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Abstract

We study a connections model where the strength of a link depends on the amount invested in it and is determined by an increasing strictly concave function. The revenue from investments in links is the information that the nodes receive through the network. First, the structures of efficient networks are characterized, and conditions for optimal investments constrained to supporting a given network are obtained. Second, assuming that links are the result of investments by the node-players involved, there is the question of stability. We introduce and characterize a notion of marginal equilibrium weaker than that of Nash equilibrium, and identify different marginally stable structures. Efficiency and stability are shown to be incompatible, but partial subsidizing is shown to be able to bridge the gap.

JEL Classification Numbers: A14, C72, D85

Key words: Networks, Connections model, Decreasing returns, Efficiency, Stability.

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[*]At the root of this paper is an unpublished working paper "A marginalist model of network formation" (Olaizola and Valenciano, 2016). However, the content of the current paper is much richer in results, and has been refined in accuracy, rigor and generality. Moreover, the study of stability is centered on an entirely new notion of "marginal" equilibrium, which was absent in the old working paper.

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1 Introduction

This paper seeks to contribute to the literature on economic models of strategic network formation. In this line of work, an increasing flow of research has been produced by game-theorists and economists in general since Myerson (1977) and Aumann and Myerson (1988).[1] In the wake of these pioneering papers in the field, two seminal influential models of network formation are Jackson and Wolinsky's (1996) connections model and Bala and Goyal's (2000) non-cooperative two-way flow model. In both models, networks are the result of creating links between pairs of individuals, by bilateral agreements in the former and unilateral decisions in the second, enabling information to flow through the resulting network. In both models, the cost of a link and its strength or quality (i.e. its decay factor) are exogenously given, giving rise to two-parameter models. The simplicity of these basic models imposes some rigidity: Necessarily bilateral formation and compulsory equal share of the fixed cost of each link in Jackson and Wolinsky's model; and unilateral formation requiring full-covering of that fixed cost by its creator in Bala and Goyal's model, and a fixed level of quality for the resulting link in both. The point of this paper is to provide and develop a more flexible model in both link-formation and link-performance.

We develop a model of network formation where links are the result of investments and the quality or strength of a link, i.e. the fidelity level of transmission through it, is never perfect and depends on the amount invested in it. A decreasing returns link-formation technology determines the quality of the resulting link as a function of the investment and is the only exogenous ingredient in the model. Formally, a decreasing returns link-formation technology is a differentiable, increasing, strictly concave function whose range is $[0, 1)$, i.e. however much is invested in a link, transmission is never perfect. The revenue from investment in links is, as in the seminal models, the information that the nodes receive through the network that results.

The question of efficiency is addressed first. It is established that the only possible non-empty efficient architectures for a decreasing returns link-formation technology are the complete network and the all-encompassing star, whose precise structures are also established. The family of decreasing returns link-formation technologies which have one of these non-empty structures as efficient is also characterized. Conditions for optimal investments constrained to supporting a given network are also obtained.

We then consider a decentralized context where links are formed according to a decreasing returns technology available to all players, and each link is the result of investments by the node-players that it connects, whose investments are assumed to be perfect substitutes. In this game-theoretic scenario the question of stability in the underlying network-formation game arises. We first examine a notion of *marginal* equilibrium of a classical flavor which is natural in this marginalist model but new in networks literature to the best of our knowledge. In a marginal equilibrium every

[1] Goyal (2007), Jackson (2008) and Vega-Redondo (2007) are excellent monographs on social and economic networks. See also Bramoullé, Galeotti and Rogers, Eds. (2015).

player is playing a *locally* best response. More precisely, an investment profile is a marginal equilibrium if the investment vector of every player in the links in which he/she is involved is locally optimal, in the sense that sufficiently small changes of these investments do not increase his/her payoff. Necessary and sufficient conditions for marginal stability are established by imposing that the marginal benefit of the investment of any player in each of his/her links must be zero. The characterizing conditions that result from this classical economic principle have a clear intuitive interpretation which permits us to identify a variety of marginally stable architectures and their precise structures. At the same time, given that marginal stability is weaker than Nash-stability, these conditions are necessary for Nash equilibrium.

A comparison of the results on efficiency and stability yields the conclusion that non-empty efficient structures are not stable, not even marginally, and vice versa. Nevertheless, it is proven that subsidizing up to half the cost of each link bridges the gap between efficiency and marginal stability.

The paper is organized as follows. Section 2 introduces basic notation and terminology. Section 3 introduces the model. Section 4 addresses the question of efficiency, first in general, characterizing efficient networks (4.1), and then the question of the efficient support of a given "infrastructure" specified by a set of feasible links (4.2). Section 5 is devoted to stability: Marginal stability (5.1) and Nash stability (5.2). Section 6 examines the incompatibility of efficiency and stability, and shows how a partial subsidy can bridge the gap. Section 7 briefly reviews some related literature. Finally, Section 8 summarizes the results and suggests some possible extensions of the model. All proofs are relegated to an Appendix.

2 Preliminaries

An *undirected weighted network* (shortened in what follows to a *network*) is a pair (N, g) where $N = \{1, 2, ..., n\}$ with $n \geq 3$ is a set of *nodes* and g is a set of *links* specified by a symmetric *adjacency matrix* $g = (g_{ij})_{i,j \in N}$ of real numbers $g_{ij} \in [0, 1)$, with $g_{ii} = 0$ for all i. Alternatively, g can be specified as a map $g : N_2 \to [0, 1)$, where N_2 denotes the set of all subsets of N with cardinality 2. When no ambiguity arises we omit N and refer to g as a network. In what follows $\underline{ij}$ stands for $\{i, j\}$ and $g_{\underline{ij}}$ for $g(\{i, j\})$ for any $\{i, j\} \in N_2$.[2] When $g_{\underline{ij}} > 0$ it is said that a *link of weight* $g_{\underline{ij}}$ connects i and j. $N^d(i; g) := \{j \in N : g_{\underline{ij}} > 0\}$ denotes the set of *neighbors* of node i, and its cardinality is the *degree* of i. A *path* connecting nodes i and j is a sequence of distinct nodes of which the first is i, the last is j, and every two consecutive nodes are connected by a link. If i and j are two consecutive nodes in a path p, we write $ij \in p$ or $\underline{ij} \in p$. $\mathcal{P}_{ij}(g)$ denotes the set of paths in g connecting i and j. $N(i; g)$ denotes the set of nodes connected to i by a path. A network is *connected* if any two

[2] The convenience of the distinction between ij and $\underline{ij}$, especially as subindexes, will be apparent in Section 5. With this convention $x_{\underline{ij}} \equiv x_{\underline{ji}}$, while $x_{ij} \neq x_{ji}$ in general.

nodes are connected by a path. A *subnetwork* of a network (N, g) is a network (N', g') s.t. $N' \subseteq N$ and $g' \subseteq g$. A *component* of a network (N, g) is a maximal connected subnetwork. An isolated node (i.e. not connected to any other) is a *trivial component*. A network has a *cycle* if there are two nodes connected by a link and also by a path of length 2 or more (the *length* of a path is the number of links that it contains, i.e. the number of nodes minus 1).

When the codomain of g is $\{0, 1\}$ instead of $[0, 1)$, i.e. $g_{\underline{ij}}$ only takes the values 0 or 1, we say that g is *a graph* and it can be specified as a set of links $S \subseteq N_2$. In particular, the non-weighted *underlying graph* S_g of a weighted network g is $S_g := \{\underline{ij} \in N_2 : g_{\underline{ij}} > 0\}$. When a given graph $S \subseteq N_2$ constrains the construction of a network which must have it as its underlying graph, we call S an *infrastructure*.

The *empty network/graph* is the one for which $g_{\underline{ij}} = 0$ for all $\underline{ij} \in N_2$. A *complete network/graph* is one where $g_{\underline{ij}} > 0$ for all $\underline{ij} \in N_2$. A *subcomplete network/graph* has only one non-trivial component which is a complete subnetwork, i.e. $g_{\underline{ij}} > 0$ if and only if $\underline{ij} \in M_2$ for some $M \subseteq N$. A *star network/graph* is one with only one non-trivial component with k nodes ($3 \leq k \leq n$) and $k-1$ links in which one node (the *center*) is connected by a link with each of the other $k-1$ nodes. A *tree network/graph* is one with only one non-trivial component and no cycles. A *circle network/graph* has only one non-trivial component with k nodes ($3 \leq k \leq n$) and k links, each of them connecting one node with the next one and the last one with the first one for a given ordering of the k nodes. A tree, a star or a circle network/graph is said to be *all-encompassing* if $k = n$.

3 The model

As in the seminal connections models of Jackson and Wolinsky (1996) and Bala and Goyal (2000), we consider a set of *nodes* or *players*, each of them endowed with an information of value $v > 0$ to any other node that receives it intact. The main difference between our model, briefly sketched in the introduction and to be formalized in detail now, and the seminal models concerns link-formation. In Olaizola and Valenciano (2020), a *link-formation technology* is a non-decreasing map $\delta : \mathbb{R}_+ \to [0, 1)$ s.t. $\delta(0) = 0$. If c is the amount invested in a link to connect two nodes, $\delta(c)$ is the level of fidelity of the transmission of information through the link. More precisely, $\delta(c)$ is the fraction of information flowing through the link that remains intact.[3] Flow occurs only through links invested in ($\delta(0) = 0$), but perfect fidelity in transmission between different nodes is never reached ($0 \leq \delta(c) < 1$). In this paper we assume a *decreasing returns* link-formation technology.

[3]Nevertheless, other interpretations are possible. For instance, the "strength of a tie" (Granovetter, 1973), i.e. a measure of the quality/intensity/value of a relationship e.g. in personal relationships, where the quality/strength of a link is a function of the investments of each of the two people involved. A link can also be a means for the flow of other goods, but we give preference here to the interpretation in terms of information.

Definition 1 *A decreasing returns link-formation technology (DR-technology for short) is a differentiable map $\delta : \mathbb{R}_+ \rightarrow [0, 1)$ s.t. $\delta(0) = 0$, and satisfies the following conditions:*

$(C.1)$ $\delta'(c) > 0$, *for all $c \geq 0$, i.e. it is increasing.*

$(C.2)$ *It is strictly concave.*

Assuming smoothness of δ makes it possible to use differential calculus, which allows for a relatively simple formal marginal analysis without getting involved in more sophisticated technical issues. $C.2$ amounts to assuming technology to be decreasing returns.

We consider the following model based on this basic ingredient. A set $N = \{1, 2, ..., n\}$ of nodes or players can be connected by links formed according to a given decreasing returns link-formation technology δ. Players can invest in links with other nodes. An *investment profile* is specified by a matrix $\mathbf{c} = (c_{ij})_{i,j \in N}$, where $c_{ij} \geq 0$ (with $c_{ii} = 0$) is the investment of player i in the link connecting players i and j, and determines a *link-investment vector* $\overline{\mathbf{c}}$:

$$\mathbf{c} \rightarrow \overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2}, \text{ where } c_{\underline{ij}} := c_{ij} + c_{ji},$$

which in turn, through the link-formation technology available, δ, yields a weighted network denoted by $g^{\mathbf{c}}$ or by $g^{\overline{\mathbf{c}}}$, where

$$g^{\mathbf{c}}_{\underline{ij}} = g^{\overline{\mathbf{c}}}_{\underline{ij}} = \delta(c_{\underline{ij}}) = \delta(c_{ij} + c_{ji}).$$

Thus players' efforts are perfect substitutes. Let $\mathcal{P}_{ik}(g^{\mathbf{c}})$ denote the set of paths in $g^{\mathbf{c}}$ connecting i and k. For a path $p \in \mathcal{P}_{ik}(g^{\mathbf{c}})$, let $\delta(p)$ denote the product of the fidelity levels through each link in that path, i.e. if $p = ii_2i_3...i_mk$, then $\delta(p) = \delta(c_{\underline{ii_2}})\delta(c_{\underline{i_2i_3}})...\delta(c_{\underline{i_mk}})$. Thus, player i values information originating from k that arrives via p by $v\delta(p)$. As in Jackson and Wolinsky (1996) and Bala and Goyal (2000), we assume that player i's valuation of the information originating from $k \neq i$, denoted by $I_{ik}(g^{\mathbf{c}})$, is that which is routed via the best possible route from k, that is

$$I_{ik}(g^{\mathbf{c}}) = \max_{p \in \mathcal{P}_{ik}(g^{\mathbf{c}})} v\delta(p) = v \max_{p \in \mathcal{P}_{ik}(g^{\mathbf{c}})} \delta(p) = v\delta(\overline{p}_{ik}),$$

where $\overline{p}_{ik}$ is an *optimal path connecting i and k*, i.e. $\overline{p}_{ik} \in \arg\max_{p \in \mathcal{P}_{ik}(g^{\mathbf{c}})} \delta(p)$ (if no path connects i and k we set $\delta(\overline{p}_{ik}) = 0$). Then i's overall revenue from $g^{\mathbf{c}}$ is

$$I_i(g^{\mathbf{c}}) = \sum_{k \in N(i;g^{\mathbf{c}})} I_{ik}(g^{\mathbf{c}}).$$

Thus, i's *payoff* is the value of the information received by i minus i's investment:

$$\Pi^{\delta}_i(\mathbf{c}) := I_i(g^{\mathbf{c}}) - C_i(\mathbf{c}) = \sum_{k \in N(i;g^{\mathbf{c}})} v\delta(\overline{p}_{ik}) - \sum_{j \in N^d(i;g^{\mathbf{c}})} c_{ij}, \tag{1}$$

and the *net value* of the network resulting is the aggregate payoff, i.e. the total value of the information received by the nodes minus the total cost of the network:

$$v(g^{\mathbf{c}}) := \sum_{i \in N} \Pi_i^{\delta}(\mathbf{c}) = \sum_{i \in N} I_i(g^{\mathbf{c}}) - \sum_{\underline{ij} \in N_2} c_{\underline{ij}} = \sum_{\underline{kl} \in N_2} 2v\delta(\overline{p}_{kl}) - \sum_{\underline{ij} \in N_2} c_{\underline{ij}}. \tag{2}$$

In this setting two main issues arise. A game in strategic form, where a *strategy* of a player i is a vector of investments ($\mathbf{c}_i = (c_{ij})_{j \in N}$, with $c_{ii} = 0$) and the *payoff function* is given by (1), is implicitly defined. Thus the question of stability arises: What structures are stable and under what conditions? The notion usually applied in a context such as this is Nash equilibrium: An investment profile is *Nash-stable* if no player has an incentive to change his/her investment vector. Nevertheless, we first devote particular attention to a weaker notion of stability new in this context: Marginal equilibrium. A second issue is the question of efficiency: What structures are efficient in the sense of maximizing the net value given by (2) and under what conditions?

