Fractional Dickey-Fuller test with or without prehistorical influence

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ABSTRACT
Recently the generalization of the standard Dickey-Fuller test to the fractional case has been proposed. The proposed test, called fractional Dickey-Fuller test can be applied to sample generated from a type I or a type II fractional process. Depending on whether the test is applied to sample generated from a type I or type II process, it is referred to as a test with or without prehistoric influence respectively. The main and the first objective of this paper is to study the impact induced by a pre-sample of the finite sample null distribution. In fact, the recently proposed test is built based on a composite null hypothesis rather than a sample one. The second objective is to highlight the theoretical justifications for the choice of the null composite hypothesis. All the theoretical results are illustrated with simulated and real data sets. Furthermore, to facilitate the reproducibility of our simulation data and figures we provide all the necessary supplementary material consisting of EViews programs.

Key Words: ARFIMA; fractional integration, Dickey-Fuller test; Fractional Dickey-Fuller test; type I and type II fractional Brownian motion.

1. INTRODUCTION

In time series analysis when we deal with the unit root test, one method is to consider the standard Dickey-Fuller test based on the following regression model,

$$y_t = \phi y_{t-1} + \varepsilon_t, \tag{1.1}$$

where ε_t is supposed to be a white noise process and y_t a time series integrated of order d, ($y_t \rightsquigarrow I(d)$). The autoregressive model (1.1) will have a unit root if $\phi = 1$. In fact, $\phi = 1$, does not mean that there is only one unit root. For example, consider the following three processes

$$(1-L)^d y_{it} = u_{d,t},\ d = 1, 2, 3. \tag{1.2}$$

where u_{1t}, u_{2t} and u_{3t} are supposed, without restrict the generality, to be a white noise process and L is lag operator. For these three processes, intuitively, we expect that if there are more than one unit root, the test for one unit root will strongly indicate that the process needs to be differenced. In the others words, for the three following models

$$y_{d,t} = \phi_d y_{d,t-1} + \varepsilon_{dt},\ t = 1, \cdots, n, \tag{1.3}$$

we have statistical evidence that $\phi_1 = 1$, $\phi_2 = 1$ and $\phi_3 = 1$. This fact is well known by all econometricians and statisticians working in the field of time series. However, as far as we known, it has not received the attention it deserves and it has never been exploited from the statistical inference point of view. Indeed, this property should have drawn attention to the possibility of expressing the null hypothesis of a unit root as a composite hypothesis rather than a simple one. Since for all models (1.3), we have $\phi = 1$ we can express the null hypothesis as $H_0 : d \geq 1$, which is equivalent to null hypothesis $H_0 : \phi = 1$. This fact is not only true for the integer values of d, but also for the fractional values. That is, consider the process $y_t \sim I(m + \delta)$ with $m \in \mathbb{N}^+$ and $0 \leq \delta < 1$ and the OLS estimator of ϕ in the regression model (1.1), since $m + \delta \geq 1$ we have statistical evidence that $\phi = 1$. This evidence comes from the fact $\widehat{\phi} \xrightarrow{p} 1$ at rate n instead of the usual rate $n^{0.5}$ and then $\widehat{\phi}$ is super-consistent estimator. The fractional values of d are well known from the works of Akonom and Gourieroux (1987), Gourieroux and Monfort (1990), Sowell (1990), Chan and Terrin (1995), Liu (1998), Wang et al (2003), and more recently Bensalma (2018). Super-consistency with order n rate convergence is true for processes with an integration order greater than or equal to one, but it can also be used for processes with an integration order less than or equal to one. In fact, if $y_t \rightsquigarrow I(d)$ with $-0.5 < d < 1$, then the differentiated process $(1 - L)^{d_0 - 1} y_t \rightsquigarrow I(d - d_0 + 1)$ with $d - d_0 + 1 \geq 1$ if $d \geq d_0$.

Remark 1: *Be careful, one should not confuse the integrated process of order $d > 1$ with the explosive process in which the coefficient ϕ is strictly greater than 1.*

To be more precise, a composite hypothesis is a hypothesis that covers a set of values from the parameter space. For example, if the parameter space $\Omega = \{0, 1, 2, 3, 4\}$, a null composite hypothesis could be $\Omega_{01} = \{1, 2, 3, 4\}$ or $\Omega_{02} = \{2, 3, 4\}$ which can be expressed, respectively as $d \geq 1$ and $d \geq 2$. For these null composite hypotheses the alternative are respectively $\Omega_{11} = \{0\}$ $(i.e., d < 1)$ or $\Omega_{12} = \{0, 1\}$ $(i.e., d < 2)$. By using a test based on a composite null rather than a simple one, we can suggest a sequential testing procedure. This can be a downward sequential procedure, which takes the largest number of unit root, (the maximum value of Ω, i.e. $d \geq d_{\max}$), under consideration as the first maintained hypothesis and then decrease the order of differencing each time the current null is rejected. This can be also an upward sequential testing procedure, which takes the smallest number of unit root, (i.e. $d \geq 1$) under consideration as the first maintained hypothesis and then increase the order of differencing each time the current alternative is accepted.

Among the non-stationary tests designed for the $ARIMA(0, d, 0)$ process, the most widely used test in the literature is the standard Dickey-Fuller test (1979) (DF in short). The DF test is designed to test the hypothesis

$$H_0 : d = 1. \tag{1.4}$$

The DF test can also be used to test the hypothesis

$$H_0 : d = m_0, \tag{1.5}$$

with $m_0 \in \mathbb{N}^*$ (see Pantula (1989) page 260). Note that test (1.4) is a special case of test (1.5). For a given series $\{y_t\}_{t=1}^n$ from $ARIMA(0, d, 0)$, with $d \in \mathbb{N}$, the hypotheses (1.4) and (1.5) are based upon testing the statistical significance of the coefficient ρ in the following regression model,

$$\Delta^{m_0} y_t = \rho \Delta^{m_0 - 1} y_{t-1} + \varepsilon_t, \tag{1.6}$$

where $\{\varepsilon_t\}_{t=1}^n$ are the residuals. Under the null hypothesis (1.4) or (1.5) the asymptotic distribution of the usual test statistic

$$DF_n = n\widehat{\rho} \quad and \quad DF_t = \frac{\widehat{\rho}}{\widehat{\sigma}_\rho} = t_\rho,$$

are respectively,

$$\frac{0.5\left[W^2(1)-1\right]}{\int_0^1 W^2(r)dr} \quad and \quad \frac{0.5\left[W^2(1)-1\right]}{\left(\int_0^1 W^2(r)dr\right)^{0.5}},$$

where $W(\cdot)$ is the standard Brownian motion. These distributions are the usual asymptotic distributions for the DF test and has been tabulated to perform the tests (1.4) and (1.5). In addition of super-consistency of the OLS estimator, another argument in favor of adopting a composite null hypothesis instead a simple one is the behavior of the limit distributions of the test statistics DF_n and DF_t in the context of $ARFIMA$ processes. Studies on the asymptotic properties of $ARIMA$ or $ARFIMA$ processes by Gourieroux, Maurel, and Manfort (1987), Silveira (1991), and Qiying Wang (2001), Gourieroux and Monfort (1990), Chan and Terrin (1988), Sowell (1990), and Wang et al (2003) show that the domains of the limit probability density function of DFn and DFt are $\mathbb{R}^-$, $\mathbb{R}$ and $\mathbb{R}^+$, respectively. Pantula (1989) gets the same results, i.e. depending on whether $d < m_0$, $d = m_0$ and $d > m_0$, the support of limit probability density of DF_n and DF_t are $\mathbb{R}^-$, $\mathbb{R}$ and $\mathbb{R}^+$ respectively. This remarkable configuration of the limiting distributions is advantageous in more than way. In particular, it is perfectly suited to the implementation and application of the basic rules of the statistical test theory when the null hypothesis is composite and corresponds perfectly to the proper implementation of upward or downward testing sequence, described above. Furthermore, similar Pantula's idea is adopted by Bensalma (2018) in the context of $ARFIMA(0, d, 0)$ process. If $\{y_t\}_{t=1}^n$ is a sample generated from an $ARFIMA(0, d, 0)$ process, with $d \in (-0.5, +\infty)$ then we can use the transformed process $(1 - L)^{d_0 - 1} y_t$ to test $H_0 : d \geq d_0$, with $d_0 \in (-0.5, +\infty)$ by using the regression model $\Delta^{d_0} y_t = \rho \Delta^{d_0 - 1} y_t + \varepsilon_t$. Bensalma (2018) show that depending on whether $d < d_0$, $d = d_0$ and $d > d_0$, the domains of limit probability density function of DF_n and DF_t are $\mathbb{R}^-$, $\mathbb{R}$ and $\mathbb{R}^+$ respectively. $(1 - L)^{d_0 - 1} y_t$ is an $I(d - d_0 + 1)$ process and then we have the following three cases

$$(d - d_0 + 1) \begin{cases} < 1 \text{ if } d < d_0, \\ = 1 \text{ if } d = d_0, \\ > 1 \text{ if } d > d_0. \end{cases}$$

It should be noted, also, that in the regression model (1.6), the process $(1-L)^{-1+m_0}y_t$ is $I(d-m_0+1)$ and then we have the following three cases

$$(d-m_0+1)\begin{cases} <1 \text{ if } d<m_0, \\ =1 \text{ if } d=m_0, \\ >1 \text{ if } d>m_0. \end{cases}$$

Note that Bensalma (2018) use the composite null hypothesis rather than a simple one, which does not imply any change in the way of applying the test. Indeed, the usual regression models can still be used as well as the usual asymptotic distributions. Therefore, we do not have to tabulate different percentiles for DF_n or DF_t. The test procedure proposed by Bensalma (2018), called fractional Dickey-Fuller test (F-DF test), will not only enable us to apprehend the case of $H_0 : d \geq 0.5$, but also the case $H_0 : d \geq d_0$, with $d_0 \in [-0.5, +\infty[$. Moreover, if we use upward testing sequence or a downward testing sequence for a set of values $d_0^1 < d_0^2 < \cdots < d_0^l$, it is possible to determine an overlap domain of the parameter d, as small as we want.

However, the literature of time series econometrics for long memory processes has adopted two distinct processes as a basis for the asymptotic analysis, the limit processes specified being known respectively as type I and type II fractional Brownian motion (see Marinucci and Robinson (1999)). Davidson and Hashimzade (2009) pointed out that the asymptotic results vary depending on the definition of fractional Brownian motions considered, which requires setting simulation experiments accordingly. In fact, the FDF test can be considered as "fractional Dickey-Fuller test based on the type II fractional Brownian motion". In this case, it is shown that the usual tabulated values of the Dickey-Fuller distributions can be used. In this paper, we address the issues of the case where we adopt a mechanism to generate a series that correspond to the type I fractional Brownian motion.

This paper is organized as follows. In section 2, preliminary concepts about the definition of the type I, type II fractional processes and their related asymptotic theory are given. We recall that these asymptotic theory are widely applied to study the behavior of the unit root test ($H_0 : d = 1$) when the data generating process is fractional. In section 3, applications of these results to the general null composite hypothesis $H_0 : d \geq d_0$, where d_0 is fractional, by using the standard Dickey-Fuller test, are presented. Section 4 examines, via simulations, the finite sample behavior of the FDF test based on type II fractional processes. We compare also, via simulation, the type I fractional process based FDF test and the type II fractional process based FDF test. The proofs of the main results presented in section 3 are left to the Appendix 1. In Appendix 2, we show how to implement the sequential Dickey-Fuller test with EViews.

2. PRELIMINARY CONCEPTS

We consider two types of the stationary/nonstationary and invertible fractional $ARFIMA(0, d, 0)$ processes. Let $d \in \,]-0.5, +\infty[$ and define the type I fractional $I(d)$ process y_t by

$$y_t = (1-L)^{-d}u_t, \quad t \in \mathbb{Z}, \tag{2.1}$$

where u_t are independent and identically distributed (i.i.d) random variables.

The type II $I(d)$ fractional process is defined as

$$y_t^* = (1-L)^{-d} u_t^*, \quad t = 1, 2, \cdots, n, \tag{2.2}$$

with initial conditions $y_t^* = 0$, if $t < 1$ and where

$$u_t^* = \begin{cases} u_t & \text{if } t \geq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

The fractional integration operator $(1-L)^{-d}$ is defined by its Maclaurin series (by its binomial expansion, if d is an integer):

$$(1-L)^{-d} = \sum_{j=0}^{\infty} \frac{\Gamma(d+j)}{\Gamma(d)\Gamma(j+1)} L^j$$

where

$$\Gamma(z) = \begin{cases} \int_0^\infty s^{z-1} e^{-z} ds & \text{If } z > 0 \\ \infty & z = 0. \end{cases}$$

If $z < 0$, $\Gamma(z)$ is defined by the recursion formula $z\Gamma(z) = \Gamma(z+1)$.

