Group-theoretic analysis of a scalar field on a square lattice

Kogure, Yosuke and Ikeda, Kiyohiro

Tohoku University

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Yosuke Kogure[1], Kiyohiro Ikeda[2]

Department of Civil and Environmental Engineering, Tohoku University, Aoba, Sendai 980-8579, Japan

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Abstract

In this paper, we offer group-theoretic bifurcation theory to elucidate the mechanism of the self-organization of square patterns in economic agglomerations. First, we consider a scalar field on a square lattice that has the symmetry described by the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ and investigate steady-state bifurcation of the spatially uniform equilibrium to steady planforms periodic on the square lattice. To be specific, we derive the irreducible representations of the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ and show the existence of bifurcating solutions expressing square patterns by two different mathematical ways: (i) using the equivariant branching lemma and (ii) solving the bifurcation equation. Second, we apply such a group-theoretic methodology to a spatial economic model with the replicator dynamics on the square lattice and demonstrate the emergence of the square patterns. We furthermore focus on a special feature of the replicator dynamics: the existence of invariant patterns that retain their spatial distribution when the value of the bifurcation parameter changes. We numerically show the connectivity between the uniform equilibrium and invariant patterns through the bifurcation. The square lattice is one of the promising spatial platforms for spatial economic models in new economic geography. A knowledge elucidated in this paper would contribute to theoretical investigation and practical applications of economic agglomerations.

Keywords: Bifurcation, group-theoretic bifurcation theory, invariant pattern, new economic geography, replicator dynamics, self-organization, spatial economic model, square lattice.

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[1]Author for correspondence: yosuke.kogure.t2@dc.tohoku.ac.jp

[2]kiyohiro.ikeda.b4@tohoku.ac.jp

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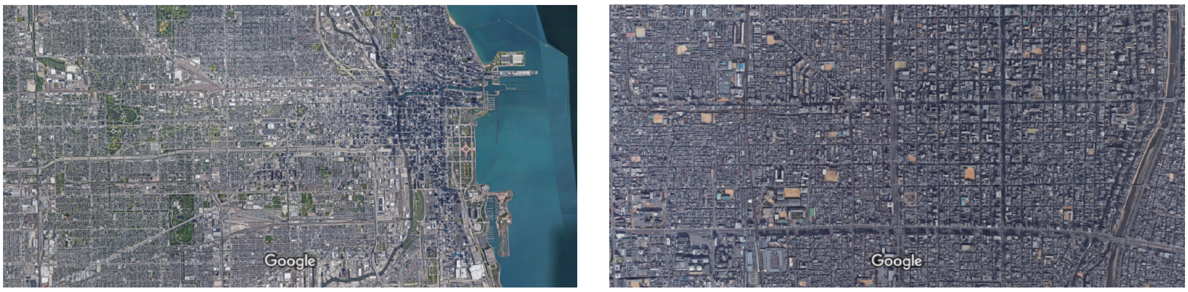


(a) Chicago (USA)

(b) Kyoto (Japan)

Figure 1.1: Satellite photographs of cities provided by Google Maps displaying square road networks

1. Introduction

Square road networks prosper worldwide. Chicago (USA) and Kyoto (Japan), for example, are well-known to accommodate such square networks historically (see Figure 1.1). This paper intends to elucidate the mechanism of economic agglomerations on such square networks as the important contribution of nonlinear mathematics to spatial economics.

In spatial economics, the mechanism of economic agglomerations is highlighted as the most important topic. After a pioneering work by Krugman, 1991 [1], bifurcation is welcomed as a catalyst to engender a core place and a peripheral place from two identical places. The study of spatial agglomerations have come to be extended from the two-region economy to the racetrack economy (one-dimension) and, in turn, to explain various polycentric agglomerations (Tabuchi and Thisse, 2011 [2]; Ikeda et al., 2012 [3]; Akamatsu et al., 2012 [4]). In economic geography, central place theory (Christaller, 1933 [5]; Lösch, 1954 [6]) envisaged the emergence of hexagonal agglomerations based on the distribution of cities and towns in Southern Germany. The existence of a hexagonal distribution of mobile production factors (e.g., firms and workers) was shown based on a microeconomic foundation (Eaton and Lipsey, 1975 [7]). To explain the mechanism of economic agglomerations in the real world, spatial platforms for spatial economic models are required to be extended to a two-dimensional spaces as conducted in this present paper.

Lattice economies, including a hexagonal lattice and a square lattice, can accommodate various two-dimensional agglomeration patterns of economic interest. Motivated by hexagonal agglomerations in central place theory, the bifurcation mechanism of spatial economic models on the hexagonal lattice has been elucidated (Ikeda and Murota, 2014 [8]). The stability of bifurcating solutions from the uniform distribution was investigated to demonstrate that theoretically predicted bifurcating solutions, including hexagonal patterns, are all unstable just after the bifurcation (Ikeda et al., 2018 [9]). Geometrical distributions that are solutions to the governing equation of a spatial economic model with the replicator dynamics, irrespective of the value of the bifurcation (transport cost) parameter, are called invariant patterns and were demonstrated to represent economic agglomerations of great economic interest (Ikeda et al., 2019 [10]).

Yet the bifurcation mechanism of spatial economic models on the square lattice is not understood to the full extent. Some studies dealt with economic agglomerations on the square lattice (Clarke and Wilson, 1983 [11]; Weidlich and Haag, 1987 [12]; Munz and Weidlich, 1990 [13]; Brakman et al., 1999 [14]) but are not based on spatial economic models. As a pioneering study that combined the square lattice with a spatial economic model, Ikeda et al., 2018 [15] investigated a break bifurcation point on the uniform distribution and indicated the occurrence of period-doubling bifurcation. This study, however, found just a fraction of bifurcating solutions and invariant patterns on the square lattice by relying on an ad hoc procedure.

That said, we aim to develop group-theoretic bifurcation theory of spatial economic models on the square lattice. Such development would enrich the application of bifurcation theory in nonlinear mathematics and would contribute to the future study in spatial economics. We rely on two perspectives of agglomeration behaviour:

- bifurcation mechanisms due to geometrical symmetry and
- the existence of invariant patterns for the replicator dynamics.

We present an exhaustive list of bifurcating solutions from the uniform distribution. We develop a systematic procedure to obtain invariant patterns as a generalization of Ikeda et al., 2018 [15] and Ikeda et al., 2019 [10]. We obtain invariant patterns exhaustively, including the uniform, monocentric, and polycentric distributions. The list of bifurcating solutions and invariant patterns advanced in this paper would be of assistance in the study of economic agglomerations.

The group-theoretic analysis in this paper proceeds as follows. We first investigate the bifurcation of a scalar field on the square lattice with periodic boundary conditions, which has the symmetry described by the finite group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. Note that many pattern-formation phenomena have been modeled by partial differential equations with group equivariance on an infinite plane. As the mathematical model of reaction-diffusion models, Navier-Stokes flow, and the Bénard problem, a system that is equivariant to the infinite group $\mathrm{D}_4 \ltimes \mathrm{T}^2$ (T^2 expresses the two-torus of translation symmetries) has been studied (Dionne et al., 1997 [16]; Golubitsky and Stewart, 2003 [17]). As for economic agglomerations described by spatial economic models, it is essential to assume a discretized finite plane. For this reason, we employ the finite group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

We next obtain invariant patterns for the replicator dynamics on the square lattice. Invariant patterns have come to be used in the analysis of spatial economic models to capture a series of agglomeration patterns of economic interest (Takayama et al., 2020 [18]; Osawa et al., 2020 [19]). We use a systematic procedure proposed for the hexagonal lattice (Ikeda et al., 2019 [10]) and obtain invariant patterns exhaustively.

We finally conduct a numerical bifurcation analysis of a spatial economic model on the square lattice. We find mesh-like bifurcation diagrams with a large number of horizontal lines and non-horizontal curves, like threads of warp and weft. Horizontal lines correspond to invariant patterns, and non-horizontal curves correspond to bifurcating solutions. Such mesh-like bifurcation diagrams are similar to those which observed for the hexagonal lattice (Ikeda et al., 2018 [9]).

As the major achievement of this paper, we elucidate the connectivity between the uniform state and invariant patterns: Population tends to be agglomerated to places with the largest positive components of a bifurcating solution from the uniform distribution, and then the spatial distribution arrives at an invariant pattern via a bifurcating curve. We futhermore pay a special attention to the fact that when two half branches at a bifurcation point are symmetric (respectively, asymmetric), they would arrive at one (respectively, two) invariant patterns. We obtain theoretical conditions for the symmetry and the asymmetry of such bifurcating half branches as another contribution of this paper.

This paper is organized as follows. Chapter 2 introduces an $n \times n$ square lattice with the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ and classifies square patterns for economic agglomerations on this lattice. Chapter 3 gives a derivation of the irreducible representations of the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. Chapter 4 provides the matrix representations of this group. Chapters 5 and 6 present a group-theoretic bifurcation analysis by using equivariant branching lemma and by solving the bifurcation equation, respectively. Chapter 7 applies the group-theoretic bifurcation analysis to a spatial economic model on the square lattice and conducts numerical simulations according to the theoretical results elucidated in Chapters 5 and 6. Chapter 8 expresses concluding remarks.

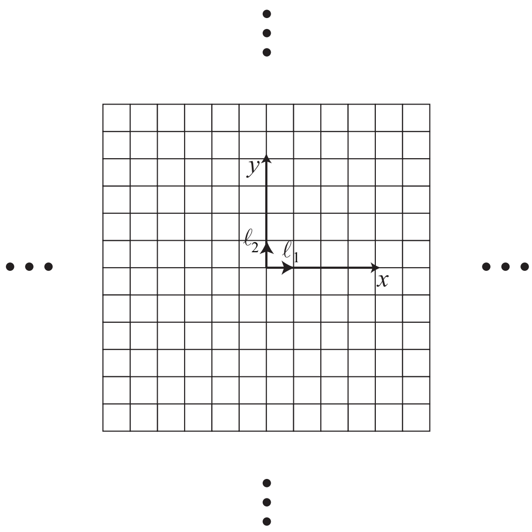


Figure 2.1: Infinite square lattice

2. Square Patterns on Square Lattice

In this chapter, we introduce an $n \times n$ finite square lattice with periodic boundary conditions comprising a system of uniformly distributed $n \times n$ places. We allocate discretized degrees-of-freedom to each node of this lattice. Periodic boundary conditions allow us to express infiniteness and uniformity and to avoid heterogeneity due to the boundaries by spatially repeating the finite lattice periodically to cover an infinite two-dimensional domain.

Using the group consisting of D_4 and $\mathbb{Z}_n \times \mathbb{Z}_n$, we express the symmetry of this lattice with discretized degrees-of-freedom. The study conducted in this chapter is purely geometric and involves no bifurcation mechanism. it forms, however, an important foundation of the group-theoretic bifurcation analysis in Chapters 5 and 6.

This chapter is organized as follows. The infinite square lattice is introduced in Section 2.1. Square patterns are described in Section 2.2. The $n \times n$ finite square lattice is given in Section 2.3. The group associated with the square lattice is given in Section 2.4.

2.1. Infinite Square Lattice

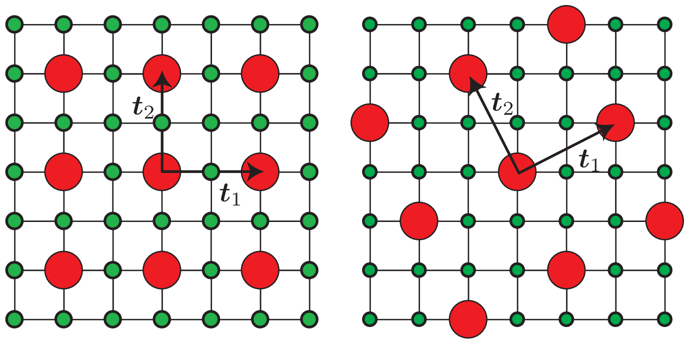
Infinite square lattice is given as a set of integer combinations of oblique *basis vectors*

$$\boldsymbol{\ell}_1 = d \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{\ell}_2 = d \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{2.1}$$

where $d > 0$ means the length of these vectors. That is, the infinite square lattice $\mathcal{H}$ is expressed as

$$\mathcal{H} = \{n_1 \boldsymbol{\ell}_1 + n_2 \boldsymbol{\ell}_2 \mid n_1, n_2 \in \mathbb{Z}\}, \tag{2.2}$$

where $\mathbb{Z}$ denotes the set of integers. Figure 2.1 depicts the infinite square lattice.



(a) $(\alpha, \beta) = (1, 0)$ (b) $(\alpha, \beta) = (1, 1)$

Figure 2.2: Square patterns on the square lattice

To represent square patterns on the lattice $\mathcal{H}$, we consider a *sublattice* spanned by

$$\boldsymbol{t}_1 = \alpha \boldsymbol{\ell}_1 + \beta \boldsymbol{\ell}_2, \quad \boldsymbol{t}_2 = -\beta \boldsymbol{\ell}_1 + \alpha \boldsymbol{\ell}_2, \tag{2.3}$$

where α and β are integer-valued parameters with $(\alpha, \beta) \neq (0, 0)$. We denote this sublattice by $\mathcal{H}(\alpha, \beta)$, that is,

$$\begin{aligned} \mathcal{H}(\alpha, \beta) &= \{ n_1 \boldsymbol{t}_1 + n_2 \boldsymbol{t}_2 \mid n_1, n_2 \in \mathbb{Z} \} \\ &= \{ (n_1 \alpha - n_2 \beta) \boldsymbol{\ell}_1 + (n_1 \beta + n_2 \alpha) \boldsymbol{\ell}_2 \mid n_1, n_2 \in \mathbb{Z} \} \\ &= \left\{ [\, \boldsymbol{\ell}_1 \ \boldsymbol{\ell}_2 \,] \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \middle| \ n_1, n_2 \in \mathbb{Z} \right\}. \end{aligned} \tag{2.4}$$

Since $\|\boldsymbol{t}_1\| = \|\boldsymbol{t}_2\|$ and the angle between $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ is $\pi/2$, the lattice $\mathcal{H}(\alpha, \beta)$ indeed represents a square pattern.

The *spatial period* L is defined to be the (common) length of the basis vectors $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$, which is given by

$$L = d\sqrt{\alpha^2 + \beta^2}. \tag{2.5}$$

We refer to

$$\frac{L}{d} = \sqrt{\alpha^2 + \beta^2} \tag{2.6}$$

as the *normalized spatial period*, which is an important index for characterizing the size of a square pattern. Although the definition here refers to the basis vectors, the spatial period L, as well as the normalized spatial period L/d, is in fact determined by the sublattice $\mathcal{H}(\alpha, \beta)$, as seen from (2.9) with (2.8) below.

The normalized spatial period L/d in (2.6) takes specific values $\sqrt{1}$, $\sqrt{2}$, $\sqrt{4}$, $\sqrt{5}$, $\ldots$ as a consequence of the fact that α and β are integers. The square pattern with $L/d = 1$ is the uniform pattern. The normalized spatial period is obtained from (2.6) as

$$\begin{aligned}\frac{L}{d} &= \sqrt{\alpha^2+\beta^2} \\ &= \sqrt{1},\ \sqrt{2},\ \sqrt{4},\ \sqrt{5},\ \sqrt{8},\ \sqrt{9},\ \sqrt{10},\ \sqrt{13},\ \sqrt{16},\ \sqrt{17},\ \sqrt{18},\ \sqrt{20},\ \sqrt{25},\ldots \\ &= \begin{cases} 1,2,3,4,5,\ldots, \\ \sqrt{2},\ \sqrt{5},\ \sqrt{8},\ \sqrt{10},\ \sqrt{13},\ \sqrt{17},\ \sqrt{18},\ \sqrt{20},\ldots. \end{cases}\end{aligned} \tag{2.7}$$

The parameter values are given as follows:

$$(\alpha,\beta) = \begin{cases} (1,0): & L/d = 1, \\ (1,1): & L/d = \sqrt{2}, \\ (2,0): & L/d = 2, \\ (2,1): & L/d = \sqrt{5}, \\ (2,2): & L/d = \sqrt{8}, \\ (3,0): & L/d = 3, \\ (3,1): & L/d = \sqrt{10}, \\ (3,2): & L/d = \sqrt{13}, \\ (4,0): & L/d = 4, \\ (4,1): & L/d = \sqrt{17}, \\ (3,3): & L/d = \sqrt{18}, \\ (4,2): & L/d = \sqrt{20}, \\ (4,3): & L/d = 5, \\ (5,0): & L/d = 5,\ldots. \end{cases}$$

Figure 2.2 depicts some square patterns.

2.2. Description of Square Patterns

Square patterns are parameterized and classified into several types.

2.2.1. Parameterization of Square Patterns

In the parameterization (α,β) of the lattice, let us note its non-uniqueness that different parameter values of (α,β) can sometimes result in the same lattice $\mathcal{H}(\alpha,\beta)$. Define

$$D = D(\alpha,\beta) = \alpha^2+\beta^2, \tag{2.8}$$

which is a positive integer for $(\alpha,\beta) \neq (0,0)$. It will be shown later in this subsection that D is an invariant in this parameterization, that is, we have the following implication:

$$\mathcal{H}(\alpha,\beta) = \mathcal{H}(\alpha',\beta') \implies D(\alpha,\beta) = D(\alpha',\beta'). \tag{2.9}$$

The converse, however, is not true, as the following example shows.

Example 2.1. For $(\alpha,\beta) = (5,0)$ and $(\alpha',\beta') = (4,3)$, we have $D(\alpha,\beta) = D(\alpha',\beta') = 25$. But the lattices $\mathcal{H}(\alpha,\beta)$ and $\mathcal{H}(\alpha',\beta')$ are different. □

Table 2.1: The values of $D(\alpha,\beta)$ for (α,β) in (2.11)

$\alpha \setminus \beta$	0	1	2	3	4	5	6	7
1	1	2						
2	4	5	8					
3	9	10	13	18				
4	16	17	20	$\boxed{25}$	32			
5	$\boxed{25}$	26	29	34	41	$\boxed{50}$		
6	36	37	40	45	52	61	72	
7	49	$\boxed{50}$	53	58	65	74	85	98

The parameter space for the square sublattices is given as follows, and the proof is given later in this subsection.

Proposition 2.1. *Square sublattices $\mathcal{H}(\alpha,\beta)$ are parameterized, one-to-one, by*

$$\{(\alpha,\beta) \in \mathbb{Z}^2 \mid \alpha > 0,\ \beta \geq 0\}. \tag{2.10}$$

Two lattices $\mathcal{H}(\alpha,\beta)$ and $\mathcal{H}(\beta,\alpha)$ are not identical in general, but are mirror images with respect to the y-axis. As such they are naturally regarded as essentially the same. Let us call two square lattices *essentially different* if they are neither identical nor mirror images with respect to the y-axis. Essentially different square sublattices are parameterized as follows, the proof being given later in this subsection.

Proposition 2.2. *Essentially different square sublattices $\mathcal{H}(\alpha,\beta)$ are parameterized, one-to-one, by*

$$\{(\alpha,\beta) \in \mathbb{Z}^2 \mid \alpha \geq \beta \geq 0,\ \alpha \neq 0\}. \tag{2.11}$$

Table 2.1 shows the values of $D = D(\alpha,\beta)$ for (α,β) with $7 \geq \alpha \geq \beta \geq 0$, $\alpha \neq 0$. It is worth noting that the values of D in this table are all distinct with the exceptions of $D(5,0) = D(4,3) = 25$ and $D(5,5) = D(7,1) = 50$. This means, in particular, that smaller square patterns (with $D < 25$) are uniquely determined by their spatial period L, which is related to D as

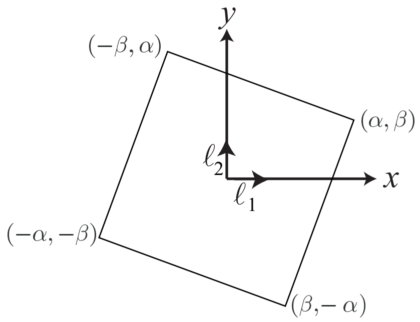
$$\frac{L}{d} = \sqrt{D} \tag{2.12}$$

by (2.6) and (2.8).

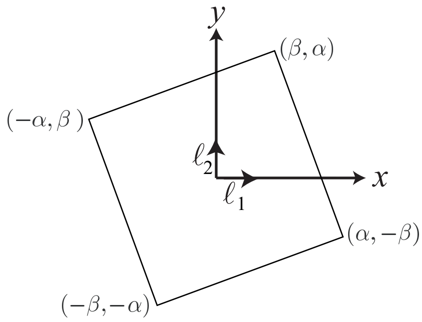
Proofs of (2.9) and Propositions 2.1 and 2.2

First, recall that $\mathcal{H}(\alpha,\beta)$ is generated by $(\boldsymbol{t}_1, \boldsymbol{t}_2) = (\boldsymbol{t}_1(\alpha,\beta), \boldsymbol{t}_2(\alpha,\beta))$ in (2.3), which can be expressed as

$$\begin{bmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\ell}_1 & \boldsymbol{\ell}_2 \end{bmatrix} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}. \tag{2.13}$$



(a) Square pattern for (2.15)



(b) Square pattern for (2.16)

Figure 2.3: Square patterns associated with (2.15) and (2.16)

The determinant of this coefficient matrix coincides with $D(\alpha,\beta)$ introduced in (2.8), i.e.,

$$D(\alpha,\beta) = \alpha^2 + \beta^2 = \det \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}. \tag{2.14}$$

If $\mathcal{H}(\alpha',\beta') \subseteq \mathcal{H}(\alpha,\beta)$, then

$$\begin{bmatrix} \alpha' & -\beta' \\ \beta' & \alpha' \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

for some integers $x_{11}, x_{12}, x_{21}, x_{22}$, and hence $D(\alpha',\beta')$ is a multiple of $D(\alpha,\beta)$. Exchanging the roles of (α,β) and (α',β'), we have (2.9).

Next, the parameter spaces (2.10) and (2.11) for $\mathcal{H}(\alpha,\beta)$ are derived. We observe geometrically (see Fig. 2.3(a)) that $\mathcal{H}(\alpha',\beta') = \mathcal{H}(\alpha,\beta)$ if and only if $\boldsymbol{t}_1' = \alpha'\boldsymbol{\ell}_1 + \beta'\boldsymbol{\ell}_2$ is obtained from $\boldsymbol{t}_1 = \alpha\boldsymbol{\ell}_1 + \beta\boldsymbol{\ell}_2$ by a rotation at an angle that is a multiple of $\pi/2$, i.e., $\boldsymbol{t}_1' = R_4{}^k\, \boldsymbol{t}_1$ with

$$R_4 = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

for some $k \in \{0, 1, 2, 3\}$. Since

$$R_4\, \boldsymbol{t}_1 = R_4(\alpha\boldsymbol{\ell}_1 + \beta\boldsymbol{\ell}_2) = \alpha(\boldsymbol{\ell}_2) + \beta(-\boldsymbol{\ell}_1) = [\, \boldsymbol{\ell}_1\ \boldsymbol{\ell}_2 \,] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

we have $\mathcal{H}(\alpha',\beta') = \mathcal{H}(\alpha,\beta)$ if and only if

$$\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

for some $k \in \{0, 1, 2, 3\}$. Therefore, we obtain the same lattice for the following four parameter values:

$$(\alpha,\beta),\ (-\beta,\alpha),\ (-\alpha,-\beta),\ (\beta,-\alpha). \tag{2.15}$$

This allows us to adopt (2.10) as the parameter space for $\mathcal{H}(\alpha,\beta)$, by which we mean that, for every $(\alpha',\beta') \neq (0,0)$ in $\mathbb{Z}^2$, the sublattice $\mathcal{H}(\alpha',\beta')$ is the same as the sublattice $\mathcal{H}(\alpha,\beta)$ for some (uniquely determined) (α,β) in (2.10). It should be mentioned, in particular, that $\mathcal{H}(0,\beta) = \mathcal{H}(\beta,0)$ by (2.15), and hence we have $\alpha > 0$ in (2.10).

Geometrically, the lattices for (α,β) and (β,α) are mirror images with respect to the line $x = y$. In this sense, we regard $\mathcal{H}(\alpha,\beta)$ and $\mathcal{H}(\beta,\alpha)$ as essentially the same. Thus, we regard the following four parameter values as essentially equivalent to (α,β):

$$(\beta,\alpha),\ (-\alpha,\beta),\ (-\beta,-\alpha),\ (\alpha,-\beta). \tag{2.16}$$

See Fig. 2.3(b) for the square pattern of (2.16). If $\beta = 0$ or $\alpha = \beta$, the set of four parameters in (2.16) is identical to the set in (2.15). This is because the lattices for $\beta = 0$ or $\alpha = \beta$ are symmetric with respect to the line $x = y$.

Thus, essentially equivalent parameter values can be summarized as follows:

$$\begin{array}{llll} (\alpha,\beta), & (-\beta,\alpha), & (-\alpha,-\beta), & (\beta,-\alpha), \\ (\beta,\alpha), & (-\alpha,\beta), & (-\beta,-\alpha), & (\alpha,-\beta). \end{array} \tag{2.17}$$

which reduces in a special case of $\beta = 0$ to

$$(\alpha,0),\quad (0,\alpha),\quad (-\alpha,0),\quad (0,-\alpha) \tag{2.18}$$

or in another special case of $\alpha = \beta$ to

$$(\alpha,\alpha),\quad (-\alpha,\alpha),\quad (-\alpha,-\alpha),\quad (\alpha,-\alpha). \tag{2.19}$$

On the basis of the observations above, (2.11) can be adopted as the parameter space for essentially different sublattices. This means that every $(\alpha,\beta) \neq (0,0)$ in $\mathbb{Z}^2$ is essentially equivalent to some (uniquely determined) member in (2.11).

2.2.2. Types of Square Patterns

The *tilt angle* φ of the sublattice $\mathcal{H}(\alpha,\beta)$ is defined as the angle between $\boldsymbol{\ell}_1$ and $\boldsymbol{t}_1$, i.e., by

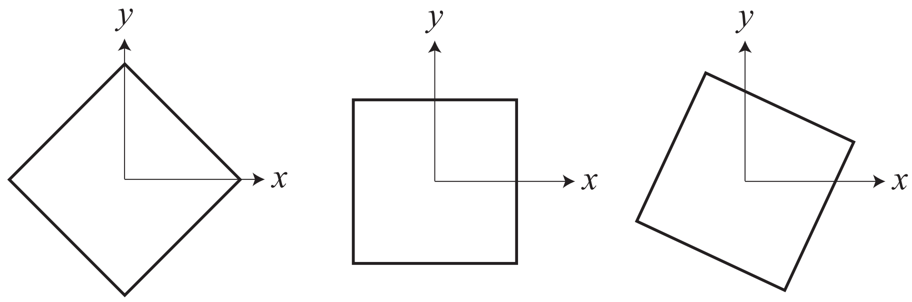
$$\cos\varphi = \frac{(\boldsymbol{\ell}_1)^\top \boldsymbol{t}_1}{\|\boldsymbol{\ell}_1\| \cdot \|\boldsymbol{t}_1\|}, \tag{2.20}$$

where (α,β) is chosen from the parameter space in (2.10) or (2.11). This is equivalent to defining φ by

$$\varphi = \arcsin\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right). \tag{2.21}$$

With reference to the tilt angle φ, square patterns can be classified into three types:

$$\begin{cases} \text{type V} & \text{if } \varphi = 0, \\ \text{type M} & \text{if } \varphi = \pi/4, \\ \text{type T} & \text{otherwise.} \end{cases} \tag{2.22}$$



(a) Type V (b) Type M (c) Type T

Figure 2.4: Square patterns of three types that are centered at the origin

Figure 2.4 depicts square patterns of these three types that are centered at the origin, where "V" indicates that the x-axis contains a vertex of the square, "M" denotes that the x-axis contains the midpoint of two neighboring vertices of that square, and "T" means "tilted." In terms of the parameter (α,β), this classification is expressed as

$$\begin{cases} \text{type V} & \text{if } (\alpha,\beta) = (\alpha,0) \quad (\alpha \geq 1), \\ \text{type M} & \text{if } (\alpha,\beta) = (\beta,\beta) \quad (\beta \geq 1), \\ \text{type T} & \text{otherwise,} \end{cases} \tag{2.23}$$

where the parameter space for type T depends on the choice of (2.10) or (2.11) as

$$\text{For (2.10)} : \{(\alpha,\beta) \mid \alpha > 0,\ \beta \geq 0,\ \alpha \neq \beta\}, \tag{2.24}$$

$$\text{For (2.11)} : \{(\alpha,\beta) \mid \alpha > \beta \geq 0\}. \tag{2.25}$$

Accordingly, the parameter spaces in (2.10) and (2.11) are divided, respectively, into three parts:

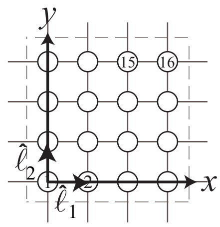
$$\{(\alpha,0) \mid \alpha \geq 1\} \ \cup\ \{(\beta,\beta) \mid \beta \geq 1\} \ \cup\ \{(\alpha,\beta) \mid \alpha > 0,\ \beta \geq 0,\ \alpha \neq \beta\}, \tag{2.26}$$

$$\{(\alpha,0) \mid \alpha \geq 1\} \ \cup\ \{(\beta,\beta) \mid \beta \geq 1\} \ \cup\ \{(\alpha,\beta) \mid \alpha > \beta \geq 0\}. \tag{2.27}$$

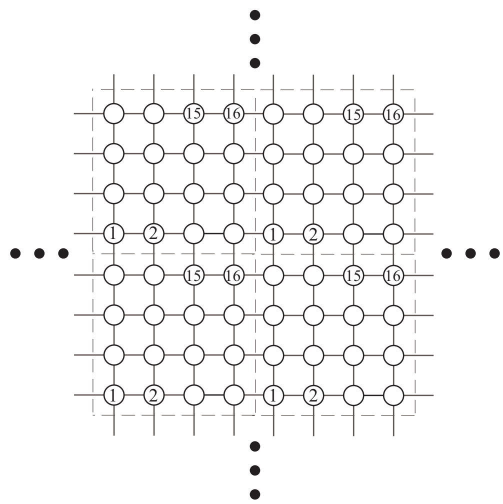
The types V, M, and T are correlated with the normalized spatial period as

$$L/d = \begin{cases} \sqrt{4},\ \sqrt{9},\ \sqrt{16},\ \sqrt{25},\ldots & : \text{type V}, \\ \sqrt{2},\ \sqrt{8},\ \sqrt{18},\ \sqrt{32},\ldots & : \text{type M}, \\ \sqrt{5},\ \sqrt{10},\ \sqrt{13},\ \sqrt{17},\ldots & : \text{type T}. \end{cases}$$

It should be emphasized, however, that the type does not always determine, nor is determined by, the spatial period. This is demonstrated by the two lattices $\mathcal{H}(5,0)$ and $\mathcal{H}(4,3)$. These lattices share the same normalized spatial period $L/d = \sqrt{25}$, but of different types; the former is of type V and the latter of type T.



(a) 4×4 square lattice



(b) Periodic boundaries

Figure 2.5: A system of places on the 4×4 square lattice with periodic boundaries

2.3. Finite Square Lattice

In the previous subsections we have considered the infinite square lattice spreading over the entire plane. We now introduce an $n \times n$ *finite square lattice* with periodic boundary conditions.

We now consider a subset $\mathcal{H}_n$ of $\mathcal{H}$ that consists of integer combinations with coefficients between 0 and $n-1$:

$$\mathcal{H}_n = \{n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2 \mid n_i \in \mathbb{Z},\ 0 \leq n_i \leq n-1\ (i = 1, 2)\}. \tag{2.28}$$

This is a finite set comprising n^2 elements, where n represents the size of $\mathcal{H}_n$. Figure 2.5(a) depicts the 4×4 square lattice.

The infinite lattice $\mathcal{H}$ is regarded as a periodic extension of $\mathcal{H}_n$ with the two-dimensional period of $(n\boldsymbol{\ell}_1, n\boldsymbol{\ell}_2)$. In other words, $\mathcal{H}$ is regarded as being covered by translations of $\mathcal{H}_n$ by vectors of the form $m_1(n\boldsymbol{\ell}_1) + m_2(n\boldsymbol{\ell}_2)$ with integers m_1 and m_2. A point $n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2$ in $\mathcal{H}$ corresponds to $n_1'\boldsymbol{\ell}_1 + n_2'\boldsymbol{\ell}_2$ in $\mathcal{H}_n$ for (n_1', n_2') given by

$$n_1' \equiv n_1 \mod n, \quad n_2' \equiv n_2 \mod n. \tag{2.29}$$

Figure 2.5(b) depicts the 4×4 square lattice with periodic boundaries.

For the sublattice $\mathcal{H}(\alpha, \beta)$ of $\mathcal{H}$ defined in (2.4), we may consider its portion $\mathcal{H}(\alpha, \beta) \cap \mathcal{H}_n$ contained in $\mathcal{H}_n$, expecting that the periodic extension of this portion coincides with $\mathcal{H}(\alpha, \beta)$ itself. If this is the case, we say that (α, β) is *compatible* with n, or n is compatible with (α, β). Using the Minkowski sum[3] of $\mathcal{H}(\alpha, \beta) \cap \mathcal{H}_n$ and $\mathcal{H}(n, 0)$, the *condition for compatibility* can be expressed as

$$(\mathcal{H}(\alpha, \beta) \cap \mathcal{H}_n) + \mathcal{H}(n, 0) = \mathcal{H}(\alpha, \beta), \tag{2.30}$$

[3]For two sets $X, Y \subseteq \mathbb{Z}^2$, their *Minkowski sum* $X + Y$ is defined as $X + Y = \{\boldsymbol{x} + \boldsymbol{y} \mid \boldsymbol{x} \in X, \boldsymbol{y} \in Y\}$.

which is equivalent to

$$\mathcal{H}(n,0) \subseteq \mathcal{H}(\alpha,\beta). \tag{2.31}$$

The compatibility condition is given as follows:

Proposition 2.3. *The size n of $\mathcal{H}_n$ is compatible with (α,β) if and only if n is a multiple of $D(\alpha,\beta)/\gcd(\alpha,\beta)$, that is,*

$$n = m\,\frac{D(\alpha,\beta)}{\gcd(\alpha,\beta)}, \quad m = 1,2,\ldots. \tag{2.32}$$

Proof. By (2.31), the size n is compatible with (α,β) if and only if

$$\begin{bmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = n \begin{bmatrix} \boldsymbol{\ell}_1 & \boldsymbol{\ell}_2 \end{bmatrix}$$

for some integers $x_{11}, x_{12}, x_{21}, x_{22}$, where $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ are defined in (2.3). Substituting

$$\begin{bmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\ell}_1 & \boldsymbol{\ell}_2 \end{bmatrix} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

into the above equation and multiplying the inverse of $\begin{bmatrix} \boldsymbol{\ell}_1 & \boldsymbol{\ell}_2 \end{bmatrix}$ from the left, we obtain

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

from which

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = n \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}^{-1} = \frac{n}{D(\alpha,\beta)} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \frac{n \gcd(\alpha,\beta)}{D(\alpha,\beta)} \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\hat{\beta} & \hat{\alpha} \end{bmatrix},$$

where $\hat{\alpha} = \alpha/\gcd(\alpha,\beta)$ and $\hat{\beta} = \beta/\gcd(\alpha,\beta)$. This shows that $x_{11}, x_{12}, x_{21}, x_{22}$ are integers if and only if n is a multiple of $D(\alpha,\beta)/\gcd(\alpha,\beta)$. □

When combined with the three types in (2.23), the compatibility condition (2.32) in Proposition 2.3 shows the following statements:

- For a pattern $\mathcal{H}(\alpha,\beta)$ of type V, parameterized by $(\alpha,\beta) = (\alpha,0)$ with $\alpha \geq 1$, a compatible n is a multiple of α.
- For a pattern $\mathcal{H}(\alpha,\beta)$ of type M, parameterized by $(\alpha,\beta) = (\beta,\beta)$ with $\beta \geq 1$, a compatible n is a multiple of 2β.
- For a pattern $\mathcal{H}(\alpha,\beta)$ of type T, with (α,β) in (2.24) or (2.25), a compatible n is a multiple of $D(\alpha,\beta)/\gcd(\alpha,\beta)$.

To sum up, we have

$$n = \begin{cases} m\alpha & (\alpha \geq 1) \quad \text{for type V}, \\ 2m\beta & (\beta \geq 1) \quad \text{for type M}, \\ mD(\alpha,\beta)/\gcd(\alpha,\beta) & \quad \text{for type T}, \end{cases} \tag{2.33}$$

where $m = 1,2,\ldots$.

2.4. *Groups Expressing the Symmetry*

The first step of the bifurcation analysis of the square pattern on the $n \times n$ square lattice is to identify the subgroup expressing the symmetry of this pattern.

2.4.1. *Symmetry of the Finite Square Lattice*

The symmetry of the $n\times n$ square lattice $\mathcal{H}_n$ in (2.28) is characterized by invariance with respect to

- r: counterclockwise rotation about the origin at an angle of $\pi/2$,
- s: reflection $y \mapsto -y$,
- p_1: periodic translation along the ℓ_1-axis (i.e., the x-axis), and
- p_2: periodic translation along the ℓ_2-axis (i.e., the y-axis).

Consequently, the symmetry of the square lattice $\mathcal{H}_n$ is described by the group

$$G = \langle r, s, p_1, p_2 \rangle, \tag{2.34}$$

which is generated by r, s, p_1, and p_2 with the fundamental relations:

$$\begin{aligned} &r^4 = s^2 = (rs)^2 = {p_1}^n = {p_2}^n = e, \quad p_2 p_1 = p_1 p_2, \\ &rp_1 = p_2 r, \quad rp_2 = p_1^{-1} r, \quad sp_1 = p_1 s, \quad sp_2 = p_2^{-1} s, \end{aligned} \tag{2.35}$$

where e is the identity element. Each element of G can be represented uniquely in the form of

$$s^l r^m {p_1}^i {p_2}^j, \quad l \in \{0, 1\}, \ m \in \{0, 1, 2, 3\}, \ i, j \in \{0, 1, \ldots, n-1\}. \tag{2.36}$$

The group G contains the *dihedral group*

$$\langle r, s \rangle \simeq \mathrm{D}_4$$

and *cyclic groups*

$$\langle p_1 \rangle \simeq \mathbb{Z}_n, \quad \langle p_2 \rangle \simeq \mathbb{Z}_n$$

as its subgroups, where $\mathbb{Z}_n$ means the cyclic group of order n, which is denoted as C_n. The group G has the structure of the semidirect product of D_4 by $\mathbb{Z}_n \times \mathbb{Z}_n$, that is, $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

Remark 2.1. A group G is said to be a *semidirect product* of a subgroup H by another subgroup A, denoted $G = A \rtimes H$, if

- A is a *normal subgroup* of G, and
- each element $g \in G$ is represented uniquely as $g = ah$ with $a \in A$ and $h \in H$.

Each element $g = ah \in G$ can also be represented uniquely in an alternative form of $g = h'a$ with $h' \in H$ and $a \in A$, since $g = ah = h(h^{-1}ah)$ and $h' = h^{-1}ah \in A$ by the normality of A. Our group $G = \langle r, s, p_1, p_2 \rangle$ is a semidirect product of $H = \mathrm{D}_4$ by $A = \mathbb{Z}_n \times \mathbb{Z}_n$, and we have $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ in accordance with $g = s^l r^m {p_1}^i {p_2}^j$ in (2.36) with $s^l r^m \in \mathrm{D}_4$ and ${p_1}^i {p_2}^j \in \mathbb{Z}_n \times \mathbb{Z}_n$. For more details on the definition of semidirect product, see Curtis and Reiner, 1962 [20]. □

2.4.2. Subgroups for Square Patterns

The symmetry of $\mathcal{H}(\alpha,\beta) \cap \mathcal{H}_n$ is described by a subgroup of $G = \langle r, s, p_1, p_2\rangle$, which we denote by $G(\alpha,\beta)$. With notations[4]

$$\Sigma(\alpha,\beta) = \langle r, s, p_1^{\alpha} p_2^{\beta}, p_1^{-\beta} p_2^{\alpha}\rangle, \tag{2.37}$$

$$\Sigma_0(\alpha,\beta) = \langle r, \ \ p_1^{\alpha} p_2^{\beta}, p_1^{-\beta} p_2^{\alpha}\rangle, \tag{2.38}$$

the subgroup $G(\alpha,\beta)$ is given as follows:

$$G(\alpha,\beta) = \begin{cases} \langle r, s, p_1^{\alpha}, p_2^{\alpha}\rangle = \Sigma(\alpha, 0) & (\alpha \geq 1, \beta = 0) \quad : \text{type V}, \\ \langle r, s, p_1^{\beta} p_2^{\beta}, p_1^{-\beta} p_2^{\beta}\rangle = \Sigma(\beta,\beta) & (\beta \geq 1, \alpha = \beta) \quad : \text{type M}, \\ \langle r, \ \ p_1^{\alpha} p_2^{\beta}, p_1^{-\beta} p_2^{\alpha}\rangle = \Sigma_0(\alpha,\beta) & (\text{otherwise}) \quad\quad\ : \text{type T}, \end{cases} \tag{2.39}$$

where the parameter (α,β) for type T runs over $\{(\alpha,\beta) \mid \alpha > 0, \ \beta \geq 0, \ \alpha \neq \beta\}$ in (2.24) or $\{(\alpha,\beta) \mid \alpha > \beta \geq 0\}$ in (2.25), depending on the adopted parameter space (2.10) or (2.11).

The parameter (α,β) must be compatible with the lattice size n via (2.33), which restricts (α,β) to stay in a bounded range. Among the square patterns of type V on the $n \times n$ lattice, we exclude those with $\Sigma(1, 0)$ from our consideration of subgroups since $\Sigma(1, 0) = \langle r, s, p_1, p_2\rangle$ represents the symmetry of the underlying $n \times n$ square lattice. That is, we consider $\Sigma(\alpha, 0)$ for $2 \leq \alpha \leq n$ since n is divisible by α by (2.33). A square pattern with the symmetry of $\Sigma(n, 0) = \mathrm{D}_4$, which lacks translational symmetry, is included here as a square of type V for theoretical consistency. As for type M, we must have $1 \leq \beta \leq n/2$ in $\Sigma(\beta,\beta)$ since n is divisible by 2β ($\beta \geq 1$) by (2.33). The parameter for type T, which is dependent on the choice of (2.10) or (2.11), must stay in the range

$$\text{For (2.10)} : \{(\alpha,\beta) \mid 1 \leq \alpha \leq n-1, \ 1 \leq \beta \leq n-1, \ \alpha \neq \beta\}, \tag{2.40}$$

$$\text{For (2.11)} : \{(\alpha,\beta) \mid 1 \leq \beta < \alpha \leq n-1\}. \tag{2.41}$$

To sum up, the relevant subgroups of our interest are given by

$$\begin{cases} \Sigma(\alpha, 0) = \langle r, s, p_1^{\alpha}, p_2^{\alpha}\rangle & (2 \leq \alpha \leq n) & : \text{type V}, \\ \Sigma(\beta,\beta) = \langle r, s, p_1^{\beta} p_2^{\beta}, p_1^{-\beta} p_2^{\beta}\rangle & (1 \leq \beta \leq n/2) & : \text{type M}, \\ \Sigma_0(\alpha,\beta) = \langle r, p_1^{\alpha} p_2^{\beta}, p_1^{-\beta} p_2^{\alpha}\rangle & ((2.40) \text{ or } (2.41)) & : \text{type T}. \end{cases} \tag{2.42}$$

Recall that (α,β) must also satisfy the compatibility condition (2.33).

[4]Subscript "0" to $\Sigma_0(\alpha,\beta)$ indicates the lack of s.

3. Irreducible Representations of the Group for Square Lattice

In the previous chapter, we introduced an $n \times n$ square lattice as a two-dimensional discretized uniform space and identified the symmetry of this lattice by the group (2.34):

$$G = \langle r, s, p_1, p_2 \rangle, \tag{3.1}$$

which is composed of the dihedral group $\langle r, s \rangle \simeq \mathrm{D}_4$ expressing local square symmetry and the group $\langle p_1, p_2 \rangle \simeq \mathbb{Z}_n \times \mathbb{Z}_n$ (direct product of two cyclic groups of order n) expressing translational symmetry in two directions. In the group-theoretic bifurcation analysis in Chapters 5 and 6, we will find bifurcating solutions for each irreducible representation of this group, as each irreducible representation is associated with possible bifurcating solutions with certain symmetries. It is, therefore, the first step of the analysis to obtain all the irreducible representations of this group.

It is not difficult to obtain all irreducible representations for groups with simple structures such as the dihedral and cyclic groups. Yet for the group in (3.1) with a far more complicated structure, it might be difficult to list all the irreducible representations in an ad hoc way. Fortunately, we can use the method of little groups in group representation theory to obtain all the irreducible representations in a systematic manner. In this chapter, we describe this method and construct a complete list of the irreducible representations of this group. It turns out that the irreducible representations over $\mathbb{R}$ are one-, two-, four-, or eight-dimensional, and all of them are absolutely irreducible. We will use the irreducible representations derived in this manner in the group-theoretic bifurcation analysis in Chapters 5 and 6 to prove the existence of square patterns.

This chapter is organized as follows. The matrix forms of the irreducible representations are listed in Section 3.1. The irreducible representations of the group are derived in Section 3.2.

3.1. List of Irreducible Representations

The irreducible representations of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ over $\mathbb{R}$ are listed in this section, whereas their derivation is given in Section 3.2.

