Why the concept of Hicks, Harrod, Solow neutral and even non-neutral augmented technical progress is flawed in principle in any economic model

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Capital and Labor Augmented Technical Progress, Growth Model, Maximum Profit Condition, Production Functions, General Technological Progress, Capital-Labor-mix, Elasticity of Substitution, Normalized CES Functions, Total Factor Productivity, DSGE Model, Solow Model, Hicks, Harris, Labor Saving

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Abstract

It is already known for several decades that the implementation of capital augmented technical progress, as is done to date, leads to the conclusion that the CES production has to be Cobb-Douglas or there exists labor augmented technical progress only. This is the so-called Cobb-Douglas labor augmented only paradox. Institutions keep on using this way of thinking in their models in spite of the theoretical inconsistency. We reject the old concept, i.e., all kind of neutral and non-neutral capital and labor augmented technical progress and introduce a new implementation of technical progress to avoid this theoretical problem. We explain the term labor saving technical progress, showing that technical progress is always relatively labor saving. We also analyze the problem on how to estimate the coefficient of elasticity of substitution. Economic growth is presented as partly exogenous, due to technical progress, and partly endogenous, due to capital growth. We introduce formulas to convert total factor productivity into economic growth to show the connection. This new theory is not limited to growth models but can be used also in DSGE models and possibly also in other areas where CES functions are useful. It will give you a different angle of view on the Solow model. And last but not least we will show the connection between Solow's growth accounting and neo-classical growth theory.

Keywords: Capital and Labor Augmented Technical Progress, Growth Model, Maximum Profit Condition, Production Functions, General Technological Progress, Capital-Labor-mix, Estimation of the Elasticity of Substitution, DSGE, Total Factor Productivity, Solow model, Hicks, Harris, Labor Saving

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Content

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We value your comments and will be available for discussions.

1. Introduction

Solow (1956) and Arrow et al. (1961) first introduced production functions with constant elasticity of substitution as an extension of the Leontief and the Cobb-Douglas production function so far used. Solow in 1956 started his paper with the following statement:

'All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive. A "crucial" assumption is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be reasonably realistic. When the results of a theory seem to flow specifically from a special crucial assumption, then if the assumption is dubious, the results are suspect.'

At that time, he was talking about the Harrod-Domar model, but his statement is still valid today and even in a much wider and general context. In line of thought with his paper, I wish to argue that this is true for capital and labor augmented technical progress, whether it is neutral or non-neutral (Brugger, 2017) (Klump, 2000).

In order to show you the details of this thought, we will start in section 2, with a description of a simple growth model. However, the model itself is not relevant. In section 3, we show the conditions of consumers behavior to reach a unique and stable equilibrium using CES functions in this growth model. Section 4 will be about wages, capital gain and the important relationship between capital share, capital-labor-mix and the capital to income ratio.

Although, the classical way of describing a CES function is as good as the normalized CES function from a mathematical point of view, each way has his advantage from an economical interpretation point of view. Section 5 will be about changing the base point, which is an important feature, that we will use together with the normalization method. To understand how we can discriminate a single CES function, we describe the class of CES functions.

In section 6, we will use this knowledge in a CES production function. We show the relation between general income and capital growth versus implementation of capital and labor augmented technical progress. General growth here is potential growth as a general progress phenomenon for capital and income.

In section 7, we introduce total factor productivity as the technical progress term, only caused by technical improvement. It contains every kind of growth that is not caused by the growth of other variables in the production function, i.e., not caused by capital and labor in our case. We will show the connection between total factor productivity, general growth, capital and labor augmented technical progress and the capital-labor-mix.

In section 8, we remind you that growth can also be influenced by consumer and producer behavior by changing to another capital to income ratio.

The known theoretical problem with regard to capital augmented technical progress leads us to the conclusion that we have to reject this type of progress as implemented to date (section 9). From here we take it a step further and have to conclude that Hicks, Harrod, Solow neutral and even (or better: especially) non-neutral implementations have to be rejected. However, we do not deny that there exists technical progress. It is the way in which it is implemented that is wrong.

In section 10, we discuss a new way of implementing technical growth, which is in fact not so new, but now at least we know why we do it. A brief introduction to the problem in estimating the capital-labor-mix, the elasticity of substitution and technical progress is captured in section 11.

We go into detail on the term labor saving progress in section 12, showing that technical progress is always relatively labor saving. Finally, in section 13 we discuss growth accounting vs. the classical growth theory. We end with some conclusions in section 14.

2. Simple growth model

As we did for the Cobb-Douglas case in De la Fonteijne (2011) we will start with the construction of our simple closed economy without government. This, however, is done for convenience and is not limiting our conclusions.

We assume that consumers have the possibility to decide to buy and consume the amount C they desire within the limits of their income. Our economy is transparent and customers tell producers the products they like to buy and producers produce exactly what is needed and the level of inventory is zero.

Producers on the other hand can decide which amount they will invest and are going to buy from capital goods producers.

Because these two purchases have to be equal to the total amount of production Y, we can write:

$$Y = C + I \tag{1}$$

This is also equal to the amount to be paid to the producers.

The producers have to pay the workers a wage w for the number of labor units L and they have to pay for the use of capital K. Direct or indirect these payments will end up with income Y

$$Y = wL + (\delta + r)K \tag{2}$$

in which δ is depreciation of capital K and r is interest on capital use.

As consumers can decide to spend C, the remaining part S of Y is saved.

$$Y = C + S \tag{3}$$

If we look at the production side, we assume the production to be dependent on K and L.

$$W = F(K, L) \tag{4}$$

This production W has the value Y

$$W = Y \tag{5}$$

As a result, we conclude that the investments I will equal the savings S.

$$I = S \tag{6}$$

Moreover, we can consider capital as accumulated labor combined with energy E (from the sun) and resources R (from mother earth). We assume this energy E and resources R are available for free and it becomes valuable once we add labor to exploit those resources. Action to preserve the environment can be thought as part of the consumption C once we agree upon this to do so, or even better to think of it as a part of depreciation to emphasize the fact that you have to consider it as costs to generate consumption C. For the sake of simplicity, we consider knowledge (human capital, research, entrepreneurial spirit, etc.) as factors responsible for technical growth concentrated in one, two or more parameters.

3. The specific case of CES production functions

We choose the production function F to be a homogeneous CES production function only for demonstration and convenience, because we can derive formulas in explicit and simple form. The philosophy stays the same if we choose an arbitrary other type of production function.

We will start with the general formula in normalized form (Klump et al., 2011) with only the production factors capital K and labor L.

$$Y = F(K, L) = Y_0[\alpha\left(\frac{K}{K_0}\right)^{\gamma} + (1-\alpha)\left(\frac{L}{L_0}\right)^{\gamma}]^{\eta/\gamma} \tag{7}$$

In which α, γ, η are parameters describing a specific production process in our economy. The interchangeability between K and L is characterized by the elasticity coefficient of substitution σ

$$\sigma = \frac{1}{1-\gamma} \tag{8a}$$

$$\gamma = \frac{\sigma-1}{\sigma} \tag{8b}$$

The parameter η defines the returns of scale. For reasons of easy explaining we take $\eta = 1$ in which case we can rewrite equation 7 per capita as

$$y = F(k, 1) = y_0[\alpha\left(\frac{k}{k_0}\right)^{\gamma} + (1-\alpha)]^{1/\gamma} \tag{9}$$

We consider a simple model with following equations:

$$Y = C + I \tag{10}$$

$$Y = Y_0[\alpha\left(\frac{K}{K_0}\right)^{\gamma} + (1-\alpha)\left(\frac{L}{L_0}\right)^{\gamma}]^{1/\gamma} \tag{11}$$

$$\dot{K} = I - \delta K \tag{12}$$

With Y is income, K is used capital, L labor needed and δ is the depreciation rate of capital K.

Per capita the equations are:

$$y = c + i \tag{13}$$

$$y = y_0[\alpha \left(\frac{k}{k_0}\right)^{\gamma} + (1-\alpha)]^{1/\gamma} \tag{14}$$

$$\dot{k} = i - \delta k \tag{15}$$

with the labor productivity

$$y = \frac{Y}{L} \tag{16}$$

the capital to labor ratio or capital deepening

$$k = \frac{K}{L} \tag{17}$$

the consumption to labor ratio

$$c = \frac{C}{L} \tag{18}$$

and the investment to labor ratio

$$i = \frac{I}{L} \tag{19}$$

If we choose

$$c = c_1 y \tag{20}$$

where c_1 is the consumer part of income y, then we can solve the equilibrium solution of equation 13-15 for k and y at every consumers choice c_1. In fact, c_1 is determined by c and i and so by consumers and producers spending. The equilibrium solution is (we use $p = y_0$ and $a_K = \frac{1}{k_0}$ interchangeable):

$$k_{c_1} = \left[\frac{1-\alpha}{\left(\frac{\delta}{p(1-c_1)}\right)^{\gamma} - \alpha a_K{}^{\gamma}}\right]^{1/\gamma} \tag{21}$$

$$y_{c_1} = \frac{\delta}{(1-c_1)}\left[\frac{1-\alpha}{\left(\frac{\delta}{p(1-c_1)}\right)^{\gamma} - \alpha a_K{}^{\gamma}}\right]^{1/\gamma} \tag{22}$$

$$c_{c_1} = c_1\frac{\delta}{(1-c_1)}\left[\frac{1-\alpha}{\left(\frac{\delta}{p(1-c_1)}\right)^{\gamma} - \alpha a_K{}^{\gamma}}\right]^{1/\gamma} \tag{23}$$

and, the capital to income ratio β at c_1

$$\beta_{c_1} = \left(\frac{k}{y}\right)_{c_1} = \frac{(1-c_1)}{\delta} \tag{24}$$

which is, with $s = (1 - c_1)$ equivalent to the well-known solutions found in literature $\beta = \frac{s}{\delta}$.

Maximizing the isoelastic utility function with risk neutrality $u(c)$, without discount, results in

$$c_{1\_opt} = 1 - \left(\left(\frac{pa_K}{\delta}\right)^\gamma \alpha\right)^{1/(1-\gamma)} \tag{25}$$

$$k_{opt} = \left[\frac{1-\alpha}{\left(\frac{\delta}{p(1-c_{1\_opt})}\right)^\gamma - \alpha a_K{}^\gamma}\right]^{1/\gamma} \tag{26}$$

$$y_{opt} = \frac{\delta}{(1-c_{1\_opt})}\left[\frac{1-\alpha}{\left(\frac{\delta}{p(1-c_{1\_opt})}\right)^\gamma - \alpha a_K{}^\gamma}\right]^{1/\gamma} \tag{27}$$

$$c_{opt} = c_{1\_opt}\frac{\delta}{(1-c_{1\_opt})}\left[\frac{1-\alpha}{\left(\frac{\delta}{p(1-c_{1\_opt})}\right)^\gamma - \alpha a_K{}^\gamma}\right]^{1/\gamma} \tag{28}$$

$$\beta_{c_{1\_opt}} = \left(\frac{k}{y}\right)_{c_{1\_opt}} = \frac{(1-c_{1\_opt})}{\delta} = \frac{1}{\delta}\left(\left(\frac{pa_K}{\delta}\right)^\gamma \alpha\right)^{1/(1-\gamma)} \tag{29}$$

For a graph of c, k, y as a function of $c1$ see fig. 1 with $\alpha = 0.3022$ and $\delta = .079$, $w = 55.7$, $a_K = .0062$ and $p = 84.4$ arbitrary chosen for the Cobb-Douglas ($\sigma = 1$) and for $\sigma = .4$.