We address the question of efficiency first, and then look at stability. Thus we deal with a model with two parameters, the number of nodes/players n and the value v of the information at each node. A third "parameter" is the link-formation technology represented by function δ.[4]

4 Efficiency

4.1 Efficient networks

In the model just described, the net value of a network $g^{\mathbf{c}}$, given by (2), that results from an investment profile $\mathbf{c} = (c_{ij})_{i,j \in N}$, depends entirely on $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2}$, where $c_{\underline{ij}} := c_{ij} + c_{ji}$. In other words, given that players' efforts are perfect substitutes, the question of efficiency depends entirely on the investments in every link, but it is immaterial who pays for them. Thus the answer to the question of efficiency is the same, regardless of whether the investments are made by node-players in a decentralized way or by a central planner. For this reason we give preference in this section to expressing results in terms of investment vectors and $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2}$ and the resulting network $g^{\overline{\mathbf{c}}}$. In Olaizola and Valenciano (2020) it is proved that *for any link-formation technology* δ*,* i.e. any δ non-decreasing and s.t. $\delta(0) = 0$, the only possibly efficient non-empty networks are the all-encompassing star, the complete network and, under certain conditions, also a whole range of intermediate particular nested split graph structures.[5]

[4] It can be assumed w.l.o.g. that $v = 1$, which slightly simplifies the presentation. However, it is preferable not to do so and to keep this otherwise hidden parameter explicit. If investments in links are made by a planner, this value can be interpreted as a subjective evaluation by the planner w.r.t. which the efficiency objective is specified. Nevertheless, the reader may choose to ignore all occurrences of v by assuming $v = 1$.

[5] A precise formulation of the conditions for this exception to occur is needed to prove the characterizing result. This is given in the proof of Theorem 1.

This conclusion thus also applies to DR-technologies. We first show the necessary conditions for a star and a complete network to be efficient for a DR-technology, based on the conditions obtained in Olaizola and Valenciano (2020), which will enable us to refine these conclusions for DR-technologies.

Proposition 1 *For a complete network $g^{\bar{\mathbf{c}}}$ to be efficient under a DR-technology δ, the following conditions are necessary:*
(i) $\delta'(0) > 1/2v$.
(ii) For all $\underline{ij} \in N_2$, $c_{\underline{ij}} = \widehat{c}_{ef}$, *where*

$$\widehat{c}_{ef} = \arg\max_{c>0}(2v\delta(c) - c), \tag{3}$$

or, equivalently,

$$\delta'(\widehat{c}_{ef}) = 1/2v. \tag{4}$$

(iii) $2v\delta(\widehat{c}_{ef})^2 \leq 2v\delta(\widehat{c}_{ef}) - \widehat{c}_{ef}$.

Therefore, in an efficient complete network *all links are of the same strength*, $\widehat{c}_{ef}$ s.t. (3) or, which is equivalent for a DR-technology, s.t. (4). Note that there is certain to be a unique $\widehat{c}_{ef} > 0$ s.t. (3) and (4) if and only if $\delta'(0) > 1/2v$.

In order to establish the structure of an efficient all-encompassing star, we first prove the necessary symmetry of an optimal star, i.e. a star with the highest net value, for *any* technology.

Lemma 1 *For any link-formation technology for which some star yields a positive net value, the optimal star is all-encompassing and all its links have the same strength.*

Proposition 2 *For an all-encompassing star $g^{\bar{\mathbf{c}}}$ to be efficient under a DR-technology δ, the following conditions are necessary:*
(i) All links receive the same investment c^*_{ef} s.t.

$$c^*_{ef} \in \arg\max_{c>0}\left(2v\delta(c) + (n-2)v\delta(c)^2 - c\right), \tag{5}$$

for which a necessary condition is

$$\delta'(c^*_{ef}) = \frac{1}{2v(1 + (n-2)\delta(c^*_{ef}))}. \tag{6}$$

(ii) Additionally, if $\max_{c>0}(2v\delta(c) - c) > 0$,

$$2v\delta(c^*_{ef})^2 \geq 2v\delta(c) - c \tag{7}$$

for all $c > 0$.

The following result shows that the existence of an optimal symmetric star is guaranteed unless the technology is "too bad" in a precise sense, but whatever the DR-technology if n is big enough.

Proposition 3 *For a DR-technology δ, there is an optimal all-encompassing star unless $\delta(c) \leq \varphi_n(c)$ for all $c \geq 0$, where*

$$\varphi_n(c) := \frac{-1 + \sqrt{1 + (n-2)c/v}}{n-2}. \tag{8}$$

For every DR-technology, there is an optimal all-encompassing star if n is big enough.

Therefore function $\varphi_n(c)$, defined by (8), sets a precise bound below which a technology is poor enough to make the formation of any star non-profitable. Notice that, as can easily be checked, $\varphi_n(0) = 0$, $\varphi_n'(c) > 0$, $\varphi_n''(c) < 0$, and consequently function φ_n meets all *but one* of the conditions for a DR-technology as per Definition 1: for a big enough c (for $c > nv$, in fact) $\varphi_n(c) > 1$. In other words, constraint $\delta(c) \leq \varphi_n(c)$ is actually active as far as $\varphi_n(c) < 1$, i.e. for $c \in (0, nv)$ (note that $\varphi_n(nv) = 1$ for all n).

Then we have a characterizing result.

Theorem 1 *Under a DR-technology δ: (i) The only non-empty possibly efficient networks are the complete network described in Proposition 1 and an all-encompassing star (as described in Proposition 2). (ii) The empty network is efficient if and only if $\delta(c) \leq \varphi_n(c)$ for all c, with φ_n given by (8). Otherwise, either the complete network or an all-encompassing star is efficient.*

Namely, *for any DR-technology worse than φ_n, i.e. whose graph is below that of φ_n, and only for such DR-technologies, no all-encompassing star and no complete network yields a positive net value.* Figure 1 illustrates this, showing the graph of function φ_n for $v = 1$ and different numbers of nodes: $n = 5, 12, 22$ and 42. The greater the number of nodes, the lower the graph of this function is, i.e. the worse the technology must be to make any star unprofitable. Two dashed lines represent the graphs of two DR-technologies: $\delta_1(c) = \frac{c}{1+c}$ and $\delta_2(c) = \frac{c}{2+2c}$. Obviously, technology δ_2 is worse than δ_1. Thus, for instance, for $n = 5$ no symmetric star or complete network yields a positive net value under δ_2, while under δ_1 there are sure to be both optimal complete and star networks. For $n = 22$ there are sure to exist both optimal complete and star networks under both technologies, δ_1 and δ_2.

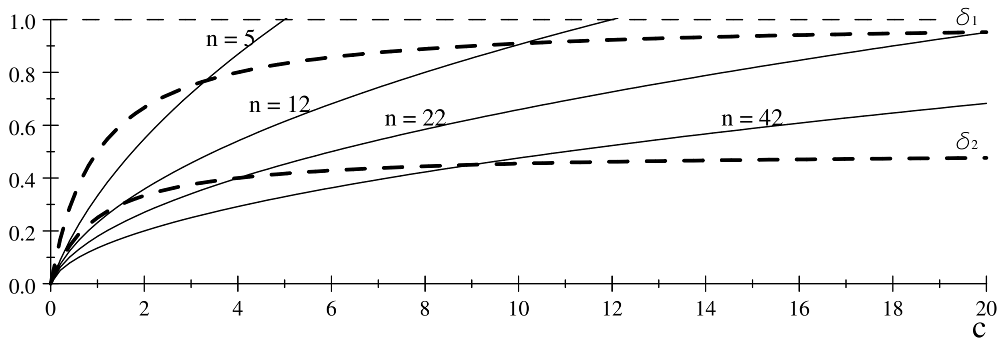

Figure 1: Graph of φ_n for $v = 1$ and $n = 5, 12, 22, 42$, and technologies δ_1 and δ_2

4.2 Efficient support of an infrastructure

Now consider the situation where a given infrastructure specified by a set of feasible links $S \subseteq N_2$ is to be supported in the most efficient way. We say that an investment profile $\mathbf{c} = (c_{ij})_{i,j \in N}$ *supports* S if the underlying graph of $g^{\overline{\mathbf{c}}}$ is S, i.e. if $c_{\underline{ij}} > 0$ if and only if $\underline{ij} \in S$.

Definition 2 *Given $S \subseteq N_2$, an investment profile $\mathbf{c}^* = (c^*_{ij})_{i,j \in N}$ supports S efficiently if it supports S and $v(g^{\overline{\mathbf{c}}^*}) \geq v(g^{\overline{\mathbf{c}}})$, for all $\mathbf{c} = (c_{ij})_{i,j \in N}$ which support S.*

That is, investments are constrained to be made in all links in S and only in them, and the function to be maximized is

$$v(g^{\overline{\mathbf{c}}}) = 2v \sum_{\underline{kl} \in N_2} \delta(\overline{p}_{kl}) - \sum_{\underline{kl} \in S} c_{\underline{kl}}. \tag{9}$$

The following result establishes necessary conditions for an investment vector to support an infrastructure S efficiently using the following notation: if $\overline{p}_{kl}$ is an optimal path connecting nodes k and l s.t. $ij \in \overline{p}_{kl}$, define:

$$\delta(\overline{p}^{ij}_{kl}) := \frac{\delta(\overline{p}_{kl})}{\delta(c_{\underline{ij}})}. \tag{10}$$

In other words, $\overline{p}^{ij}_{kl}$ can be seen as a path from k to l which results from replacing link $\underline{ij}$ in path $\overline{p}_{kl}$ by a "perfect" link with no decay. In particular, if $\{k, l\} = \{i, j\}$, $\delta(\overline{p}^{ij}_{ij}) := 1$.

Proposition 4 *Let δ be a DR-technology. For a link-investment vector $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij} \in N_2}$ that supports an infrastructure $S \subseteq N_2$ to do so efficiently the following conditions are* *necessary. For any two connected nodes in $g^{\overline{\mathbf{c}}}$ there must be a unique optimal path connecting them, and for each $\underline{ij} \in S$,*

$$\delta'(c_{\underline{ij}}) = \frac{1}{2v \sum_{\underline{kl} \in N_2 \ s.t. \ \underline{ij} \in \overline{p}_{kl}} \delta(\overline{p}_{kl}^{ij})}. \tag{11}$$

Two comments are worth making here. First, note that (11) has a clear interpretation. The denominator of its right-hand side is the total amount of information that crosses link $\underline{ij}$ (subject to a decay $\delta(c_{\underline{ij}})$), i.e. between all pairs of nodes whose optimal connecting path contains link $\underline{ij}$. The greater this amount the greater the denominator is and the smaller the quotient, i.e. the smallest $\delta'(c_{\underline{ij}})$ and consequently the greater its strength $\delta(c_{\underline{ij}})$. Second, as the investments in an efficient complete network and in an efficient all-encompassing star both efficiently support the infrastructure of their underlying graph, (4) and (6) are particular cases of (11).

5 Stability

Whether investments are made by a planner or in a decentralized way by node-players is immaterial in addressing the question of efficiency, but now we consider the situation where nodes are *players* who form links by investing in them and using an available DR-technology. An investment profile $\mathbf{c} = (c_{ij})_{i,j \in N}$ (an $n \times n$ matrix with zeros in the main diagonal) where $c_{ij} \geq 0$ is the investment of player i in the link connecting players i and j, actually represents a strategy profile, where its i-row, $\mathbf{c}_i = (c_{ij})_{j \in N}$ with $c_{ii} = 0$, is the strategy of player i, whose payoff is given by[6]

$$\Pi_i^\delta(\mathbf{c}) = v \sum_{k \in N(i;g^{\mathbf{c}})} \delta(\overline{p}_{ik}) - \sum_{j \in N^d(i;g^{\mathbf{c}})} c_{ij}. \tag{12}$$

This situation raises the question of stability. We first consider a weak form of stability which is, as far as we know, new in network literature, but quite natural in the context of this "marginalist" model. Moreover, in addition to its interest per se, its characterization provides necessary conditions for stronger notions of stability.

5.1 Marginal stability

If $\mathbf{c} = (c_{ij})_{i,j \in N}$ is an investment profile and $\mathbf{c}'_i = (c'_{ij})_{j \in N}$ an investment vector of player i, let $(\mathbf{c}_{-i}; \mathbf{c}'_i)$ denote the investment profile that results from replacing row i in $\mathbf{c}$ by $\mathbf{c}'_i$.

Definition 3 *An investment profile $\mathbf{c} = (c_{ij})_{i,j \in N}$ is marginally stable (or a marginal equilibrium) if for some $\varepsilon > 0$ the following holds: for all $i \in N$ and all $\mathbf{c}'_i = (c'_{ik})_{k \in N}$ s.t. $c'_{ik} > 0$ only if $c_{\underline{ik}} > 0$ and $|c_{ik} - c'_{ik}| < \varepsilon$ for all k, $\Pi_i^\delta(\mathbf{c}) \geq \Pi_i^\delta(\mathbf{c}_{-i}; \mathbf{c}'_i)$.*

[6] In what follows whenever we refer to an investment vector $\mathbf{c}_i = (c_{ij})_{j \in N}$ the condition $c_{ii} = 0$ is always assumed although omitted for the sake of brevity.

In other words, an investment profile is marginally stable if the investments of every node in its links are *locally* optimal, in the sense that sufficiently small changes in its investments in the links in which it is involved do not increase its payoff.