3.1.1. Number of Irreducible Representations

The irreducible representations of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ over $\mathbb{R}$ are one-, two-, four-, or eight-dimensional. The number $N_d$of d-dimensional irreducible representations of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ depends on n, as shown below:

$n \setminus d$	1	2	4	8
	N_1	N_2	N_4	N_8
$2m$	8	6	$3(n-2)$	$(n^2-6n+8)/8$
$2m-1$	4	1	$2(n-1)$	$(n^2-4n+3)/8$

(3.2)

where m denotes a positive integer. For some values of n, the concrete numbers N_d of the d-dimensional irreducible representations are listed in Table 3.1. This table for $n = 1$ shows that $\mathrm{D}_4 \ltimes (\mathbb{Z}_1 \times \mathbb{Z}_1)$, being isomorphic to D_4, has four one-dimensional irreducible representations and one two-dimensional ones. Four-dimensional irreducible representations exist for $n \geq 3$ and eight-dimensional ones appear for $n \geq 5$.

Table 3.1: Number N_d of d-dimensional irreducible representations of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$

$n \setminus d$	1	2	4	8	
	N_1	N_2	N_4	N_8	$\sum N_d$
1	4	1	0	0	5
2	8	6	0	0	14
3	4	1	4	0	9
4	8	6	6	0	20
5	4	1	8	1	14
6	8	6	12	1	27
7	4	1	12	3	20
8	8	6	18	3	35
9	4	1	16	6	27
10	8	6	24	6	44
11	4	1	20	10	35
12	8	6	30	10	54

$n \setminus d$	1	2	4	8	
	N_1	N_2	N_4	N_8	$\sum N_d$
13	4	1	24	15	44
14	8	6	36	15	65
15	4	1	28	21	54
16	8	6	42	21	77
17	4	1	32	28	65
18	8	6	48	28	90
19	4	1	36	36	77
20	8	6	54	36	104
21	4	1	40	45	90
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
42	8	6	120	190	324

We have the relation

$$\sum_d d^2 N_d = 1^2 N_1 + 2^2 N_2 + 4^2 N_4 + 8^2 N_8 = 8n^2, \tag{3.3}$$

which is a special case of the well-known general identity for the number of irreducible representations over $\mathbb{C}$. This formula applies since all the irreducible representations over $\mathbb{R}$ of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ are absolutely irreducible (see Section 3.2).

In Sections 3.1.2–3.1.5, the matrix forms of the irreducible representations of respective dimensions are shown together with their characters. Table 3.2 is a preview summary, referring to names of irreducible representations, such as $(1; +, +, +)$ and $(8; k, \ell)$, to be introduced in the following subsections.

3.1.2. One-Dimensional Irreducible Representations

The group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ has eight one-dimensional irreducible representations. They are labeled $(1; +, +, +)$, $(1; +, -, +)$, $(1; -, +, +)$, $(1; -, -, +)$, $(1; +, +, -)$, $(1; +, -, -)$, $(1; -, +, -)$, $(1; -, -, -)$ and are given by

$$\begin{array}{llll}
T^{(1;+,+,+)}(r) = 1, & T^{(1;+,+,+)}(s) = 1, & T^{(1;+,+,+)}(p_1) = 1, & T^{(1;+,+,+)}(p_2) = 1, \\
T^{(1;+,-,+)}(r) = 1, & T^{(1;+,-,+)}(s) = -1, & T^{(1;+,-,+)}(p_1) = 1, & T^{(1;+,-,+)}(p_2) = 1, \\
T^{(1;-,+,+)}(r) = -1, & T^{(1;-,+,+)}(s) = 1, & T^{(1;-,+,+)}(p_1) = 1, & T^{(1;-,+,+)}(p_2) = 1, \\
T^{(1;-,-,+)}(r) = -1, & T^{(1;-,-,+)}(s) = -1, & T^{(1;-,-,+)}(p_1) = 1, & T^{(1;-,-,+)}(p_2) = 1, \\
T^{(1;+,+,-)}(r) = 1, & T^{(1;+,+,-)}(s) = 1, & T^{(1;+,+,-)}(p_1) = -1, & T^{(1;+,+,-)}(p_2) = -1, \\
T^{(1;+,-,-)}(r) = 1, & T^{(1;+,-,-)}(s) = -1, & T^{(1;+,-,-)}(p_1) = -1, & T^{(1;+,-,-)}(p_2) = -1, \\
T^{(1;-,+,-)}(r) = -1, & T^{(1;-,+,-)}(s) = 1, & T^{(1;-,+,-)}(p_1) = -1, & T^{(1;-,+,-)}(p_2) = -1, \\
T^{(1;-,-,-)}(r) = -1, & T^{(1;-,-,-)}(s) = -1, & T^{(1;-,-,-)}(p_1) = -1, & T^{(1;-,-,-)}(p_2) = -1.
\end{array} \tag{3.4}$$

Table 3.2: Irreducible representations of $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$

$n \setminus d$	1	2	4	8
$2m$	$(1;+,+,+), (1;+,-,+)$	$(2;+), (2;-)$	$(4;k,0,+), (4;k,0,-)$	$(8;k,\ell)$
	$(1;+,-,+), (1;-,-,+)$	$(2;+,+), (2;+,-)$	$(4;k,k,+), (4;k,k,-)$	
	$(1;+,+,-), (1;+,-,-)$	$(2;-,+), (2;-,-)$	$(4;n/2,\ell,+), (4;n/2,\ell,-)$	
	$(1;-,+,-), (1;-,-,-)$			
$2m-1$	$(1;+,+,+), (1;+,-,+)$	$(2;+)$	$(4;k,0,+), (4;k,0,-)$	$(8;k,\ell)$
	$(1;+,-,+), (1;-,-,+)$		$(4;k,k,+), (4;k,k,-)$	

$(4;k,0;+), (4;k,0;-)$ with $1 \leq k \leq \lfloor (n-1)/2 \rfloor$ in (3.13);
$(4;k,k;+), (4;k,k;-)$ with $1 \leq k \leq \lfloor (n-1)/2 \rfloor$ in (3.14);
$(4;n/2,\ell;+), (4;n/2,\ell;-)$ with $1 \leq \ell \leq \lfloor (n-1)/2 \rfloor$ in (3.15);
$(8;k,\ell)$ with $1 \leq \ell \leq k-1$, $2 \leq k \leq \lfloor (n-1)/2 \rfloor$ in (3.25)

The characters $\chi^{\mu}(g) = \operatorname{Tr} T^{\mu}(g)$, which are equal to $T^{\mu}(g)$ for one-dimensional representations, are given as follows for $\mu = (1;+,+,+)$, $(1;+,-,+)$, $(1;-,+,+)$, $(1;-,-,+)$, $(1;+,+,-)$, $(1;+,-,-)$, $(1;-,+,-)$, $(1;-,-,-)$:

g	$\chi^{(1;+,+,+)}(g)$	$\chi^{(1;+,-,+)}(g)$	$\chi^{(1;-,+,+)}(g)$	$\chi^{(1;-,-,+)}(g)$
${p_1}^i {p_2}^j$	1	1	1	1
$r{p_1}^i {p_2}^j$	1	1	-1	-1
$r^2{p_1}^i {p_2}^j$	1	1	1	1
$r^3{p_1}^i {p_2}^j$	1	1	-1	-1
$sr^m{p_1}^i {p_2}^j$ (m : even)	1	-1	1	-1
(m : odd)	1	-1	-1	1

(3.5)

g	$\chi^{(1;+,+,-)}(g)$	$\chi^{(1;+,-,-)}(g)$	$\chi^{(1;-,+,-)}(g)$	$\chi^{(1;-,-,-)}(g)$
${p_1}^i {p_2}^j$	$(-1)^{i+j}$	$(-1)^{i+j}$	$(-1)^{i+j}$	$(-1)^{i+j}$
$r{p_1}^i {p_2}^j$	$(-1)^{i+j}$	$(-1)^{i+j}$	$-(-1)^{i+j}$	$-(-1)^{i+j}$
$r^2{p_1}^i {p_2}^j$	$(-1)^{i+j}$	$(-1)^{i+j}$	$(-1)^{i+j}$	$(-1)^{i+j}$
$r^3{p_1}^i {p_2}^j$	$(-1)^{i+j}$	$(-1)^{i+j}$	$-(-1)^{i+j}$	$-(-1)^{i+j}$
$sr^m{p_1}^i {p_2}^j$ (m : even)	$(-1)^{i+j}$	$-(-1)^{i+j}$	$(-1)^{i+j}$	$-(-1)^{i+j}$
(m : odd)	$(-1)^{i+j}$	$-(-1)^{i+j}$	$-(-1)^{i+j}$	$(-1)^{i+j}$

(3.6)

where $i, j = 0, 1, \ldots, n-1$ and $m = 0, 1, 2, 3$.

3.1.3. Two-Dimensional Irreducible Representations

The group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ has six or one two-dimensional irreducible representations depending on whether n is even or odd. Two two-dimensional irreducible representations, denoted as $(2;\sigma)$ $(\sigma \in \{+,-\})$, exist for n even and are defined by

$$T^{(2;\sigma)}(r) = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}, \quad T^{(2;\sigma)}(s) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \tag{3.7}$$

$$T^{(2,\sigma)}(p_1) = T^{(2;\sigma)}(p_2) = \sigma \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \tag{3.8}$$

whereas $(2;-)$ is absent for n odd. The other four two-dimensional irreducible representations, denoted as $(2;\sigma_r,\sigma_s)$ $(\sigma_r,\sigma_s \in \{+,-\})$, exist when n is even and are defined by

$$T^{(2;\sigma_r,\sigma_s)}(r) = \begin{bmatrix} & \sigma_r \\ 1 & \end{bmatrix}, \quad T^{(2;\sigma_r,\sigma_s)}(s) = \sigma_s \begin{bmatrix} 1 & \\ & \sigma_r \end{bmatrix}, \tag{3.9}$$

$$T^{(2;\sigma_r,\sigma_s)}(p_1) = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}, \quad T^{(2;\sigma_r,\sigma_s)}(p_2) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}. \tag{3.10}$$

The characters $\chi^{\mu}(g) = \mathrm{Tr}\, T^{\mu}(g)$ are given as follows for $\mu = (2;+)$, $(2;-)$, $(2;+,+)$, $(2;+,-)$, $(2;-,+)$, $(2;-,-)$. For the representations $\mu = (2;+), (2;-)$ in (3.7) and (3.8), we have

g	$\chi^{(2;+)}(g)$	$\chi^{(2;-)}(g)$
${p_1}^i {p_2}^j$	2	$(-1)^{i+j}2$
$r{p_1}^i {p_2}^j$	0	0
$r^2{p_1}^i {p_2}^j$	-2	$-(-1)^{i+j}2$
$r^3{p_1}^i {p_2}^j$	0	0
$sr^m{p_1}^i {p_2}^j$	0	0

(3.11)

where $i, j = 0, 1, \ldots, n-1$ and $m = 0, 1, 2, 3$. For the representations $\mu = (2;+,+)$, $(2;+,-)$, $(2;-,+)$, $(2;-,-)$ in (3.9) and (3.10), we have

g	$\chi^{(2;+,+)}(g)$	$\chi^{(2;+,-)}(g)$	$\chi^{(2;-,+)}(g)$	$\chi^{(2;-,-)}(g)$
${p_1}^i {p_2}^j$	$(-1)^i + (-1)^j$	$(-1)^i + (-1)^j$	$(-1)^i + (-1)^j$	$(-1)^i + (-1)^j$
$r{p_1}^i {p_2}^j$	0	0	0	0
$r^2{p_1}^i {p_2}^j$	$(-1)^i + (-1)^j$	$(-1)^i + (-1)^j$	$-(-1)^i - (-1)^j$	$-(-1)^i - (-1)^j$
$r^3{p_1}^i {p_2}^j$	0	0	0	0
$s{p_1}^i {p_2}^j$	$(-1)^i + (-1)^j$	$-(-1)^i - (-1)^j$	$(-1)^i - (-1)^j$	$-(-1)^i + (-1)^j$
$sr{p_1}^i {p_2}^j$	0	0	0	0
$sr^2{p_1}^i {p_2}^j$	$(-1)^i + (-1)^j$	$-(-1)^i - (-1)^j$	$-(-1)^i + (-1)^j$	$(-1)^i - (-1)^j$
$sr^3{p_1}^i {p_2}^j$	0	0	0	0

(3.12)

where $i, j = 0, 1, \ldots, n-1$ and $m = 0, 1, 2, 3$.

3.1.4. Four-Dimensional Irreducible Representations

The group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ with $n \geq 3$ has 4-dimensional irreducible representations. We can designate them by

$$(4; k, 0, \sigma) \;\text{ with }\; 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \; \sigma \in \{+, -\}; \tag{3.13}$$

$$(4; k, k, \sigma) \;\text{ with }\; 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \; \sigma \in \{+, -\}; \tag{3.14}$$

$$(4; n/2, \ell, \sigma) \;\text{ with }\; 1 \leq \ell \leq \frac{n}{2} - 1, \; \sigma \in \{+, -\}. \tag{3.15}$$

Here $(4; n/2, \ell, \sigma)$ exists only for n even and $\lfloor x \rfloor$ denotes the largest integer not larger than x for a real number x. The number of 4-dimensional irreducible representations is given by

$$N_4 = \begin{cases} 3n - 6 & (n = 2m), \\ 2n - 2 & (n = 2m - 1). \end{cases} \tag{3.16}$$

The irreducible representation $(4; k, 0, \sigma)$ is given by

$$T^{(4;k,0,\sigma)}(r) = \begin{bmatrix} & S \\ I & \end{bmatrix}, \quad T^{(4;k,0,\sigma)}(s) = \sigma \begin{bmatrix} I & \\ & S \end{bmatrix}, \tag{3.17}$$

$$T^{(4;k,0,\sigma)}(p_1) = \begin{bmatrix} R^k & \\ & I \end{bmatrix}, \quad T^{(4;k,0,\sigma)}(p_2) = \begin{bmatrix} I & \\ & R^k \end{bmatrix}, \tag{3.18}$$

where

$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}. \tag{3.19}$$

The irreducible representation $(4; k, k, \sigma)$ is given by

$$T^{(4;k,k,\sigma)}(r) = \begin{bmatrix} & S \\ I & \end{bmatrix}, \quad T^{(4;k,k,\sigma)}(s) = \sigma \begin{bmatrix} & S \\ S & \end{bmatrix}, \tag{3.20}$$

$$T^{(4;k,k,\sigma)}(p_1) = \begin{bmatrix} R^k & \\ & R^{-k} \end{bmatrix}, \quad T^{(4;k,k,\sigma)}(p_2) = \begin{bmatrix} R^k & \\ & R^k \end{bmatrix}. \tag{3.21}$$

The irreducible representation $(4; n/2, \ell, \sigma)$ is given by

$$T^{(4;n/2,\ell,\sigma)}(r) = \begin{bmatrix} & S \\ I & \end{bmatrix}, \quad T^{(4;n/2,\ell,\sigma)}(s) = \sigma \begin{bmatrix} S & \\ & I \end{bmatrix}, \tag{3.22}$$

$$T^{(4;n/2,\ell,\sigma)}(p_1) = \begin{bmatrix} -I & \\ & R^{-\ell} \end{bmatrix}, \quad T^{(4;n/2,\ell,\sigma)}(p_2) = \begin{bmatrix} R^\ell & \\ & -I \end{bmatrix}. \tag{3.23}$$

The characters $\chi^{(4;k,0,\sigma)}(g) = \operatorname{Tr} T^{(4;k,0,\sigma)}(g)$, $\chi^{(4;k,k,\sigma)}(g) = \operatorname{Tr} T^{(4;k,k,\sigma)}(g)$, and $\chi^{(4;n/2,\ell,\sigma)}(g) =$

$\operatorname{Tr} T^{(4;n/2,\ell,\sigma)}(g)$ for $\sigma \in \{+,-\}$ are given as follows:

$$
\begin{array}{c|ccc}
g & \chi^{(4;k,0,\sigma)}(g) & \chi^{(4;k,k,\sigma)}(g) & \chi^{(4;n/2,\ell,\sigma)}(g) \\
\hline
{p_1}^i{p_2}^j & \begin{array}{l} 2[\cos(ki\theta) \\ \quad + \cos(kj\theta)] \end{array} & \begin{array}{l} 2[\cos(k(i+j)\theta) \\ \quad + \cos(k(i-j)\theta)] \end{array} & \begin{array}{l} 2[(-1)^i\cos(\ell j\theta) \\ \quad + (-1)^j\cos(\ell i\theta)] \end{array} \\
\hline
\begin{array}{c} r^m{p_1}^i{p_2}^j \\ (m=1,2,3) \end{array} & 0 & 0 & 0 \\
\hline
s{p_1}^i{p_2}^j & 2\sigma\cos(ki\theta) & 0 & 2\sigma(-1)^j\cos(\ell i\theta) \\
sr{p_1}^i{p_2}^j & 0 & 2\sigma\cos(k(i-j)\theta) & 0 \\
sr^2{p_1}^i{p_2}^j & 2\sigma\cos(kj\theta) & 0 & 2\sigma(-1)^i\cos(\ell j\theta) \\
sr^3{p_1}^i{p_2}^j & 0 & 2\sigma\cos(k(i+j)\theta) & 0 \\
\hline
\end{array}
\tag{3.24}
$$

where $\theta = 2\pi/n$ and $i, j = 0, 1, \dots, n-1$.

3.1.5. Eight-Dimensional Irreducible Representations

The group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ with $n \geq 5$ has eight-dimensional irreducible representations. We can designate them by $(8; k, \ell)$ with

$$1 \leq \ell \leq k-1, \quad 2 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{3.25}$$

The number of eight-dimensional irreducible representations is given by

$$N_8 = \begin{cases} (n^2 - 6n + 8)/8 & (n = 2m), \\ (n^2 - 4n + 3)/8 & (n = 2m-1). \end{cases} \tag{3.26}$$

The irreducible representation $(8; k, \ell)$ is defined as

$$T^{(8;k,\ell)}(r) = \left[\begin{array}{cc|cc} & S & & \\ I & & & \\ \hline & & & I \\ & & S & \end{array}\right], \quad T^{(8;k,\ell)}(s) = \left[\begin{array}{cc|cc} & & I & \\ & & & I \\ \hline I & & & \\ & I & & \end{array}\right], \tag{3.27}$$

$$T^{(8;k,\ell)}(p_1) = \left[\begin{array}{cc|cc} R^k & & & \\ & R^{-\ell} & & \\ \hline & & R^k & \\ & & & R^{-\ell} \end{array}\right], \quad T^{(8;k,\ell)}(p_2) = \left[\begin{array}{cc|cc} R^\ell & & & \\ & R^k & & \\ \hline & & R^{-\ell} & \\ & & & R^{-k} \end{array}\right] \tag{3.28}$$

with

$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}. \tag{3.29}$$

The characters $\chi^{(8;k,\ell)}(g) = \operatorname{Tr} T^{(8;k,\ell)}(g)$ are given as follows. For $g = {p_1}^i{p_2}^j$, being free from r and s, we have

$$\chi^{(8;k,\ell)}({p_1}^i{p_2}^j) = 2[\cos((ki + \ell j)\theta) + \cos((-\ell i + kj)\theta) + \cos((ki - \ell j)\theta) + \cos((-\ell i - kj)\theta)], \tag{3.30}$$

where $\theta = 2\pi/n$ and $i, j = 0, 1, \dots, n-1$. For other g, we have $\chi^{(8;k,\ell)}(g) = 0$.

3.2. Derivation of Irreducible Representations

The group $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ for the $n \times n$ square lattice is a semidirect product of $H = \langle r, s\rangle = \mathrm{D}_4$ by an abelian group $A = \langle p_1, p_2\rangle = \mathbb{Z}_n \times \mathbb{Z}_n$. The irreducible representations over $\mathbb{C}$ of such a group can be constructed in a systematic manner by the method of little groups of Wigner and Mackey. It turns out that the irreducible representations over $\mathbb{C}$ are one-, two-, four-, or eight-dimensional, and they are representable over $\mathbb{R}$.

3.2.1. Method of Little Groups

A systematic method, called the *method of little groups*, for constructing irreducible representations of a general group with the structure of semidirect product by an abelian group is described in this subsection. For more details, see Section 8.2 of Serre, 1977 [21].

Let G be a group that is a semidirect product of a group H and an abelian group A. This means that A is a normal subgroup of G, and each element $g \in G$ is represented uniquely as $g = ah$ with $a \in A$ and $h \in H$.

Since A is abelian, every irreducible representation of A over $\mathbb{C}$ is one-dimensional, and is identified with its character χ. Accordingly, the set of all irreducible representations of A over $\mathbb{C}$ can be denoted as

$$X = \{\chi^i \mid i \in R(A)\} \tag{3.31}$$

with a suitable index set $R(A)$. For $\chi \in X$ and $g \in G$, we define a function ${}^g\chi$ on A by

$${}^g\chi(a) = \chi(g^{-1}ag), \quad a \in A, \tag{3.32}$$

which is also a character of A, belonging to X. This defines an action of G on X.

With reference to the action of G on X, we classify the elements of X into orbits. It should be noted that, for $g = bh$ with $b \in A$ and $h \in H$, we have

$${}^g\chi(a) = \chi((bh)^{-1}a(bh)) = \chi(h^{-1}ah) = {}^h\chi(a), \quad a \in A,$$

in which $b^{-1}ab = a$ since A is abelian. Hence, the orbits can in fact be obtained by the action of the subgroup H on X, instead of that of G. Denote by

$$\{\chi^i \mid i \in R(A)/H\} \tag{3.33}$$

a system of representatives from the orbits, where $R(A)/H$ is an index set, or the set of "names" of the orbit. This means that

- $\chi^i \in X$ for each $i \in R(A)/H$,
- for distinct i and j in $R(A)/H$, $\chi^i \neq {}^h(\chi^j)$ for any $h \in H$, and
- for each $\chi \in X$, there exist some $i \in R(A)/H$ and $h \in H$ such that $\chi = {}^h(\chi^i)$.

For each $i \in R(A)/H$, we define

$$H^i = \{h \in H \mid {}^h(\chi^i) = \chi^i\}, \tag{3.34}$$

which is a subgroup of H associated with the orbit i, and

$$G^i = \{ah \mid a \in A,\ h \in H^i\}, \tag{3.35}$$

which is a subgroup of G, called the *little group*. Noting that each element of G^i can be represented as ah with $a \in A$ and $h \in H^i$, we define a function $\tilde{\chi}^i$ on G^i by

$$\tilde{\chi}^i(ah) = \chi^i(a), \quad a \in A,\ h \in H^i, \tag{3.36}$$

which is a one-dimensional representation (a character of degree one) of G^i.

Let T^μ be an irreducible representation of H^i over $\mathbb{C}$ indexed by $\mu \in R(H^i)$. Then the matrix-valued function $T^{(i,\mu)}$ defined on G^i of (3.35) by

$$T^{(i,\mu)}(ah) = \chi^i(a)T^\mu(h), \quad a \in A,\ h \in H^i \tag{3.37}$$

is an irreducible representation of G^i. Denote by $\tilde{T}^{(i,\mu)}$ the induced representation of G obtained from $T^{(i,\mu)}$ (see Remark 3.1 below). Then $\tilde{T}^{(i,\mu)}$ is an irreducible representation of G. Moreover, all the irreducible representations of G can be obtained in this manner, and $\tilde{T}^{(i,\mu)}$'s are mutually inequivalent for different (i,μ). Thus, the irreducible representations of G are indexed by (i,μ), i.e.,

$$R(G) = \{(i,\mu) \mid i \in R(A)/H,\ \mu \in R(H^i)\} \tag{3.38}$$

and

$$\{\tilde{T}^{(i,\mu)} \mid i \in R(A)/H,\ \mu \in R(H^i)\} \tag{3.39}$$

gives a complete list of irreducible representations of G over $\mathbb{C}$.

Remark 3.1. The induced representation is explained here. Let G be a group, G' be a subgroup of G, and T' be a representation of G' of dimension N'. Consider the *coset decomposition*

$$G = g_1G' + g_2G' + \cdots + g_mG', \tag{3.40}$$

where $j = 1,\dots,m$ and $m = |G|/|G'|$. Each $g \in G$ causes a permutation of $(g_1, g_2, \dots, g_m)$ to $(g_{\pi(1)}, g_{\pi(2)}, \dots, g_{\pi(m)})$ according to the equation

$$gg_j = g_{\pi(j)}f_j, \quad f_j \in G' \tag{3.41}$$

for $j = 1,\dots,m$. Note that the choice of $(g_1, g_2, \dots, g_m)$ is not unique, but once this is fixed, f_j is uniquely determined for each g.

Define $\tilde{T}(g)$ to be an $mN' \times mN'$ matrix with rows and columns partitioned into m blocks of size N' such that the $(\pi(j), j)$-block of $\tilde{T}(g)$ equals $T'(f_j)$, whereas the (i, j)-block of $\tilde{T}(g)$ equals O if $i \neq \pi(j)$. Note that this is well-defined, since f_j and $\pi(j)$ are uniquely determined from g, and $T'(f_j)$ for $j = 1,\dots,m$ are assumed to be given. The family of matrices $\{\tilde{T}(g) \mid g \in G\}$ is a representation of G of dimension mN', called the *induced representation*. For example, if $m = 3$, $(\pi(1), \pi(2), \pi(3)) = (2, 3, 1)$, we have

$$\tilde{T}(g) = \begin{bmatrix} & & T'(f_3) \\ T'(f_1) & & \\ & T'(f_2) & \end{bmatrix}.$$

We shall apply this construction to $T' = T^{(i,\mu)}$ on $G' = G^i$ to obtain $\tilde{T} = \tilde{T}^{(i,\mu)}$, where the dimension N' of $T^{(i,\mu)}$ is equal to that of T^μ by (3.37). □

3.2.2. Orbit Decomposition and Little Groups

We apply the method of little groups in Section 3.2.1 to

$$A = \mathbb{Z}_n \times \mathbb{Z}_n = \langle p_1 \rangle \times \langle p_2 \rangle, \quad H = \mathrm{D}_4 = \langle r, s \rangle$$

to obtain the irreducible representations of $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

As the first step we determine the system of representatives (3.33) in the orbit decomposition of X. Since $A = \mathbb{Z}_n \times \mathbb{Z}_n$ is an abelian group, all the irreducible representations are one-dimensional. The set X of irreducible representations of $A = \mathbb{Z}_n \times \mathbb{Z}_n$ is indexed by

$$R(A) = \{(k, \ell) \mid 0 \le k \le n-1,\ 0 \le \ell \le n-1\}, \tag{3.42}$$

where (k, ℓ) denotes a one-dimensional representation (or character) $\chi^{(k,\ell)}$ defined by

$$\chi^{(k,\ell)}(p_1) = \omega^k, \quad \chi^{(k,\ell)}(p_2) = \omega^\ell \tag{3.43}$$

with

$$\omega = \exp(2\pi \mathrm{i}/n). \tag{3.44}$$

We extend the notation (k, ℓ) for any integers, to designate the element (k', ℓ') of $R(A)$ with $k' \equiv k \mod n$ and $\ell' \equiv \ell \mod n$.

For the orbit decomposition of X by H, we compute $h^{-1}p_1h$ and $h^{-1}p_2h$ for $h \in H$, to obtain

h	e	r	r^2	r^3	s	sr	sr^2	sr^3
$h^{-1}p_1h$	p_1	p_2^{-1}	p_1^{-1}	p_2	p_1	p_2^{-1}	p_1^{-1}	p_2
$h^{-1}p_2h$	p_2	p_1	p_2^{-1}	p_1^{-1}	p_2^{-1}	p_1^{-1}	p_2	p_1

(3.45)

For example, for $h = s$, we have $(h^{-1}p_1h, h^{-1}p_2h) = (p_1, p_2^{-1})$, and we see, by (3.43), that the action of s in (3.32) is given as ${}^s\chi^{(k,\ell)} = \chi^{(k,-\ell)}$, which is expressed symbolically as $(k, \ell) \Rightarrow (k, -\ell)$. In this manner, we can obtain the following orbit containing (k, ℓ):

$$\boxed{\begin{array}{ccc} (\ell, -k) & \leftarrow & (-k, -\ell) \\ \downarrow & & \uparrow \\ (k, \ell) & \rightarrow & (-\ell, k) \\ \Downarrow & & \\ (k, -\ell) & \rightarrow & (-\ell, -k) \\ \uparrow & & \downarrow \\ (\ell, k) & \leftarrow & (-k, \ell) \end{array}} \tag{3.46}$$

where "$\Downarrow$" means the action of s, and "$\rightarrow$" (or "$\leftarrow$", "$\uparrow$", "$\downarrow$") means the action of r. It should be clear that $(\ell, -k)$, for example, is understood as $(\ell \bmod n, -k \bmod n)$. The orbit (3.46) is illustrated in Fig. 3.1.

The system of representatives in (3.33) in the orbit decomposition of X with respect to the action of G is given as follows. In view of Fig. 3.1, it is natural to take

$$R(A)/H = \{(k, \ell) \mid 0 \le \ell \le k \le \lfloor (n-1)/2 \rfloor\}, \tag{3.47}$$

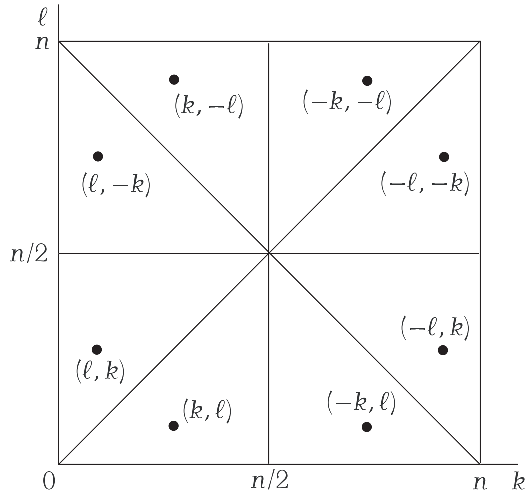


Figure 3.1: Orbit of (k, ℓ) in (3.46)

which corresponds to the set of integer lattice points (k, ℓ) contained in the triangle with vertices at $(k, \ell) = (0, 0), (n/2, 0), (n/2, n/2)$, where the points on the edges of the triangle are included.

The subgroup $H^i = H^{(k,\ell)}$ in (3.34) for $i = (k, \ell)$, which is expressed as

$$\begin{aligned} H^{(k,\ell)} &= \{h \in \mathrm{D}_4 \mid {}^h(\chi^{(k,\ell)}) = \chi^{(k,\ell)}\} \\ &= \{h \in \mathrm{D}_4 \mid \chi^{(k,\ell)}(h^{-1}ah) = \chi^{(k,\ell)}(a) \text{ for all } a \in \mathbb{Z}_n \times \mathbb{Z}_n\}, \end{aligned}$$

is obtained with reference to (3.43) and (3.45). For $h \in \mathrm{D}_4$, we have $h \in H^{(k,\ell)}$ if and only if

$$\chi^{(k,\ell)}(hp_1h^{-1}) = \chi^{(k,\ell)}(p_1), \quad \chi^{(k,\ell)}(hp_2h^{-1}) = \chi^{(k,\ell)}(p_2).$$

For $(k, \ell) = (0, 0)$, for example, this condition is satisfied by all $h \in \mathrm{D}_4$, and hence $H^{(0,0)} = \langle r, s\rangle$. In this manner, we obtain

$$H^{(k,\ell)} = \begin{cases} \langle r, s\rangle & \text{for } (k,\ell) = (0,0), \\ \langle r, s\rangle & \text{for } (k,\ell) = (n/2, n/2) \quad \text{if } n \text{ is even}, \\ \langle r^2, s\rangle & \text{for } (k,\ell) = (n/2, 0) \quad \text{if } n \text{ is even}, \\ \{e, s\} & \text{for } (k,\ell) = (k, 0) \quad (1 \le k \le \lfloor \frac{n-1}{2} \rfloor), \\ \{e, sr^3\} & \text{for } (k,\ell) = (k, k) \quad (1 \le k \le \lfloor \frac{n-1}{2} \rfloor), \\ \{e, sr^2\} & \text{for } (k,\ell) = (n/2, \ell) \quad (1 \le \ell \le \lfloor \frac{n-1}{2} \rfloor) \quad \text{if } n \text{ is even}, \\ \{e\} & \text{for } (k,\ell) \quad (1 \le \ell \le k-1, 2 \le k \le \lfloor \frac{n-1}{2} \rfloor). \end{cases} \tag{3.48}$$

The little group $G^i = G^{(k,\ell)}$ in (3.35) for $i = (k, \ell)$ is obtained as the semidirect product of $H^{(k,\ell)}$ by $A = \langle p_1, p_2\rangle$.

Example 3.1. For $n = 3, 4, 7, 8, 9$, the system of representatives $R(A)/H$ and the associated subgroups $H^{(k,\ell)}$ in (3.48) are given as follows:

$n=3$	
(k,ℓ)	$H^{(k,\ell)}$
$(0,0)$	$\langle r, s\rangle$
$(1,0)$	$\{e, s\}$
$(1,1)$	$\{e, sr^3\}$

$n=4$	
(k,ℓ)	$H^{(k,\ell)}$
$(0,0)$	$\langle r, s\rangle$
$(2,2)$	$\langle r, s\rangle$
$(2,0)$	$\langle r^2, s\rangle$
$(1,0)$	$\{e, s\}$
$(1,1)$	$\{e, sr^3\}$
$(2,1)$	$\{e, sr^2\}$

$n=7$	
(k,ℓ)	$H^{(k,\ell)}$
$(0,0)$	$\langle r, s\rangle$
$(1,0), (2,0), (3,0)$	$\{e, s\}$
$(1,1), (2,2), (3,3)$	$\{e, sr^3\}$
$(2,1), (3,1), (3,2)$	$\{e\}$

$n=8$	
(k,ℓ)	$H^{(k,\ell)}$
$(0,0)$	$\langle r, s\rangle$
$(4,4)$	$\langle r, s\rangle$
$(4,0)$	$\langle r^2, s\rangle$
$(1,0), (2,0), (3,0)$	$\{e, s\}$
$(1,1), (2,2), (3,3)$	$\{e, sr^3\}$
$(4,1), (4,2), (4,3)$	$\{e, sr^2\}$
$(2,1), (3,1), (3,2)$	$\{e\}$

$n=9$	
(k,ℓ)	$H^{(k,\ell)}$
$(0,0)$	$\langle r, s\rangle$
$(1,0), (2,0), (3,0), (4,0)$	$\{e, s\}$
$(1,1), (2,2), (3,3), (4,4)$	$\{e, sr^3\}$
$(2,1), (3,1), (3,2)$	$\{e\}$
$(4,1), (4,2), (4,3)$	$\{e\}$

□

3.2.3. Induced Irreducible Representations

The procedure for constructing irreducible representations of $G = \langle r, s, p_1, p_2\rangle$ using the orbit decomposition and little groups in Section 3.2.2 is as follows.

For each $(k,\ell) \in R(A)/H$, we have the associated subgroup $H^{(k,\ell)}$ in (3.48). Let T^{μ} be an irreducible representation of $H^{(k,\ell)}$ indexed by $\mu \in R(H^{(k,\ell)})$, and define $T^{(k,\ell,\mu)}$ by

$$T^{(k,\ell,\mu)}({p_1}^i {p_2}^j h) = \chi^{(k,\ell)}({p_1}^i {p_2}^j) T^{\mu}(h) = \omega^{ki+\ell j} T^{\mu}(h), \quad 0 \le i, j \le n-1, \ h \in H^{(k,\ell)}, \tag{3.49}$$

which is an irreducible representation of the little group $G^{(k,\ell)}$.

The coset decomposition (3.40) takes the form of

$$G = g_1 G^{(k,\ell)} + g_2 G^{(k,\ell)} + \cdots + g_m G^{(k,\ell)} \tag{3.50}$$

with $m = |G|/|G^{(k,\ell)}| = |\mathrm{D}_4|/|H^{(k,\ell)}| = 8/|H^{(k,\ell)}|$. Since $G^{(k,\ell)} \supseteq \langle p_1, p_2\rangle$, we may assume that $g_j \in \langle r, s\rangle$ for $j = 1, \ldots, m$ and $g_1 = e$.

The induced representation $\tilde{T}^{(k,\ell,\mu)}(g)$ is determined by its values at $g = p_1, p_2, r, s$ that generate the group G. Hence, it suffices to consider $g = p_1, p_2, r, s$ in the equation (3.41):

$$g g_j = g_{\pi(j)} f_j, \tag{3.51}$$

where $\pi(j)$ and $f_j \in G^{(k,\ell)}$ are to be found for $j = 1, \ldots, m$. The induced representation $\tilde{T}^{(k,\ell,\mu)}$ is an irreducible representation of dimension $mN^{\mu} = 8N^{\mu}/|H^{(k,\ell)}|$ over $\mathbb{C}$, where N^{μ} denotes the dimension of T^{μ}.

Table 3.3: Induced irreducible representations of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$

(k,ℓ)	$H^{(k,\ell)}$	m	Induced irreducible representations
$(0,0)$	$\langle r, s\rangle$	1	$(1;+,+,+),(1;+,-,+),(1;-,+,+),(1;-,-,+),(2;+)$
$(n/2,n/2)$	$\langle r, s\rangle$	1	$(1;+,+,-),(1;+,-,-),(1;-,+,-),(1;-,-,-),(2;-)$
$(n/2,0)$	$\langle r^2, s\rangle$	2	$(2;+,+),(2;+,-),(2;-,+),(2;-,-)$
$(k,0)$	$\{e,s\}$	4	$(4;k,0,+),(4;k,0,-)$
(k,k)	$\{e,sr^3\}$	4	$(4;k,k,+),(4;k,k,-)$
$(n/2,\ell)$	$\{e,sr^2\}$	4	$(4;n/2,\ell,+),(4;n/2,\ell,-)$
(k,ℓ)	$\{e\}$	8	$(8;k,\ell)$

$(k,\ell) = (n/2,n/2)$ and $(n/2,0)$ exist if n is even;
$(k,0)$ for $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ in (3.13);
(k,k) for $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ in (3.14);
$(n/2,\ell)$ for $1 \leq \ell \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ in (3.15);
(k,ℓ) for $1 \leq \ell \leq k-1$, $2 \leq k \leq \lfloor (n-1)/2 \rfloor$ in (3.25)

According to the general theory, $\tilde{T}^{(k,\ell,\mu)}$ obtained in this manner is not a representation over $\mathbb{R}$ but over $\mathbb{C}$, as is evident from the fact that ω appearing in (3.49) is a complex number defined by (3.44). Fortunately, however, all irreducible representations thus obtained are representable over $\mathbb{R}$. We can thus determine a complete list of irreducible representations over $\mathbb{R}$ of the group $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. Table 3.3 is a summary of the derivations below.

Case of $(k,\ell) = (0,0)$

For $(k,\ell) = (0,0)$, $\chi^{(k,\ell)}$ is the unit representation by (3.43), and therefore

$$H^{(k,\ell)} = \langle r, s\rangle = \mathrm{D}_4,$$

as is shown in (3.48). D_4 has four one-dimensional irreducible representations $\mu = (+,+,+)$, $(+,-,+)$, $(-,+,+)$, and $(-,-,+)$, and one two-dimensional irreducible representation $\mu = (2;+)$ (e.g., see Kim, 1999 [22] and Kettle, 2008 [23]).

Since $G^{(k,\ell)} = G$, the coset decomposition (3.50) is trivial with $m = 1$ and $g_1 = e$, and the equation (3.51) reads $g \cdot g_1 = g_1 \cdot g$ for every $g \in G$. For each μ, the induced representation $\tilde{T}^{(0,0,\mu)}(g)$ for $g = {p_1}^i {p_2}^j h$ with $h \in \mathrm{D}_4$ is given by (3.49) as

$$\tilde{T}^{(0,0,\mu)}(g) = \tilde{T}^{(0,0,\mu)}({p_1}^i {p_2}^j h) = \chi^{(0,0)}({p_1}^i {p_2}^j) T^{\mu}(h) = T^{\mu}(h).$$

With this result, we have the one-dimensional irreducible representations $(1;+,+,+)$, $(1;+,-,+)$, $(1;-,+,+)$, $(1;-,-,+)$ in Section 3.1.2, and the two-dimensional irreducible representation $(2;+)$ in Section 3.1.3 as the irreducible representations for the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

Case of $(k,\ell) = (n/2,n/2)$

In this case, $\chi = \chi^{(k,\ell)} = \chi^{(n/2,n/2)}$ is given by (3.43) as $\chi(p_1) = \chi(p_2) = \omega^{n/2} = -1$. For $(k,\ell) = (n/2,n/2)$, we have

$$H^{(k,\ell)} = \langle r, s\rangle = \mathrm{D}_4,$$

as is shown in (3.48). Hence we have the one-dimensional irreducible representations $(1;+,+,-)$, $(1;+,-,-)$, $(1;-,+,-)$, $(1;-,-,-)$ in Section 3.1.2, and the two-dimensional irreducible representation $(2;-)$ in Section 3.1.3.

Case of $(k,\ell) = (n/2, 0)$

The case of $(k,\ell) = (n/2,0)$ occurs when n is even. In this case, $\chi = \chi^{(k,\ell)} = \chi^{(n/2,0)}$ is given by (3.43) as $\chi(p_1) = -1$ and $\chi(p_2) = 1$, and therefore

$$H^{(k,\ell)} = \{e, r^2, s, sr^2\} = \langle r^2, s\rangle \simeq \mathrm{D}_2,$$

as is shown in (3.48). This group has four one-dimensional irreducible representations, say, $\mu = (\sigma_r, \sigma_s) = (+,+), (+,-), (-,+), (-,-)$ defined by

$$T^\mu(r^2) = \sigma_r = \pm 1, \quad T^\mu(s) = \sigma_s = \pm 1.$$

Since $G^{(k,\ell)} = \langle r^2, s, p_1, p_2\rangle$, the coset decomposition in (3.50) is given by

$$G = g_1 G^{(k,\ell)} + g_2 G^{(k,\ell)} = e \cdot \langle r^2, s, p_1, p_2\rangle + r \cdot \langle r^2, s, p_1, p_2\rangle$$

with $m = 2$, $g_1 = e$ and $g_2 = r$. The equation (3.51) for $g = p_1, p_2, r, s$ reads as follows:

$p_1 \cdot g_j = g_{\pi(j)} \cdot f_j$	$p_2 \cdot g_j = g_{\pi(j)} \cdot f_j$	$r \cdot g_j = g_{\pi(j)} \cdot f_j$	$s \cdot g_j = g_{\pi(j)} \cdot f_j$
$p_1 \cdot e = e \cdot p_1$	$p_2 \cdot e = e \cdot p_2$	$r \cdot e = r \cdot e$	$s \cdot e = e \cdot s$
$p_1 \cdot r = r \cdot p_2^{-1}$	$p_2 \cdot r = r \cdot p_1$	$r \cdot r = e \cdot r^2$	$s \cdot r = r \cdot sr^2$

For the one-dimensional representation $\mu = (\sigma)$ with $\sigma \in \{+,-\}$, the induced representation $\tilde{T} = \tilde{T}^{(n/2,0,\mu)}$ is given by

$$\tilde{T}(p_1) = \begin{bmatrix} \chi(p_1)T^\mu(e) & \\ & \chi(p_2^{-1})T^\mu(e) \end{bmatrix} = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix},$$
$$\tilde{T}(p_2) = \begin{bmatrix} \chi(p_2)T^\mu(e) & \\ & \chi(p_1)T^\mu(e) \end{bmatrix} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix},$$
$$\tilde{T}(r) = \begin{bmatrix} & \chi(e)T^\mu(r^2) \\ \chi(e)T^\mu(e) & \end{bmatrix} = \begin{bmatrix} & \sigma_r \\ 1 & \end{bmatrix},$$
$$\tilde{T}(s) = \begin{bmatrix} \chi(e)T^\mu(s) & \\ & \chi(e)T^\mu(sr^2) \end{bmatrix} = \sigma_s \begin{bmatrix} 1 & \\ & \sigma_r \end{bmatrix},$$

where (3.49) is used and the nonzero blocks here are determined with reference to $\pi(j)$ and f_j computed above (Remark 3.1 in Section 3.2.1).

