Proportional Tax under Ambiguity

Dong, Xueqi and Liu, Shuo Li

Newcastle University, Harvard University

May 2021

Proportional Tax under Ambiguity [*]

Xueqi Dong[†1] and Shuo Li Liu [‡2]

[1]Newcastle University
[2]Harvard University

May 10, 2021

Abstract

This paper studies how investment can be influenced by common tax and monetary policies, where investment is measured by the proportion of an investor' wealth invested in the asset that pays a random return. We further prove that all risk-uncertainty averse individuals will increase investments if and only if a type of proportional tax with full loss offset (Domar and Musgrave 1944, QJE) is imposed. This result holds: 1. under ambiguity, that is when the probability distribution of an asset's return is unknown; 2. when borrowing in the safe asset is allowed.

1 Introduction

Suppose a government wants to encourage investment in an economy, what are the common policies available and to what extent these policies can achieve the intended goal? First, an interest rate cut is widely used as the main monetary policy instrument to encourage investments and stimulate an economy as it is expected that lowering return for safe assets would encourage more investment in risky assets. While historically the effectiveness of interest rate is supported by empirical evidence, the effectiveness has been questioned in recent years [1].

While the diverge of reality from a theory can be due to the overlooked complexity in a real world, we want to point out that the widely-accepted theory that a rate cut leads to re-balancing towards riskier assets is flawed. This point has been made as early as in Arrow (1971). The intuition is as follows. In general, as people get poor, they lose scope to bear the risk and become more prudent in the safe assets they currently hold. Interest rate cut further impoverishes people, which makes them penalise risk more heavily in the trade-off between risk and return. It then results in less risk-taking. This can be viewed as a case when the income effect dominates the substitution effect[2].

[*]The authors thank Chris Chambers, Michael Richter and Wessel Vermeulen for valuable comments.

[†]xueqi.dong@ncl.ac.uk

[‡]shuoliliu@g.harvard.edu

[1]Boucinha and Burlon (2020) find that only 1/6 of eurozone GDP growth can be attributed to interest rate cut between 2014 and 2019. Eggertsson, Juelsrud, Summers and Wold (2019) even argue the negative interest rate may have a contractive effect on the economy. Some literature attribute the ineffectiveness to the capital structure of banks, which serve as the transmission channel from rate cute to the economy . For example, Dell'Aricciaa, Laevena and Marquez (2014) demonstrate theoretically that highly levered banks can decrease its risk taking in response to an interest rate cut.

[2]this is illustrated in Figure 2a on Page 9.

Since the effect of an interest rate cut is debatable, what about taxation? Ever since Domar and Musgrave (1944), academics are informed that a *proportional income tax with full loss offset*(PRIT) can increase individual risk taking, which is measured the proportion of capital invested in the risky assets. PRIT taxes the upside return of a risky asset and compensates its downside return using the same proportion. This effectively move away some of the risk of a risky asset hence can increase more risk taking. Common corporate profits tax features PRIT since loss can usually be carried forward as tax relief. However, more risk taking under PRIT can not be guaranteed when the interest rate is not assumed to be 0. Sandmo (1969) later points out this monotone effect of the PRIT can be maintained if it is levied on the excess return of the risky asset over the constant interest rate[3]. Though the effect of PRIT has been challenged when applied in a general equilibrium model (Konrad 1991), its partial effect on individual risky taking is generally accepted.

However, previous studies on the effect of taxation are usually in a setting when the outcomes of the risky assets are describable by objectively known(or agreed on) probability distributions(termed risk), which is either in an Arrow-Debreu framework with known state probabilities or in von Neumann- Morgenstern framework. The more realistic setting when such probability distributions are unknown is usually termed as *ambiguity*. Ever since Ellsberg Paradox (Ellsberg, 1961), a large body of research has found there is an extra layer of preferences over ambiguity. Relevance of ambiguity attitude has been evidenced in experiments, insurance demand, medical decisions (see Machina and Siniscalchi 2013 for a survey) and portfolio choice (among others, see Bossaerts *et al* (2010)for an experimental study, Dimmock *et al* (2016) for a study using US household portfolio choice data).

How investors react to a policy under ambiguity has attracted little attention. Therefore, our paper aims to fill in this gap. We consider the standard portfolio choice problem of choosing the proportions of a fixed capital to be invested in an uncertain asset that pays a state-dependent return and to be kept in one interest-bearing safe asset. We use a standard Anscombe-Aumann framework to model ambiguity. Investment is measured by the proportion of capital invested in the uncertain asset. Uncertainty aversion, which is a general notion of ambiguity aversion that many theories of ambiguity is built on[4], is assumed. We then prove that all *strictly risk averse* and *uncertainty averse* investors will increase investment in response to a change in the distribution of an assets' return **if and only if** PRIT is implemented.

We further prove that this result still stands when the safe asset is allowed to be short. We think this is relevant as it is common that households or institutions borrow to invest, especially during a low interest rate environment.

2 Historical Notes

Previous studies on the effect of interest rate typically assume Expected Utility(EU) preferences and adopt a portfolio choice problem of one risky asset and one safe asset. Arrow (1971) shows that the demand for the safe asset is monotone with interest rate if an EU maximiser has decreasing absolute risk aversion and increasing relative risk aversion.

[3] Sandmo made this point as a minor result in a two-period model.

[4] For example, Schmeidler's (1989)'s Choquet Expected Utility with convex capacity, Gilboa and Schmeidlerl's (1989) max-min Expected Utility, Maccheroni, Marinacci, and Rustichini's (2006) Variational Preference, Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio's (2011) penalization representation, Strzalecki (2011) Multiplier Preferences.

However, Kubler *et al* (2013) show that this result does not hold when short-selling of the safe asset is allowed.