It is worth emphasizing the interest of this weak notion of equilibrium per se. In this model, Nash equilibrium poses computational and informational difficulties. Apart from the computational difficulties of calculating best responses in a complex network, it requires a huge amount of information. Moreover, if the network is the means of transmission of information, how do players know about the revenue from links with players with whom they are not directly or even indirectly connected? If players are only aware of the marginal contribution of their investments *in links in which they are actually involved* (a much weaker assumption about their information) a marginal equilibrium means that no player receives signals inducing him/her to change his/her investments and the situation will remain unchanged. Note that the clause "s.t. $c'_{ik} > 0$ only if $c_{\underline{ik}} > 0$" restricts responses to existing links. In other words, the creation of new links is *not* a response w.r.t. which a marginal equilibrium must be immune. A stronger variant of Definition 3 closer to Nash equilibrium, but still weaker, is obtained by eliminating this clause.

Definition 4 *An investment profile* $\mathbf{c} = (c_{ij})_{i,j \in N}$ *is strongly marginally stable* (*or a strong marginal equilibrium*) *if for some* $\varepsilon > 0$ *the following holds: for all* $i \in N$ *and all* $\mathbf{c}'_i = (c'_{ik})_{k \in N}$ *s.t.* $|c_{ik} - c'_{ik}| < \varepsilon$ *for all* k, $\Pi_i^\delta(\mathbf{c}) \geq \Pi_i^\delta(\mathbf{c}_{-i}; \mathbf{c}'_i)$.

Although strictly speaking one should refer to stability of investment profiles, we often express our results in terms of the resulting networks. Thus a "(strongly) marginally stable network" should be read as a *weighted network that results from a* (*strongly*) *marginally stable investment profile*. The following lemma shows that the two notions are equivalent for connected networks.

Lemma 2 *Let* $\mathbf{c} = (c_{ij})_{i,j \in N}$ *be an investment profile. If* $g^{\mathbf{c}}$ *is connected, then* $\mathbf{c}$ *is strongly marginally stable if and only if it is a marginal equilibrium.*

The following result establishes a necessary and sufficient condition for the *empty network* to be (strongly) marginally stable.

Proposition 5 *Let* δ *be a DR-technology. The empty network is marginally stable whatever the technology, and is strongly marginally stable if and only if*

$$\delta'(0) \leq 1/v.$$

It is convenient to introduce some notation in order to formulate and prove the following characterization establishing necessary and sufficient conditions for an investment profile to be marginally stable. Note that expression (12) of the payoff of a player i, involves the choice of an optimal path $\overline{p}_{ik}$ for each $k \in N(i; g^{\mathbf{c}})$. We denote by

$\overline{\boldsymbol{p}}_i = \{\overline{p}_{ik} : k \in N(i; g^{\mathbf{c}})\}$ any particular choice of such optimal paths. We make use of a special case of notation (10): $\delta(\overline{p}_{ik}^{ij}) := \delta(\overline{p}_{ik})/\delta(c_{\underline{ij}})$ whenever $\overline{p}_{ik}$ is an optimal path that contains link ij. If $C_{i,j}^g$ is the set of nodes that are connected with i in g through optimal paths that contain link ij, i.e.

$$C_{i,j}^g := \{k \in N : \exists \overline{p}_{ik} \ s.t.\ ij \in \overline{p}_{ik}\},$$

then, choose for each $k \in C_{i,j}^g$ an optimal path $\overline{p}_{ik}$ s.t. $ij \in \overline{p}_{ik}$ and define

$$K_{i,j}^g := \sum_{k \in C_{i,j}^g} \delta(\overline{p}_{ik}^{ij}).$$

Note that $K_{i,j}^g$ does not depend on the choice of the $\overline{p}_{ik}$'s such that $ij \in \overline{p}_{ik}$ because if $\overline{p}_{ik}$ and $\overline{q}_{ik}$ are two different optimal paths containing ij, then $\delta(\overline{p}_{ik}) = \delta(\overline{q}_{ik})$ and consequently $\delta(\overline{p}_{ik}^{ij}) = \delta(\overline{q}_{ik}^{ij})$. We have then the following result making use of Kuhn-Tucker's conditions:

Theorem 2 *Under a DR-technology δ, for an investment profile $\mathbf{c}^* = (c_{ij}^*)_{i,j \in N}$ to be marginally stable the following conditions are necessary and sufficient. For all $i, j \in N$ $(i \neq j)$ s.t. $c_{\underline{ij}}^* > 0$,*
(i) If $c_{ij}^* > 0$ any optimal path connecting i and k that contains link ij is the only path connecting them and

$$\delta'(c_{\underline{ij}}^*) = \frac{1}{vK_{i,j}^{g^{\mathbf{c}^*}}}. \tag{13}$$

(ii) If $c_{ij}^* = 0$,

$$\delta'(c_{\underline{ij}}^*) \leq \frac{1}{vK_{i,j}^{g^{\mathbf{c}^*}}}. \tag{14}$$

Part (i) establishes that, in a marginal equilibrium, if i sees k through an optimal path in which he/she invests it cannot be the case that i sees k also through another optimal path. In other words, in a marginally stable profile, the optimal paths in which a player invests form a unique tree rooted at that node.

As to (13), it is the result of requiring the *marginal benefit of the investment of any player in each of the links that he/she invests in to be zero.* The resulting condition when $c_{ij}^* > 0$ is

$$\delta'(c_{\underline{ij}}^*) = \frac{1}{vK_{i,j}^{g^{\mathbf{c}^*}}} = \frac{1}{v\sum_{k \in N(i;g^{\mathbf{c}^*})\ s.t.\ \underline{ij} \in \overline{p}_{ik}} \delta(\overline{p}_{ik}^{ij})}.$$

This has a clear interpretation: If player i invests in a link with j, the denominator of the fraction in formula (13) that yields $\delta'(c_{\underline{ij}}^*)$ is v times the sum of the fidelity levels through all subpaths up to j of optimal paths containing link $\underline{ij}$ through which player i receives information. In other words, $vK_{i,j}^{g^{\mathbf{c}^*}}$ is the actual *amount of information that* *reaches j on its optimal way to i.* Thus this sum is a measure of the importance of link $\underline{ij}$ *to player i*: the greater this amount, the smaller $\delta'(c^*_{\underline{ij}})$, i.e. the greater $c^*_{\underline{ij}}$ and $\delta(c^*_{\underline{ij}})$. Condition (ii) means a lack of incentives to invest in a link entirely supported by the other player. Condition (14) ensures that not investing in link $\underline{ij}$ is optimal for i because player j is investing in the link the amount that player i would be willing to invest for all the information that he/she can receive through link $\underline{ij}$ or even more.[7]

Equation (13) and inequality (14), which are necessary conditions for an investment profile $\mathbf{c}^*$ to be marginally stable, involve only the resulting investment vector $\overline{\mathbf{c}}^*$, not directly the investment profile $\mathbf{c}^*$. Nevertheless parts (i) and (ii) actually involve $\mathbf{c}$, because which condition ((13) or (14)), applies for a link $\underline{ij}$, depends on whether $c^*_{ij} > 0$ or $c^*_{ij} = 0$. A direct consequence of these conditions is the following important conclusion.

Corollary 1 *Under a DR-technology δ, if two players are connected by a link in the network resulting from a marginally stable investment profile but do not receive the same amount of information through that link, all the investment in that link is made by the player who receives more information through it.*

As for strong marginal equilibrium we have

Proposition 6 *Under a DR-technology δ, an investment profile $\mathbf{c}^* = (c^*_{ij})_{i,j\in N}$ is a strong marginal equilibrium if and only if in addition to conditions (i) and (ii) of Theorem 2, either $\delta'(0) \leq \frac{1}{v+K}$, where K is the information received by the node that receives the greatest amount of information in $g^{\mathbf{c}^*}$, or $g^{\mathbf{c}^*}$ is connected.*

The rest of this section is devoted to show how Theorem 2 and Corollary 1 can be applied to establish that certain graph architectures, as subcomplete, complete, star, tree or circle graphs, are the result of investment profiles marginally stable and characterize such profiles.

Definition 5 *Given a graph $S \subseteq N_2$, an investment profile $\mathbf{c} = (c_{ij})_{i,j\in N}$ sustains S in marginal equilibrium if it supports S and $\mathbf{c}$ is a marginal equilibrium. When such $\mathbf{c}$ does exist we say that graph S is sustainable in marginal equilibrium.*

The following two propositions refer to subcomplete and star networks.

[7] Alternatively, Theorem 2 can be reformulated like this:
Theorem 2 (reformulated) *Under a DR-technology δ, an investment profile $\mathbf{c}^* = (c^*_{ij})_{i,j\in N}$ is marginally stable if and only if for all $i, j \in N$ $(i \neq j)$ s.t. $c^*_{\underline{ij}} > 0$,*

$$\delta'(c^*_{\underline{ij}}) \leq \frac{1}{vK^{g^{\mathbf{c}^*}}_{i,j}} \quad \textit{and} \quad c^*_{ij}(\delta'(c^*_{\underline{ij}}) - \frac{1}{vK^{g^{\mathbf{c}^*}}_{i,j}}) = 0$$

and whenever $c^*_{ij} > 0$ any optimal path connecting i and k that contains link ij is the only optimal path connecting them.

Proposition 7 *Let δ be a DR-technology and let $\mathbf{c} = (c_{ij})_{i,j \in N}$ be an investment profile such that $g^{\mathbf{c}}$ is subcomplete, then $\mathbf{c}$ is marginally stable if and only if $\delta'(0) > 1/v$ and all links receive the same joint investment $\widehat{c}_{eq} > 0$, such that*

$$\delta'(\widehat{c}_{eq}) = 1/v. \tag{15}$$

If $g^{\mathbf{c}}$ is complete these conditions are necessary and sufficient also for $\mathbf{c}$ to be strongly marginally stable.

Proposition 8 *Let δ be a DR-technology and let $\mathbf{c} = (c_{ij})_{i,j \in N}$ be an investment profile such that $g^{\mathbf{c}}$ is a star connecting p nodes $(3 \leq p \leq n)$, then $\mathbf{c}$ is marginally stable if and only if $g^{\mathbf{c}}$ is a periphery-sponsored star where all peripheral players invest the same amount $c^*_{p,eq}$ in the only link in which each of them is involved s.t.*

$$\delta'(c^*_{p,eq}) = \frac{1}{v(1 + (p-2)\delta(c^*_{p,eq}))}. \tag{16}$$

If $p = n$ (i.e. the star is all-encompassing), these conditions are necessary and sufficient also for $g^{\mathbf{c}}$ to be strongly marginally stable.

In particular, Propositions 7 and 8 establish necessary and sufficient conditions for the only two non-empty architectures that can be efficient, i.e. the complete and the all-encompassing star networks (case $p = n$), to be sustainable in marginal equilibrium. In the case of the complete graph the *existence* of an investment profile satisfying these conditions is guaranteed if the technology satisfies the condition $\delta'(0) > 1/v$. Before addressing the question of existence of $c^*_{p,eq}$ s.t. (16) for a star to be marginally stable we establish a result relative to *any tree* based on a fixed point argument.

Proposition 9 *Under a DR-technology δ continuously differentiable s.t. $\delta'(0) > \frac{1}{v}$, any tree is sustainable in marginal equilibrium, and any all-encompassing tree graph is sustainable in strong marginal equilibrium.*

As mentioned in the proof, by Corollary 1, in the marginally stable profile $\mathbf{c}$ s.t. $g^{\mathbf{c}}$ is a tree, peripheral or terminal nodes must pay the full cost of their links, and the cost of any link where the nodes that it connects do not receive the same amount of information through it must be paid for fully by the node who receives more information through it.

Proposition 10 *Under a DR-technology δ continuously differentiable s.t. $\delta'(0) > \frac{1}{v}$, any star graph of p nodes is sustainable in marginal equilibrium, and also when $\delta'(0) \leq 1/v$ for a sufficiently large p. By contrast, for a fixed p, there is no $c^*_{p,eq}$ s.t. (16) if $\delta'(0) \leq \frac{1}{v(1+(p-2)\delta(\infty))}$, where $\delta(\infty)$ denotes $\lim_{c \to \infty} \delta(c)$.*

Thus even if condition $\delta'(0) > \frac{1}{v}$ does not hold, a star graph continues to be sustainable in marginal equilibrium for a *sufficiently big p*. Compared with this result for the star, condition $\delta'(0) > \frac{1}{v}$ seems rather strong relative to trees. In fact, this condition enables the results of Proposition 9 (first part) and Proposition 10 to be proved for *any number of nodes*[8]. The symmetry of the star enables the precise smaller bound $\delta'(0) \leq \frac{1}{v(1+(p-2)\delta(\infty))}$ to be calculated, while a similar refinement of this bound for an arbitrary tree would require specific study.

The following proposition shows that a circle graph also can be sustained in marginal equilibrium.

Proposition 11 *Under a DR-technology δ continuously differentiable s.t. $\delta'(0) > \frac{1}{v}$, any circle graph of k nodes $(3 \leq k \leq n)$ can be sustained in marginal equilibrium with all links of the same strength, $\delta(\overline{c})$, if k is odd, given by*

$$\delta'(\overline{c}) = \frac{1}{v(1 + \delta(\overline{c}) + \delta(\overline{c})^2 + ... + \delta(\overline{c})^{\frac{k-3}{2}})}; \tag{17}$$

and with links alternating two levels of strength, $\delta(\overline{c})$ and $\delta(\overline{\overline{c}})$, if k is even, given by

$$\delta'(\overline{\overline{c}}) = \frac{1}{v(1 + \delta(\overline{c}) + \delta(\overline{c})\delta(\overline{\overline{c}}) + \delta(\overline{c})^2\delta(\overline{\overline{c}}) + ... + \delta(\overline{\overline{c}})^{\frac{k-4}{4}}\delta(\overline{c})^{\frac{k}{4}})}, \tag{18}$$

and

$$\delta'(\overline{c}) = \frac{1}{v(1 + \delta(\overline{\overline{c}}) + \delta(\overline{c})\delta(\overline{\overline{c}}) + \delta(\overline{\overline{c}})^2\delta(\overline{c}) + ... + \delta(\overline{\overline{c}})^{\frac{k-4}{4}}\delta(\overline{c})^{\frac{k-4}{4}})}. \tag{19}$$

If $k = n$ (i.e. an all-encompassing circle graph) it can be sustained in strong marginal equilibrium.