Case of $(k,\ell) = (k,0)$*,* (k,k)*, or* $(n/2,\ell)$

For $(k,\ell) = (k,0)$ in (3.13), we have $\chi^{(k,\ell)}(p_1) = \omega^k$ and $\chi^{(k,\ell)}(p_2) = 1$ by (3.43), and therefore

$$H^{(k,\ell)} = \{e, s\},$$

as is shown in (3.48). For $(k,\ell)=(k,k)$ in (3.14), we have $\chi^{(k,\ell)}(p_1)=\chi^{(k,\ell)}(p_2)=\omega^k$, and therefore

$$H^{(k,\ell)}=\{e, sr^3\}.$$

For $(k,\ell)=(n/2,\ell)$ in (3.15), we have $\chi^{(k,\ell)}(p_1)=-1$ and $\chi^{(k,\ell)}(p_2)=\omega^\ell$, and therefore

$$H^{(k,\ell)}=\{e, sr^2\}.$$

Let $h_0=s$ for $(k,\ell)=(k,0)$, $h_0=sr^3$ for $(k,\ell)=(k,k)$, and $h_0=sr^2$ for $(k,\ell)=(n/2,\ell)$. In either case $H^{(k,\ell)}=\{e,h_0\}$ is isomorphic to D_1 and has two one-dimensional irreducible representations, say, $\mu=\mu_1,\mu_2$ defined by

$$T^{\mu_1}(h_0)=1, \quad T^{\mu_2}(h_0)=-1.$$

That is, $T^\mu(h_0)=\sigma^\mu$ with $\sigma^{\mu_1}=1$ and $\sigma^{\mu_2}=-1$. The notation is summarized as follows:

(k,ℓ)	$H^{(k,\ell)}$	h_0	$T^{\mu_1}(h_0)$	$T^{\mu_2}(h_0)$
$(k,0)$	$\{e,s\}$	sr	1	-1
(k,k)	$\{e,sr^3\}$	sr^3	1	-1
$(n/2,\ell)$	$\{e,sr^2\}$	sr^2	1	-1

The coset decomposition in (3.50) is given by $G^{(k,\ell)}=\langle h_0,p_1,p_2\rangle$, $m=4$, and $g_j=r^{j-1}$ for $j=1,\ldots,4$. The equation (3.51) for $g=p_1,p_2,r,s$ reads as follows (see (3.45) for p_1 and p_2):

$p_1\cdot g_j=g_{\pi(j)}\cdot f_j$	$p_2\cdot g_j=g_{\pi(j)}\cdot f_j$	$r\cdot g_j=g_{\pi(j)}\cdot f_j$
$p_1\cdot e=e\cdot p_1$	$p_2\cdot e=e\cdot p_2$	$r\cdot e=r\cdot e$
$p_1\cdot r=r\cdot p_2^{-1}$	$p_2\cdot r=r\cdot p_1$	$r\cdot r=r^2\cdot e$
$p_1\cdot r^2=r^2\cdot p_1^{-1}$	$p_2\cdot r^2=r^2\cdot p_2^{-1}$	$r\cdot r^2=r^3\cdot e$
$p_1\cdot r^3=r^3\cdot p_2$	$p_2\cdot r^3=r^3\cdot p_1$	$r\cdot r^3=e\cdot e$

$s\cdot g_j=g_{\pi(j)}\cdot f_j$		
$(k,0)$	(k,k)	$(n/2,\ell)$
$s\cdot e=e\cdot s$	$s\cdot e=r^3\cdot sr^3$	$s\cdot e=r^2\cdot sr^2$
$s\cdot r=r^3\cdot s$	$s\cdot r=r^2\cdot sr^3$	$s\cdot r=r\cdot sr^2$
$s\cdot r^2=r^2\cdot s$	$s\cdot r^2=r\cdot sr^3$	$s\cdot r^2=e\cdot sr^2$
$s\cdot r^3=r\cdot s$	$s\cdot r^3=e\cdot sr^3$	$s\cdot r^3=r^3\cdot sr^2$

For $(k,\ell)=(k,0),(k,k),(n/2,\ell)$ and $\mu=\mu_1,\mu_2$, the induced representation $\tilde{T}^{(k,\ell,\mu)}$ is given, with $\omega=\exp(2\pi\mathrm{i}/n)$, by

$$\tilde{T}^{(k,\ell,\mu)}(p_1)=\mathrm{diag}(\chi(p_1),\ \chi(p_2^{-1}),\ \chi(p_1^{-1}),\ \chi(p_2))=\mathrm{diag}(\omega^k,\ \omega^{-\ell},\ \omega^{-k},\ \omega^\ell),$$
$$\tilde{T}^{(k,\ell,\mu)}(p_2)=\mathrm{diag}(\chi(p_2),\ \chi(p_1),\ \chi(p_2^{-1}),\ \chi(p_1^{-1}))=\mathrm{diag}(\omega^\ell,\ \omega^k,\ \omega^{-\ell},\ \omega^{-k}),$$

$$\tilde{T}^{(k,\ell,\mu)}(r) = T^{\mu}(e)\begin{bmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix} = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix},$$

and

$$\tilde{T}^{(k,0,\mu)}(s) = T^{\mu}(s)\begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{bmatrix} = \sigma^{\mu}\begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{bmatrix},$$

$$\tilde{T}^{(k,k,\mu)}(s) = T^{\mu}(sr^3)\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} = \sigma^{\mu}\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

$$\tilde{T}^{(n/2,\ell,\mu)}(s) = T^{\mu}(sr^2)\begin{bmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{bmatrix} = \sigma^{\mu}\begin{bmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{bmatrix}.$$

The above representation over $\mathbb{C}$ can be transformed to a real representation. By permuting the rows and columns as $(1,3,2,4)$, we obtain

$$\hat{T}^{(k,\ell,\mu)}(p_1) = \left[\begin{array}{cc|cc} \omega^{k} & & & \\ & \omega^{-k} & & \\ \hline & & \omega^{-\ell} & \\ & & & \omega^{\ell} \end{array}\right], \quad \hat{T}^{(k,\ell,\mu)}(p_2) = \left[\begin{array}{cc|cc} \omega^{\ell} & & & \\ & \omega^{-\ell} & & \\ \hline & & \omega^{k} & \\ & & & \omega^{-k} \end{array}\right], \quad \hat{T}^{(k,\ell,\mu)}(r) = \left[\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline 1 & & & \\ & 1 & & \end{array}\right],$$

$$\hat{T}^{(k,0,\mu)}(s) = \sigma^{\mu}\left[\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right], \quad \hat{T}^{(k,k,\mu)}(s) = \sigma^{\mu}\left[\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right], \quad \hat{T}^{(n/2,\ell,\mu)}(s) = \sigma^{\mu}\left[\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right].$$

It is apparent that these representations are equivalent, respectively, to the four-dimensional real irreducible representations $(4; k, 0, \sigma)$ and $(4; k, k, \sigma)$ in Section 3.1.4 with $\sigma = \sigma^{\mu}$.

Case of General (k, ℓ)

For (k,ℓ) in (3.25), $\chi = \chi^{(k,\ell)}$ is given by (3.43), and $H^{(k,\ell)} = \{e\}$. The unit representation μ is the only irreducible representation of $H^{(k,\ell)}$.

The coset decomposition in (3.50) is given by $G^{(k,\ell)} = \langle p_1, p_2 \rangle$, $m = 8$, and

$$g_1 = e, \quad g_2 = r, \quad g_3 = r^2, \quad g_4 = r^3, \quad g_5 = s, \quad g_6 = sr, \quad g_7 = sr^2, \quad g_8 = sr^3.$$

The equation (3.51) for $g = p_1, p_2, r, s$ reads as follows:

$p_1 \cdot g_j = g_{\pi(j)} \cdot f_j$	$p_2 \cdot g_j = g_{\pi(j)} \cdot f_j$	$r \cdot g_j = g_{\pi(j)} \cdot f_j$	$s \cdot g_j = g_{\pi(j)} \cdot f_j$
$p_1 \cdot e = e \cdot p_1$	$p_2 \cdot e = e \cdot p_2$	$r \cdot e = r \cdot e$	$s \cdot e = s \cdot e$
$p_1 \cdot r = r \cdot p_2^{-1}$	$p_2 \cdot r = r \cdot p_1$	$r \cdot r = r^2 \cdot e$	$s \cdot r = sr \cdot e$
$p_1 \cdot r^2 = r^2 \cdot p_1^{-1}$	$p_2 \cdot r^2 = r^2 \cdot p_2^{-1}$	$r \cdot r^2 = r^3 \cdot e$	$s \cdot r^2 = sr^2 \cdot e$
$p_1 \cdot r^3 = r^3 \cdot p_2$	$p_2 \cdot r^3 = r^3 \cdot p_1^{-1}$	$r \cdot r^3 = e \cdot e$	$s \cdot r^3 = sr^3 \cdot e$
$p_1 \cdot s = s \cdot p_1$	$p_2 \cdot s = s \cdot p_2^{-1}$	$r \cdot s = sr^3 \cdot e$	$s \cdot s = e \cdot e$
$p_1 \cdot sr = sr \cdot p_2^{-1}$	$p_2 \cdot sr = sr \cdot p_1^{-1}$	$r \cdot sr = s \cdot e$	$s \cdot sr = r \cdot e$
$p_1 \cdot sr^2 = sr^2 \cdot p_1^{-1}$	$p_2 \cdot sr^2 = sr^2 \cdot p_2$	$r \cdot sr^2 = sr \cdot e$	$s \cdot sr^2 = r^2 \cdot e$
$p_1 \cdot sr^3 = sr^3 \cdot p_2$	$p_2 \cdot sr^3 = sr^3 \cdot p_1$	$r \cdot sr^3 = sr^2 \cdot e$	$s \cdot sr^3 = r^3 \cdot e$

The induced representation $\tilde{T} = \tilde{T}^{(k,\ell,\mu)}$, of dimension 8, is given in terms of $\omega = \exp(2\pi \mathrm{i}/n)$ as follows:

$$\begin{aligned}
\tilde{T}(p_1) &= \mathrm{diag}(\omega^k, \omega^{-\ell}, \omega^{-k}, \omega^{\ell}, \omega^k, \omega^{-\ell}, \omega^{-k}, \omega^{\ell}), \\
\tilde{T}(p_2) &= \mathrm{diag}(\omega^{\ell}, \omega^k, \omega^{-\ell}, \omega^{-k}, \omega^{-\ell}, \omega^{-k}, \omega^{\ell}, \omega^k), \\
\tilde{T}(r) &= \begin{bmatrix} C & O \\ O & C^\top \end{bmatrix}, \quad \tilde{T}(s) = \begin{bmatrix} O & I \\ I & O \end{bmatrix}
\end{aligned}$$

with

$$C = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix}, \quad I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

The above representation over $\mathbb{C}$ can be transformed to a real representation. By permuting the rows and columns as $(1,3,2,4,5,7,6,8)$, we obtain

$$\hat{T}(p_1) = \begin{bmatrix} \Omega_1 & \\ & \Omega_1 \end{bmatrix}, \quad \hat{T}(p_2) = \begin{bmatrix} \Omega_2 & \\ & \Omega_3 \end{bmatrix}, \quad \hat{T}(r) = \begin{bmatrix} D & \\ & D^\top \end{bmatrix}, \quad \hat{T}(s) = \begin{bmatrix} & I \\ I & \end{bmatrix},$$

where

$$\Omega_1 = \left[\begin{array}{cc|cc} \omega^k & & & \\ & \omega^{-k} & & \\ \hline & & \omega^{-\ell} & \\ & & & \omega^{\ell} \end{array}\right], \quad \Omega_2 = \left[\begin{array}{cc|cc} \omega^{\ell} & & & \\ & \omega^{-\ell} & & \\ \hline & & \omega^k & \\ & & & \omega^{-k} \end{array}\right],$$

$$\Omega_3 = \left[\begin{array}{cc|cc} \omega^{-\ell} & & & \\ & \omega^{\ell} & & \\ \hline & & \omega^{-k} & \\ & & & \omega^{k} \end{array}\right], \quad D = \left[\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline 1 & & & \\ & 1 & & \end{array}\right].$$

This representation is easily seen to be equivalent to the eight-dimensional real irreducible representation $(8; k, \ell)$ in Section 3.1.5.

4. Matrix Representation for Square Lattice

In preparation for the group-theoretic analysis in Chapters 5 and 6, we found the irreducible representations of the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ in Chapter 3. Note that not all the irreducible representations are involved in mathematical models on the square lattice. The consideration of relevant irreducible representations is essential in a group-theoretic analysis that provides accurate information about bifurcating solutions.

In this chapter, we first identify the irreducible representations μ that are relevant to our analysis on the square lattice. For this purpose, we derive the explicit form of the permutation representation $T(g)$ of the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ and investigate the irreducible decomposition of this permutation representation. We can exclude irreducible representations that are not contained in $T(g)$ from consideration in search of square bifurcating patterns in Chapters 5 and 6. It turns out that the only some of the one-, two-, and four-dimensional ones are relevant, and all of the eight-dimensional ones are relevant.

We next present the transformation matrix Q for irreducible decomposition. Since the irreducible representations are multiplicity-free ($a^\mu = 1$ or 0), the orthogonal transformation of the Jacobian matrix J of $\boldsymbol{F}$ takes a diagonal form

$$Q^{-1} J Q = \mathrm{diag}(e_1, \dots, e_N).$$

This diagonal form is useful in the eigenanalysis of the computational bifurcation analysis on the square lattice.

This chapter is organized as follows. The permutation representation for the square lattice is investigated in Section 4.1. The irreducible decomposition of the permutation representation is presented in Section 4.2. Transformation matrices for block-diagonalization are derived in Section 4.3.

4.1. Representation Matrix

In our study of a system of $N = n^2$ places on the $n \times n$ square lattice, each element g of $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ acts as a permutation of place numbers $(1, \dots, N)$. Consequently, the representation matrix $T(g)$ is a permutation matrix for each g. By definition, $T(g)$ has "1" at the (i, j) entry if place j is moved to place i by the action of g.

The representation matrix $T(g)$ for general n can be determined as follows. The coordinate of a place on the $n \times n$ square lattice is given by

$$\boldsymbol{x} = n_1 \boldsymbol{\ell}_1 + n_2 \boldsymbol{\ell}_2, \quad n_1, n_2 = 0, 1, \dots, n-1$$

with $\boldsymbol{\ell}_1 = d(1,0)^\top$, $\boldsymbol{\ell}_2 = d(0,1)^\top$ in (2.1), where d means the length of these vectors. Thus, the n^2 places are indexed by (n_1, n_2), and so are the rows and columns of the representation matrix $T(g)$. The action of r is expressed as

$$r \cdot \boldsymbol{\ell}_1 = \boldsymbol{\ell}_2, \quad r \cdot \boldsymbol{\ell}_2 = -\boldsymbol{\ell}_1.$$

Hence, we have

$$r \cdot \boldsymbol{x} = n_1 (r \cdot \boldsymbol{\ell}_1) + n_2 (r \cdot \boldsymbol{\ell}_2) = n_1 (\boldsymbol{\ell}_2) + n_2 (-\boldsymbol{\ell}_1) = (-n_2) \boldsymbol{\ell}_1 + n_1 \boldsymbol{\ell}_2,$$

which means that the action of r on (n_1, n_2) is given by

$$r \cdot (n_1, n_2) \equiv (-n_2,\ n_1) \mod n. \tag{4.1}$$

Then, the column of $T(r)$ indexed by (n_1, n_2) has "1" in the row indexed by $(-n_2 \mod n, n_1)$. Similarly, the actions of s, p_1, and p_2 are expressed as

$$s \cdot (n_1, n_2) \equiv (n_1,\ -n_2) \mod n, \tag{4.2}$$
$$p_1 \cdot (n_1, n_2) \equiv (n_1 + 1,\ n_2) \mod n, \tag{4.3}$$
$$p_2 \cdot (n_1, n_2) \equiv (n_1,\ n_2 + 1) \mod n. \tag{4.4}$$

The permutation representation $T(g)$ is specified by (4.1)–(4.4) above.

Example 4.1. The permutation representation for the 4×4 square lattice is given by (4.1)–(4.4) as follows:

$$T(r) = \left[\begin{array}{cccc|cccc|cccc|cccc}
1&&&&&&&&&&&&&&&\\
&&&&&&&&&&&&1&&&\\
&&&&&&&&1&&&&&&&\\
&&&&1&&&&&&&&&&&\\
\hline
&1&&&&&&&&&&&&&&\\
&&&&&&&&&&&&&1&&\\
&&&&&&&&&1&&&&&&\\
&&&&&1&&&&&&&&&&\\
\hline
&&1&&&&&&&&&&&&&\\
&&&&&&&&&&&&&&1&\\
&&&&&&&&&&1&&&&&\\
&&&&&&1&&&&&&&&&\\
\hline
&&&1&&&&&&&&&&&&\\
&&&&&&&&&&&&&&&1\\
&&&&&&&&&&&1&&&&\\
&&&&&&&1&&&&&&&&
\end{array}\right], \quad
T(s) = \left[\begin{array}{cccc|cccc|cccc|cccc}
1&&&&&&&&&&&&&&&\\
&1&&&&&&&&&&&&&&\\
&&1&&&&&&&&&&&&&\\
&&&1&&&&&&&&&&&&\\
\hline
&&&&&&&&&&&&1&&&\\
&&&&&&&&&&&&&1&&\\
&&&&&&&&&&&&&&1&\\
&&&&&&&&&&&&&&&1\\
\hline
&&&&&&&&1&&&&&&&\\
&&&&&&&&&1&&&&&&\\
&&&&&&&&&&1&&&&&\\
&&&&&&&&&&&1&&&&\\
\hline
&&&&1&&&&&&&&&&&\\
&&&&&1&&&&&&&&&&\\
&&&&&&1&&&&&&&&&\\
&&&&&&&1&&&&&&&&
\end{array}\right],$$

$$T(p_1) = \left[\begin{array}{cccc|cccc|cccc|cccc}
&&&1&&&&&&&&&&&&\\
1&&&&&&&&&&&&&&&\\
&1&&&&&&&&&&&&&&\\
&&1&&&&&&&&&&&&&\\
\hline
&&&&&&&1&&&&&&&&\\
&&&&1&&&&&&&&&&&\\
&&&&&1&&&&&&&&&&\\
&&&&&&1&&&&&&&&&\\
\hline
&&&&&&&&&&&1&&&&\\
&&&&&&&&1&&&&&&&\\
&&&&&&&&&1&&&&&&\\
&&&&&&&&&&1&&&&&\\
\hline
&&&&&&&&&&&&&&&1\\
&&&&&&&&&&&&1&&&\\
&&&&&&&&&&&&&1&&\\
&&&&&&&&&&&&&&1&
\end{array}\right], \quad
T(p_2) = \left[\begin{array}{cccc|cccc|cccc|cccc}
&&&&&&&&&&&&1&&&\\
&&&&&&&&&&&&&1&&\\
&&&&&&&&&&&&&&1&\\
&&&&&&&&&&&&&&&1\\
\hline
1&&&&&&&&&&&&&&&\\
&1&&&&&&&&&&&&&&\\
&&1&&&&&&&&&&&&&\\
&&&1&&&&&&&&&&&&\\
\hline
&&&&1&&&&&&&&&&&\\
&&&&&1&&&&&&&&&&\\
&&&&&&1&&&&&&&&&\\
&&&&&&&1&&&&&&&&\\
\hline
&&&&&&&&1&&&&&&&\\
&&&&&&&&&1&&&&&&\\
&&&&&&&&&&1&&&&&\\
&&&&&&&&&&&1&&&&
\end{array}\right].$$

□

4.2. Irreducible Decomposition

The irreducible decomposition of the permutation representation $T(g)$ for the $n{\times}n$ square lattice is now investigated. The multiplicities of irreducible representations in this decomposition are determined. It is to be emphasized that irreducible representations lacking in the decomposition of $T(g)$ can be excluded from consideration in the search for square bifurcating patterns in Chapters 5 and 6.

4.2.1. Simple Examples

Prior to the analysis for general n we present the results for $n = 3$ and $n = 4$.

We begin with the case of $n = 3$. The group $\mathrm{D}_4 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)$ has nine irreducible representations (see Section 3.1):

$$\begin{aligned} R(\mathrm{D}_4 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)) = \{&(1;+,+,+),(1;+,-,+),(1;-,+,+),(1;-,-,+),\\ &(2;+),(4;1,0,+),(4;1,0,-),(4;1,1,+),(4;1,1,-)\}. \end{aligned}$$

Among these nine irreducible representations, only three of them, $(1;+,+,+)$, $(4;1,0,+)$, and $(4;1,1,+)$, are contained in $T(g)$ with multiplicity 1, whereas the others are missing in $T(g)$. Indeed we will see in Section 4.3 in a general setting that

$$Q^{-1}T(g)Q = T^{(1;+,+,+)}(g) \oplus T^{(4;1,0,+)}(g) \oplus T^{(4;1,1,+)}(g), \quad g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3) \tag{4.5}$$

for some orthogonal matrix Q. Accordingly, the multiplicities a^{μ} for $\mu \in R(\mathrm{D}_4 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3))$ are given as follows:

$$\begin{aligned} &a^{(1;+,+,+)} = 1, \quad a^{(1;+,-,+)} = 0, \quad a^{(1;-,+,+)} = 0, \quad a^{(1;-,-,+)} = 0;\\ &a^{(2;+)} = 0; \quad a^{(4;1,0,+)} = 1, \quad a^{(4;1,0,-)} = 0, \quad a^{(4;1,1,+)} = 1, \quad a^{(4;1,1,-)} = 0. \end{aligned}$$

We next show the case of $n = 4$. Recall the permutation representation $T(g)$ for $n = 4$ from Example 4.1. The group $\mathrm{D}_4 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$ has 20 irreducible representations (see Section 3.1):

$$\begin{aligned} R(\mathrm{D}_4 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)) = \{&(1;+,+,+),(1;+,-,+),(1;-,+,+),(1;-,-,+),\\ &(1;+,+,-),(1;+,-,-),(1;-,+,-),(1;-,-,-),\\ &(2;+),(2;-),(2;+,+),(2;+,-),(2;-,+),(2;-,-),\\ &(4;1,0,+),(4;1,0,-),(4;1,1,+),(4;1,1,-),(4;2,1,+),(4;2,1,-)\}. \end{aligned}$$

Among these 20 irreducible representations, only six of them, $(1;+,+,+)$, $(1;+,+,-)$, $(2;+,+)$, $(4;1,0,+)$, $(4;1,1,+)$, and $(4;2,1,+)$, are contained in $T(g)$ with multiplicity 1, whereas the others are missing in $T(g)$, as we will see in Section 4.3 in a general setting. This means that

$$\begin{aligned} Q^{-1}T(g)Q = T^{(1;+,+,+)}(g) \oplus T^{(1;+,+,-)}(g) \oplus T^{(2;+,+)}(g) \oplus T^{(4;1,0,+)}(g) \oplus T^{(4;1,1,+)}(g) \oplus T^{(4;2,1,+)}(g),\\ g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4) \end{aligned} \tag{4.6}$$

Table 4.1: The values of character χ of the permutation representation T

g		$\chi(g)$	
e		n^2	
${p_1}^i{p_2}^j$	$((i, j) \neq (0,0))$	0	
$r{p_1}^i{p_2}^j$	$(i + j = 2k)$	2	1
	$(i + j \neq 2k)$	0	1
		$(n = 2m)$	$(n \neq 2m)$
$r^2{p_1}^i{p_2}^j$	$(i, j$: even$)$	4	1
	(other (i, j))	0	1
		$(n = 2m)$	$(n \neq 2m)$
$r^3{p_1}^i{p_2}^j$	$(i + j = 2k)$	2	1
	$(i + j \neq 2k)$	0	1
		$(n = 2m)$	$(n \neq 2m)$

g		$\chi(g)$	
$s{p_1}^i{p_2}^j$	$(i = 0,\ j = 2k)$	$2n$	n
	$(i = 0,\ j \neq 2k)$	0	n
	$(i \neq 0)$	0	0
		$(n = 2m)$	$(n \neq 2m)$
$sr{p_1}^i{p_2}^j$	$(i = j)$	n	
	$(i \neq j)$	0	
$sr^2{p_1}^i{p_2}^j$	$(j = 0,\ i = 2k)$	$2n$	n
	$(j = 0,\ i \neq 2k)$	0	n
	$(j \neq 0)$	0	0
		$(n = 2m)$	$(n \neq 2m)$
$sr^3{p_1}^i{p_2}^j$	$(i = n - j)$	n	
	$(i \neq n - j)$	0	

$0 \leq i, j \leq n - 1;\ k, m$: integers

for some orthogonal matrix Q, the concrete form of which is given in Example 4.2 in Section 4.3.1. Accordingly, the multiplicities a^μ for $\mu \in R(\mathrm{D}_4 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4))$ are given as follows:

$$
\begin{aligned}
&a^{(1;+,+,+)} = 1, \quad a^{(1;+,-,+)} = 0, \quad a^{(1;-,+,+)} = 0, \quad a^{(1;-,-,+)} = 0,\\
&a^{(1;+,+,-)} = 1, \quad a^{(1;+,-,-)} = 0, \quad a^{(1;-,+,-)} = 0, \quad a^{(1;-,-,-)} = 0,\\
&a^{(2;+)} = 0, \quad a^{(2;-)} = 0, \quad a^{(2;+,+)} = 1, \quad a^{(2;+,-)} = 0,\\
&a^{(2;-,+)} = 0, \quad a^{(2;-,-)} = 0,\\
&a^{(4;1,0,+)} = 1, \quad a^{(4;1,0,-)} = 0, \quad a^{(4;1,1,+)} = 1, \quad a^{(4;1,1,-)} = 0,\\
&a^{(4;2,1,+)} = 1, \quad a^{(4;2,1,-)} = 0.
\end{aligned}
$$

4.2.2. Analysis for the Finite Square Lattice

For general n, the permutation representation $T(g)$ is specified by (4.1)–(4.4). We determine the irreducible decomposition of $T(g)$ with the aid of characters. Let $\chi(g)$ be the character of $T(g)$, which is defined by

$$\chi(g) = \mathrm{Tr}\, T(g), \quad g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n). \tag{4.7}$$

Table 4.1 shows the values of $\chi(g)$ for all $g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$, which are dependent on whether n is even or odd. For example, the action of $rp_1^i p_2^j$ reads

$$rp_1^i p_2^j \cdot (n_1, n_2) = r \cdot (n_1 + i, n_2 + j) = (-n_2 - j, n_1 + i).$$

Table 4.2: The values of irreducible characters χ^μ appearing in (4.10)

g	$\chi^{(1;+,+,+)}$	$\chi^{(4;k,0,+)}$	$\chi^{(4;k,k,+)}$	$\chi^{(8;k,\ell)}$	$\chi^{(1;+,+,-)}$ ($n=2m$)	$\chi^{(2;+,+)}$ ($n=2m$)	$\chi^{(4;n/2,\ell,+)}$ ($n=2m$)
${p_1}^i{p_2}^j$	1	$2[\cos(ki\theta)$ $+\cos(kj\theta)]$	$2[\cos(k(i+j)\theta)$ $+\cos(k(i-j)\theta)]$	(3.30)	$(-1)^{i+j}$	$(-1)^i+(-1)^j$	$2[(-1)^i\cos(\ell j\theta)$ $+(-1)^j\cos(\ell i\theta)]$
$r{p_1}^i{p_2}^j$	1	0	0	0	$(-1)^{i+j}$	0	0
$r^2{p_1}^i{p_2}^j$	1	0	0	0	$(-1)^{i+j}$	$(-1)^i+(-1)^j$	0
$r^3{p_1}^i{p_2}^j$	1	0	0	0	$(-1)^{i+j}$	0	$(-1)^i+(-1)^j$
$s{p_1}^i{p_2}^j$	1	$2\cos(ki\theta)$	0	0	$(-1)^{i+j}$	$(-1)^i+(-1)^j$	$2(-1)^j\cos(\ell i\theta)$
$sr{p_1}^i{p_2}^j$	1	0	$2\cos(k(i-j)\theta)$	0	$(-1)^{i+j}$	0	0
$sr^2{p_1}^i{p_2}^j$	1	$2\cos(kj\theta)$	0	0	$(-1)^{i+j}$	$(-1)^i+(-1)^j$	$2(-1)^i\cos(\ell j\theta)$
$sr^3{p_1}^i{p_2}^j$	1	0	$2\cos(k(i+j)\theta)$	0	$(-1)^{i+j}$	0	0

$\theta = 2\pi/n$; (3.30) reads:

$$\chi^{(8;k,\ell)}({p_1}^i{p_2}^j) = 2[\,\cos((ki+\ell j)\theta)+\cos((-\ell i+kj)\theta)+\cos((ki-\ell j)\theta)+\cos((-\ell i-kj)\theta)\,]$$

Invariant points (n_1,n_2) are those which satisfying $(n_1,n_2)\equiv(-n_2-j,n_1+i) \pmod n$. The number of these points, which depend on $i+j$ and n, gives $\chi(rp_1^i p_2^j)$.

In terms of characters, the irreducible decomposition of $T(g)$ can be expressed as

$$\chi(g) = \sum_\mu a^\mu \chi^\mu(g), \quad g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_n\times\mathbb{Z}_n), \tag{4.8}$$

where χ^μ is the character of $\mu \in R(\mathrm{D}_4 \ltimes (\mathbb{Z}_n\times\mathbb{Z}_n))$, and the multiplicity a^μ of μ can be determined by the formula

$$a^\mu = \frac{1}{8n^2}\sum_{g\in\mathrm{D}_4\ltimes(\mathbb{Z}_n\times\mathbb{Z}_n)} \chi(g)\chi^\mu(g). \tag{4.9}$$

In the case of $n=2m$, for example, we obtain

$$\begin{aligned}\chi(g) = \chi^{(1;+,+,+)}(g) + \chi^{(1;+,+,-)}(g) + \chi^{(2;+,+)}(g) + \sum_{k:(3.15)} \chi^{(4;n/2,\ell,+)}(g)\\ + \sum_{k:(3.13)}\chi^{(4;k,0,+)}(g) + \sum_{k:(3.14)}\chi^{(4;k,k,+)}(g) + \sum_{(k,\ell):(3.25)}\chi^{(8;k,\ell)}(g)\end{aligned}$$

as the decomposition (4.8). The terms $\chi^{(1;+,+,-)}(g)$, $\chi^{(2;+,+)}(g)$, and $\chi^{(4;n/2,\ell,+)}(g)$ appear only when n

is even. Hence we may represent this succinctly as

$$\begin{aligned}\chi(g) = \chi^{(1;+,+,+)}(g) &\left[+\chi^{(1;+,+,-)}(g) + \chi^{(2;+,+)}(g) + \sum_{k:(3.15)} \chi^{(4;n/2,\ell,+)}(g)\right]_{\text{if } n=2m} \\ &+ \sum_{k:(3.13)} \chi^{(4;k,0,+)}(g) + \sum_{k:(3.14)} \chi^{(4;k,k,+)}(g) + \sum_{(k,\ell):(3.25)} \chi^{(8;k,\ell)}(g), \\ &g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n), \end{aligned} \tag{4.10}$$

where $[\ \cdot\ ]_{\text{if } n=2m}$ means that the term is included when n is even. Table 4.2 shows the values of the irreducible characters $\chi^{\mu}(g)$ appearing on the right-hand side of (4.10) (see Section 3.1 for details about $\chi^{\mu}(g)$). The equality in (4.10) can be verified with the aid of Tables 4.1 and 4.2.

The decomposition (4.10) of the character $\chi(g)$ of $T(g)$ means that some orthogonal matrix Q exists such that

$$\begin{aligned}Q^{-1}T(g)Q = T^{(1;+,+,+)}(g) &\left[\oplus T^{(1;+,+,-)}(g) \oplus T^{(2;+,+)}(g) \oplus \bigoplus_{k:(3.15)} T^{(4;n/2,\ell,+)}(g)\right]_{\text{if } n=2m} \\ &\oplus \bigoplus_{k:(3.13)} T^{(4;k,0,+)}(g) \oplus \bigoplus_{k:(3.14)} T^{(4;k,k,+)}(g) \oplus \bigoplus_{(k,\ell):(3.25)} T^{(8;k,\ell)}(g), \\ &g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n). \end{aligned} \tag{4.11}$$

This gives the irreducible decomposition of $T(g)$. Accordingly, the multiplicities a^{μ} in the irreducible decomposition of $T(g)$ are given as follows:

$$\begin{aligned}
&a^{(1;+,+,+)} = 1, \quad a^{(1;+,-,+)} = 0, \quad a^{(1;-,+,+)} = 0, \quad a^{(1;-,-,+)} = 0, \\
&a^{(1;+,+,-)} = 1, \quad a^{(1;+,-,-)} = 0, \quad a^{(1;-,+,-)} = 0, \quad a^{(1;-,-,-)} = 0, \\
&a^{(2;+)} = 0, \quad a^{(2;-)} = 0, \\
&a^{(2;+,+)} = \begin{cases} 1 & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}, \end{cases} \\
&a^{(2;+,-)} = 0, \quad a^{(2;-,+)} = 0, \quad a^{(2;-,-)} = 0, \\
&a^{(4;k,0,+)} = 1, \quad a^{(4;k,0,-)} = 0, \quad 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \\
&a^{(4;k,k,+)} = 1, \quad a^{(4;k,k,-)} = 0, \quad 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \\
&a^{(4;n/2,\ell,+)} = \begin{cases} 1 & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}, \end{cases} \\
&a^{(4;n/2,\ell,-)} = 0, \quad 1 \le \ell \le \frac{n}{2} - 1, \\
&a^{(8;k,\ell)} = 1, \quad 1 \le \ell \le k-1, \ 2 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor.
\end{aligned}$$

Table 4.3: Irreducible representations contained in the permutation representation T

$n \setminus d$	1	2	4	8
$2m$	$(1;+,+,+),(1;+,+,-)$	$(2;+,+)$	$(4;k,0;+),(4;k,k;+),(4;n/2,\ell,+)$	$(8;k,\ell)$
$2m-1$	$(1;+,+,+)$		$(4;k,0;+),(4;k,k;+)$	$(8;k,\ell)$

$(4;k,0;+)$ for k with $1 \leq k \leq \lfloor (n-1)/2 \rfloor$;
$(4;k,k;+)$ for k with $1 \leq k \leq \lfloor (n-1)/2 \rfloor$;
$(4;n/2,\ell;+)$ for k with $1 \leq \ell \leq n/2-1$;
$(8;k,\ell)$ for (k,ℓ) with $1 \leq \ell \leq k-1,\ 2 \leq k \leq \lfloor (n-1)/2 \rfloor$

Table 4.4: Number $\tilde{N}_d$ of d-dimensional irreducible representations of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ contained in the permutation representation T for the square lattice

$n \setminus d$	1	2	4	8	
	$\tilde{N}_1$	$\tilde{N}_2$	$\tilde{N}_4$	$\tilde{N}_8$	$\sum \tilde{N}_d$
1	1				1
2	2	1			3
3	1		2		3
4	2	1	3		6
5	1		4	1	6
6	2	1	6	1	10
7	1		6	3	10
8	2	1	9	3	15

$n \setminus d$	1	2	4	8	
	$\tilde{N}_1$	$\tilde{N}_2$	$\tilde{N}_4$	$\tilde{N}_8$	$\sum \tilde{N}_d$
9	1		8	6	15
10	2	1	12	6	21
11	1		10	10	21
12	2	1	15	10	28
13	1		12	15	28
14	2	1	18	15	36
15	1		14	21	36
16	2	1	21	21	45

$n \setminus d$	1	2	4	8	
	$\tilde{N}_1$	$\tilde{N}_2$	$\tilde{N}_4$	$\tilde{N}_8$	$\sum \tilde{N}_d$
17	1		16	28	45
18	2	1	24	28	55
19	1		18	36	55
20	2	1	27	36	66
21	1		20	45	66
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
42	2	1	30	190	223

It is noteworthy that the multiplicity is either 0 or 1 for each irreducible representation, that is, the permutation representation $T(g)$ in (4.1)–(4.4) is *multiplicity-free* (see Remark 4.1). Table 4.3 shows a summary.

By $\tilde{N}_d$, we denote the number of d-dimensional irreducible representations of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ that exist in the permutation representation $T(g)$. We have the following expressions for $\tilde{N}_d$:

$n \setminus d$	1	2	4	8
	$\tilde{N}_1$	$\tilde{N}_2$	$\tilde{N}_4$	$\tilde{N}_8$
$2m$	2	1	$3(n-2)/2$	$(n^2-6n+8)/8$
$2m-1$	1	0	$n-1$	$(n^2-4n+3)/8$

(4.12)

whereas Table 4.4 shows the values of $\tilde{N}_d$ for several n. Also note the relation

$$\sum_d d\tilde{N}_d = n^2. \tag{4.13}$$

Remark 4.1. It is a basic fact that a permutation representation $T(g)$ representing the action of a group G on a finite set P is multiplicity-free if there exists some $g \in G$ such that $g \cdot p = q$ and $g \cdot q = p$ (e.g., see Proposition 1.4.8 of Ceccherini-Silberstein et al., 2010 [24]). The permutation representation $T(g)$ in (4.1)–(4.4) satisfies this condition as follows. By (4.1), (4.3), and (4.4), we have

$$r^2 {p_1}^i {p_2}^j \cdot (n_1, n_2) \equiv (-n_1 - i, n_2 - j) \mod n.$$

Hence, any pair of (n_1, n_2) and (n_1', n_2') can be rewritten as

$$g \cdot (n_1, n_2) \equiv (n_1', n_2') \mod n, \quad g \cdot (n_1', n_2') \equiv (n_1, n_2) \mod n$$

by $g = r^2 {p_1}^i {p_2}^j$ with $i = n_1 - n_1'$ and $j = n_2 - n_2'$. □

4.3. Transformation Matrix for Irreducible Decomposition

Transformation matrix Q for the irreducible decomposition is derived for the square lattice, and examples of this matrix Q are presented.

4.3.1. Formulas for Transformation Matrix

For the $n \times n$ square lattice with the symmetry of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$, we derive the transformation matrix

$$Q = (Q^\mu \mid \mu \in \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)) \tag{4.14}$$

for the irreducible decomposition. Note that the column set of Q is partitioned into blocks, each associated with an irreducible representation μ contained in $T(g)$ (see Table 4.3). Since such μ has $a^\mu = 1$ (multiplicity-free), we have the relation

$$T(g)Q^\mu = Q^\mu T^\mu(g), \quad g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n), \tag{4.15}$$

where $T(g)$ is the permutation representation given in Section 4.1.

The vector $\boldsymbol{\lambda}$ expressing population pattern is defined as

$$\begin{aligned}\boldsymbol{\lambda} &= (\lambda_1, \dots, \lambda_N)^\top \\ &= (\lambda_{00}, \dots, \lambda_{n-1,0};\ \lambda_{01}, \dots, \lambda_{n-1,1};\ \dots\ ;\ \lambda_{0,n-1}, \dots, \lambda_{n-1,n-1})^\top \\ &= (\lambda_{n_1 n_2} \mid n_1, n_2 = 0, \dots, n-1),\end{aligned}$$

where $N = n^2$ and $(\lambda_{n_1 n_2} \mid n_1, n_2 = 0, \dots, n-1)$ is an N-dimensional column vector. For a vector on this lattice with the (n_1, n_2)-component $g(n_1, n_2)$, we express its normalization as[5]

$$\langle g(n_1, n_2) \rangle = (g(n_1, n_2)/(\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} g(i, j)^2)^{1/2} \mid n_1, n_2 = 0, \dots, n-1). \tag{4.16}$$

Recall that the permutation representation $T(g)$ is specified by (4.1)–(4.4) above. The action of r on (n_1, n_2), for example, is expressed by

$$r \cdot (n_1, n_2) \equiv (-n_2,\ n_1) \mod n$$

in (4.1), which shows that the column of $T(r)$ indexed by (n_1, n_2) has "1" in the row indexed by $(-n_2, n_1) \mod n$. For the present purpose, however, it is convenient to consider $T(g)$ row-wise. It is seen that the row of $T(r)$ indexed by (n_1, n_2) has "1" at the column indexed by $(n_2, -n_1) \mod n$, since

$$(n_1', n_2') \equiv (-n_2,\ n_1) \mod n$$

can be solved for (n_1, n_2) as

$$(n_1, n_2) \equiv (n_2', -n_1') \mod n.$$

We denote this as

$$r * (n_1, n_2) \equiv (n_2, -n_1) \mod n. \tag{4.17}$$

For s, p_1, and p_2, a similar argument based on (4.2)–(4.4) yields

$$s * (n_1, n_2) \equiv (n_1,\ -n_2) \mod n, \tag{4.18}$$

$$p_1 * (n_1, n_2) \equiv (n_1 - 1,\ n_2) \mod n, \tag{4.19}$$

$$p_2 * (n_1, n_2) \equiv (n_1,\ n_2 - 1) \mod n. \tag{4.20}$$

The submatrices Q^μ for μ are given by the following proposition, where the notation $\langle \cdot \rangle$ for normalization in (4.16) is used.

[5] The notation $\langle \cdot \rangle$ here should not be confused with that for the generators of a group.

Proposition 4.1. *The submatrices Q^{μ} of the transformation matrix Q on the $n \times n$ square lattice are given by*

$$Q^{(1;+,+,+)} = \frac{1}{n}(1,\ldots,1)^{\top} = \langle 1 \rangle, \tag{4.21}$$

$$Q^{(1;+,+,-)} = \begin{cases} [\ \langle \cos(\pi(n_1 - n_2)) \rangle\ ] & \text{if } n \text{ is even,} \\ \text{missing} & \text{if } n \text{ is odd,} \end{cases} \tag{4.22}$$

$$Q^{(2;+,+)} = \begin{cases} [\ \langle \cos(\pi n_1) \rangle,\ \langle \cos(\pi n_2) \rangle\ ] & \text{if } n \text{ is even,} \\ \text{missing} & \text{if } n \text{ is odd,} \end{cases} \tag{4.23}$$

$$\begin{aligned} Q^{(4;k,0,+)} &= [\ \langle \cos(2\pi k\, n_1/n) \rangle,\ \langle \sin(2\pi k\, n_1/n) \rangle,\ \langle \cos(2\pi k n_2/n) \rangle,\ \langle \sin(2\pi k n_2/n) \rangle\ ] \\ &\qquad \text{for } 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \end{aligned} \tag{4.24}$$

$$\begin{aligned} Q^{(4;k,k,+)} &= [\ \langle \cos(2\pi k(n_1 + n_2)/n) \rangle,\ \langle \sin(2\pi k(n_1 + n_2)/n) \rangle, \\ &\quad \langle \cos(2\pi k(-n_1 + n_2)/n) \rangle,\ \langle \sin(2\pi k(-n_1 + n_2)/n) \rangle\ ] \\ &\qquad \text{for } 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \end{aligned} \tag{4.25}$$

$$Q^{(4;n/2,\ell,+)} = \begin{cases} [\ \langle \cos(\pi n_1 + 2\pi \ell n_2/n) \rangle,\ \langle \sin(\pi n_1 + 2\pi \ell n_2/n) \rangle, & \\ \quad \langle \cos(-2\pi \ell n_1/n + \pi n_2) \rangle,\ \langle \sin(-2\pi \ell n_1/n + \pi n_2) \rangle\ ] & \\ \qquad \text{for } 1 \le \ell \le \frac{n}{2} - 1 & \text{if } n \text{ is even,} \\ \text{missing} & \text{if } n \text{ is odd,} \end{cases} \tag{4.26}$$

$$\begin{aligned} Q^{(8;k,\ell)} &= [\ \langle \cos(2\pi(k n_1 + \ell n_2)/n) \rangle,\ \langle \sin(2\pi(k n_1 + \ell n_2)/n) \rangle, \\ &\quad \langle \cos(2\pi(-\ell n_1 + k n_2)/n) \rangle,\ \langle \sin(2\pi(-\ell n_1 + k n_2)/n) \rangle, \\ &\quad \langle \cos(2\pi(k n_1 - \ell n_2)/n) \rangle,\ \langle \sin(2\pi(k n_1 - \ell n_2)/n) \rangle, \\ &\quad \langle \cos(2\pi(-\ell n_1 - k n_2)/n) \rangle,\ \langle \sin(2\pi(-\ell n_1 - k n_2)/n) \rangle\ ] \\ &\qquad \text{for } 1 \le \ell \le k-1,\ 2 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor. \end{aligned} \tag{4.27}$$

Proof. Proof is given in Section 4.3.2. □

An example of the transformation matrix Q for $n = 4$ is presented below by assembling submatrices Q^{μ} in Proposition 4.1.

Example 4.2. The transformation matrix Q for the 4×4 square lattice reads

$$
\begin{aligned}
Q &= [Q^{(1;+,+,+)}, Q^{(1;+,+,-)}, Q^{(2;+,+)}, Q^{(4;1,0,+)}, Q^{(4;1,1,+)}, Q^{(4;2,1,+)}] \\
&= [\ \langle 1 \rangle \mid \langle \cos(\pi(n_1 - n_2)) \rangle \mid \langle \cos(\pi n_1) \rangle,\ \langle \cos(\pi n_2) \rangle \ \mid \\
&\qquad \langle \cos(\pi\, n_1/2) \rangle,\ \langle \sin(\pi\, n_1/2) \rangle,\ \langle \cos(\pi n_2/2) \rangle,\ \langle \sin(\pi n_2/2) \rangle \mid \\
&\qquad \langle \cos(\pi(n_1 + n_2)/2) \rangle,\ \langle \sin(\pi(n_1 + n_2)/2) \rangle,\ \langle \cos(\pi(-n_1 + n_2)/2) \rangle,\ \langle \sin(\pi(-n_1 + n_2)/2) \rangle \mid \\
&\qquad \langle \cos(\pi n_1 + \pi n_2/2) \rangle,\ \langle \sin(\pi n_1 + \pi n_2/2) \rangle,\ \langle \cos(-\pi n_1/2 + \pi n_2) \rangle,\ \langle \sin(-\pi n_1/2 + \pi n_2) \rangle\ ] \\
&= \frac{1}{4}\left[\begin{array}{c|c|cc|cccc|cccc|cccc}
1 & 1 & 1 & 1 & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & \\
1 & -1 & -1 & 1 & & \sqrt{2} & \sqrt{2} & & & \sqrt{2} & & -\sqrt{2} & -\sqrt{2} & & & -\sqrt{2} \\
1 & 1 & 1 & 1 & -\sqrt{2} & & \sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & \sqrt{2} & & -\sqrt{2} & \\
1 & -1 & -1 & 1 & & -\sqrt{2} & \sqrt{2} & & & -\sqrt{2} & & \sqrt{2} & -\sqrt{2} & & & \sqrt{2} \\
\hline
1 & -1 & 1 & -1 & \sqrt{2} & & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & -\sqrt{2} & \\
1 & 1 & -1 & -1 & & \sqrt{2} & & \sqrt{2} & -\sqrt{2} & & \sqrt{2} & & & -\sqrt{2} & & \sqrt{2} \\
1 & -1 & 1 & -1 & -\sqrt{2} & & & \sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & \sqrt{2} & \sqrt{2} & \\
1 & 1 & -1 & -1 & & -\sqrt{2} & & \sqrt{2} & \sqrt{2} & & -\sqrt{2} & & & -\sqrt{2} & & -\sqrt{2} \\
\hline
1 & 1 & 1 & 1 & \sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & \sqrt{2} & \\
1 & -1 & -1 & 1 & & \sqrt{2} & -\sqrt{2} & & & -\sqrt{2} & & \sqrt{2} & \sqrt{2} & & & -\sqrt{2} \\
1 & 1 & 1 & 1 & -\sqrt{2} & & -\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & \\
1 & -1 & -1 & 1 & & -\sqrt{2} & -\sqrt{2} & & & \sqrt{2} & & -\sqrt{2} & \sqrt{2} & & & \sqrt{2} \\
\hline
1 & -1 & 1 & -1 & \sqrt{2} & & & -\sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & -\sqrt{2} & \\
1 & 1 & -1 & -1 & & \sqrt{2} & & -\sqrt{2} & \sqrt{2} & & -\sqrt{2} & & & \sqrt{2} & & \sqrt{2} \\
1 & -1 & 1 & -1 & -\sqrt{2} & & & -\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & -\sqrt{2} & \sqrt{2} & \\
1 & 1 & -1 & -1 & & -\sqrt{2} & & -\sqrt{2} & -\sqrt{2} & & \sqrt{2} & & & \sqrt{2} & & -\sqrt{2}
\end{array}\right].
\end{aligned}
$$

□

4.3.2. Proof of Proposition 4.1

We will now show that the relation $T(g)Q^{\mu} = Q^{\mu}T^{\mu}(g)$ in (4.15) is satisfied by Q^{μ} in Proposition 4.1 for r, s, p_1, and p_2 that generate the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. Recall the actions of r, s, p_1, and p_2 given in (4.17)–(4.20). We demonstrate the proof for $\mu = (2;+,+)$ and $(8;k,\ell)$, and the other cases can be treated similarly.