Instead of studying the asset demand (the actual amounts of capital invested in assets), finance-oriented studies adopt the portfolio choice problem with the objective of choice being the proportions of capital allocated into assets. Fishburn and Porter (1976) prove that an increase in the interest rate guarantees a decrease in the optimal proportion invested in the risky asset when an EU maximiser's absolute risk aversion is nondecreasing or her proportional risk aversion never exceeds unity. Harder and Seo (1990) extend their study to the case when there is more than one risky asset. Meyer and Ormiston (1994) further extend it to the case when the return distributions of assets are dependent.

To direct study how tax affects investment behaviour is an important field. Since Domar and Musgrave (1944), it is widely accepted by academia that proportional income tax with full loss offset (PRIT) can increase risk taking for some investors. Tobin (1958) pioneered the study of tax effect in a portfolio choice problem under a risk averse EU framework. He shows that a PRIT that scaling down each realisation of the return of a risky asset cause all investors to increase investments when either utility function is restricted to a quadratic function form or the probability distribution of investment return can be characterised by only mean and variance. Feldstein (1969) demonstrates that Tobin' result relies on the assumption that the safe asset has zero yield[5]. Allowing a positive yield on the risk-free asset, Stiglize (1969) derives the conditions on risk aversion measures so that PRIT is still sufficient to cause a monotone effect.

As a tax, PRIT works by effectively change the probability distribution of the risky asset return $\mathbf{f}$. More recent studies focus on directly deriving conditions on the change of the return distribution to ensure monotone change on portfolio choice. Gollier (1995) provides the necessary and sufficient condition on the change of the return distribution of the risky asset f such that all risk averse EU maximisers will increase risk taking. He terms the condition "Greater Central Riskiness" (GCR), which means there exists a scalar γ such that the Cumulative Density Function of new distribution $\int_{-\infty}^{x} s d\mathbf{f'}(s) \geq \gamma \int_{-\infty}^{x} s d\mathbf{f}(s)$. Being sufficient and necessary condition, GCR is the least constraining condition. Several preceding studies had provided special cases of GCR that are sufficient to guarantee all risk averse EU investors to take more risk (For example, see Fishburn and Porter (1976) and Milgrom (1981). PRIT gives $\mathbf{f}' - \mathbf{r} = (1-t)(\mathbf{f} - \mathbf{r})$ where $t \in (0,1)$. This then gives $\int_{-\infty}^{x} s d\mathbf{f'}(s) = (1-t)\int_{-\infty}^{x} s d\mathbf{f}(s) + \int_{-\infty}^{x} rt ds$ where the second part is positive. So we have $\int_{-\infty}^{x} s\mathbf{f'}(s) \geq (1-t)\int_{-\infty}^{x} s d\mathbf{f}(s)$. This demonstrates PRIT is a special case of GCR where we assign γ to be $(1-t)$.

While most studies focus on the EU, Ormiston and Schlee (2001) argue mean-variance (MV) preferences should be addressed. They prove that all MV preferences investors increase risk taking in response to a change in the distribution of that asset's return if and only if the change reduces both the mean and standard deviation of the risky asset's return by the same percentage. Note in their MV model, the preference functional is defined in general over the mean and variance instead of only restricts to a linear form of mean and variance, which can be considered as an EU preference takes a quadratic utility function. Since this MV model is not a special case of EU, GCR and its special cases (including PRIT) can not automatically guarantee more risk taking for MV investors.

[5]Feldstein shows that Tobin's result only holds when one of the assets has zero yield and zero risk. He argues that the assumption of the existence of a safe asset is too restrictive due to inflation risk. Subsequent studies in general accepts this assumption. Besides, given most central banks in developed countries are targeting low and stable inflation rates, we think assuming the existence of an safe asset is acceptable, especially our goal is to study the effect of a policy, which is usually targeting a short term.

PRIT is more effective and easier to implement compared to the preceding two cases. It is not restricted to the type of investors of EU or MV preferences or in the setting of risk in general. Ormiston and Schlee(2001)'s tax is not easy to implement as it is difficult to envisage how to implement a policy based on variance from a practical standpoint.

More recently, Gollier (2011) adopts a static two-asset portfolio problem with one safe asset and one uncertain one. The ambiguity of the uncertain asset is expressed by a second-order prior probability distribution over the set of first-order distributions for the excess return. His purpose is to apply a popular decision theory under ambiguity (Klibanoff, Marinacci and Mukerji (2005)) to a portfolio choice model for studying its comparative statics under ambiguity aversion. Our goal is to draw insight with a general notion of ambiguity aversion without restricting to a particular theory. To our humble knowledge, although the tax policy under objective risk has been extensively studied, there exists extremely sparse literature on the formal analysis of tax policy under Knightian uncertainty[6].

3 The Model

Since the main insight can be drawn from a model of one uncertain asset and one safe asset, we consider this simple model here and provide the analysis for multiple assets in Appendix D. Consider the typical one-period portfolio choice problem: an investor allocates a proportion $\alpha \in \mathbb{R}_+$ of her capital w_0 to an uncertain asset and the remaining proportion $(1-\alpha)$ to a sage asset. $\alpha > 1$ is allowed as investors can borrow to invest. Denote the set of states by S that the outcome of the uncertain asset will depend on, and let $s \in S$ denote an individual state. Suppose there is a finite number of states that is also denoted by S. The gross rate of return (hereafter return) of the uncertain asset is $f : S \to X$ that maps states into a linear space X.

We will mainly study the case when the linear space $X = \Delta(Z)$, where $Z = \mathbb{R}_+$ is the monetary outcome space and Δ is a probability simplex. The typical element in $\Delta(Z)$ is a probability distribution over monetary outcomes, which is termed as an *(objective) lottery* in decision theory literature. The return of the uncertain asset in state s is denoted by f_s, which is a lottery. And hence, ambiguity is expressed in this way: the subjective uncertainty (states) will solve and, depend on how it resolves the return of the uncertain asset is an objective lottery. While the information about the probability of the subjective uncertainty is not available, the specification and the parameters of the lottery can be estimated using statistics [7].