The literature of time series econometrics for long memory processes has adopted two distinct processes, (2.4) and (2.5) as a basis for asymptotic analysis. Given that $d = m + \delta$, for $m \geq 0$ and $\delta \in]-0.5, 0.5[$, and defining the partial sums,

$$Y_n(r) = \frac{\sum_{i=1}^{[nr]} y_t}{\sqrt{Var\left(\sum_{i=1}^{[nr]} y_t\right)}}, \tag{2.4}$$

$$Y_n^*(r) = \frac{\sum_{i=1}^{[nr]} y_t^*}{\sqrt{Var\left(\sum_{i=1}^{[nr]} y_t^*\right)}}. \tag{2.5}$$

it is well known that for $d = \delta$, $Y_n(r) \Longrightarrow B_\delta(r)$ and for $d > 0.5$, $Y_n^*(r) \Longrightarrow W_d(r)$, where $\Longrightarrow$ denotes the weak convergence in the space of measures on $D[0,1]$, $B_\delta(r)$ is the type I fractional Brownian motion and $W_d(r)$ is the type II fractional Brownian motion (Marinucci and Robinson (1999), Davidson and Hashimzade (2009)). A general procedure for generating a type I fractional $I(d)$ series of length (n) is to apply, for $t = 1, 2, \cdots, n$ and some fixed (k) large enough, the formula

$$y_t = \sum_{j=0}^{k+t-1} \frac{\Gamma(d+j)}{\Gamma(j+1)\Gamma(d)} u_{t-j}, \tag{2.6}$$

where $\{u_{1-k}, u_{2-k}, \cdots, u_n\}$ is a random sequence of suitable type. On the other hand, choosing $k = 0$ in formula (2.6) we generate a type II fractional $I(d)$ series of length (n), (Davidson and Hashimzade (2009)).

The asymptotic theory based on type I fractional Brownian motion needs to treat the case $d = m + \delta$ with $m \in \mathbb{N}$ and $\delta \in]-0.5, 0.5[$ separately from the case $d = m + \frac{1}{2}$, with $m \in \mathbb{N}$ (see Liu, 2003 for more details). Whereas, the asymptotic theory based on type II fractional Brownian motion needs to treat the case $d = \frac{1}{2}$ separately from the case $d > \frac{1}{2}$ (see Tanaka, 1999 for more details). Because of this we restrict ourselves to present the asymptotic theory of $ARFIMA$ processes on the domain $d = m + \delta$ with $m \in \mathbb{N}$ and $\delta \in]-0.5, 0.5[$ for the type I fractional processes. For the type II we present the asymptotic theory for $d > 0.5$. This does not restrict generality and allows the use of homogeneous normalization constants throughout this article.

2.1. ASYMPTOTIC BASED ON TYPE I FRACTIONAL PROCESSES

Liu (1998) has derived a functional limit theorem for nonstationary $ARFIMA(0, d, 0)$ processes defined by (2.1). First, Liu (1998) state the functional limit theorem for lower order integration (i.e. $m = 0$, $\delta \in]-0.5, 0.5[$). The lower order convergence results are given in Theorem 1 below.

Theorem 1: *If* $y_t \rightsquigarrow I(\delta)$, $\delta \in]-0.5, 0.5[$, u_t *satisfy the assumption that* $E|u_t|^\alpha < \infty$ *for* $\alpha \geq \max\left\{4, \frac{-8\delta}{1+2\delta}\right\}$ *and* $\sigma_n^2 = Var\left(\sum_{t=1}^n y_t\right)$ *then*

1) $\frac{\sigma_n^2}{n^{1+2\delta}\sigma_u^2 A(\delta)} \longrightarrow 1$ *if* $\delta \in]-0.5, 0.5[$
2) For $r \in (0, 1)$,
$$\left[\sigma_n^{-1} \sum_{t=1}^{[nr]} y_t \Longrightarrow \frac{1}{\sqrt{A(\delta)}} B_\delta(r)\right] \Longleftrightarrow \left[n^{-0.5-\delta} \sum_{t=1}^{[nr]} y_t \Longrightarrow \sigma_u B_\delta(r)\right], \text{ if } \delta \in]-0.5, 0.5[$$

where $B_\delta(r)$ *is the type I fractional Brownian motion defined by stochastic integral*

$$B_\delta(r) = \frac{1}{\Gamma(1+\delta)} \left\{ \int_0^r (r-x)^\delta dB(x) + \int_{-\infty}^0 \left[(r-x)^\delta - (-x)^\delta\right] dB(x) \right\}$$

and

$$A(\delta) = \frac{1}{\Gamma^2(1+\delta)} \left\{ \frac{1}{1+2\delta} + \int_0^\infty \left[(1+\tau)^\delta - \tau^\delta\right] d\tau \right\}$$
$$= \frac{\Gamma(1-2\delta)}{(1+2\delta)\Gamma(1+\delta)\Gamma(1-\delta)}.$$

$B(r)$ *is standard Brownian motion.*■

Based on lower-order convergence results of Theorem 1, Liu (1998) establishes the convergence result when $d > 0.5$ by applying the continuous mapping Theorem. The following Theorem 2 gives the convergence results when $d > 0.5$.

Theorem 2: (Liu (1998) *Let* y_t *satisfy (2.1) with* $d = m + \delta$, $\delta \in]-0.5, 0.5[$ $m \geq 1$ and u_t *satisfy the assumption that* $E|u_t|^\alpha < \infty$ *for* $\alpha \geq \max\left\{4, \frac{-8\delta}{1+2\delta}\right\}$. *Then, for* $r \in (0, 1)$ *we have*

1) $\frac{1}{n^{-0.5+(\delta+m)}} y_{[nr]} \Longrightarrow \sigma_u B_{\delta,m}(r)$,
2) $\frac{1}{n^{0.5+(\delta+m)}} \sum_{t=1}^{[nr]} y_t \Longrightarrow \sigma_u B_{\delta,m+1}(r) = \sigma_u \int_0^r B_{\delta,m}(x) dx$,
3) $\frac{1}{n^{2(\delta+m)}} \sum_{t=1}^{[nr]} y_t^2 \Longrightarrow \sigma_u^2 \int_0^r \left[B_{\delta,m}(x)\right]^2 dx$.

$$\text{where } B_{\delta,m}(r) = \begin{cases} B_{\delta,1}(r) = B_\delta(r), & \text{if } m = 1, \\ \int_0^r \int_0^{r_{m-1}} \cdots \int_0^{r_2} B_\delta(r_1) dr_1 dr_2 \cdots dr_{m-1}, & \text{if } m \geq 2. \end{cases}$$ ■

If $\delta = 0$ and $m \geq 1$ the convergence results of Theorem 2 agree with those of Chan and Wei (1988). Given that

$$B_{0,1}(r) = \frac{1}{\Gamma(1+\delta)} \left\{ \int_0^r (r-x)^\delta dB(x) + \int_{-\infty}^0 \left[(r-x)^\delta - (-x)^\delta \right] dB(x) \right\} = \int_0^r dB(x) = B(r),$$

and

$$B_{0,m}(r) = \int_0^r \int_0^{r_{m-1}} \cdots \int_0^{r_2} B(r_1) dr_1 dr_2 \cdots dr_{m-1} = \frac{1}{\Gamma(m)} \int_0^r (r-s)^{m-1} dB(s)$$

therefore the following Corollary is a direct consequence of Theorem 2.

Corollary 1: *Let* y_t satisfy (2.1) with $\delta = 0$ and $m \geq 1$. *Under the assumption that* u_t *is a class of i.i.d Gaussian processes we have*

1) $n^{0.5-m} y_{[nr]} \Longrightarrow \sigma_u B_{0,m}(r) \left(\equiv \frac{\sigma_u}{\Gamma(m)} \int_0^r (r-s)^{m-1} dB(s) \right).$
2) $n^{-0.5-m} \sum_{t=1}^{[nr]} y_t \Longrightarrow \sigma_u B_{0,m+1}(r) = \sigma_u \int_0^r B_{0,m}(s) ds \left(\equiv \frac{\sigma_u}{\Gamma(m+1)} \int_0^r (r-s)^m dB(s) \right).$
3) $n^{-2m} \sum_{t=1}^{[nr]} y_t^2 \Longrightarrow \sigma_u^2 \int_0^r \left[B_{0,m}(s) \right]^2 ds.$■

2.2. ASYMPTOTIC BASED ON TYPE II FRACTIONAL PROCESSES

Investigation of asymptotic behavior of partial sum (2.5) when $d > 0.5$ was considered by Akonom and Gourieroux (1987), Gourieroux, Maurel and Manfort (1987), Silveira (1991), Qiying Wang (2001) and Wang, Lin and Gulati (2002). Investigation including the case $d = 0.5$ was considered by Tanaka (1999).

Theorem 3: *If* $y_t^* = (1-L)^{-d} u_t^*$, $d > 0.5$, u_t^* *defined by (2.3), satisfy the assumption that* $E\left|u_t\right|^r < \infty$ *for* $r \geq \max\left\{2, \frac{2}{2d+1}\right\}$ *and given that* $\sigma_n^{*2} = Var\left[(1-L)^{-d} u_n^*\right]$ *then*

1) $\frac{\sigma_n^{*2}}{n^{2d-1}\sigma_u^2 A^*(d)} \longrightarrow 1$ *if* $d > \frac{1}{2}$.
2) If $d > 0.5$, for $r \in (0,1)$,
$$\left[\left(\sigma_n^*\right)^{-1} y_{[nr]}^* \Longrightarrow \frac{1}{\sqrt{A^*(d)}} W_{d-1}(r) \right] \Longleftrightarrow \left[n^{0.5-d} y_{[nr]}^* \Longrightarrow \sigma_u W_d(r) \right],$$

where

$$W_d(r) = \frac{1}{\Gamma(d)} \int_0^r (r-s)^{d-1} dW(s),$$
$$A^*(d) = \frac{1}{\Gamma^2(d)(2d-1)},$$

and

$W(r) = B(r)$ *is standard Brownian motion.*■

Theorem 4: *If* $y_t^* = (1-L)^{-d} u_t^*$, $d > 0.5$, u_t^* *defined by (2.3), satisfy the assumption that* $E\left|u_t\right|^r < \infty$ *for* $r \geq \max\left\{2, \frac{2}{2d+1}\right\}$ *and* $\left(\sigma_n^*\right)^2 = Var\left(\sum_{t=1}^{[nr]} y_t^*\right)$ *then*

1) $\dfrac{\left(\sigma_n^*\right)^2}{n^{2d+1}\sigma_u^2 A^*(d+1)} \longrightarrow 1.$
2) $n^{-0.5-d} \sum_{t=1}^{[nr]} y_t^* \Longrightarrow \sigma_u W_{d+1}(r)$, $r \in (0,1)$.
3) $n^{-2d} \sum_{t=1}^{[nr]} \left(y_t^*\right)^2 \Longrightarrow \sigma_u^2 \int_0^r W_d^2(s) ds$, $r \in (0,1)$.

where $A^*(d)$ *and* $W_d(r)$ *are defined as in Theorem 3.*■

Remark 2: *By noting that,* $W_0(r) = \int_0^r dW(s) = W(r) = B(r) = B_{0,1}(r)$ *and that for the integer values of* $d = m$*, we have* $W_m(r) = B_{0,m}(r) = \frac{1}{\Gamma(m)} \int_0^r (r-s)^{m-1} dB(s)$ (see Tanaka (1999) page 554)*, then Corollary 1 can also be deduced from Theorem 3 and 4. In the other words, the definitions of type I and type II fractional Brownian motion coincide when* d *is an integer. We can find another intuitive explanation in Davidson and Hashimzade (2008) remark: "An* $I(1)$ *model cannot be allowed to have an infinitely remote starting date, but must be conceived as a cumulation of increments initiated at date* $t = 1$*, with an initial condition* y_0 *that must be generated by a different mechanism. The view that this construction should apply to the whole class of* $I(d)$ *models leads naturally to the type II framework".*

The above fractional functional limit theorems have been applied to statistical tests in regression models under the nonstationarity assumption (see Gourieroux, Maurel and Monfort (1987), Sowell (1990), Tanaka (1999), Wang et al. (2003)). All these applications focus more specifically on the null hypothesis test

$$H_0 : d = 1.$$

Furthermore, these applications are also based only on type I fractional Brownian motion or type II. Depending on whether one works within the framework of type I or type II fractional Brownian motion, the same remarkable result is demonstrated concerning the asymptotic distributions of the test statistics DF_n and DF_t. That is, the domains of limit probability density function of DF_n and DF_t, according to weather $d < 1$, $d = 0$ and $d > 1$ are $\mathbb{R}^-$, $\mathbb{R}$ and $\mathbb{R}^+$ respectively. This remarkable property can be exploited to test the composite null $H_0 : d \geq 1$ rather than the simple one (Bensalma (2016)).