Notice that by putting these equations per labor unit will force capital k to be used to its full capacity to generate y and the part not used for consumption is invested. Except for capital no stocks exist, which means that these stocks cannot influence the dynamic behavior.

By choosing $c = c_1 y$ we introduce the consumers and producers behavior with respect to the dynamics of the system.

If we rewrite eq. 13-15 this results in:

$$\dot{k} = p[\alpha(a_K k)^\gamma + (1-\alpha)]^{1/\gamma} - \delta k - c \tag{30}$$

We choose $c = c_1 y$ and linearize around k_{c_1}, using Taylor expansion at c_1 gives

$$\dot{k} = \frac{(1-c_1)}{\left(\frac{k}{y}\right)_{c_1}}\frac{\alpha(a_K k_{c_1})^\gamma}{\alpha(a_K k_{c_1})^\gamma + (1-\alpha)}k - \delta k = \left(\frac{\alpha(a_K k_{c_1})^\gamma}{\alpha(a_K k_{c_1})^\gamma + (1-\alpha)} - 1\right)\delta k = (1 - ks)\delta k \tag{31}$$

The eigen value of this equation is λ

$$\lambda = (1 - ks)\delta < 0 \tag{32}$$

which holds for $\forall\ c_1 \epsilon (0,1)$

This means that this system is stable and will converge towards the equilibrium at c_1, starting from arbitrary initial condition $k_0 > 0$. The time constant τ is

$$\tau = \left|\frac{1}{\lambda}\right| \tag{33}$$

Lemma: If consumer's behavior is $c = c_1 y$ under maximizing profit with wages fixed or under maximum profit conditions, then all choices c_1 will result in a unique and stable equilibrium.

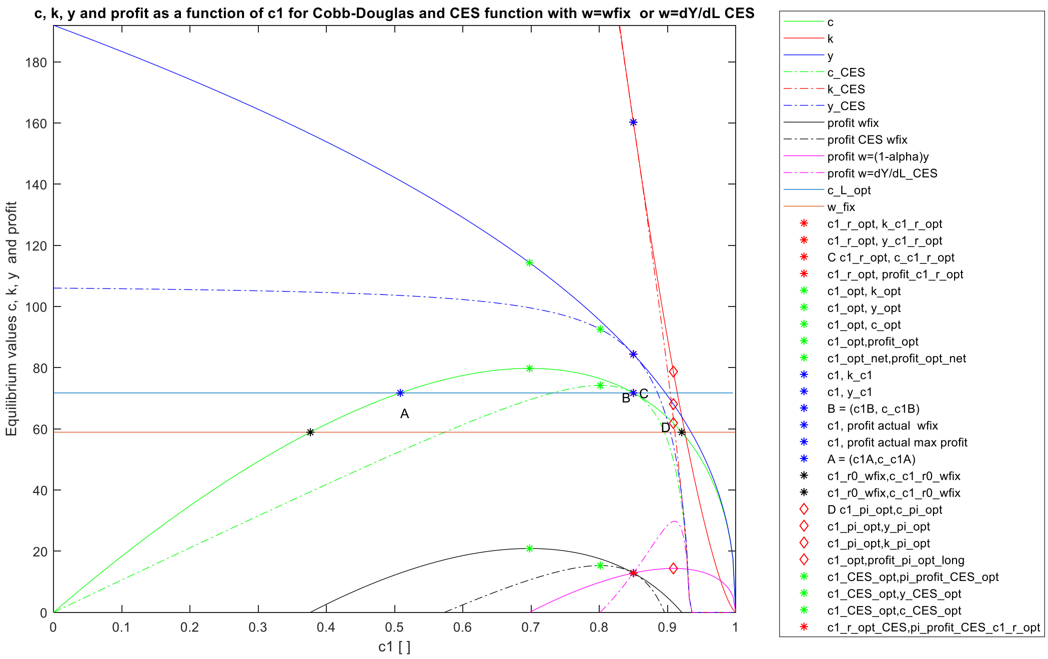

Fig. 1 CES production functions, the Cobb-Douglas case ($\sigma = 1$) with $p = 84.4$, $a_K = .0062$, $\delta = 0.079$, $\alpha = 0.3022$ and a CES case with $\sigma = .4$. Also shown are the profit curves for fixed wage $w = 55.7$ and for wages under maximum profit conditions. Curves are tangent at $c_1 = .85$.

4. Wages, capital share, labor share and net profitability rate in CES production functions

First, we maximize profit per work unit keeping wages fixed and then with variable wages under maximum profit conditions.

At this point, we introduce labor to understand what this will mean for the number of labor units required. So far, we have examined long-term profit maximization. On the short term, firms consider capital K and wages w as fixed and optimize with respect to the workforce they hire.

We use equations 13-15 and

$$Y = wL + (r + \delta)K \tag{34}$$

Equation 34 divided by L results in

$$y = w + (r+\delta)k \quad (35)$$

We calculate profit π for fixed w as

$$\pi = y - w - \delta k \quad (36)$$

Maximum profit is at $\frac{d\pi}{dk} = \frac{dy}{dk} - \delta = 0$ (37)

$$\frac{dy}{dk} = \alpha p^{\gamma} y^{1-\gamma} a_K{}^{\gamma} k^{\gamma-1} = \alpha p^{\gamma} a_K{}^{\gamma} \beta^{\gamma-1} = \delta \quad (38)$$

Taking the second derivative with respect to k together with equation 24 gives us

$$\frac{d^2\pi}{dk^2} = \frac{d^2 y}{dk^2} = \alpha p^{\gamma} y^{1-\gamma} a_K{}^{\gamma}(\gamma-1)k^{\gamma-2} + \alpha p^{\gamma}(1-\gamma)y^{-\gamma}\frac{dy}{dk} a_K{}^{\gamma} k^{\gamma-1} =$$

$$= \alpha y k^{-2} p^{\gamma} a_K{}^{\gamma} \beta^{\gamma}(\gamma-1)(1-\delta\beta) = \alpha p^{\gamma} y k^{-2} a_K{}^{\gamma} \beta^{\gamma}(\gamma-1)c_1 < 0 \quad (39)$$

which is the condition that we deal with a maximum.

From eq. 37 we calculate with eq. 24 the maximum profit at c_1

$$c_{1\_opt\_wfix} = 1 - \left(\left(\frac{pa_K}{\delta}\right)^{\gamma} \alpha\right)^{1/(1-\gamma)} = 1 - \left(\left(\frac{1}{\beta_0 \delta}\right)^{\gamma} \alpha\right)^{1/(1-\gamma)} \text{ for } \forall w \text{ fixed.} \quad (40)$$

Notice that $c_{1\_opt\_wfix} = c_{1\_opt}$ for $\forall w$ fixed. (41)

Notice that maximum profit for fixed wages w coincides with maximum utility $u(c) = c$ at $c_1 = c_{1\_opt}$.

Maximum profit, capital, income and consumption per capita can be calculated from equation 36, 21, 22 and 23.

Under maximum profit condition we have

$$\frac{\partial Y}{\partial L} = w \quad (42)$$

and

$$\frac{\partial Y}{\partial K} = r + \delta \quad (43)$$

which yields the following equations:

$$\frac{(1-\alpha)}{\alpha(a_K k)^{\gamma} + (1-\alpha)} y = w \quad (44)$$

$$\frac{\alpha(a_K k)^{\gamma}}{\alpha(a_K k)^{\gamma} + (1-\alpha)} \frac{y}{k} = \delta + r \quad (45)$$

For each c_1 we can calculate a corresponding w and r for which maximum profit conditions holds.

We rewrite equation 44 and 45 as labor share *ws* and capital share *ks*

$$ws = \frac{w}{y} = \frac{(1-\alpha)}{\alpha(a_K k)^\gamma + (1-\alpha)} = 1 - \alpha\left(\frac{\beta}{\beta_0}\right)^\gamma \tag{46}$$

$$ks = \frac{k}{y}(\delta + r) = \beta(\delta + r) = \frac{\alpha(a_K k)^\gamma}{\alpha(a_K k)^\gamma + (1-\alpha)} = \frac{\alpha(a_K k)^\gamma}{\left(\frac{y}{p}\right)^\gamma} = \alpha\left(\frac{\frac{k}{y}}{\frac{k_0}{y_0}}\right)^\gamma = \alpha\left(\frac{\beta}{\beta_0}\right)^\gamma \tag{47}$$

Of course, labor share plus capital share adds up to one.

$$ws + ks = 1 \tag{48}$$

We are left with one degree of freedom in our system. We can choose e.g. one of the following: c_1, w, β, y, k, ws, ks, r or time preference and optimize a desirable consumer utility function. In our opinion the time preference is not a suitable tool as also stated in De la Fonteijne (2015a).

Suppose we choose ratio c_1 with known parameters δ, p, a_K, α, σ. For a sustainable solution to exist we choose $c_{1\_opt} < c_1 < c_{1\_max}$.

The value of *k, y, c* and β can be calculated from equation 21, 22, 23 and 24.

Wage *w* and wage share *ws* can be calculated from equation 24, 44, and 46 as

$$ws = 1 - \alpha\left(\frac{\beta}{\beta_0}\right)^\gamma = 1 - \alpha\left(\frac{(1-c_1)}{(1-c_{1\_0})}\right)^\gamma \tag{49}$$

$$w = y\left(1 - \alpha\left(\frac{\beta}{\beta_0}\right)^\gamma\right) = y\left(1 - \alpha\left(\frac{(1-c_1)}{(1-c_{1\_0})}\right)^\gamma\right) \tag{50}$$

and capital share *ks* from equation 48 and 49 as

$$ks = 1 - ws \quad = \alpha\left(\frac{\beta}{\beta_0}\right)^\gamma = \alpha\left(\frac{(1-c_1)}{(1-c_{1\_0})}\right)^\gamma \tag{51}$$

Net profitability rate *r* follows from equation 45

$$r = \frac{ks}{\beta} - \delta = \frac{\alpha\left(\frac{\beta}{\beta_0}\right)^\gamma}{\beta} - \delta = \left(\frac{\alpha\left(\frac{(1-c_1)}{(1-c_{1_0})}\right)^\gamma}{(1-c_1)} - 1\right)\delta \tag{52}$$

where c_{1_0} is the corresponding value for c_1 at the base point in equilibrium. Or we write equivalent

$$r = \frac{1-ws}{\beta} - \delta = \frac{\alpha^{\frac{\sigma}{\sigma-1}}}{\beta_0}(1-ws)^{\frac{1}{1-\sigma}} - \delta \tag{53}$$

or the inverse

$$ws = 1 - \beta_0(r+\delta)^{1-\sigma}\alpha^\sigma \tag{54}$$

Fig. 2 shows the labor work share as a function of c_1. The graph is characteristic for all CES functions.