Recent decision models under ambiguity are often (see Marchina and Siniscalchi 2014 for a survey) built on this type of Anscombe-Aumann (AA) framework, where $f : S \to \Delta(Z)$ is called an AA act or a two-stage horse-roulette act (hereafter act). The classic Expected utility model is maintained for preferences over primitive lotteries. This objective-subjective approach provides a framework for representing uncertain prospects that involve both objective and subjective uncertainty. In this set-up, ambiguity aversion attitudes featured in the Ellsberg paradox can be incorporated.

[6]For legal analysis, see Chorvat (2001); for the analysis of other policies under ambiguity, see Manski (2014).

[7]For example, consider how investors may formulate the effect of international travel restrictions on an airline company' return: in state 1 (with an travel restriction), the return is uniformly distributed on the region of two times the standard deviation of 0.2 around the mean of 0.4 ; state 2 (without travel restriction), the region is then two times the standard deviation of 0.2 around a higher mean of 1.4. While the mean and standard deviations can be calculated based on statistical data, there is not enough information to the probability of the event of a travel restrictions being imposed.

The safe asset pays a fixed return of $r \in \mathbb{R}_+$ in all states. With slightly abusing the use of notation we also let r denote the certainty act where the outcome on each state is the constant $r \in \mathbb{R}_+$. Therefore, the final wealth of the investor's portfolio in state s is

$$x_s = w_0(\alpha f_s + (1-\alpha)r).$$

and the generic form of the final wealth of the investor' portfolio is written as

$$x = w_0(\alpha f + (1-\alpha)r). \tag{1}$$

Since $X = \Delta(Z)$, the addition operation "$+$" in (1) (hereafter called *portfolio mix*) is different from the algebra addition in a classic portfolio choice model within Arrow-Debreu framework. It is also different from the operation of *probability mixture* that is often used in decision theory literature [8]. It is similar to the one in Gollier (2013)'s portfolio model. Consider f_s as a probability distribution of a random variable, which is independent of r. x_s can be considered the probability distribution of the weighted sum of the two independent random variables f_s and r. x is the statewise sum. The formal mathematical definition of the addition operation "$+$" in (1) is defined in Appendix A.

Preferences $\succsim$ are defined on final wealth x. We assume investors are *rational*, that is $\succsim$ satisfies *Completeness* and *Transitivity*. We assume investors are *risk-uncertainty averse*, which is implied by the standard assumption of *strictly risk aversion* and *uncertainty aversion* (Schmeidler 1989)[9]. The proof is demonstrated in Appendix B. Risk-uncertainty averse can be interpreted as investors are risk averse when they are dealing with objective risk and uncertainty averse when ambiguity is involved. Formally, it is defined as follows.

Assumption 1. A preferences $\succsim$ is *(Strictly) Risk-Uncertainty Averse* if it satisfies the following: for two different acts f and g, $f \succsim g$ implies $\lambda f + (1-\lambda)g \succ g$ for any $\lambda \in (0,1)$.

Strictly risk-uncertainty averse is equivalent to the standard definition of strictly convex preferences here and hence implies the existence of an unique optimum, which is useful in the next steps [10].

Since S is finite, we let the vectors denote acts, for example, $\mathbf{f} = (f_1, ..., f_S)$ represents the act f. Let $\mathbf{F}$ denote the return matrix $(\mathbf{f}, \mathbf{r})$. The budget set

$$B(\mathbf{F}, w_0) = \{\mathbf{x} \in X^S : \mathbf{x} = w_0(\alpha\mathbf{f} + (1-\alpha)\mathbf{r}), \alpha \geq 0\} \tag{2}$$

is a compact and closed set. Then the investor's portfolio choice problem can be represented by the following Utility Maximisation Problem

$$\max\ u(\mathbf{x}) \quad s.t.\ x \in B(\mathbf{F}, w_0) \tag{3}$$

[8]To see the difference, for instance, note $1-\alpha$ can be negative here while it is insensible in the probability mixture. Figure 5 in Appendix A provides an example of the comparison.

[9]Strictly risk aversion means that the marginal utility of money is decreasing. Uncertainty aversion implies that the decision maker prefer a risky alternative to an ambiguous alternative in Ellsberg's paradox. As a vital axiom, uncertainty aversion appears in many recent preference functional under uncertainty and ambiguity. For example, Schmeidler's (1989)'s Choquet Expected Utility with convex capacity, Gilboa and Schmeidlerl's (1989) max-min Expected Utility, Maccheroni, Marinacci, and Rustichini's (2006) Variational Preference, Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio's (2011) penalization representation, Strzalecki (2011) Multiplier Preferences. Alternatively, risk-uncertainty averse can also be implied by variance averse and uncertainty averse. This is demonstrated in Appendix C

[10]The standard *strictly convexity* definition is if $y \succsim x$, $z \succsim x$ and $y \neq z$ then $\alpha y + (1-\alpha)z \succ x$ for all $\alpha \in (0,1)$. If we let $z = x$, *strictly convexity* immediately implies *Risk-Uncertainty Averse* . Now we show the opposite. Take $y \succsim x$, $z \succsim x$ and $y \neq z$. By completeness, at least one of the following stands: $z \succsim y$ or $y \succsim z$. When $z \succsim y$, by *Risk-Uncertainty Averse* , $\alpha y + (1-\alpha)z \succ y$. Then by transitivity: $\alpha y + (1-\alpha)z \succ x$. When $z \succsim y$, by *Risk-Uncertainty Averse* , $(1-\alpha)z + \alpha y \succ z$. By transitivity: $(1-\alpha)z + \alpha y \succ x$.

where $u(\cdot) : X \to \mathbb{R}$ is strictly quasiconcave. It then follows there exists an unique demand

$$\mathbf{x}^* = \{\mathbf{x} \in B : \forall \mathbf{y} \in B, u(\mathbf{x}^*) \geq u(\mathbf{y})\}$$

and the corresponding *investment* is $\alpha^* = (\frac{\mathbf{x}^*}{w_0} - \mathbf{r})/(\mathbf{f} - \mathbf{r})$, which is the proportion allocated to f under $\mathbf{x}^*$.