3. APPLICATION TO THE STANDARD FRACTIONAL DICKEY-FULLER TEST

3.1. FUNCTIONAL LIMIT THEOREMS FOR THE GENERAL NULL HYPOTHESIS

In this section, we apply the main results presented in section 2 to more general composite null hypothesis, namely,

$$H_0 : d \geq d_0. \tag{3.1}$$

To test the null hypothesis (3.1) we use the following standard autoregression model

$$\Delta^{d_0} y_t = \rho \Delta^{d_0 - 1} y_{t-1} + \varepsilon_t. \tag{3.2}$$

Note that for $d_0 = 1$, the autoregression model (3.2) is the standard model used for the unit root test. In order to extend the studies of the behavior of asymptotic distributions of the DF_n and DF_t test statistics deduced from the more general model (3.2), we must derive functional limit theorems for the transformed processes $\Delta^{d_0} y_t$ and $\Delta^{d_0-1} y_t$. It is straightforward to deduce, from the above asymptotic theory, the corresponding asymptotic theory of $\Delta^{d_0} y_t$ and $\Delta^{d_0-1} y_t$. This asymptotic theory is given by three corollaries, the first corresponds to the type I fractional process, the second corresponds to the type II fractional process and the third corresponds to the case where d is an integer.

COROLLARY 2 . *Let consider* $x_t = (1-L)^{d_0-1}y_t$, $d_0 \in \mathbb{R}^*$, *where* y_t *satisfy (2.1) with* $d-d_0 = m+\delta$, $m \in \mathbb{N}$ *and* $\delta \in]-0.5, 0.5[$. u_t *satisfy the assumption that* $E\left|u_t\right|^{\alpha} < \infty$ *for* $\alpha \geq \max\left\{4, \frac{-8\delta}{1+2\delta}\right\}$. *Then, for* $0 \leq r \leq 1$, x_t *has the following asymptotic properties,*

1) *For* $m > 0$ *and* $-0.5 < \delta < 0.5)$,
   - (a) $n^{-0.5-(d-d_0)}x_{[nr]} \Longrightarrow \sigma_u B_{\delta,m+1}(r)$.
   - (b) $n^{-\frac{3}{2}-(d-d_0)}\sum_{j=1}^{[nr]} x_j \Longrightarrow \sigma_u \int_0^r B_{\delta,m+1}(s)ds$.
   - (c) $n^{-2(d-d_0+1)}\sum_{j=1}^{[nr]} x_j^2 \Longrightarrow \sigma_u^2 \int_0^r B_{\delta,m+1}^2(s)ds$.
2) *For* $m > 1$ *and* $-0.5 < \delta < 0.5)$,
   - (a) $n^{0.5-(d-d_0)}\Delta x_{[nr]} \Longrightarrow \sigma_u B_{\delta,m}(r)$.
   - (b) $n^{-0.5-(d-d_0)}\sum_{j=1}^{[nr]} \Delta x_j \Longrightarrow \sigma_u \int_0^r B_{\delta,m}(s)ds$.
   - (c) $n^{-2(d-d_0)}\sum_{j=1}^{[nr]} (\Delta x_j)^2 \Longrightarrow \sigma_u^2 \int_0^r B_{\delta,m}^2(s)ds.\blacksquare$

COROLLARY 3 . *Let consider* $x_t^* = (1-L)^{d_0-1}y_t^*$, $d_0 \in \mathbb{R}^*$, *where* y_t^* *satisfy (2.2) with* $d-d_0 = m+\delta$, $m \in \mathbb{N}$ *and* $\delta \in]-0.5, 0.5[$. u_t^* *defined by (2.3), satisfy the assumption that* $E\left|u_t\right|^{r} < \infty$ *for* $r \geq \max\left\{2, \frac{2}{2d+1}\right\}$. *Then, for* $0 \leq r \leq 1$, x_t^* *has the following asymptotic properties,*

1) *For* $m > 0$ *and* $-0.5 < \delta < 0.5)$,
   - (a) $n^{-0.5-(d-d_0)}x_{[nr]}^* \Longrightarrow \sigma_u W_{d-d_0+1}(r)$.
   - (b) $n^{-\frac{3}{2}-(d-d_0)}\sum_{j=1}^{[nr]} x_j^* \Longrightarrow \sigma_u W_{d-d_0+2}(r)$.
   - (c) $n^{-2(d-d_0+1)}\sum_{j=1}^{[nr]} \left(x_j^*\right)^2 \Longrightarrow \sigma_u^2 \int_0^r W_{d-d_0+1}^2(s)ds$.
2) *For* $m > 1$ *and* $-0.5 < \delta < 0.5$,
   - (a) $n^{0.5-(d-d_0)}\Delta x_{[nr]}^* \Longrightarrow \sigma_u W_{d-d_0+1}(r)$.
   - (b) $n^{-0.5-(d-d_0)}\sum_{j=1}^{[nr]} \Delta x_j^* \Longrightarrow \sigma_u W_{d-d_0}(r)$.
   - (c) $n^{-2(d-d_0)}\sum_{j=1}^{[nr]} \left(\Delta x_j^*\right)^2 \Longrightarrow \sigma_u^2 \int_0^r W_{d-d_0}^2(s)ds.\blacksquare$

COROLLARY 4. *Let consider* $x_t = \Delta^{m_0-1}y_t$, $m_0 \in \mathbb{N}^*$, *where* y_t *satisfy (2.1) with* $d = m$ $(\delta = 0)$ *and* $m - m_0 \geq 1$. *Under the assumption that* u_t is a class of i.i.d Gaussian processes we have. *Then, for* $0 \leq r \leq 1$, x_t *has the following asymptotic properties, depending on whether* $m < m_0$, $m \geq m_0$,

1) *for* $m - m_0 > 0$ *we have*
   - (a) $n^{-0.5-(m-m_0)}x_{[nr]} \Longrightarrow \sigma_u B_{0,m-m_0+1}(r)$ $(\equiv \sigma_u W_{m-m_0+1}(r))$,
   - (b) $n^{-1.5-(m-m_0)}\sum_{t=1}^{[nr]} x_j \Longrightarrow \sigma_u \int_0^r B_{0,m-m_0+1}(s)ds$ $(\equiv \sigma_u W_{m-m_0+2}(r))$,
   - (c) $n^{-2(m-m_0+1)}\sum_{t=1}^{[nr]} x_j^2 \Longrightarrow \sigma_u^2 \int_0^r B_{0,m-m_0+1}^2(s)ds$ $(\equiv \sigma_u^2 \int_0^r W_{m-m_0+1}^2(s)ds)$,
2) *for* $m - m_0 \geq 1$ *we have*
   - (a) $n^{0.5-(m-m_0)}\Delta x_{[nr]} \Longrightarrow \sigma_u B_{0,m-m_0}(r)$ $(\equiv \sigma_u W_{m-m_0}(r))$,
   - (b) $n^{-0.5-(m-m_0)}\sum_{t=1}^{[nr]} \Delta x_j \Longrightarrow \sigma_u \int_0^r B_{0,m-m_0}(s)ds = \sigma_u B_{0,m-m_0+1}(r)$ $(\equiv \sigma_u W_{m-m_0+1}(r))$,
   - (c) $n^{-2(m-m_0)}\sum_{t=1}^{[nr]} (\Delta x_j)^2 \Longrightarrow \sigma_u^2 \int_0^r B_{0,m-m_0}^2(s)ds$ $(\equiv \sigma_u^2 \int_0^r W_{m-m_0}^2(s)ds)$

where $B_{0,m}(r) = W_m(r) = \frac{1}{\Gamma(m)} \int_0^r (r-s)^{m-1} dB(s)$.■

Remark 3: Corollary 4 is only mentioned here to highlight that

$$B_{0,m}(r) = W_m(r) \quad \text{but} \quad B_{\delta,m}(r) \neq W_{m+\delta}(r), \text{ for } \delta \neq 0$$

3.2. THE ASYMPTOTIC NULL AND ALTERNATIVE DISTRIBUTIONS

In this subsection we use the Corollaries 2 and 3 to derive the asymptotic distribution of DF_n and DF_t statistics under the null hypothesis $d \geq d_0$ when the data generating processes are defined by (2.1) and (2.2) respectively (see Theorems 5 and 6).

Theorem 5 : *Let* $\{y_t^*\}$ *be generated according to the data generating process (2.2) with* $d \in (-0.5, +\infty)$. *If regression model* (3.2) *is fitted to a sample of size* n *then, as* $n \to \infty$, *for* $d - d_0 = m + \delta$, *with* $m \geq 0$ *and* $\delta \in (-0.5; 0.5)$ *we have*

1) $DF_n \to -\infty$ and $DF_t \to -\infty$ if $d < d_0$.
2) $DF_n \Longrightarrow \frac{0.5\left[W^2(1)-1\right]}{\int_0^1 W^2(r)dr}$ and $DF_t \Longrightarrow \frac{0.5\left[W^2(1)-1\right]}{\left(\int_0^1 W^2(r)dr\right)^{0.5}}$ if $d = d_0$.
3) $DF_n \Longrightarrow \frac{\frac{1}{2}W_{d-d_0}^2(1)}{\int_0^1 W_{d-d_0+1}^2(r)dr}$ and $DF_t \Longrightarrow +\infty$ if $d > d_0$.

where $W(\cdot)$ *is a standard Brownian motion and* $W_d(\cdot)$ *defined as in Theorem 3.*■

Proof *: See Appendix.*

Theorem 6 : *Let* $\{y_t\}$ *be generated according to data generating process (2.1) with* $d \in (-0.5, +\infty)$. *If regression model* (3.2) *is fitted to a sample of size* n *then, as* $n \to \infty$, *for* $d - d_0 = m + \delta$, *with* $m \geq 0$ *and* $\delta \in (-0.5; 0.5)$ *we have*

1) $DF_n \to -\infty$ and $DF_t \to -\infty$ if $d < d_0$.
2) $DF_n \Longrightarrow \frac{0.5\left[B_{0,1}^2(1)-1\right]}{\int_0^1 B_{0,1}^2(r)dr}$ and $DF_t \Longrightarrow \frac{0.5\left[B_{0,1}^2(1)-1\right]}{\left(\int_0^1 B_{0,1}^2(r)dr\right)^{0.5}}$ if $d = d_0$.
3) $DF_n \Longrightarrow \frac{\frac{1}{2}B_{\delta,m+1}^2(1)}{\int_0^1 B_{\delta,m+1}^2(r)dr}$ and $DF_t \Longrightarrow +\infty$ if $d > d_0$

where $B_{0,1}(\cdot)$ *is a standard Brownian motion and* $B_{\delta,m}(\cdot)$ *defined as in Theorem 2.*■

Proof *: The proof is similar to that of Theorem 5.*

Chan and Wei (1988) earlier obtained the preliminary tools for the study of test statistics in Theorem 5 and 6, for $I(d)$ processes with (d) an integer, although their expressions for the limiting distributions are slightly different from those given here. The advantage of the present formulation is that demonstrates how all these three cases are extremely useful for our study.