Two-Dimensional Irreducible Representation

We shall prove that

$$
Q^{(2;+,+)} = [\ \langle \cos(\pi n_1) \rangle,\ \langle \cos(\pi n_2) \rangle\ ] \tag{4.28}
$$

satisfies (4.15) for $\mu = (2;+,+)$. Recall that $(2;+,+)$ exists when n is even and $T^{(2;+,+)}(g)$ is defined by (3.9) and (3.10).

The action of r on the wave numbers (n_1, n_2) in (4.28) is given, by a formal calculation using (4.17), as

$$
r * (n_1, n_2) = (r * n_1, r * n_2) \equiv (n_2, -n_1 \mod n).
$$

In the matrix form, this gives

$$
\begin{aligned}
T(r)Q^{(2;+,+)} &= [\ \langle \cos(\pi n_2) \rangle,\ \langle \cos(-\pi n_1) \rangle\ ] \\
&= [\ \langle \cos(\pi n_2) \rangle,\ \langle \cos(\pi n_1) \rangle\ ] \\
&= [\ \cos(\pi n_1),\ \cos(\pi n_2)\ ] \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \\
&= Q^{(2;+,+)}T^{(2;+,+)}(r).
\end{aligned}
$$

The action of p_1 on the wave numbers (n_1, n_2) is given by (4.19) as

$$p_1 * (n_1, n_2) \equiv (n_1 - 1 \mod n, n_2),$$

which, in the matrix form, yields

$$\begin{aligned} T(p_1)Q^{(2;+,+)} &= [\ \langle \cos(\pi(n_1-1))\rangle,\ \langle \cos(\pi n_2)\rangle\ ] \\ &= [\ \langle -\cos(\pi n_1)\rangle,\ \langle \cos(\pi n_2)\rangle\ ] \\ &= [\ \langle \cos(\pi n_1)\rangle,\ \langle \cos(\pi n_2)\rangle\ ] \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} \\ &= Q^{(2;+,+)}T^{(2;+)}(p_1). \end{aligned}$$

The cases of s and p_2 can be treated similarly. Thus, we have

$$T(g)Q^{(2;+,+)} = Q^{(2;+,+)}T^{(2;+,+)}(g), \quad g = r, s, p_1, p_2.$$

This completes the proof for $\mu = (2; +, +)$.

Eight-Dimensional Irreducible Representations

We shall prove that

$$\begin{aligned} Q^{(8;k,\ell)} = [\ &\langle \cos(2\pi(kn_1 + \ell n_2)/n)\rangle,\ \langle \sin(2\pi(kn_1 + \ell n_2)/n)\rangle, \\ &\langle \cos(2\pi(-\ell n_1 + kn_2)/n)\rangle,\ \langle \sin(2\pi(\ell n_1 + kn_2)/n)\rangle, \\ &\langle \cos(2\pi(kn_1 - \ell n_2)/n)\rangle,\ \langle \sin(2\pi(kn_1 - \ell n_2)/n)\rangle, \\ &\langle \cos(2\pi(-\ell n_1 - kn_2)/n)\rangle,\ \langle \sin(2\pi(-\ell n_1 - kn_2)/n)\rangle\ ] \\ &\text{for } 1 \le \ell \le k-1,\ 2 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor \end{aligned} \tag{4.29}$$

satisfies (4.15) for $(8; k, \ell)$ where $n \geq 5$. Recall the definition of $T^{(8;k,\ell)}(g)$ for $g = r, s, p_1, p_2$ in (3.27) and (3.28), as well as the notations

$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$

The action of r on the four wave numbers in (4.29) is given by (4.17) as

$$r * \begin{bmatrix} kn_1 + \ell n_2 \\ -\ell n_1 + kn_2 \\ kn_1 - \ell n_2 \\ -\ell n_1 - kn_2 \end{bmatrix} \equiv \begin{bmatrix} -\ell n_1 + kn_2 \\ -(kn_1 + \ell n_2) \\ -(-\ell n_1 - kn_2) \\ kn_1 - \ell n_2 \end{bmatrix} \mod n,$$

which permutes and changes the sign of the column vectors of $Q^{(8;k,\ell)}$ in (4.29) as

$$T(r)Q^{(8;k,\ell)} = Q^{(8;k,\ell)} \left[\begin{array}{cc|cc} & S & & \\ I & & & \\ \hline & & & I \\ & & S & \end{array}\right] = Q^{(8;k,\ell)}T^{(8;k,\ell)}(r).$$

The action of s on the four wave numbers in (4.29) is given by (4.18) as

$$s * \begin{bmatrix} kn_1 + \ell n_2 \\ -\ell n_1 + kn_2 \\ kn_1 - \ell n_2 \\ -\ell n_1 - kn_2 \end{bmatrix} \equiv \begin{bmatrix} kn_1 - \ell n_2 \\ -\ell n_1 - kn_2 \\ kn_1 + \ell n_2 \\ -\ell n_1 + kn_2 \end{bmatrix} \mod n,$$

which gives

$$T(s)Q^{(8;k,\ell)} = Q^{(8;k,\ell)} \left[\begin{array}{cc|cc} & & I & \\ & & & I \\ \hline I & & & \\ & I & & \end{array}\right] = Q^{(8;k,\ell)} T^{(8;k,\ell)}(s).$$

The action of p_1 on the four wave numbers in (4.29) is given by (4.19) as

$$p_1 * \begin{bmatrix} kn_1 + \ell n_2 \\ -\ell n_1 + kn_2 \\ kn_1 - \ell n_2 \\ -\ell n_1 - kn_2 \end{bmatrix} \equiv \begin{bmatrix} kn_1 + \ell n_2 - k \\ -\ell n_1 + kn_2 + \ell \\ kn_1 - \ell n_2 - k \\ -\ell n_1 - kn_2 + \ell \end{bmatrix} \mod n,$$

which gives

$$T(p_1)Q^{(8;k,\ell)} = Q^{(8;k,\ell)} \left[\begin{array}{cc|cc} R^k & & & \\ & R^{-\ell} & & \\ \hline & & R^k & \\ & & & R^{-\ell} \end{array}\right] = Q^{(8;k,\ell)} T^{(8;k,\ell)}(p_1).$$

The action of p_2 on the four wave numbers in (4.29) is given by (4.20) as

$$p_2 * \begin{bmatrix} kn_1 + \ell n_2 \\ -\ell n_1 + kn_2 \\ kn_1 - \ell n_2 \\ -\ell n_1 - kn_2 \end{bmatrix} \equiv \begin{bmatrix} kn_1 + \ell n_2 - \ell \\ -\ell n_1 + kn_2 - k \\ kn_1 - \ell n_2 + \ell \\ -\ell n_1 - kn_2 + k \end{bmatrix} \mod n,$$

which gives

$$T(p_2)Q^{(8;k,\ell)} = Q^{(8;k,\ell)} \left[\begin{array}{cc|cc} R^\ell & & & \\ & R^{k} & & \\ \hline & & R^{-\ell} & \\ & & & R^{-k} \end{array}\right] = Q^{(8;k,\ell)} T^{(8k,\ell)}(p_2).$$

Thus, we have the following relation to complete the proof for $\mu = (8; k, \ell)$:

$$T(g)Q^{(8;k,\ell)} = Q^{(8;k,\ell)} T^{(8;k,\ell)}(g), \quad g = r, s, p_1, p_2.$$

5. Square Patterns: Using Equivariant Branching Lemma

We presented fundamental facts about the square lattice in Chapters 2–4. We introduced the $n \times n$ square lattice with periodic boundary conditions as a spatial platform for agglomeration (Chapter 2). We labeled the symmetry of this lattice by the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$, and obtained the irreducible representations of this group (Chapter 3). We decomposed the representation matrix for the square lattice into irreducible components to determine the multiplicity a^μ of each irreducible representation μ (Chapter 4).

We would like to investigate the existence of square patterns as bifurcating solutions on the square lattice. For each irreducible representation μ with $a^\mu \geq 1$, we study bifurcation from a critical point associated with μ by using group-theoretic bifurcation analysis procedures under group symmetry. The following two different methods are available.

(i) The equivariant branching lemma is applied to the bifurcation equation associated with μ to show the existence of bifurcating solutions with a specified symmetry. This analysis is algebraic or group-theoretic, which focuses on the symmetry of solutions. The concrete form of the bifurcation equation need not be derived, and isotropy subgroups play a key role in the analysis.

(ii) The bifurcation equation is obtained in the form of power series expansions and is solved asymptotically. This method is more complicated, treating nonlinear terms directly, but is more informative, giving asymptotic forms of the bifurcating solutions and their directions in addition to their existence.

In this chapter, we apply the first method (i), using the equivariant branching lemma, to the economy on the $n \times n$ square lattice with the symmetry of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. We obtain possible bifurcating square patterns and associated lattice sizes for all the irreducible representations, which are related to group-theoretic critical points with multiplicity $M = 1, 2, 4$, and 8.

The second method (ii), solving the bifurcation equation, is not based on the equivariant branching lemma and capable of capturing all bifurcating solutions by dealing with the bifurcation equation explicitly. The first method conducted in this chapter demands less analytical effort than this method and fits to pinpoint the targeted square patterns among many other bifurcating solutions.

This chapter is organized as follows. Theoretically-predicted bifurcating square patterns are previewed in Section 5.1. Fundamentals of bifurcation analysis are recapitulated in Section 5.2. Bifurcation points of multiplicity $M = 1, 2, 4$, and 8 are respectively studied in Sections 5.3–5.6.

5.1. Theoretically-Predicted Bifurcating Square Patterns

A possible bifurcation mechanism that can produce square patterns is presented as a preview of the group-theoretic bifurcation analysis in Sections 5.4–5.6. Note that all critical points are assumed to be group-theoretic as explained in Section 5.2.

5.1.1. Symmetry of Bifurcating Square Patterns

Recall first that the symmetry of the $n \times n$ square lattice is labeled by the group

$$G = \langle r, s, p_1, p_2 \rangle = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) \tag{5.1}$$

in (2.34) with the fundamental relations (2.35):

$$\begin{aligned} r^4 = s^2 = (rs)^2 = {p_1}^n = {p_2}^n = e, \quad p_2 p_1 = p_1 p_2, \\ r p_1 = p_2 r, \quad r p_2 = p_1^{-1} r, \quad s p_1 = p_1 s, \quad s p_2 = p_2^{-1} s, \end{aligned} \tag{5.2}$$

where e is the identity element.

We consider an equilibrium equation of the form

$$\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \mathbf{0}, \tag{5.3}$$

where $\boldsymbol{\lambda} = (\lambda, \ldots, \lambda_N)^\top$ with $N = n^2$ is an N-dimensional independent variable vector and ϕ is the bifurcation parameter. Among many possible solutions $\boldsymbol{\lambda}$ to the equilibrium equation (5.3), we are particularly interested in those bifurcating solutions that represent the square patterns.

To describe the square patterns, we introduced in (2.4) a sublattice

$$\begin{aligned} \mathcal{H}(\alpha,\beta) &= \{n_1(\alpha \boldsymbol{\ell}_1 + \beta \boldsymbol{\ell}_2) + n_2(-\beta \boldsymbol{\ell}_1 + \alpha \boldsymbol{\ell}_2) \mid n_1, n_2 \in \mathbb{Z}\} \\ &= \{\begin{bmatrix} \boldsymbol{\ell}_1 & \boldsymbol{\ell}_2 \end{bmatrix} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \mid n_1, n_2 \in \mathbb{Z}\}, \end{aligned} \tag{5.4}$$

where

$$\boldsymbol{\ell}_1 = d \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{\ell}_2 = d \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5.5}$$

are basis vectors of length d of the underlying infinite square lattice

$$\mathcal{H} = \{n_1 \boldsymbol{\ell}_1 + n_2 \boldsymbol{\ell}_2 \mid n_1, n_2 \in \mathbb{Z}\} \tag{5.6}$$

in (2.2). In this chapter we adopt the parameter space

$$\{(\alpha,\beta) \in \mathbb{Z}^2 \mid \alpha > 0,\ \beta \geq 0\} \tag{5.7}$$

in (2.10) of Proposition 2.1, instead of $\{(\alpha,\beta) \in \mathbb{Z}^2 \mid \alpha \geq \beta \geq 0,\ \alpha \neq 0\}$ in (2.11), unless otherwise stated. The size of the square patterns in $\mathcal{H}(\alpha,\beta)$ is characterized in (2.8) by

$$D = D(\alpha,\beta) = \alpha^2 + \beta^2. \tag{5.8}$$

We recall the $n \times n$ square lattice

$$\mathcal{H}_n = \{n_1 \boldsymbol{\ell}_1 + n_2 \boldsymbol{\ell}_2 \mid n_i \in \mathbb{Z},\ 0 \leq n_i \leq n-1\ (i = 1, 2)\} \tag{5.9}$$

in (2.28) and describe the symmetry of a square pattern $\mathcal{H}(\alpha,\beta)\cap\mathcal{H}_n$ on this lattice by the subgroup $G(\alpha,\beta)$. This subgroup is classified in (2.39) into three types:

$$\begin{array}{lll} \langle r, s, p_1^{\alpha}, p_2^{\alpha}\rangle = \Sigma(\alpha,0) & (\alpha \geq 1, \beta = 0) & : \text{type V}, \\ \langle r, s, p_1^{\beta}p_2^{\beta}, p_1^{-\beta}p_2^{\beta}\rangle = \Sigma(\beta,\beta) & (\beta \geq 1, \alpha = \beta) & : \text{type M}, \\ \langle r, \ \ p_1^{\alpha}p_2^{\beta}, p_1^{-\beta}p_2^{\alpha}\rangle = \Sigma_0(\alpha,\beta) & (\text{otherwise}) & : \text{type T}. \end{array} \tag{5.10}$$

Here it is convenient to introduce a convention

$$\Sigma_0(0,0) = \langle r\rangle, \quad \Sigma(0,0) = \langle r, s\rangle, \quad \Sigma(1,0) = \langle r, s, p_1, p_2\rangle. \tag{5.11}$$

Recall the compatibility condition (2.33) between (α,β) and n given as

$$n = \begin{cases} m\alpha & \text{for type V}, \\ 2m\beta & \text{for type M}, \\ mD(\alpha,\beta)/\gcd(\alpha,\beta) & \text{for type T}, \end{cases} \tag{5.12}$$

where $m = 1, 2, \ldots$.

The objective of this chapter is to look for a solution $\boldsymbol{\lambda}$ to (5.3) such that the isotropy subgroup $\Sigma(\boldsymbol{\lambda})$ for the symmetry of $\boldsymbol{\lambda}$ coincides with one of the subgroups in (5.10).

5.1.2. Square Patterns Engendered by Direct Bifurcations

The main message of this chapter is that bifurcating solutions for square patterns do arise from the mathematical model on the square lattice with pertinent lattice sizes, and therefore these patterns can be understood within the framework of group-theoretic bifurcation theory. The major results to be derived in Sections 5.4–5.6, are summarized below.

Proposition 5.1. *A bifurcating solution with the square symmetry expressed by the subgroup in* (5.10) *exists for pertinent lattice sizes n. More specifically, we have the following, where m denotes a positive integer.*

- *For* $(\alpha,\beta;n) = (\alpha,0;\alpha m)$ $(2 \leq \alpha \leq n)$*, a square pattern of type V with symmetry* $\Sigma(\alpha,0)$ *branches at a bifurcation point with multiplicity* $M = 2$ $(\alpha = 2)$*,* $M = 4$ $(\alpha \geq 3)$*, or* $M = 8$ $(\alpha \geq 5)$.
- *For* $(\alpha,\beta;n) = (\beta,\beta;2\beta m)$ $(1 \leq \beta \leq n/2)$*, a square pattern of type M with symmetry* $\Sigma(\beta,\beta)$ *branches at a bifurcation point with multiplicity* $M = 1$ $(\beta = 1)$*,* $M = 4$ $(\beta \geq 2)$*, or* $M = 8$ $(\beta \geq 4)$.
- *For* $(\alpha,\beta;n) = (\alpha,\beta;mD(\alpha,\beta)/\gcd(\alpha,\beta))$*, where* $1 \leq \alpha \leq n-1$*,* $1 \leq \beta \leq n-1$*, and* $\alpha \neq \beta$*, a square pattern of type T with symmetry* $\Sigma_0(\alpha,\beta)$ *branches at a bifurcation point with multiplicity* $M = 8$.

Proof. This is proved in Sections 5.4–5.6. □

Possible square patterns for each value of $(\alpha,\beta;n)$ in Proposition 5.1 are summarized as follows:

$(\alpha,\beta;n)$		M	Type
	$\alpha=2$	2	
$(\alpha,0;\alpha m)$	$\alpha\geq 3$	4	V
	$\alpha\geq 5$	8	
	$\beta=1$	1	
$(\beta,\beta;2\beta m)$	$\beta\geq 2$	4	M
	$\beta\geq 4$	8	
$(\alpha,\beta;\dfrac{mD(\alpha,\beta)}{\gcd(\alpha,\beta)})$	$1\leq\alpha\leq n-1,\ 1\leq\beta\leq n-1,\ \alpha\neq\beta$	8	T

where $m=1,2,\ldots$.

The following proposition plays a pivotal role in the search for square patterns.

Proposition 5.2. *The existence of square patterns depends on the divisors of the lattice size n as follows:*

(i) *If n has a divisor α $(2\leq\alpha\leq n)$, a square pattern of type V with symmetry $\Sigma(\alpha,0)$ exists.*

(ii) *If n has a divisor 2β $(1\leq\beta\leq n/2)$, a square pattern of type M with symmetry $\Sigma(\beta,\beta)$ exists.*

(iii) *If n has a divisor $D(\alpha,\beta)/\gcd(\alpha,\beta)$, where $1\leq\alpha\leq n-1$, $1\leq\beta\leq n-1$, and $\alpha\neq\beta$, a square pattern of type T with symmetry $\Sigma_0(\alpha,\beta)$ exists.*

Proof. This follows from Proposition 5.1. □

Possible square patterns emerging via direct bifurcations for several values of n, obtained from Proposition 5.2, are listed in Tables 5.1 and 5.2.

5.2. Procedure of Theoretical Analysis

A bifurcation analysis procedure resorting to the equivariant branching lemma is summarized.

5.2.1. Bifurcation and Symmetry of Solutions

Let us consider the system of equilibrium equations

$$\boldsymbol{F}(\boldsymbol{\lambda},\phi)=\boldsymbol{0} \tag{5.13}$$

endowed with the symmetry of, or *equivariance* to, $G=\mathrm{D}_4\ltimes(\mathbb{Z}_n\times\mathbb{Z}_n)$ formulated as

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda},\phi)=\boldsymbol{F}(T(g)\boldsymbol{\lambda},\phi),\quad g\in G. \tag{5.14}$$

Recall that ϕ serves as a bifurcation parameter, $\boldsymbol{\lambda}\in\mathbb{R}^N$ is an independent variable vector of dimension $N=n^2$ expressing a pattern of mobile population, $\boldsymbol{F}:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}^N$ is the nonlinear function, and T is the N-dimensional permutation representation in Section 4.1 of the group $G=\mathrm{D}_4\ltimes(\mathbb{Z}_n\times\mathbb{Z}_n)$.

Let $(\boldsymbol{\lambda}_{\mathrm{c}},\phi_{\mathrm{c}})$ be a critical point of multiplicity M (≥ 1), at which the Jacobian matrix of $\boldsymbol{F}$ has a rank deficiency M. The critical point $(\boldsymbol{\lambda}_{\mathrm{c}},\phi_{\mathrm{c}})$ is assumed to be G-symmetric in the sense of

$$T(g)\boldsymbol{\lambda}_{\mathrm{c}}=\boldsymbol{\lambda}_{\mathrm{c}},\quad g\in G. \tag{5.15}$$

Table 5.1: Possible square patterns for several lattice sizes n (n = 2–17)

n	(α,β)	D	Type	$G(\alpha,\beta)$	M
2	$(2,0)$	4	V	$\Sigma(2,0)$	2
	$(1,1)$	2	M	$\Sigma(1,1)$	1
3	$(3,0)$	9	V	$\Sigma(3,0)$	4
4	$(2,0)$	4	V	$\Sigma(2,0)$	2
	$(4,0)$	16	V	$\Sigma(4,0)$	4
	$(1,1)$	2	M	$\Sigma(1,1)$	1
	$(2,2)$	8	M	$\Sigma(2,2)$	4
5	$(5,0)$	25	V	$\Sigma(5,0)$	4 or 8
	$(2,1)$	5	T	$\Sigma_0(5,0)$	4
6	$(2,0)$	4	V	$\Sigma(2,0)$	2
	$(3,0)$	9	V	$\Sigma(3,0)$	4
	$(6,0)$	36	V	$\Sigma(6,0)$	4 or 8
	$(1,1)$	2	M	$\Sigma(1,1)$	1
	$(3,3)$	18	M	$\Sigma(3,3)$	4
7	$(7,0)$	49	V	$\Sigma(7,0)$	4 or 8
8	$(2,0)$	4	V	$\Sigma(2,0)$	2
	$(4,0)$	16	V	$\Sigma(4,0)$	4
	$(8,0)$	64	V	$\Sigma(8,0)$	4 or 8
	$(1,1)$	2	M	$\Sigma(1,1)$	1
	$(2,2)$	8	M	$\Sigma(2,2)$	4
	$(4,4)$	32	M	$\Sigma(4,4)$	4 or 8
9	$(3,0)$	9	V	$\Sigma(3,0)$	4
	$(9,0)$	81	V	$\Sigma(9,0)$	4 or 8
10	$(2,0)$	4	V	$\Sigma(2,0)$	2
	$(5,0)$	25	V	$\Sigma(5,0)$	4 or 8
	$(10,0)$	100	V	$\Sigma(10,0)$	4 or 8
	$(1,1)$	2	M	$\Sigma(1,1)$	1
	$(5,5)$	50	M	$\Sigma(5,5)$	4 or 8
	$(2,1)$	5	T	$\Sigma_0(2,1)$	8
	$(3,1)$	10	T	$\Sigma_0(3,1)$	8
	$(4,2)$	20	T	$\Sigma_0(4,2)$	8

n	(α,β)	D	Type	$G(\alpha,\beta)$	M
11	$(11,0)$	121	V	$\Sigma(11,0)$	4 or 8
12	$(2,0)$	4	V	$\Sigma(2,0)$	2
	$(3,0)$	9	V	$\Sigma(3,0)$	4
	$(4,0)$	16	V	$\Sigma(4,0)$	4
	$(6,0)$	36	V	$\Sigma(6,0)$	4 or 8
	$(12,0)$	144	V	$\Sigma(12,0)$	4 or 8
	$(1,1)$	2	M	$\Sigma(1,1)$	1
	$(2,2)$	8	M	$\Sigma(2,2)$	4
	$(3,3)$	18	M	$\Sigma(3,3)$	4
	$(6,6)$	72	M	$\Sigma(6,6)$	4 or 8
13	$(13,0)$	169	V	$\Sigma(13,0)$	4 or 8
	$(3,2)$	13	T	$\Sigma_0(3,2)$	8
14	$(2,0)$	4	V	$\Sigma(2,0)$	2
	$(7,0)$	49	V	$\Sigma(7,0)$	4 or 8
	$(14,0)$	196	V	$\Sigma(14,0)$	4 or 8
	$(1,1)$	2	M	$\Sigma(1,1)$	1
	$(7,7)$	98	M	$\Sigma(7,7)$	8 or 8
15	$(3,0)$	9	V	$\Sigma(3,0)$	4
	$(5,0)$	25	V	$\Sigma(5,0)$	4 or 8
	$(15,0)$	225	V	$\Sigma(15,0)$	4 or 8
	$(2,1)$	5	T	$\Sigma_0(2,1)$	8
	$(6,3)$	45	T	$\Sigma_0(6,3)$	8
16	$(2,0)$	4	V	$\Sigma(2,0)$	2
	$(4,0)$	16	V	$\Sigma(4,0)$	4
	$(8,0)$	64	V	$\Sigma(8,0)$	4 or 8
	$(16,0)$	256	V	$\Sigma(16,0)$	4 or 8
	$(1,1)$	2	M	$\Sigma(1,1)$	1
	$(2,2)$	8	M	$\Sigma(2,2)$	4
	$(4,4)$	32	M	$\Sigma(4,4)$	4 or 8
	$(8,8)$	128	M	$\Sigma(8,8)$	4 or 8
17	$(17,0)$	289	V	$\Sigma(17,0)$	4 or 8
	$(4,1)$	17	T	$\Sigma_0(4,1)$	8

Table 5.2: Possible square patterns for several lattice sizes n (n = 18–30)

n	(α, β)	D	Type	$G(\alpha, \beta)$	M
18	$(2,0)$	4	V	$\Sigma(2,0)$	2
	$(3,0)$	9	V	$\Sigma(3,0)$	4
	$(6,0)$	36	V	$\Sigma(6,0)$	4 or 8
	$(9,0)$	81	V	$\Sigma(9,0)$	4 or 8
	$(18,0)$	324	V	$\Sigma(18,0)$	4 or 8
	$(1,1)$	2	M	$\Sigma(1,1)$	1
	$(3,3)$	18	M	$\Sigma(3,3)$	4
	$(9,9)$	162	M	$\Sigma(9,9)$	4 or 8
19	$(19,0)$	361	V	$\Sigma(19,0)$	4 or 8
20	$(2,0)$	4	V	$\Sigma(2,0)$	2
	$(4,0)$	16	V	$\Sigma(4,0)$	4
	$(5,0)$	25	V	$\Sigma(5,0)$	4 or 8
	$(10,0)$	100	V	$\Sigma(10,0)$	4 or 8
	$(20,0)$	400	V	$\Sigma(20,0)$	4 or 8
	$(1,1)$	2	M	$\Sigma(1,1)$	1
	$(2,2)$	8	M	$\Sigma(2,2)$	4
	$(5,5)$	50	M	$\Sigma(5,5)$	4 or 8
	$(10,10)$	200	M	$\Sigma(10,10)$	4 or 8
	$(2,1)$	5	T	$\Sigma_0(2,1)$	8
	$(3,1)$	10	T	$\Sigma_0(3,1)$	8
	$(4,2)$	20	T	$\Sigma_0(4,2)$	8
	$(6,2)$	40	T	$\Sigma_0(6,2)$	8
	$(8,4)$	80	T	$\Sigma_0(8,4)$	8
21	$(3,0)$	9	V	$\Sigma(3,0)$	4
	$(7,0)$	49	V	$\Sigma(7,0)$	4 or 8
	$(21,0)$	441	V	$\Sigma(21,0)$	4 or 8
22	$(2,0)$	4	V	$\Sigma(2,0)$	2
	$(11,0)$	121	V	$\Sigma(11,0)$	4 or 8
	$(22,0)$	484	V	$\Sigma(22,0)$	4 or 8
	$(1,1)$	2	M	$\Sigma(1,1)$	1
	$(11,11)$	242	M	$\Sigma(11,11)$	4 or 8
23	$(23,0)$	529	V	$\Sigma(23,0)$	4 or 8
24	$(2,0)$	4	V	$\Sigma(2,0)$	2
	$(3,0)$	9	V	$\Sigma(3,0)$	4
	$(4,0)$	16	V	$\Sigma(4,0)$	4
	$(6,0)$	36	V	$\Sigma(6,0)$	4 or 8
	$(12,0)$	144	V	$\Sigma(12,0)$	4 or 8
	$(24,0)$	576	V	$\Sigma(24,0)$	4 or 8
	$(1,1)$	2	M	$\Sigma(1,1)$	1
	$(2,2)$	8	M	$\Sigma(2,2)$	4
	$(3,3)$	18	M	$\Sigma(3,3)$	4
	$(4,4)$	32	M	$\Sigma(4,4)$	4 or 8
	$(6,6)$	72	M	$\Sigma(6,6)$	4 or 8
	$(12,12)$	288	M	$\Sigma(12,12)$	4 or 8
25	$(5,0)$	25	V	$\Sigma(5,0)$	4 or 8
	$(25,0)$	625	V	$\Sigma(25,0)$	4 or 8
	$(2,1)$	5	T	$\Sigma(2,1)$	8
	$(4,3)$	25	T	$\Sigma(4,3)$	8
	$(10,5)$	125	T	$\Sigma(10,5)$	8
26	$(2,0)$	4	V	$\Sigma(2,0)$	2
	$(13,0)$	169	V	$\Sigma(13,0)$	4 or 8
	$(26,0)$	676	V	$\Sigma(26,0)$	4 or 8
	$(1,1)$	2	M	$\Sigma(1,1)$	1
	$(13,13)$	338	M	$\Sigma(13,13)$	4 or 8
	$(3,2)$	13	T	$\Sigma_0(3,2)$	8
	$(5,1)$	26	T	$\Sigma_0(5,1)$	8
	$(6,4)$	52	T	$\Sigma_0(6,4)$	8
27	$(3,0)$	9	V	$\Sigma(3,0)$	4
	$(9,0)$	81	V	$\Sigma(9,0)$	4 or 8
	$(27,0)$	729	V	$\Sigma(27,0)$	4 or 8
28	$(2,0)$	4	V	$\Sigma(2,0)$	2
	$(4,0)$	16	V	$\Sigma(4,0)$	4
	$(7,0)$	49	V	$\Sigma(7,0)$	4 or 8
	$(14,0)$	196	V	$\Sigma(14,0)$	4 or 8
	$(28,0)$	784	V	$\Sigma(28,0)$	4 or 8
	$(1,1)$	2	M	$\Sigma(1,1)$	1
	$(2,2)$	8	M	$\Sigma(2,2)$	4
	$(7,7)$	98	M	$\Sigma(7,7)$	4 or 8
	$(14,14)$	392	M	$\Sigma(14,14)$	4 or 8
29	$(29,0)$	841	V	$\Sigma(29,0)$	4 or 8
	$(5,2)$	29	T	$\Sigma_0(5,2)$	8
30	$(2,0)$	4	V	$\Sigma(2,0)$	2
	$(3,0)$	9	V	$\Sigma(3,0)$	4
	$(5,0)$	25	V	$\Sigma(5,0)$	4 or 8
	$(6,0)$	36	V	$\Sigma(6,0)$	4 or 8
	$(10,0)$	100	V	$\Sigma(10,0)$	4 or 8
	$(15,0)$	225	V	$\Sigma(15,0)$	4 or 8
	$(30,0)$	900	V	$\Sigma(30,0)$	4 or 8
	$(1,1)$	2	M	$\Sigma(1,1)$	1
	$(3,3)$	18	M	$\Sigma(3,3)$	4
	$(5,5)$	50	M	$\Sigma(5,5)$	4 or 8
	$(15,15)$	450	M	$\Sigma(15,15)$	4 or 8
	$(2,1)$	5	T	$\Sigma_0(2,1)$	8
	$(3,1)$	10	T	$\Sigma_0(3,1)$	8
	$(4,2)$	20	T	$\Sigma_0(4,2)$	8
	$(6,3)$	45	T	$\Sigma_0(6,3)$	8
	$(9,3)$	90	T	$\Sigma_0(9,3)$	8
	$(12,6)$	180	T	$\Sigma_0(12,6)$	8

Moreover, it is assumed to be *group-theoretic*, which means, by definition, that the M-dimensional kernel space of the Jacobian matrix at $(\boldsymbol{\lambda}_{\rm c}, \phi_{\rm c})$ is irreducible with respect to the representation T. Then the critical point $(\boldsymbol{\lambda}_{\rm c}, \phi_{\rm c})$ is associated with an irreducible representation μ of G, and the multiplicity M corresponds to the dimension of the irreducible representation μ. We denote the representation matrix for μ by $T^{\mu}(g)$.

By the Liapunov–Schmidt reduction with symmetry,[6] the full system of equilibrium equations (5.13) is reduced, in a neighborhood of the critical point $(\boldsymbol{\lambda}_{\rm c}, \phi_{\rm c})$, to a system of bifurcation equations

$$\widetilde{\boldsymbol{F}}(\boldsymbol{w}, \widetilde{\phi}) = \boldsymbol{0} \tag{5.16}$$

in $\boldsymbol{w} \in \mathbb{R}^M$, where $\widetilde{\boldsymbol{F}} : \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^M$ is a function and $\widetilde{\phi} = \phi - \phi_{\rm c}$ denotes the increment of ϕ. In this reduction process, the equivariance (5.14) of the full system is inherited by the reduced system (5.16). With the use of the representation matrix $T^{\mu}(g)$ for the associated irreducible representation μ, the equivariance of the bifurcation equation can be expressed as

$$T^{\mu}(g)\widetilde{\boldsymbol{F}}(\boldsymbol{w}, \widetilde{\phi}) = \widetilde{\boldsymbol{F}}(T^{\mu}(g)\boldsymbol{w}, \widetilde{\phi}), \quad g \in G. \tag{5.17}$$

This inherited symmetry plays a key role in determining the symmetry of bifurcating solutions.

The reduced equation (5.16) can possibly admit multiple solutions $\boldsymbol{w} = \boldsymbol{w}(\widetilde{\phi})$ with $\boldsymbol{w}(0) = \boldsymbol{0}$, since $(\boldsymbol{w}, \widetilde{\phi}) = (\boldsymbol{0}, 0)$ is a singular point of (5.16). This gives rise to bifurcation. Each $\boldsymbol{w}$ uniquely determines a solution $\boldsymbol{\lambda}$ to the full system (5.13), and moreover the symmetry of $\boldsymbol{w}$ is identical with that of $\boldsymbol{\lambda}$. Indeed, we have the following relation:

$$G^{\mu} \subseteq \Sigma^{\mu}(\boldsymbol{w}) = \Sigma(\boldsymbol{\lambda}), \tag{5.18}$$

where G^{μ} is a subgroup of G as

$$G^{\mu} = \{g \in G \mid T^{\mu}(g) = I\}, \tag{5.19}$$

and $\Sigma(\boldsymbol{\lambda})$ and $\Sigma^{\mu}(\boldsymbol{w})$ are isotropy subgroups defined respectively as

$$\Sigma(\boldsymbol{\lambda}) = \Sigma(\boldsymbol{\lambda}; G, T) = \{g \in G \mid T(g)\boldsymbol{\lambda} = \boldsymbol{\lambda}\}, \tag{5.20}$$
$$\Sigma^{\mu}(\boldsymbol{w}) = \Sigma(\boldsymbol{w}; G, T^{\mu}) = \{g \in G \mid T^{\mu}(g)\boldsymbol{w} = \boldsymbol{w}\}. \tag{5.21}$$

The significance of the relation (5.18) is twofold. First, unless a subgroup Σ is large enough to contain G^{μ}, no bifurcating solution $\boldsymbol{\lambda}$ exists such that $\Sigma = \Sigma(\boldsymbol{\lambda})$. Second, the symmetry of a bifurcating solution $\boldsymbol{\lambda}$ is known as $\Sigma(\boldsymbol{\lambda}) = \Sigma^{\mu}(\boldsymbol{w})$ through the analysis of the bifurcation equation in $\boldsymbol{w}$.

Remark 5.1. We define the variables $\boldsymbol{w} = (w_1, \dots, w_M)^\top$ in the bifurcation equation (5.16) with the matrix Q derived in Section 4.3. That is, the components of $\boldsymbol{w} = (w_1, \dots, w_M)^\top$ are assumed to correspond to the column vectors of $Q^{\mu} = [\boldsymbol{q}_1^{\mu}, \dots, \boldsymbol{q}_M^{\mu}]$. Then, the equivariance condition (5.17) holds for the matrix representations T^{μ} of the irreducible representations μ derived in Section 3.2. □

[6] For more details on the Liapunov–Schmidt reduction, see Sattinger, 1979 [25] and Golubitsky et al., 1988 [26].

Table 5.3: Irreducible representations of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ to be considered in bifurcation analysis

$n \setminus d$	1	2	4	8
$2m$	$(1;+,+,+), (1;+,+,-)$	$(2;+,+)$	$(4;k,0;+), (4;k,k;+), (4;n/2,\ell,+)$	$(8;k,\ell)$
$2m-1$	$(1;+,+,+)$		$(4;k,0;+), (4;k,k;+)$	$(8;k,\ell)$

$(4;k,0;+)$ for k with $1 \leq k \leq \lfloor (n-1)/2 \rfloor$;
$(4;k,k;+)$ for k with $1 \leq k \leq \lfloor (n-1)/2 \rfloor$;
$(4;n/2,\ell;+)$ for k with $1 \leq \ell \leq \lfloor (n-1)/2 \rfloor$;
$(8;k,\ell)$ for (k,ℓ) with $1 \leq \ell \leq k-1,\ 2 \leq k \leq \lfloor (n-1)/2 \rfloor$

5.2.2. Use of Equivariant Branching Lemma

Equivariant branching lemma is a useful mathematical means to prove the existence of a bifurcating solution with a specified symmetry without actually solving the bifurcation equation in (5.16). By the equivariant branching lemma, we shall demonstrate the emergence of square patterns.

Bifurcation Equation and the Associated Irreducible Representation

To investigate the existence of a bifurcating solution $\boldsymbol{\lambda}$ with a specified symmetry Σ to the equilibrium equation $\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{0}$ in (5.13), it suffices to apply the equivariant branching lemma to the bifurcation equation $\widetilde{\boldsymbol{F}}(\boldsymbol{w}, \widetilde{\phi})$ in (5.16). This is justified by the fact that the isotropy subgroup $\Sigma(\boldsymbol{\lambda})$ expressing the symmetry of a bifurcating solution $\boldsymbol{\lambda}$ is identical to the isotropy subgroup $\Sigma^{\mu}(\boldsymbol{w})$ of the corresponding solution $\boldsymbol{w}$ for the bifurcation equation, i.e., $\Sigma(\boldsymbol{\lambda}) = \Sigma^{\mu}(\boldsymbol{w})$ as shown in (5.18).

The bifurcation equation is associated with an irreducible representation μ of $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ as in (5.17). The associated irreducible representation μ is restricted to

$$\begin{aligned}\mu = {} & (1;+,+,+),\ (1;+,+,-),\ (2;+,+),\\ & (4;k,0;+),\ (4;k,k;+),\ (4;n/2,\ell;+),\ (8;k,\ell)\end{aligned}$$

with k for $(4;k,0;+)$ in (3.13), k for $(4;k,k;+)$ in (3.14), ℓ for $(4;n/2,\ell;+)$ in (3.15), and (k,ℓ) for $(8;k,\ell)$ in (3.25), as a consequence of the irreducible decomposition (4.11) of the permutation representation T for the economy on the $n \times n$ square lattice. The unit representation $(1;+,+,+)$ has been excluded since it does not correspond to a symmetry-breaking bifurcation point. Thus we have to deal with critical points of multiplicity $M = 1$, 2, 4, and 8. As a modified form of Table 4.3, therefore, we obtain Table 5.3, where the multiplicity M of a critical point is equal to the dimension d of the associated irreducible representation.

Isotropy Subgroup and Fixed-Point Subspace

In the analysis by the equivariant branching lemma, the isotropy subgroup of $\boldsymbol{w}$ with respect to T^{μ}:

$$\Sigma^{\mu}(\boldsymbol{w}) = \{g \in G \mid T^{\mu}(g)\boldsymbol{w} = \boldsymbol{w}\}$$

introduced in (5.21), and the fixed-point subspace of Σ for T^{μ}:

$$\mathrm{Fix}^{\mu}(\Sigma) = \{\boldsymbol{w} \in \mathbb{R}^{M} \mid T^{\mu}(g)\boldsymbol{w} = \boldsymbol{w} \text{ for all } g \in \Sigma\} \tag{5.22}$$

play the major roles. The following facts, though immediate from the definitions, are important and useful.

- By definition, Σ is an isotropy subgroup if and only if $\Sigma = \Sigma^{\mu}(\boldsymbol{w})$ for some $\boldsymbol{w} \neq \boldsymbol{0}$.
- If $\Sigma = \Sigma^{\mu}(\boldsymbol{w})$, then $\boldsymbol{w} \in \mathrm{Fix}^{\mu}(\Sigma)$ and $\dim \mathrm{Fix}^{\mu}(\Sigma) \geq 1$.
- Not every Σ with the property of $\dim \mathrm{Fix}^{\mu}(\Sigma) \geq 1$ is an isotropy subgroup.
- $\Sigma \subseteq \Sigma^{\mu}(\boldsymbol{w})$ for every $\boldsymbol{w} \in \mathrm{Fix}^{\mu}(\Sigma)$.
- Σ is an isotropy subgroup if and only if $\Sigma = \Sigma^{\mu}(\boldsymbol{w})$ for some $\boldsymbol{w} \in \mathrm{Fix}^{\mu}(\Sigma)$ with $\boldsymbol{w} \neq \boldsymbol{0}$.
- Unless Σ is an isotropy subgroup, there exists no bifurcating solution $\boldsymbol{w}$ with symmetry Σ.

Analysis Procedure Using Equivariant Branching Lemma

The analysis for the $n \times n$ square lattice based on the equivariant branching lemma follows the steps below.

1. Specify an irreducible representation μ of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ in Table 5.3.
2. Specify a subgroup Σ as a candidate of an isotropy subgroup of a possible bifurcating solution.
3. Obtain the fixed-point subspace $\mathrm{Fix}^{\mu}(\Sigma)$ in (5.22) for the subgroup Σ with respect to the irreducible representation μ.
4. Search for some $\boldsymbol{w} \in \mathrm{Fix}^{\mu}(\Sigma)$ such that $\Sigma^{\mu}(\boldsymbol{w}) = \Sigma$. If no such $\boldsymbol{w}$ exists, then Σ is not an isotropy subgroup, and hence there exists no solution with the specified symmetry Σ for the bifurcation equation associated with μ. If such $\boldsymbol{w}$ exists, then we can ensure that Σ is an isotropy subgroup, and can proceed to the next step.
5. Calculate the dimension $\dim \mathrm{Fix}^{\mu}(\Sigma)$ of the fixed-point subspace.
6. If $\dim \mathrm{Fix}^{\mu}(\Sigma) = 1$, a bifurcating solution with symmetry Σ is guaranteed to exist generically by the equivariant branching lemma. If $\dim \mathrm{Fix}^{\mu}(\Sigma) \geq 2$, no definite conclusion can be reached by means of the equivariant branching lemma.

Remark 5.2. The equivariant branching lemma assumes two technical conditions: i) absolute irreducibility and ii) genericity (see Section 2.4.5 of Ikeda et al., 2014 [8]). The former condition is satisfied by the group $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ since all the irreducible representations over $\mathbb{R}$ of this group are absolutely irreducible (see Section 3.2). The latter condition is a matter of modeling, and we assume this condition throughout this paper. For more details on the equivariant branching lemma, see Cicogna, 1981 [27], Vanderbauwhede, 1982 [28], and Golubitsky et al., 1988 [26]. □

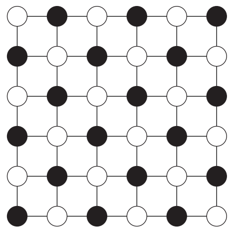


Figure 5.1: Pattern on the 6×6 square lattice expressed by the column vector of $Q^{(1;+,+,-)}$. A black circle denotes a positive component and a white circle denotes a negative component.

5.3. *Bifurcation Point of Multiplicity 1*

As shown by Table 5.3 in Section 5.2.2, a critical point of multiplicity 1 is associated with the two-dimensional irreducible representation $(1;+,+,-)$, which exists only when n is even. Recall from (3.4) that this irreducible representation is given by

$$T^{(1;+,+,-)}(r) = 1, \quad T^{(1;+,+,-)}(s) = 1, \quad T^{(1;+,+,-)}(p_1) = -1, \quad T^{(1;+,+,-)}(p_2) = -1. \tag{5.23}$$

In view of Remark 5.1 in Section 5.2.1, let us assume that the variable $\boldsymbol{w} = w$ for the bifurcation equation (5.16) corresponds to the column vectors of

$$Q^{(1;+,+,-)} = [\boldsymbol{q}] = [\ \langle \cos(\pi(n_1 - n_2)) \rangle\ ] \tag{5.24}$$

in (4.22). The spatial pattern for this vector is depicted in Fig. 5.1 for $n = 6$. This is the smallest square pattern.