Our goal is to study how policies affect investment α^*. A *policy* can either be a tax policy that effectively changes the return of the uncertain asset $\mathbf{f}$ or a monetary policy that changes the interest rate r. PRIT is a policy that works through effectively scaling down the excess return of uncertain asset over the risk free rate with the same rate under all states. Formally, it is defined as follows.

Definition 1. A *Proportional Income Tax with Full Loss Offset* (PRIT) of a tax rate $t, 0 < t < 1$ changes f to f' such that $f' - r = (1-t)(f - r)$.

Or we can also write $f' = (1-t)f + tr$. There are two interpretations for PRIT (Sandmo 1979). The first one is to consider it as a proportional income tax with deductible interest. The wealth before tax at the end of the period is $w_1 = w_0(\alpha f + (1-\alpha)r)$ and wealth after tax is $w_1^t = w_0(\alpha f' + (1-\alpha)r)$. Income before tax is $I = w_1 - w_0 = w_0\alpha(f-r) + w_0(r-1)$. Let T denote the amount of tax under PRIT, then $T = w_1 - w_1^t = tw_0\alpha(f-r)$. If the investor borrows the initial capital w_0, then there is an interest worth $w_0(r-1)$ needs to be repaid at the end of the period. The taxable income becomes $I' = I - w_0(r-1) = w_0\alpha(f-r)$. Comparing I' with T, it means PRIT is a proportional tax on the taxable income: $T = tI'$. Alternatively, we can consider $w_0(r-1)$ as the opportunity cost and consider $I' = I - w_0(r-1)$ as the excess income of investing in the uncertain asset.

4 Effect of Proportional Income Tax with Full Loss Offset (PRIT)

In this section, we provide our results in two propositions. The first proposition states PRIT is the sufficiency and necessary condition on the change of an asset's return to guarantee all investors to increase investment when borrowing the safe asset is allowed. In the second proposition, borrowing is not allowed while the same result is maintained. We normalise the initial wealth w_0 to be 1 since we do not study the effect of change in wealth on investment behaviour.

Proposition 1. *All investors raise investments in response to a change in an asset's return if and only if PRIT is implemented.*

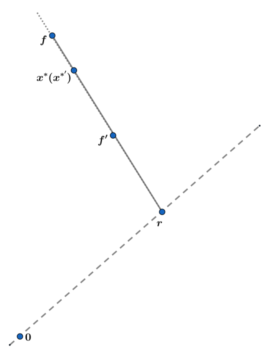
Proof. Here we illustrate the proof using a simple case where there are two assets and two states. The formal proof that considers multiple assets and multiple states is presented in Appendix D. Since $w_0 = 1$, $\alpha > 1$ means the investor is borrowing in the safe asset to buy the uncertain asset.

Sufficiency:

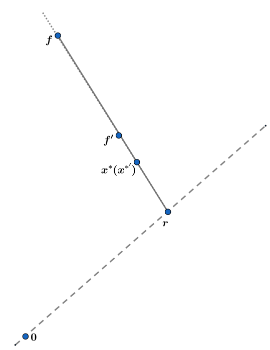
The proof of sufficiency is to show that all investors would increase more in the uncertain asset after the tax is imposed. In Figure 1, $\mathbf{f}$ and $\mathbf{r}$ denote the return for the uncertain asset and the safe asset respectively. $\mathbf{0}$ is a certainty act where the outcome on each state is a certainty 0. Then the budget constraint B is the line segment that starts with $\mathbf{r}$

and crosses **f**. Let $\overline{\mathbf{fr}}$ denote the segment with endpoints **f** and **r**. After imposing PRIT, **f** moves to the position as indicated by $\mathbf{f}'$. When borrowing in the risk-free asset is allowed, PRIT does not change the budget set and hence the new optimum $\mathbf{x}^{*'}$is still the same as in initial optimum $\mathbf{x}^*$. There are three types of investors to discuss based on the position of the initial optimum $\mathbf{x}^*$.

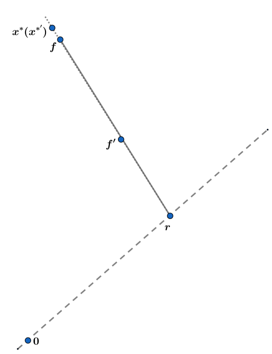
We first consider the type of investors who do not borrow initially, that is $\alpha^* \leq 1$ and the optimum $\mathbf{x}^*$ is within $\overline{\mathbf{fr}}$. Depending on the relative positions of $\mathbf{x}^*$ to $\mathbf{f}'$, we further have two cases to analyse. In Figure 1a, $\mathbf{x}^*$ is in the northwest of f'. This means the investors borrow some risk-free asset to buy the uncertain asset after PRIT. This is clearly an increase in investment. In Figure 1b, $\mathbf{x}^*$ is in the southeast of $\mathbf{f}'$. Since $\mathbf{x}^*$ is relatively close to $\mathbf{f}'$ than to **f**, this also means the investment has increased.



(a) No borrow initially, borrow after PRIT
Initially $\alpha^* \leq 1$; after PRIT, to obtain the same optimum $\mathbf{x}^*$ the investor needs to borrow $\alpha^{*'} > 1$ so $\alpha^{*'} > 1 > \alpha^*$.