3.3. COMMENTS ON LIMIT DISTRIBUTIONS OF THE TESTS STATISTICS

Before giving a simple reading or interpretation of Theorems 5 and 6, we recall some basic rules of the statistical testing theory, when we deal with the composite null hypothesis. Consider the general testing problem:

$$H_0 : d \in \Omega_0 \quad \text{against} \quad H_1 : d \in \Omega_1 \tag{2.4}$$

Ω_0 denote the possible values of d under H_0 and Ω_1 denote the possible values of d under H_1. Furthermore, we have $\Omega_0 \cap \Omega_1 = \emptyset$. Let $T(y) = T(y_1, y_2, \cdots, y_n)$ the test statistic corresponding to (2.4). Generally, the distribution of $T(y)$, under H_0, is analyzed in order to define a rejection region C. For one sided test, C is typically of the form $(-\infty, c_0]$ or $[c_1, +\infty)$. c_0 and c_1 are critical values obtained from quantiles of the null distribution. Under the composite null there is a wide class of distribution F. For each distribution, we can define the test function $\Phi_F(\cdot)$ by

$$\Phi_F(T) = \begin{cases} 1 & \text{if } \; T(y) \in C, \\ 0 & \text{otherwise.} \end{cases}$$

If we observed $\Phi_F(T) = 1$, we reject H_0, while if $\Phi_F(T) = 0$, we accept.
The criterion for deciding whether one test is better than another is to compare their power. The power function of a test Φ_F is defined to be

$$\Pi_F(d) = Prob\left[\text{Reject } H_0\right] = E_d\left(\Phi_F(T)\right)$$

defined for all $d \in \Omega$. A good test of a given size α is one which makes $\Pi_F(d)$ as large as possible for $d \in \Omega_1$ while satisfying the constraint $\Pi_F(d) \leq \alpha$ for all $d \in \Omega_0$.
A uniformly most powerful (UMP) test of size α is a test $\Phi_{F_0}(\cdot)$ which

- $E_d\left[\Phi_{F_0}(T(y)\right] \leq \alpha$ for all $d \in \Omega_0$.
- Given any other test $\Phi_F(\cdot)$ for which $E_d\left[\Phi_F(T(y)\right] \leq \alpha$ for all $d \in \Omega_0$, we have $E_d\left[\Phi_{F_0}(T(y)\right] \geq E_d\left[\Phi_F(T(y)\right]$ for all $d \in \Omega_1$.

In practice, when the null is composite, these basic rules of statistical testing theory are used as follows: for a given size α, the probability of type I error is controlled by imposing this constraint on it:

$$P\left[type\ I\ error\right] = \underset{d \in \Omega_0}{Sup} P\left[T(y) \in C/d \in \Omega_0\right] \leq \alpha$$

In the light of this theoretical recall, we shall interpret the last theorems set out above. Since the results of all theorems are similar, we interpret Theorem 5 only through the statistical distributions of DF_n. The conclusions to be drawn from this interpretation are valid for the statistics of DF_t and the theorem 7.
To allow a simple reading or interpretation of theorem 5 and to fix ideas suppose that the true value of d is found in the set $\{0, 1, 2, 3\}$. To test the null hypothesis $H_0 : d \geq 1$ against the alternative $H_0 : d < 1$, (i.e. $m_0 = 1$) we use the regression model $\Delta y_t = \rho y_{t-1} + \varepsilon_t$. It is easy to see, from Theorem 5, that under the null hypothesis, the limit distributions of the normalized least squares estimator, $n\widehat{\rho} \equiv DF_n$, according to the value of d under H_0 are:

$$DF_n^1 = n\widehat{\rho} \Longrightarrow \frac{0.5\left[W^2(1)-1\right]}{\int_0^1 W^2(r)dr}, \qquad \text{if } d=1$$

$$DF_n^2 = n\widehat{\rho} \Longrightarrow \frac{0.5\left[W_1(1)\right]^2}{\int_0^1 W_1^2(r)dr}, \qquad \text{if } d=2$$

$$DF_n^3 = n\widehat{\rho} \Longrightarrow \frac{0.5\left[W_2(1)\right]^2}{\int_0^1 W_2^2(r)dr}, \qquad \text{if } d=3$$

where $W(\cdot)$ is a standard Brownian motion on $C\,[0,1]$, i.e. the space of continuous functions on the unit interval, and $W_1(r) = \int_0^r W(s)ds$, $W_2(r) = \int_0^r W_1(s)ds$. For a given level of significance, α, we can build three statistical tests,

$$\Phi_{d=i}(n\widehat{\rho}) = \begin{cases} 1 & \text{if } n\widehat{\rho} \geq cv_i \\ 0 & \text{if } n\widehat{\rho} < cv_i \end{cases} \quad i = 1,2,3,$$

where cv_1, cv_2 and cv_3 are the lower-tail critical values of DF_n^1, DF_n^2 and DF_n^3 respectively. Critical regions that are most powerful against a particular alternative at a given level of significance are identified for the testing of a composite hypothesis. For the alternative hypothesis, $H_1 : d < 1$, we consider the lower-tail critical regions of the form.

$$C_i(\alpha) = \{n\widehat{\rho} < cv_i, \text{ for } i = 1, 2 \text{ and } 3\}.$$

Since DF_n^1 has real line support, whereas DF_n^2 and DF_n^3 have positive support, it is easy to show that, for any given critical value (cv),

$$\underset{d\in\{1,2,3\}}{Sup}\; P(n\widehat{\rho} < cv) = P_{d=1}(n\widehat{\rho} < cv).$$

The best region is therefore

$$C_1(\alpha) = \{n\widehat{\rho} < cv_1\}.$$

This remarkable arrangement of the limit distributions of $n\widehat{\rho}$ and $t_{\widehat{\rho}}$, depending on whether $d < m_0$, $d = m_0$ and $d < m_0$, is advantageous in more than one way. First, it enables the test to be consistent. Second, this arrangement allows the upward or downward sequential testing procedure to be considered, not only without having to tabulate different percentiles for DF_n or DF_t, but also using the same tabulated critical value at each step. Intuitively, consistency is based on the following facts:

- In the case of $d > m_0$, we must expect that the percentage of rejection of H_0 is close to zero (less than the nominal size α: $E_d(\Phi\,(n\widehat{\rho})) \leq \alpha$).
- When $d = m_0$, we can expect that the rejection percentage of H_0 is close to nominal size α.
- When $d < m_0$, we can expect that the rejection percentage of H_0 is between α and 100.

These facts can be summarized as the following,

$$\lim_{n\to\infty} P\,[\text{Rejecting } H_0] \begin{cases} \leq \alpha, & d > d_0, \\ = \alpha, & d = d_0, \\ = 1, & d < d_0, \end{cases} \tag{A}$$

where (d) and (d_0) can be any real numbers. (A) denotes that, by employing a test based on a null composite hypothesis, we can propose a sequential test procedure that begins with the largest order of integration, which is considered the first maintained hypothesis, and gradually decreases the order of integration as the current null is rejected. We can also suggest a sequential test procedure that takes the smallest order of integration (for example. $d \geq 0$) into consideration as the first maintained hypothesis and then increases the order of integration each time the current alternative is rejected.

Another important result suggested by Theorems above, given that

$$B_{0,m}(r) = W_m(r) \quad \text{but} \quad B_{\delta,m}(r) \neq W_{m+\delta}(r), \text{ for } \delta \neq 0,$$

is that asymptotically we have

$$DF_n \Longrightarrow \frac{0.5\big[B_{0,1}^2(1)-1\big]}{\int_0^1 B_{0,1}^2(r)dr} = \frac{0.5\big[W^2(1)-1\big]}{\int_0^1 W^2(r)dr} \quad \text{and} \quad DF_t \Longrightarrow \frac{0.5\big[B_{0,1}^2(1)-1\big]}{\big(\int_0^1 B_{0,1}^2(r)dr\big)^{0.5}} = \frac{0.5\big[W^2(1)-1\big]}{\big(\int_0^1 W^2(r)dr\big)^{0.5}} \quad \text{if } d = d_0.$$

This means that we can use the tabulated values of Dickey and Fuller's finite sample standard distributions. This is true for the type II fractional processes-based test, because the class of $I(d)$ processes with (d) an integer is particular case of the class of type II fractional processes. But what about the fractional process-based test of type I? Finding an answer to this question is our main concern in the next section.

4. MONTE-CARLO EVIDENCES

The simulations we carried out are divided into two parts. In the first part, we evaluate the fractional Dickey-Fuller test within the general framework of the fractional processes of type II, defined by (2.2). By construction, the processes $I(d)$ with d integer represent a particular class of that defined by (2.2). The main objective is to verify if the theoretical results (A) and (B) are verified for small or medium sample sizes. In addition, these simulations show that the approximate distributions of the test statistic (DF_n or DF_t) for the same sample size, low or moderate, when $d = 1$ is comparable to $d = d_0$, with d_0 fractional. That is, in the general context of fractional type II processes, we can use the critical values of the standard Dickey-Fuller distribution. In the second part, we address the question whether we can use the standard Dickey-Fuller distribution when we perform the type I fractional process based test. For that, we compare the type I fractional process based test, with different pre-sample size, with the type II fractional process based test without pre-sample influence. This comparison will be made only in the case where $d = d_0$, because theorems 5, 6 and 7 imply that in this case, the limit distributions are the same and coincide with the standard Dickey-Fuller distribution.

4.1. MONTE CARLO EVIDENCE FOR TYPE II FRACTIONAL PROCESS BASED TEST

In the following, we examine the analytical findings numerically when d and d_0 are integer. For integration order $d \in \{0, 1, 2, 3, 4\}$, from the data generating process (DGP) (2.2) (i.e. the type *II* $I(d)$ fractional process), we simulate a sample of 50 observations. For each value of d we perform the test hypothesis (3.1) via the autoregression model (3.2) for $m_0 \in \{1,2,3,4\}$. The simulation is based on 10000 Monte-Carlo replications. The results are reported in the form of a matrix in Table 1. The main diagonal of table 1 shows clearly that P_{m_0} [Rejecting H_0] $\simeq \alpha$, for $d = m_0$. All the entries below the main diagonal of the table 1 show that P_{m_0} [Rejecting H_0] $< \alpha$ for $d > m_0$. Finally, all the entries above the main diagonal in table 1 show that P_{m_0} [Rejecting H_0] $\simeq 1$ for $d < m_0$. The critical values used to perform the DF test are those of the Dickey-Fuller distribution.

Table 1: Percentage of rejection of the null $H_0 : d \geq m_0$ ($P[DF_t < cv_\alpha]$) when the DGP is (2.2)

		m_0	**1**	**2**	**3**	**4**
d	cv_α	α				
	-2.57	1%	100	100	100	100
0	-1.95	5%	100	100	100	100
	-1.61	10%	100	100	100	100
	-2.57	1%	**1.20**	100	100	100
1	-1.95	5%	**5.15**	100	100	100
	-1.61	10%	**9.98**	100	100	100
	-2.57	1%	0	**1.07**	100	100
2	-1.95	5%	0	**4.69**	100	100
	-1.61	10%	0	**9.74**	100	100
	-2.57	1%	0	0	**1.25**	100
3	-1.95	5%	0	0	**5.21**	100
	-1.61	10%	0	0	**10.01**	100
	-2.57	1%	0	0	0	**1.12**
4	-1.95	5%	0	0	0	**4.98**
	-1.61	10%	0	0	0	**10.13**

The results are so perfect, they do not seem random in the cases $d > m_0$ and $d < m_0$. This can be explained by the too great difference between d and m_0 (which is at least equal to 1). Table 2 shows that the results will be much less nuanced when d is fractional and the difference between d and d_0 is less than one. Indeed, to view how the DF test performs in term of level and power in the fractional case, the simulations are conducted as follows. For integration order $d \in \{0.2; 0.4; 0.6; 0.8; 1; 1.2\}$, from the data generating process (2.2), we simulate a sample of 50 observations. For each value of d we perform the test hypothesis ($H_0 : d \geq d_0$) via the autoregression model (3.2) for $d_0 \in \{0.2; 0.4; 0.6; 0.8; 1; 1.2\}$. The simulation is based on 10000 Monte-Carlo replications. The results are shown in the table 2. The table 2, shows that the DF test still has a good performance since P_{d_0} [Rejecting H_0] $\simeq \alpha$ for $d = d_0$, P_{d_0} [Rejecting H_0] $< \alpha$ for $d > d_0$ and P_{d_0} [Rejecting H_0] $\simeq 1$ for $d < d_0$. However, these performances are far from being as perfect as those of table 1, especially when $d - d_0 = -0.2$. The three central diagonal lines represent the differences between $d - d_0$ equal to (+0.2), (0) and (-0.2) respectively. The critical values used to perform the FDF test are those of the Dickey-Fuller distribution. Supplementary material 2 contains the simulation program with Eviews upon which these results are based.

