competing forecasts according to their correlations with the target variable.

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8. Appendix

A.1 Forecasting commodities with commodity-currencies.

(1)	*(2) Australia*	*(3) Canada*	*(4) Chile*	*(5) NewZealand*	*(6) SouthAfrica*	*(7) AR(1)*	*(8) RW*	*(9) DRW*
				Copper				
Correlation	*0.008*	*-0.001*	*0.137*	*-0.267*	*-0.082*	*0.192*	*0.002*	*-*
RMSPE	*0.083*	*0.081*	*0.081*	*0.085*	*0.082*	*0.081*	*0.082*	*0.081*
				Gold				
Correlation	*-0.112*	*-0.075*	*-0.171*	*0.027*	*-0.169*	*0.046*	*0.061*	*-*
RMSPE	*0.052*	*0.051*	*0.052*	*0.051*	*0.052*	*0.051*	*0.050*	*0.051*
				Lead				
Correlation	*0.148*	*0.062*	*0.138*	*0.129*	*0.051*	*0.075*	*0.048*	*-*
RMSPE	*0.094*	*0.095*	*0.094*	*0.094*	*0.095*	*0.095*	*0.094*	*0.094*
				Nickel				
Correlation	*0.030*	*-0.056*	*0.012*	*-0.116*	*-0.015*	*0.045*	*0.031*	*-*
RMSPE	*0.103*	*0.104*	*0.104*	*0.105*	*0.103*	*0.103*	*0.101*	*0.101*
				Tin				
Correlation	*0.037*	*-0.068*	*0.034*	*-0.074*	*-0.005*	*0.109*	*0.120*	*-*
RMSPE	*0.077*	*0.077*	*0.076*	*0.078*	*0.076*	*0.076*	*0.075*	*0.075*
				Zinc				
Correlation	*-0.151*	*-0.060*	*0.030*	*-0.234*	*-0.091*	*-0.078*	*-0.037*	*-*
RMSPE	*0.090*	*0.088*	*0.090*	*0.089*	*0.088*	*0.090*	*0.089*	*0.087*

Notes: RMSPE stands for Root MSPE. Source: Author's elaboration

A.2 Forecasting exchange rates with commodities.

	(1) FX4cast 3	(2) Rolling	(3) AR(p)	(4) LMEX	(5) GSCI	(6) Oil WTI
			Australia h=3			
Correlation	0.001	-0.063	0.031	-0.029	0.100	0.093
RMSPE	0.036	0.033	0.032	0.033	0.034	0.034
			Canada h=3			
Correlation	0.028	0.013	0.087	0.058	0.046	0.023
RMSPE	0.024	0.023	0.022	0.024	0.024	0.025
			Chile h=3			
Correlation	0.101	-0.070	-0.046	-0.094	-0.175	-0.134
RMSPE	0.033	0.034	0.033	0.037	0.035	0.035
			New Zealand h=3			
Correlation	-0.044	-0.058	-0.009	-0.045	0.010	0.070
RMSPE	0.042	0.037	0.040	0.038	0.037	0.037
			South Africa h=3			
Correlation	0.087	-0.098	-0.012	-0.099	-0.040	0.002
RMSPE	0.048	0.048	0.048	0.048	0.048	0.048

Notes: RMSPE stands for Root MSPE, *h* is the forecasting horizon and FX4cast 3 is the 3 months-ahead forecast. "Rolling" simply represents the forecast using the historical mean. Source: Author's elaboration.

A.3 Illustration of the Paradox zone in a framework with equal bias and $\boldsymbol{V(Z) > V(X)}$.

Here we consider a case that relates to the bias-variance tradeoff. Suppose that forecasts X and Z are "equally biased" and consider the case V(Z)>V(X). In this environment, according to Corollary 3, the Paradox zone is given by

$$\Delta \equiv [Corr(X,Y) - Corr(Z,Y)] \in (0; \frac{[V(X) - V(Z)]}{2\sqrt{V(Y)V(X)}} - \frac{Corr(Y,Z)\left(\sqrt{V(X)} - \sqrt{V(Z)}\right)}{\sqrt{V(X)}}]$$

In order to show that this interval may be a non-empty region, we carry out the following simulation: We start by setting $EZ = EX = -1.5$, $EYZ = 2.5040$, $EXZ = 2.4549$, $EY = -1$, $E(Y^2) = 5$, $E(Z^2) = 3.3$ and $E(X^2) = 2.5$. This implies the following values: $V(Y) = 4$; $\sqrt{V(Y)} = 2$; $V(X) = 0.25$; $\sqrt{V(X)} = 0.5$; $V(Z) = 1.05$; $\sqrt{V(Z)} = 1.0247$; Corr(Y,Z)=0.4899 and $EY(EX - EZ) = 0$.

Then using expression (5) we have

$$\Delta MSPE = \mathrm{MSPE_x} - MSPE_z = 0.2282 - 2 * \Delta$$

Keeping $EZ, EYZ, EXZ, EY, EX, E(Y^2), E(Z^2)$ and $E(X^2)$ constant, our decomposition is just a linear function between $\Delta MSPE$ and Δ with an approximate slope of -2 and an approximate intercept of 0.2282. In order to draw this linear function without affecting its slope and intercept, we generate different values of Δ just by changing EYX but keeping in mind that the following covariance matrix Ω_1 must be positive definite:

$$\Omega_1 = \begin{pmatrix} 5 - 1^2 & EYX - 1.5 * 1 & 2.504 - 1.5 * 1 \\ EYX - 1.5 * 1 & 2.5 - 1.5\text{^}2 & 2.4549 - 1.5 * 1.5 \\ 2.504 - 1.5 * 1 & 2.4549 - 1.5 * 1.5 & 3.3 - 1.5^2 \end{pmatrix}$$

We consider EYX = 1.9 + δ, where δ is a sequence of small positive increments of 0.001. Notice that the slope and the intercept of our linear function remain unaltered in this simulation. In this case, the Paradox zone is given by $Corr(X, Y) - Corr(Z, Y) \in [0; 0.114)$. The following Figure A1 provides a visual representation of this Paradox zone:

Figure A1: Paradox zone in a framework with equal bias and $V(Z) > V(X)$.