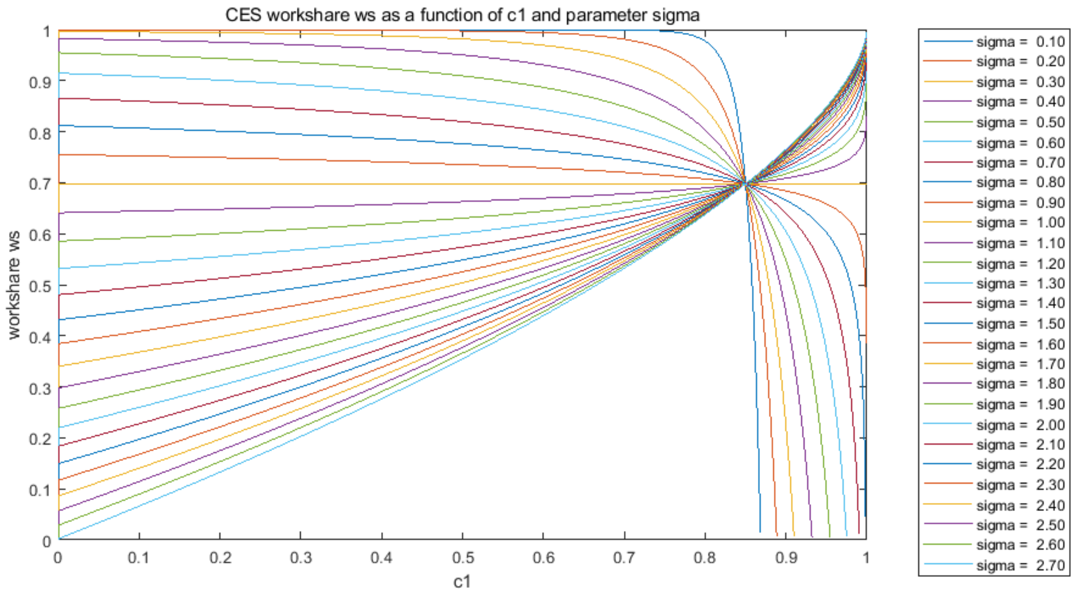

Fig. 2 Workshare as a function of c_1 with parameter σ. For the base point c_1=.85.

5. Changing the base point in CES production functions

We will show that for a change in base point there exists a parameter transformation, which leaves the elasticity of substitution invariant.

Recall that the production function is

$$y = y_0 \left[\alpha_0 \left(\frac{k}{k_0}\right)^\gamma + (1-\alpha_0)\right]^{1/\gamma} \tag{55}$$

Suppose that our system is in equilibrium in the base point $k = k_0$ and $y = y_0$, with $\alpha = \alpha_0$ at $c_1 = c_{1_0}$.

If we change c_1 to $c_1 = c_{1_0}{}'$ then using eq. 21 and eq. 22 k will move to $k = k_1$ and y will move to $y = y_1$. Rewriting eq. 55 as

$$y = y_0 \left[\alpha_0 \left(\frac{k_1}{k_0}\right)^\gamma \left(\frac{k}{k_1}\right)^\gamma + (1-\alpha_0)\right]^{1/\gamma} \tag{56}$$

and normalizing with $\alpha_0 \left(\frac{k_1}{k_0}\right)^\gamma + (1-\alpha_0)$ results in

$$y = y_0 \left(\alpha_0 \left(\frac{k_1}{k_0}\right)^\gamma + (1-\alpha_0)\right)^{1/\gamma} \left[[\frac{\alpha_0\left(\frac{k_1}{k_0}\right)^\gamma}{\alpha_0\left(\frac{k_1}{k_0}\right)^\gamma + (1-\alpha_0)} \left(\frac{k}{k_1}\right)^\gamma + \frac{(1-\alpha_0)}{\alpha_0\left(\frac{k_1}{k_0}\right)^\gamma + (1-\alpha_0)}\right]^{1/\gamma} \tag{57}$$

which we can write as

$$y = y_0 \left(\alpha_0 \left(\frac{k_1}{k_0}\right)^\gamma + (1-\alpha_0) \right)^{1/\gamma} \left[[\alpha_1 \left(\frac{k}{k_1}\right)^\gamma + (1-\alpha_1) \right]^{1/\gamma} \tag{58}$$

where

$$\alpha_1 = \frac{\alpha_0 \left(\frac{k_1}{k_0}\right)^\gamma}{\alpha_0 \left(\frac{k_1}{k_0}\right)^\gamma + (1-\alpha_0)} \tag{59}$$

For trivial reason, using equation 55

$$\left(\alpha_0 \left(\frac{k_1}{k_0}\right)^\gamma + (1-\alpha_0) \right)^{1/\gamma} = \frac{y_1}{y_0} \tag{60}$$

Combining equation 58 and 60 results in

$$y = y_1 \left[[\alpha_1 \left(\frac{k}{k_1}\right)^\gamma + (1-\alpha_1) \right]^{1/\gamma} \tag{61}$$

which is exactly the equation for our production function expressed in the new base point values. It is leaving σ unchanged. This means that the coefficient of elasticity of substitution σ is invariant under a shift in basepoint.

Lemma: In a CES production function the coefficient of elasticity of substitution σ is invariant under a shift in basepoint.

We will use this property to shed some light on the sense and non-sense of capital and labor augmented technical progress.

To describe the class of CES function we use equation 47

$$\alpha = ks = \alpha_0 \left(\frac{\beta}{\beta_0}\right)^\gamma = a\beta^\gamma \tag{62}$$

with

$$a = \frac{\alpha_0}{{\beta_0}^\gamma} \tag{63}$$

which is a constant for each coefficient of substitution σ for a specific CES production function.

The class of all CES functions can be described by the formula of a CES function for all combination a and σ.

Lemma: The class of CES production functions can be described by

$y = y_0[\alpha \left(\frac{k}{k_0}\right)^\gamma + (1-\alpha)]^{1/\gamma}$ for $\forall a, \sigma \in \mathcal{R}^+$,

where $a = \frac{\alpha_0}{{\beta_0}^\gamma}$ and σ is the coefficient of elasticity of substitution.

The coefficient $a = \frac{\alpha_0}{{\beta_0}^\gamma}$ is the same as in $y = [ak^\gamma + b]^{1/\gamma}$, the coefficient b is equal to $b = (1-\alpha_0){y_0}^\gamma$. With respect to the end result and conclusions it will make no difference which representation of the production function you use. They are mathematically equivalent. Each has his own advantages for economic interpretation.

6. General technological progress and the capital-labor mix and the trouble with capital and labor augmented technical progress

In literature, labor and capital augmented technical progress, ξ_{LT} and ξ_K, is incorporated in models as an additional multiplier factor for labor L and capital K, respectively. The idea is that both labor and capital augmented technical progress represents the evolution of technical progress, due to inventions, education etc. The nature of augmented technical progress (Klump et al., 2011) remains vague and results are not always conclusive, especially regarding the theoretical trouble with capital augmented technical progress. To escape from it, Jones (2003) e.g. introduced a short term CES function in combination with a long term Cobb-Douglas function.

We will examine labor and capital augmented technological progress. Both, ξ_{LT} and ξ_K are function of time and, if you wish, functions of the determinants, i.e. inventions, education, etc. We use the same implementation as is done in literature. This gives us the per capita production function in normalized form

$$y = y_0\left[\alpha_0\left(\frac{\xi_K k}{k_0}\right)^{\gamma} + (1-\alpha_0){\xi_{LT}}^{\gamma}\right]^{1/\gamma} \qquad (64)$$

We rewrite equation 64 by taking out a mutual general progress part ξ_g (so returning for $k = \xi_g k_0$ to the same capital to income ratio as we started with)

$$y = y_0\xi_g\left[\alpha_0{\xi_K}^{\gamma}\left(\frac{k}{\xi_g k_0}\right)^{\gamma} + (1-\alpha_0)\left(\frac{\xi_{LT}}{\xi_g}\right)^{\gamma}\right]^{1/\gamma} \qquad (65)$$

Notice that by choosing a mutual general progress part ξ_g, we indirect assume that the capital to income ratio is constant.

Normalizing to the new α_2 should yield

$$y = y_0\xi_g\left[\alpha_2\left(\frac{k}{\xi_g k_0}\right)^{\gamma} + (1-\alpha_2)\right]^{1/\gamma} \qquad (66)$$

This can be achieved if we take

$$\alpha_2 = \frac{\alpha_0{\xi_K}^{\gamma}}{\alpha_0{\xi_K}^{\gamma} + (1-\alpha_0)\left(\frac{\xi_{LT}}{\xi_g}\right)^{\gamma}} = \alpha_0{\xi_K}^{\gamma} \qquad (67)$$

Equation 67 is defining a relation between ξ_g, ξ_{LT} and ξ_K as

$$\alpha_0{\xi_K}^{\gamma} + (1-\alpha_0)\left(\frac{\xi_{LT}}{\xi_g}\right)^{\gamma} = 1 \qquad (68)$$

And the result for ξ_g is

$$\xi_g = \left(\frac{(1-\alpha_0){\xi_{LT}}^{\gamma}}{1-\alpha_0{\xi_K}^{\gamma}}\right)^{1/\gamma} \qquad (69)$$

We will leave k_0 and y_0 unchanged, leaving the original base point unchanged.

The general description for the production function is then

$$y = y_0 \xi_g \left[\alpha_2 \left(\frac{k}{\xi_g k_0} \right)^{\gamma} + (1 - \alpha_2) \right]^{1/\gamma} \tag{70}$$

Equation 67 is limiting the value of ξ_K. We require $\alpha_2 \leq 1$, because otherwise $1 - \alpha_2 < 0$, in which case it is better not to use labor at all, to avoid the negative influence of labor in the production function. This holds mutatis mutandis for capital at the lower bound of α_2.