Table 2: Percentage of rejection of the null $H_0 : d \geq d_0$ ($P\left[DF_t < cv_\alpha\right]$) when the DGP is (2.2)

d	cv_α	α	d_0: 0.2	0.4	0.6	0.8	1	1.2
0.2	-2.57	1%	**1.15**	10.3	49.95	95.17	99.99	100
	-1.95	5%	**5.14**	27.26	75.38	99.45	100	100
	-1.61	10%	**10.08**	40.39	85.73	99.87	100	100
0.4	-2.57	1%	0.06	**1.14**	10.72	50.06	95.7	99.98
	-1.95	5%	0.69	**5.16**	27.06	75.08	99.52	100
	-1.61	10%	1.58	**10.36**	40.60	86.71	99.87	100
0.6	-2.57	1%	0.01	0.08	**1.21**	10.83	49.37	95.24
	-1.95	5%	0.07	0.61	**4.97**	26.71	75.41	99.41
	-1.61	10%	0.20	1.70	**10.02**	39.77	86.39	99.96
0.8	-2.57	1%	0	0	0.02	**1.18**	10.39	49.70
	-1.95	5%	0	0.01	0.57	**4.92**	26.65	75.75
	-1.61	10%	0.02	0.15	1.57	**10.17**	40.12	86.49
1	-2.57	1%	0	0	0	0.04	**1.08**	10.45
	-1.95	5%	0	0	0.03	0.61	**5.06**	26.54
	-1.61	10%	0	0.02	0.18	1.64	**9.67**	39.97
1.2	-2.57	1%	0	0	0	0	0.1	**1.02**
	-1.95	5%	0	0	0.01	0.07	0.51	**4.87**
	-1.61	10%	0	0.01	0.03	0.25	1.52	**9.83**

The results of Tables 1 and 2 demonstrate what the theoretical results in (A) predict. In addition, it should be noted that we used the same critical values (cv_α) taken from the Dickey-Fuller tables, regardless of the values of d and d_0. It should also be noted that at each time, when $d = d_0$, we find that the estimated probability of rejection of H_0 is very similar to the nominal value α.
All these results show that fractionally integrated type II processes are a natural extension of the class of processes $I(d)$ where d is an integer.

4.2. MONTE CARLO EVIDENCE FOR TYPE I FRACTIONAL PROCESS BASED TEST

To facilitate replication of the current Monte Carlo study, we provide all necessary material in the form of a single Eviews program (see supplementary material 3). The F-DF test described in subsection 4.2, can be considered as "fractional Dickey-Fuller test without pre-historical influence" because in our simulation study, for generating a fractionally integrated series of length (n) we apply, for $t = 1, \cdots, n$, the formula (2.6) by choosing $k = 0$. In the following, we address the issues of the case where the formula (2.6) is used, with $k \neq 0$. Can we in this case use the usual tabulated values? To respond to this question we proceed as the following.

1) For a series $\{u_{1-k}, u_{2-k}, \cdots, u_{50}\}$ generated from a Gaussian $i.i.d(0,1)$ process, 4 samples from $ARFIMA(0, 0.5, 0)$ processes were generated by using the formula (2.6) for $k \in \{0, 50, 100, 150\}$, namely,

$$y_t = \sum_{j=1}^{k+t-1} \frac{\Gamma(d+j)}{\Gamma(d)\Gamma(j+1)} u_{t-j}, \text{ with } k \in \{0, 50, 100, 150\}$$

2) For each sample we estimate the autoregression model

$$(1-L)^{0.5} y_t = \rho(1-L)^{0.5-1} y_{t-1} + \varepsilon_t,\ t = 1, \cdots, 50$$

3) From the step 2 $DF_n = 50\widehat{\rho}$ and $DF_t = t_{\widehat{\rho}}$ are calculated.

Remark: *We draw the reader's attention that we are interested, here, only in the simple null hypothesis* $H_0 : d = d_0 = 0.5$.

To estimate the densities of DF_n and DF_t 10000 replication realization of step 1 to 3 are made. The estimated densities for $k = 0$, 50, 100 and 150 are represented in the same graph. The kernel densities estimate of DF_n and DF_t are presented in Figures 1 and 2 respectively.

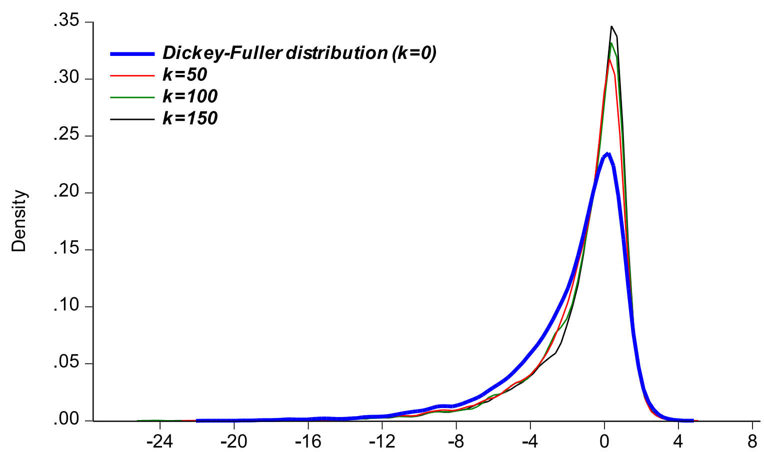

Fig. 1: Kernel estimate of DF_n for different values of k

Under the null hypothesis $H_0 : d = 0.5$ our simulations show that when k varies the distribution of the test statistic DF_n and DF_t also varies (see Figure 1 and 2). Therefore, we face a situation where we do not know the actual data generating process. In this situation, in testing statistical hypothesis theory a level α is defined as a largest value, among several possible densities of the test statistic under the null. Then, for a given critical value "cv", we have

$$\alpha = \underset{k \in \mathbb{N}}{SupP} \left[\text{Reject } H_0 \,|d = d_0 \right]$$

Our simulation results show that

$$\begin{aligned} \alpha &= \underset{k \in \mathbb{N}}{SupP} \left[\text{Reject } H_0 \,|d = d_0 \right] \\ &= P \left[\text{Reject } H_0 \,|d = d_0,\ k = 0 \right] \end{aligned}$$

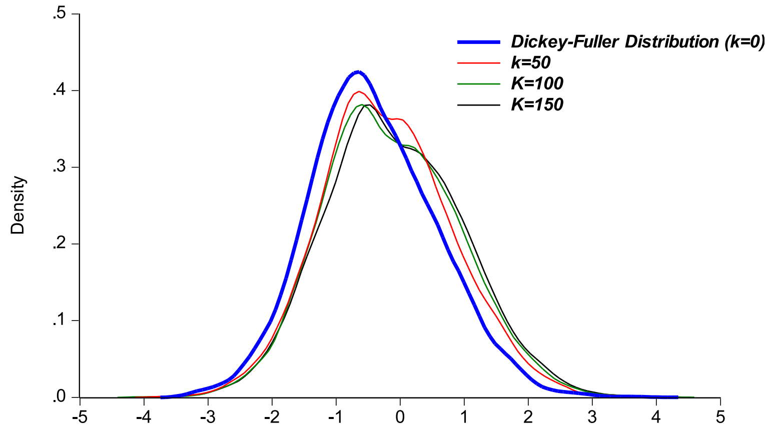

Fig. 2: Kernel estimate of DF_t for different values of k

In the following the distributions of the two test statistics corresponding to a particular value of k are denoted by $DF_n^{(k)}$ and $DF_t^{(k)}$.
From the 10000 values of $DF_n^{(k)}$ the percentage of rejection of the null for a given critical value (-12.9), (-7.7) and (-5.5), which are the quantiles of order $\alpha = 1\%$, 5% and 10% respectively (see page 762 Hamilton (1994)) are given in table 3.

Table 3 : Percentage of rejection of the null $H_0 : d = 0.5$
$\left(P\left[DF_n^{(k)} < cv(\alpha)\right]\right)$ for $k = 0; 50; 100; 150$ and $n = 50$

	$cv(1\%)$	$cv(5\%)$	$cv(10\%)$
	-12.9	-7.7	-5.5
k			
0	0.89	5.15	10.22
50	0.62	3.62	7.26
100	0.59	3.20	6.70
150	0.58	3.23	6.70

From the 10000 values of $DF_t^{(k)}$ the percentage of rejection of the null for a given critical value (-2.62), (-1.95) and (-1.61), which are the quantiles of order $\alpha = 1\%$, 5% and 10% respectively (see page 763 Hamilton (1994)), are given in table 4.

Table 4 : Percentage of rejection of the null $H_0 : d = 0.5$ $\left(P\left[DF_t^{(k)} < cv(\alpha)\right]\right)$ for $k = 0; 50; 100; 150$ and $n = 50$

	$cv(1\%)$	$cv(5\%)$	$cv(10\%)$
	-2.62	-1.95	-1.61
k			
0	0.9	5.04	10.29
50	0.62	3.72	7.63
100	0.61	3.20	6.97
150	0.62	3.26	6.97

Table 3 show that

$$\underset{k\in\{0,50,100,150\}}{Sup} P\left[DF_n^{(k)} < cv(\alpha)\,|d = 0.5\right] = P\left[DF_n^{(0)} < cv(\alpha)\,|d = 0.5\right]$$

$$= \begin{cases} 0.89\% & \text{for } \alpha = 1\% \\ 5.15\% & \text{for } \alpha = 5\% \\ 10.22\% & \text{for } \alpha = 10\% \end{cases}$$

and table 4 show that

$$\underset{k\in\{0,50,100,150\}}{Sup} P\left[DF_t^{(k)} < cv(\alpha)\,|d = 0.5\right] = P\left[DF_t^{(0)} < cv(\alpha)\,|d = 0.5\right]$$

$$= \begin{cases} 0.9\% & \text{for } \alpha = 1\% \\ 5.04\% & \text{for } \alpha = 5\% \\ 10.29\% & \text{for } \alpha = 10\% \end{cases}$$

If we combine the results of section 4.1 and section 4.2, then under the null hypothesis $H_0 : d \geq d_0$ we have,

$$\underset{k\in\mathbb{N},\ d\geq d_0}{Sup} P\left[DF_n^{(k)} < cv(\alpha)\right] = P\left[DF_n^{(0)} < cv(\alpha)\,|d = d_0\right]$$

and

$$\underset{k\in\mathbb{N},\ d\geq d_0}{Sup} P\left[DF_t^{(k)} < cv(\alpha)\right] = P\left[DF_t^{(0)} < cv(\alpha)\,|d = d_0\right]$$

Therefore, assuming that the real data generating process is indeed a fractional process, whether it is type 1 or type 2 and in view of our ignorance, we can apply the well-known principle of statistical test theory described above. This principle leads us to the possibility of using the tabulated values of the Dickey-Fuller distributions.

5. EMPIRICAL APPLICATION

In this section, we considered the well-known series for annual minima of the Nile, as studied by Hurst (1951) and reproduced in Beran (1994). The sample size of this series is 633 annual

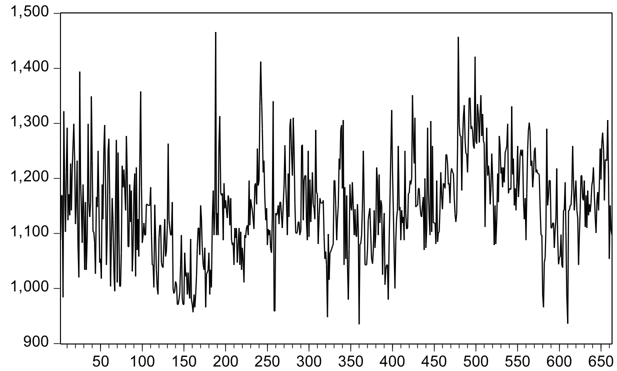

Fig. 3: Nile series

observations (622-1284 AD) and the Figure 3 is the time plot of it. We denote this series by $\{y_t,\, t = 1, \cdots, 633\}$.

For sake of illustration, the implementation of a sequential procedure of the standard FDF test is made on the demeaned series $Nile_t$, namely

$$y_t = Nile_t - \overline{Nile}$$

where $\overline{Nile} = \frac{\sum_{t=1}^{633} Nile_t}{633}$. We use upward testing sequence for a set of values

$$d_0^i \in \{0,\ 0.1,\, 0.2,\, 0.3,\, 0.4,\, 0.5,\, 0.6,\, 0.7,\, 0.8,\, 0.9,\, 1\}\,.$$

For a particular value d_0^i from this latest set, we test the hypothesis

$$H_{0i} : d \geq d_0^i$$

by using the $t_{\widehat{\rho}_i} = DF_t$ calculated via the estimation of the following autoregression model

$$(1 - L)^{d_0^i} y_t = (1 - L)^{d_0^i - 1} y x_{t-1} + \varepsilon_t.$$

The Table 5 summarize the sequential upward testing procedure of the standard FDF test on the x_t series (See supplementary 1, where we show how to perform the sequential procedure of FDF test with Eviews).