(b) No borrow initially, no borrow after PRIT
Initially $\alpha^* \leq 1$; after PRIT, $\mathbf{x}^*$ is still inside **f'r** but it becomes closer to $\mathbf{f}'$ compared to **f**. This means the investor needs to borrow so $1 \geq \alpha^{*'} > \alpha^*$



(c) Borrow initially(implies also borrow after PRIT)
Initially $\alpha^* > 1$; after PRIT, $\mathbf{x}^*$ become further away from $\mathbf{f}'$ compared to **f**. This means the investor needs to borrow more so $\alpha^{*'} > \alpha^* > 1$.

Figure 1: Sufficiency (PRIT increases investment)
f and **r** denote the return for the uncertain asset and the risk free asset respectively. **0** is a certainty act where the outcome on each state is a certainty 0. PRIT does not change the budget set hence the new optimum $\mathbf{x}^{*'}$ is the same as the original one $\mathbf{x}^*$.These three figures show different positions of $\mathbf{x}^*$. that representing three types of investors.

Now consider the type of investors that borrow to invest initially, that is $\alpha^* > 1$ as shown in Figure 1c. Given $\mathbf{x}^*$ is further away from $\mathbf{f}'$ than from $\mathbf{f}$, we have $\alpha^{*'} > \alpha^*$ which means this type of investors would borrow even more to invest after the Tax. Some algebra can also show the preceding arguments. The new budge line after PRIT is $B(f, r) = \{\alpha\mathbf{f}' + (1-\alpha)\mathbf{r} : \alpha \in \mathbb{R}^+\}$ which can be simplified as $B'(\mathbf{F}, \mathbf{r}) = \{(1-t)\alpha\mathbf{f} + (1-(1-t)\alpha)\mathbf{r} : (1-t)\alpha \in \mathbb{R}^+\}$ after substituting $\mathbf{f}' = (1-t)\mathbf{f} + t\mathbf{r}, 0 < t < 1$. The two budget sets are the same. So we have $\mathbf{x}^* = \mathbf{x}^{*'}$, which gives $\alpha^*\mathbf{f} + (1-\alpha^*)\mathbf{r} = \alpha^{*'}\mathbf{f}' + (1-\alpha^{*'})\mathbf{r}$. By simplification we can get $\alpha^{*'} = \alpha^*/(1-t) \geq \alpha^*, 0 < t < 1$. The new investment is multiplied by $1/(1-t) > 1$.

Necessity

The necessity proof is to show that if a policy makes all the investors invest more in the uncertain asset then the policy must be PRIT. It suffices to prove that there exist some investors for whom any policy except PRIT would cause them to invest less.

We choose the investors that there are two states $S = 2$ for the uncertain asset and the probability distribution for both lotteries follow a Dirac delta function:

$$\delta(x, f_s) = \begin{cases} +\infty & x = f_s \\ 0 & x \neq f_s \end{cases}$$

where $f_s \in \mathbb{R}_+$ for $s = 1, 2$. We are now dealing with a Euclidean space. Since there are only two states, one of the assets will be statewise dominated if the return of the uncertain asset is higher(or lower) than the safe asset in both states. This is hardly interesting and does not exist in arbitrage-free market, therefore will be excluded in the analysis. Without loss of generality, we let $f_1 < r$ and $f_2 > r$.

Suppose there is an investor for whom the initial investment $\alpha^* \in (0, 1)$ and the new investment is reduced $\alpha^{*'} \in (0, \alpha^*)$. We will show that this type of investors exist by demonstrating these choices can be represented by maximising a strictly concave utility function as in equation (3). We use a generalised version of Strong Axiom of Revealed Preference theory (Matzkin and Richter 1991, hereafter MR) to obtain strict concavity.

Consider the following Walrasian budget set

$$B(\mathbf{p}, m) = \{\mathbf{x} \in R_+^2 : \mathbf{px} \leq m\} \tag{4}$$

where $\mathbf{p} >> 0$. The set of chosen bundles is denoted by the correspondence $h(p, m) \subseteq B(\mathbf{p}, m)$. MR say that h is *exhaustive* if the all chosen bundles satisfy

$$\mathbf{px} = m.$$

Theorem 2 in MR states that for an exhaustive demand function h, if the chosen bundles satisfy Strong Axiom of Revealed Preference , then there exists a continuous and strictly concave function u rationalising h.

Now we show how we apply MR's theorem to our problem. Let $\mathbf{p} = (f_2 - r, r - f_1)$ denote the "prices" and $m = (f_2 - f_1)r$ denote the "income". Note the value of m is assigned such that for any $\alpha \in R$, there is $m = \mathbf{px}$ where $\mathbf{x} = \alpha\mathbf{f} + (1-\alpha)\mathbf{r}$. Therefore, h is automatically exhaustive in our problem. Next, we just need to show that SARP is obeyed.

Slightly abusing the use of notations, let $\mathbf{p}, m$ also denote the prices and income before the policy. And let $\mathbf{p}', m'$ denote their equivalence after a policy. The two chosen bundles can be written as $(\mathbf{p}, \mathbf{x}^*)$ and $(\mathbf{p}', \mathbf{x}^{*'})$. Depends on which asset a policy is implemented on, $\mathbf{p'}$ has different values. Next, we omit the symbol of $*$ in $\mathbf{x}$ and α to simplify notations.