The limits for capital augmented technical progress ξ_K are

$$0 < \xi_K < \left(\frac{1}{\alpha_0} \right)^{\gamma} \quad \text{for } \sigma > 1 \tag{71}$$

and

$$\left(\frac{1}{\alpha_0} \right)^{\gamma} < \xi_K \qquad \text{for } \sigma < 1 \tag{72}$$

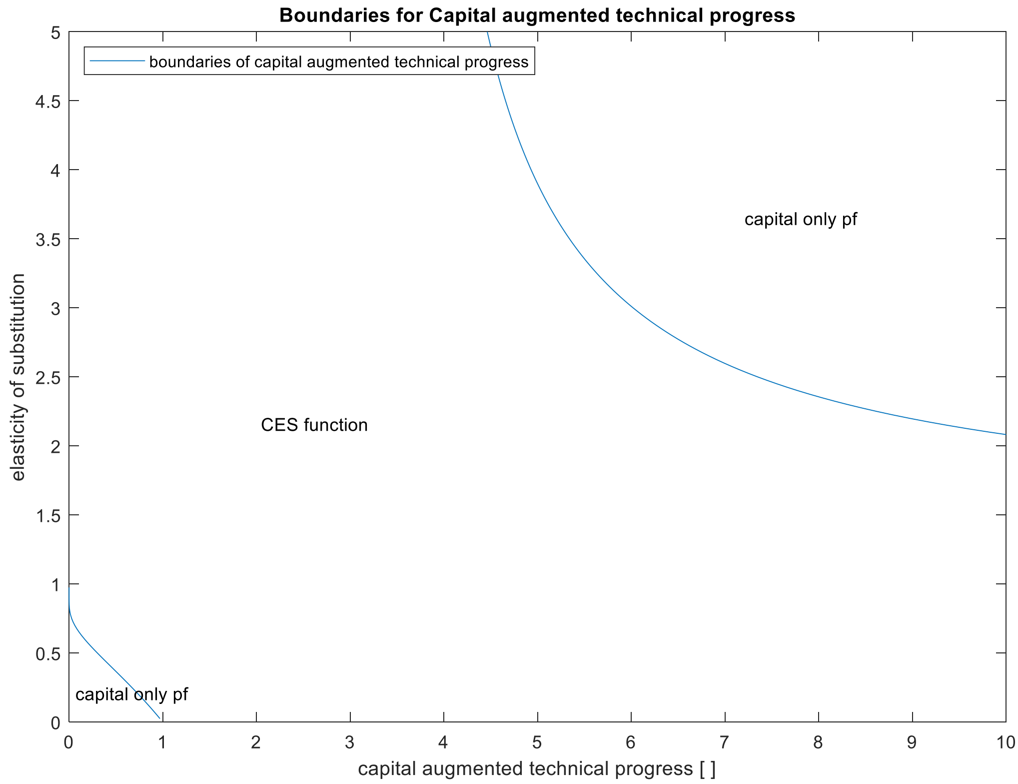

Fig. 3 Boundary for capital augmented technical progress ξ_K as a function of the coefficient of elasticity of substitution σ. The drawn line is representing the boundary for which the capital-labor-mix α_2 exceeds 1, $\alpha_0 = .5$. The boundaries for $\sigma > 1$ are $0 < \xi_K < \left(\frac{1}{\alpha_0}\right)^{\gamma}$, which results in a capital production function if ξ_K exceeds the maximum value. For $\sigma < 1$ we have $\left(\frac{1}{\alpha_0}\right)^{\gamma} < \xi_K$, which also results in a capital production function if ξ_K is lower than the minimum value.

Beyond these boundaries, we have capital or labor production functions only. To be more specific, assuming that capital augmented technical progress will continue to grow, then this growth will turn the CES production function into a capital only production function if the coefficient of substitution is greater than one at the boundary value of ξ_K. And it will turn into a labor only production function if the coefficient of substitution is lower than one and ξ_K goes to infinity.

Uzawa (1961-1) realized that capital augmented technical progress introduced a problem for steady state growth. To solve the problem, he came up with his labor augmented technical progress only theorem.

In case of a neo classical steady state Jones (2004) gave an alternative proof of the labor augmented technical progress only theorem introduced by Uzawa, where he proved that a steady state can only exist if technical progress is labor augmenting technical progress only (capital and labor augmented technical progress implemented as is done since the 1960's). Later we will show that the labor technical progress term he used in fact was total growth of income and capital.

Uzawa (1961-1) in the same paper also proved that the equilibria were unique and stable. We can not agree on this last item. He assumed that capital never depreciates, but without depreciation, you simply cannot prove it, because the eigen value of the differential equal is zero. I agree on his final differential equation $\frac{\dot{z}(t)}{z(t)} = f_k[z(t)] - \lambda - \mu$, but not on his conclusion with respect to stability. Uniqueness is forced by putting $f_k[z(t)] = \lambda + \mu$, allowing only one capital to income ratio, but then uniqueness is trivial. Moreover, a limitation, by letting depreciation zero, is not realistic and in addition we showed that uniqueness and stability is influenced by consumers behavior (De la Fonteijne, 2011). If you would take the wages smaller than $w < y - (\lambda + \mu)k$ you can even get continuous growth without technical growth.

We have showed that labor and capital augmented technical progress can be expressed in a general progress term and a change in the capital-labor-mix. Both are influencing general technical progress and only ξ_K is responsible for the change in the new capital-labor-mix α.

The capital-labor-mix α is a factor in the production function influencing the output. Under maximum profit conditions α is equal to the capital share of income ks in the base point for CES production functions.

Suppose that k_0 is the initial base point and is not varying in time. If y is growing in time then so is k at the same speed along a balanced (Jones, 2005) growth path. The capital-labor-mix factor α depends on capital augmented technical progress ξ_K (Jones, 2003) (Acemoglu, 2003). Interesting to see that consumers and producers can decide what to do with this change by choosing c_1. In literature a balanced growth path is referring to a stabilized capital to income ratio over time in combination with a not changing shape of the production function, ruling out any other progress then general progress. If, however, the shape is changing we still have the possibility that capital progress can be compensated by an adaption of c_1 to keep the capital to income ratio constant.

Normalizing with respect to general progress ξ_g, with $y^* = \frac{y}{\xi_g}$ and $k^* = \frac{k}{\xi_g}$ will give us the general progress independent solution

$$y^* = y_0 \left[\alpha \left(\frac{k^*}{k_0}\right)^{\gamma} + (1-\alpha)\right]^{1/\gamma}. \qquad (73)$$

Realize that α can fluctuate over time to express the change in the capital-labor mix.

Lemma: Capital and labor augmented technical progress in a CES production function can be expressed by the terms general technical progress and a change in the capital-labor mix α.

An even more explicit way of demonstrating and motivating this approach is by expressing the CES production function in terms of the capital to income ratio by dividing equation 70 by y and rewrite it as

$$y = y_0 \xi_g \left[\frac{1-\alpha}{1-\alpha\left(\frac{\beta}{\beta_0}\right)^{\gamma}} \right]^{1/\gamma} = y_0 \xi_g \left[\frac{1-\alpha}{1-ks} \right]^{1/\gamma} \qquad (74)$$

with a general growth term ξ_g and a form factor not dependent on general growth

$$\left[\frac{1-\alpha}{1-\alpha\left(\frac{\beta}{\beta_0}\right)^{\gamma}} \right]^{1/\gamma} = \left[\frac{1-\alpha}{1-ks} \right]^{1/\gamma} \qquad (75)$$

Lemma: It is convenient to split up a production function into a general progress term and a shape term, the parameters of which may vary in time.

In fact, we can use the same idea for any arbitrary production function.

Technical improvement can also result in a price chance for the production factors. In a micro economic setting this would been taken care of by adapting the prices in the cost equation. In a macro economic setting wage w and capital cost rate $(r + \delta)$ is already fixed by the fact that we assume that we operate under maximum profit principle and by the choice of c_1. So it is already included in our model.

Lemma: Price changes of the production factors due to technical progress is already taking care of by means of the choice of c_1 and the fact that we operate under maximum profit principle.

7. Total Factor Productivity

To examine the same process from another angle we split the process of growth in two part:

- income growth from technical progress only
- income growth due to the raise in capital

Again, we start with the per capita production function including the augmenting technical progress factors

$$y = y_0 \left[\alpha_0 {\xi_K}^{\gamma} \left(\frac{k}{k_0}\right)^{\gamma} + (1-\alpha_0){\xi_{LT}}^{\gamma} \right]^{1/\gamma} \qquad (76)$$

Introducing total factor productivity and normalizing to the new α yields

$$y = y_0 \xi_{TFP} \left[\alpha_1 \left(\frac{k}{k_0}\right)^{\gamma} + (1-\alpha_1) \right]^{1/\gamma} \qquad (77)$$

where

$$\xi_{TFP} = [\alpha_0 \xi_K{}^\gamma + (1-\alpha_0)\xi_{LT}{}^\gamma]^{1/\gamma} \tag{78}$$

and

$$\alpha_1 = \frac{\alpha_0 \xi_K{}^\gamma}{\alpha_0 \xi_K{}^\gamma + (1-\alpha_0)\xi_{LT}{}^\gamma} = \alpha_0 \left(\frac{\xi_K}{\xi_{TFP}}\right)^\gamma \tag{79}$$

Due to this increase in technical growth there is an instantaneous growth in productivity y, which we like to refer to as total factor productivity growth ξ_{TFP}. This part has only to do with the production function. The next step will involve the maximum profit system as a whole. Due to the increase in productivity, capital is not in equilibrium. Assuming that c_1 is kept constant then capital will go to the level k^*, while the capital to income ratio β will return to its original level β_0. Realize, however, that this two-step experiment of thought will take place in one go and will lower consumption $c = c_1 y$ by the investment needed to increase capital. Furthermore, we limit ourselves to CES functions and only α_0 is allowed to change over time.

$$k^* = \beta_0\, y^* = \beta_0 y_0 \xi_{TFP} \left[\alpha_1 \left(\frac{k^*}{k_0}\right)^\gamma + (1-\alpha_1)\right]^{1/\gamma} \tag{80}$$

Solving k^* yields

$$k^* = \left(\frac{(1-\alpha_1)\xi_{TFP}{}^\gamma}{1-\alpha_1 \xi_{TFP}{}^\gamma}\right)^{1/\gamma} k_0 \tag{81}$$

Which means that the growth of k and, of course, of y in total is

$$\xi_g = \left(\frac{(1-\alpha_1)\xi_{TFP}{}^\gamma}{1-\alpha_1 \xi_{TFP}{}^\gamma}\right)^{1/\gamma} \tag{82}$$

Changing to the new base point and normalizing to the new α yields

$$y = y^* \left[\alpha_2 \left(\frac{k}{k^*}\right)^\gamma + (1-\alpha_2)\right]^{1/\gamma} \tag{83}$$

where

$$\alpha_2 = \frac{\alpha_1 \xi_g{}^\gamma}{\alpha_1 \xi_g{}^\gamma + (1-\alpha_1)}, \tag{84}$$

$y^* = y_0 \xi_g$ and $k^* = k_0 \xi_g$.