Proposition 5.3. *When n is even, a bifurcating solution in the direction of $\boldsymbol{q}$ with the symmetry of $\langle r, s, p_1p_2, p_1^{-1}p_2\rangle$ arises from a critical point of multiplicity 1 associated with the irreducible representation* $(1;+,+,-)$.

Proof. The general procedure in Section 5.2.2 is applied to $\mu = (1;+,+,-)$ and $\Sigma = \langle r, s\rangle \ltimes \langle p_1p_2, p_1^{-1}p_2\rangle$. We have

$$\mathrm{Fix}^{(1;+,+,-)}(\Sigma) = \{w \in \mathbb{R}\}$$

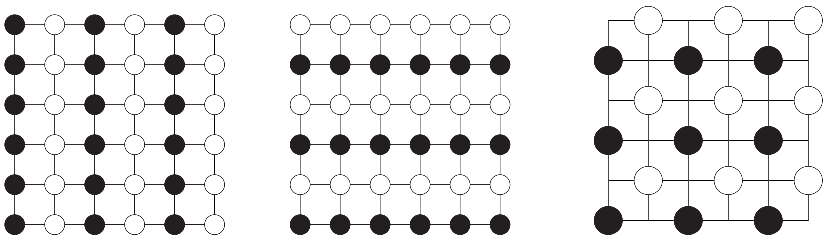
since

$$T^{(1;+,+,-)}(r)w = w, \quad T^{(1;+,+,-)}(s)w = w, \quad T^{(1;+,+,-)}(p_1p_2)w = w, \quad T^{(1;+,+,-)}(p_1^{-1}p_2)w = w$$

by (5.23). Thus the targeted symmetry Σ is an isotropy subgroup with

$$\dim \mathrm{Fix}^{(1;+,+,-)}(\Sigma) = 1.$$

The equivariant branching lemma then guarantees the existence of a bifurcating path with symmetry Σ. $\square$



(a) $\boldsymbol{q}_1$ $(\langle r^2, s, p_1^2, p_2\rangle)$ (b) $\boldsymbol{q}_2$ $(\langle r^2, s, p_1, p_2^2\rangle)$ (c) $\boldsymbol{q}_1 + \boldsymbol{q}_2$ $(\langle r, s, p_1^2, p_2^2\rangle)$

Figure 5.2: Patterns on the 6×6 square lattice expressed by the column vectors of $Q^{(2;+,+)}$. A black circle denotes a positive component and a white circle denotes a negative component.

5.4. Bifurcation Point of Multiplicity 2

As shown by Table 5.3 in Section 5.2.2, a critical point of multiplicity 2 is associated with the two-dimensional irreducible representation $(2;+,+)$, which exists only when n is even. Recall from (3.9) and (3.10) that this irreducible representation is given by

$$T^{(2;+,+)}(r) = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \quad T^{(2;+,+)}(s) = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \tag{5.25}$$

$$T^{(2;+,+)}(p_1) = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}, \quad T^{(2;+,+)}(p_2) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}. \tag{5.26}$$

In view of Remark 5.1 in Section 5.2.1, let us assume that the variable $\boldsymbol{w} = (w_1, w_2)^\top$ for the bifurcation equation (5.16) corresponds to the column vectors of

$$Q^{(2;+,+)} = [\boldsymbol{q}_1, \boldsymbol{q}_2] = [\ \langle\cos(\pi n_1)\rangle,\ \langle\cos(\pi n_2)\rangle\ ] \tag{5.27}$$

in (4.23). The spatial patterns for these vectors are depicted in Fig. 5.2 for $n = 6$. The vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ represent stripe patterns but $\boldsymbol{q}_1 + \boldsymbol{q}_2$ expresses a square pattern.

Proposition 5.4. *When n is even, bifurcating solutions from a critical point of multiplicity 2 associated with the irreducible representation* $(2;+,+)$ *exist in the following directions:*
(i) $\boldsymbol{q}_1 + \boldsymbol{q}_2$ *with the symmetry of* $\langle r, s, p_1^2, p_2^2\rangle$,
(ii) $\boldsymbol{q}_1$ *with the symmetry of* $\langle r^2, s, p_1^2, p_2\rangle$, *and*
(iii) $\boldsymbol{q}_2$ *with the symmetry of* $\langle r^2, s, p_1, p_2^2\rangle$.

Proof. (i) The general procedure in Section 5.2.2 is applied to $\mu = (2;+,+)$ and $\Sigma = \langle r, s\rangle \ltimes \langle p_1^2, p_2^2\rangle$. Note

$$\mathrm{Fix}^{(2;+,+)}(\Sigma) = \mathrm{Fix}^{(2;+,+)}(\langle r\rangle) \ \cap\ \mathrm{Fix}^{(2;+,+)}(\langle s, p_1^2, p_2^2\rangle).$$

Here we have

$$\mathrm{Fix}^{(2;+,+)}(\langle r\rangle) = \{c(1,1)^\top \mid c \in \mathbb{R}\}$$

since $T^{(2;+,+)}(r)(w_1, w_2)^\top = (w_2, w_1)^\top$ by (5.25), whereas

$$\mathrm{Fix}^{(2;+,+)}(\langle s, p_1^2, p_2^2\rangle) = \mathbb{R}^2$$

since $T^{(2;+,+)}(s) = T^{(2;+,+)}(p_1^2) = T^{(2;+,+)}(p_2^2) = I$ by (5.25) and (5.26). Therefore,

$$\mathrm{Fix}^{(2;+,+)}(\Sigma) = \{c(1,1)^\top \mid c \in \mathbb{R}\},$$

that is, $\Sigma = \Sigma^{(2;+,+)}(\boldsymbol{w}_0)$ for $\boldsymbol{w}_0 = (1,1)^\top$. Thus the targeted symmetry Σ is an isotropy subgroup with

$$\dim \mathrm{Fix}^{(2;+,+)}(\Sigma) = 1.$$

The equivariant branching lemma then guarantees the existence of a bifurcating path with symmetry Σ.

(ii) Next the general procedure is applied to $\mu = (2;+,+)$ and $\Sigma = \langle r^2, s, p_1^2, p_2\rangle$. Note

$$\mathrm{Fix}^{(2;+,+)}(\Sigma) = \mathrm{Fix}^{(2;+,+)}(\langle p_2\rangle) \ \cap\ \mathrm{Fix}^{(2;+,+)}(\langle r^2, s, p_1^2\rangle).$$

Here we have

$$\mathrm{Fix}^{(2;+,+)}(\langle p_2\rangle) = \{c(1,0)^\top \mid c \in \mathbb{R}\}$$

since $T^{(2;+,+)}(p_2)(w_1, w_2)^\top = (w_1, -w_2)^\top$ by (5.25), whereas

$$\mathrm{Fix}^{(2;+,+)}(\langle r^2, s, p_1^2\rangle) = \mathbb{R}^2$$

since $T^{(2;+,+)}(r^2) = T^{(2;+,+)}(s) = T^{(2;+,+)}(p_1^2) = I$ by (5.25) and (5.26). Therefore,

$$\mathrm{Fix}^{(2;+,+)}(\Sigma) = \{c(1,0)^\top \mid c \in \mathbb{R}\},$$

that is, $\Sigma = \Sigma^{(2;+,+)}(\boldsymbol{w}_0)$ for $\boldsymbol{w}_0 = (1,0)^\top$. Thus the targeted symmetry Σ is an isotropy subgroup with

$$\dim \mathrm{Fix}^{(2;+,+)}(\Sigma) = 1.$$

The equivariant branching lemma then guarantees the existence of a bifurcating path with symmetry Σ. The case of (iii) can be treated similarly. □

5.5. Bifurcation Point of Multiplicity 4

Square patterns branching from bifurcation points of multiplicity 4 are investigated.

5.5.1. Representation in Complex Variables

As shown by Table 5.3 in Section 5.2.2, a critical point of multiplicity 4 is associated with one of the four-dimensional irreducible representations

$$(4;k,0,+) \ \text{with} \ 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{5.28}$$

$$(4;k,k,+) \ \text{with} \ 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{5.29}$$

$$(4;n/2,\ell,+) \ \text{with} \ 1 \le \ell \le \frac{n}{2} - 1, \tag{5.30}$$

where $n \ge 3$ and $(4;n/2,\ell,+)$ exists only when n is even.

The irreducible representation $(4; k, 0, +)$ is given by

$$T^{(4;k,0,+)}(r) = \begin{bmatrix} & S \\ I & \end{bmatrix}, \quad T^{(4;k,0,+)}(s) = \begin{bmatrix} I & \\ & S \end{bmatrix}, \tag{5.31}$$

$$T^{(4;k,0,+)}(p_1) = \begin{bmatrix} R^k & \\ & I \end{bmatrix}, \quad T^{(4;k,0,+)}(p_2) = \begin{bmatrix} I & \\ & R^k \end{bmatrix}, \tag{5.32}$$

where

$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}.$$

The irreducible representation $(4; k, k, +)$ is given by

$$T^{(4;k,k,+)}(r) = \begin{bmatrix} & S \\ I & \end{bmatrix}, \quad T^{(4;k,k,+)}(s) = \begin{bmatrix} & S \\ S & \end{bmatrix}, \tag{5.33}$$

$$T^{(4;k,k,+)}(p_1) = \begin{bmatrix} R^k & \\ & R^{-k} \end{bmatrix}, \quad T^{(4;k,k,+)}(p_2) = \begin{bmatrix} R^k & \\ & R^k \end{bmatrix}. \tag{5.34}$$

The irreducible representation $(4; n/2, \ell, +)$ is given by

$$T^{(4;n/2,\ell,+)}(r) = \begin{bmatrix} & S \\ I & \end{bmatrix}, \quad T^{(4;n/2,\ell,+)}(s) = \begin{bmatrix} S & \\ & I \end{bmatrix}, \tag{5.35}$$

$$T^{(4;n/2,\ell,+)}(p_1) = \begin{bmatrix} -I & \\ & R^{-\ell} \end{bmatrix}, \quad T^{(4;n/2,\ell,+)}(p_2) = \begin{bmatrix} R^\ell & \\ & -I \end{bmatrix}. \tag{5.36}$$

Let us assume that, for $(4; k, 0, +)$, the variable $\boldsymbol{w} = (w_1, w_2, w_3, w_4)^\top$ for the bifurcation equation (5.16) corresponds to the column vectors of

$$Q^{(4;k,0,+)} = [\, \langle \cos(2\pi k\, n_1/n) \rangle,\ \langle \sin(2\pi k\, n_1/n) \rangle,\ \langle \cos(2\pi k n_2/n) \rangle,\ \langle \sin(2\pi k n_2/n) \rangle \,] \tag{5.37}$$

in (4.24). The variables $\boldsymbol{w}$ for $(4; k, k, +)$ and $(4; n/2, \ell, +)$ can be defined similarly. Examples of the spatial patterns for these vectors are depicted in Fig. 5.3 for $n = 6$.

Using complex variables

$$(z_1, z_2) = (w_1 + \mathrm{i}w_2, w_3 + \mathrm{i}w_4),$$

we can express the actions in $(4; k, 0, +)$, given in (5.31) and (5.32) for the 4-dimensional vectors $(w_1, \ldots, w_4)$, as

$$\begin{aligned} r &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_2 \\ z_1 \end{bmatrix}, & s &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ \bar{z}_2 \end{bmatrix}, \\ p_1 &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ z_2 \end{bmatrix}, & p_2 &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ \omega^k z_2 \end{bmatrix}, \end{aligned} \tag{5.38}$$

where $\omega = \exp(\mathrm{i}2\pi/n)$. The actions in $(4; k, k, +)$, given in (5.33) and (5.34), are

$$\begin{aligned} r &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_2 \\ z_1 \end{bmatrix}, & s &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_2 \\ \bar{z}_1 \end{bmatrix}, \\ p_1 &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^{-k} z_2 \end{bmatrix}, & p_2 &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^k z_2 \end{bmatrix}. \end{aligned} \tag{5.39}$$

The actions in $(4; n/2, \ell, +)$, given in (5.35) and (5.36), are

$$
\begin{aligned}
&r: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z}_2 \\ z_1 \end{bmatrix}, \qquad s: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z}_1 \\ z_2 \end{bmatrix}, \\
&p_1: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} -z_1 \\ \omega^{-\ell} z_2 \end{bmatrix}, \quad p_2: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^{\ell} z_1 \\ -z_2 \end{bmatrix}.
\end{aligned}
\tag{5.40}
$$

The actions of p_1 and p_2 in $(4; k, \ell, +)$ are expressed in a unified form as

$$
p_1: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^{k} z_1 \\ \omega^{-\ell} z_2 \end{bmatrix}, \quad p_2: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^{\ell} z_1 \\ \omega^{k} z_2 \end{bmatrix}. \tag{5.41}
$$

5.5.2. Isotropy Subgroups

To apply the method of analysis in Section 5.2.2, we identify isotropy subgroups for $(4; k, 0, +)$, $(4; k, k, +)$, and $(4; n/2, \ell, +)$ that are relevant to square patterns. We denote the isotropy subgroup of $z = (z_1, z_2)$ and the fixed-point subspace of Σ with respect to $T^{(4;k,\ell,+)}$ with $\ell \in \{0, k\}$ as

$$
\Sigma^{(4;k,\ell,+)}(z) = \{g \in G \mid T^{(4;k,\ell,+)}(g) \cdot z = z\}, \tag{5.42}
$$

$$
\mathrm{Fix}^{(4;k,\ell,+)}(\Sigma) = \{z \mid T^{(4;k,\ell,+)}(g) \cdot z = z \text{ for all } g \in \Sigma\}, \tag{5.43}
$$

where $T^{(4;k,\ell,+)}(g) \cdot z$ means the action of $g \in G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ on z given in (5.38) and (5.39). We also define

$$
\check{n} = \frac{n}{\gcd(n,k)}, \quad \check{k} = \frac{k}{\gcd(n,k)}, \quad \tilde{n} = \frac{n}{\gcd(n,\ell)}, \quad \tilde{\ell} = \frac{\ell}{\gcd(n,\ell)}, \tag{5.44}
$$

where $\gcd(\cdot,\cdot)$ means the greatest common divisor of the integers therein.

The symmetries of $\langle r \rangle$ and $\langle r, s \rangle$ and the translational symmetry of $p_1^a p_2^b$ are dealt with in Propositions 5.5, 5.6, and 5.7 below. In this connection, the isotropy subgroups of $z = (z_1, z_2) = (1, 1)$ (i.e., $\boldsymbol{w} = (1, 0, 1, 0)^\top$) play a crucial role. Remark 5.3 given later should be consulted with regard to the geometrical interpretation of the propositions below.

Proposition 5.5. *For* $(4; k, 0, +)$ *in* (3.13)*, we have the following statements:*

(i) $\mathrm{Fix}^{(4;k,0,+)}(\langle r \rangle) = \mathrm{Fix}^{(4;k,0,+)}(\langle r, s \rangle) = \{c(1,1) \mid c \in \mathbb{R}\}$ *for each* k*.*

(ii) $p_1^a p_2^b \in \Sigma^{(4;k,0,+)}((1,1))$ *if and only if*

$$
\check{k}a \equiv 0, \quad \check{k}b \equiv 0 \mod \check{n}. \tag{5.45}
$$

(iii) $\Sigma^{(4;k,0,+)}((1,1)) = \Sigma(\check{n}, 0)$ *and* $\mathrm{Fix}^{(4;k,0,+)}(\Sigma(\check{n},0)) = \{c(1,1) \mid c \in \mathbb{R}\}$*. That is,* $\Sigma(\check{n}, 0)$ *is the isotropy subgroup of* $z = (1,1)$ *with* $\dim \mathrm{Fix}^{(4;k,0,+)}(\Sigma(\check{n},0)) = 1$*.*

(iv) *If* $\Sigma(\alpha,\beta)$ *is an isotropy subgroup (for some* z*), then* $(\alpha,\beta) = (\check{n}, 0)$ *and it is the isotropy subgroup of* $z = (1,1)$*.*

(v) $\Sigma_0(\alpha,\beta)$ *is not an isotropy subgroup (for any* z*) for any value of* (α,β)*.*

Proof. (i) By (5.38), $z = (z_1, z_2)$ is invariant to r if and only if $(\overline{z}_2, z_1) = (z_1, z_2)$, which is equivalent to $z_1 = z_2 \in \mathbb{R}$. Such z is also invariant to s.

(ii) By (5.41) for $(4; k, \ell, +)$, the invariance of $z = (1, 1)$ to $p_1^a p_2^b$ is expressed as

$$ka + \ell b \equiv 0, \quad -\ell a + kb \equiv 0 \mod n, \tag{5.46}$$

For $\ell = 0$, this condition reduces to

$$ka \equiv 0, \quad kb \equiv 0 \mod n,$$

which is equivalent to (5.45).

(iii) (a, b) satisfies (5.45) if and only if both a and b are multiples of $\check{n}$. The subgroup of G generated by $p_1^a p_2^b$ for such (a, b), together with r and s, coincides with $\Sigma(\check{n}, 0)$.

(iv) This follows from (i) and (iii).

(v) This follows from (v). □

Proposition 5.6. *For* $(4; k, k, +)$ *in* (3.14)*, we have the following statements:*

(i) $\mathrm{Fix}^{(4;k,k,+)}(\langle r \rangle) = \mathrm{Fix}^{(4;k,k,+)}(\langle r, s \rangle) = \{c(1, 1) \mid c \in \mathbb{R}\}$ *for each* k*.*

(ii) $p_1^a p_2^b \in \Sigma^{(4;k,k,+)}((1, 1))$ *if and only if*

$$\check{k}(a + b) \equiv 0, \quad \check{k}(-a + b) \equiv 0 \mod \check{n}. \tag{5.47}$$

(iii) *If* $\check{n}$ *is even, then we have*

$$\Sigma^{(4;k,k,+)}((1, 1)) = \Sigma(\check{n}/2, \check{n}/2),$$
$$\mathrm{Fix}^{(4;k,k,+)}(\Sigma(\check{n}/2, \check{n}/2)) = \{c(1, 1) \mid c \in \mathbb{R}\};$$

that is, $\Sigma(\check{n}/2, \check{n}/2)$ *is the isotropy subgroup of* $z = (1, 1)$ *with* $\dim \mathrm{Fix}^{(4;k,k,+)}(\Sigma(\check{n}/2, \check{n}/2)) = 1$*. If* $\check{n}$ *is odd, then we have*

$$\Sigma^{(4;k,k,+)}((1, 1)) = \Sigma(\check{n}, 0),$$
$$\mathrm{Fix}^{(4;k,k,+)}(\Sigma(\check{n}, 0)) = \{c(1, 1) \mid c \in \mathbb{R}\};$$

that is, $\Sigma(\check{n}, 0)$ *is the isotropy subgroup of* $z = (1, 1)$ *with* $\dim \mathrm{Fix}^{(4;k,k,+)}(\Sigma(\check{n}, 0)) = 1$*.*

(iv) *If* $\Sigma(\alpha, \beta)$ *is an isotropy subgroup (for some* z*), then*

$$(\alpha, \beta) = \begin{cases} (\check{n}/2, \check{n}/2) & \text{if } \check{n} \text{ is even,} \\ (\check{n}, 0) & \text{if } \check{n} \text{ is odd.} \end{cases}$$

(v) $\Sigma_0(\alpha, \beta)$ *is not an isotropy subgroup (for any* z*) for any value of* (α, β)*.*

Proof. (i) By (5.39), $z = (z_1, z_2)$ is invariant to r if and only if $(\overline{z}_2, z_1) = (z_1, z_2)$, which is equivalent to $z_1 = z_2 \in \mathbb{R}$. Such z is also invariant to s.

(ii) The condition (5.46) for $\ell = k$ reduces to

$$k(a + b) \equiv 0, \quad k(-a + b) \equiv 0 \mod n,$$

which is equivalent to (5.47).

(iii) The condition (5.47) is equivalent to the existence of integers p and q such that

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \check{n}\begin{bmatrix} p \\ q \end{bmatrix}.$$

Hence a and b satisfy (5.47) if and only if they are integers expressed as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \check{n}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} p \\ q \end{bmatrix} = \frac{\check{n}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix}$$

for some integers p and q. When $\check{n}$ is odd, this is equivalent to $(a, b) = \check{n}(p', q')$ for integers p' and q'. Therefore, the subgroup of G generated by $p_1^a p_2^b$ with such (a, b), together with r and s, is given by $\Sigma(\check{n}, 0)$ with $(p', q') = (1, 0)$ or $\Sigma(\check{n}/2, \check{n}/2)$ with $(p, q) = (1, 0)$ according to whether $\check{n}$ is odd or even .

(iv) This follows from (i) and (iii).

(v) This follows from (i). □

Proposition 5.7. *For* $(4; n/2, \ell, +)$ *in* (3.15)*, we have the following statements.*

(i) $\mathrm{Fix}^{(4;n/2,\ell,+)}(\langle r\rangle) = \mathrm{Fix}^{(4;n/2,\ell,+)}(\langle r, s\rangle) = \{c(1, 1) \mid c \in \mathbb{R}\}$ *for each* ℓ*.*

(ii) $p_1^a p_2^b \in \Sigma^{(4;n/2,\ell,+)}((1, 1))$ *if and only if*

$$\frac{1}{2}\tilde{n}a + \tilde{\ell}b \equiv 0, \quad -\tilde{\ell}a + \frac{1}{2}\tilde{n}b \equiv 0 \mod \tilde{n}. \tag{5.48}$$

(iii) *If* $\tilde{n}$ *is odd, then we have*

$$\Sigma^{(4;n/2,\ell,+)}((1, 1)) = \Sigma(2\tilde{n}, 0),$$
$$\mathrm{Fix}^{(4;n/2,\ell,+)}(\Sigma(2\tilde{n}, 0)) = \{c(1, 1) \mid c \in \mathbb{R}\};$$

that is, $\Sigma(2\tilde{n}, 0)$ *is the isotropy subgroup of* $z = (1, 1)$ *with* $\dim \mathrm{Fix}^{(4;n/2,\ell,+)}(\Sigma(2\tilde{n}, 0)) = 1$*. If* $\tilde{n}$ *is even and* $\tilde{n}/2$ *is odd, then we have*

$$\Sigma^{(4;n/2,\ell,+)}((1, 1)) = \Sigma(\tilde{n}/2, \tilde{n}/2),$$
$$\mathrm{Fix}^{(4;n/2,\ell,+)}(\Sigma(\tilde{n}/2, \tilde{n}/2)) = \{c(1, 1) \mid c \in \mathbb{R}\};$$

that is, $\Sigma(\tilde{n}/2, \tilde{n}/2)$ *is the isotropy subgroup of* $z = (1, 1)$ *with* $\dim \mathrm{Fix}^{(4;n/2,\ell,+)}(\Sigma(\tilde{n}/2, \tilde{n}/2)) = 1$*. If* $\tilde{n}$ *is even and* $\tilde{n}/2$ *is even, then we have*

$$\Sigma^{(4;n/2,\ell,+)}((1, 1)) = \Sigma(\tilde{n}, 0),$$
$$\mathrm{Fix}^{(4;n/2,\ell,+)}(\Sigma(\tilde{n}, 0)) = \{c(1, 1) \mid c \in \mathbb{R}\};$$

that is, $\Sigma(\tilde{n}, 0)$ *is the isotropy subgroup of* $z = (1, 1)$ *with* $\dim \mathrm{Fix}^{(4;n/2,\ell,+)}(\Sigma(\tilde{n}, 0)) = 1$.

(iv) *If* $\Sigma(\alpha, \beta)$ *is an isotropy subgroup (for some* z*), then*

$$(\alpha, \beta) = \begin{cases} (2\tilde{n}, 0) & \text{if } \tilde{n} \text{ is odd,} \\ (\tilde{n}, 0) & \text{if } \tilde{n} \text{ is even and } \tilde{n}/2 \text{ is even,} \\ (\tilde{n}/2, \tilde{n}/2) & \text{if } \tilde{n} \text{ is even and } \tilde{n}/2 \text{ is odd.} \end{cases}$$

(v) $\Sigma_0(\alpha, \beta)$ *is not an isotropy subgroup (for any* z*) for any value of* (α, β)*.*

Proof. (i) By (5.40), $z = (z_1, z_2)$ is invariant to r if and only if $(\overline{z}_2, z_1) = (z_1, z_2)$, which is equivalent to $z_1 = z_2 \in \mathbb{R}$. Such z is also invariant to s.

(ii) The condition (5.46) for $k = n/2$ reduces to

$$\frac{n}{2}a + \ell b \equiv 0, \quad -\ell a + \frac{n}{2}b \equiv 0 \mod n,$$

which is equivalent to (5.48).

(iii) When $\tilde{n}$ is odd, (5.49) gives p and q are even, that is, $(p, q) = (2p', 2q')$ for integers p' and q'. Then, we have $(a, b) = \tilde{n}(2p', 2q') = 2\tilde{n}(p', q')$. When $\tilde{n}$ is even, from (5.48), we have $(a, b) = (\tilde{n}/2)(p, q)$ for integers p and q and this equation is rewritten as

$$\frac{\tilde{n}}{2}p + \tilde{\ell}q \equiv 0, \quad -\tilde{\ell}p + \frac{\tilde{n}}{2}q \equiv 0 \mod 2. \tag{5.49}$$

When $\tilde{n}$ is even and $\tilde{n}/2$ is even, we have $\tilde{\ell}$ odd and $(p, q) = (2p'', 2q'')$. Hence, we have $(a, b) = (\tilde{n}/2)(2p'', 2q'') = \tilde{n}(p'', q'')$ for integers p'' and q''. When $\tilde{n}$ is even and $\tilde{n}/2$ is odd ($\tilde{\ell}$ odd), we have $(a, b) = (\tilde{n}/2)(p, q)$ for $p + q$ even. Therefore, the subgroup of G generated by $p_1^a p_2^b$ with such (a, b), together with r and s, is given by $\Sigma(2\tilde{n}, 0)$ with $(p', q') = (1, 0)$, $\Sigma(\tilde{n}, 0)$ with $(p'', q'') = (1, 0)$, or $\Sigma(\tilde{n}/2, \tilde{n}/2)$ with $(p, q) = (1, 1)$, according to whether $\tilde{n}$ is odd, $\tilde{n}/2$ is even, or $\tilde{n}/2$ is odd.

(iv) This follows from (i) and (iii).

(v) This follows from (i). □

The above propositions show that, in either case of $(4; k, 0, +)$, $(4; k, k, +)$, and $(4; n/2, \ell, +)$, any isotropy subgroup Σ containing $\langle r \rangle$, which is of our interest, can be represented as $\Sigma = \Sigma^{(4;k,\ell,+)}(z)$ for $z = (1, 1)$ and that $\dim \mathrm{Fix}^{(4;k,\ell,+)}(\Sigma) = 1$. On the basis of this fact, we will investigate possible occurrences of square patterns for each of the three types V, M, and T in Sections 5.5.3–5.5.5.

Remark 5.3. The four-dimensional space of $\boldsymbol{w} = (w_1, w_2, w_3, w_4)^\top$ for the bifurcation equation (5.16) is spanned by the column vectors of

$$Q^{(4;k,\ell,+)} = [\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4], \tag{5.50}$$

the concrete form of which is given in (4.24)–(4.26). For example, the spatial patterns for these vectors with $n = 6$ are depicted in Fig. 5.3. The two vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_3$ represent stripe patterns in different directions. The sum $\boldsymbol{q}_{\mathrm{sum}} = \boldsymbol{q}_1 + \boldsymbol{q}_3$ of these two vectors, which is associated with $z = (1, 1)$, represents square patterns. □

5.5.3. Square Patterns of Type V

Square patterns of type V are here shown to branch from critical points of multiplicity 4. Recall that a square pattern of type V is characterized by the symmetry of $\Sigma(\alpha, 0)$ with $2 \leq \alpha \leq n$ compatible with n (see (5.10) and (5.12)) and that $D(\alpha, 0) = \alpha^2$.

The following proposition is concerned with the square patterns of type V.

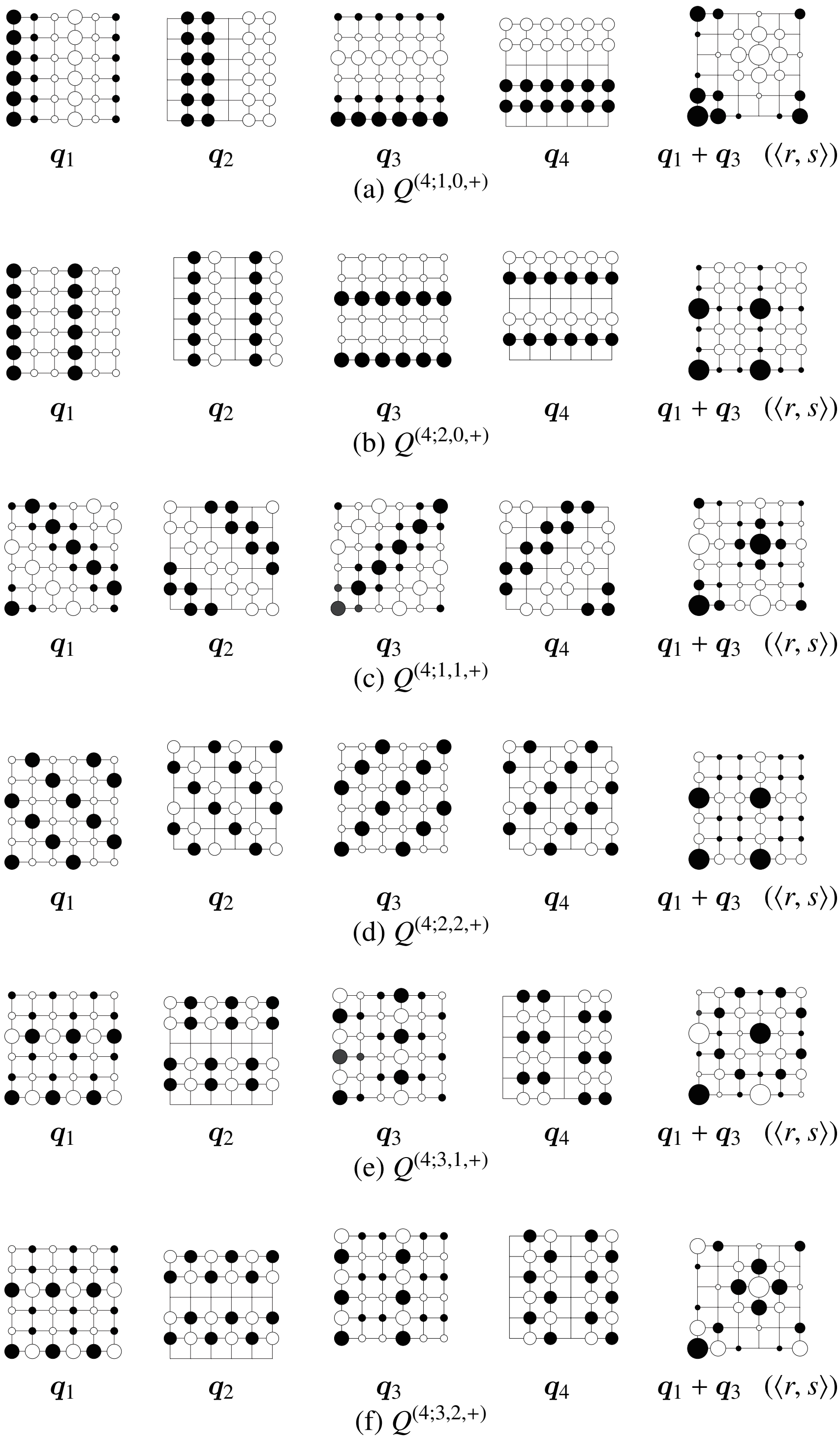


Figure 5.3: Patterns on the 6×6 square lattice expressed by the column vectors of $Q^{(4;1,0,+)}$, $Q^{(4;2,0,+)}$, $Q^{(4;1,1,+)}$, $Q^{(4;2,2,+)}$, $Q^{(4;3,1,+)}$, and $Q^{(4;3,2,+)}$. A black circle denotes a positive component and a white circle denotes a negative component.

Proposition 5.8. *Square patterns of type V with the symmetry of* $\Sigma(\alpha, 0)$ $(\alpha \geq 3)$ *arise as bifurcating solutions from critical points of multiplicity 4 for specific values of n and associated irreducible representations given by*

$$\begin{array}{llll l}
(\alpha,\beta) & D & n & (k,\ell) \text{ in } (4;k,\ell,+) & \\
\hline
(\alpha,0) & \alpha^2 & \alpha m & (pm,0) & \\
(\alpha,0) & \alpha^2 & \alpha m & (pm,pm) & (\alpha \text{ is odd}) \\
(\alpha,0) & \alpha^2 & \alpha m & (\alpha m/2,pm) & (\alpha \text{ is even and } \alpha/2 \text{ is even}) \\
(\alpha,0) & \alpha^2 & \alpha m & (\alpha m/2,2p'm) & (\alpha \text{ is even and } \alpha/2 \text{ is odd}) \\
\hline
\end{array} \tag{5.51}$$

where $m \geq 1$ *and*

$$\gcd(p, \alpha) = 1, \quad 1 \leq p < \alpha/2, \tag{5.52}$$

$$\gcd(p', \alpha/2) = 1, \quad 1 \leq p' < \alpha/4. \tag{5.53}$$

Proof. By Propositions 5.5, 5.6, and 5.7, we have three possibilities: $(4; k, 0, +)$, $(4; k, k, +)$, and $(4; n/2, \ell)$. For $(4; k, 0, +)$, we fix α and look for (k, n) that satisfies (5.28) and $\check{n} = \alpha$. For such (k, n), $\Sigma(\alpha, 0) = \Sigma(\check{n}, 0)$ is an isotropy subgroup with $\dim \mathrm{Fix}^{(4;k,0,+)}(\Sigma(\alpha, 0)) = 1$ by Proposition 5.5. Then the equivariant branching lemma (Section 5.2.2) guarantees the existence of a bifurcating solution with symmetry $\Sigma(\alpha, 0)$.

For $(4; k, k, +)$, we fix α that is odd and look for (k, n) that satisfies (5.29) and $\tilde{n} = \alpha$, and proceed in a similar manner using Proposition 5.6.

For $(4; n/2, \ell, +)$, we fix α that is even and look for (ℓ, n) that satisfies (5.30) and $\tilde{n} = \alpha/2$ for $\alpha/2$ odd and $\tilde{n} = \alpha$ for $\alpha/2$ even, and proceed in a similar manner using Proposition 5.7.

Suppose that (k, n) for $(k, \ell) = (pm, 0)$ and (pm, pm) is given by (5.51) with (5.52). Then $m = \gcd(k, n)$ by $\gcd(p, \alpha) = 1$ and $\check{n} = n/\gcd(k, n) = n/m = \alpha$. We have $k = pm \geq 1$ and $k/n = p/\alpha < 1/2$, thereby showing $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ in (5.28) for $(4; pm, 0, +)$ and (5.29) for $(4; pm, pm, +)$.

Suppose that (ℓ, n) for $(k, \ell) = (\alpha m/2, pm)$ is given by (5.51) with (5.52). Then $m = \gcd(n, \ell)$ by $\gcd(p, \alpha) = 1$ and $\tilde{n} = n/\gcd(\ell, n) = n/m = \alpha$. We have $\ell = pm \geq 1$ and $\ell/n = p/\alpha < 1/2$, thereby showing (5.30).

Suppose that (ℓ, n) for $(k, \ell) = (\alpha m/2, 2p'm)$ is given by (5.51) with (5.53). Then $2m = \gcd(n, \ell)$ by $\gcd(p', \alpha/2) = 1$ and $\tilde{n} = n/\gcd(\ell, n) = n/(2m) = \alpha/2$. We have $\ell = 2p'm \geq 1$ and $\ell/n = 2p'/\alpha < 1/2$, thereby showing (5.30).

Conversely, suppose that (k, n) satisfies $\check{n} = \alpha$, and (5.28) or (5.29). Then we have $\alpha = \check{n} = n/\gcd(k, n)$, which shows $\gcd(k, n) = n/\alpha$ is an integer, say m. We also have $k = \check{k} \gcd(k, n) = mp$ for $p = \check{k}$. Then $\gcd(p, \alpha) = \gcd(\check{k}, \check{n}) = 1$, $p = \check{k} \geq 1$, and $p/\alpha = k/n < 1/2$ by (5.28) or (5.29), thereby showing (5.52).

Suppose that $\alpha/2$ is even and (ℓ, n) satisfies $\tilde{n} = \alpha$, and (5.30). Then we have $\alpha = \tilde{n} = n/\gcd(\ell, n)$, which shows $\gcd(\ell, n) = n/\alpha$ is an integer, say m. We also have $\ell = \tilde{\ell} \gcd(\ell, n) = mp$ for $p = \tilde{\ell}$. Then $\gcd(p, \alpha) = \gcd(\tilde{\ell}, \tilde{n}) = 1$, $p = \tilde{\ell} \geq 1$, and $p/\alpha = \ell/n < 1/2$ by (5.30), thereby showing (5.52).

Suppose that $\alpha/2$ is odd and (ℓ, n) satisfies $2\tilde{n} = \alpha$ and (5.30). Then we have $\alpha = 2\tilde{n} = 2n/\gcd(\ell, n)$, which shows $\gcd(\ell, n) = 2n/\alpha$ is an even integer, say $2m$. We also have $\ell =$

$\tilde{\ell}\,\mathrm{gcd}(\ell, n) = 2mp'$ for $p' = \tilde{\ell}$. Then $\mathrm{gcd}(p', \alpha/2) = \mathrm{gcd}(\tilde{\ell}, \tilde{n}) = 1$, $p' = \tilde{\ell} \geq 1$, and $p'/\alpha = \ell/(2n) < 1/4$ by (5.30), thereby showing (5.53).

The above argument is in fact valid for $\alpha \geq 2$. For $\alpha = 2$, however, the condition $1 \leq p < \alpha/2$ or $1 \leq p' < \alpha/4$ is already a contradiction, which proves the nonexistence of the square pattern with $D = 4$ ($\alpha = 2$). □

Example 5.1. The parameter values of (5.51) in Proposition 5.8 give

(α,β)	D	n	(k,ℓ) in $(4;k,\ell,+)$		
$(3,0)$	9	$3m$	$(m,0)$;	(m,m)	
$(4,0)$	16	$4m$	$(m,0)$;		$(2m,m)$
$(5,0)$	25	$5m$	$(m,0),(2m,0)$;	$(m,m),(2m,2m)$	
$(6,0)$	36	$6m$	$(m,0)$;		$(3m,2m)$
$(7,0)$	49	$7m$	$(m,0),(2m,0),(3m,0)$;	$(m,m),(2m,2m),(3m,3m)$	
$(8,0)$	64	$8m$	$(m,0),(3m,0)$;		$(4m,m),(4m,3m)$

where $m \geq 1$. For each $\alpha \geq 3$, there exists at least one eligible (k, n) for $(4; k, 0, +)$ in (5.51); for instance, $(k, n) = (m, \alpha m)$, which corresponds to $p = 1$. □

5.5.4. Square Patterns of Type M

Square patterns of type M are shown here to branch from critical points of multiplicity 4. Recall that a square pattern of type M is characterized by the symmetry of $\Sigma(\beta,\beta)$ with $1 \leq \beta \leq n/2$ compatible with n (see (5.10) and (5.12)) and $D(\beta,\beta) = 2\beta^2$.

The following proposition is concerned with the square patterns of type M.

Proposition 5.9. *Square patterns of type M with the symmetry of* $\Sigma(\beta,\beta)$ $(\beta \geq 2)$ *arise as bifurcating solutions from critical points of multiplicity 4 for specific values of n and associated irreducible representations given by*

(α,β)	D	n	(k,ℓ) in $(4;k,\ell,+)$	
(β,β)	$2\beta^2$	$2\beta m$	(pm, pm)	
(β,β)	$2\beta^2$	$2\beta m$	$(\beta m, pm)$	(β *is odd*)

(5.54)

where $m \geq 1$ *and*

$$\mathrm{gcd}(p, 2\beta) = 1, \quad 1 \leq p < \beta. \tag{5.55}$$

Proof. By Propositions 5.5, 5.6, and 5.7, we have two possibilities: $(4; k, k, +)$ and $(4; n/2, \ell)$ and look for (k, n) that satisfies (5.29) or (5.30) and the condition that

$$\check{n} \text{ is even and } \beta = \check{n}/2 \quad \text{for } (4; k, k, +), \tag{5.56}$$

$$\check{n} \text{ is even, } \tilde{n}/2 \text{ is odd, and } \beta = \tilde{n}/2 \quad \text{for } (4; n/2, \ell). \tag{5.57}$$

For such parameter value (k, n) in (5.56) for $(4; k, k, +)$, $\Sigma(\beta,\beta) = \Sigma(\check{n}/2, \check{n}/2)$ is an isotropy subgroup with $\dim \mathrm{Fix}^{(4;k,k,+)}(\Sigma(\beta,\beta)) = 1$ by Proposition 5.6. For such parameter value (ℓ, n) in (5.57) for $(4; n/2, \ell, +)$, $\Sigma(\beta,\beta) = \Sigma(\tilde{n}/2, \tilde{n}/2)$ is an isotropy subgroup with $\dim \mathrm{Fix}^{(4;n/2,\ell,+)}(\Sigma(\beta,\beta)) = 1$

by Proposition 5.7. Then the equivariant branching lemma (Section 5.2.2) guarantees the existence of a bifurcating solution with symmetry $\Sigma(\beta,\beta)$ for both $(4;k,k,+)$ and $(4;n/2,\ell,+)$.

For $(4;k,k,+)$, suppose that (k,n) is given by (5.54). Then $m = \gcd(k,n)$ by $\gcd(p,2\beta) = 1$, and $\check{n} = n/\gcd(k,n) = 2\beta$, which shows (5.56). As for the condition (5.29), we first observe that $k/n = p/(2\beta) < 1/2$, which shows $k < n/2$. The case of $(4;n/2,\ell,+)$ can be treated similarly.

Conversely, for $(4;k,k,+)$, suppose that (k,n) satisfies (5.29) and (5.56). Put $m' = \gcd(k,n)$ to obtain $n = m'\check{n} = 2m'\beta$ and $k = m'\check{k} = m'p$ for $p = \check{k}$. Hence we have $(k,n) = (pm', 2\beta m')$, where $\gcd(p,2\beta) = \gcd(\check{k},\check{n}) = 1$ and $p/(2\beta) = k/n < 1/2$, thereby showing (5.55). The case of $(4;n/2,\ell,+)$ can be treated similarly.

The above argument is valid also for $\beta = 1$. For $\beta = 1$, however, no p satisfies $1 \le p < \beta$. This proves the nonexistence of the square pattern with $D = 2$. □

Example 5.2. The parameter values of (5.54) in Proposition 5.9 give

(α,β)	D	n	(k,ℓ) in $(4;k,\ell,+)$	
$(2,2)$	8	$4m$	(m,m)	
$(3,3)$	18	$6m$	(m,m);	$(3m,m)$
$(4,4)$	32	$8m$	$(m,m),(3m,3m)$,	
$(5,5)$	50	$10m$	$(m,m),(3m,3m)$;	$(5m,m),(5m,3m)$
$(6,6)$	72	$12m$	$(m,m),(5m,5m)$	

where $m \ge 1$. For each $\beta \ge 2$, there exists at least one eligible (k,n) in (5.54); for instance, $(k,n) = (m, 2\beta m)$, which corresponds to $p = 1$. □

5.5.5. Square Patterns of Type T

It is shown that square patterns of type T do not appear from critical points of multiplicity 4. Recall that a square pattern of type T is characterized by the symmetry of $\Sigma_0(\alpha,\beta)$ with $1 \le \alpha \le n-1,\ 1 \le \beta \le n-1,\ \alpha \ne \beta$ (see (2.42)). The following proposition denies the existence of square patterns of type T.

Proposition 5.10. *Square patterns of type T with the symmetry of* $\Sigma_0(\alpha,\beta)$ $1 \le \alpha \le n-1,\ 1 \le \beta \le n-1,\ \alpha \ne \beta$ *do not arise as bifurcating solutions from critical points of multiplicity 4 for any n.*

Proof. By Propositions 5.5, 5.6, and 5.7, $\Sigma_0(\alpha,\beta)$ is not an isotropy subgroup with respect to neither $(4;k,0,+)$, nor $(4;k,k,+)$, nor $(4;,n/2,\ell)$. □

5.5.6. Possible Square Patterns for Several Lattice Sizes

We have investigated possible occurrences of square patterns for each of the three types V, M, and T, and enumerated all possible parameter values of n for the lattice size and k for the associated irreducible representations $(4;k,0,+)$, $(4;k,k,+)$, and/or $(4;n/2,\ell,+)$. By compiling the obtained facts, we can capture, for each n, all square patterns that can potentially arise from critical points of multiplicity 4. The result is given in Tables 5.4–5.7 for several lattice sizes n.