Remarks: (i) Note that even though the preceding results refer to graphs with only one non-trivial component, be it complete or subcomplete, a tree, a circle or a star graph, it follows immediately that any graph which has trees, circles, stars and subcomplete graphs as non-trivial components also can be sustained in marginal equilibrium if the conditions of Propositions 7-11 hold.[9]

(ii) A property of marginal equilibria worth noting is its resilience in response to shocks such as deletion of nodes under certain conditions. For instance, a marginally stable star network ceases to be so if a spoke node vanishes. Nevertheless, by diminishing the investments of the remaining spoke nodes a new marginal equilibrium sustaining the new star with one arm less can be obtained surely if $\delta'(0) > \frac{1}{v}$ or, otherwise, if the number of nodes is big enough. A similar situation occurs by the elimination of a

[8] Even in the extreme case $n = 2$, where this condition is also necessary.

[9] Moreover, as the reader may check, structures which underlie a marginal equilibrium other than the ones shown here can be obtained by combining them. For instance, a circle in which each node is connected with the same number of peripheral nodes, each of them supporting its link.

node in a tree network. This yields a network with a number of components equal to the degree of the node eliminated: namely, in general, some isolated nodes and some tree networks of smaller diameters. In a circle network the elimination of a node yields a line (particular case of a tree). In all these cases some of the resulting components can possibly be sustained in marginal equilibrium by readjusting the investments of the nodes. In the case of a marginally stable subcomplete network the elimination of a node yields a new marginally stable subcomplete network with a node less.

(iii) The variety of graph architectures that have been shown to be sustainable in marginal equilibrium can be misleading, conveying the impression that every graph is sustainable. This is not so as the following example shows. Consider a star with sufficient number, p, of nodes and a technology δ such that $\delta(\widehat{c}_{eq})$, s.t. (15), and $\delta(c^*_{p-1,eq})$, s.t. (16), verify

$$\delta(\widehat{c}_{eq}) < \delta(c^*_{p-1,eq})^2.$$

Then, the graph that results from adding to a star graph of p nodes a link connecting two spoke nodes is *not* sustainable in marginal equilibrium. The reason is clear: whatever the investments of those two spoke nodes in the link connects them, both nodes would have an incentive to diminish their investment in that link.

5.2 Nash-stability

An investment profile is Nash-stable if no player is interested in changing his/her investments unilaterally. Formally:

Definition 6 *An investment profile* $\mathbf{c} = (c_{ij})_{i,j\in N}$ *is Nash-stable if for all* i *and* $\mathbf{c}'_i = (c'_{ij})_{j\in N}$

$$\Pi_i^\delta(\mathbf{c}) \geq \Pi_i^\delta(\mathbf{c}_{-i};\mathbf{c}'_i).$$

Obviously the notion of marginal equilibrium (strong or not) is weaker than that of Nash equilibrium. Consequently, the characterizing conditions for marginal stability established in Theorem 2 are *necessary* conditions for Nash stability. Thus we have the following results in part as a corollary of the results in the previous section.

Proposition 12 *Let* δ *be a DR-technology. The empty network is a Nash network if and only if*

$$\delta'(0) \leq 1/v.$$

As established in Section 4, the only non-empty possibly efficient networks are the all-encompassing star and the complete network. This raises the question of Nash-stability conditions for these architectures. We first examine the Nash-stability of a complete network focusing on the symmetric one, i.e. a complete network where the cost of each link is equally shared.

Proposition 13 *Let* δ *be a DR-technology and let* $\widehat{\mathbf{c}}_n$ *be an investment profile such that* $g^{\widehat{\mathbf{c}}_n}$ *is complete, symmetric and marginally stable, i.e.* $c_{\underline{ij}} = \widehat{c}_{eq}$ *and* $c_{ij} = \frac{\widehat{c}_{eq}}{2}$*, with* $\delta'(\widehat{c}_{eq}) = 1/v$*. Then* $\widehat{\mathbf{c}}_n$ *is Nash-stable, if and only if for all* $k = 2, ..., n-1$ *:*

$$v\delta(\widehat{c}_{eq}) - \frac{\widehat{c}_{eq}}{2} \geq \frac{\delta(\overline{c}_k)v(1+(k-1)\delta(\widehat{c}_{eq})) - (\overline{c}_k - \frac{\widehat{c}_{eq}}{2})}{k}, \tag{20}$$

where $\widehat{c}_{eq}$ *is s.t.* (15) *and* $\overline{c}_k$ *is s.t.*

$$\delta'(\overline{c}_k) = \frac{1}{v(1+(k-1)\delta(\widehat{c}_{eq}))}. \tag{21}$$

Note that the structure of a Nash complete network is unique (all its links are $\widehat{c}_{eq}$-links), but there may exist *different investment profiles that support it* depending on how the cost of each link is shared.[10] The conditions for Nash equilibrium for a non-symmetric investment profile $\mathbf{c} = (c_{ij})_{i,j \in N}$, s.t. $c_{\underline{ij}} = \widehat{c}_{eq}$ for all $i \neq j$ are much more complicated.

Proposition 14 *Let* δ *be a DR-technology and let* $\mathbf{c}_n^*$ *be an investment profile such that* $g^{\mathbf{c}_n^*}$ *is an all-encompassing periphery-sponsored star which is marginally stable, i.e. the investment in each of its links is* $c_{eq}^* = c_{n,eq}^*$ *s.t.* (16)*. Then if* $\delta'(0) \leq 1/v$*,* $\mathbf{c}_n^*$ *is Nash-stable, while if* $\delta'(0) > 1/v$*,* $\mathbf{c}_n^*$ *is Nash-stable if and only if for all* $k = 1, 2, ..., n-2$ *:*

$$v\delta(c_{eq}^*)^2 - (v\delta(\widehat{c}_{eq}) - \widehat{c}_{eq}) \geq \frac{((n-k-2)\delta(c_{eq}^*)+1)v(\delta(c_{eq}^*) - \delta(\overline{c}_k)) - (c_{eq}^* - \overline{c}_k)}{k}, \tag{22}$$

where $\widehat{c}_{eq}$ *is s.t.* (15) *and* $\overline{c}_k$ *is s.t.*

$$\delta'(\overline{c}_k) = \frac{1}{v(1+(n-k-2)\delta(c_{eq}^*))}. \tag{23}$$

The complexity of conditions (20) and (22) for Nash-stability, even for so simple structures as that of the symmetric complete network and that of a symmetric all-encompassing star, may seem somewhat disappointing compared with the simplicity of the conditions for their marginal stability established in Propositions 7 and 8. This corroborates the computational difficulties that the notion of Nash equilibrium involves, particularly in a richer model like this, and enhances the tractability of marginal equilibrium.

[10] Only if $v(\delta(\widehat{c}_{eq}) - \delta(\widehat{c}_{eq})^2) = \widehat{c}_{eq}/2$, in equilibrium every player would have to invest exactly $\widehat{c}_{eq}/2$ in each of his/her links.

6 Efficiency vs. equilibrium

In view of the results on efficiency and on stability we have the following.

Proposition 15 *Under a DR-technology, efficiency and Nash stability or even marginal stability are incompatible, unless $\delta'(0) \leq \frac{1}{2v}$, in which case the empty network is both efficient and Nash-stable. Similarly, efficient support and stable support of an infrastructure are incompatible.*

The reason is clear: From the results in Sections 4 and 5 it follows that $\widehat{c}_{eq} < \widehat{c}_{ef}$ and $c^*_{eq} < c^*_{ef}$, and consequently a non-empty efficient network requires link-investments which are not stable because they give players the opportunity of free riding by taking advantage of externalities, even if responses are restricted to being profitable only marginally. The same occurs in the seminal discrete models of Jackson and Wolinsky (1996) and Bala and Goyal (2000). The robustness of this incompatibility, now in a much more flexible model, may seem somewhat surprising. Nevertheless, the reason is clear. Similarity and difference between (4) and (15), and between (6) and (16), both stem from the same source. Conditions for optimality and marginal stability are based on the same economic principle: Imposing zero marginal benefit, but *social* (i.e. aggregate) benefit for efficiency, and *individual* benefit for stability. From the point of view of efficiency the strength of a link must maximize its contribution to the aggregate payoff, while from the point of view of either player involved in its support it must maximize his/her payoff. Hence the incompatibility.

Nevertheless, in a mixed environment, if a central planner offers to pay for *half* the investment to every *player,* or, more precisely, subsidizing each dollar invested by a player with another dollar, efficiency can be sustained in marginal equilibrium. This can be shown as follows. In this situation, if the actual investment of each player in link $\underline{ij}$ is $c_{ij}/2$, (27) becomes

$$\Pi_i^\delta(\mathbf{c}) = \sum_{j \in N^d(i,g^{\mathbf{c}})} \left(v\delta(c_{ij} + c_{ji}) \sum_{k \in N(i;g^{\mathbf{c}}) \text{ s.t. } \underline{ij} \in \overline{p}_{ik}} \delta(\overline{p}_{ik}^{ij}) - \frac{c_{ij}}{2} \right), \qquad (24)$$

which by an argument identical to that which that leads to (13) and (14), leads to the necessary conditions for equilibrium:

$$\delta'(c_{\underline{ij}}) = \frac{1}{2v \sum_{k \in N(i;g^{\mathbf{c}}) \text{ s.t. } \underline{ij} \in \overline{p}_{ik}} \delta(\overline{p}_{ik}^{ij})} \text{ (whenever } c_{ij} > 0).$$

$$\delta'(c_{\underline{ij}}) \leq \frac{1}{2v \sum_{k \in N(i;g^{\mathbf{c}}) \text{ s.t. } \underline{ij} \in \overline{p}_{ik}} \delta(\overline{p}_{ik}^{ij})} \text{ (whenever } c_{ij} = 0 \text{ and } c_{\underline{ij}} > 0).$$

In particular, if $g^{\mathbf{c}}$ is a complete network only the first one applies and becomes

$$\delta'(c_{\underline{ij}}) = \frac{1}{2v} \text{ (for all } \underline{ij} \in N_2);$$

while for an all-encompassing star whose center is player 1, it yields,

$$\delta'(c_{i1}^*) = \frac{1}{2v(1+(n-2)\delta(c_{eq}^*))} \quad \text{(for all } i \neq 1\text{)}.$$

That is, under this subsidy efficiency can be sustained in marginal equilibrium.

Notice that from the point of view of players the effect of this subsidy is like replacing the actual technology δ by a better technology β, s.t. $\beta(c) = \delta(2c)$, also a DR-technology. In fact, more generally, subsidies of the form $\beta(c) = \delta((1+\lambda)c)$, i.e. of λ dollars per dollar invested, with λ ranging from 0 to 1, bridge the gulf between equilibrium and efficiency. Figure 2 shows the graphs of technology $\delta(c) = \frac{c}{2c+2}$ and that of $\beta(c) = \delta(2c) = \frac{c}{2c+1}$, superimposed over those of φ_n for $n = 5, 12, 22$ and 42 (as in Figure 1).

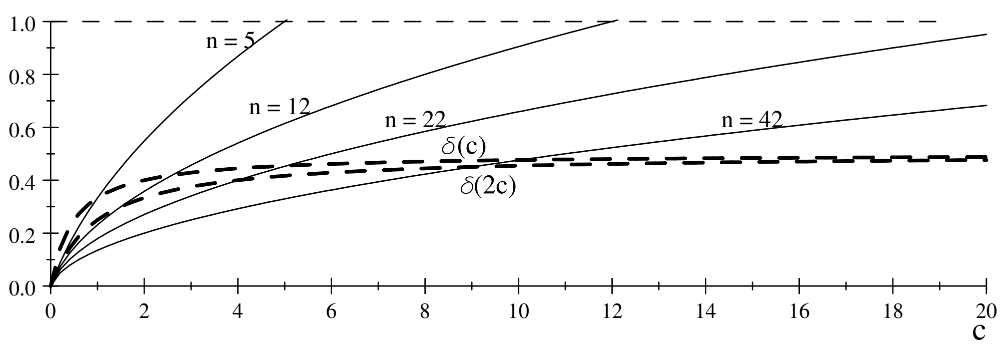

Figure 2: Graph of φ_n for $v = 1$ and $n = 5, 12, 22, 42$, and $\delta(c)$ and $\delta(2c)$

7 Related literature

In this brief review we concentrate mainly on papers published after the seminal connections models of Jackson and Wolinsky (1996) and Bala and Goyal (2000), where agents derive utility from their direct *and indirect* connections, and focus on those most closely related to the model studied in this paper.[11] Apart from other differences between our model and those commented below, there is one that applies *to all of them*: in our approach to stability the central concept is that of marginal equilibrium, a weaker notion than Nash equilibrium.

Bloch and Dutta (2009) introduce endogenous link strength in a connections model by replacing Jackson and Wolinsky's discrete technology by an additively separable *convex* function of players' investments in a link that determines its strength, i.e. they assume *non-decreasing* returns. They also assume that players' investments are limited

[11] This means leaving aside a number of important papers, such as those in the wake of Ballester, Calvó-Armengol and Zenou (2006).

by a unit of resources. We instead assume technology to be a *concave* function of the joint investments of the players (i.e. we assume *decreasing* returns), whose efforts are assumed to be perfect substitutes, and have no budget constraint, hence the different results. Bloch and Dutta prove that in their model the star is the only Nash-stable architecture and the only efficient one. Deroian (2009) studies a similar model, but with *directed* communication, i.e. where links are directed, and proves that, as in Bloch and Dutta (2009), in equilibrium agents concentrate their investment on a single link and the complete wheel is the only efficient architecture and the unique Nash-stable architecture. Also in the wake of Bloch and Dutta (2009), So (2016) assumes that technology is an additively separable function of players' investments, which are limited by a budget. But unlike Bloch and Dutta, So assumes that the strength of a link connecting i and j where i invests x_i^j and j invests x_j^i is $\phi(x_i^j) + \phi(x_j^i)$, with ϕ increasing and strictly concave; while in our model the strength is a function of $x_i^j + x_j^i$, that is, players's efforts (i.e. investments) are perfect substitutes. She obtains sufficient conditions for the symmetric complete network to dominate all star networks and for the symmetric star and the complete network to be Nash-stable, but no characterization is provided.