Substituting equation 82 in equation 84 results in

$$\alpha_2 = \alpha_1 \xi_{TFP}{}^\gamma \tag{85}$$

With equation 78 and 79 equation 85 changes to

$$\alpha_2 = \alpha_0 \xi_K{}^\gamma \tag{86}$$

and equation 82 changes to

$$\xi_g = \left(\frac{(1-\alpha_0)\xi_{LT}{}^\gamma}{1-\alpha_0 \xi_K{}^\gamma}\right)^{1/\gamma} \tag{87}$$

Notice that the equations 86 and 87 are the same as 67 and 69.

Using equation 86 and 78 we can express ξ_g (equation 87) in term of α_2 and ξ_{TFP} only

$$\xi_g = \left(\frac{\xi_{TFP}{}^{\gamma} - \alpha_2}{1-\alpha_2}\right)^{1/\gamma} \tag{88}$$

For Cobb-Douglas, $\sigma = 1$ ($\gamma = 0$) equation 88 reduces to

$$\xi_g = \xi_{TFP}{}^{1/(1-\alpha_0)} \tag{89}$$

for all choices of ξ_K and ξ_{LT}, similar to what we described in De la Fonteijne (2011) and what can be found in literature with respect to Cobb-Douglas production functions (Acemoglu, 2008) (Jones, 2013).

For Harrod, Hicks and Solow neutrality, we will use the same definitions as is done by Klump et al. (2011) and as is indicated in the following text.

Notice that if in equation 87 $\xi_K = 1$, then $\xi_g = \xi_{LT}$ (Harrod neutral as labor augmented technical progress only), i.e. labor augmented technical progress ξ_{LT} is equal to the increase in productivity and not to total factor productivity. Total factor productivity is

$$\xi_{TFP} = \xi_{Harrod} = [\alpha_0 + (1-\alpha_0)\xi_{LT}{}^{\gamma}]^{1/\gamma}. \tag{90}$$

The form of the production function has not changed, because the base point stayed the same and the capital-labor-mix did not change $\alpha_2 = \alpha_0$, while keeping the capital to income ratio constant. Total productivity growth is

$$\xi_g = \xi_{LT} = \left(\frac{\xi_{Harrod}{}^{\gamma} - \alpha_0}{1-\alpha_0}\right)^{1/\gamma} \tag{91}$$

Notice that if in equation 78 we take $\xi_K = \xi_{LT}$ (Hicks neutral as equally capital and labor augmenting technical progress), then

$$\xi_{TFP} = \xi_{Hicks} = \xi_K = \xi_{LT}\,, \tag{92}$$

i.e. labor augmented technical progress is equal to total factor productivity. The form of the production function has changed, because the base point stayed the same and the capital-labor-mix changed from α_0 to $\alpha_2 = \alpha_0\xi_K{}^{\gamma} = \alpha_0\xi_{Hicks}{}^{\gamma}$, while keeping the capital to income ratio constant. Total productivity growth is

$$\xi_g = \left(\frac{\xi_{Hicks}{}^{\gamma} - \alpha_2}{1-\alpha_2}\right)^{1/\gamma} = \xi_{Hicks}\left(\frac{1-\alpha_0}{1-\alpha_0\xi_{Hicks}{}^{\gamma}}\right)^{1/\gamma}. \tag{93}$$

Notice that if in equation 86 $\xi_{LT} = 1$ (Solow neutral as capital augmenting technical progress only), that $\alpha_2 = \alpha_0\xi_K{}^{\gamma}$. The form of the production function has changed, because the base point stayed the same and the capital-labor-mix changed from α_0 to $\alpha_2 = \alpha_0\xi_K{}^{\gamma}$, while keeping the capital to income ratio constant. Total factor productivity is

$$\xi_{TFP} = \xi_{Solow} = [\alpha_0\xi_K{}^{\gamma} + (1-\alpha_0)]^{1/\gamma}. \tag{94}$$

Total productivity growth is

$$\xi_g = \left(\frac{\xi_{Solow}{}^{\gamma} - \alpha_2}{1-\alpha_2}\right)^{1/\gamma}. \tag{95}$$

Growth of capital per capita is expressed by ξ_g and growth of income per capita takes place in two steps, the first direct part due to the total of technical progress ξ_{TFP} and the second indirect part ξ_{yk} due to the growth of capital. In total the growth of income per capita ξ_g is the same as the growth of capital per capita, i.e. if the capital to income ratio is kept at a constant value.

$$\xi_g = \xi_{yk}\xi_{TFP} \qquad (96)$$

Capital growth is not a direct result of labor or capital technical progress itself. Capital growth is a result of the mathematical process by which technical progress allows the economy to use more capital. This is the true nature of capital growth.

Lemma: Capital growth is not a direct result of labor or capital technical progress itself, but it is a result of the mathematical process by which technical progress allows the economy to use more capital if there is general progress. This is the true nature of capital growth.

Lemma: Income growth is partly a direct result of labor or capital technical progress itself and the second part is caused by the use of more capital. The first part is ξ_{TFP}, which is considered in most cases as exogenous and the second part is ξ_{yk}, which is endogenous, in total $\xi_g = \xi_{yk}\xi_{TFP}$.

The labor part of ξ_{TFP} is due to the improvement of skills, education, etc. and the capital part of ξ_{TFP} is due to the technical improvement of existing capital as well as of new capital, both under the name of investments.

Later we will calculate the part due to TFP increase in total income increase.

8. Consumers behavior

Growth of capital also depends on consumers behavior i.e. the choice of c_1 in $c = c_1 y$. If c_1 is constant then the desired capital to income ratio is constant and the economy and capital will grow, under maximum profit conditions, up to the total of the general progress term ξ_g. If e.g. the new c_1 is chosen smaller than the original one, then GDP and capital will grow until the new equilibrium with a changed capital to income ratio is reached as described in section 3.

9. The consequence of ongoing capital augmented technical progress

From the previous sections we conclude that if there exists an ongoing capital augmented technical progress $\xi_K > 1$, then the production function must be Cobb-Douglas as reported in literature.

Lemma: If, using a CES production function, there exists an ongoing capital augmented technical progress $\xi_K > 1$ as implemented, then the production function must be Cobb-Douglas.

To compensate for the change in the capital-labor-mix we can adapt the capital to income ratio.

If $\xi_K > 1$ and $\sigma < 1$ then ${\xi_K}^{\gamma} < 1$, which means that α_2<α_0. This can be compensated by changing to the new capital to income ratio. Recall that equation 63 holds for a particular CES production function. In our case there is first a change to a_2 and then a change in β to β_2

$a_2 = \frac{\alpha_2}{{\beta_0}^{\gamma}} = \frac{\alpha_0}{{\beta_2}^{\gamma}}$, where a_2 is a constant. (97)

With use of equation 86 we can calculate β_2 as

$\beta_2 = \frac{\beta_0}{\xi_K}$ (98)

This means, assuming $\xi_K > 1$, that β_2 has to be taken smaller than β_0 by choosing the new c_1 greater. It holds for all σ. A lower β_2 means a lower capital k and a lower income y, quite contrary to the 'capital is back' explanation of Piketty (2014). Income and capital are lower than in base point operation, and so is consumption.

Lemma: If there exists an ongoing capital augmented technical progress $\xi_K > 1$ as implemented, with the capital share held constant, then the capital to income ratio will continue to decrease.

If $\xi_K = 1$, i.e. progress is labor augmenting only, then due to the increase of ξ_{TFP}, the capital-labor-mix $\alpha_1$is changing from α_0 to α_1 and due to the increase of capital α_1 is changing back to α_0 (equations 79 and 84). Realize that only capital augmented technical progress can change the shape of production function, because then a in equation 63 is changing. When capital is growing, we are moving over the same CES production function, while the capital-labor-mix is changing. Compensating and changing back to the original capital-labor-mix by changing c_1 also leaves the shape of the production function unchanged.

We exclude the possibility on the long run $\xi_K \leq 1$, because it is not plausible. There is a clear evidence that capital as well as labor improve over time and we do not see an end to this process. If $\xi_K > 1$, then the production function has to be Cobb-Douglas as is known from literature. We end up with an unresolved dispute how to solve this paradox.

At this point, we have to draw a conclusion, because there clearly is a theoretical contradiction. What might cause this contradiction? What do we have so far:

- two economic identities
- a CES production function, with a certain elasticity of substitution $\sigma > 0$
- increasing capital and labor technical progress
- the implementation of capital and labor augmented technical progress

We have no reasons to doubt the first item. Concerning the second item, we do not see any theoretical or practical evidence that the elasticity of substitution should be exactly one. The third item we feel reasonable comfortable with. This bring us to the idea that we might have a problem with the implementation of augmented technical progress.

In his book Acemoglu (2008, p. 59) describes this situation as troubling, which seems to be an understatement.

A possible (and probably the only) solution could be that we have to reject the way we have implemented capital and labor augmented technical progress. And, in fact, that is what we will do. What we find bothering, is the combination of capital and labor technical progress factor together with capital and labor as a multiplication factor. It seems logical, but it is not. Suppose you want to measure the effect of technical progress directly, then the only way is, to measure the changed output y, which can be determined for each factor. However, what counts is the combination of capital and labor progress on output y, which leads us to a combined output factor, say total factor productivity ξ_{TFP}. This total factor productivity ξ_{TFP} might be a more complicated function of the level of capital, labor and other determinants, with the risk that the production function is not homogeneous of degree one. But that would be more a theoretical problem than a practical problem in a numerical world.

Lemma: If there exists an ongoing capital augmented technical progress $\xi_K > 1$ as implemented and we allow CES functions in general, then the way of implementation of capital and labor augmented technical progress has to be rejected. There exists an ongoing improvement of technical progress. So, we reject the way capital and labor progress is implemented.

Altogether this means that we have to come up with a new way of implementing technical progress. We will go into more detail in the next section.