Table 5.4: Square patterns arising from critical points of multiplicity 4 for several lattice sizes n ($\check{n}$ is given for $(4;k,0,+)$ and $(4;k,k,+)$, and $\tilde{n}$ is given for $(4;n/2,\ell,+)$)

n	(k,ℓ) in $(4;k,\ell,+)$	$\check{n}$	$\tilde{n}$	(α,β)	D	Type
3	$(1,0)$	3		$(3,0)$	9	V
	$(1,1)$					
4	$(1,0)$	4		$(4,0)$	16	V
	$(2,1)$		4			
	$(1,1)$	4		$(2,2)$	8	M
5	$(1,0),(2,0)$	5		$(5,0)$	25	V
	$(1,1),(2,2)$					
6	$(2,0)$	3		$(3,0)$	9	V
	$(2,2)$					
	$(1,0)$	6		$(6,0)$	36	V
	$(3,2)$		3			
	$(1,1)$	6		$(3,3)$	18	M
	$(3,1)$		6			
7	$(1,0),(2,0),(3,0)$	7		$(7,0)$	49	V
	$(1,1),(2,2),(3,3)$					
8	$(2,0)$	4		$(4,0)$	16	V
	$(4,2)$		4			
	$(1,0),(3,0)$	8		$(8,0)$	64	V
	$(4,1),(4,3)$		8			
	$(2,2)$	4		$(2,2)$	8	M
	$(1,1),(3,3)$	8		$(4,4)$	32	M
9	$(3,0)$	3		$(3,0)$	9	V
	$(3,3)$					
	$(1,0),(2,0),(4,0)$	9		$(9,0)$	81	V
	$(1,1),(2,2),(4,4)$					
10	$(2,0),(4,0)$	5		$(5,0)$	25	V
	$(2,2),(4,4)$					
	$(1,0),(3,0)$	10		$(10,0)$	100	V
	$(5,2),(5,4)$		5			
	$(1,1),(3,3)$	10		$(5,5)$	50	M
	$(5,1),(5,3)$		10			
11	$(1,0),(2,0),(3,0),(4,0),(5,0)$	11		$(11,0)$	121	V
	$(1,1),(2,2),(3,3),(4,4),(5,5)$					
12	$(4,0)$	3		$(3,0)$	9	V
	$(4,4)$					
	$(3,0)$	4		$(4,0)$	16	V
	$(6,3)$		4			
	$(2,0)$	6		$(6,0)$	36	V
	$(6,4)$		3			
	$(1,0),(5,0)$	12		$(12,0)$	144	V
	$(6,1),(6,5)$		12			
	$(3,3)$	4		$(2,2)$	8	M
	$(2,2)$	6		$(3,3)$	18	M
	$(6,2)$		6			
	$(1,1),(5,5)$	12		$(6,6)$	72	M
13	$(1,0),(2,0),(3,0),(4,0),(5,0),(6,0)$	13		$(13,0)$	169	V
	$(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)$					

Table 5.5: Square patterns arising from critical points of multiplicity 4 for several lattice sizes n ($\check{n}$ is given for $(4;k,0,+)$ and $(4;k,k,+)$, and $\tilde{n}$ is given for $(4;n/2,\ell,+)$)

n	(k,ℓ) in $(4;k,\ell,+)$	$\check{n}$	$\tilde{n}$	(α,β)	D	Type
14	$(2,0),(4,0),(6,0)$	7		$(7,0)$	49	V
	$(2,2),(4,4),(6,6)$					
	$(1,0),(3,0),(5,0)$	14		$(14,0)$	196	V
	$(7,2),(7,4),(7,6)$		7			
	$(1,1),(3,3),(5,5)$	14		$(7,7)$	98	M
	$(7,1),(7,3),(7,5)$		14			
15	$(5,0)$	3		$(3,0)$	9	V
	$(5,5)$					
	$(3,0),(6,0)$	5		$(5,0)$	25	V
	$(3,3),(6,6)$					
	$(1,0),(2,0),(4,0),(7,0)$	15		$(15,0)$	225	V
	$(1,1),(2,2),(4,4),(7,7)$					
16	$(4,0)$	4		$(4,0)$	16	V
	$(8,4)$		4			
	$(2,0),(6,0)$	8		$(8,0)$	64	V
	$(8,2),(8,6)$		8			
	$(1,0),(3,0),(5,0),(7,0)$	16		$(16,0)$	256	V
	$(8,1),(8,3),(8,5),(8,7)$		16			
	$(4,4)$	4		$(2,2)$	8	M
	$(2,2),(6,6)$	8		$(4,4)$	32	M
	$(1,1),(3,3),(5,5),(7,7)$	16		$(8,8)$	72	M
17	$(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0),(8,0)$	17		$(17,0)$	289	V
	$(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8)$					
18	$(6,0)$	3		$(3,0)$	9	V
	$(6,6)$					
	$(3,0)$	6		$(6,0)$	36	V
	$(9,6)$		3			
	$(2,0),(4,0),(8,0)$	9		$(9,0)$	81	V
	$(2,2),(4,4),(8,8)$					
	$(1,0),(5,0),(7,0)$	18		$(18,0)$	324	V
	$(9,2),(9,4),(9,8)$		9			
	$(3,3)$	6		$(3,3)$	18	M
	$(9,3)$		6			
	$(1,1),(5,5),(7,7)$	18		$(9,9)$	162	M
	$(9,1),(9,5),(9,7)$		18			
19	$(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0),(8,0),(9,0)$	19		$(19,0)$	361	V
	$(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9)$					
20	$(5,0)$	4		$(4,0)$	16	V
	$(10,5)$		4			
	$(4,0),(8,0)$	5		$(5,0)$	25	V
	$(4,4),(8,8)$					
	$(2,0),(6,0)$	10		$(10,0)$	100	V
	$(10,4),(10,8)$		5			
	$(1,0),(3,0),(7,0),(9,0)$	20		$(20,0)$	400	V
	$(10,1),(10,3),(10,7),(10,9)$		20			
	$(5,5)$	4		$(2,2)$	8	M
	$(2,2),(6,6)$	10		$(5,5)$	50	M
	$(10,2),(10,6)$		10			
	$(1,1),(3,3),(7,7),(9,9)$	20		$(10,10)$	200	M

Table 5.6: Square patterns arising from critical points of multiplicity 4 for several lattice sizes n ($\check{n}$ is given for $(4; k, 0, +)$ and $(4; k, k, +)$, and $\tilde{n}$ is given for $(4; n/2, \ell, +)$)

n	(k, ℓ) in $(4; k, \ell, +)$	$\check{n}$	$\tilde{n}$	(α, β)	D	Type
21	(7, 0)	3		(3, 0)	9	V
	(7, 7)					
	(3, 0), (6, 0), (9, 0)	7		(7, 0)	49	V
	(3, 3), (6, 6), (9, 9)					
	(1, 0), (2, 0), (4, 0), (5, 0), (8, 0), (10, 0)	21		(21, 0)	441	V
	(1, 1), (2, 2), (4, 4), (5, 5), (8, 8), (10, 10)					
22	(2, 0), (4, 0), (6, 0), (8, 0), (10, 0)	11		(11, 0)	121	V
	(2, 2), (4, 4), (6, 6), (8, 8), (10, 10)					
	(1, 0), (3, 0), (5, 0), (7, 0), (9, 0)	22		(22, 0)	484	V
	(11, 2), (11, 4), (11, 6), (11, 8), (11, 10)		11			
	(1, 1), (3, 3), (5, 5), (7, 7), (9, 9)	22		(11, 11)	242	M
	(11, 1), (11, 3), (11, 5), (11, 7), (11, 9)		22			
23	(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0), (8, 0), (9, 0), (10, 0)	23		(23, 0)	529	V
	(11, 0)					
	(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9), (10, 10)					
	(11, 11)					
24	(8, 0)	3		(3, 0)	9	V
	(8, 8)					
	(6, 0)	4		(4, 0)	16	V
	(12, 6)		4			
	(4, 0)	6		(6, 0)	36	V
	(12, 8)		3			
	(3, 0), (9, 0)	8		(8, 0)	64	V
	(12, 3), (12, 9)		8			
	(2, 0), (10, 0)	12		(12, 0)	144	V
	(12, 2), (12, 10)		12			
	(1, 0), (5, 0), (7, 0), (11, 0)	24		(24, 0)	576	V
	(12, 1), (12, 5), (12, 7)(12, 11)		24			
	(6, 6)	4		(2, 2)	8	M
	(4, 4)	6		(3, 3)	18	M
	(12, 4)		6			
	(3, 3), (9, 9)	8		(4, 4)	32	M
	(2, 2), (10, 10)	12		(6, 6)	72	M
	(1, 1), (5, 5), (7, 7), (11, 11)	24		(12, 12)	288	M
25	(5, 0), (10, 0)	5		(5, 0)	25	V
	(5, 5), (10, 10)					
	(1, 0), (2, 0), (3, 0), (4, 0), (6, 0), (7, 0), (8, 0), (9, 0), (11, 0), (12, 0)	25		(25, 0)	625	V
	(1, 1), (2, 2), (3, 3), (4, 4), (6, 6), (7, 7), (8, 8), (9, 9), (11, 11), (12, 12)					
26	(2, 0), (4, 0), (6, 0), (8, 0), (10, 0), (12, 0)	13		(13, 0)	169	V
	(2, 2), (4, 4), (6, 6), (8, 8), (10, 10), (12, 12)					
	(1, 0), (3, 0), (5, 0), (7, 0), (9, 0), (11, 0)	26		(26, 0)	676	V
	(13, 2), (13, 4), (13, 6), (13, 8), (13, 10), (13, 12)		13			
	(1, 1), (3, 3), (5, 5), (7, 7), (9, 9), (11, 11)	26		(13, 13)	338	M
	(13, 1), (13, 3), (13, 5), (13, 7), (13, 9), (13, 11)		26			

Table 5.7: Square patterns arising from critical points of multiplicity 4 for several lattice sizes n ($\check{n}$ is given for $(4;k,0,+)$ and $(4;k,k,+)$, and $\tilde{n}$ is given for $(4;n/2,\ell,+)$)

n	(k,ℓ) in $(4;k,\ell,+)$	$\check{n}$	$\tilde{n}$	(α,β)	D	Type
27	$(9,0)$	3		$(3,0)$	9	V
	$(9,9)$					
	$(3,0),(6,0),(12,0)$	9		$(9,0)$	81	V
	$(3,3),(6,6),(12,12)$					
	$(1,0),(2,0),(4,0),(5,0),(7,0),(8,0),(10,0),(11,0),(13,0)$	27		$(27,0)$	729	V
	$(1,1),(2,2),(4,4),(5,5),(7,7),(8,8),(10,10),(11,11),(13,13)$		27			
28	$(7,0)$	4		$(4,0)$	16	V
	$(14,7)$		4			
	$(4,0),(8,0),(12,0)$	7		$(7,0)$	49	V
	$(4,4),(8,8),(12,12)$					
	$(2,0),(6,0),(10,0)$	14		$(14,0)$	392	V
	$(14,4),(14,8),(14,12)$		7			
	$(1,0),(3,0),(5,0),(9,0),(11,0),(13,0)$	28		$(28,0)$	784	V
	$(14,1),(14,3),(14,5),(14,9),(14,11),(14,13)$		28			
	$(7,7)$	24		$(2,2)$	8	M
	$(2,2),(6,6),(10,10)$	14		$(7,7)$	98	M
	$(14,2),(14,6),(14,10)$		14			
	$(1,1),(3,3),(5,5),(9,9),(11,11),(13,13)$	28		$(14,14)$	392	M
29	$(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0),(8,0),(9,0),(10,0)$	29		$(29,0)$	841	V
	$(11,0),(12,0),(13,0),(14,0)$					
	$(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9),(10,10)$					
	$(11,11),(12,12),(13,13),(14,14)$					
30	$(10,0)$	3		$(3,0)$	9	V
	$(10,10)$					
	$(6,0),(12,0)$	5		$(5,0)$	25	V
	$(6,6),(12,12)$					
	$(5,0)$	6		$(6,0)$	36	V
	$(15,10)$		3			
	$(3,0),(9,0)$	10		$(10,0)$	100	V
	$(15,6),(15,12)$		5			
	$(2,0),(4,0),(8,0),(14,0)$	15		$(15,0)$	225	V
	$(2,2),(4,4),(8,8),(14,14)$					
	$(1,0),(7,0),(11,0),(13,0)$	30		$(30,0)$	900	V
	$(15,2),(15,4),(15,8),(15,14)$		15			
	$(5,5)$	6		$(3,3)$	18	M
	$(15,5)$		6			
	$(3,3),(9,9)$	10		$(5,5)$	50	M
	$(15,3),(15,9)$		10			
	$(1,1),(7,7),(11,11),(13,13)$	30		$(15,15)$	450	M
	$(15,1),(15,7),(15,11),(15,13)$		30			

5.6. Bifurcation Point of Multiplicity 8

Square patterns branching from critical points of multiplicity 8 are investigated. The emergence of tilted square patterns of type T is the most phenomenal finding of this chapter. In addition, larger square patterns of type V and M also branch.

5.6.1. Representation in Complex Variables

As shown by Table 5.3 in Section 5.2.2, a critical point of multiplicity 8 is associated with the eight-dimensional irreducible representation $(8; k, \ell)$ with

$$1 \leq \ell \leq k-1, \quad 2 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{5.58}$$

where $n \geq 5$.

Recall from (3.27)–(3.28) that the irreducible representation $(8; k, \ell)$ is given by

$$T^{(8;k,\ell)}(r) = \left[\begin{array}{cc|cc} & S & & \\ I & & & \\ \hline & & & I \\ & & S & \end{array}\right], \quad T^{(8;k,\ell)}(s) = \left[\begin{array}{cc|cc} & & I & \\ & & & I \\ \hline I & & & \\ & I & & \end{array}\right], \tag{5.59}$$

$$T^{(8;k,\ell)}(p_1) = \left[\begin{array}{cc|cc} R^k & & & \\ & R^{-\ell} & & \\ \hline & & R^k & \\ & & & R^{-\ell} \end{array}\right], \quad T^{(8;k,\ell)}(p_2) = \left[\begin{array}{cc|cc} R^\ell & & & \\ & R^k & & \\ \hline & & R^{-\ell} & \\ & & & R^{-k} \end{array}\right] \tag{5.60}$$

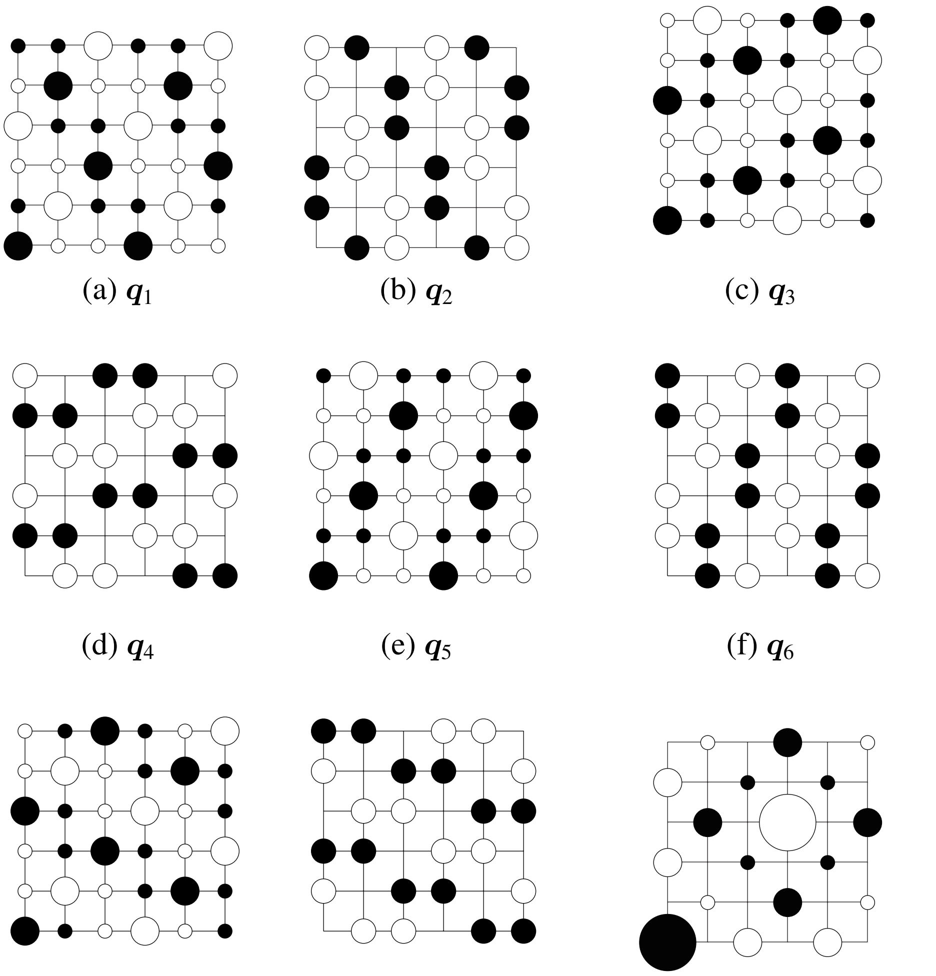
with

$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}. \tag{5.61}$$

Let us assume that the variable $\boldsymbol{w} = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8)^\top$ for the bifurcation equation (5.16) corresponds to the column vectors of

$$\begin{aligned} Q^{(8;k,\ell)} = [\ & \langle\cos(2\pi(kn_1 + \ell n_2)/n)\rangle,\ \langle\sin(2\pi(kn_1 + \ell n_2)/n)\rangle, \\ & \langle\cos(2\pi(-\ell n_1 + kn_2)/n)\rangle,\ \langle\sin(2\pi(-\ell n_1 + kn_2)/n)\rangle, \\ & \langle\cos(2\pi(kn_1 - \ell n_2)/n)\rangle,\ \langle\sin(2\pi(kn_1 - \ell n_2)/n)\rangle, \\ & \langle\cos(2\pi(-\ell n_1 - kn_2)/n)\rangle,\ \langle\sin(2\pi(-\ell n_1 - kn_2)/n)\rangle\ ] \\ & \qquad \text{for } 1 \leq \ell \leq k-1,\ 2 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor. \end{aligned}$$

Examples of the spatial patterns for these vectors are depicted in Fig. 5.4 for $n = 6$.



(a) $\boldsymbol{q}_1$ (b) $\boldsymbol{q}_2$ (c) $\boldsymbol{q}_3$

(d) $\boldsymbol{q}_4$ (e) $\boldsymbol{q}_5$ (f) $\boldsymbol{q}_6$

(g) $\boldsymbol{q}_7$ (h) $\boldsymbol{q}_8$ (i) $\boldsymbol{q}_1 + \boldsymbol{q}_3 + \boldsymbol{q}_5 + \boldsymbol{q}_7$ $(\langle r, s\rangle)$

Figure 5.4: Patterns on the 6×6 square lattice expressed by the column vectors of $Q^{(8;2,1)}$. A black circle denotes a positive component and a white circle denotes a negative component.

The action given in (5.59) and (5.60) on 8-dimensional vectors $(w_1, \dots, w_8)$ can be expressed for complex variables $z_j = w_{2j-1} + \mathrm{i}w_{2j}$ $(j = 1, \dots, 4)$ as

$$r : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z}_2 \\ z_1 \\ z_4 \\ \overline{z}_3 \end{bmatrix}, \quad s : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} z_3 \\ z_4 \\ z_1 \\ z_2 \end{bmatrix}, \tag{5.62}$$

$$p_1 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k \, z_1 \\ \omega^{-\ell} \, z_2 \\ \omega^k \, z_3 \\ \omega^{-\ell} \, z_4 \end{bmatrix}, \quad p_2 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} \omega^\ell \, z_1 \\ \omega^k \, z_2 \\ \omega^{-\ell} \, z_3 \\ \omega^{-k} \, z_4 \end{bmatrix}, \tag{5.63}$$

where $\omega = \exp(\mathrm{i}2\pi/n)$.

5.6.2. Outline of Analysis

The major ingredients of our analysis for critical points of multiplicity 8 associated with $(8; k, \ell)$ are previewed.

We denote the isotropy subgroup of $z = (z_1, \dots, z_4)$ with respect to $(8; k, \ell)$ as

$$\Sigma^{(8;k,\ell)}(z) = \{g \in G \mid T^{(8;k,\ell)}(g) \cdot z = z\}, \tag{5.64}$$

where $T^{(8;k,\ell)}(g) \cdot z$ means the action of $g \in G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ on z given in (5.62) and (5.63). It turns out that the isotropy subgroup of $z = (1, 1, 0, 0)$ plays a crucial role in our analysis and that

$$\Sigma^{(8;k,\ell)}((1, 1, 0, 0)) = \Sigma_0(\alpha, \beta) \tag{5.65}$$

for a uniquely determined (α, β) with $0 \le \beta \le \alpha \le n$ (see Proposition 5.17 in Section 5.6.3). We denote this correspondence $(k, \ell) \mapsto (\alpha, \beta) = (\alpha(k, \ell, n), \beta(k, \ell, n))$ by

$$\Phi(k, \ell, n) = (\alpha, \beta). \tag{5.66}$$

In a sense, (k, ℓ) and (α, β) are dual to each other; (k, ℓ) prescribes the action of the translations p_1 and p_2, and (α, β) describes the symmetry preserved under this action.[7]

Whereas the concrete form of the correspondence Φ is discussed in detail in Section 5.6.9, the following proposition shows the most fundamental formulas connecting (k, ℓ) and (α, β). We use the notations:

$$\hat{k} = \frac{k}{\gcd(k, \ell, n)}, \quad \hat{\ell} = \frac{\ell}{\gcd(k, \ell, n)}, \quad \hat{n} = \frac{n}{\gcd(k, \ell, n)}, \tag{5.67}$$

where $\gcd(k, \ell, n)$ means the greatest common divisor of k, ℓ, and n.

Proposition 5.11. *Let* $(\alpha, \beta) = \Phi(k, \ell, n)$.

(i)

$$\hat{n} = \frac{D(\alpha, \beta)}{\gcd(\alpha, \beta)}. \tag{5.68}$$

[7] In an analogy with physics we may compare (k, ℓ) to frequency and (α, β) to wave length.

irred. rep. (k, ℓ)	$\rightarrow(\Phi)\rightarrow$	symmetry (α,β)	$\rightarrow$(Eq. Br. Lem.)$\rightarrow$	bifurcating solution

Figure 5.5: Two stages of bifurcation analysis at a critical point of multiplicity 8.

(ii)

$$\frac{\hat{n}}{\gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n})} = \gcd(\alpha,\beta). \tag{5.69}$$

Proof. The proof is given in Section 5.6.10; see Propositions 5.35(ii) and 5.36. It is mentioned here that the proof relies on the Smith normal form for integer matrices. □

Our analysis of bifurcation consists of two stages (see Fig. 5.5):

1. Connect the irreducible representation (k, ℓ) to the associated symmetry represented by (α,β) by obtaining the function $\Phi : (k, \ell) \mapsto (\alpha,\beta)$.
2. Connect the symmetry represented by (α,β) to the existence of bifurcating solutions on the basis of the equivariant branching lemma.

Proposition 5.12 below is a preview of a major result (Proposition 5.20 in Section 5.6.4) in a simplified form. For classification of bifurcation into several cases, we consider the condition

$$\textbf{GCD-div} : 2\gcd(\hat{k}, \hat{\ell}) \text{ is not divisible by } \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}), \tag{5.70}$$

and the negation of this condition is referred to as $\overline{\textbf{GCD-div}}$. The set of even integers is denoted by $2\mathbb{Z}$ below.

Proposition 5.12. *For a critical point of multiplicity 8, let* $(8; k, \ell)$ *be the associated irreducible representation and* $(\alpha,\beta) = \Phi(k, \ell, n)$. *The bifurcation at this point is classified as follows.*

Case 1: **GCD-div** *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 2\mathbb{Z}$: *A bifurcating solution with symmetry* $\Sigma(\hat{n}, 0)$ *exists. This solution is of type V.*

Case 2: **GCD-div** *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}$: *A bifurcating solution with symmetry* $\Sigma(\hat{n}/2, \hat{n}/2)$ *exists. This solution is of type M.*

Case 3: $\overline{\textbf{GCD-div}}$ *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 2\mathbb{Z}$: *Bifurcating solutions with symmetries* $\Sigma(\hat{n}, 0)$, $\Sigma_0(\alpha,\beta)$, *and* $\Sigma_0(\beta, \alpha)$ *exist.*[8] *The first solution is of type V, and the other two solutions are of type T.*

Case 4: $\overline{\textbf{GCD-div}}$ *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}$: *Bifurcating solutions with symmetries* $\Sigma(\hat{n}/2, \hat{n}/2)$, $\Sigma_0(\alpha,\beta)$, *and* $\Sigma_0(\beta, \alpha)$ *exist. The first solution is of type M, and the other two solutions are of type T.*

The classification criteria for the above four cases become more transparent when expressed in terms of (α,β) $(= \Phi(k, \ell, n))$ rather than (k, ℓ). The expressions in terms of (α,β) can be obtained

[8] To be precise, $\Sigma_0(\beta, \alpha)$ should be denoted as $\Sigma_0(\alpha', \beta')$ with (α', β') in (5.81), which lies in the parameter space of (5.7).

Table 5.8: Classification of bifurcation at a critical point associated with $(8; k, \ell)$ with $(\alpha, \beta) = \Phi(k, \ell, n)$

	$\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 2\mathbb{Z}$	$\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}$
	$\hat{D} \notin 2\mathbb{Z}$	$\hat{D} \in 2\mathbb{Z}$
GCD-div	Case 1:	Case 2:
$\beta = 0$ or $\alpha = \beta$	type V	type M
$\overline{\textbf{GCD-div}}$	Case 3:	Case 4:
$\beta \neq 0$ and $\alpha \neq \beta$	type V and type T	type M and type T

from Proposition 5.13 below, where

$$\hat{\alpha} = \frac{\alpha}{\gcd(\alpha, \beta)}, \quad \hat{\beta} = \frac{\beta}{\gcd(\alpha, \beta)}, \tag{5.71}$$

$$\hat{D} = \hat{\alpha}^2 + \hat{\beta}^2 = \frac{D(\alpha, \beta)}{(\gcd(\alpha, \beta))^2}. \tag{5.72}$$

It is noted in passing that an alternative expression

$$\hat{D} = \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}) \tag{5.73}$$

results from (5.68), (5.69), and (5.72).

Proposition 5.13. *Let* $(\alpha, \beta) = \Phi(k, \ell, n)$.

(i) $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z} \iff \hat{D} \in 2\mathbb{Z}$.

(ii) **GCD-div** *in* (5.70) $\iff \beta = 0$ *or* $\alpha = \beta$.

Proof. The proof is given in Section 5.6.10; see Proposition 5.35(i) and Proposition 5.40. It is mentioned here that the proof of the equivalence in (ii) relies on the Smith normal form for integer matrices and the integer analogue of the Farkas lemma. □

Propositions 5.12 and 5.13 together yield Table 5.8 that summarizes the classification of bifurcation phenomena into the four cases in terms of both (k, ℓ) and (α, β).

An important observation here is that the classification into the four cases in Proposition 5.12, as well as in Table 5.8, can also be described in terms of the subgroup $\Sigma_0(\alpha, \beta)$. The following proposition shows how the conditions "$\beta = 0$ or $\alpha = \beta$" and "$\hat{D} \in 2\mathbb{Z}$" can be replaced by conditions for $\Sigma_0(\alpha, \beta)$.

Proposition 5.14.

(i) $\Sigma_0(\alpha, \beta) = \Sigma_0(\beta, \alpha) \iff \beta = 0$ *or* $\alpha = \beta$.

(ii)

$$\Sigma_0(\alpha, \beta) \cap \Sigma_0(\beta, \alpha) = \begin{cases} \Sigma_0(\alpha'', 0) & \text{if } \hat{D} \notin 2\mathbb{Z}, \\ \Sigma_0(\beta'', \beta'') & \text{if } \hat{D} \in 2\mathbb{Z} \end{cases} \tag{5.74}$$

with

$$\alpha'' = \frac{D(\alpha, \beta)}{\gcd(\alpha, \beta)}, \quad \beta'' = \frac{D(\alpha, \beta)}{2 \gcd(\alpha, \beta)}. \tag{5.75}$$

Table 5.9: Bifurcation at a critical point associated with $(8; k, \ell)$ classified in terms of the subgroup $\Sigma_0(\alpha,\beta)$ for $(\alpha,\beta) = \Phi(k, \ell, n)$

	$\Sigma_0(\alpha,\beta) \cap \Sigma_0(\beta,\alpha)$ $= \Sigma_0(\alpha'', 0)$	$\Sigma_0(\alpha,\beta) \cap \Sigma_0(\beta,\alpha)$ $= \Sigma_0(\beta'',\beta'')$
$\Sigma_0(\alpha,\beta) = \Sigma_0(\beta,\alpha)$	Case 1: type V	Case 2: type M
$\Sigma_0(\alpha,\beta) \neq \Sigma_0(\beta,\alpha)$	Case 3: type V and type T	Case 4: type M and type T

Proof. (i) This is obvious from the definition of $\Sigma_0(\alpha,\beta)$ in (5.10).

(ii) The proof is given in Proposition 5.32 in Section 5.6.10. □

By Proposition 5.14 above, we can rewrite Table 5.8 as Table 5.9. In particular, solutions of type T exist if and only if $\Sigma_0(\alpha,\beta)$ is asymmetric in the sense of $\Sigma_0(\alpha,\beta) \neq \Sigma_0(\beta,\alpha)$. Not only is this statement intuitively appealing, but it plays a crucial role in our technical arguments in Section 5.6.10.

Remark 5.4. Some comments are in order about (5.74) in each case corresponding to type V, type M, or type T.

- If $\beta = 0$, we have $\hat{D} = 1$ and $\alpha'' = D(\alpha, 0)/\gcd(\alpha, 0) = \alpha^2/\alpha = \alpha$.
- If $\alpha = \beta$, we have $\hat{D} = 2$ and $\beta'' = D(\beta,\beta)/(2\gcd(\beta,\beta)) = (2\beta^2)/(2\beta) = \beta$.
- For (α,β) with $1 \le \beta < \alpha$, we have $\hat{D} = 5, 10, 13, 17, 20$, and so on, some of which satisfy $\hat{D} \in 2\mathbb{Z}$, while others do not.

It should be also mentioned that the identity (5.74) is purely geometric in that it is valid for all (α,β) that may or may not be related to irreducible representation $(8; k, \ell)$. If (α,β) is associated with $(8; k, \ell)$, we have $\alpha'' = \hat{n}$ and $\beta'' = \hat{n}/2$ by (5.68) and (5.75), respectively. □

5.6.3. Isotropy Subgroups

To apply the method of analysis described in Section 5.2.2, we identify isotropy subgroups for $(8; k, \ell)$ related to square patterns.

We denote the fixed-point subspace of Σ in terms of $z = (z_1, \ldots, z_4)$ as

$$\mathrm{Fix}^{(8;k,\ell)}(\Sigma) = \{z \mid T^{(8;k,\ell)}(g) \cdot z = z \text{ for all } g \in \Sigma\}, \tag{5.76}$$

where $T^{(8;k,\ell)}(g) \cdot z$ means the action of $g \in G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ on z given in (5.62) and (5.63). Also recall from (5.64) the notation $\Sigma^{(8;k,\ell)}(z)$ for the isotropy subgroup of z.

The symmetries of $\langle r \rangle$ and $\langle r, s \rangle$ are dealt with in Proposition 5.15 below, and the translational symmetry $p_1^a p_2^b$ is considered thereafter. Remark 5.8 below should be consulted with regard to the geometrical interpretation of the following discussion.

Proposition 5.15.
(i) $\mathrm{Fix}^{(8;k,\ell)}(\langle r\rangle) = \{c(1,1,0,0) + c'(0,0,1,1) \mid c, c' \in \mathbb{R}\}$.
(ii) $\mathrm{Fix}^{(8;k,\ell)}(\langle r, s\rangle) = \{c(1,1,1,1) \mid c \in \mathbb{R}\}$.

Proof. (i) By (5.62), z is invariant to r if and only if $(\overline{z}_2, z_1, z_4, \overline{z}_3) = (z_1, z_2, z_3, z_4)$, which is equivalent to $z_1 = z_2 \in \mathbb{R}$ and $z_3 = z_4 \in \mathbb{R}$.
(ii) By (5.62), z is invariant to s if and only if $(z_3, z_4, z_1, z_2) = (z_1, z_2, z_3, z_4)$, which is equivalent to $z_1 = z_3$ and $z_2 = z_4$. Hence z is invariant to both r and s if and only if $z_1 = z_2 = z_3 = z_4 \in \mathbb{R}$. □

The above proposition implies that any isotropy subgroup Σ containing $\langle r\rangle$, which is of our interest, can be represented as $\Sigma = \Sigma^{(8;k,\ell)}(z)$ for some vector z of the form

$$z = c(1,1,0,0) + c'(0,0,1,1), \quad c, c' \in \mathbb{R}, \tag{5.77}$$

and that $\dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) \leq 2$.
We now turn to the invariance to the translational symmetry $p_1^a p_2^b$.

Proposition 5.16.
(i) $p_1^a p_2^b \in \Sigma^{(8;k,\ell)}((1,1,0,0))$ *if and only if*

$$\hat{k}a + \hat{\ell}b \equiv 0, \quad \hat{\ell}a - \hat{k}b \equiv 0 \mod \hat{n}. \tag{5.78}$$

(ii) $p_1^a p_2^b \in \Sigma^{(8;k,\ell)}((0,0,1,1))$ *if and only if*

$$\hat{k}a - \hat{\ell}b \equiv 0, \quad \hat{\ell}a + \hat{k}b \equiv 0 \mod \hat{n}. \tag{5.79}$$

Proof. (i) By (5.63), the invariance of $z = (1,1,0,0)$ to $p_1^a p_2^b$ is expressed as

$$ka + \ell b \equiv 0, \quad \ell a - kb \equiv 0 \mod n,$$

which is equivalent to (5.78) with the notations in (5.67).
(ii) By (5.63) the invariance of $z = (0,0,1,1)$ to $p_1^a p_2^b$ is expressed as

$$ka - \ell b \equiv 0, \quad \ell a + kb \equiv 0 \mod n,$$

which is equivalent to (5.79). □

The isotropy subgroup of $z = c(1,1,0,0) + c'(0,0,1,1)$ of the form of (5.77) is identified in the following two propositions: the case with $cc' = 0$ in Proposition 5.17 and the case with $cc' \neq 0$ in Proposition 5.18.

Proposition 5.17.
(i) *For each* (k, ℓ)*, we have*

$$\Sigma^{(8;k,\ell)}((1,1,0,0)) = \Sigma_0(\alpha, \beta) \tag{5.80}$$

for a uniquely determined (α, β) *with* $0 \leq \beta < n$*,* $0 < \alpha \leq n$*.*

(ii) *For the* (α,β) *associated with* (k,ℓ) *as in* (i) *above, define*

$$(\alpha',\beta') = \begin{cases} (\beta,\alpha) & \text{if } \beta > 0, \\ (\alpha,0) & \text{if } \beta = 0. \end{cases} \tag{5.81}$$

Then we have

$$\Sigma^{(8;k,\ell)}((0,0,1,1)) = \Sigma_0(\alpha',\beta'). \tag{5.82}$$

Proof. (i) By (5.62), $\Sigma^{(8;k,\ell)}((1,1,0,0))$ contains r and not s. To investigate the translation symmetry, denote by $\mathcal{A}(k,\ell,n)$ the set of all (a,b) satisfying (5.78). That is,

$$\mathcal{A}(k,\ell,n) = \{(a,b) \in \mathbb{Z}^2 \mid \hat{k}a + \hat{\ell}b \equiv 0,\ \hat{\ell}a - \hat{k}b \equiv 0 \mod \hat{n}\}. \tag{5.83}$$

Then $\mathcal{A}(k,\ell,n)$ is closed under integer combination, i.e., if (a_1,b_1), $(a_2,b_2) \in \mathcal{A}(k,\ell,n)$, then $n_1(a_1,b_1) + n_2(a_2,b_2) \in \mathcal{A}(k,\ell,n)$ for any $n_1,n_2 \in \mathbb{Z}$. Next, if $(a,b) \in \mathcal{A}(k,\ell,n)$, then $(a',b') = (-b,a)$ also belongs to $\mathcal{A}(k,\ell,n)$ since

$$\begin{aligned} \hat{k}a' + \hat{\ell}b' &= \hat{k}(-b) + \hat{\ell}a = \hat{\ell}a - \hat{k}b \equiv 0 \mod \hat{n}, \\ \hat{\ell}a' - \hat{k}b' &= \hat{\ell}(-b) - \hat{k}a = -(\hat{k}a + \hat{\ell}b) \equiv 0 \mod \hat{n}. \end{aligned}$$

The above argument shows that $\mathcal{A}(k,\ell,n)$ coincides with a set of the form

$$\mathcal{L}(\alpha,\beta) = \{(a,b) \in \mathbb{Z}^2 \mid (a,b) = n_1(\alpha,\beta) + n_2(-\beta,\alpha),\ n_1,n_2 \in \mathbb{Z}\} \tag{5.84}$$

for some appropriately chosen integers α and β. For such (α,β) we have

$$\Sigma^{(8;k,\ell)}((1,1,0,0)) = \langle r \rangle \ltimes \langle p_1^{\alpha} p_2^{\beta}, p_1^{-\beta} p_2^{\alpha} \rangle = \Sigma_0(\alpha,\beta).$$

To see the uniqueness of (α,β) we note the obvious correspondence between $\mathcal{L}(\alpha,\beta)$ and the square sublattice $\mathcal{H}(\alpha,\beta)$ in (2.4). By Proposition 2.1, $\mathcal{H}(\alpha,\beta)$ is uniquely parameterized by (α,β) with $0 \le \beta < \alpha$. Furthermore, we have $\alpha \le n$ as a consequence of the fact that $\mathcal{L}(\alpha,\beta)$ contains no point (a,b) of the form of $(a,b) = x(\alpha,\beta) + y(-\beta,\alpha)$ with $0 < x < 1$ and $0 < y < 1$, which lies in the interior of the parallelogram formed by its basis vectors (α,β) and $(-\beta,\alpha)$. To prove this by contradiction, suppose that $\alpha > n$ and consider the point $(a,b) = (\alpha - n,\beta)$. This point belongs to $\mathcal{L}(\alpha,\beta)$, satisfying the defining conditions in of $\mathcal{A}(k,\ell,n)$ in (5.83), whereas the corresponding (x,y) satisfies $0 < x < 1$ and $0 < y < 1$, which is a contradiction.

(ii) Since

$$(0,0,1,1) = T^{(8;k,\ell)}(s) \cdot (1,1,0,0),$$

it follows using the relation for the orbit $\Sigma(T(g)\boldsymbol{u}) = g \cdot \Sigma(\boldsymbol{u}) \cdot g^{-1}$ $(g \in G)$, (5.80), (2.35), and (5.81) in this order that

$$\Sigma^{(8;k,\ell)}((0,0,1,1)) = s \cdot \Sigma^{(8;k,\ell)}((1,1,0,0)) \cdot s^{-1} = s \cdot \Sigma_0(\alpha,\beta) \cdot s^{-1} = \Sigma_0(\beta,\alpha) = \Sigma_0(\alpha',\beta').$$

□

We denote the correspondence $(k,\ell) \mapsto (\alpha,\beta) = (\alpha(k,\ell,n),\beta(k,\ell,n))$ defined by (5.80) in Proposition 5.17 as

$$\Phi(k,\ell,n) = (\alpha,\beta). \tag{5.85}$$

Remark 5.5. A preliminary explanation is presented here about how the value of $(\alpha,\beta) = \Phi(k,\ell,n)$ can be determined, whereas a systematic method is given in Section 5.6.9.

The condition for $(a,b) \in \mathcal{A}(k,\ell,n)$ in (5.83) is equivalent to the existence of integers p and q satisfying

$$\begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \hat{n} \begin{bmatrix} p \\ q \end{bmatrix}. \tag{5.86}$$

Hence a pair of integers (a,b) belongs to $\mathcal{A}(k,\ell,n)$ if and only if

$$\begin{bmatrix} a \\ b \end{bmatrix} = \hat{n} \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \end{bmatrix} = \frac{\hat{n}}{\hat{k}^2 + \hat{\ell}^2} \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \tag{5.87}$$

for some integers p and q. There are two cases to consider.

- If $\hat{n}/(\hat{k}^2+\hat{\ell}^2)$ is an integer, a simpler method works. In this case, the right-hand side of (5.87) gives a pair of integers for any integers p and q. Therefore, we set $(p,q) = (1,0)$ to obtain an integer vector

  $$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{\hat{n}}{\hat{k}^2 + \hat{\ell}^2} \begin{bmatrix} \hat{k} \\ \hat{\ell} \end{bmatrix} \tag{5.88}$$

  and note that the vectors $(a,b)^\top$ of integers satisfying (5.86) form a lattice spanned by $(\alpha,\beta)^\top$ and $(\beta,-\alpha)^\top$. For $(k,\ell,n) = (3,1,20)$, for example, we have $(\hat{k},\hat{\ell},\hat{n}) = (3,1,20)$ and $\hat{n}/(\hat{k}^2 + \hat{\ell}^2) = 20/10 = 2$, and hence (5.87) reads

  $$\begin{bmatrix} a \\ b \end{bmatrix} = 2 \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

  This shows $\Phi(3,1,20) = (\alpha,\beta) = (6,2)$, corresponding to $(p,q) = (1,0)$.

- If $\hat{n}/(\hat{k}^2 + \hat{\ell}^2)$ is not an integer, number-theoretic considerations are needed to determine $(\alpha,\beta) = \Phi(k,\ell,n)$. For $(k,\ell,n) = (18,2,42)$, for instance, we have $(\hat{k},\hat{\ell},\hat{n}) = (9,1,21)$ and $\hat{k}^2 + \hat{\ell}^2 = 82$, and (5.87) reads

  $$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{21}{82} \begin{bmatrix} 9 & 1 \\ 1 & -9 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

  With some inspection we could arrive at $\Phi(18,2,42) = (\alpha,\beta) = (21,0)$, which corresponds to $(p,q) = (9,1)$. A systematic procedure based on the Smith normal form is given in Section 5.6.9.

□

Remark 5.6. In the following arguments we shall make use of Propositions 5.11, 5.13, and 5.14 presented in Section 5.6.2. The readers may take these propositions for granted in the first reading, but those who are interested in mathematical issues are advised to have a look at their proofs given in Section 5.6.10. □

Proposition 5.18. *Let* $(\alpha,\beta) = \Phi(k,\ell,n)$*, and define*

$$\alpha'' = \frac{D(\alpha,\beta)}{\gcd(\alpha,\beta)}, \quad \beta'' = \frac{D(\alpha,\beta)}{2\gcd(\alpha,\beta)}. \tag{5.89}$$

For distinct nonzero real numbers c *and* c' $(c \neq c')$*, we have the following statements:*

(i)

$$\Sigma^{(8;k,\ell)}((c,c,c,c)) = \begin{cases} \Sigma(\alpha'',0) & \text{if } \hat{D} \notin 2\mathbb{Z}, \\ \Sigma(\beta'',\beta'') & \text{if } \hat{D} \in 2\mathbb{Z}, \end{cases}$$

where $\hat{D}$ *is defined in* (5.72) *and* $\hat{D} \in 2\mathbb{Z}$ *means that* $\hat{D}$ *is even.*

(ii)

$$\Sigma^{(8;k,\ell)}((c,c,c',c')) = \begin{cases} \Sigma_0(\alpha'',0) & \text{if } \hat{D} \notin 2\mathbb{Z}, \\ \Sigma_0(\beta'',\beta'') & \text{if } \hat{D} \in 2\mathbb{Z}. \end{cases}$$

Proof. We first prove (ii). By (5.62), $\Sigma^{(8;k,\ell)}((c,c,c',c'))$ contains r and not s. We have

$$\begin{aligned} \Sigma^{(8;k,\ell)}((c,c,c',c')) &= \Sigma^{(8;k,\ell)}((1,1,0,0)) \cap \Sigma^{(8;k,\ell)}((0,0,1,1)) \\ &= \Sigma_0(\alpha,\beta) \cap \Sigma_0(\alpha',\beta'), \end{aligned}$$

where the second equality is due to Proposition 5.17. Then the claim follows from Proposition 5.14(ii).

Next we prove (i). By (5.62), $\Sigma^{(8;k,\ell)}((c,c,c,c))$ contains both r and s. We can proceed in a similar manner as above while including the element s. Therefore

$$\Sigma^{(8;k,\ell)}((c,c,c,c)) = \Sigma(\alpha,\beta) \cap \Sigma(\alpha',\beta'),$$

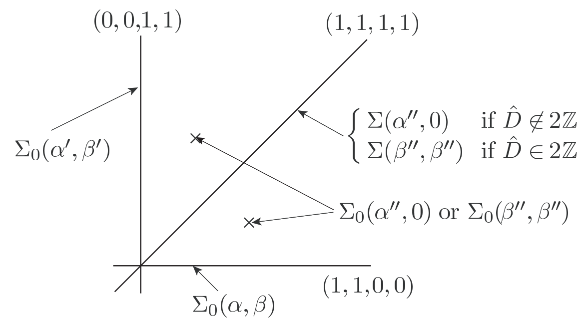
which implies the claim. □

In Proposition 5.19, we can present the isotropy subgroups containing $\langle r \rangle$, with a classification of the irreducible representations $(8;k,\ell)$ in terms of $(\alpha,\beta) = \Phi(k,\ell,n)$. See Fig. 5.6 for the classification.