Other models with endogenous link strength less closely related to ours are the following. In Cabrales, Calvo and Zenou (2011) players choose a level of socialization effort which is distributed across all possible bilateral interactions in proportion to the partner's socialization effort. In Feri and Meléndez-Jiménez's (2013) dynamic model the choice of whom to link to and a coordination game determine the strength of the links. In Harmsen-van Hout, Herings and Dellaert's (2013) model individuals derive social value from direct connections and informational value from direct and indirect connections, but the more links an individual sustains the weaker they are. Boucher (2015) considers a model where individuals with a limited budget derive utility from self-investment and from *direct* connections, assuming the utility of a direct link to be a convex function of the investments of the two players involved, whose distance also enters as an argument in their utility. In Salonen (2015), Baumann (2019) and Griffith (2019) individuals with limited resources derive utility from self-investment and from direct connections, but assuming that the *utility* of a link is a strictly concave function of the investments of the two players. Ding (2019) considers a constant elasticity of substitution link-formation technology that nests unilateral and bilateral network formation.

8 Concluding remarks

We have developed a marginalist decreasing returns connections model which is a natural extension of the seminal discrete connections models of Jackson and Wolinsky (1996) and Bala and Goyal (2000). The basic logic is the same, *payoff* = *information* − *investment*, but it is based on a non-discrete, smooth decreasing returns link-formation technology, which is the only exogenous ingredient in the model.

The characterization of efficient networks for DR-technologies is solved by Theorem 1, which establishes that the only possible non-empty efficient structures are symmetric all-encompassing stars and complete networks, and characterizes the family of DR-technologies which admit one of these non-empty structures as efficient. This result shows the somewhat surprising robustness of the result on efficiency in the seminal discreet two-parameter connections model of Jackson and Wolinsky (1996).

As to stability, we introduce a notion of marginal equilibrium, natural in this marginalist model and new in the networks literature to the best of our knowledge, and obtain necessary and sufficient characterizing conditions for this weak notion of stability (Theorem 2). In a marginal equilibrium, the optimal paths or channels for information which each player pays for form a well-defined tree, i.e. a multiplicity of such optimal paths is incompatible with marginal stability. Moreover, this along with the other characterizing conditions (Theorem 2 and Corollary 1) enables a variety of graph architectures sustainable in marginal equilibria to be identified, such as subcomplete graphs, stars, trees and circles, and determine the investment profiles that sustain them in marginal equilibrium (Propositions 7-11). A feature worth noting of marginal equilibrium is its resilience in response to shocks, such as deletion of nodes. Although no dynamic model has been provided, it is clear that nodes sensitive to the marginal revenue of its links can readjust to a new marginal equilibrium after a node vanishes by responding to such changes in many cases.

As to Nash-stability, no characterization has been obtained, only for a symmetric complete network and all-encompassing stars. Nevertheless, given that marginal stability is necessary for it, Proposition 12 gives necessary and sufficient conditions for the empty network to be Nash-stable, Proposition 13 gives necessary and sufficient conditions for a complete network to be Nash-stable and Proposition 14 for an all-encompassing star to be Nash-stable. In a decentralized context, the comparison from Bala and Goyal (2000) on stability issues is pertinent here. In this respect, a salient difference with Bala and Goyal (2000) is that, under a DR-technology, a Nash-stable (marginally stable) all-encompassing star is necessarily *periphery-sponsored*.[12]

Finally, the conditions for efficiency (Theorem 1) and stability, even if only marginal (Theorem 2), lead to the conclusion that they are incompatible and make it transparent why. Conditions for efficiency and for marginal stability are based on the same economic principle: imposing zero marginal benefit, but *social* (i.e. aggregate) benefit for efficiency, and *individual* benefit for stability. Nevertheless, it is shown that subsidizing up to a dollar per dollar invested by each player would bridge the gap between efficiency and marginal stability.

There are three lines of further research that might be of particular interest. First, although Theorem 2 gives necessary and sufficient conditions for marginal equilibrium, no complete characterization of the architectures sustainable in marginal equilibrium

[12] In Bala and Goyal (2000) two-way flow model with decay stars non-necessarily center-sponsored can be strict Nash.

has been provided. Second, exploring the impact of assuming heterogeneity, in technology and/or in individual values. Third, enriching the model by introducing some dynamics. This seems especially desirable related to marginal stability. If nodes are only sensitive to the marginal value of their investments in actual links how does a network form? This calls for a random ingredient, be it in the prior formation of an infrastructure or in that of the network itself, where stochastic stability can be studied (see Feri (2007)).

Appendix

Proposition 1

Proof. (i) and (ii): For a complete network $g^{\bar{\mathbf{c}}}$ to be efficient every link must be used only by the pair of nodes that it connects. Therefore, for all $\underline{ij} \in N_2$, $c_{\underline{ij}}$ must maximize $2v\delta(c_{\underline{ij}}) - c_{\underline{ij}}$ and yield a positive value, i.e.

$$2v\delta(c_{\underline{ij}}) - c_{\underline{ij}} = \max_{c>0}(2v\delta(c) - c) > 0$$

must hold. As δ is a strictly concave differentiable function, this implies $\delta'(0) > 1/2v$ (otherwise $\max_{c>0}(2v\delta(c)-c) = 0$) and $\arg\max_{c>0}(2v\delta(c)-c)$ is necessarily a singleton, namely, the only c s.t. $2v\delta'(c) - 1 = 0$, hence (4).

(iii) If $2v\delta(\widehat{c}_{ef})^2 > 2v\delta(\widehat{c}_{ef}) - \widehat{c}_{ef}$, the net value of the network would increase by deleting a link. ■

Lemma 1

Proof. Let δ be any link-formation technology for which a star g of, say, $m+1$ nodes and $m \leq n-1$ links is optimal, and assume w.l.o.g. that node 1 is the center. Let i and j be any two spoke nodes connected to the center by links of strengths $\delta(c_i)$ and $\delta(c_j)$ (denoting $c_i := c_{\underline{1i}}$). Its net value is

$$v(g) = 2v(\delta(c_i) + \delta(c_j))(1 + K) + K' + 2v\delta(c_i)\delta(c_j) - c_i - c_j,$$

where

$$K = \sum_{k\neq 1,i,j} \delta(c_k) \text{ and } K' = 2vK + 2v \sum_{k,l\neq 1,i,j(k\neq l)} \delta(c_k)\delta(c_l) - \sum_{k\neq 1,i,j} c_k.$$

Let g' (g'') be the star that results from replacing the c_j-link by a c_i-link (the c_i-link by a c_j-link). Then

$$\begin{aligned} v(g') &= 2v2\delta(c_i)(1+K) + K' + 2v\delta(c_i)^2 - 2c_i, \\ v(g'') &= 2v2\delta(c_j)(1+K) + K' + 2v\delta(c_j)^2 - 2c_j. \end{aligned}$$

Therefore, as g is optimal,

$$v(g) - v(g') = 2v(\delta(c_j) - \delta(c_i))(1 + \delta(c_i) + K) - c_j + c_i \geq 0,$$

$$v(g) - v(g'') = 2v(\delta(c_i) - \delta(c_j))(1 + \delta(c_j) + K) - c_i + c_j \geq 0,$$

which yield

$$2v(\delta(c_i) - \delta(c_j))(1 + \delta(c_i) + K) \leq c_i - c_j$$

and

$$2v(\delta(c_i) - \delta(c_j))(1 + \delta(c_j) + K) \geq c_i - c_j.$$

Assume w.l.o.g. $c_i \geq c_j$, then $\delta(c_i) \geq \delta(c_j)$, and consequently

$$c_i - c_j \geq 2v(\delta(c_i) - \delta(c_j))(1 + \delta(c_i) + K) \geq 2v(\delta(c_i) - \delta(c_j))(1 + \delta(c_j) + K) \geq c_i - c_j,$$

which implies that all three expressions must have the same value, from which it follows that

$$(\delta(c_i) - \delta(c_j))\delta(c_i) = (\delta(c_i) - \delta(c_j))\delta(c_j),$$

which implies $\delta(c_i) = \delta(c_j)$. Therefore all links in g necessarily have the same strength.

Finally, if a star all of whose links have the same strength yields a positive net value but is not all-encompassing, its net value would increase by connecting any other node with a link of the same strength to the center. ■

Proposition 2

Proof. (i) By Lemma 1, all links in an efficient all-encompassing star must have the same investment, c^*, which must maximize its net value, i.e. such that (5). Condition (6) stems from the first order condition for an extreme of

$$(n-1)\left(2v\delta(c) + (n-2)v\delta(c)^2 - c\right),$$

which is the net value of an all-encompassing star with n nodes and $n-1$ links of strength $\delta(c)$.

(ii) Otherwise, if $2v\delta(c_{ef}^*)^2 < 2v\delta(c) - c$ for some c, connecting two spoke nodes would increase the net value of the network. ■

Proposition 3

Proof. As $\delta(c) < 1$ for all $c \geq 0$, the net value of a symmetric all-encompassing star of c-links, denoted by $g^{c\text{-}star}$, is

$$v(g^{c\text{-}star}) = (n-1)(2v\delta(c) + (n-2)v\delta(c)^2 - c) < (n-1)(nv - c),$$

and $(n-1)(nv-c) > 0$ if and only if $c < nv$. In other words, an all-encompassing star of c-links yields a positive net value only if $c < nv$. Therefore,

$$\arg\max v(g^{c\text{-}star}) \subseteq [0, nv],$$

and such a maximum exists because $v(g^{c\text{-}star})$ is continuous on c. Moreover, that maximum is > 0 (i.e. an optimal symmetric star actually does exist) *unless* $v(g^{c\text{-}star}) \leq 0$ for all $c > 0$, i.e. unless

$$2v\delta(c) + (n-2)v\delta(c)^2 - c \leq 0 \quad (\forall c \geq 0),$$

or, equivalently,

$$\left(\delta(c)-\frac{-1+\sqrt{1+\frac{(n-2)c}{v}}}{n-2}\right)\left(\delta(c)-\frac{-1-\sqrt{1+\frac{(n-2)c}{v}}}{n-2}\right)\leq 0 \quad (\forall c\geq 0).$$

This in turn is equivalent to requiring $\delta(c)$ to remain within the interval

$$\delta(c)\in\left[\frac{-1-\sqrt{1+\frac{(n-2)c}{v}}}{n-2},\frac{-1+\sqrt{1+\frac{(n-2)c}{v}}}{n-2}\right],$$

but note that its lower extreme is <0, while the other is >0 and $\delta(c)\geq 0$. Therefore, this condition is equivalent to

$$0\leq\delta(c)\leq\frac{-1+\sqrt{1+\frac{(n-2)c}{v}}}{n-2} \quad (\forall c\geq 0).$$

Summing up, unless this condition holds there is always an optimal symmetric all-encompassing star.

Finally, note that the limit of the upper bound for each $c>0$ when $n\to\infty$ is 0. In other words, for n big enough $\delta(c)$ is sure to be outside this interval for some c and consequently there is sure to exist an optimal symmetric all-encompassing star. ■

Theorem 1

Proof. (i) Theorem 1 in Olaizola and Valenciano (2020) establishes that for any technology (i.e. any non-decreasing map $\delta:\mathbb{R}_+\to[0,1)$ s.t. $\delta(0)=0$) the only non-empty architectures of possibly efficient networks are the all-encompassing star and the complete network unless a "supertie" occurs, i.e. unless

$$2v\delta(\widehat{c}_{ef})-\widehat{c}_{ef}=2v\delta(c^*_{ef})-c^*_{ef}=2v\delta(c^*_{ef})^2. \tag{25}$$

However, this cannot occur under a DR-technology. Proposition 1 shows that the set $\arg\max_{c>0}(2v\delta(c)-c)$ is a singleton $\widehat{c}_{ef}>0$ s.t. (4) if and only if $\delta'(0)>1/2v$. But from (4) and (6), it follows that $\delta'(c^*_{ef})<\delta'(\widehat{c}_{ef})$, which implies that $\widehat{c}_{ef}<c^*_{ef}$. Thus necessarily $2v\delta(\widehat{c}_{ef})-\widehat{c}_{ef}>2v\delta(c^*_{ef})-c^*_{ef}$, because $\arg\max_{c>0}(2v\delta(c)-c)=\{\widehat{c}_{ef}\}$ (a singleton), which excludes the possibility of (25).

(ii) By Proposition 3, if $\delta(c)\leq\varphi_n(c)$ for all c, no symmetric all-encompassing star yields a positive net value. Note also that, as $\varphi'_n(0)=1/2v$, the upper bound $\delta(c)\leq\varphi_n(c)$ (for all c) imposes $\delta'(0)\leq 1/2v$, and consequently (Proposition 1) no complete network yields a positive net value. Therefore, for any technology whose graph is below that of φ_n, no symmetric all-encompassing star and nor complete network yields positive net value. Therefore, only the empty network is efficient. By contrast, if $\delta(c)>\varphi_n(c)$ for some c, an optimal symmetric all-encompassing star is sure to exist and yield a positive net value, and consequently there is sure to be an efficient star or complete network. ■

Proposition 4

Proof. Let $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij}\in N_2}$ be a link-investment vector s.t. $c_{\underline{ij}} > 0$ if and only if $\underline{ij} \in S$. Assume that $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij}\in N_2}$ efficiently supports S and $\underline{ij} \in S$. Link $\underline{ij}$ is thus a necessary part of at least one optimal path in $g^{\overline{\mathbf{c}}}$, the one connecting i and j, because otherwise $\overline{\mathbf{c}}$ would not be efficient. The contribution of link $\underline{ij}$, i.e. of investment $c_{\underline{ij}}$, to the net value $v(g^{\overline{\mathbf{c}}})$ given by (9) for a choice of optimal paths $(\overline{p}_{kl})_{\underline{kl}\in N_2}$ is

$$2v \sum_{\substack{\underline{kl}\in N_2 \ s.t. \\ \underline{ij}\in\overline{p}_{kl}}} \delta(\overline{p}_{kl}) - c_{\underline{ij}} = 2v\delta(c_{\underline{ij}}) \sum_{\substack{\underline{kl}\in N_2 \ s.t. \\ \underline{ij}\in\overline{p}_{kl}}} \delta(\overline{p}^{ij}_{kl}) - c_{\underline{ij}}. \tag{26}$$

Thus for investment $c_{\underline{ij}}$ to be optimal it must maximize

$$2v\delta(c) \sum_{\underline{kl}\in N_2 \ s.t. \ \underline{ij}\in\overline{p}_{kl}} \delta(\overline{p}^{ij}_{kl}) - c,$$

for which it is necessary that

$$\left.\frac{d}{dc}\right|_{c=c_{\underline{ij}}} (2v\delta(c) \sum_{\substack{\underline{kl}\in N_2 \ s.t. \\ \underline{ij}\in\overline{p}_{kl}}} \delta(\overline{p}^{ij}_{kl}) - c) = 2v\delta'(c_{\underline{ij}}) \sum_{\substack{\underline{kl}\in N_2 \ s.t. \\ \underline{ij}\in\overline{p}_{kl}}} \delta(\overline{p}^{ij}_{kl}) - 1 = 0,$$

which yields (11). A non-null derivative w.r.t. c at $c_{\underline{ij}}$ means that by slightly increasing (if it is > 0) or decreasing (if it is < 0) the investment in link $\underline{ij}$ the aggregate payoff through those paths would surely increase, which contradicts the efficiency of $\overline{\mathbf{c}}$.