10. A new way of implementing technical progress

To be clear, we skip the augmented progress terms for each production factor separately. The reason for that is, the disturbing influence of ξ_K on the form parameter a or, which is the same, on the capital-labor-mix at constant capital to income ratio β. To our opinion an improvement of capital or labor can have an increasing or decreasing effect on the form parameter a. For now, we simply accept, that we have no clear understanding how this influence on parameter a takes place. When there is no effect on parameter a, we stay on a CES function with the same form. This means, if income increase instantaneously because of ξ_{TFP}, then simultaneously α changes to $\alpha_1 = \alpha_0 \left(\frac{1}{\xi_{TFP}}\right)^{\gamma}$. If at the same time α_0 change to α_2 then the production function is

$$y = y_0 \xi_{TFP} \left[\alpha_1 \left(\frac{k}{k_0}\right)^{\gamma} + (1-\alpha_1)\right]^{1/\gamma} \tag{99}$$

and

$$\alpha_1 = \alpha_2 \left(\frac{1}{\xi_{TFP}}\right)^{\gamma} \tag{100}$$

Notice that equation 77 holds and the result is the same as using labor augmented technical progress only, albeit now including capital augmented progress, but without the trouble. The rest of the procedure stays the same. Due to ξ_{TFP} the economy per capita can grow in total with ξ_g and α is changed to α_2 or in case there is no change in the capital-labor-mix back again to $\alpha_2 = \alpha_0$. This solves the paradox.

In fact, we can extend this thought to all parameters of the CES function (or arbitrary production function), making all parameters a function of time when technical progress evolves over time.

Lemma: A new way of implementing capital and labor technical progress is adding a total factor productivity ξ_{TFP} and to adapt the new capital-labor-mix α to α_2 and calculate α_1 from equation 85

$\alpha_1 = \alpha_2 \left(\frac{1}{\xi_{TFP}}\right)^{\gamma}$. The new production function is

$$y = y_0 \xi_{TFP} \left[\alpha_1 \left(\frac{k}{k_0}\right)^{\gamma} + (1 - \alpha_1)\right]^{1/\gamma}$$

This procedure solves a long existing dispute on the 'Cobb-Douglas or labor augmented technical progress only' paradox.

Fortunately, we end up with the same way of adapted interpretation of technical progress as we used before, but now without the burden.

This means that the increase in income per capita ξ_g is the result of total factor productivity ξ_{TFP} and capital increase. The capital-labor-mix α_2 is allowed to fluctuate up or down and will be dictated by the interaction of capital and labor, for which we have to find out the determinants. If you choose or measure α_2, then α_1 has to be calculated from equation 85.

Lemma: In total the economy per capita will grow with $\xi_g = \left(\frac{\xi_{TFP}{}^{\gamma} - \alpha_2}{1 - \alpha_2}\right)^{1/\gamma}$. The new production function is $y = y_0 \xi_g \left[\alpha_2 \left(\frac{k}{\xi_g k_0}\right)^{\gamma} + (1 - \alpha_2)\right]^{1/\gamma}$. The capital-labor-mix α_2 (at constant capital to income ratio β) can be adapted accordingly measured or estimated values, resulting in an adapted form parameter a.

To estimate the coefficient of substitution all investigators, I am aware of, used Hicks, Harrod or Solow neutrality or some kind of Box-Cox weighing function (Klump et al., 2007, 2011). But as long as you are using the same system and assuming that there is no end to capital augmented technical progress, you will always explicitly frustrate the 'Cobb-Douglas or labor augmented technical progress only' paradox.

With regards to the Box-Cox method (Klump et al., 2011) to express the decaying and stabilizing effect on the capital augmented technical progress we argue that this is also some kind of capital augmented technical progress and therefore in principle not suitable. In particular to the formula used, we argue that to express a decaying effect, there exists simpler and more transparent formulas to achieve the same goal. You better use a simple exponential function, which is also more flexible to adjust, e.g.

$$E(t) = a + (1 - a)e^{-bt} \,. \tag{101}$$

The labor part of ξ_{TFP} is due to the improvement of skills, education, etc. and the capital part of ξ_{TFP} is due to the technical improvement of capital, new invested or existing.

It remains interesting to understand where the growth of total factor productivity comes from (Donselaar, 2011). Donselaar referred to and used the formulas of Solow, but fortunately limited himself to the Cobb-Douglas case including human capital, and so avoiding the problem. I.e., if you agree that Cobb-Douglas is the right choice and by not referring to ξ_K or ξ_{LT}. Donselaar spend (was struggling with the inconsistency in theory) a substantial part of his thesis on neo classical growth theory vs. growth accounting and I wonder if parts have to be reconsidered regarding terms as labor saving technical growth, embodied and disembodied technological growth. We will come back on labor saving progress and growth accounting later.

Until so far, we derived formulas per capita. To arrive at the extensive production function, we only have to multiply with the workforce at that time and equation 99 changes to

$$Y = Y_0 \xi_{TFP} \left[\alpha_1 \left(\frac{K}{K_0}\right)^\gamma + (1-\alpha_1)\left(\frac{L}{L_0}\right)^\gamma\right]^{1/\gamma} \tag{102}$$

where

$$\alpha_1 = \alpha_2 \left(\frac{1}{\xi_{TFP}}\right)^\gamma \tag{103}$$

The same equation but now expressed in terms of growth of GDP $\xi_G = \xi_g \xi_L$, where ξ_L represents growth of the workforce

$$Y = Y_0 \xi_G \left[\alpha_2 \left(\frac{K}{\xi_G K_0}\right)^\gamma + (1-\alpha_2)\left(\frac{L}{\xi_L L_0}\right)^\gamma\right]^{1/\gamma} \tag{104}$$

We can calculate the part of TFP in total potential growth under constant capital to income ratio as

$$\eta_{TFP} = \frac{\frac{\dot{\xi}_{TFP}}{\xi_{TFP}}}{\frac{\dot{\xi}_g}{\xi_g}} = \lim_{\xi_{TFP}\to 1} \frac{\xi_{TFP}-1}{\xi_g-1} = \lim_{\xi_{TFP}\to 1} \frac{\xi_{TFP}-1}{\left(\frac{\xi_{TFP}{}^\gamma-\alpha_2}{1-\alpha_2}\right)^{1/\gamma}-1} = 1-\alpha_2 \text{ for all } \sigma. \tag{105}$$

It is simple to prove that this result is valid not only for CES functions, but for all production functions.

We remind you that there no longer exists a relation between α_2 and ξ_K. We do not use ξ_K. If it turns out that there is a relation between α_2 and ξ_{TFP} then equation 105 has to be adapted accordingly.

The ratio η_{TFP} is calculated under the condition that the capital to income ratio is constant. It is also possible to take another criterium, e.g. leave the profit rate r constant. If there is a change in the capital-labor-mix due to the technological improvement from α_0 to α_2, then the capital to income ratio has to be adapted in the same way $\beta_2 = \frac{\alpha_2}{\alpha_0}\beta_0$. This will, of course, also change c_1. The ratio η_{TFP} will change because the total increase in income is different.

Equation 80 has to be adapted to

$$k^* = \beta_2 y^* = \frac{\alpha_2}{\alpha_0}\beta_0 y_0 \xi_{TFP}\left[\alpha_1\left(\frac{k^*}{k_0}\right)^\gamma + (1-\alpha_1)\right]^{1/\gamma} \tag{106}$$

and the growth of k^* is

$$\xi_{k^*} = \left(\frac{\xi_{TFP}{}^\gamma - \alpha_2}{\left(\frac{\alpha_2}{\alpha_0}\right)^\gamma - \alpha_2}\right)^{1/\gamma} \tag{107}$$

and the growth of $y^*$is

$$\xi_{y^*} = \left(\frac{\xi_{TFP}{}^{\gamma} - \alpha_2}{1 - \alpha_2 \left(\frac{\beta_0}{\beta_2} \right)^{\gamma}} \right)^{1/\gamma} \tag{108}$$

resulting in an alternative definition for η_{TFP} at constant profitability

$$\eta_{TFP} = \lim_{\xi_{TFP} \to 1} \frac{\xi_{TFP} - 1}{\left(\frac{\xi_{TFP}{}^{\gamma} - \alpha_2}{1 - \alpha_2 \left(\frac{\beta_0}{\beta_2} \right)^{\gamma}} \right)^{1/\gamma} - 1} = 0 \quad \text{for all } \sigma. \tag{109}$$

So, in case we take the limit the result is not useful. However, without the limit it will give you the actual value of η_{TFP}. Notice that the growth of k and y are not equal.

In case $\alpha_2 = \alpha_0$ then there is no difference between the two definitions. If we assume that ξ_{TFP} and α_2 are slowly moving functions in time, then it is allowed to take $\alpha_2 = \alpha_0$ which results in the momentarily value for η_{TFP} at k_0 and t_0

$$\eta_{TFP} = \lim_{t \to t_0} \frac{\xi_{TFP} - 1}{\left(\frac{\xi_{TFP}{}^{\gamma} - \alpha_2}{1 - \alpha_2 \left(\frac{\beta_0}{\beta_2} \right)^{\gamma}} \right)^{1/\gamma} - 1} = 1 - \alpha_0 \text{ for all } \sigma. \tag{110}$$

This seems a reasonable and general useful definition.

Lemma: The technical improvement part (TFP) in total potential growth is equal to one minus the capital-labor-mix at the considered constant capital to income ratio.

η_{TFP}=$1 - \alpha_0$ for all σ and for all production functions $f(k)$

In the next section, we will use this new way of technical progress when analyzing the problem of estimating the parameters of a CES production function.

These parameters may change over time then the parameters are also a function of time, i.e.
$\xi_G = \xi_G(k_0(t), t)$, $\sigma = \sigma(k_0(t), t)$, $\alpha = \alpha(k_0(t), t)$. (111)

11. Analysis of the problem in estimating the coefficient of substitution

In this section we show you, as a first step, how you can estimate the elasticity coefficient of substitution following a slightly different way than you can find in most of the literature.

We saw already in equation 70 that the workshare for a CES function is

$$ws = \frac{w}{y} = 1 - \alpha \left(\frac{\beta}{\beta_0} \right)^{\gamma} \tag{112}$$

and the elasticity coefficient of substitution σ is

$$\sigma = \frac{1}{1-\gamma} \tag{113}$$

Partial differentiating ws with respect to β gives

$$\frac{\partial(ws)}{\partial\beta} = -\frac{(1-ws)}{\beta}\frac{(\sigma-1)}{\sigma} \tag{114}$$

which implies that if α is constant

$$\text{if and only if } \sigma < 1 \text{ then } \frac{\partial(ws)}{\partial\beta} > 0, \tag{115}$$

$$\text{if and only if } \sigma = 1 \text{ then } \frac{\partial(ws)}{\partial\beta} = 0, \tag{116}$$

and

$$\text{if and only if } \sigma > 1 \text{ then } \frac{\partial(ws)}{\partial\beta} < 0. \tag{117}$$

Partial differentiating ws with respect to α gives

$$\frac{\partial(ws)}{\partial\alpha} = -\left(\frac{\beta}{\beta_0}\right)^{\gamma} = -\frac{(1-ws)}{\alpha} \tag{118}$$

The derivative is independent of σ and in the base point $\frac{\partial(ws)}{\partial\alpha} = -1$.