Policy implemented through changing interest rate r

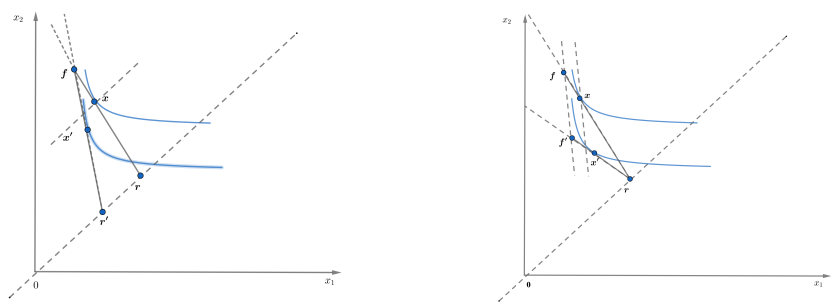
Let r' denote the new interest rate, hence $\mathbf{p'} = (f_2 - r', r' - f_1)$. Then $\mathbf{px} - \mathbf{px'} = (f_2 - f_1)(r - r')(1-\alpha')$ and $\mathbf{p'x} - \mathbf{p'x'} = (f_2 - f_1)(r - r')(1-\alpha)$. Since $\alpha' < \alpha < 1$, the two equations would have the same sign. That is either $\mathbf{px} > \mathbf{px'}, \mathbf{p'x} > \mathbf{p'x'}$ or $\mathbf{p'x'} > \mathbf{p'x}, \mathbf{px'} > \mathbf{px}$. SARP is obeyed.

Figure 2a illustrates the case of an interest rate cut $r' < r$. We can find a pair of $\mathbf{x}^{*'}$ and $\mathbf{x}^*$ such that $\mathbf{x}^{*'}$ that is below the dashed line that is paralleling to 45-degree line and crossing $\mathbf{x}^*$, which implies $\alpha^{*'} < \alpha^*$. Since $\mathbf{x}'$ and $\mathbf{x}$ do not violate SARP, it can be represented by equation (3). Similarly, we can also find such kind of investor in the case when interest rate increases.

Tax Policy implemented through changing f

Let $\mathbf{f}' = (f_1', f_2')$ denote the new return then $\mathbf{p'} = (f_2' - r, r - f_1')$. Then $\mathbf{px} - \mathbf{px'} = \alpha'(f_1 f_2' - f_2 f_1' - f_1 r + f_2 r + f_1' r - f_2' r)$ and $\mathbf{p'x} - \mathbf{p'x'} = \alpha(f_1 f_2' - f_2 f_1' - f_1 r + f_2 r + f_1' r - f_2' r)$. For any non-PRIT tax policy[11], we have $(f_2' - r)/(f_1' - r) \neq (f_2 - r)/(f_1 - r)$. This then guarantees that $f_1 f_2' - f_2 f_1' - f_1 r + f_2 r + f_1' r - f_2' r \neq 0$. Since both α' and α are positive, the two expressions have the same sign. SARP is obeyed.

Figure 2b shows the case when a non-PRIT policy reduces f_2 and increases f_1. Again we can find a pair of $\mathbf{x}^{*'}$ and $\mathbf{x}^*$ gives rise to $\alpha^{*'} < \alpha^*$ and do not violate SARP.



(a) Policy on risk-free asset: Interest Rate Cut (b) Non-PRIT Policy on Uncertain Asset f

Figure 2: Necessity (Non-PRIT policy gives rise to reduction in investment)
The pair of $\mathbf{x}^{*'}$ and $\mathbf{x}^*$ gives rise to $\alpha^{*'} < \alpha^*$ and satisfy SARP. This mean we find investors would reduce investment in reaction to these policies.

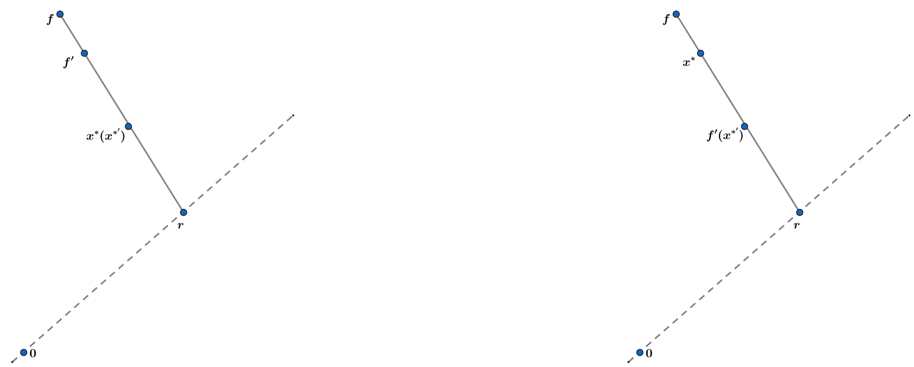
□

Proposition 2. *When borrowing the safe asset is not allowed, all investors raise investments in response to a change in an asset's return distribution if and only if PRIT is implemented.*

Proof. Differing from the Proposition 1, the budget set will not be the same after PRIT when borrowing risk-free asset is not allowed.

[11] We skip the analysis for the case when $\mathbf{f} - r = (1-t)(\mathbf{f} - \mathbf{r}$ where $t \geq 1$ or $t \leq 0$ as this case carries little economic meaning given we are discussing tax policy. However, its technical proof can be easily constructed in a similar way.

Sufficiency:



(a) Optimum Does Not Change
The original optimum $\mathbf{x}^*$ is obtainable in the new budget Set. Since it is closer between $\mathbf{x}^{*'}$ and f' than between $\mathbf{x}^*$ and $\mathbf{f}$, we have $\alpha^{*'} > \alpha^*$.

(b) Optimum Moves to the Boundary
The original optimum $\mathbf{x}^*$ is unobtainable. By (Assumption 1) convexity the new optimum $\mathbf{x}^{*'}$ must be at the end point of $\mathbf{f}'$. This implies $\alpha^{*'} = 1 \geq \alpha^*$

Figure 3: Sufficiency (PRIT increases investment when borrowing is not allowed)
The budget set is shortened after PRIT of borrowing safe asset is not allowed.