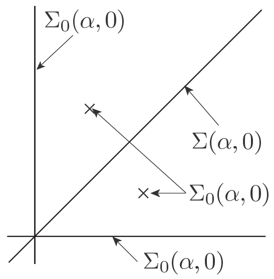
Proposition 5.19. *For an irreducible representation* $(8;k,\ell)$*, let* $(\alpha,\beta) = \Phi(k,\ell,n)$*, and define* (α',β')*,* α'' *and* β'' *by* (5.81) *and* (5.89)*, respectively. Then the isotropy subgroups containing* $\langle r \rangle$ *are given by* Σ *listed below.*

Case 1: $(\alpha,\beta) = (\alpha,0)$ *with* $1 \leq \alpha \leq n$*.*

$$\begin{cases} \text{(a) } \Sigma = \Sigma(\alpha,0) = \Sigma^{(8;k,\ell)}((1,1,1,1)), \\ \qquad \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c,c) \mid c \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(b) } \Sigma = \Sigma_0(\alpha,0) = \Sigma^{(8;k,\ell)}((c,c,c',c')) \ (c \neq c', c \neq 0, c' \neq 0), \\ \qquad \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c',c') \mid c,c' \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 2. \end{cases}$$



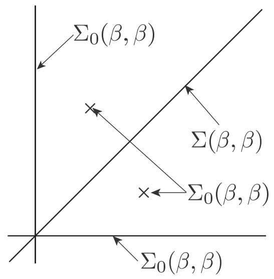
General result



Case 1. $(\alpha,\beta) = (\alpha,0)$

dim Fix$\Sigma(\alpha,0) = 1$: type V, $z = (1,1,1,1)$

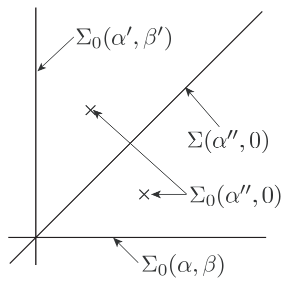
dim Fix$\Sigma_0(\alpha,0) = 2$: non-targeted



Case 2. $(\alpha,\beta) = (\beta,\beta)$

dim Fix$\Sigma(\beta,\beta) = 1$: type M, $z = (1,1,1,1)$

dim Fix$\Sigma_0(\beta,\beta) = 2$: non-targeted



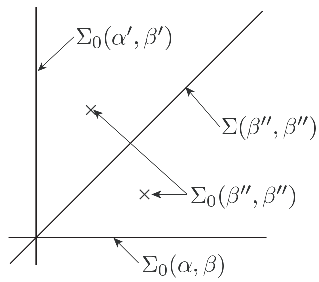
Case 3. (α,β) : $\alpha \neq \beta$,
$1 \leq \alpha \leq n-1,\ 1 \leq \beta \leq n-1,\ \hat{D} \notin 2\mathbb{Z}$

dim Fix$\Sigma(\alpha'',0) = 1$: type V, $z = (1,1,1,1)$

dim Fix$\Sigma_0(\alpha,\beta) = 1$: type T, $z = (1,1,0,0)$

dim Fix$\Sigma_0(\alpha',\beta') = 1$: type T, $z = (0,0,1,1)$

dim Fix$\Sigma_0(\alpha'',0) = 2$: non-targeted



Case 4. (α,β) : $\alpha \neq \beta$,
$1 \leq \alpha \leq n-1,\ 1 \leq \beta \leq n-1,\ \hat{D} \in 2\mathbb{Z}$

dim Fix$\Sigma(\beta'',\beta'') = 1$: type M, $z = (1,1,1,1)$

dim Fix$\Sigma_0(\alpha,\beta) = 1$: type T, $z = (1,1,0,0)$

dim Fix$\Sigma_0(\alpha',\beta') = 1$: type T, $z = (0,0,1,1)$

dim Fix$\Sigma_0(\beta'',\beta'') = 2$: non-targeted

Figure 5.6: Isotropy subgroups for $(8;k,\ell)$ with $(\alpha,\beta) = \Phi(k,\ell,n)$, (α',β') in (5.81), and (α'',β'') in (5.89).

Case 2: $(\alpha,\beta) = (\beta,\beta)$ *with* $1 \le \beta \le n/2$.

$$\begin{cases} \text{(a) } \Sigma = \Sigma(\beta,\beta) = \Sigma^{(8;k,\ell)}((1,1,1,1)), \\ \qquad \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c,c) \mid c \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(b) } \Sigma = \Sigma_0(\beta,\beta) = \Sigma^{(8;k,\ell)}((c,c,c',c')) \ (c \ne c', c \ne 0, c' \ne 0), \\ \qquad \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c',c') \mid c,c' \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 2. \end{cases}$$

Case 3: (α,β) *with* $1 \le \alpha \le n-1,\ 1 \le \beta \le n-1,\ \alpha \ne \beta$ *and* $\hat{D} \notin 2\mathbb{Z}$.

$$\begin{cases} \text{(a) } \Sigma = \Sigma(\alpha'',0) = \Sigma^{(8;k,\ell)}((1,1,1,1)), \\ \qquad \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c,c) \mid c \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(b) } \Sigma = \Sigma_0(\alpha,\beta) = \Sigma^{(8;k,\ell)}((1,1,0,0)), \\ \qquad \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,0,0) \mid c \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(c) } \Sigma = \Sigma_0(\alpha',\beta') = \Sigma^{(8;k,\ell)}((0,0,1,1)), \\ \qquad \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = \{(0,0,c',c') \mid c' \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(d) } \Sigma = \Sigma_0(\alpha'',0) = \Sigma^{(8;k,\ell)}((c,c,c',c')) \ (c \ne c', c \ne 0, c' \ne 0), \\ \qquad \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c',c') \mid c,c' \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 2. \end{cases}$$

Case 4: (α,β) *with* $1 \le \alpha \le n-1,\ 1 \le \beta \le n-1,\ \alpha \ne \beta$ *and* $\hat{D} \in 2\mathbb{Z}$.

$$\begin{cases} \text{(a) } \Sigma = \Sigma(\beta'',\beta'') = \Sigma^{(8;k,\ell)}((1,1,1,1)), \\ \qquad \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c,c) \mid c \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(b) } \Sigma = \Sigma_0(\alpha,\beta) = \Sigma^{(8;k,\ell)}((1,1,0,0)), \\ \qquad \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,0,0) \mid c \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(c) } \Sigma = \Sigma_0(\alpha',\beta') = \Sigma^{(8;k,\ell)}((0,0,1,1)), \\ \qquad \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = \{(0,0,c',c') \mid c' \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(d) } \Sigma = \Sigma_0(\beta'',\beta'') = \Sigma^{(8;k,\ell)}((c,c,c',c')) \ (c \ne c', c \ne 0, c' \ne 0), \\ \qquad \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c',c') \mid c,c' \in \mathbb{R}\}, \quad \dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 2. \end{cases}$$

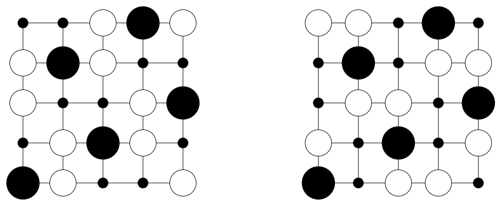
Proof. With an observation that $\Sigma_0(\alpha,\beta) \ne \Sigma_0(\alpha',\beta')$ in Cases 3 and 4, the above classification follows immediately from Propositions 5.17 and 5.18. □

Remark 5.7. In Case 1 of Proposition 5.19, we may have $\alpha = n$, in which case $\Sigma(\alpha,0) = \Sigma(0,0) = \langle r, s\rangle$ and $\Sigma_0(\alpha,0) = \Sigma_0(0,0) = \langle r\rangle$, and the translational symmetry is absent. □

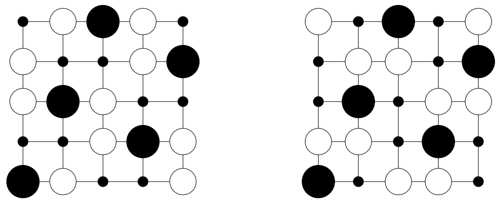
Remark 5.8. The isotropy subgroups in Proposition 5.19 can be understood quite naturally with reference to the column vectors of the matrix

$$Q^{(8;k,\ell)} = [\boldsymbol{q}_1, \dots, \boldsymbol{q}_8]$$

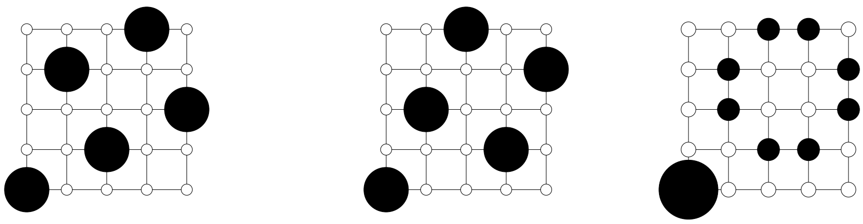
given in (4.27). The spatial patterns for these vectors are depicted in Fig. 5.7, for example, for $(8;2,1)$ with $n = 5$. Although the four vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_3$, $\boldsymbol{q}_5$, and $\boldsymbol{q}_7$ do not represent square patterns (Figs. 5.7(a)–(f)), the sum of these four vectors, which is associated with $z = (1,1,1,1)$ ($\boldsymbol{w} = (1,0,1,0,1,0,1,0)^\top$), represents a square pattern of type V with $D = 25$ (Fig. 5.7(g)). Moreover, the sum $\boldsymbol{q}_1 + \boldsymbol{q}_3$, which is associated with $z = (1,1,0,0)$, represents square pattern of type T with $D = 5$ (Fig. 5.7(e)). On the other hand, the pattern in Fig. 5.7(f), which is associated with $z = (0,0,1,1)$, represents another square pattern of type T with $D = 5$. □



(a) $\boldsymbol{q}_1$ $(\langle p_1^2 p_2, p_1^{-1} p_2^2 \rangle)$ (b) $\boldsymbol{q}_3$ $(\langle p_1^2 p_2, p_1^{-1} p_2^2 \rangle)$



(c) $\boldsymbol{q}_5$ $(\langle p_1 p_2^2, p_1^{-2} p_2 \rangle)$ (d) $\boldsymbol{q}_7$ $(\langle p_1 p_2^2, p_1^{-2} p_2 \rangle)$



(e) $\boldsymbol{q}_1 + \boldsymbol{q}_3$ $(\Sigma_0(2,1) = \langle r, p_1^2 p_2, p_1^{-1} p_2^2 \rangle)$ (f) $\boldsymbol{q}_5 + \boldsymbol{q}_7$ $(\Sigma_0(1,2) = \langle r, p_1 p_2^2, p_1^{-2} p_2 \rangle)$ (g) $\sum_{i=1,3,5,7} \boldsymbol{q}_i$ $(\mathrm{D}_4 = \langle r, s \rangle)$

Figure 5.7: Patterns on the 5×5 square lattice expressed by the column vectors of $Q^{(8;2,1)}$. A white circle denotes a positive component and a black circle denotes a negative component.

5.6.4. Existence of Bifurcating Solutions

A combination of Proposition 5.19 with the equivariant branching lemma (Section 5.2.2) shows the existence of solutions with the targeted symmetry bifurcating from a critical point associated with $(8; k, \ell)$.

Bifurcating solutions can be classified in accordance with number-theoretic properties of (k, ℓ). To be specific, it depends on the following two properties:

$$2 \gcd(\hat{k}, \hat{\ell}) \text{ is divisible by } \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}), \tag{5.90}$$

$$\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}. \tag{5.91}$$

We refer to the condition (5.90) as **GCD-div** and its negation as $\overline{\textbf{GCD-div}}$. It should be mentioned that a simplified version of the following proposition has already been presented as Proposition 5.12 in Section 5.6.2. See also Table 5.8.

Proposition 5.20. *From a critical point associated with the irreducible representation* $(8; k, \ell)$*, solutions with the following symmetries emerge as bifurcating solutions, where* $(\alpha,\beta) = \Phi(k, \ell, n)$ *and* (α',β') *is defined in* (5.81)*. We have four cases.*

Case 1: **GCD-div** *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 2\mathbb{Z}$: *We have* $\Phi(k, \ell, n) = (\alpha,\beta) = (\hat{n}, 0)$*. A bifurcating solution with symmetry* $\Sigma(\hat{n}, 0)$*, which corresponds to* $z^{(1)} = c(1, 1, 1, 1)$*, exists. This solution is of type V.*

Case 2: **GCD-div** *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}$: *We have* $\Phi(k, \ell, n) = (\alpha,\beta) = (\hat{n}/2, \hat{n}/2)$*. A bifurcating solution with symmetry* $\Sigma(\hat{n}/2, \hat{n}/2)$*, corresponding to* $z^{(1)} = c(1, 1, 1, 1)$*, exists. This solution is of type M.*

Case 3: $\overline{\textbf{GCD-div}}$ *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 2\mathbb{Z}$: *We have* $\Phi(k, \ell, n) = (\alpha,\beta)$ *with* $1 \le \alpha \le n - 1$, $1 \le \beta \le n - 1$, $\alpha \neq \beta$*, and* $\hat{D} \notin 2\mathbb{Z}$*. Bifurcating solutions with symmetries* $\Sigma(\hat{n}, 0)$*,* $\Sigma_0(\alpha,\beta)$*, and* $\Sigma_0(\alpha',\beta')$*, corresponding to* $z^{(1)} = c(1, 1, 1, 1)$*,* $z^{(2)} = c(1, 1, 0, 0)$*, and* $z^{(3)} = c(0, 0, 1, 1)$*, respectively, exist. The first solution is of type V, and the other two solutions are of type T.*

Case 4: $\overline{\textbf{GCD-div}}$ *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}$: *We have* $\Phi(k, \ell, n) = (\alpha,\beta)$ *with* $1 \le \alpha \le n - 1$, $1 \le \beta \le n - 1$, $\alpha \neq \beta$*, and* $\hat{D} \in 2\mathbb{Z}$*. Bifurcating solutions with symmetries* $\Sigma(\hat{n}/2, \hat{n}/2)$*,* $\Sigma_0(\alpha,\beta)$*, and* $\Sigma_0(\alpha',\beta')$*, corresponding to* $z^{(1)} = c(1, 1, 1, 1)$*,* $z^{(2)} = c(1, 1, 0, 0)$*, and* $z^{(3)} = c(0, 0, 1, 1)$*, respectively, exist. The first solution is of type M, and the other two solutions are of type T.*

Proof. By Proposition 5.13, as well as Remark 5.4 in Section 5.6.2, the above four cases correspond to those in Proposition 5.19. In all cases, the relevant subgroup Σ is an isotropy subgroup with $\dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 1$ by Proposition 5.19. Then the equivariant branching lemma (Section 5.2.2) guarantees the existence of a bifurcating solution with symmetry Σ. □

Remark 5.9. The subgroup $\Sigma = \Sigma_0(\alpha, 0)$, $\Sigma_0(\beta,\beta)$, $\Sigma_0(\hat{n}, 0)$ or $\Sigma_0(\hat{n}/2, \hat{n}/2)$ appearing in Proposition 5.19 is an isotropy subgroup with $\dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 2$, for which the equivariant branching lemma is not effective. It is emphasized that Proposition 5.20 does not assert the nonexistence of solutions of these symmetries. Nonetheless, we do not have to deal with these subgroups since none of these symmetries corresponds to square patterns (see (5.10)). □

5.6.5. Square Patterns of Type V

Square patterns of type V (with $D \geq 25$) are predicted to branch from critical points of multiplicity 8, whereas smaller square patterns of type V with $D = 4, 9,$ and 16 do not exist. Recall that a square pattern of type V is characterized by the symmetry of $\Sigma(\alpha, 0)$ with $2 \leq \alpha \leq n$ (see (5.10)) and that $D(\alpha, 0) = \alpha^2$.

The following propositions show such nonexistence and existence of square patterns of type V.

Proposition 5.21. *Square patterns of type V with $D = 4, 9,$ or 16 do not arise as bifurcating solutions from critical points of multiplicity 8 for any n.*

Proof. The proof is given at the end of the proof of Proposition 5.22. □

Proposition 5.22. *Square patterns of type V with the symmetry of $\Sigma(\alpha, 0)$ $(5 \leq \alpha \leq n)$ arise as bifurcating solutions from critical points of multiplicity 8 for specific values of n and irreducible representations given by*

(α,β)	D	n	(k,ℓ) in $(8;k,\ell)$
$(\alpha,0)$	α^2	αm	$((p+q)m, qm)$

(5.92)

with $m \geq 1$ and

$$p \geq 1, \quad q \geq 1, \quad \gcd(p,q,\alpha) = 1, \quad \gcd(p,\alpha) \notin 2\mathbb{Z}, \qquad \begin{cases} 2(p+q+1) \leq \alpha & \text{if } n \text{ is even and } m = 1, \\ 2(p+q)+1 \leq \alpha & \text{otherwise.} \end{cases} \tag{5.93}$$

Proof. Type V occurs in Case 1 and Case 3 in Proposition 5.20, characterized by the condition of $\gcd(\hat{k}-\hat{\ell},\hat{n}) \notin 2\mathbb{Z}$. Put $\hat{k} = p+q$ and $\hat{\ell} = q$ for some $p, q \in \mathbb{Z}$ and note $\hat{n} = \alpha$. Since $\gcd(\hat{k}-\hat{\ell},\hat{n}) = \gcd(p,\alpha)$, the condition $\gcd(\hat{k}-\hat{\ell},\hat{n}) \notin 2\mathbb{Z}$ holds if and only if $\gcd(p,\alpha) \notin 2\mathbb{Z}$. We have $(k,\ell,n) = ((p+q)m, qm, \alpha m)$ for $m = \gcd(k,\ell,n)$. Here we must have

$$1 = \gcd(\hat{k},\hat{\ell},\hat{n}) = \gcd(p+q,q,\alpha) = \gcd(p,q,\alpha).$$

The inequality constraint in (5.58) is translated as

$$1 \leq \ell \leq k-1, \ 2 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \iff$$
$$p \geq 1, \ q \geq 1, \ \begin{cases} 2(p+q+1) \leq \alpha & \text{if } n \text{ is even and } m = 1, \\ 2(p+q)+1 \leq \alpha & \text{otherwise.} \end{cases}$$

Proposition 5.22 is thus obtained.

To prove Proposition 5.21, we note that, for $\alpha = 2, 3, 4$, no (p,q) satisfies (5.93), which proves the nonexistence of the smaller square patterns claimed in Proposition 5.21. □

Example 5.3. The parameter values of (5.92) in Proposition 5.22 give Table 5.10. Here, the asterisk $(\cdot)^*$ indicates coexistence of type T (see (5.96)), i.e., Case 3 of Proposition 5.20, whereas unmarked cases correspond to Case 1 of Proposition 5.20, where no solution of type T coexists. □

Remark 5.10. In all cases in (5.92), the compatibility condition (5.12) is satisfied for $\Sigma(\alpha, 0)$ as $n = m\alpha$ with $m = \gcd(k,\ell,n)$, since we have

$$\gcd(k,\ell,n) = ((p+q)m, qm, \alpha m) = m \gcd(p+q,q,\alpha) = m \gcd(p,q,\alpha) = m$$

by (5.92) and (5.93). □

Table 5.10: Correspondence of irreducible representation $(8; k, \ell)$ to (α,β) for square patterns of type V

(α,β)	D	n	(k,ℓ) in $(8;k,\ell)$
$(5,0)$	25	$5m$	$(2m,m)^*$
$(6,0)$	36	$6m$	$(2m,m)$
$(7,0)$	49	$7m$	$(2m,m),(3m,m),(3m,2m)$
$(8,0)$	64	$8m$	$(2m,m),(3m,2m)$
$(9,0)$	81	$9m$	$(2m,m),(3m,m),(3m,2m),(4m,m),(4m,2m),(4m,3m)$
$(10,0)$	100	$10m$	$(2m,m)^*,(3m,2m),(4m,m),(4m,3m)^*$
$(11,0)$	121	$11m$	$(2m,m),(3m,m),(3m,2m),(4m,m),(4m,2m),(4m,3m),$ $(5m,m),(5m,2m),(5m,3m),(5m,4m)$
$(12,0)$	144	$12m$	$(2m,m),(3m,2m),(4m,m),(4m,3m),(5m,2m),(5m,4m)$

$m = 1, 2, \ldots$; $(\cdot)^*$ indicates coexistence of type T (Case 3)

5.6.6. Square Patterns of Type M

Larger square patterns of type M (with $D \geq 32$) are predicted to branch from critical points of multiplicity 8, whereas smaller square patterns of type M with $D = 2$, 8, and 18 do not exist. Recall that a square pattern of type M is characterized by the symmetry of $\Sigma(\beta,\beta)$ with $1 \leq \beta \leq n/2$ (see (5.10)) and that $D(\beta,\beta) = 2\beta^2$.

The following propositions show such nonexistence and existence of square patterns of type M.

Proposition 5.23. *Square patterns of type M with $D = 2$, 8, or 18 do not arise as bifurcating solutions from critical points of multiplicity 8 for any n.*

Proof. The proof is given at the end of the proof of Proposition 5.24. □

Proposition 5.24. *Square patterns of type M with the symmetry of $\Sigma(\beta,\beta)$ $(4 \leq \beta \leq n/2)$ arise as bifurcating solutions from critical points of multiplicity 8 for specific values of n and irreducible representations given by*

(α,β)	D	n	(k,ℓ) in $(8;k,\ell)$
(β,β)	$2\beta^2$	$2\beta m$	$((2p+q)m, qm)$

(5.94)

where $m \geq 1$ and

$$p \geq 1, \quad q \geq 1, \quad 2p + q \leq \beta - 1, \quad q \notin 2\mathbb{Z}, \quad \gcd(p, q, \beta) = 1. \tag{5.95}$$

Proof. Type M occurs in Case 2 and Case 4 in Proposition 5.20, characterized by the condition of $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}$. For $\hat{k} - \hat{\ell} \in 2\mathbb{Z}$ to be true, we can put $\hat{k} = 2p + q$ and $\hat{\ell} = q$ for some $p, q \in \mathbb{Z}$. Then $(k, \ell, n) = ((2p + q)m, qm, 2\beta m)$ for $m = \gcd(k, \ell, n)$. Since

$$1 = \gcd(\hat{k}, \hat{\ell}, \hat{n}) = \gcd(2p + q, q, 2\beta) = \gcd(2p, q, 2\beta),$$

we must have $q \notin 2\mathbb{Z}$ and $\gcd(p, q, \beta) = 1$. The inequality constraint in (5.58) is translated as

$$1 \le \ell \le k - 1,\ 2 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor \iff p \ge 1,\ q \ge 1,\ 2p + q \le \beta - 1.$$

Proposition 5.24 is thus proved.

Finally, for $\beta = 1, 2, 3$, no (p, q) satisfies (5.95), which proves the nonexistence of the smaller square patterns claimed in Proposition 5.23. □

Example 5.4. The parameter values of (5.94) in Proposition 5.24 give the following:

(α,β)	D	n	(k,ℓ) in $(8;k,\ell)$
$(4,4)$	32	$8m$	$(3m,m)$
$(5,5)$	50	$10m$	$(3m,m)^*$
$(6,6)$	72	$12m$	$(3m,m), (5m,m), (5m,3m)$
$(7,7)$	98	$14m$	$(3m,m), (5m,m), (5m,3m)$
$(8,8)$	128	$16m$	$(3m,m), (5m,m), (5m,3m), (7m,m), (7m,3m), (7m,5m)$
$(9,9)$	162	$18m$	$(3m,m), (5m,m), (5m,3m), (7m,m), (7m,3m), (7m,5m)$
$(10,10)$	162	$20m$	$(3m,m)^*, (5m,m), (5m,3m), (7m,m)^*, (7m,3m), (7m,5m),$ $(9m,m), (9m,3m)^*, (9m,5m), (9m,7m)^*$
$(11,11)$	242	$22m$	$(3m,m), (5m,m), (5m,3m), (7m,m), (7m,3m), (7m,5m),$ $(9m,m), (9m,3m), (9m,5m), (9m,7m)$
$(12,12)$	288	$24m$	$(3m,m), (5m,m), (5m,3m), (7m,m), (7m,3m), (7m,5m),$ $(9m,m), (9m,5m), (9m,7m), (11m,m), (11m,3m), (11m,5m),$ $(11m,7m), (11m,9m)$

where $m \ge 1$. The asterisk $(\cdot)^*$ indicates the coexistence of type T (see (5.96)), i.e., Case 4 of Proposition 5.20. The other (unmarked) cases correspond to Case 2 of Proposition 5.20, where no solution of type T coexists. The coexistence of type T is a relatively rare event; it does not occur for $n = 8m$, $12m$, and $14m$, but it recurs for $n = 10m$. □

Remark 5.11. In all cases in (5.94), the compatibility condition (5.12) for $\Sigma(\beta,\beta)$ is satisfied as $n = 2m\beta$ with $m = \gcd(k, \ell, n)$, since

$$\gcd(k, \ell, n) = \gcd((2p + q)m, qm, 2\beta m) = m \gcd(2p + q, q, 2\beta) = m \gcd(2p, q, 2\beta) = m$$

by (5.94) and (5.95). □

5.6.7. Square Patterns of Type T

Square patterns of type T are shown here to branch from critical points of multiplicity 8. Recall that a square pattern of type T is characterized by the symmetry of $\Sigma_0(\alpha,\beta)$ with $1 \le \alpha \le n - 1$, $1 \le \beta \le n - 1$, and $\alpha \ne \beta$ (see (5.10)).

The following proposition is concerned with the five square patterns of type T with $D = 5$, 10, 13, 17 and 20 among ten smallest square patterns.

Proposition 5.25. *Square patterns of type T with* $D = 5$, 10, 13, 17, *and* 20 *arise as bifurcating solutions from critical points of multiplicity 8 for specific values of n and irreducible representations given by*

(α,β)	D	n	(k,ℓ) in $(8;k,\ell)$	
			$z^{(2)} = c(1,1,0,0)$	$z^{(3)} = c(0,0,1,1)$
$(2,1)$	5	$5m$	$(2m,m)$	none
$(1,2)$			none	$(2m,m)$
$(3,1)$	10	$10m$	$(3m,m)$	none
$(1,3)$			none	$(3m,m)$
$(3,2)$	13	$13m$	$(3m,m),(6m,4m)$	$(5m,m)$
$(2,3)$			$(5m,m)$	$(3m,m),(6m,4m)$
$(4,1)$	17	$17m$	$(4m,m),(7m,6m),(8m,2m)$	$(5m,3m)$
$(1,4)$			$(5m,3m)$	$(4m,m),(7m,6m)(8m,2m)$
$(4,2)$	20	$20m$	$(4m,2m)$	$(8m,6m)$
$(2,4)$			$(8m,6m)$	$(4m,2m)$

(5.96)

where $m \geq 1$ *is an integer.*

Proof. By Proposition 5.20 (Case 3 and 4), a bifurcating solution with symmetry $\Sigma_0(\alpha,\beta)$ exists for (k,ℓ) such that $\Phi(k,\ell,n) = (\alpha,\beta)$, where the bifurcating solution corresponds to $z = c(1,1,0,0)$. For such (k,ℓ), another bifurcating solution exists, which corresponds to $z = c(0,0,1,1)$ and is endowed with the symmetry $\Sigma_0(\alpha',\beta')$ for (α',β') given by (5.81). The list of parameters in (5.96) is obtained by searching for such (k,ℓ) in the range of (5.58) using the method given in Section 5.6.9, which was previewed in Remark 5.5 in Section 5.6.3. Alternatively, we can search for such (k,ℓ) in the range of (5.58) satisfying (5.78) for a given (a,b). $\square$

For square patterns of type T, in general, the above statement extends as follows.

Proposition 5.26. *Assume* $1 \leq \alpha \leq n-1$, $1 \leq \beta \leq n-1$, *and* $\alpha \neq \beta$ *for* (α,β).

(i) *Square patterns of type T with the symmetry of* $\Sigma_0(\alpha,\beta)$ *arise as bifurcating solutions from critical points of multiplicity 8 associated with the irreducible representation* $(8;k,\ell)$ *such that* $\Phi(k,\ell,n) = (\alpha,\beta)$ *or* (α',β'), *where* (α',β') *is defined by* (5.81).

(ii) *Some* (k,ℓ,n) *exist such that* $\Phi(k,\ell,n) = (\alpha,\beta)$ *or* (α',β').

Proof. (i) The proof is the same as the proof of Proposition 5.25.

(ii) We can assume $\alpha > \beta$ by replacing (α,β) by (α',β') if necessary. Take $(k,\ell,n) = m(\hat{\alpha},\hat{\beta},D(\alpha,\beta)/\mathrm{gcd}(\alpha,\beta$ for instance. Then $m = \mathrm{gcd}(k,\ell,n)$ and $(\hat{k},\hat{\ell},\hat{n}) = (\hat{\alpha},\hat{\beta},D(\alpha,\beta)/\mathrm{gcd}(\alpha,\beta))$, and therefore

$$\hat{k}^2 + \hat{\ell}^2 = \hat{\alpha}^2 + \hat{\beta}^2 = \hat{n}/\,\mathrm{gcd}(\alpha,\beta).$$

This shows that the simpler method of computing $\Phi(k,\ell,n)$, described in Remark 5.5 in Section 5.6.3, is applicable. The right-hand side of (5.88) is calculated as

$$\frac{\hat{n}}{\hat{k}^2+\hat{\ell}^2}\begin{bmatrix}\hat{k}\\ \hat{\ell}\end{bmatrix} = \mathrm{gcd}(\alpha,\beta)\begin{bmatrix}\hat{\alpha}\\ \hat{\beta}\end{bmatrix} = \begin{bmatrix}\alpha\\ \beta\end{bmatrix},$$

which shows $\Phi(k, \ell, n) = (\alpha, \beta)$.

We also note that the chosen parameter (k, ℓ) lies in the range of (5.58). The inequality $1 \leq \ell \leq k-1$ is immediate from $\beta \geq 1$ and $\alpha > \beta$, whereas $2 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ is shown as follows. The inequality $k \geq 2$ holds since $\hat{\alpha} \geq 2$. When n is odd,

$$\begin{aligned}
\frac{2}{m}\left(\left\lfloor \frac{n-1}{2} \right\rfloor - k\right) &= \frac{1}{m}(n-1-2k) = \gcd(\alpha,\beta)(\hat{\alpha}^2 + \hat{\beta}^2) - \frac{1}{m} - 2\hat{\alpha} \\
&\geq (\hat{\alpha}^2 + \hat{\beta}^2) - 1 - 2\hat{\alpha} = \hat{\alpha}(\hat{\alpha}-2) + \hat{\beta}^2 - 1 \geq 0.
\end{aligned}$$

where $\hat{\alpha} \geq 2$ and $\hat{\beta} \geq 1$ is used in the last inequality. When n is even,

$$\frac{2}{m}\left(\left\lfloor \frac{n-1}{2} \right\rfloor - k\right) = \frac{1}{m}(n-2-2k) = \gcd(\alpha,\beta)(\hat{\alpha}^2 + \hat{\beta}^2) - \frac{2}{m} - 2\hat{\alpha}.$$

If $\hat{n}$ is odd, we have m even since n is even and

$$\frac{2}{m}\left(\left\lfloor \frac{n-1}{2} \right\rfloor - k\right) \geq (\hat{\alpha}^2 + \hat{\beta}^2) - 1 - 2\hat{\alpha} = \hat{\alpha}(\hat{\alpha}-2) + \hat{\beta}^2 - 1 \geq 0.$$

If $\hat{n}$ is even,

$$\frac{2}{m}\left(\left\lfloor \frac{n-1}{2} \right\rfloor - k\right) \geq (\hat{\alpha}^2 + \hat{\beta}^2) - 2 - 2\hat{\alpha} = [\hat{\alpha}(\hat{\alpha}-2) + \hat{\beta}^2 - 1] - 1 \geq 0$$

because $[\hat{\alpha}(\hat{\alpha}-2) + \hat{\beta}^2 - 1 \geq 1$ as $(\hat{\alpha}, \hat{\beta}) = (2, 1)$, which gives $\hat{n} = 5$, is excluded by $\hat{n}$ even. □

Square patterns of type T appear in Cases 3 and 4 in Proposition 5.20, and these two cases are characterized by a single condition

$$\overline{\textbf{GCD-div}} : 2\gcd(\hat{k}, \hat{\ell}) \text{ is not divisible by } \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}). \tag{5.97}$$

This observation yields the following statement.

Proposition 5.27. *A bifurcating solution of type T exists if and only if* $\overline{\textbf{GCD-div}}$ *holds.*

In addition, we have the following statement for some concrete cases.

Proposition 5.28. *A bifurcating solution of type T does not exist for the cases* $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{k}, \hat{k}, \hat{\ell})$*,* $(4\hat{\ell}, \hat{k}, \hat{\ell})$*, and* $(2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$*.*

Proof. First, we show that $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{k}, \hat{k}, \hat{\ell})$ contradicts the condition $\overline{\textbf{GCD-div}}$ in (5.97). Let $\gcd(\hat{k}, \hat{\ell}) = \alpha$. Then, we have $\hat{n} = 4\hat{k} = 4\alpha(\hat{k}/\alpha)$. Recall that $\gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1$. If $\alpha \neq 1$, then $\hat{n}, \hat{k}$, and $\hat{\ell}$ have a common divisor $\alpha \geq 2$. This contradicts $\gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1$. Hence, we have $\gcd(\hat{k}, \hat{\ell}) = \alpha = 1$. Thus, we rewrite (5.97) as

$$\overline{\textbf{GCD-div}} \text{ for } (\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{k}, \hat{k}, \hat{\ell}) : 2 \text{ is not divisible by } \gcd(\hat{k}^2 + \hat{\ell}^2, 4\hat{k}). \tag{5.98}$$

This condition is equivalent to that $\hat{k}^2 + \hat{\ell}^2$ and $4\hat{k}$ have 4 or a prime number $m \geq 3$ as a common divisor.

- For the case that $\hat{k}^2 + \hat{\ell}^2$ and $4\hat{k}$ have 4 as a common divisor, we have

$$\hat{k}^2 + \hat{\ell}^2 = 4p. \tag{5.99}$$

Here, p is a positive integer. Using $\hat{k}^2 + \hat{\ell}^2 = (\hat{k} - \hat{\ell})^2 + 2\hat{k}\hat{\ell}$, we have

$$(\hat{k} - \hat{\ell})^2 + 2\hat{k}\hat{\ell} = 4p. \tag{5.100}$$

Recall that $\gcd(\hat{k}, \hat{\ell}) = 1$. Hence, either $\hat{k}$ or $\hat{\ell}$, or both are odd. When we consider either $\hat{k}$ or $\hat{\ell}$ is odd, we see that $\hat{k} - \hat{\ell}$ is odd. Hence, $(\hat{k} - \hat{\ell})^2$ is odd. Thus, $(\hat{k} - \hat{\ell})^2 + 2\hat{k}\hat{\ell}$ is odd. This contradicts (5.100). On the other hand, when we consider both $\hat{k}$ and $\hat{\ell}$ are odd, we see that $\hat{k} - \hat{\ell}$ is even. Hence, $(\hat{k} - \hat{\ell})^2$ is divisible by 4. Since $\hat{k}\hat{\ell}$ is odd, $2\hat{k}\hat{\ell}$ is not divisible by 4. Hence, $(\hat{k} - \hat{\ell})^2 + 2\hat{k}\hat{\ell}$ is not divisible by 4. This contradicts (5.100).

- For the case that $\hat{k}^2 + \hat{\ell}^2$ and $4\hat{k}$ have a prime number $m \geq 3$ as a common divisor, we have

$$\hat{k}^2 + \hat{\ell}^2 = mp, \tag{5.101}$$

$$4\hat{k} = mq. \tag{5.102}$$

Here, p and q are positive integers. Multiplying the both sides of (5.101) by q, we have

$$q(\hat{k}^2 + \hat{\ell}^2) = mpq. \tag{5.103}$$

Multiplying the both sides of (5.102) by p, we have

$$4p\hat{k} = mpq. \tag{5.104}$$

Combining (5.103) and (5.104), we have $q(\hat{k}^2 + \hat{\ell}^2) = 4p\hat{k}$. Hence, we have $q\hat{\ell}^2 = \hat{k}(4p - q\hat{k})$. Since $\gcd(\hat{k}, \hat{\ell}) = 1$, q is divisible by $\hat{k}$. Hence, we have $q = r\hat{k}$ with some positive integer r. Substituting this into (5.102), we have $m = 4/r$. Recall that $m \geq 3$. From this, we have $r = 1$. Hence, we have $m = 4/r = 4$. Thus, we have $q = r\hat{k} = \hat{k}$. Substituting this into (5.103), we have $\hat{k}^2 + \hat{\ell}^2 = 4p$. Using $\hat{k}^2 + \hat{\ell}^2 = (\hat{k} - \hat{\ell})^2 + 2\hat{k}\hat{\ell}$, we have

$$(\hat{k} - \hat{\ell})^2 + 2\hat{k}\hat{\ell} = 4p. \tag{5.105}$$

This condition is equivalent to (5.100) in the above case. Hence, we have contradiction in a similar manner to the above case.

Thus, we see that $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{k}, \hat{k}, \hat{\ell})$ contradicts $\overline{\textbf{GCD-div}}$. In the same way, we can see that $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{\ell}, \hat{k}, \hat{\ell})$ contradicts $\overline{\textbf{GCD-div}}$.

Next, we show that $(\hat{n}, \hat{k}, \hat{\ell}) = (2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$ contradicts the condition $\overline{\textbf{GCD-div}}$ in (5.97). Let $\gcd(\hat{k}, \hat{\ell}) = \alpha$. Then, $\hat{n} = 2\hat{k} + 2\hat{\ell} = 2\alpha(\hat{k} + \hat{\ell})/\alpha$. Recall that $\gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1$. If $\alpha \neq 1$, then $\hat{n}, \hat{k}$, and $\hat{\ell}$ have a common divisor α. This contradicts $\gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1$. Hence, we have $\gcd(\hat{k}, \hat{\ell}) = \alpha = 1$. Thus, we rewrite $\overline{\textbf{GCD-div}}$ as

$$\overline{\textbf{GCD-div}} \text{ for } (\hat{n}, \hat{k}, \hat{\ell}) = (2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell}) : 2 \text{ is not divisible by } \gcd(\hat{k}^2 + \hat{\ell}^2, 2\hat{k} + 2\hat{\ell}). \tag{5.106}$$

This condition is equivalent to that $\hat{k}^2 + \hat{\ell}^2$ and $2\hat{k} + 2\hat{\ell}$ have 4 or a prime number $m \geq 3$ as a common divisor.

- For the case that $\hat{k}^2 + \hat{\ell}^2$ and $2\hat{k} + 2\hat{\ell}$ have 4 as a common divisor, we have

$$\hat{k}^2 + \hat{\ell}^2 = 4p, \tag{5.107}$$
$$2\hat{k} + 2\hat{\ell} = 4q. \tag{5.108}$$

  Here, p and q are positive integers. From (5.108), we have $\hat{k} + \hat{\ell} = 2q$. Since $\gcd(\hat{k}, \hat{\ell}) = \alpha = 1$, $\hat{k}$ and $\hat{\ell}$ are not both even. Hence, we have

$$\hat{k} = 2r + 1, \tag{5.109}$$
$$\hat{\ell} = 2s + 1. \tag{5.110}$$

  Here, r and s are positive integers. Substituting (5.109) and (5.110) into (5.107), we have $(2r + 1)^2 + (2s + 1)^2 = 4p$. Rearranging this, we have

$$p - r(r + 1) - s(s + 1) = 1/2. \tag{5.111}$$

  This equality has contradiction since $p - r(r + 1) - s(s + 1)$ is an integer.

- For the case that $\hat{k}^2 + \hat{\ell}^2$ and $2\hat{k} + 2\hat{\ell}$ have a prime number $m \geq 3$ as a common divisor, we have

$$\hat{k}^2 + \hat{\ell}^2 = mp, \tag{5.112}$$
$$2\hat{k} + 2\hat{\ell} = mq. \tag{5.113}$$

  Here, p and q are positive integers. Using $\hat{k}^2 + \hat{\ell}^2 = (\hat{k} + \hat{\ell})^2 - 2\hat{k}\hat{\ell}$, we have

$$(\hat{k} + \hat{\ell})^2 - 2\hat{k}\hat{\ell} = mp. \tag{5.114}$$

  Substituting (5.113) into (5.114), we have $q^2m^2/4 - 2\hat{k}\hat{\ell} = mp$. Rearranging this, we have

$$8\hat{k}\hat{\ell}/m = -4p + mq^2. \tag{5.115}$$

  Hence, $\hat{k}/m$ or $\hat{\ell}/m$ is an integer. When we consider $\hat{k}/m$ is an integer, we have $\hat{k} = mr$ with some positive integer r. From (5.113), we have $\hat{\ell} = m(q - r)$. Hence, $\hat{\ell}$ and $\hat{k}$ has m as a common divisor. This contradicts $\gcd(\hat{k}, \hat{\ell}) = 1$. When we consider $\hat{\ell}/m$ is an integer, we have contradiction in a similar manner.

Thus, we see that $(\hat{n}, \hat{k}, \hat{\ell}) = (2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$ contradicts $\overline{\textbf{GCD-div}}$. □

Remark 5.12. The compatibility condition (5.12) for $\Sigma_0(\alpha, \beta)$ is satisfied as

$$n = m\,\frac{D(\alpha, \beta)}{\gcd(\alpha, \beta)}$$

with $m = \gcd(k, \ell, n)$ by (5.68) with (5.67). □

Table 5.11: Square patterns of types V, M, and T arising from critical points of multiplicity 8 for the $n \times n$ square lattices with $n = 5, 6, 10, 13, 17$, and 18 ($\hat{D}$ is defined in (5.72))

n	(k, ℓ) in $(8; k, \ell)$	$\hat{n}$	z	(α, β)	D	$\hat{D}$	Type
5	$(2, 1)$	5	$z^{(1)}$	$(5, 0)$	25	1	V
		5	$z^{(2)}$	$(2, 1)$	5	5	T
		5	$z^{(3)}$	$(1, 2)$	5	5	T
6	$(2, 1)$	6	$z^{(1)}$	$(6, 0)$	36	1	V
10	$(4, 2)$	5	$z^{(1)}$	$(5, 0)$	25	1	V
			$z^{(2)}$	$(2, 1)$	5	5	T
			$z^{(3)}$	$(1, 2)$	5	5	T
	$(3, 1)$	10	$z^{(1)}$	$(5, 5)$	50	2	M
			$z^{(2)}$	$(3, 1)$	10	10	T
			$z^{(3)}$	$(1, 3)$	10	10	T
	$(2, 1)$	10	$z^{(1)}$	$(10, 0)$	100	1	V
			$z^{(2)}$	$(4, 2)$	20	20	T
			$z^{(3)}$	$(2, 4)$	20	20	T
	$(4, 3)$	10	$z^{(1)}$	$(10, 0)$	100	1	V
			$z^{(2)}$	$(2, 4)$	20	20	T
			$z^{(3)}$	$(4, 2)$	20	20	T
	$(3, 2), (4, 1)$	10	$z^{(1)}$	$(10, 0)$	100	1	V
13	$(3, 2), (6, 4)$	13	$z^{(1)}$	$(13, 0)$	169	1	V
			$z^{(2)}$	$(3, 2)$	13	13	T
			$z^{(3)}$	$(2, 3)$	13	13	T
	$(5, 1)$	13	$z^{(1)}$	$(13, 0)$	169	1	V
			$z^{(2)}$	$(2, 3)$	13	13	T
			$z^{(3)}$	$(3, 2)$	13	13	T
	other (k, ℓ)'s	13	$z^{(1)}$	$(13, 0)$	169	1	V
17	$(4, 1), (7, 6), (8, 2)$	17	$z^{(1)}$	$(17, 0)$	17^2	1	V
			$z^{(2)}$	$(4, 1)$	17	17	T
			$z^{(3)}$	$(1, 4)$	17	17	T
	$(5, 3)$	17	$z^{(1)}$	$(17, 0)$	17^2	1	V
			$z^{(2)}$	$(1, 4)$	17	17	T
			$z^{(3)}$	$(4, 1)$	17	17	T
	other (k, ℓ)'s	17	$z^{(1)}$	$(17, 0)$	17^2	1	V
18	$(6, 3)$	6	$z^{(1)}$	$(6, 0)$	36	1	V
	$(4, 2), (6, 2), (6, 4), (8, 2), (8, 4), (8, 6)$	9	$z^{(1)}$	$(9, 0)$	81	1	V
	$(2, 1), (3, 2), (4, 1), (4, 3), (5, 2), (5, 4), (6, 1), (6, 5)$	18	$z^{(1)}$	$(18, 0)$	18^2	1	V
	$(7, 2), (7, 4), (7, 6), (8, 1), (8, 3), (8, 5), (8, 7)$						
	$(3, 1), (5, 1), (5, 3), (7, 1), (7, 3), (7, 5)$	18	$z^{(1)}$	$(9, 9)$	162	2	M

Table 5.12: Square patterns of types V, M, and T arising from critical points of multiplicity 8 for the $n \times n$ square lattice with $n = 20$ and 24 ($\hat{D}$ is defined in (5.72))

n	(k, ℓ) in $(8; k, \ell)$	$\hat{n}$	z	(α, β)	D	$\hat{D}$	Type
20	$(8, 4)$	5	$z^{(1)}$	$(5, 0)$	25	1	V
			$z^{(2)}$	$(2, 1)$	5	5	T
			$z^{(3)}$	$(1, 2)$	5	5	T
	$(6, 2)$	10	$z^{(1)}$	$(5, 5)$	50	2	M
			$z^{(2)}$	$(3, 1)$	10	10	T
			$z^{(3)}$	$(1, 3)$	10	10	T
	$(4, 2)$	10	$z^{(1)}$	$(10, 0)$	100	1	V
			$z^{(2)}$	$(4, 2)$	20	20	T
			$z^{(3)}$	$(2, 4)$	20	20	T
	$(8, 6)$	10	$z^{(1)}$	$(10, 0)$	100	1	V
			$z^{(2)}$	$(2, 4)$	20	20	T
			$z^{(3)}$	$(4, 2)$	20	20	T
	$(3, 1), (9, 3)$	20	$z^{(1)}$	$(10, 10)$	200	2	M
			$z^{(2)}$	$(6, 2)$	40	40	T
			$z^{(3)}$	$(2, 6)$	40	40	T
	$(7, 1), (9, 7)$	20	$z^{(1)}$	$(10, 10)$	200	2	M
			$z^{(2)}$	$(2, 6)$	40	40	T
			$z^{(3)}$	$(6, 2)$	40	40	T
	$(4, 3), (7, 4), (8, 1), (9, 8)$	20	$z^{(1)}$	$(20, 0)$	400	1	V
			$z^{(2)}$	$(8, 4)$	80	80	T
			$z^{(3)}$	$(4, 8)$	80	80	T
	$(2, 1), (6, 3), (7, 6), (9, 2)$	20	$z^{(1)}$	$(20, 0)$	400	1	V
			$z^{(2)}$	$(4, 8)$	80	80	T
			$z^{(3)}$	$(8, 4)$	80	80	T
	$(6, 4), (8, 2)$	10	$z^{(1)}$	$(10, 0)$	100	1	V
	$(3, 2), (4, 1), (5, 2), (5, 4), (6, 1), (6, 5)$	20	$z^{(1)}$	$(20, 0)$	400	1	V
	$(7, 2), (8, 3), (8, 5), (8, 7), (9, 4), (9, 6)$						
	$(5, 1), (5, 3), (7, 3), (7, 5), (9, 1), (9, 5)$	10	$z^{(1)}$	$(10, 10)$	200	2	M
24	$(8, 4)$	6	$z^{(1)}$	$(6, 0)$	36	1	V
	$(6, 3), (9, 6)$	8	$z^{(1)}$	$(8, 0)$	64	1	V
	$(4, 2), (6, 4), (8, 2), (8, 6), (10, 4), (10, 8)$	12	$z^{(1)}$	$(12, 0)$	144	1	V
	$(2, 1), (3, 2), (4, 1), (4, 3), (5, 2), (5, 4), (6, 1), (6, 5)$	24	$z^{(1)}$	$(24, 0)$	24^2	1	V
	$(7, 2), (7, 4), (7, 6), (8, 1), (8, 3), (8, 5), (8, 7), (9, 2)$						
	$(9, 4), (9, 8), (10, 1), (10, 3), (10, 5), (10, 7), (10, 9)$						
	$(11, 2), (11, 4), (11, 6, (11, 8), (11, 10)$						
	$(9, 3)$	8	$z^{(1)}$	$(4, 4)$	32	2	M
	$(6, 2), (10, 2), (10, 6)$	12	$z^{(1)}$	$(6, 6)$	72	2	M
	$(3, 1), (5, 1), (5, 3), (7, 1), (7, 3), (7, 5), (9, 1)$	24	$z^{(1)}$	$(12, 12)$	288	2	M
	$(9, 5), (9, 7), (11, 1), (11, 3), (11, 5), (11, 7), (11, 9)$						

5.6.8. Possible Square patterns for Several Lattice Sizes

In Sections 5.6.5–5.6.7 we have investigated possible occurrences of square patterns for each of the three types V, M, and T, and have enumerated all possible combinations of lattice size n and irreducible representation $(8; k, \ell)$ that can potentially engender square patterns. By compiling these results, we can capture, for each n, all square patterns that can potentially arise from critical points of multiplicity 8. The results are given in Tables 5.11 and 5.12 for several lattice sizes. The results are also incorporated in Table 5.1. Recall from Proposition 5.20 in Section 5.6.4 that bifurcating square patterns are associated with

$$z = \begin{cases} z^{(1)} = c(1,1,1,1) & \text{corresponding to type V or type M,} \\ z^{(2)} = c(1,1,0,0) & \text{corresponding to type T,} \\ z^{(3)} = c(0,0,1,1) & \text{corresponding to type T.} \end{cases}$$

For $n = 5$, square patterns of type T exist for the irreducible representation $(8; k, \ell) = (8; 2, 1)$ with $\Sigma_0(\alpha, \beta) = \Sigma_0(2, 1)$ and $\Sigma_0(1, 2)$. No hexagon of type M or T exists for the lattice sizes of $n = 6$. For a composite number $n = 20$ with several divisors, square patterns of various kinds exist. Subgroups of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ expressing square patterns satisfy the inclusion relations given below.