Assume now that two nodes r and s are connected by two different optimal paths in $g^{\overline{\mathbf{c}}}$. Then there is at least one link, say $\underline{ij}$, that is part of one of these paths but not of the other. Then the right-hand side of (9) admits at least *two* different expressions: One where the optimal path between any pair of nodes k, l is $\overline{p}_{kl}$, and another where it is $\overline{q}_{kl}$, and such that for any pair k, l different from pair r, s, $\overline{p}_{kl} = \overline{q}_{kl}$, while for r and s the optimal path is different, i.e. $\overline{p}_{rs} \neq \overline{q}_{sr}$, and only the first one contains $\underline{ij}$. In that case,

$$\frac{1}{2v\sum_{\underline{kl}\in N_2 \ s.t. \ \underline{ij}\in\overline{p}_{kl}} \delta(\overline{p}^{ij}_{kl})} \neq \frac{1}{2v\sum_{\underline{kl}\in N_2 \ s.t. \ \underline{ij}\in\overline{q}_{kl}} \delta(\overline{q}^{ij}_{kl})}$$

because

$$\sum_{\underline{kl}\in N_2 \ s.t. \ \underline{ij}\in\overline{p}_{kl}} \delta(\overline{p}^{ij}_{kl}) - \sum_{\underline{kl}\in N_2 \ s.t. \ \underline{ij}\in\overline{q}_{kl}} \delta(\overline{q}^{ij}_{kl}) = \delta(\overline{p}_{rs}) > 0,$$

which leads to a contradiction because (11) yields two different values for $\delta'(c_{\underline{ij}})$. ■

Lemma 2

Proof. It is obvious that strong marginal stability implies marginal stability. Assume that $g^{\mathbf{c}}$ is connected and $\mathbf{c}$ is marginally stable. Let $\mathbf{c}'_i = (c'_{ik})_{k\in N}$, and let $\mathbf{c}''_i$ be given by

$$c''_{ij} = \begin{cases} c'_{ij} \text{ if } c_{\underline{ij}} \neq 0, \\ 0 \text{ if } c_{\underline{ij}} = 0, \end{cases}$$

and $\mathbf{c}'' = (\mathbf{c}_{-i}; \mathbf{c}''_i)$. For $\mathbf{c}'_i$ sufficiently close to $\mathbf{c}_i$, the underlying graphs of $g^{\mathbf{c}}$ and $g^{\mathbf{c}''}$ are the same and s. t. $\Pi_i^\delta(\mathbf{c}) \geq \Pi_i^\delta(\mathbf{c}_{-i}; \mathbf{c}''_i)$ because $\mathbf{c}$ is marginally stable. And for all j s.t. $c_{\underline{ij}} = 0$ and $c'_{ij} \neq 0$, as i and j are indirectly connected in $g^{\mathbf{c}}$ and $g^{\mathbf{c}''}$, i and j receive an amount of information from each other through a path in $g^{\mathbf{c}''}$. Thus, a sufficiently small investment c'_{ij} in link $\underline{ij}$ (namely, as far as $\delta(c'_{ij})$ is smaller than the decay along that path) is sure to be unprofitable. Therefore, for $\mathbf{c}'_i$ sufficiently close to $\mathbf{c}_i$,

$$\Pi_i^\delta(\mathbf{c}) \geq \Pi_i^\delta(\mathbf{c}_{-i}; \mathbf{c}''_i) \geq \Pi_i^\delta(\mathbf{c}_{-i}; \mathbf{c}'_i).$$

■

Proposition 5

Proof. The empty network has no actual links, so it satisfies trivially marginal stability. Assume now that the empty network, i.e. $c_{ij} = 0$ for all $i, j \in N$, is strongly marginally stable. Then for some $\varepsilon > 0$, for all c s.t. $0 < c < \varepsilon$, it holds that $v\delta(c) - c \leq 0$. Or, equivalently, $\frac{\delta(c)}{c} \leq 1/v$ for all $c < \varepsilon$. Then taking limits, $\lim_{c\to 0} \frac{\delta(c)}{c} = \delta'(0) \leq 1/v$. Assume now that the empty network is not strongly marginally stable,i.e. for every $\varepsilon > 0$ there exists $c < \varepsilon$ s.t. $v\delta(c) - c > 0$. But then $\delta'(0) > \frac{\delta(c)}{c} > 1/v$. ■

Theorem 2

Proof. (*Necessity*) Let δ be a DR-technology and $\mathbf{c}^* = (c^*_{ij})_{i,j\in N}$ a marginally stable investment profile.

(i) Assume $c^*_{\underline{ij}} > 0$, then at least one of them, i or j, say i, invests in that link, i.e. $c^*_{ij} > 0$. Then link $\underline{ij}$ is part of at least one optimal path in $g^{\mathbf{c}^*}$ for i's information, the one connecting i and j, otherwise i would increase payoff by diminishing investment in it. Fix one of the, in principle, possible different but equivalent expressions on the right-hand side of (12). Then i's payoff for this particular choice of optimal paths $\overline{\boldsymbol{p}}_i$ is given by the right-hand side of (12), which can be rewritten like this:

$$\Pi_i^\delta(\mathbf{c}^*; \overline{\boldsymbol{p}}_i) = \sum_{j\in N^d(i;g^{\mathbf{c}^*})} \left(v\delta(c^*_{ij} + c^*_{ji}) \sum_{\overline{p}_{ik}\in\overline{\boldsymbol{p}}_i \ \&\ ij\in\overline{p}_{ik}} \delta(\overline{p}^{ij}_{ik}) - c^*_{ij} \right). \tag{27}$$

From the point of view of player i, with the investments by the other players $j \neq i$ taken as given, the right-hand side of (27) depends on i's admissible strategy $\mathbf{c}_i$, and it is a differentiable function of as many variables as i has neighbors, $(c_{ij})_{j\in N^d(i;g^{\mathbf{c}^*})}$. Namely,

$$\Pi_i^\delta(\mathbf{c}^*_{-i}; \mathbf{c}_i; \overline{\boldsymbol{p}}_i) = \sum_{j\in N^d(i;g^{\mathbf{c}^*})} \left(v\delta(c_{ij} + c^*_{ji}) \sum_{\overline{p}_{ik}\in\overline{\boldsymbol{p}}_i \ \&\ ij\in\overline{p}_{ik}} \delta(\overline{p}^{ij}_{ik}) - c_{ij} \right). \tag{28}$$

Thus the terms in (28) where c_{ij} enters are

$$v\delta(c_{ij} + c_{ji}^*)K_{i,j}(\overline{\boldsymbol{p}}_i) - c_{ij},$$

with

$$K_{i,j}(\overline{\boldsymbol{p}}_i) := \sum_{\overline{p}_{ik} \in \overline{\boldsymbol{p}}_i \ \& \ ij \in \overline{p}_{ik}} \delta(\overline{p}_{ik}^{ij}) \leq K_{i,j}^{g^{\mathbf{c}^*}}.$$

Therefore, for the strategy of player i, $(c_{ij}^*)_{j\in N}$, to be marginally stable given the investments made by the other players (which fully determine $\delta(\overline{p}_{ik}^{ij})$ for all $j \in N^d(i; g^{\mathbf{c}^*})$ and all $k \in N(i; g^{\mathbf{c}^*})$ s.t. $\underline{ij} \in \overline{p}_{ik}$) the following must hold

$$\frac{\partial}{\partial c_{ij}} \Pi_i^\delta(\mathbf{c}_{-i}^*; (c_{ij}^*)_{j\in N}; \overline{\boldsymbol{p}}_i) = 0.$$

A non-null partial derivative w.r.t. c_{ij} of (27) at $\mathbf{c}^*$ means that slightly increasing (if it is > 0) or decreasing (if it is < 0) investment by i in link $\underline{ij}$ would increase i's payoff (through the same available paths), which contradicts the marginal stability of $\mathbf{c}^*$. Therefore[13]

$$\frac{\partial}{\partial c_{ij}} \Pi_i^\delta(\mathbf{c}_{-i}^*; \mathbf{c}_i; \overline{\boldsymbol{p}}_i) = v\delta'(c_{\underline{ij}})K_{i,j}(\overline{\boldsymbol{p}}_i) - 1 = 0,$$

must hold at $c_{\underline{ij}}^*$, that is,

$$\delta'(c_{\underline{ij}}^*) = \frac{1}{vK_{i,j}(\overline{\boldsymbol{p}}_i)} = \frac{1}{v\sum_{\overline{p}_{ik} \in \overline{\boldsymbol{p}}_i \ \& \ ij \in \overline{p}_{ik}} \delta(\overline{p}_{ik}^{ij})}. \tag{29}$$

By construction, $K_{i,j}(\overline{\boldsymbol{p}}_i) \leq K_{i,j}^{g^{\mathbf{c}^*}}$, but note that it must be $K_{i,j}(\overline{\boldsymbol{p}}_i) = K_{i,j}^{g^{\mathbf{c}^*}}$. Otherwise, a different choice of optimal paths $\overline{\boldsymbol{p}}'_i = (\overline{p}'_{ik})_{k\in N(i;g^{\mathbf{c}^*})}$ would yield $K_{i,j}(\overline{\boldsymbol{p}}_i) \neq K_{i,j}(\overline{\boldsymbol{p}}'_i)$ and then (29) would lead to a contradiction. Therefore (13) must hold.

This means that any optimal path $\overline{p}_{ik}$ containing a link $\underline{ij}$ s.t. $c_{ij}^* > 0$ has a positive impact on its cost, because $\delta(\overline{p}_{ik}^{ij})$ is a summand in $K_{i,j}^{g^{\mathbf{c}^*}}$, the denominator in (13), so that $\delta'(c_{\underline{ij}}^*)$ decreases and $c_{\underline{ij}}$ increases. Then, if an optimal path $\overline{p}_{ik}$ contains $\underline{ij}$ and $\delta(\overline{p}_{ik}) = \delta(\overline{q}_{ik})$ for some other optimal path $\overline{q}_{ik}$ in $g^{\mathbf{c}^*}$, the optimality of $\overline{p}_{ik}$ would be superfluous because its marginal revenue for i is $v\delta(\overline{p}_{ik}) = v\delta(\overline{q}_{ik})$ at a cost that can be spared given that it is also received through $\overline{q}_{ik}$, i.e. a small decrease in $c_{\underline{ij}}^*$ would increase i's payoff, contradicting the marginal stability of $\mathbf{c}^*$. Thus, every optimal path that contains link $\underline{ij}$ and connects node i with another node must be the only optimal

[13] Just note that by the chain rule

$$\frac{\partial}{\partial c_{ij}}(\delta(c_{ij} + c_{ji}^*))\Big|_{c_{ij}=c_{ij}^*} = \delta'(c_{ij} + c_{ji}^*) \cdot 1\big|_{c_{ij}=c_{ij}^*} = \delta'(c_{\underline{ij}}^*).$$

path connecting them. In other words, the optimal paths connecting one node with other nodes in which a node invests form a well-defined tree rooted at that node.

(ii) Assume now that $c^*_{\underline{ij}} > 0$ and $c_{ij} = 0$, i.e. the link $\underline{ij}$ is entirely supported by j. A similar argument to the one used to prove part (i) leads in this case to the conclusion that

$$v\delta'(c_{\underline{ij}})K_{i,j}(\overline{\boldsymbol{p}}_i) - 1 \leq 0$$

must hold at $\mathbf{c}^*_i$, whatever the choice of optimal paths $\overline{\boldsymbol{p}}_i = (\overline{p}_{ik})_{k \in N(i;g^{\mathbf{c}^*})}$. Otherwise player i's payoff increases by investing in link $\underline{ij}$, which yields $\delta'(c^*_{\underline{ij}}) \leq \frac{1}{vK_{i,j}(\overline{\boldsymbol{p}}_i)}$. And choosing $\overline{\boldsymbol{p}}_i$ s.t. $K_{i,j}(\overline{\boldsymbol{p}}_i)$ is maximal, i.e. $K_{i,j}(\overline{\boldsymbol{p}}_i) = K^{g^*}_{i,j}$, we have $\delta'(c^*_{\underline{ij}}) \leq \frac{1}{vK^{g^*}_{i,j}}$.

Thus conditions (i) and (ii) are necessary for $\mathbf{c}^*$ to be marginally stable.