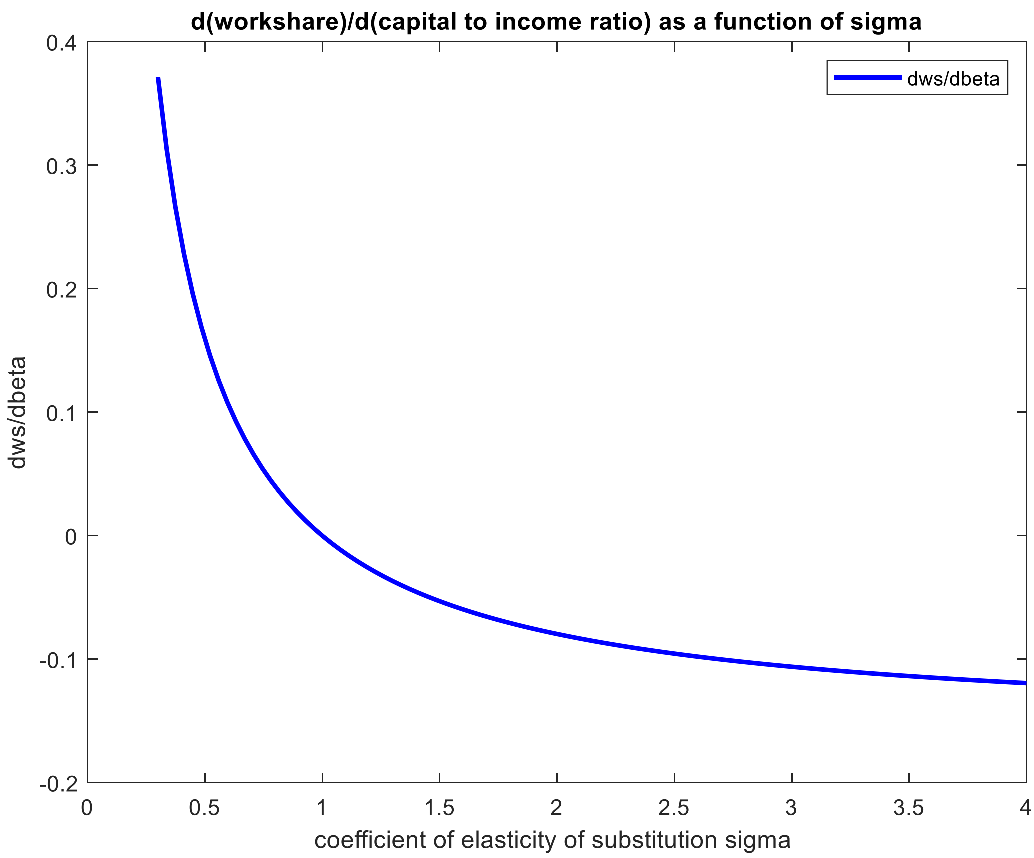

Fig. 4 Derivative of workshare with respect to the capital to income ratio as a function of the elasticity of substitution for a 1-sector CES model with the parameters time independent, i.e. σ and α are constant.

Before going any further on the consequences of these formulas, I like to refer to Piketty (2014), where he is arguing that the capital to income ratio will rise in the future and that the coefficient of elasticity might have turned into a value above 1. For an extensive treatment, see my critics on Piketty in De la Fonteijne (2016). In his blog, Jacobs (2014) was more precise in his arguments and formulas but was not very conclusive, nor was Krugman, where Jacobs is referring to.

The weak point of Piketty and Jacobs is their a priori assumption of $\alpha = constant$ to estimate σ smaller or greater than one.

Here is the problem, we do not have the possibility to measure the needed differentials momentarily, i.e. at one point in time. We need at least a few points in time where β is changing to be able to determine σ from equation 112 and 113, under the assumption that parameter α is not changing. If α is not changing then $\alpha = ks$ in the basepoint and is known if we take one of the measured point as our basepoint. In total, we need at least two points to calculate σ from equation 112 and 113, i.e. from

$$\gamma = \frac{log(ks) - log\,(\alpha)}{log(\beta) - log\,(\beta_0)} \quad \text{and} \tag{119}$$

$$\sigma = \frac{1}{1-\gamma} \tag{120}$$

With more data points available, we can fit the data to this non-linear equation by e.g. a least square fit.

If α is a function of time then the solution is not so straightforward.

We can write equation 70 as

$$\alpha = \frac{1 - \left(\frac{y}{\xi_g y_0}\right)^{\gamma}}{1 - \left(\frac{k}{\xi_g k_0}\right)^{\gamma}} \tag{121}$$

Be aware of the fact that for this purpose we are not using formulas that depend on c_1, because those formulas only hold in equilibrium. Equation 119 and 121 are derived from the two economic identities available and from the maximum profit conditions, which means that they hold at all times.

Diamond and McFadden (1965, 1978) proved that σ could not be resolved without additional a priori knowledge or assumptions.

Klump et al. (2011) and many others find in their investigations that there is a strong evidence that the elasticity of substitution is below 1, or more in particular, is between .4 and .7.

As we argued in section 6, we will use a general progress term as is done in the equations 119-121 or alternatively equations 119, 120 and 99. In addition σ, α, ξ_G and ξ_{TFP} are functions of time in general.

With n measured data points we have $3(n-1)$ unknown variables and $2(n-1)$ equations. It is clear that we cannot resolve the variables. However, we can reduce the number of variables by taking e.g. the elasticity of substitution σ a constant and letting the capital-labor-mix α a linear or quadratic function of time. In the linear case for α we have $1+1+(n-1)$ variables and $2(n-1)$ equations, which we can solve if we have enough data points with a minimum of $n=3$ (the base point inclusive). For the quadratic case we have $3+(n-1)$ variables and $2(n-1)$ equations, which we can solve if we have enough data points with a minimum of $n=4$ (the base point inclusive).

Upfront it is not clear where we can reduce the parameter space. It seems plausible that the capital-labor-mix α is moving slowly in time, allowing a linear or quadratic approximation.

A point of concern is the sensitivity for errors on determining σ. You can see from equation 119 that if α and ks are close together then the error should be smaller than the difference between α and ks. The same is true for β and β_0. Partial differentiating equation 119 with respect to α clearly show this sensitivity.

$$\frac{\partial\sigma}{\partial\alpha}=\frac{\sigma(\sigma-1)}{\alpha\log\left(\frac{\alpha}{ks}\right)} \qquad (122)$$

or relative

$$\frac{\partial\sigma}{\sigma}=\frac{\partial\alpha}{\alpha}\frac{(\sigma-1)}{\log\left(\frac{\alpha}{ks}\right)} \qquad (123)$$

Notice that for $\sigma\approx 1$ the power to discriminate decreases.

Equation 123 clearly shows the accuracy of the measured data needed to be able to estimate the elasticity of substitution.

12. Labor or capital saving technical progress

In literature we encounter in many places the term labor saving progress. Unfortunately, is it not always clear what is exactly meant by this term. As a consequence of our new way of describing technical progress we will have to reconsider it. For that we take a closer look at labor saving and start with a microeconomic definition. We call a new production process labor saving if we can achieve the same output with less labor under a certain budget constraint. For simplicity we will only consider labor and capital production factors. As is common knowledge, we know that depending on the elasticity of substitution the equilibrium between labor and capital will change if prices of the factors will change.

However, from a macroeconomic perspective the reasoning is a little bit different. Prices are indirectly determined by consumer and producer behavior, by the depreciation rate, by the elasticity of substitution and by the capital-labor-ratio.

Suppose for a moment that these parameters do not change and that there is only ξ_{TFP} due to technical progress, which will generate ξ_g as we have seen, then the capital deepening will increase with the factor ξ_g. The capital deepening is the ratio of used capital and labor. One could say that this is relatively labor saving. That is, relative to capital. But it will not save labor in the sense that there will be less people needed resulting in unemployment, because from a theoretical point of view we can always achieve full employment on each level of capital.

Altogether this means that technical progress ξ_{TFP} itself, always will be relatively labor saving. The other direction is not very likely to happen, because we will not allow changes in technical progress if this will lower our productivity. The other parameters have the ability to introduce relatively labor saving as well as more relatively labor consuming changes. Total factor productivity change ξ_{TFP} is the only factor causing ongoing growth.

> Lemma: Total factor productivity change ξ_{TFP} is the only factor causing an ongoing increase in capital deepening and growth. It is always relatively labor saving, meaning that capital deepening is increasing.

This brings us to an interesting practical case. Recently it is argued, that to keep medical health care possible with an aging population L, with the number of workers L_w, it would be wise to focus on labor saving measures regarding the health care if the ratio workforce and total population $r_L = \frac{L_w}{L}$ is decreasing. First of all, health care is a field which is labor intensive, especially regarding the needed one to one attention for people. It will be hard or impossible to increase technical progress in health care to compensate for the lack of employees. In fact, it is probably easier to focus on technical progress in general for the economy in total and reduce the problem to a money reallocation issue and shift more workers to health care organizations. One of the easiest ways, which is totally under our control, is to increase hours L_h worked per worker or increase number of years worked (so shifting pension age) or a combination of the two in order to keep productivity per inhabitant y_L at the same level.

In formula form

$$Y = L_w L_h y_h \tag{124}$$

with y_h is productivity per worked hour. Needed capital will become

$$k_h = \beta y_h \tag{125}$$

Productivity per inhabitant is expressed by

$$y_L = \frac{Y}{L} = r_L L_h y_h = r_L y_w \tag{126}$$

with y_w is the productivity per worker.

If the depreciation rate δ, the coefficient of substitution σ, the capital-labor-mix α and c_1 are not changing then the capital to income ratio β is constant. As can be seen from the formulas, a decrease of r_L can be compensated by an increase of L_h or an increase ξ_g of productivity y_h, which is the same as an increase ξ_g of capital deepening k_h due to an increase in total factor productivity ξ_{TFP}.

If the depreciation rate δ, the coefficient of substitution σ and the capital-labor-mix α are changing then we have to adapt c_1 to the desired firm profitability is reached again, or to another desired criterium.