Table 5: Sequential FDF test applied to Nile series				
d_0^i	FDF test on	$t_{\widehat{\rho}_i} = DF_t$	Dickey-Fuller $cv(5\%)$	Reject or Accept $H_{0i} : d \geq d_0^i$
0	$\Delta^{-1} y_t$	-0.1986	-1.94	accept
0.1	$\Delta^{-0.9} y_t$	-0.2746	-1.94	accept
0.2	$\Delta^{-0.8} y_t$	-0.4201	-1.94	accept
0.3	$\Delta^{-0.7} y_t$	-0.7148	-1.94	accept
0.4	$\mathbf{\Delta^{-0.6} y_t}$	$\mathbf{-1.2174}$	$\mathbf{-1.94}$	**accept**
0.5	$\mathbf{\Delta^{-0.5} y_t}$	$\mathbf{-2.0029}$	$\mathbf{-1.94}$	**reject**
0.6	$\Delta^{-0.4} y_t$	-3.2481	-1.94	reject
0.7	$\Delta^{-0.3} y_t$	-4.9185	-1.94	reject
0.8	$\Delta^{-0.2} y_t$	-7.1955	-1.94	reject
0.9	$\Delta^{-0.1} y_t$	-10.0671	-1.94	reject
1	x_t	-13.3303	-1.94	reject
Conclusion	$0.4 \leq d < 0.5$			

The table 5 show that we can apply a downward testing sequence, in this case we take the largest value, (the maximum value of d_0^i, i.e. $d \geq 1$), under consideration as the first maintained hypothesis and then decrease the order of differenced each time the current null is rejected. The table 5 show, also, that an upward testing sequence can be applied. In this case, we take the smallest value of d_0^i, (i.e. $d \geq 0$) under consideration as the first maintained hypothesis and then increase the order of differenced each time the current alternative is accepted. An Eviews program to perform the sequential fractional Dickey-Fuller test to the Nile series can be found in supplementary material 1 (see Appendix 2).

6. CONCLUDING REMARKS

In this article, we are particularly interested in the case where the data generating process is an $ARFIMA(0, d, 0)$. This type of process can be defined by (2.1) or by (2.2). This leads us to ask the question of whether the limiting distributions of the usual Dickey-Fuller test statistics must be a function of type I or type II fractional Brownian motion. Faced with our ignorance on how ”nature” chose to construct the sample of observations, we proceeded to a comparative study. The goal of this comparative study is to determine if in both cases we can use the critical values of the Dickey-Fuller distributions to test the null hypothesis $H_0 : d = d_0$ (or more general null hypothesis $H_0 : d \geq d_0$), where d_0 is any given real value greater or equal 0.5.

Although Bensalma (2018) has already proposed the idea of the generalization of the standard Dickey-Fuller test to the fractional case, this article provides new elements concerning this generalization. The idea of the test is simple, to test the null hypothesis $H_0 : d = d_0$ on the basis of a sample of real data, $\{y_t,\ t = 1, \cdots, n\}$, assumed to be generated by an $ARFIMA(0, d, 0)$

process, it suffices to test that the process $x_t = (1-L)^{1-d_0} y_t$ is integrated of order 1. Based on the estimation of the regression

$$(1-L)^{d_0} y_t = \rho(1-L)^{1-d_0} y_{t-1} + \varepsilon_t, \ t = 1, \cdots, n$$

we focus, in this article, in the limit and estimated distributions in finite sample of $DF_n = n\widehat{\rho}$ and $DF_t = t_{\widehat{\rho}}$. We suppose that $\{y_t, \ t = 1, \cdots, n\}$ is generated from the following formula,

$$\begin{cases} y_t = \sum_{j=1}^{k+t-1} \frac{\Gamma(d+j)}{\Gamma(d)\Gamma(j+1)} u_{t-j}, & t = 1, \cdots, n \\ \text{where } u_t \text{ is white noise process} \end{cases}$$

We were interested in three types of processes.

1) Unit root process : $y_t = \sum_{j=1}^{t-1} u_{t-j}, \ (d=1)$.
2) Fractional type II process : $\sum_{j=1}^{t-1} \frac{\Gamma(d+j)}{\Gamma(d)\Gamma(j+1)} u_{t-j}, \ (d=0.5)$.
3) Fractional type I process :$\sum_{j=1}^{k+t-1} \frac{\Gamma(d+j)}{\Gamma(d)\Gamma(j+1)} u_{t-j}, \ (k \neq 0, \ d=0.5)$.

We compare the limit and finite sample distributions of the DF test statistics when the null hypothesis is either (a), (b) or (c)

$$H_0 : d = 1, \tag{a}$$

$$\begin{cases} H_0 : d = d_{0,} \ (d_0 \text{ fractional}) \\ y_t \text{ is Fractional type II process,} \end{cases} \tag{b}$$

$$\begin{cases} H_0 : d = d_{0,} \ (d_0 \text{ fractional}) \\ y_t \text{ is Fractional type I process,} \end{cases} \tag{c}$$

- For the (a), (b) and (c) cases, we showed that the DF statistics have the same limits distribution.
- For the case (a) and (b), we showed, by simulation, that the DF statistics have the same finite sample distribution.
- For the case (b) and (c), with $d = 0.5$, we showed, by simulation, that the DF statistics have not the same finite sample distribution.

This last comparison may lead one to believe that one cannot use the critical tabulated valuesof the standard Dickey-Fuller distributions when the data generating process is the fractional type I process. In fact, by a simple statistical argument, often used in test theory, (see section 4) we have shown that it is possible to use the tabulated values of the Dickey-Fuller distributions even when the process generating the data is that fractional processes of type I. On the other hand, it has also been shown that, limit distributions of DF_n and DF_t are defined on $\mathbb{R}^-$, $\mathbb{R}$ and $\mathbb{R}^+$ when $d < d_0$, $d = d_0$ or $d > d_0$, respectively. This remarkable arrangement is exploited to propose to test the null hypothesis $H_0 : d \geq d_{0,}$. For this composite null hypothesis, to control the type I error (α), we have

$$\alpha = \underset{d \geq d_0}{Sup} P\left[DF_t < cv(\alpha)\right] = P\left[DF_t < cv(\alpha) \left| d = d_0 \right.\right].$$

This means that to test the composite null hypothesis the tabulated values of the Dickey-Fuller distributions can be used.

To generalize our results to the case where the data generating process is an $ARFIMA(p,d,q)$, it suffices to exploit the idea of Pantula (1989) who showed that to test the null hypothesis

$$H_0 : d = m_0$$

where $m_0 \in \mathbb{N}^*$ we can use the ADF test by using the regression model

$$\Delta^{m_0} y_t = \rho \Delta^{m_0 - 1} y_{t-1} + \sum_{j=1}^{k} \lambda_j \Delta^{m_0} y_{t-j} + \varepsilon_t,\ t = 1, \cdots, n$$

Pantula (1989) shows that the limiting distributions of ADF_n and ADF_t are

$$ADF_n = \frac{n\widehat{\rho}}{1-\sum \gamma_j} \xrightarrow{L} \frac{0.5\left[W^2(1)-1\right]}{\int_0^1 W^2(r)dr} \quad \text{et} \quad ADF_t = t_{\widehat{\rho}} \xrightarrow{L} \frac{0.5\left[W^2(1)-1\right]}{\left(\int_0^1 W^2(r)dr\right)^{0.5}}$$

Moreover, he shows that the domains of limit probability density function of DF_n and DF_t, according to weather $d < m_0$, $d = m_0$ and $d > m_0$ are $\mathbb{R}^-$, $\mathbb{R}$ and $\mathbb{R}^+$ respectively. Then, we get

$$\lim_{n\to\infty} P\left[\text{Rejecting } H_0\right] \begin{cases} \leq \alpha & d > m_0 \\ = \alpha & d = m_0 \\ = 1 & d < m_0 \end{cases}$$

For the fractional case, replacing, m_0 by d_0 the results of Pantula (1989) remain valid.

REFERENCES

[1] Akonom, J. and Gourieroux, C. (1987) 'A functional central limit theorem for fractional processes', Discussion Paper #8801,CEPREMAP, Paris.

[2] Bensalma, A. (2016) 'A consistent test for unit root against fractional alternative', International Journal of Operational Research, Vol. 27, Nos. 1/2, pp.252–274.

[3] Bensalma, A. (2018) 'Testing the fractional integration parameter revisited: a fractional Dickey-Fuller test', International Journal of Mathematics in Operational Research, Vol. 12, No. 4, pp. 471-506.

[4] Bensalma, A**.,** (2021) 'An Eviews program to perform the fractional Dickey-Fuller test'. Supplementary material 1

[5] Bensalma, A., (2021)'A farctional Dickey-Fuller test: An Eviews program to evaluate the size and power of a type II fractional process based test'. Supplementary material 2

[6] Bensalma, A**.,**(2021)**,** 'Fractional Dickey-Fuller test: A simulation program with Eviews to compare the simple null distribution of the type I fractional process based test and the type II fractional process based test. Supplementary material 3.

[7] Chan, N.H. and Terrin, N. (1995) 'Inference for unstable long-memory processes with applications to fractional unit root autoregressions', The annals of Statistics, Vol. 23, No. 5, pp. 1662-1683.

[8] Chan, N.H. and Wei, C.Z. (1988) 'Limiting distributions of least squares estimates of unstable autoregressive processes', The annals of Statistics, Vol. 16, No. 1, pp. 367-401.

[9] Davidson, J. and Hashimzade, N. (2009) 'Type I and type II fractional Brownian motions: A reconsideration' Computational Statistics & Data Analysis, Vol. 53, No. 6, pp. 2089-2106.

[10] Dickey, D.A. and Fuller, W.A. (1979) 'Distribution of the estimators for autoregressive time series with a unit root', Journal of the American Statistical Association, Vol. 74, No. 366a, pp.427–431.

[11] Dickey, D.A. and Fuller, W.A. (1981) 'Likelihood ratio tests for autoregressive time series with a unit root', Econometrica, Vol. 49, No. 4, pp.1057–1072.

[12] Dickey, D.A. and Pantula, S.G. (1987) 'Determining the order of differencing in autoregressive processes', Journal of Business and Economic Statistics, Vol. 15, No. 4, pp.455–461.

[13] Gourieroux, C., F. Maurel, and A. Monfort (1987), 'Regression and nonstationarity', CREST document no. 8708.

[14] Gourieroux, C. and A. Monfort (1997), 'Times series and dynamic models', CAMBRIDGE UNIVERSITY PRESS.

[15] Granger, C.W.J. and Joyeux, R. (1980) 'An introduction to long memory time series models and fractional differencing', Journal of Time Series Analysis, Vol. 1, No. 1, pp.15–29.

[16] Hamilton, J.D. (1994) 'Time series analysis', Princeton University Press.

[17] Hosking, J.R.M. (1981) 'Fractional differencing', Biométrika, Vol. 68, No. 1, pp.165–176.

[18] Liu, M. (1998) 'Asymptotics of nonstationary fractional integrated series', Econometric Theory, Vol. 14, No. 5, pp.641–662.

[19] Marinucci, D. and P.M. Robinson (1999) 'Alternative forms of fractional Brownian motion', Journal of Statistical Inference and Planning, Vol. 80, pp.111-122.

[20] Mandelbrot, B. and Van Ness, J. (1968).'Fractional Brownian motions, fractional noises and applications'. S.I.A.M. Review. Vol. 10,pp. 422-437.

[21] Park, J.Y. and Phillips, C.B. (1988) 'Statistical inference in regressions with integrated processes: Part 1', Econometric Theory, Vol. 4, pp. 468-497.

[22] Park, J.Y. and Phillips, C.B. (1989) 'Statistical inference in regressions with integrated processes: Part 2', Econometric Theory, Vol. 5, pp. 95-131.

[23] Pantula, S.G. (1989) 'Testing for unit root in time series data', Econometric Theory, Vol. 5,No 2, pp 256-271.

[24] Schwert, W.S. (1989) 'Tests for unit roots: A monte carlo investigation', Journal of business & Economic Statistics, Vol.7, No.2, pp. 147-159.

[25] Silveira, G. (1991) 'Contributions to strong approximations in time series with applications in nonparametric statistics and functional central limit theorems', PHD Thesis, University of London.

[26] Sowell, F.B. (1990) 'The fractional unit root distribution', Econometrica, Vol. 58, No. 2, pp.494–505.

[27] Tanaka, K (1999) 'The nonstationary fractional unit root', Econometric Theory, Vol. 15, No. 4, pp.549–582.

[28] Tanaka, K (1996) 'Time series analysis: Nonstationary and Noninvertible distribution Theory', New York : Wiley.

[29] Wang, Q., Lin Y. & Gulati, C.M. (2003) 'Asymptotic for general fractionally integrated processes with applications to unit root tests', Econometric Theory, Vol. 19, pp. 143-164.