(*Sufficiency*) Assume conditions (i) and (ii) hold for an investment profile $\mathbf{c}^* = (c^*_{ij})_{i,j \in N}$. Then i's payoff for a particular choice of optimal paths $\overline{\boldsymbol{p}}_i$ is given by (27), that is

$$\Pi^\delta_i(\mathbf{c}^*; \overline{\boldsymbol{p}}_i) = \sum_{j \in N^d(i;g^{\mathbf{c}^*})} \left(v\delta(c^*_{ij} + c^*_{ji})K_{i,j}(\overline{\boldsymbol{p}}_i) - c^*_{ij} \right).$$

If $\mathbf{c}_i = (c_{ij})_{j \in N}$ is an alternative admissible strategy of player i s.t. $c_{ij} \neq 0$ only if $c^*_{\underline{ij}} \neq 0$, then i's payoff through the same paths is given by (28), that is,

$$\Pi^\delta_i(\mathbf{c}^*_{-i}; \mathbf{c}_i; \overline{\boldsymbol{p}}_i) = \sum_{j \in N^d(i;g^{\mathbf{c}^*})} \left(v\delta(c_{ij} + c^*_{ji})K_{i,j}(\overline{\boldsymbol{p}}_i)) - c_{ij} \right),$$

which is a differentiable concave function of $(c_{ij})_{j \in N^d(i;g^{\mathbf{c}^*})}$[14]. Moreover, the Kuhn-Tucker conditions for a maximum of $\Pi^\delta_i(\mathbf{c}^*_{-i}; \mathbf{c}_i; \overline{\boldsymbol{p}}_i)$ constrained by $c_{ij} \geq 0$ are

$$\begin{cases} \frac{\partial}{\partial c_{ij}}\Pi^\delta_i(\mathbf{c}^*_{-i}; \mathbf{c}_i; \overline{\boldsymbol{p}}_i) + \lambda_j = v\delta'(c_{\underline{ij}})K_{i,j}(\overline{\boldsymbol{p}}_i) - 1 + \lambda_j = 0, & \text{(K-T.1)} \\ \lambda_j c_{ij} = 0, & \text{(K-T.2)} \\ c_{ij} \geq 0, & \text{(K-T.3)} \\ \lambda_j \geq 0, & \text{(K-T.4)} \end{cases}$$

for all $j \in N^d(i; g^{\mathbf{c}^*})$. Now if $c^*_{ij} > 0$ and (13) holds, given that any optimal path containing ij and connecting i with any node k is necessarily the only path connecting them, it must be $K_{i,j}(\overline{\boldsymbol{p}}_i) = K^{g^{\mathbf{c}^*}}_{i,j}$. Then whenever $c^*_{ij} > 0$ condition (K-T.3) holds, condition (K-T.1) becomes $v\delta'(c_{\underline{ij}})K^{g^{\mathbf{c}^*}}_{i,j} - 1 + \lambda_j = 0$, which holds along with (K-T.2) and (K-T.4) with $\lambda_j = 0$. Whereas if $c^*_{ij} = 0$ and $\delta'(c^*_{\underline{ij}}) < \frac{1}{vK^{g^*}_{i,j}}$, (K-T.2) and (K-T.3) hold, while (K-T.1) and (K-T.4) hold with $\lambda_j = -(v\delta'(c_{\underline{ij}})K^{g^{\mathbf{c}^*}}_{i,j} - 1) > 0$. Thus Kuhn-Tucker conditions for a maximum of $\Pi^\delta_i(\mathbf{c}^*_{-i}; \mathbf{c}_i; \overline{\boldsymbol{p}}_i)$ constrained by $c_{ij} \geq 0$ hold at $\mathbf{c}^*_i$. Given that $\Pi^\delta_i(\mathbf{c}^*_{-i}; \mathbf{c}_i; \overline{\boldsymbol{p}}_i)$ is concave, these conditions are also *sufficient* for a maximum. In short, the necessary conditions guarantee that *whatever the choice*

[14] Sum of a positive linear combination of concave functions and a linear function.

of optimal paths $\overline{\boldsymbol{p}}_i$ through which player i receives information, the investments of each player in his/her actual links are optimal to receive it trough them. Therefore, a sufficiently small change of investments of any player is necessarily non-profitable. ■

Corollary 1

Proof. Let $\mathbf{c}^*$ be a marginally stable investment profile and assume $c^*_{\underline{ij}} > 0$. Then if both invest in link $\underline{ij}$ condition (13) must hold for i and j, and j and i, i.e. i and j can interchange roles in (13), which yields two expressions for $\delta'(c^*_{\underline{ij}})$:

$$\frac{1}{v\sum_{k\in N(i;g^{\mathbf{c}^*})\ s.t.\ \underline{ij}\in\overline{p}_{ik}}\delta(\overline{p}^{ij}_{ik})} = \delta'(c^*_{\underline{ij}}) = \frac{1}{v\sum_{k\in N(j;g^{\mathbf{c}^*})\ s.t.\ \underline{ji}\in\overline{p}_{jk}}\delta(\overline{p}^{ji}_{jk})}.$$

But this is possible only if the sums in both denominators are equal, in other words, only if both players, i and j, receive the same amount of information through link $\underline{ij}$. Otherwise, the two conditions are incompatible and stability is possible only if the player who receives more information through link $\underline{ij}$, say i, covers the whole investment, so that

$$\frac{1}{v\sum_{k\in N(i;g^{\mathbf{c}^*})\ s.t.\ \underline{ij}\in\overline{p}_{ik}}\delta(\overline{p}^{ij}_{ik})} = \delta'(c^*_{\underline{ij}}) < \frac{1}{v\sum_{k\in N(j;g^{\mathbf{c}^*})\ s.t.\ \underline{ji}\in\overline{p}_{jk}}\delta(\overline{p}^{ji}_{jk})}.$$

In this way *both* conditions (13) and (14) hold. ■

Proposition 6

Proof. Conditions (i) and (ii) of Theorem 2 are equivalent to marginal stability. Assume now that $\mathbf{c}^*$ is strongly marginally stable, but $g^{\mathbf{c}^*}$ is not connected and $\delta'(0) > \frac{1}{v+K}$, where K is the information received by the node, say i, that receives the maximal amount of information. Then, if j is any node in a different component, any sufficiently small investment of j in a link with i is sure to increase j's payoff, contradicting $\mathbf{c}^*$'s strong marginal stability.

Reciprocally, if i and j are in different components and $\delta'(0) \leq \frac{1}{v+K}$, where K is the information received by the node that receives the maximal amount of information, then $\frac{\delta(c)}{c} < \delta'(0) \leq \frac{1}{v+K}$ for all c, i.e. $(v+K)\delta(c) - c < 0$, and also replacing K by the information received by the node that receives the maximal amount of information in the component of i or that of j. Consequently any investment in a link connecting them is not profitable for either of them. Whereas if $g^{\mathbf{c}^*}$ is connected, by Lemma 2 it is also strongly marginally stable. ■

Proposition 7

Proof. Assume $\mathbf{c} = (c_{ij})_{i,j\in N}$ is s.t. $c_{\underline{ij}} \neq 0$ if and only if $\underline{ij} \in M_2$ for some $M \subseteq N$. Then, for all $\underline{ij} \in M_2$, $\overline{p}_{ij} = \underline{ij}$, and by Theorem 2, (13) is necessary for marginal stability, i.e. $\delta'(c_{\underline{ij}}) = 1/v$ for all $i, j \in M$ $(i \neq j)$. That is, for all $i, j \in M$ $(i \neq j)$ $c_{ij} + c_{ij} = \widehat{c}_{eq}$, s.t. $\delta'(\widehat{c}_{eq}) = 1/v$; which is feasible only if $\delta'(0) > 1/v$. Moreover, such a $\widehat{c}_{eq}$ is unique by strict concavity of δ. But this is also sufficient because then (14) is also satisfied however the cost $\widehat{c}_{eq}$ of each link is shared.

In particular, if $M = N$, then $g^{\mathbf{c}}$ is connected and, by Lemma 2, also strongly marginally stable. ■

Proposition 8

Proof. Assume that $\mathbf{c}$ is marginally stable and $g^{\mathbf{c}}$ is a star of $p-1$ links ($3 \leq p \leq n$). Assume w.l.o.g. that node 1 is the central player, connected with $2, 3, ..., p$, and $M = \{1, 2, ..., p\}$. Obviously, in a star there is only one path connecting any two nodes. By Corollary 1, the star must be periphery-sponsored, that is, for any spoke player i, $c_{\underline{i1}} = c_{i1}$. The payoff of spoke player i is then

$$\Pi_i^{\delta}(\mathbf{c}) = v\delta(c_{i1})(1 + \sum_{k \in M \backslash \{1,i\}} \delta(c_{k1})) - c_{i1}.$$

We show first that in marginal equilibrium all spoke nodes invest the same amount in the link that each of them supports and receive the same payoff. Assume two spoke nodes, say i and j, invest different amounts in their links, and assume that $\Pi_i^{\delta}(\mathbf{c}) \geq \Pi_j^{\delta}(\mathbf{c})$. Then

$$\Pi_i^{\delta}(\mathbf{c}) - \Pi_j^{\delta}(\mathbf{c}) = (v\delta(c_{i1}) - v\delta(c_{j1}))(1 + \sum_{k \in M \backslash \{1,i,j\}} \delta(c_{k1})) + (c_{j1} - c_{i1}) \geq 0,$$

which implies

$$\frac{\delta(c_{j1}) - \delta(c_{i1})}{c_{j1} - c_{i1}} = \frac{\delta(c_{i1}) - \delta(c_{j1})}{c_{i1} - c_{j1}} \geq \frac{1}{v(1 + \sum_{k \in M \backslash \{1,i,j\}} \delta(c_{k1}))}. \tag{30}$$

Assume $c_{i1} > c_{j1}$. As δ is differentiable and strictly concave,

$$\delta(c_{i1}) - \delta(c_{j1}) < \delta'(c_{j1})(c_{i1} - c_{j1})$$

must hold, i.e.

$$\delta'(c_{j1}) > \frac{\delta(c_{i1}) - \delta(c_{j1})}{c_{i1} - c_{j1}}.$$

Which along with (30) yields

$$\delta'(c_{j1}) > \frac{1}{v(1 + \sum_{k \in M \backslash \{1,i,j\}} \delta(c_{k1}))},$$

which is a contradiction, given that, by (13),

$$\delta'(c_{j1}) = \frac{1}{v(1 + \delta(c_{i1}) + \sum_{k \in M \backslash \{1,i,j\}} \delta(c_{k1}))}.$$

Through the same steps, $c_{j1} > c_{i1}$ leads to a similar contradiction for $\delta'(c_{i1})$. Therefore marginal equilibrium implies that the star must be entirely symmetric: All spoke nodes invest the same amount $c^*_{p,eq}$, which must satisfy (13), which in this case becomes precisely (16), and receive the same payoff. And the by being periphery sponsored, condition (14) is also satisfied. Then such a periphery-sponsored star satisfies all conditions in Theorem 2 necessary and sufficient to be marginally stable. Moreover, if the star is all-encompassing, by Lemma 2 it is also strong marginal equilibrium. ■

Proposition 9

Proof. Let $\mathbf{c} = (c_{ij})_{i,j\in N}$ be an investment profile s.t. $g^{\mathbf{c}}$ is a tree and let $T \subset N_2$ be the underlying graph. By Theorem 2, for $\mathbf{c}$ to be marginally stable condition (13) must hold. That is, for each $\underline{ij} \in T$ s.t. $c_{ij} > 0$,

$$\delta'(c_{\underline{ij}}) = \frac{1}{v\sum_{k\in N(i;g^{\mathbf{c}})\ s.t.\ \underline{ij}\in\overline{p}_{ik}} \delta(\overline{p}^{ij}_{ik})}.$$

If this condition holds and *all links for which the two nodes that it connects are paid for by the node that receives more information through it*, then condition (ii) of Theorem 2 is also satisfied. The first part of condition (i) of Theorem 2 holds necessarily due to the structure of a tree, where any two connected nodes are connected by only one path. Therefore it is enough to prove that there exists an investment *vector* $\overline{\mathbf{c}}$ whose underlying graph is T and s.t. condition (13) holds. As δ is continuously differentiable, strictly concave and increasing, δ' is continuous and decreasing, and consequently invertible, moreover its inverse δ'^{-1} is continuous. Let then

$$\Phi : [\delta'^{-1}(\frac{1}{v}), \delta'^{-1}(\frac{1}{v(n-1)})]^T \longrightarrow [\frac{1}{v(n-1)}, \frac{1}{v}]^T$$

be the function that maps any investment vector $\overline{\mathbf{c}} = (c_{\underline{ij}})_{\underline{ij}\in N_2}$ whose underlying graph is T and s.t. $\delta'^{-1}(\frac{1}{v}) \leq c_{\underline{ij}} \leq \delta'^{-1}(\frac{1}{v(n-1)})$ (i.e. $\frac{1}{v(n-1)} \leq \delta'(c_{\underline{ij}}) \leq 1/v$) for all $\underline{ij} \in T$, into a T-vector $\Phi(\overline{\mathbf{c}}) = (\Phi_{\underline{ij}}(\overline{\mathbf{c}}))_{\underline{ij}\in T}$ defined for each $\underline{ij} \in T$ by

$$\Phi_{\underline{ij}}(\overline{\mathbf{c}}) = \min\left\{\frac{1}{v\sum_{k\in N(i;g^{\mathbf{c}})\ s.t.\ \underline{ij}\in\overline{p}_{ik}} \delta(\overline{p}^{ij}_{ik})}, \frac{1}{v\sum_{k\in N(j;g^{\mathbf{c}})\ s.t.\ \underline{ij}\in\overline{p}_{jk}} \delta(\overline{p}^{ji}_{jk})}\right\}.$$

Then Φ is continuous and $\Phi_{\underline{ij}}(\overline{\mathbf{c}}) \in [\frac{1}{(n-1)v}, \frac{1}{v}]$ for all $\underline{ij} \in T$, because

$$1 < \sum_{k\in N(i;g^{\mathbf{c}})\ s.t.\ \underline{ij}\in\overline{p}_{ik}} \delta(\overline{p}^{ij}_{ik}) < n-1.$$

Note that for each pair $i, j \in N$ connected by the tree there is only one path connecting them in $g^{\mathbf{c}}$, and consequently $\delta(\overline{p}^{ij}_{ik})$ is a product $\Pi_{lm\in\overline{p}_{jk}}\delta(c_{\underline{lm}})$ of continuous functions. Thus each $\Phi_{\underline{ij}}(\overline{\mathbf{c}})$ is continuous and so is Φ consequently. Note that although investments are not bounded, in a network with n nodes no link can support a flow of information greater than $(n-1)v$. Consequently, in a marginal equilibrium no link receives an investment c s.t. $\delta'(c) < 1/(n-1)v$. On the other hand, as no link transmits less than v, in marginal equilibrium no link receives an investment c s.t. $\delta'(c) > 1/v$. Thus denote by D^{-1} the continuous function

$$D^{-1} : [\frac{1}{v(n-1)}, \frac{1}{v}]^T \longrightarrow [\delta'^{-1}(\frac{1}{v}), \delta'^{-1}(\frac{1}{v(n-1)})]^T$$

where $D^{-1}_{\underline{kl}}((x_{\underline{ij}})_{\underline{ij} \in T}) = \delta'^{-1}(x_{\underline{kl}})$. Then the composition $D^{-1} \circ \Phi$ is a continuous function that maps compact convex set $[\delta'^{-1}(\frac{1}{v}), \delta'^{-1}(\frac{1}{v(n-1)})]^T$ to itself, so there must be a fixed-point $\overline{\mathbf{c}} = (\overline{c}_{ij})_{i,j \in N}$ s.t. $D^{-1}(\Phi(\overline{\mathbf{c}})) = \overline{\mathbf{c}}$, that is, s.t.

$$\Phi_{\underline{ij}}(\overline{\mathbf{c}}) = \delta'(\overline{c}_{\underline{ij}})$$

for all $\underline{ij} \in T$. Therefore, the investment profile where each link $\underline{ij}$ in the tree receives an investment of $c_{\underline{ij}}$ and links for which the two nodes that it connects are paid for by the node that receives more information through it is a marginally stable investment whose underlying graph is T. By Lemma 2, if the tree is all-encompassing it will also be strongly marginally stable. ■

Proposition 10

Proof. If $\delta'(0) > 1/v$, any star graph can be sustained in marginal equilibrium as a corollary of Proposition 9. Now assume $\delta'(0) \leq 1/v$. Let $p \geq 3$ and define $\varphi(c) := \frac{1}{v(1+(p-2)\delta(c)}$. We prove that for a p sufficiently big there is sure to be $c > 0$ s.t. $\varphi(c) = \delta'(c)$, i.e. condition (16) of Proposition 8 holds and consequently a periphery-sponsored star of p nodes is marginally stable. Given $\varphi(c) > \frac{1}{v(1+(p-2)\delta(\infty)}$ for all $c > 0$, and $\lim_{c \to \infty} \delta'(c) = 0$, it is sufficient to prove that for n sufficiently big $\varphi(1) \leq \delta'(1)$. But it is easy to check that this is equivalent to

$$p - 2 \geq \frac{1 - 2v\delta'(1)}{v\delta(1)\delta'(1)},$$

which is sure to hold for p a big enough.