As shown in Figure 3, the budget line segment will be shortened after $\mathbf{f}$ moves to $\mathbf{f}'$. Depending on if $\mathbf{x}^*$ is still obtainable in the new budget set, there are two cases to discuss.

In Figure 3a, the shortened budget line still contains $\mathbf{x}^*$. Therefore the new optimum $\mathbf{x}^{*'} = \mathbf{x}^*$. Since the relative position of $\mathbf{x}^{*'}$ to $\mathbf{f}'$ is closer than the position of $\mathbf{x}^*$ to $\mathbf{f}$, $\alpha^{*'} > \alpha^*$. In Figure 3b, $\mathbf{x}^*$ is unattainable in the new budget line. The new optimum $\mathbf{x}^{*'}$ must be at the end point $\mathbf{f}'$, which means $\alpha^{*'} = 1 \geq \alpha^*$. The proof of of the preceding point is as follows. Suppose that the new optimum $\mathbf{x}^{*'}$ is not $\mathbf{f}'$, then $\mathbf{x}^{*'} \succ \mathbf{f}'$. We also have $\mathbf{x}^* \succ \mathbf{x}^{*'}$ as $\mathbf{x}^*$ is chosen when $\mathbf{x}^{*'}$ is obtainable. Since $\mathbf{f}'$ between $\mathbf{x}^*$ and $\mathbf{x}^* \succ \mathbf{x}^{*'}$, there is $f' \succ \mathbf{x}^{*'}$ by Assumption 1. This is a contradiction.

Necessity:

This shares the same proof of Proposition 1. In that necessity proof, we have demonstrated that there exist some investors who would reduce investment for all non-PRIT policy. Those investors are also valid for proof of Proposition 2 since they do not borrow the safe assets (Recall that $\alpha^* \in (0,1)$ and $\alpha^{*'} \in (0,1)$ were assumed).

□

5 Concluding Comments

We provide the first study on taxation effect on individuals' investment behaviour under ambiguity. In a one-period static portfolio choice model, we have proved that a proportional tax with full loss offset (PRIT) is the necessary and sufficient condition on an asset's return to guarantee all risk-uncertainty investors to increase investment, without assuming the uncertain asset's return probability distribution is known.

Though we have focused on the Anscombe-Aumann (AA) framework, the results stand for any linear space X. One can also apply the classic Arrow-Debreu framework of subjective uncertainty by assigning $X = \mathbb{R}_+$. Therefore the results also apply to the preferences such as Choquent Expected Utility with convex capacity (Gilboa, 1987) and Maxmin Expected Utility over Savage acts (Casadesus-Masanellab, Klibanoff and Ozdenorenare, 2000).

The key assumption of *(strict) risk-uncertainty aversion* is implied by *strict risk aversion* and *uncertainty aversion* and it guarantees the existence of an unique optimum. If we relax *strict risk aversion* to *risk aversion* then we get the weak version of strict risk-uncertainty aversion [12]. It then implies that there can be multiple optima and the existence of demand correspondence. We then need to first define a strong set-ordering, and then define the concept of "increase in investment" in the sense of strong set-ordering. In this case, our main results still hold. The necessity proof should use a generalised version of GARP (For example, see Cherchye, Demuynck, and De Rock (2014)).

Since our model is an individual decision making model, it provides a partial insight in terms of taxation effect on individual investor. Future study could extend the analysis to an aggregated level by incorporating a game-theoretic or a general equilibrium model.

[12] The weak version can be defined by replacing the strictly preferred $\succ$ with $\succsim$ in Assumption 1.

References

Boucinha and Burlon (2020). "Negative rates and the transmission of monetary policy". In: *Economic Bulletin Articles* 3.

Brunnermeier, Markus K and Yann Koby (2016). "The reversal interest rate: An effective lower bound on monetary policy". In: *unpublished paper, Princeton University*.

Buch, Claudia M, Sandra Eickmeier, and Esteban Prieto (2014). "In search for yield? Survey-based evidence on bank risk taking". In: *Journal of Economic Dynamics and Control* 43, pp. 12–30.

Casadesus-Masanell, Ramon, Peter Klibanoff, and Emre Ozdenoren (2000). "Maxmin expected utility over Savage acts with a set of priors". In: *journal of economic theory* 92.1, pp. 35–65.

Cerreia-Vioglio, Simone et al. (2011). "Uncertainty averse preferences". In: *Journal of Economic Theory* 146.4, pp. 1275–1330.

Cherchye, Laurens, Thomas Demuynck, and Bram De Rock (2014). "Revealed preference analysis for convex rationalizations on nonlinear budget sets". In: *Journal of economic theory* 152, pp. 224–236.

Chorvat, Terrence R (2001). "Ambiguity and income taxation". In: *Cardozo L. Rev.* 23, p. 617.

De Nicolò, Gianni et al. (2010). "Monetary policy and bank risk taking". In: *Available at SSRN 1654582*.

Debreu, Gerard (1972). "Smooth preferences". In: *Econometrica: Journal of the Econometric Society*, pp. 603–615.

DellAriccia, Giovanni, Luc Laeven, and Robert Marquez (2014). "Real interest rates, leverage, and bank risk-taking". In: *Journal of Economic Theory* 149, pp. 65–99.

Dimmock, Stephen G et al. (2016). "Ambiguity aversion and household portfolio choice puzzles: Empirical evidence". In: *Journal of Financial Economics* 119.3, pp. 559–577.

Domar, Evsey D and Richard A Musgrave (1944). "Proportional income taxation and risk-taking". In: *The Quarterly Journal of Economics* 58.3, pp. 388–422.

Eggertsson, Gauti B et al. (2019). *Negative nominal interest rates and the bank lending channel*. Tech. rep. National Bureau of Economic Research.