[30] Wang, Q., Lin Y. & Gulati, C.M. (2002) 'Asymptotics for general nonstationary fractionally integrated processes without prehistoric influence', Journal of applied mathematics and decision sciences, Vol. 6, No. 4, pp. 255-269 pp.

7. APPENDICES

7.1. APPENDIX 1: PROOF OF THEOREM 5

By denoting $\Delta^{-1+d_0}y_t^* = x_t^*$, the OLS estimator of ρ and its t-ratio for the auxiliary regression model (2.2), are given by the usual squares expressions

$$\widehat{\rho}_n = \frac{\sum_{t=1}^n (\Delta x_t^*)\left(x_{t-1}^*\right)}{\sum_{t=1}^n \left(x_{t-1}^*\right)^2},$$

$$t_{\widehat{\rho}_n} = \frac{\sum_{t=1}^n (\Delta x_t^*)\left(x_{t-1}^*\right)}{\left\{s_n^2 \sum_{t=1}^n \left(x_{t-1}^*\right)^2\right\}^{1/2}},$$

where the variance of the residuals, s_n^2 is given by

$$s_n^2 = n^{-1}\sum_{t=1}^n \left(\Delta x_t^* - \widehat{\rho} x_{t-1}^*\right)^2.$$

See that, $x_{t-1}^* \sim FI(1+d-d_0)$ and $\Delta x_t^* \sim FI(d-d_0)$, we have the following:

For the $\sum_{t=1}^n \left(x_{t-1}^*\right)^2$ term, it follows from Corollary (3), item (1, c) we have

$$n^{-2-2(d-d_0)}\sum_{t=1}^n \left(x_{t-1}^*\right)^2 \Rightarrow \sigma_u^2 \int_0^1 W_{d-d_0+1}^2(r)dr, \text{ if } -0.5 < d-d_0 < 0, \quad \text{(A1)}$$

$$n^{-2}\sum_{t=1}^n \left(x_{t-1}^*\right)^2 \Rightarrow \sigma_u^2 \int_0^1 W^2(r)dr, \text{ if } d-d_0 = 0, \quad \text{(A2)}$$

$$n^{-2-2(d-d_0)}\sum_{t=1}^n \left(x_{t-1}^*\right)^2 \Rightarrow \sigma_u^2 \int_0^1 W_{d-d_0+1}^2(r)dr, \text{ if } 0 < d-d_0 < \frac{1}{2}, \quad \text{(A3)}$$

$$n^{-2-2(d-d_0)}\sum_{t=1}^n \left(x_{t-1}\right)^2 \Rightarrow \sigma_u^2 \int_0^1 W_{d-d_0+1}^2(r)dr, \text{ if } d-d_0 = m+\delta > \frac{1}{2}. \quad \text{(A4)}$$

For the $\sum_{t=1}^n \left[\Delta x_t^*\right]\left[x_{t-1}^*\right]$ term, we have that

$$\sum_{t=1}^n \left[\Delta x_t^*\right]\left[x_{t-1}^*\right] = \frac{1}{2}\left(x_n^*\right)^2 - \frac{1}{2}\sum_{t=1}^n \left(\Delta x_t^*\right)^2$$

For the first term,it follows from Corollary 3, item (1, a)

$$\frac{n^{-1-2\delta}}{2}\left(x_n^*\right)^2 \Rightarrow \frac{1}{2}\sigma_u^2 W_{\delta+1}^2(1), \text{ if } -0.5 < (d-d_0=\delta) < 0, \quad \text{(A5)}$$

$$\frac{1}{2n}\left(x_n^*\right)^2 \Rightarrow \frac{1}{2}\sigma_u^2 W^2(1), \text{ if } d-d_0 = 0, \quad \text{(A6)}$$

$$\frac{n^{-1-2\delta}}{2}\left(x_n^*\right)^2 \Rightarrow \frac{\sigma_u^2}{2} W_{\delta+1}^2(1), \text{ if } 0 < (d-d_0=\delta) < 0.5, \quad \text{(A7)}$$

$$\frac{n^{-1-2(m+\delta)}}{2}\left(x_n^*\right)^2 \Rightarrow \frac{1}{2}\sigma_u^2 W_{m+\delta+1}^2(1), \text{ if } d-d_0 = m+\delta > \frac{1}{2}. \quad \text{(A8)}$$

For the second term, we have:

Given that Δx_t^* is stationary fractionally integrated of order $\delta \in]-0.5, 0.5[$ and the ergodic theorem (note that here $d - d_0 = \delta$), and by noting that $\gamma_j = E\left(\Delta x_t^* \Delta x_{t-j}^*\right)$

When $-0.5 < d - d_0 < 0$,

$$-\frac{1}{2n}\sum_{t=1}^{n}\left(\Delta x_t^*\right)^2 \xrightarrow{p} -\frac{1}{2}var(\Delta x_t^*) = -\frac{\gamma_0}{2}. \tag{A9}$$

When $d - d_0 = 0$,

$$-\frac{1}{2n}\sum_{t=1}^{n}\left(\Delta x_t^*\right)^2 \xrightarrow{p} -\frac{1}{2}var(\Delta x_t^*) = -\frac{\sigma_u^2}{2}. \tag{A10}$$

When $0 < d - d_0 < 0.5$,

$$-\frac{1}{2n}\sum_{t=1}^{n}\left(\Delta x_t^*\right)^2 \xrightarrow{p} -\frac{1}{2}var(\Delta x_t^*) = -\frac{\gamma_0}{2}. \tag{A11}$$

When $m > 1$ and $d - d_0 = m + \delta > 0.5$, from Corollary 3, item (c)

$$n^{-2(d-d_0)}\sum_{t=1}^{n}\left(\Delta x_t^*\right)^2 \Rightarrow \sigma_u^2 \int_0^1 W_{d-d_0}^2(r)dr. \tag{A12}$$

Therefore, when $-0.5 < d - d_0 < 0$, by using $(A5)$ and $(A9)$, we have

$$n^{-1}\sum_{t=1}^{n}\left[\Delta x_t^*\right]\left[x_{t-1}^*\right] \xrightarrow{p} -\frac{\gamma_0}{2}. \tag{A13}$$

When $d - d_0 = 0$, by using $(A6)$ and $(A10)$, we have

$$n^{-1}\sum_{t=1}^{n}\left[\Delta x_t^*\right]\left[x_{t-1}^*\right] \Rightarrow -\frac{\sigma_u^2}{2}\left\{W^2(1) - 1\right\}. \tag{A14}$$

When $0 < (d - d_0 = \delta) < 0.5$, by using $(A7)$ and $(A11)$, we have

$$n^{-1-2\delta}\sum_{t=1}^{n}\left[\Delta x_t\right]\left[x_{t-1}\right] \Rightarrow \frac{\sigma_u^2}{2}W_{\delta+1}^2(1). \tag{A15}$$

When $m \geq 1$ and $d - d_0 = m + \delta > 0.5$, by using $(A8)$ and $A(12)$

$$n^{-1-2(d-d_0)}\sum_{t=1}^{n}\left[\Delta x_t\right]\left[x_{t-1}\right] \Rightarrow \frac{\sigma_u^2}{2}W_{d-d_0+1}^2(1). \tag{A16}$$

Hence, using $(A1, A13)$, $(A2, A14)$, $(A3, A15)$, $(A4, A16)$ respectively and the continuous mapping theorem, we obtain that

if $-0.5 < d - d_0 < 0$,

$$n^{1+2(d-d_0)}\widehat{\rho}_n = \frac{n^{-1}\sum_{t=1}^{n}\left[\Delta x_t\right]\left[x_{t-1}\right]}{n^{-2-2(d-d_0)}\sum_{t=1}^{n}\left[x_{t-1}\right]^2} \Rightarrow \frac{-\frac{\gamma_0}{2}}{\sigma_u^2\int_0^1 W_{d-d_0+1}^2(r)dr}, \tag{A17}$$

if $d - d_0 = 0$.

$$n\widehat{\rho}_n = \frac{n^{-1}\sum_{t=1}^{n}\left[\Delta x_t\right]\left[x_{t-1}\right]}{n^{-2}\sum_{t=1}^{n}\left[x_{t-1}\right]^2} \Rightarrow \frac{\frac{1}{2}\left\{W^2(1) - 1\right\}}{\int_0^1 W^2(r)dr}, \tag{A18}$$

if $0 < d - d_0 < 0.5$,

$$n\widehat{\rho}_n = \frac{n^{-1-2(d-d_0)}\sum_{t=1}^{n}\left[\Delta x_t\right]\left[x_{t-1}\right]}{n^{-2-2(d-d_0)}\sum_{t=1}^{n}\left[x_{t-1}\right]^2} \Rightarrow \frac{\frac{1}{2}W_{d-d_0+1}^2(1)}{\int_0^1 W_{d-d_0+1}^2(r)dr}, \tag{A19}$$

if $m \geq 1$ and $d - d_0 = m + \delta > 0.5$

$$n\widehat{\rho}_n = \frac{n^{-1-2(d-d_0)}\sum_{t=1}^{n}\left[\Delta x_t\right]\left[x_{t-1}\right]}{n^{-2-2(d-d_0)}\sum_{t=1}^{n}\left[x_{t-1}\right]^2} \Rightarrow \frac{\frac{1}{2}W_{d-d_0+1}^2(1)}{\int_0^1 W_{d-d_0+1}^2(r)dr}(\equiv \rho_\infty). \tag{A20}$$

Now consider the t-statistic, first notice that

$$\widehat{\sigma}^2 = n^{-1}\sum_{t=1}^{n}\left(\Delta x_t^* - \widehat{\rho}_n x_{t-1}^*\right)^2 = n^{-1}\left(\sum_{t=1}^{n}\left(\Delta x_t^*\right)^2 + \widehat{\rho}_n^2\sum_{t=1}^{n}\left(x_{t-1}^*\right)^2 - 2\widehat{\rho}_n\sum_{t=1}^{n}\left[\Delta x_t^*\right]\left[x_{t-1}^*\right]\right).$$

Hence,

When $-0.5 < d - d_0 < 0$, by using $A1$, $A9$, $A13$ and $A17$, it follows

$$\widehat{\sigma}^2 \xrightarrow{p} var(\Delta x_t) = \frac{\sigma_u^2\Gamma(1-2\delta)}{\Gamma^2(1-\delta)}. \tag{A21}$$

When $d - d_0 = 0$, by using $A2$, $A10$, $A14$ and $A18$, it follows

$$\widehat{\sigma}^2 \xrightarrow{p} var(\Delta x_t) = \sigma_u^2. \tag{A22}$$

When $0 < d - d_0 < 0.5$, by using $A3$, $A11$, $A15$ and $A19$, it follows

$$\widehat{\sigma}^2 \xrightarrow{p} var(\Delta x_t) = \frac{\sigma_u^2\Gamma(1-2\delta)}{\Gamma^2(1-\delta)}. \tag{A23}$$

When $m \geq 1$ and $d - d_0 = m + \delta > 0.5$, by using $A4$, $A12$, $A16$ and $A20$, it follows

$$n^{1-2(d-d_0)}\widehat{\sigma}^2 \Rightarrow \int_0^1 W_{d-d_0}^2(r)dr + \rho_\infty^2\int_0^1 W_{d-d_0+1}^2(r)dr - \rho_\infty W_{d-d_0+1}^2(1). \tag{A24}$$