Profitability in the base point is

$$r_0 = \frac{\alpha_0}{\beta_0} - \delta_0 \qquad (127)$$

with

$$\beta_0 = \frac{1-c_{1_0}}{\delta_0} \qquad (128)$$

Profitability in the new point is

$$r_1 = \frac{\alpha_1}{\beta_1} - \delta_1 \qquad (129)$$

with

$$\beta_1 = \frac{1-c_{1_0}}{\delta_1} \qquad (130)$$

In order to keep $r = r_0$ we have to adapt c_1 to c_{1_2}. Profitability will become

$$r_2 = \frac{\alpha_1\left(\frac{\beta_2}{\beta_1}\right)^{\gamma}}{\beta_2} - \delta_1 = \frac{\delta_1\alpha_1\left(\frac{1-c_{1_2}}{1-c_{1_0}}\right)^{\gamma}}{1-c_{1_2}} - \delta_1 \qquad (131)$$

For $r_2 = r_0$ the new c_1 will result in c_{1_2}

$$c_{1_2} = 1 - \left(\frac{\delta_1\alpha_1}{(r_0+\delta_1)(1-c_{1_0})}\right)^{\sigma}(1-c_{1_0}) \qquad (132)$$

In the special case that $\delta_1 = \delta_0$ then c_{1_2} reduces to

$$c_{1_2} = 1 - \left(\frac{\alpha_1}{\alpha_0}\right)^{\sigma}(1-c_{1_0}) \qquad (133)$$

Altogether this result in a shift of k_h which also has to be compensated for in the way we discussed before.

13. Growth accounting vs. neo-classical growth theory

Solow started his theory on growth accounting with the production function formula

$$Y = A\,F(K,L)\,. \qquad (134)$$

He stated explicitly that he used neutral capital and labor technical progress by using

$$A\,F(K,L) = F(AK,AL)\,. \qquad (135)$$

Assuming that $F(K, L)$ is homogeneous of degrees one equation 135 is correct, but as we have explained in this paper calling factor A neutral capital and labor technical progress is not allowed.

This means that the starting principles of growth accounting are wrong.

To show where this will lead us, we will start with equation 99 and derive equivalent formulas for the same production function in the intensive form

$$y = y_0 \xi_{TFP} \left[\alpha_1 \left(\frac{k}{k_0}\right)^{\gamma} + (1 - \alpha_1)\right]^{1/\gamma} \tag{136}$$

$$y = y_0 \xi_g \left[\alpha_2 \left(\frac{k}{\xi_g k_0}\right)^{\gamma} + (1 - \alpha_2)\right]^{1/\gamma} \tag{137}$$

$$y = \xi_g \left[a \left(\frac{k}{\xi_g}\right)^{\gamma} + b\right]^{1/\gamma} \tag{138}$$

and in the extensive form

$$Y = Y_0 \xi_{TFP} \left[\alpha_1 \left(\frac{K}{K_0}\right)^{\gamma} + (1 - \alpha_1) \left(\frac{L}{L_0}\right)^{\gamma}\right]^{1/\gamma} \tag{139}$$

$$Y = Y_0 \xi_G \left[\alpha_2 \left(\frac{K}{\xi_G K_0}\right)^{\gamma} + (1 - \alpha_2) \left(\frac{L}{\xi_L L_0}\right)^{\gamma}\right]^{1/\gamma} \tag{140}$$

$$Y = \xi_G \left[a \left(\frac{K}{\xi_G}\right)^{\gamma} + b \left(\frac{L}{\xi_L}\right)^{\gamma}\right]^{1/\gamma} \tag{141}$$

where α_0 holds in the base point, $\alpha_1 = \alpha_2 \left(\frac{1}{\xi_{TFP}}\right)^{\gamma}$ and α_2 is the changed capital-labor mix from α_0 to α_2.

If we change to a general format, we can write

$$Y = \xi_G \, F(\frac{K}{\xi_G}, \frac{L}{\xi_L}) \, . \tag{142}$$

Dividing by Y and using its homogeneous degree one property equation 142 results in

$$1 = F\left(\beta, \frac{\xi_G}{Y} \frac{L}{\xi_L}\right) = F\left(\beta, \frac{\xi_g}{y}\right). \tag{143}$$

This equation shows that under the condition of constant capital to income ratio, the productivity has to be divided by its increase ξ_g in order to satisfy the formula.

Equivalent we can write

$$Y = F(K, \xi_g L) \tag{144}$$

from which it is clear that the same is achieved by multiplying the labor production factor L with the increase in productivity ξ_g. It looks like Harrod Neutral, but keep in mind that we do not deal with any kind of capital or labor neutrality any longer. We change the principle 'technical progress is Harrod neutral or else the production function has to be Cobb-Douglas' into 'technical progress in a homogeneous degree one production function can be represented by multiplying the labor production factor L with the productivity factor ξ_g (equation 144), corresponding with the change in total factor productivity ξ_{TFP} at constant capital to income ratio'. For an arbitrary production function use equation 142.

Lemma: The principle 'technical progress is Harrod neutral or else the production function has to be Cobb-Douglas' is no longer valid and has to be changed into 'technical progress in a homogeneous degree one production function can be represented by multiplying the labor production factor L with the productivity factor ξ_g (equation 144), corresponding with the change in total factor productivity ξ_{TFP} at constant capital to income ratio'. For an arbitrary production function use equation 142.

Apart from ξ_g, as time is evolving, the production function $Y = F(K, \xi_g L)$ will change and result in not easy to implement formulas, that is why, in that case, we will approximate the production function piece wise by CES production functions.

For point (Y_0, K_0, L_0) we assume that we also know the changed parameter α_2 and σ. We can use equation 139 as the approximated production function.

To see the connection with Solow's growth accounting formula, we will consider equation 144 and write it in the same form as Solow did (equation 134), now assuming that ξ_g is the only change in the production function, leaving α_0 unchanged under constant capital to income ratio

$$Y = \xi_{TFP} F\left(\frac{K}{\xi_{TFP}}, \frac{\xi_g L}{\xi_{TFP}}\right) \tag{145}$$

Notice that equation 139 is satisfying equation 145, which is easy to show using equation 88.

We have to conclude that equation 134 with factor A is equal to total factor productivity ξ_{TFP}

$$Y = \xi_{TFP}\, F(K, L) \tag{146}$$

is not equal to our derivation (equation 145) of the production function after technical progress has taken place, nor can factor A be adjusted to accomplish that equality.

As a check we change, in our CES example, to the new base point $(Y_1, K_1, L_1) = (\xi_G Y_0, \xi_G K_0, \xi_L L_0)$ with unchanged $\alpha_2 = \alpha_0$. Equation 139 or 140 will convert to

$$Y = Y_0 \xi_{TFP} \left[\alpha_1 \left(\frac{K}{K_0}\right)^{\gamma} + (1 - \alpha_1)\left(\frac{L}{L_0}\right)^{\gamma}\right]^{1/\gamma} = Y_1 \left[\alpha_0 \left(\frac{K}{K_1}\right)^{\gamma} + (1 - \alpha_0)\left(\frac{L}{L_1}\right)^{\gamma}\right]^{1/\gamma} \tag{147}$$

which has the same form as the production function in point (Y_0, K_0, L_0) and so representing the same production function, albeit on a scaled level.

If we use equation 144 as a start for growth accounting we can write

$$\frac{dY}{Y} = \frac{\partial F}{\partial \xi_g}\frac{\xi_g}{Y}\frac{d\xi_g}{\xi_g} + \frac{\partial F}{\partial K}\frac{K}{Y}\frac{dK}{K} + \frac{\partial F}{\partial L}\frac{L}{Y}\frac{dL}{L} \tag{148}$$

In point $(Y_1, K_1, L_1) = (\xi_G Y_0, \xi_G K_0, \xi_L L_0)$ this reduces to the growth rate equation

$$g_Y = (1-\alpha_2)g_g + \alpha_2 g_K + (1-\alpha_2)g_L = g_{TFP} + \alpha_2 g_K + (1-\alpha_2)g_L \tag{149}$$

where we used equation 89, because this formula also holds in general for infinitesimal small changes of ξ_{TFP}, from which we deduct that $(1-\alpha_2)g_g = g_{TFP}$.

Equation 149 is the standard basic equation from which growth accounting is evaluated.

If we use Solow's equation 146 in (Y_0, K_0, L_0) we arrive at a similar expression

$$g_Y = g_{TFP} + \alpha_0 g_K + (1-\alpha_0)g_L \tag{150}$$

We conclude that this leads to the same result.

It is obvious that equation 146 holds for the Cobb-Douglas case. Near the base point all production functions with the same labor-capital-mix α_0 are nearly the same and can be approximated by e.g. a Cobb-Douglas production function. Only in the Cobb-Douglas case it is a good description of the entire production function. However, we can use this procedure to estimate the total productivity factor ξ_{TFP} in the base point.

> Lemma: If we forget the origin of Solow's formula, i.e. some kind of capital or labor neutrality, the formulas for growth accounting are still valid.
>
> $g_Y = g_{TFP} + \alpha_0 g_K + (1-\alpha_0)g_L$

Altogether this means that the presented formulas hold in general for infinitesimal small variations. The fact that we started with a wrong equation 134 was not harmful, because it is (by coincidence?) the right formula in case of a Cobb-Douglas production function, for which equation 150 always holds. Notice that A is total factor productivity. In general, however, if you are not considering small variations, you have to do the exact inverse calculation to resolve A.

14. Conclusions

You might think, if it turns out in a special case, that the elasticity of substitution is approximately $\sigma \approx 1$ then we could use the Cobb-Douglas case as a good approximation. With equation 78 reduced to $\xi_{TFP} = {\xi_K}^{\alpha}{\xi_{LT}}^{1-\alpha}$, we still can use Cobb-Douglas. This will, however, introduce a new problem because then you have to explain why it does not matter how progress ξ_{TFP} in total factor productivity TFP is achieved, by ξ_K or ξ_{LT} or a combination of the two. And that is why we also have to reject capital and labor augmented technical progress in this case. Although, ignoring it would be a solution. And this is what is done in growth accounting for many decades and also how it is used by Donselaar (2011). So as long as you do not hold ξ_K and/or ξ_{LT} accountable for ξ_{TFP} you are on the right track. And it is exactly that what we introduced with our new philosophy in section 10, but now for all production functions. In the case of CES functions capital and labor technical progress is responsible for changes in ξ_{TFP} and can also introduce changes in the capital-labor-mix as well as in the elasticity of substitution. And to be real, what else had you expected if we have one scale factor and two form factors available with regard to CES production functions. In total this leads to:

- We have rejected the old way of implementing capital and labor augmented technical progress as is done since it was introduced/used/described by Hicks, Solow, Harrod, Uzawa, Inada, Jones, Acemoglu, Barro, Krusell (2014) and Stiglitz, to name a few.
- We introduced an alternative for capital and labor progress in the form of total factor productivity and other possible parameter changes.
- It solves the long existing theoretical problem, labor augmented technical progress only or Cobb-Douglas.
- We showed the relation between total factor productivity and total growth per capita.
- We analyzed the difficulties of determining the elasticity of substitution.
- The analysis presented gives you an alternative view on the Solow growth model.
- Growth accounting and neo-classical growth theory are in perfect harmony.
- We have the impression that these changes will create new possibilities of describing capital and labor progress in an even better way.
- It also may pave the way to new forms of endogenous technical progress.

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