Finally, if inequality $\delta'(0) \leq \frac{1}{v(1+(p-2)\delta(\infty))}$ holds for a fixed p, then the graphs of $\varphi(c)$ and $\delta'(c)$ do not intersect, because

$$\varphi(c) = \frac{1}{v(1+(p-2)\delta(c))} > \frac{1}{v(1+(p-2)\delta(\infty))} \geq \delta'(0) > \delta'(c)$$

for all c. Note that there is no contradiction with the preceding result: whatever the value of $\delta'(0)$, for p *big enough* $\delta'(0) > \frac{1}{v(1+(p-2)\delta(\infty))}$. ■

Proposition 11

Proof. Let $T \subset N_2$ be the graph of a circle of k nodes where nodes are numbered so that only consecutive nodes and 1 and k are linked, i.e. $T = \{\underline{12}, \underline{23}, \underline{34}, ..\underline{k-1k}, \underline{k1}\}$. There are two cases:

Case k is *odd*: Let $\mathbf{c} = (c_{ij})_{i,j \in N}$ be an investment profile where all links in T receive the same investment, i.e. $c_{\underline{ij}} = \overline{c}$ if $\underline{ij} \in T$ and $c_{\underline{ij}} = 0$ otherwise. Then note that the optimal path connecting any two nodes is the shortest, and at least one of the two nodes that each link connects invests in it, through which the information that it receives is

$$v(\delta(\overline{c}) + \delta(\overline{c})^2 + \delta(\overline{c})^3 + ... + \delta(\overline{c})^{\frac{k-1}{2}}).$$

Thus condition (14) applies and becomes (17). If (17) holds, however players share the cost of each link all conditions of Theorem 2 hold and $\mathbf{c}$ is marginally stable.

Case k is *even*: Let $\mathbf{c} = (c_{ij})_{i,j \in N}$ be an investment profile where $c_{\underline{ij}} = \overline{c}$ or $\overline{\overline{c}}$, alternating $\overline{c}$-links and $\overline{\overline{c}}$-links so that

$$c_{\underline{ij}} = \begin{cases} \overline{\overline{c}}, \text{ if } \underline{ij} \in T \text{ and } \min\{i,j\} \text{ is odd,} \\ \overline{c}, \text{ if } \underline{ij} \in T \text{ and } \min\{i,j\} \text{ is even,} \\ 0 \text{ otherwise,} \end{cases}$$

i.e. $c_{\underline{12}} = \overline{\overline{c}}$, $c_{\underline{23}} = \overline{c}$, $c_{\underline{34}} = \overline{\overline{c}}$, etc., with $\overline{c} < \overline{\overline{c}}$. Note that if $k/2$ is even, the optimal path is the shortest when there is only one, while when there are two of the same length the information through the one containing the $\overline{\overline{c}}$-link that the player is involved in is

$$v(\delta(\overline{\overline{c}}) + \delta(\overline{\overline{c}})\delta(\overline{c}) + \delta(\overline{\overline{c}})\delta(\overline{c})\delta(\overline{\overline{c}}) + \delta(\overline{\overline{c}})^2\delta(\overline{c})^2 + ... + \delta(\overline{\overline{c}})^{\frac{k}{4}}\delta(\overline{c})^{\frac{k}{4}}),$$

while through the one containing the $\overline{c}$-link that the player is involved in, is

$$v(\delta(\overline{c}) + \delta(\overline{c})\delta(\overline{\overline{c}}) + \delta(\overline{c})\delta(\overline{\overline{c}})\delta(\overline{c}) + \delta(\overline{\overline{c}})^2\delta(\overline{c})^2 + ... + \delta(\overline{\overline{c}})^{\frac{k}{4}}\delta(\overline{c})^{\frac{k}{4}}).$$

A term to term comparison shows that the optimal path is the one containing the $\overline{\overline{c}}$-link that the player is involved in. Thus condition (14) applies to $\delta'(\overline{\overline{c}})$ becoming (18).Then the actual information a node receives through the path containing the $\overline{c}$-link that that node is involved in is

$$v(\delta(\overline{c}) + \delta(\overline{c})\delta(\overline{\overline{c}}) + \delta(\overline{c})\delta(\overline{\overline{c}})\delta(\overline{c}) + \delta(\overline{\overline{c}})^2\delta(\overline{c})^2 + ... + \delta(\overline{\overline{c}})^{\frac{k-4}{4}}\delta(\overline{c})^{\frac{k}{4}}).$$

Thus condition (14) applies to $\delta'(\overline{c})$ and becomes (19). Therefore if (18) and (19) hold, $\mathbf{c}$ is marginally stable. A similar argument leads to the same conclusion if $k/2$ is odd. Again a fixed-point argument proves that the existence of such $\overline{c}$ and such $\overline{\overline{c}}$ is guaranteed if $\delta'(\overline{c}) > 1/v$. Finally, the result for $k = n$ follows from Lemma 2. ■

Proposition 12

Proof. The necessity is a corollary of Proposition 5. As for sufficiency, let δ be a DR-technology and $g^{\mathbf{0}}$ the empty network, i.e. $c_{ij} = 0$ for all $i, j \in N$. In these conditions a player has an incentive to invest $c > 0$ in a link with another (or any number of them) only if $v\delta(c) - c > 0$. But, by the assumptions on technology δ, if $\delta'(0) \leq 1/v$ and $c > 0$, then $\delta(c) < c\delta'(0) \leq c/v$, i.e. $v\delta(c) - c \leq 0$ for all c. ■

Proposition 13

Proof. Let $\widehat{\mathbf{c}}_n$ be an investment profile such that $g^{\widehat{\mathbf{c}}_n}$ is complete, symmetric and marginally stable, i.e. $c_{\underline{ij}} = \widehat{c}_{eq}$ and $c_{ij} = \frac{\widehat{c}_{eq}}{2}$, with $\delta'(\widehat{c}_{eq}) = 1/v < \delta'(0)$ by Proposition

7. The possible best responses of a node i consist of withdrawing support from $k-1$, links (with $2 \leq k \leq n-1$) *and* replacing one of the remaining $\widehat{c}_{eq}$-links that connects it with a node connected with them by $\widehat{c}_{eq}$-links by a $\overline{c}_k$-link that optimizes the benefit of this indirect connection with these $k-1$ nodes (see Figure 3[15]), i.e. by a $\overline{c}_k$-link s.t. (21) by investing $\overline{c}_k - \frac{\widehat{c}_{eq}}{2}$ in it.[16] Then, if $\mathbf{a}^k$ denotes the response described, $\widehat{\mathbf{c}}_n$ is Nash-stable if and only if

$$\Pi_i^\delta(\widehat{\mathbf{c}}_n) - \Pi_i^\delta(\mathbf{a}^k) \geq 0, \text{ for all } k = 3, ..., n,$$

i.e., if and only if

$$k(v\delta(\widehat{c}_{eq}) - \widehat{c}_{eq}/2) - \delta(\overline{c}_k)(v + (k-1)\delta(\widehat{c}_{eq})v) - (\overline{c}_k + \widehat{c}_{eq}/2) \geq 0,$$

which yields (20). ■

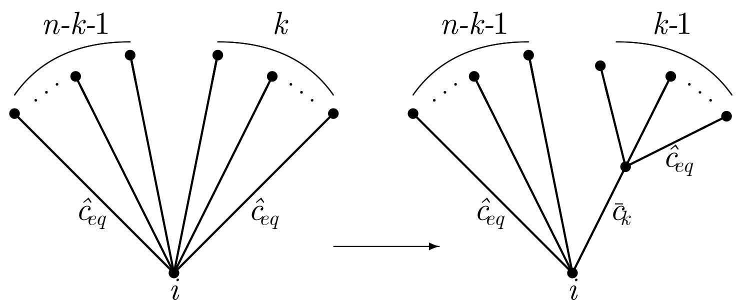

Figure 3: A possible best response to a complete network

Proposition 14
Proof. Let $\mathbf{c}_n^*$ be an investment profile such that $g^{\mathbf{c}_n^*}$ is a marginally stable all-encompassing star. By Proposition 8, $g^{\mathbf{c}_n^*}$ must be a periphery sponsored star of c_{eq}^*-links. The central node is obviously playing its best response. If $\delta'(0) > 1/v$ the possible best responses of a spoke node i consist of connecting k spoke nodes (with $1 \leq k \leq n-2$) by $\widehat{c}_{eq}$-links and replacing the link c_{eq}^*-link that connects it with the center by a link that optimizes the benefit of connecting with the center of the star formed by the remaining $n-k-2$ c_{eq}^*-links (see Figure 4), i.e. by a $\overline{c}_k$-link s.t. (23). Then, if $\mathbf{b}^k$ denotes the response described, we have

$$\Pi_i^\delta(\mathbf{c}_n^*) = v(1 + (n-2)\delta(c_{eq}^*))\delta(c_{eq}^*) - c_{eq}^*,$$

$$\Pi_i^\delta(\mathbf{b}^k) = v(1 + (n-k-2)\delta(c_{eq}^*))\delta(\overline{c}_k) - \overline{c}_k + k(v\delta(\widehat{c}_{eq}) - \widehat{c}_{eq}).$$

[15]Note that only the links in which node i is involved and the paths through which it receives information after the response are represented. The same applies to Figure 4.

[16]Although there are other possible best responses by making a similar change for several sets of links, all are covered by this case for the different values of k.

Then $\mathbf{c}_n^*$ is Nash-stable if and only if for all $k = 1, 2, ..., n-2$:

$$\Pi_i^\delta(\mathbf{c}_n^*) - \Pi_i^\delta(\mathbf{b}^k) \geq 0,$$

i.e.

$$((n-k-2)\delta(c_{eq}^*) + 1)v(\delta(c_{eq}^*) - \delta(\bar{c}_k)) - (c_{eq}^* - \bar{c}_k) + k(v\delta(c_{eq}^*)^2 - (v\delta(\widehat{c}_{eq}) - \widehat{c}_{eq})) \geq 0,$$

which yields (22).

Note that if $\delta'(0) \leq 1/v$, the only best response of a spoke node is to keep the investment unchanged. ■

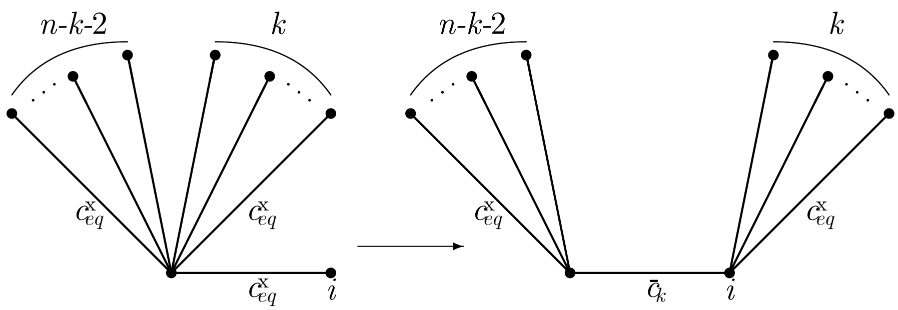

Figure 4: A possible best response to the star

Proposition 15
Proof. In view of Theorem 1, the empty network is efficient if and only if $\delta(c) \leq \varphi_n(c)$ for all c, with φ_n given by (8), and by Propositions 5 and 12 the empty network is strongly marginally stable and Nash-stable if and only if $\delta'(0) \leq 1/v$. But if $\delta(c) \leq \varphi_n(c)$ for all c, then $\delta'(0) \leq 1/2v < 1/v$. Thus whenever the empty network is efficient it is also strongly marginally and Nash-stable, but not reciprocally: whenever $1/2v < \delta'(0) \leq 1/v$, the empty network is stable in both senses but not efficient.

In any other case, Propositions 1 and 2 establish necessary conditions for efficiency under a DR-technology for the only non-empty structures that are proven in Theorem 1 to be possibly efficient for any DR-technology: all-encompassing stars and complete networks. Propositions 7 and 8 provide necessary and sufficient conditions for these structures to be marginally stable. Comparing (4) and (15), with (6) and (16), makes it clear that efficient all-encompassing stars and efficient complete networks are *not* Nash-stable, or even marginally stable. This incompatibility extends to the case of an investment supporting an infrastructure efficiently and in marginal equilibrium according to Definitions 2 and 6. This follows immediately from a comparison between (11) and (13). ■

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