Ellsberg, Daniel (1961). "Risk, ambiguity, and the Savage axioms". In: *The quarterly journal of economics*, pp. 643–669.

Feldstein, Martin S (1969). "The effects of taxation on risk taking". In: *Journal of Political Economy* 77.5, pp. 755–764.

Fishburn, Peter C and R Burr Porter (1976). "Optimal portfolios with one safe and one risky asset: Effects of changes in rate of return and risk". In: *Management science* 22.10, pp. 1064–1073.

Forges, Francoise and Enrico Minelli (2009). "Afriat's theorem for general budget sets". In: *Journal of Economic Theory* 144.1, pp. 135–145.

Gilboa, Itzhak (1987). "Expected utility with purely subjective non-additive probabilities". In: *Journal of mathematical Economics* 16.1, pp. 65–88.

Gilboa, Itzhak and David Schmeidler (1989). "Maxmin expected utility with non-unique prior". In: *Journal of mathematical economics* 18.2, pp. 141–153.

Gollier, Christian (1995). "The comparative statics of changes in risk revisited". In: *Journal of Economic Theory* 66.2, pp. 522–535.

— (2011). "Portfolio choices and asset prices: The comparative statics of ambiguity aversion". In: *The Review of Economic Studies* 78.4, pp. 1329–1344.

Hadar, Josef and William R Russell (1969). "Rules for ordering uncertain prospects". In: *The American economic review* 59.1, pp. 25–34.

Hadar, Josef and Tae Kun Seo (1990). “The effects of shifts in a return distribution on optimal portfolios”. In: *International Economic Review*, pp. 721–736.
Hart, Oliver D (1975). “Some negative results on the existence of comparative statics results in portfolio theory”. In: *The Review of Economic Studies* 42.4, pp. 615–621.
Jakobsen, Katrine et al. (2020). “Wealth taxation and wealth accumulation: Theory and evidence from Denmark”. In: *The Quarterly Journal of Economics* 135.1, pp. 329–388.
Klibanoff, Peter, Massimo Marinacci, and Sujoy Mukerji (2005). “A smooth model of decision making under ambiguity”. In: *Econometrica* 73.6, pp. 1849–1892.
Konrad, Kai A (1991). “Risk taking and taxation in complete capital markets”. In: *The Geneva Papers on Risk and Insurance Theory* 16.2, pp. 167–177.
Kreps, David (1988). *Notes on the Theory of Choice*. Westview press.
Kubler, Felix, Larry Selden, and Xiao Wei (2013). “Inferior good and Giffen behavior for investing and borrowing”. In: *American Economic Review* 103.2, pp. 1034–53.
Landais, Camille, Emmanuel Saez, and Gabriel Zucman (2020). “A progressive European wealth tax to fund the European COVID response”. In: *VoxEU*.
Lanier, Joshua (2020). “Risk, ambiguity, and Giffen assets”. In: *Journal of Economic Theory* 186, p. 104976.
Maccheroni, Fabio, Massimo Marinacci, and Aldo Rustichini (2006). “Ambiguity aversion, robustness, and the variational representation of preferences”. In: *Econometrica* 74.6, pp. 1447–1498.
Machina, Mark J and Marciano Siniscalchi (2014). “Ambiguity and ambiguity aversion”. In: *Handbook of the Economics of Risk and Uncertainty*. Vol. 1. Elsevier, pp. 729–807.
Manski, Charles F (2014). “Choosing size of government under ambiguity: Infrastructure spending and income taxation”. In: *The Economic Journal* 124.576, pp. 359–376.
Mas-Colell, Andreu, Michael Dennis Whinston, Jerry R Green, et al. (1995). *Microeconomic theory*. Vol. 1. Oxford university press New York.
Matzkin, Rosa L and Marcel K Richter (1991). “Testing strictly concave rationality”. In: *Journal of Economic Theory* 53.2, pp. 287–303.
Meyer, Jack and Michael B Ormiston (1985). “Strong increases in risk and their comparative statics”. In: *International Economic Review*, pp. 425–437.
— (1994). “The effect on optimal portfolios of changing the return to a risky asset: The case of dependent risky returns”. In: *International Economic Review*, pp. 603–612.
Milgrom, Paul R (1981). “Good news and bad news: Representation theorems and applications”. In: *The Bell Journal of Economics*, pp. 380–391.
Mintz, Jack M (1981). “Some additional results on investment, risk taking, and full loss offset corporate taxation with interest deductibility”. In: *The Quarterly Journal of Economics* 96.4, pp. 631–642.
Mossin, Jan (1968). “Taxation and risk-taking: an expected utility approach”. In: *Economica* 35.137, pp. 74–82.
Ormiston, Michael B and Edward E Schlee (2001). “Mean-variance preferences and investor behaviour”. In: *The Economic Journal* 111.474, pp. 849–861.
Publishing, OECD. (2018). *The Role and Design of Net Wealth Taxes in the OECD*. OECD.
Sandmo, Agnar (1969). “Capital risk, consumption, and portfolio choice”. In: *Econometrica: Journal of the Econometric Society*, pp. 586–599.
Schmeidler, David (1989). “Subjective probability and expected utility without additivity”. In: *Econometrica: Journal of the Econometric Society*, pp. 571–587.
Stiglitz, Joseph E (1969). “The effects of income, wealth, and capital gains taxation on risk-taking”. In: *The Quarterly Journal of Economics* 83.2, pp. 263–283.
Streeten, Paul (1953). “The effect of taxation on risk-taking”. In: *Oxford Economic Papers* 5.3, pp. 271–287.
Strzalecki, Tomasz (2011). “Axiomatic foundations of multiplier preferences”. In: *Econometrica* 79.1, pp. 47–73.

Tobin, James (1958). “Liquidity preference as behavior towards risk”. In: *The review of economic studies* 25.2, pp. 65–86.