Finally, by using respectively $(A1, A13, A21)$, $(A2, A14, A22)$, $(A3, A15, A23)$, $(A4, A16, A24)$ we obtain for the t-statistic

$$t_{\widehat{\rho}_n} = \frac{n^{-(d-d_0)}\left(n^{-1}\sum_{t=1}^{n}\left[\Delta x_t\right]\left[x_{t-1}\right]\right)}{\widehat{\sigma}\left(n^{-2-2(d-d_0)}\sum_{t=1}^{n}\left[x_{t-1}\right]^2\right)^{0.5}} \xrightarrow{p} -\infty, \text{ and } t_{\widehat{\rho}_n} = O_p(n^{-(d-d_0)}) \text{ if } -0.5 < d - d_0 < 0.$$

$$t_{\widehat{\rho}_n} = \frac{n^{-1}\sum_{t=1}^{n}\left[\Delta x_t\right]\left[x_{t-1}\right]}{\widehat{\sigma}\left(n^{-2}\sum_{t=1}^{n}\left[x_{t-1}\right]^2\right)^{0.5}} \Rightarrow \frac{\frac{1}{2}\left[W^2(1)-1\right]}{\left[\int_0^1 W^2(r)dr\right]^{0.5}}, \text{ and } t_{\widehat{\rho}_n} = O_p(1) \text{ if } d - d_0 = 0.$$

$$t_{\widehat{\rho}_n} = \frac{n^{d-d_0}\left(n^{-1-2(d-d_0)}\sum_{t=1}^{n}\left[\Delta x_t\right]\left[x_{t-1}\right]\right)}{\widehat{\sigma}\left(n^{-2-2(d-d_0)}\sum_{t=1}^{n}\left[x_{t-1}\right]^2\right)^{0.5}} \xrightarrow{p} +\infty, \text{ and } t_{\widehat{\rho}_n} = O_p(n^{d-d_0}) \text{ if } 0 < d - d_0 < 0.5.$$

$$t_{\widehat{\rho}_n} = \frac{n^{0.5}\left(n^{-1-2(d-d_0)}\sum_{t=1}^{n}\left[\Delta x_t\right]\left[x_{t-1}\right]\right)}{\left\{n^{0.5-(d-d_0)}\widehat{\sigma}\right\}\left(n^{-2-2(d-d_0)}\sum_{t=1}^{n}\left[x_{t-1}\right]^2\right)^{0.5}} \xrightarrow{p} +\infty,$$

and $t_{\widehat{\rho}_n} = O_p(n^{0.5})$ if $m \geq 1$ and $d - d_0 = m + \delta > 0.5$.■

APPENDIX 2: HOW TO IMPLEMENT THE SEQUENTIAL FDF TEST IN EViews by using fracdiff package

In EViews 9, a general procedure to compute the fractional difference of a given series $\{y_t,\ t = 1, \cdots, n\}$ is to apply the formula

$$x_t = (1 - L)^d y_t = \sum_{j=0}^{t-1} \frac{\Gamma(-d - j)}{\Gamma(-d)\Gamma(j + 1)} y_{t-j}. \tag{1.2}$$

For example, to compute the fractional difference of the demeaned Nile data with $d = 0.3$, we can use the following command lines,

1) series y=Nile-@mean(Nile)
2) y.fracdiff(d=0.3)

The second command line, simply take the difference of order 0.3 and saves the output as y_diff.

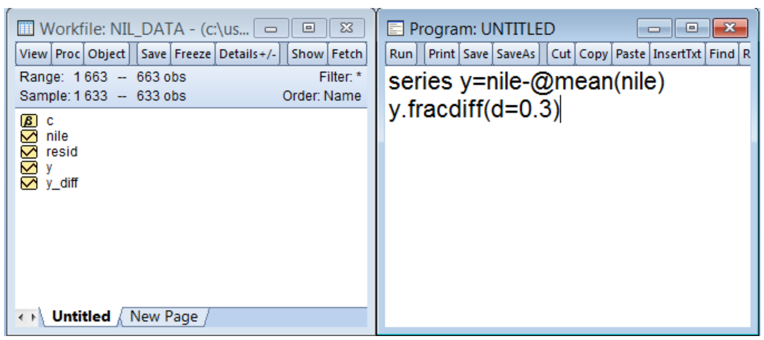

Fig. 4: Compute fractional difference of the Nile series

For another naming output, we can use the third command line,

3. rename y_diff x

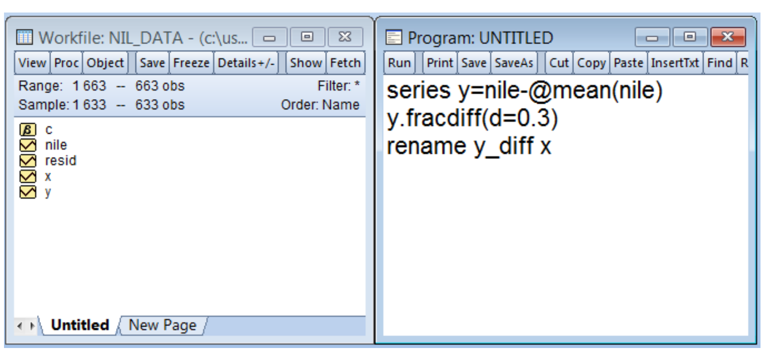

Fig. 5: Compute fractional difference of the Nile series

If one want to compute many fractional difference series of the Nile data for a sequence of different values of d, for example, $d_1 = 0.1$, $d_2 = 0.2$, $d_3 = 0.3$, $\cdots$, $d_{10} = 1$ we can use the following command lines

1.	series y=nile-@mean(nile)
2.	for !i=1 to 10
3.	!d=0.1*!i
4.	y.fracdiff(d=!d)
5.	rename y_diff x!i
6.	next

The output is x_1, x_2, $\cdots$, x_{10}, where

$$x_{i,t} = (1 - L)^{0.1*i} y_t$$

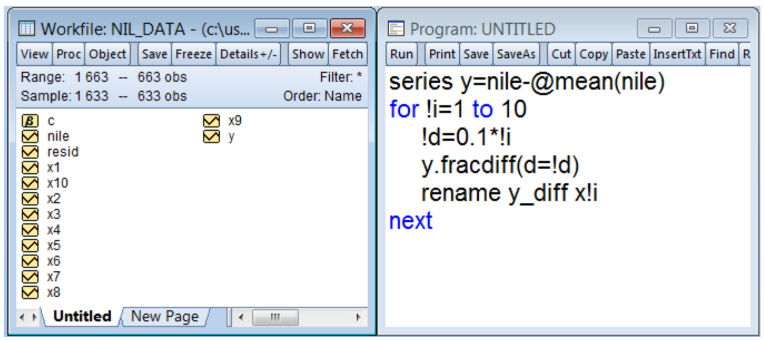

Fig. 6: Compute many different fractional difference of Nile series

We give in the following the program that run this sequential testing procedure on the Nile series. The first 30 lines constitute the main program. The rest of the program consists of an understandable formatting of the results in a table. The figure 5, show how the results are displayed in Eviews after the execution of the EViews sequential FDF program.

****

* Sequential fractional Dickey-Fuller test with Eviews *

****

series y=nile-@mean(nile)

vector (10) Accept_Reject_H0

'______________________________

' Set of sequential values of d_0

'______________________________

for !i=1 to 10

!d0=0.1*!i

!d=-1+0.1*!i

'_____________________________-

'Compute $x(t) = (1 - L)^{d_0 - 1} y(t)$

'_____________________________-

```
y.fracdiff(d=!d, )
rename y_diff x!i
'____________________________________________________________
'Testing the null H0 : d ≥ d0 by means the t-stat of c(1)
'coefficient in the model (1 − L)^d0 y_t = c(1) ∗ (1 − L)^(d0−1) y_(t−1) + ε_t
'____________________________________________________________
equation eq!i.ls d(x!i) x!i(-1)
if eq!i.@tstat(1)>-1.94 then Accept_Reject_H0(!i)=1 else Accept_Reject_H0(!i)=0
endif
delete x!i
next
'________________________________________
'Find the lower and upper bound of d
'________________________________________
for !i= 1 to 9
if Accept_Reject_H0(!i)=1 and Accept_Reject_H0(!i+1)=0 then
scalar Lower_bound_of_d=0.1*!i
scalar Upper_bound_of_d=0.1*(!i+1)
endif
next
'____________________
' display results in table
'____________________
table tab1
setcolwidth(tab1,1,20)
tab1(1,1)="Table 1"
tab1(2,1)="Sequential testing procedure,"
tab1(3,1)="of the standard Dickey-Fuller test"
tab1(4,1)="applied to the nile series: H0 : d >= d0,i"
tab1(4,2)=" "
tab1(4,3)=" "
tab1(4,4)=" "
tab1(4,5)=" "
setline(tab1,5)
tab1(6,1)="d0,i"
tab1(6,2)="d0,i-1"
tab1(6,3)="DFt"
tab1(6,4)="DF cv(5%)" 'critical value of the Dickey-Fuller distribution
tab1(6,5)="Accept_Reject_H0"
```

```
setline(tab1,7)
tab1(8,1)="0.1"
tab1(8,2)="0.9"
tab1(8,3)=eq1.@tstat(1)
tab1(8,4)="-1.94"
tab1(8,5)=Accept_Reject_H0(1)
setline(tab1,9)
tab1(10,1)="0.2"
tab1(10,2)="0.8"
tab1(10,3)=eq2.@tstat(1)
tab1(10,4)="-1.94"
tab1(10,5)=Accept_Reject_H0(2)
setline(tab1,11)
tab1(12,1)="0.3"
tab1(12,2)="0.7"
tab1(12,3)=eq3.@tstat(1)
tab1(12,4)="-1.94"
tab1(12,5)=Accept_Reject_H0(3)
setline(tab1,13)
tab1(14,1)="0.4"
tab1(14,2)="0.6"
tab1(14,3)=eq4.@tstat(1)
tab1(14,4)="-1.94"
tab1(14,5)=Accept_Reject_H0(4)
setline(tab1,15)
tab1(16,1)="0.5"
tab1(16,2)="0.5"
tab1(16,3)=eq5.@tstat(1)
tab1(16,4)="-1.94"
tab1(16,5)=Accept_Reject_H0(5)
setline(tab1,17)
tab1(18,1)="0.6"
tab1(18,2)="0.4"
tab1(18,3)=eq6.@tstat(1)
tab1(18,4)="-1.94"
tab1(18,5)=Accept_Reject_H0(6)
setline(tab1,19)
tab1(20,1)="0.7"
tab1(20,2)="0.3"
```

```
tab1(20,3)=eq7.@tstat(1)
tab1(20,4)=”-1.94”
tab1(20,5)=Accept_Reject_H0(7)
setline(tab1,21)
tab1(22,1)=”0.8”
tab1(22,2)=”0.2”
tab1(22,3)=eq8.@tstat(1)
tab1(22,4)=”-1.94”
tab1(22,5)=Accept_Reject_H0(8)
setline(tab1,23)
tab1(24,1)=”0.9”
tab1(24,2)=”0.1”
tab1(24,3)=eq9.@tstat(1)
tab1(24,4)=”-1.94”
tab1(24,5)=Accept_Reject_H0(9)
setline(tab1,25)
tab1(26,1)=”1”
tab1(26,2)=”0”
tab1(26,3)=eq10.@tstat(1)
tab1(26,4)=”-1.94”
tab1(26,5)=Accept_Reject_H0(10)
setline(tab1,27)
tab1(28,1)=”Lower bound of d=”+@str(Lower_bound_of_d)
tab1(29,1)=”Upper bound of d=”+@str(Upper_bound_of_d)
tab1(30,1)=@str(Lower_bound_of_d)+”=< d <”+@str(Upper_bound_of_d)
show tab1
```

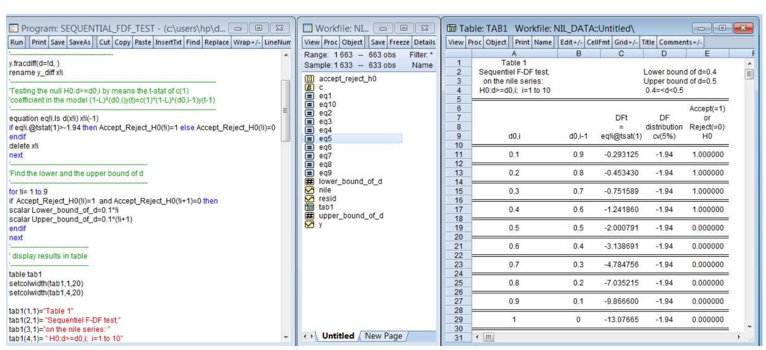

Fig. 7: Output of the execution of the Eviews sequential FDF program

Table 1 Sequentiel F-DF test, on the nile series: H0:d>=d0,i; i=1 to 10			Lower bound of d=0.4 Upper bound of d=0.5 0.4=<d<0.5	
d0,i	d0,i-1	DFt = eq!i@tsat(1)	DF distribution cv(5%)	Accept or Reject H0
0.1	0.9	−0.293125	-1.94	1.000000
0.2	0.8	−0.453430	-1.94	1.000000
0.3	0.7	−0.751589	-1.94	1.000000
0.4	0.6	−1.241860	-1.94	1.000000
0.5	0.5	−2.000791	-1.94	0.000000
0.6	0.4	−3.138691	-1.94	0.000000
0.7	0.3	−4.784756	-1.94	0.000000
0.8	0.2	−7.035215	-1.94	0.000000
0.9	0.1	−9.866600	-1.94	0.000000
1	0	−13.07665	-1.94	0.000000

TABLE I: Output of the execution of the Eviews sequential FDF program