Example 5.5. For $n = 20$, possible square patterns are of type V, M, and T. Subgroups for square patterns of type T have inclusion relations

$$\Sigma_0(2,1) \supset \left\{ \begin{array}{ccc} \Sigma_0(1,3) & \supset & \Sigma_0(2,6) \\ \Sigma_0(4,2) & \supset & \Sigma_0(8,4) \end{array} \right\} \supset \Sigma_0(20,0) = \langle r \rangle,$$

$$\Sigma_0(1,2) \supset \left\{ \begin{array}{ccc} \Sigma_0(3,1) & \supset & \Sigma_0(6,2) \\ \Sigma_0(2,4) & \supset & \Sigma_0(4,8) \end{array} \right\} \supset \Sigma_0(20,0) = \langle r \rangle,$$

and satisfy

$$\begin{aligned} &\Sigma_0(3,1) \cap \Sigma_0(1,3) = \Sigma_0(5,5), \\ &\Sigma_0(4,2) \cap \Sigma_0(2,4) = \Sigma_0(10,0), \\ &\Sigma_0(6,2) \cap \Sigma_0(2,6) = \Sigma_0(10,10), \\ &\Sigma_0(8,4) \cap \Sigma_0(4,8) = \Sigma_0(20,0) = \langle r \rangle. \end{aligned}$$

In addition, subgroups for square patterns of type V and type M satisfy

$$\begin{aligned} \Sigma(1,0) \supset &\left\{ \begin{array}{ccccccccc} \Sigma(1,1) & \supset & \Sigma(2,0) & \supset & \Sigma(2,2) & \supset & \Sigma(4,0) & \supset & \Sigma(4,4) \\ \Sigma(5,0) & \supset & \Sigma(5,5) & \supset & \Sigma(10,0) & \supset & \Sigma(10,10) & & \end{array} \right\} \\ &\supset \Sigma(20,0) = \langle r, s \rangle. \end{aligned}$$

□

In particular, possible square patterns for $n = 18$, 20, and 24 for critical points of all kinds of multiplicity ($M = 1, 2, 4, 8$) are classified in Tables 5.13 and 5.14.

5.6.9. Appendix: Construction of the Function Φ

A systematic construction procedure of the function Φ in (5.85) is given here.

Table 5.13: Square patterns of types V and M arising from for critical points of all kinds of multiplicity ($M = 1, 2, 4, 8$) for the $n \times n$ square lattice with $n = 18$ and 24

n	μ or (k, ℓ) in $(4; k, \ell)$ or (k, ℓ) in $(8; k, \ell)$	(α, β)	D	Type	M
18	$(1; +, +, -)$	$(1, 1)$	2	M	1
	$(2; +, +)$	$(2, 0)$	4	V	2
	$(6, 0)$	$(3, 0)$	9	V	4
	$(6, 6)$				
	$(3, 0)$	$(6, 0)$	36	V	
	$(9, 6)$				
	$(2, 0), (4, 0), (8, 0)$	$(9, 0)$	81	V	
	$(2, 2), (4, 4), (8, 8)$				
	$(1, 0), (5, 0), (7, 0)$	$(18, 0)$	324	V	
	$(9, 2), (9, 4), (9, 8)$				
	$(3, 3)$	$(3, 3)$	18	M	
	$(9, 3)$		18	M	
	$(1, 1), (5, 5), (7, 7)$	$(9, 9)$	162	M	
	$(9, 1), (9, 5), (9, 7)$		162	M	
	$(6, 3)$	$(6, 0)$	36	V	8
	$(4, 2), (6, 2), (6, 4), (8, 2), (8, 4), (8, 6)$	$(9, 0)$	81	V	
	$(2, 1), (3, 2), (4, 1), (4, 3), (5, 2), (5, 4), (6, 1), (6, 5)$	$(18, 0)$	18^2	V	
	$(7, 2), (7, 4), (7, 6), (8, 1), (8, 3), (8, 5), (8, 7)$				
	$(3, 1), (5, 1), (5, 3), (7, 1), (7, 3), (7, 5)$	$(9, 9)$	162	M	
24	$(1; +, +, -)$	$(1, 1)$	2	M	1
	$(2; +, +)$	$(2, 0)$	4	V	2
	$(8, 0)$	$(3, 0)$	9	V	4
	$(8, 8)$				
	$(6, 0)$	$(4, 0)$	16	V	
	$(12, 6)$				
	$(4, 0)$	$(6, 0)$	36	V	
	$(12, 8)$				
	$(3, 0), (9, 0)$	$(8, 0)$	64	V	
	$(12, 3), (12, 9)$				
	$(2, 0), (10, 0)$	$(12, 0)$	144	V	
	$(12, 2), (12, 10)$				
	$(1, 0), (5, 0), (7, 0), (11, 0)$	$(24, 0)$	576	V	
	$(12, 1), (12, 5), (12, 7)(12, 11)$				
	$(6, 6)$	$(2, 2)$	8	M	
	$(4, 4)$	$(3, 3)$	18	M	
	$(12, 4)$				
	$(3, 3), (9, 9)$	$(4, 4)$	32	M	
	$(2, 2), (10, 10)$	$(6, 6)$	72	M	
	$(1, 1), (5, 5), (7, 7), (11, 11)$	$(12, 12)$	288	M	
	$(8, 4)$	$(6, 0)$	36	V	8
	$(6, 3), (9, 6)$	$(8, 0)$	64	V	
	$(4, 2), (6, 4), (8, 2), (8, 6), (10, 4), (10, 8)$	$(12, 0)$	144	V	
	$(2, 1), (3, 2), (4, 1), (4, 3), (5, 2), (5, 4), (6, 1), (6, 5)$	$(24, 0)$	24^2	V	
	$(7, 2), (7, 4), (7, 6), (8, 1), (8, 3), (8, 5), (8, 7), (9, 2)$				
	$(9, 4), (9, 8), (10, 1), (10, 3), (10, 5), (10, 7), (10, 9)$				
	$(11, 2), (11, 4), (11, 6), (11, 8), (11, 10)$				
	$(9, 3)$	$(4, 4)$	32	M	
	$(6, 2), (10, 2), (10, 6)$	$(6, 6)$	72	M	
	$(3, 1), (5, 1), (5, 3), (7, 1), (7, 3), (7, 5), (9, 1)$	$(12, 12)$	288	M	
	$(9, 5), (9, 7), (11, 1), (11, 3), (11, 5), (11, 7), (11, 9)$				

Table 5.14: Square patterns of types V, M, and T arising from for critical points of all kinds of multiplicity ($M = 1, 2, 4, 8$) for the $n \times n$ square lattice with $n = 20$

n	μ or (k, ℓ) in $(4; k, \ell)$ or (k, ℓ) in $(8; k, \ell)$	(α, β)	D	Type	M
20	$(1; +, +, -)$	$(1, 1)$	2	M	1
	$(2; +, +)$	$(2, 0)$	4	V	2
	$(5, 0)$	$(4, 0)$	16	V	4
	$(10, 5)$				
	$(4, 0), (8, 0)$	$(5, 0)$	25	V	
	$(4, 4), (8, 8)$				
	$(2, 0), (6, 0)$	$(10, 0)$	100	V	
	$(10, 4), (10, 8)$				
	$(1, 0), (3, 0), (7, 0), (9, 0)$	$(20, 0)$	400	V	
	$(10, 1), (10, 3), (10, 7), (10, 9)$				
	$(5, 5)$	$(2, 2)$	8	M	
	$(2, 2), (6, 6)$	$(5, 5)$	50	M	
	$(10, 2), (10, 6)$				
	$(1, 1), (3, 3), (7, 7), (9, 9)$	$(10, 10)$	200	M	
	$(8, 4)$	$(5, 0)$	25	V	8
		$(2, 1)$	5	T	
		$(1, 2)$	5	T	
	$(6, 2)$	$(5, 5)$	50	M	
		$(3, 1)$	10	T	
		$(1, 3)$	10	T	
	$(4, 2)$	$(10, 0)$	100	V	
		$(4, 2)$	20	T	
		$(2, 4)$	20	T	
	$(8, 6)$	$(10, 0)$	100	V	
		$(2, 4)$	20	T	
		$(4, 2)$	20	T	
	$(3, 1), (9, 3)$	$(10, 10)$	200	M	
		$(6, 2)$	40	T	
		$(2, 6)$	40	T	
	$(7, 1), (9, 7)$	$(10, 10)$	200	M	
		$(2, 6)$	40	T	
		$(6, 2)$	40	T	
	$(4, 3), (7, 4), (8, 1), (9, 8)$	$(20, 0)$	400	V	
		$(8, 4)$	80	T	
		$(4, 8)$	80	T	
	$(2, 1), (6, 3), (7, 6), (9, 2)$	$(20, 0)$	400	V	
		$(4, 8)$	80	T	
		$(8, 4)$	80	T	
	$(6, 4), (8, 2)$	$(10, 0)$	100	V	
	$(3, 2), (4, 1), (5, 2), (5, 4), (6, 1), (6, 5)$	$(20, 0)$	400	V	
	$(7, 2), (8, 3), (8, 5), (8, 7), (9, 4), (9, 6)$				
	$(5, 1), (5, 3), (7, 3), (7, 5), (9, 1), (9, 5)$	$(10, 10)$	200	M	

Basic Facts about Integer Matrices

We present here some basic facts about integer matrices[9] that are used in the construction of the correspondence Φ and in the proofs in Section 5.6.10.

A square integer matrix U is called *unimodular* if its determinant is equal to ± 1; U is unimodular if and only if its inverse U^{-1} exists and is an integer matrix. For an integer matrix A, the kth *determinantal divisor*, denoted $d_k(A)$, is the greatest common divisor of all $k \times k$ minors (subdeterminants) of A. By convention we put $d_0(A) = 1$.

The first theorem states that every integer matrix can be brought to the *Smith normal form* by a bilateral unimodular transformation.

Theorem 5.1. *Let A be an $m \times n$ integer matrix. There exist unimodular matrices U and V such that*

$$UAV = \left[\begin{array}{ccc|c} \alpha_1 & & 0 & \\ & \ddots & & 0_{r,n-r} \\ 0 & & \alpha_r & \\ \hline & 0_{m-r,r} & & 0_{m-r,n-r} \end{array}\right], \tag{5.116}$$

where $r = \mathrm{rank}\, A$ and $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_r$ are positive integers with the divisibility property:[10]

$$\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_r.$$

Such integers $\alpha_1, \alpha_2, \ldots, \alpha_r$ are uniquely determined by A, and are expressed as

$$\alpha_k = \frac{d_k(A)}{d_{k-1}(A)}, \quad k = 1, \ldots, r,$$

in terms of the determinantal divisors $d_1(A), d_2(A), \ldots, d_r(A)$ of A.

The second theorem gives a solvability criterion for a system of linear equations in unknown integer vectors.

Theorem 5.2. *Let A be an $m \times n$ integer matrix and $\boldsymbol{b}$ an m-dimensional integer vector. The following two conditions* (a) *and* (b) *are equivalent.*

(a) *The system of equations $A\boldsymbol{x} = \boldsymbol{b}$ admits an integer solution $\boldsymbol{x}$.*
(b) *Two matrices A and $[A \mid \boldsymbol{b}]$ share the same determinantal divisors, i.e.,* $\mathrm{rank}\, A = \mathrm{rank}\, [A \mid \boldsymbol{b}]$ *and $d_k(A) = d_k([A \mid \boldsymbol{b}])$ for all k.*

As a corollary of Theorem 5.2 we can obtain the following facts.

Proposition 5.29. *Let $a_1, \ldots, a_n$ be integers.*

(i) $\gcd(a_1, \ldots, a_n) = 1$ *if and only if there exist some integers $x_1, \ldots, x_n$ such that $a_1x_1 + \cdots + a_nx_n = 1$.*

(ii) *An integer b is divisible by* $\gcd(a_1, \ldots, a_n)$ *if and only if there exist some integers $x_1, \ldots, x_n$ such that $a_1x_1 + \cdots + a_nx_n = b$.*

[9]See Schrijver, 1986 [29] for more details on integer matrices.

[10]Notation "$a \mid b$" means that a divides b, that is, b is a multiple of a.

The third theorem is a kind of duality theorem, which is sometimes referred to as the *integer analogue of the Farkas lemma*.

Theorem 5.3. *Let A be an $m \times n$ integer matrix and $\boldsymbol{b}$ an m-dimensional integer vector. The following two conditions* (a) *and* (b) *are equivalent.*

(a) *The system of equations $A\boldsymbol{x} = \boldsymbol{b}$ admits an integer solution $\boldsymbol{x}$.*
(b) *We have "$\boldsymbol{y}^\top A \in \mathbb{Z}^n \implies \boldsymbol{y}^\top \boldsymbol{b} \in \mathbb{Z}$" for any m-dimensional vector $\boldsymbol{y}$.*

Construction of Φ via the Smith Normal Form

The correspondence $\Phi : (k, \ell) \mapsto (\alpha, \beta)$ can be constructed with the aid of the Smith normal form. Recall notations

$$\hat{k} = \frac{k}{\gcd(k,\ell,n)}, \quad \hat{\ell} = \frac{\ell}{\gcd(k,\ell,n)}, \quad \hat{n} = \frac{n}{\gcd(k,\ell,n)}$$

in (5.67), for which

$$\gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1. \tag{5.117}$$

By the definition of the correspondence Φ of (5.85) in Proposition 5.17, we have

$$\mathcal{A}(k,\ell,n) = \mathcal{L}(\alpha,\beta) \quad \text{for} \quad (\alpha,\beta) = \Phi(k,\ell,n), \tag{5.118}$$

where

$$\mathcal{A}(k,\ell,n) = \{(a,b) \in \mathbb{Z}^2 \mid \hat{k}a + \hat{\ell}b \equiv 0,\ \hat{\ell}a - \hat{k}b \equiv 0 \mod \hat{n}\}, \tag{5.119}$$

$$\mathcal{L}(\alpha,\beta) = \{(a,b) \in \mathbb{Z}^2 \mid (a,b) = n_1(\alpha,\beta) + n_2(-\beta,\alpha),\ n_1, n_2 \in \mathbb{Z}\}. \tag{5.120}$$

The condition in the definition of $\mathcal{A}(k,\ell,n)$ can be rewritten in a matrix form as

$$\begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mod \hat{n}. \tag{5.121}$$

We define matrices K and A as

$$K = \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} \end{bmatrix}, \quad A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \tag{5.122}$$

which play the key role in our analysis. Note that

$$\mathcal{L}(\alpha,\beta) = \{(a,b) \mid \begin{bmatrix} a \\ b \end{bmatrix} = A \begin{bmatrix} n_1 \\ n_2 \end{bmatrix};\ n_1, n_2 \in \mathbb{Z}\} \tag{5.123}$$

by (5.120).

The condition for $\mathcal{A}(k,\ell,n)$ in (5.121) is equivalent to the existence of integers p and q such that

$$\left[\begin{array}{cc|cc} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{array}\right] \begin{bmatrix} a \\ b \\ p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.124}$$

Since the determinantal divisors d_1 and d_2 of this 2×4 coefficient matrix are

$$\begin{aligned} d_1 &= \gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1, \\ d_2 &= \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{k}\hat{n}, \hat{\ell}\hat{n}, \hat{n}^2) = \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}\gcd(\hat{k}, \hat{\ell}, \hat{n})) \\ &= \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}), \end{aligned}$$

the Smith normal form of that matrix is given (see Theorem 5.1) as

$$U \left[\begin{array}{cc|cc} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{array} \right] V = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \end{array} \right], \tag{5.125}$$

where U and V are unimodular matrices and

$$\kappa = \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}). \tag{5.126}$$

The 4×4 matrix V for the Smith normal form in (5.125) affords an explicit representation of the correspondence Φ that is defined rather implicitly by the relationship in (5.118). As stated in the following proposition, the correspondence $(\alpha, \beta) = \Phi(k, \ell, n)$ is encoded in the upper-right block of a suitably chosen matrix V. Partition the matrix V into 2×2 submatrices as

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix},$$

and recall the matrix A in (5.122) that is parameterized by (α, β).

Proposition 5.30. *We can take V such that $V_{12} = A$ for some (α, β) with $\alpha > \beta \geq 0$. Then $\Phi(k, \ell, n) = (\alpha, \beta)$.*

Proof. Putting

$$\boldsymbol{a} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \boldsymbol{p} = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = V^{-1} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{p} \end{bmatrix}$$

and using (5.125), we can rewrite (5.124) as

$$U[K \mid -\hat{n}I]V \cdot V^{-1} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{p} \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \end{array} \right] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \mathbf{0}.$$

This shows that $\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{y}$ is free. Therefore, the solutions of (5.124) are given as

$$\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{p} \end{bmatrix} = V \begin{bmatrix} \mathbf{0} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y} \in \mathbb{Z}^2.$$

This means, by (5.118), that

$$\mathcal{L}(\alpha, \beta) = \{\boldsymbol{a} = (a, b)^\top \mid \boldsymbol{a} = V_{12}\boldsymbol{y},\ \boldsymbol{y} \in \mathbb{Z}^2\}.$$

By comparing this with (5.123), we see that the column vectors of V_{12} and those of A are both basis vectors of the same lattice. As is well-known, this implies that the matrices V_{12} and A are related as $V_{12}W = A$ for some unimodular matrix W. Therefore,

$$\tilde{V} = V \begin{bmatrix} I & O \\ O & W \end{bmatrix} = \begin{bmatrix} \tilde{V}_{11} & \tilde{V}_{12} \\ \tilde{V}_{21} & \tilde{V}_{22} \end{bmatrix}$$

is also a valid choice for the Smith normal form (5.125), with the property that $\tilde{V}_{12} = A$. □

In what follows we assume $V_{12} = A$, i.e.,

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} V_{11} & A \\ V_{21} & V_{22} \end{bmatrix}. \tag{5.127}$$

Remark 5.13. In Remark 5.5 in Section 5.6.3, we indicated a simpler construction of Φ that works when $\hat{n}/(\hat{k}^2 + \hat{\ell}^2)$ is an integer. This simpler construction can also be understood in the framework of the general method here. Let U and V_{11} be some unimodular matrices that transform the matrix K in (5.122) to its Smith normal form: $UKV_{11} = \mathrm{diag}\,(1, \kappa)$. By choosing

$$V_{12} = \frac{\hat{n}}{\hat{k}^2 + \hat{\ell}^2} \begin{bmatrix} \hat{k} & -\hat{\ell} \\ \hat{\ell} & \hat{k} \end{bmatrix}, \quad V_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad V_{22} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

in (5.127), we obtain a unimodular matrix V since $|\det V| = |\det V_{11}| \cdot |\det V_{22}| = 1$. Then we have (5.125), and therefore $(\alpha, \beta) = \Phi(k, \ell, n)$ is obtained from the first column of V_{12}, i.e., $(\alpha, \beta) = m(\hat{k}, \hat{\ell})$ with $m = \hat{n}/(\hat{k}^2 + \hat{\ell}^2)$. □

The use of the Smith normal form is demonstrated below when $\hat{n}/(\hat{k}^2 + \hat{\ell}^2)$ is not an integer, whereas when $\hat{n}/(\hat{k}^2 + \hat{\ell}^2)$ is an integer, the simpler method of construction in Remark 5.5 in Section 5.6.3 is used.

The example is a case with a solution of type V and without one of type T.

Example 5.6. [Case 1 of Proposition 5.19] For $(k, \ell, n) = (2m, m, 6m)$ with $m \geq 1$, we have $(\hat{k}, \hat{\ell}, \hat{n}) = (2, 1, 6)$, $\hat{k}^2 + \hat{\ell}^2 = 5$, and $\kappa = \gcd(5, 6) = 1$. The transformation to the Smith normal form in (5.125) is given as

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \left[\begin{array}{cc|cc} 2 & 1 & -6 & 0 \\ 1 & -2 & 0 & -6 \end{array}\right] \left[\begin{array}{cc|cc} 2 & 1 & 6 & 0 \\ 1 & -2 & 0 & 6 \\ \hline 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

This shows $\mathcal{A}(2m, m, 6m) = \mathcal{L}(6, 0)$, i.e., $\Phi(2m, m, 6m) = (6, 0) = (\alpha, \beta)$. We have $\alpha = \hat{n} = 6$ and $(\alpha', \beta') = (6, 0)$ by (5.81). This is a case of $(\alpha, \beta) = (\alpha', \beta')$, and we have

$$\Sigma_0(\alpha, \beta) = \Sigma_0(\alpha', \beta') = \Sigma_0(\alpha, \beta) \cap \Sigma_0(\alpha', \beta') = \Sigma_0(6, 0).$$

When $m = 1$, $\Sigma_0(6, 0)$ reduces to $\langle r \rangle$. We have $(\hat{\alpha}, \hat{\beta}) = (1, 0)$, $\hat{D} = 1 \notin 2\mathbb{Z}$, $\gcd(\hat{k} - \hat{\ell}, \hat{n}) = \gcd(1, 6) = 1 \notin 2\mathbb{Z}$, and **GCD-div** since $2\gcd(\hat{k}, \hat{\ell}) = 2\gcd(2, 1) = 2$ is divisible by $\kappa = 1$. □

5.6.10. Appendix: Proofs of Propositions 5.11, 5.13, and 5.14

In this section we establish a series of propositions, which together serve as the proofs of Propositions 5.11, 5.13, and 5.14 presented in Section 5.6.2.

We first focus on Proposition 5.14.

Proposition 5.31.

(i) $\gcd(\hat{\alpha} + \hat{\beta}, \hat{\alpha} - \hat{\beta}) \in \{1, 2\}$.

(ii) $\gcd(\hat{\alpha} + \hat{\beta}, \hat{\alpha} - \hat{\beta}) = 2 \iff \hat{D} \in 2\mathbb{Z}$.

(iii) $\gcd(\hat{\alpha} + \hat{\beta}, \hat{\alpha} - \hat{\beta}) = 1 \iff \hat{D} \notin 2\mathbb{Z}$.

Proof. (i) Since $\gcd(\hat{\alpha}, \hat{\beta}) = 1$, Proposition 5.29(i) implies the existence of integers x and y such that $x\hat{\alpha} + y\hat{\beta} = 1$. For $p = x + y$, $q = x - y$, we have

$$p(\hat{\alpha} + \hat{\beta}) + q(\hat{\alpha} - \hat{\beta}) = 2(x\hat{\alpha} + y\hat{\beta}) = 2.$$

Then Proposition 5.29(ii) shows that 2 is divisible by $\gcd(\hat{\alpha} + \hat{\beta}, \hat{\alpha} - \hat{\beta})$, which is equivalent to the statement of (i) of this proposition.

(ii) We have $\{1, 2\} \ni \gcd(\hat{\alpha} + \hat{\beta}, \hat{\alpha} - \hat{\beta}) = \gcd(\hat{\alpha} + \hat{\beta}, 2\hat{\alpha})$. Therefore, $\gcd(\hat{\alpha} + \hat{\beta}, \hat{\alpha} - \hat{\beta}) = 2$ if and only if $\hat{\alpha} + \hat{\beta} \in 2\mathbb{Z}$. Finally we note a simple identity $\hat{D} = (\hat{\alpha} + \hat{\beta})^2 - 2\hat{\alpha}\hat{\beta}$ to see that $\hat{\alpha} + \hat{\beta} \in 2\mathbb{Z}$ if and only if $\hat{D} \in 2\mathbb{Z}$.

(iii) This is obvious from (i) and (ii) above. □

Proposition 5.32.

$$\Sigma_0(\alpha, \beta) \cap \Sigma_0(\beta, \alpha) = \begin{cases} \Sigma_0(\alpha'', 0) & \text{if } \hat{D} \notin 2\mathbb{Z}, \\ \Sigma_0(\beta'', \beta'') & \text{if } \hat{D} \in 2\mathbb{Z} \end{cases} \tag{5.128}$$

with

$$\alpha'' = \frac{D(\alpha, \beta)}{\gcd(\alpha, \beta)}, \quad \beta'' = \frac{D(\alpha, \beta)}{2\gcd(\alpha, \beta)}. \tag{5.129}$$

Proof. First note that $\Sigma_0(\alpha, \beta) \cap \Sigma_0(\beta, \alpha)$ is the subgroup generated by r and $p_1^a p_2^b$ for $(a, b) \in \mathcal{L}(\alpha, \beta) \cap \mathcal{L}(\beta, \alpha)$. In considering $\mathcal{L}(\alpha, \beta)$ of (5.120), it is convenient to have $\mathcal{H}(\alpha, \beta)$ of (5.4) in mind, as it has a natural correspondence with $\mathcal{L}(\alpha, \beta)$. The set $\mathcal{H}(\alpha, \beta) \cap \mathcal{H}(\beta, \alpha)$ is a square sublattice with the reflection symmetry with respect to the x-axis, and hence it can be represented as $\mathcal{H}(\alpha'', 0)$ or $\mathcal{H}(\beta'', \beta'')$ for some α'' or β''. Such α'' is determined as the minimum α'' satisfying $\mathcal{L}(\alpha'', 0) \subseteq \mathcal{L}(\alpha, \beta)$, and β'' as the minimum β'' satisfying $\mathcal{L}(\beta'', \beta'') \subseteq \mathcal{L}(\alpha, \beta)$. Then $\mathcal{L}(\alpha, \beta) \cap \mathcal{L}(\beta, \alpha)$ coincides with the larger of $\mathcal{L}(\alpha'', 0)$ and $\mathcal{L}(\beta'', \beta'')$.

The parameter α'' is determined as follows. The inclusion $\mathcal{L}(\alpha'', 0) \subseteq \mathcal{L}(\alpha, \beta)$ holds if and only if integers n_1 and n_2 exist such that

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} \alpha'' \\ 0 \end{bmatrix}.$$

By the solvability criterion using determinantal divisors given in Theorem 5.2, this holds if and only if

$$d_1\left(\left[\begin{array}{cc|c} \alpha & -\beta & \alpha'' \\ \beta & \alpha & 0 \end{array} \right] \right) \quad \text{equals} \quad d_1\left(\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \right) = \gcd(\alpha, \beta),$$

$$d_2\left(\left[\begin{array}{cc|c} \alpha & -\beta & \alpha'' \\ \beta & \alpha & 0 \end{array} \right] \right) \quad \text{equals} \quad d_2\left(\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \right) = D(\alpha, \beta).$$

The former condition is equivalent to α'' being a multiple of $\gcd(\alpha, \beta)$, and the latter to α'' being a multiple of $D(\alpha, \beta)/\gcd(\alpha, \beta)$. Hence we have $\alpha'' = D(\alpha, \beta)/\gcd(\alpha, \beta)$, which is a multiple of $\gcd(\alpha, \beta)$ since $D(\alpha, \beta)/\gcd(\alpha, \beta) = \hat{D}\gcd(\alpha, \beta)$.

The parameter β'' is determined as follows. The inclusion $\mathcal{L}(\beta'',\beta'') \subseteq \mathcal{L}(\alpha,\beta)$ holds if and only if integers n_1 and n_2 exist such that

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} \beta'' \\ \beta'' \end{bmatrix}.$$

Again by Theorem 5.2, this holds if and only if

$$d_1\left(\left[\begin{array}{cc|c} \alpha & -\beta & \beta'' \\ \beta & \alpha & \beta'' \end{array}\right]\right) \quad \text{equals} \quad d_1\left(\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}\right) = \gcd(\alpha,\beta),$$
$$d_2\left(\left[\begin{array}{cc|c} \alpha & -\beta & \beta'' \\ \beta & \alpha & \beta'' \end{array}\right]\right) \quad \text{equals} \quad d_2\left(\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}\right) = D(\alpha,\beta).$$

The former condition is equivalent to β'' being a multiple of $\gcd(\alpha,\beta)$, and the latter to β'' being a multiple of

$$\frac{D(\alpha,\beta)}{\gcd(\alpha+\beta,\alpha-\beta)} = \frac{D(\alpha,\beta)}{\gcd(\alpha,\beta)\gcd(\hat{\alpha}+\hat{\beta},\hat{\alpha}-\hat{\beta})}.$$

Then by Proposition 5.31, we obtain

$$\beta'' = \begin{cases} D(\alpha,\beta)/\gcd(\alpha,\beta) & \text{if } \hat{D} \notin 2\mathbb{Z}, \\ D(\alpha,\beta)/(2\gcd(\alpha,\beta)) & \text{if } \hat{D} \in 2\mathbb{Z}. \end{cases}$$

We have $\mathcal{L}(\alpha'',0) \supset \mathcal{L}(\beta'',\beta'')$ (with $\beta'' = \alpha''$) if $\hat{D} \notin 2\mathbb{Z}$, and $\mathcal{L}(\beta'',\beta'') \supset \mathcal{L}(\alpha'',0)$ (with $\beta'' = \alpha''/2$) if $\hat{D} \in 2\mathbb{Z}$. This completes the proof. □

Next we focus on Proposition 5.11(i). With this aim in mind, we rephrase (5.128) in Proposition 5.32 in terms of (k,ℓ) instead of (α,β).

Proposition 5.33.
(i) $\gcd(\hat{k}+\hat{\ell},\hat{k}-\hat{\ell},\hat{n}) \in \{1,2\}$.
(ii) $\gcd(\hat{k}+\hat{\ell},\hat{k}-\hat{\ell},\hat{n}) = 2 \iff \gcd(\hat{k}-\hat{\ell},\hat{n}) \in 2\mathbb{Z}$.
(iii) $\gcd(\hat{k}+\hat{\ell},\hat{k}-\hat{\ell},\hat{n}) = 1 \iff \gcd(\hat{k}-\hat{\ell},\hat{n}) \notin 2\mathbb{Z}$.

Proof. (i) Since $\gcd(\hat{k},\hat{\ell},\hat{n}) = 1$, Proposition 5.29(i) implies the existence of integers a, b, and c such that $a\hat{k} + b\hat{\ell} + c\hat{n} = 1$. For $p = a+b$, $q = a-b$, $r = 2c$, we have

$$p(\hat{k}+\hat{\ell}) + q(\hat{k}-\hat{\ell}) + r\hat{n} = 2(a\hat{k} + b\hat{\ell} + c\hat{n}) = 2.$$

Then Proposition 5.29(ii) shows that 2 is divisible by $\gcd(\hat{k}+\hat{\ell},\hat{k}-\hat{\ell},\hat{n})$, which is equivalent to the claim in (i).

(ii) We have $\{1,2\} \ni \gcd(\hat{k}+\hat{\ell},\hat{k}-\hat{\ell},\hat{n}) = \gcd(\hat{k}-\hat{\ell},2\hat{\ell},\hat{n})$. Hence follows the claim.

(iii) This is obvious from (i) and (ii) above. □

Proposition 5.34.

$$\Sigma_0(\alpha,\beta) \cap \Sigma_0(\beta,\alpha) = \begin{cases} \Sigma_0(\hat{n},0) & \text{if } \gcd(\hat{k}-\hat{\ell},\hat{n}) \notin 2\mathbb{Z}, \\ \Sigma_0(\hat{n}/2,\hat{n}/2) & \text{if } \gcd(\hat{k}-\hat{\ell},\hat{n}) \in 2\mathbb{Z}. \end{cases} \tag{5.130}$$

Proof. Recall the notation $\mathcal{A}(k,\ell,n)$ in (5.119). By the same argument as in the proof of Proposition 5.32, we compute the minimum α'' satisfying $(\alpha'',0) \in \mathcal{A}(k,\ell,n)$ and the minimum β'' satisfying $(\beta'',\beta'') \in \mathcal{A}(k,\ell,n)$. Then $\mathcal{L}(\alpha,\beta) \cap \mathcal{L}(\beta,\alpha)$ coincides with the larger of $\mathcal{L}(\alpha'',0)$ and $\mathcal{L}(\beta'',\beta'')$.

By the definition of $\mathcal{A}(k,\ell,n)$ in (5.119) we have $(\alpha'',0) \in \mathcal{A}(k,\ell,n)$ if and only if

$$\hat{k}\alpha'' \equiv 0, \quad \hat{\ell}\alpha'' \equiv 0 \mod \hat{n}.$$

Since $\gcd(\hat{k},\hat{\ell},\hat{n}) = 1$, the smallest α'' satisfying this condition is given by $\alpha'' = \hat{n}$. As for β'', we have $(\beta'',\beta'') \in \mathcal{A}(k,\ell,n)$ if and only if

$$(\hat{k}+\hat{\ell})\beta'' \equiv 0, \quad (\hat{k}-\hat{\ell})\beta'' \equiv 0 \mod \hat{n}.$$

The smallest β'' satisfying this condition is given by

$$\beta'' = \frac{\hat{n}}{\gcd(\hat{k}+\hat{\ell},\hat{k}-\hat{\ell},\hat{n})} = \begin{cases} \hat{n} & \text{if } \gcd(\hat{k}-\hat{\ell},\hat{n}) \notin 2\mathbb{Z}, \\ \hat{n}/2 & \text{if } \gcd(\hat{k}-\hat{\ell},\hat{n}) \in 2\mathbb{Z}, \end{cases}$$

where Proposition 5.33 is used. We finally note $\mathcal{L}(\hat{n},\hat{n}) \subset \mathcal{L}(\hat{n},0)$ and $\mathcal{L}(\hat{n},0) \subset \mathcal{L}(\hat{n}/2,\hat{n}/2)$ if $\hat{n} \in 2\mathbb{Z}$. This completes the proof. □

Proposition 5.35.

(i) $\gcd(\hat{k}-\hat{\ell},\hat{n}) \in 2\mathbb{Z} \iff \hat{D} \in 2\mathbb{Z}$.

(ii)

$$\hat{n} = \frac{D(\alpha,\beta)}{\gcd(\alpha,\beta)}. \tag{5.131}$$

Proof. This follows from a comparison of Proposition 5.32 with Proposition 5.34. □

We now focus on the second statement of Proposition 5.11.

Proposition 5.36.

$$\frac{\hat{n}}{\gcd(\hat{k}^2+\hat{\ell}^2,\hat{n})} = \gcd(\alpha,\beta). \tag{5.132}$$

Proof. We rely on the representation of Φ given in Proposition 5.30 in terms of the transformation matrix V in the Smith normal form of $[K \mid -\hat{n}I]$ in (5.125) with (5.122). Let

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

be the inverse of the matrix V in (5.125). We have $|\det V| = 1$ since V is unimodular. By a well-known formula in linear algebra and $V_{12} = A$ in (5.127), we have

$$|\det W_{12}| = |\det V_{12}|/|\det V| = |\det A| = D(\alpha,\beta). \tag{5.133}$$

On the other hand, it follows from (5.125) with $V = W^{-1}$ that

$$U\left[\begin{array}{cc|cc} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \end{array}\right]\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$

This implies

$$-\hat{n}U = \begin{bmatrix} 1 & 0 \\ 0 & \kappa \end{bmatrix} W_{12},$$

which shows

$$\hat{n}^2 = \kappa \, |\det W_{12}| \tag{5.134}$$

since $|\det U| = 1$.

Combining (5.133) and (5.134) with the expression (5.126) of κ, we obtain

$$\hat{n}^2 = \kappa \, D(\alpha,\beta) = \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}) \cdot D(\alpha,\beta).$$

By eliminating $D(\alpha,\beta)$ using (5.131), we obtain (5.132). □

Propositions 5.37–5.40 below are concerned with the symmetry of $\mathcal{A}(k,\ell,n)$ of (5.119), or that of $\Sigma_0(\alpha,\beta)$. Interestingly, such symmetry consideration leads to the proof of Proposition 5.13 of duality nature.

Proposition 5.37. *The four conditions* (a), (b), (c), *and* (d) *below are equivalent.*

(a) $(u_1, u_2) \in \mathbb{Z}^2$ *exists such that*

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} \end{bmatrix} \equiv \begin{bmatrix} \hat{\ell} & \hat{k} \end{bmatrix} \mod \hat{n}. \tag{5.135}$$

(b) *An integer matrix U exists such that*

$$U \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} \end{bmatrix} \equiv \begin{bmatrix} \hat{\ell} & \hat{k} \\ \hat{k} & -\hat{\ell} \end{bmatrix} \mod \hat{n}. \tag{5.136}$$

(c) $\gcd(\hat{k}^2 - \hat{\ell}^2, \ 2\hat{k}\hat{\ell})$ is divisible by $\gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n})$.

(d) **GCD-div** *in* (5.70)*:*

$$2\gcd(\hat{k}, \hat{\ell}) \text{ is divisible by } \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}) .$$

Proof. First, we show (a) ⇔ (b). For $(u_1, u_2) \in \mathbb{Z}^2$ satisfying (5.135), the matrix $U = \begin{bmatrix} u_1 & u_2 \\ -u_2 & u_1 \end{bmatrix}$ is an integer matrix that satisfies (5.136). This shows (a) ⇒ (b), whereas (b) ⇒ (a) is obvious.

Next, we show (a) ⇔ (c). The condition (a) is equivalent to the existence of integers u_1, u_2, p, and q that satisfy

$$\begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ p \\ q \end{bmatrix} = \begin{bmatrix} \hat{\ell} \\ \hat{k} \end{bmatrix}.$$

By the solvability criterion using determinantal divisors given in Theorem 5.2, this holds if and only if

$$d_1\left(\left[\begin{array}{cccc|c} \hat{k} & \hat{\ell} & -\hat{n} & 0 & \hat{\ell} \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} & \hat{k} \end{array} \right] \right) \quad \text{equals} \quad d_1\left(\begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{bmatrix} \right) = 1,$$

$$d_2\left(\left[\begin{array}{cccc|c} \hat{k} & \hat{\ell} & -\hat{n} & 0 & \hat{\ell} \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} & \hat{k} \end{array} \right] \right) \quad \text{equals} \quad d_2\left(\begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{bmatrix} \right).$$

The former condition imposes nothing and the latter reduces to (c). We have thus shown (a) $\Leftrightarrow$ (c).

Finally, we show (c) $\Leftrightarrow$ (d). Since $\hat{k}^2 + \hat{\ell}^2$ is a multiple of $\kappa = \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n})$, $\hat{k}^2 - \hat{\ell}^2$ is divisible by κ if and only if $(\hat{k}^2 - \hat{\ell}^2) + (\hat{k}^2 + \hat{\ell}^2) = 2\hat{k}^2$ is divisible by κ. Therefore, $\gcd(\hat{k}^2 - \hat{\ell}^2,\ 2\hat{k}\hat{\ell})$ is divisible by κ if and only if $\gcd(2\hat{k}^2,\ 2\hat{k}\hat{\ell}) = 2\hat{k}\gcd(\hat{k}, \hat{\ell})$ is divisible by κ. Since $\gcd(\hat{k}, \hat{n}) = 1$, $\gcd(\hat{k}^2 - \hat{\ell}^2,\ 2\hat{k}\hat{\ell})$ is divisible by κ if and only if $2\gcd(\hat{k}, \hat{\ell})$ is divisible by κ. $\square$

Proposition 5.38. *The following two conditions are equivalent.*

(a) $\mathcal{A}(k, \ell, n) = \mathcal{A}(\ell, k, n)$.

(b) $(a, b) \in \mathcal{A}(k, \ell, n) \implies (b, a) \in \mathcal{A}(k, \ell, n)$.

Proof. The defining equations in (5.119) for $\mathcal{A}(k, \ell, n)$ are invariant under the change of variables $(a, b, k, \ell) \mapsto (b, a, \ell, k)$, and therefore, $\mathcal{A}(\ell, k, n) = \{(b, a) \mid (a, b) \in \mathcal{A}(k, \ell, n)\}$. This shows the equivalence of (a) and (b). $\square$

Proposition 5.39. *The following two conditions are equivalent.*

(a) $\mathcal{A}(k, \ell, n) = \mathcal{A}(\ell, k, n)$.

(b) *An integer matrix U exists such that* (5.136) *holds.*

Proof. Although the claim is intuitively obvious from symmetry, we provide here a rigorous proof on the basis of Theorem 5.3 (the integer analogue of the Farkas lemma).

As in the proof of Proposition 5.37, the condition (b) is equivalent to the existence of integer tuples (u_1, u_2, p, q) and (u_1', u_2', p', q') such that

$$\begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ p \\ q \end{bmatrix} = \begin{bmatrix} \hat{\ell} \\ \hat{k} \end{bmatrix}, \quad \begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ p' \\ q' \end{bmatrix} = \begin{bmatrix} \hat{k} \\ -\hat{\ell} \end{bmatrix}.$$

By Theorem 5.3, the existence of such (u_1, u_2, p, q) is equivalent to the following condition:

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{bmatrix} \in \mathbb{Z}^4 \quad \Longrightarrow \quad \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} \hat{\ell} \\ \hat{k} \end{bmatrix} \in \mathbb{Z},$$

which can be rewritten as

$$[\hat{k}y_1 + \hat{\ell}y_2,\ \hat{\ell}y_1 - \hat{k}y_2,\ -\hat{n}y_1,\ -\hat{n}y_2\ ] \in \mathbb{Z}^4 \quad \Longrightarrow \quad \hat{\ell}y_1 + \hat{k}y_2 \in \mathbb{Z}.$$

Integrality condition for the third and fourth components allows us to put $y_1 = a/\hat{n}$ and $y_2 = b/\hat{n}$ with integers a and b. Then we can rewrite the above as

$$\hat{k}a + \hat{\ell}b \equiv 0, \quad \hat{\ell}a - \hat{k}b \equiv 0 \mod \hat{n} \quad \Longrightarrow \quad \hat{\ell}a + \hat{k}b \equiv 0 \mod \hat{n}.$$

Similarly, the existence of (u_1', u_2', p', q') above is equivalent to the following condition:

$$\hat{k}a + \hat{\ell}b \equiv 0, \quad \hat{\ell}a - \hat{k}b \equiv 0 \mod \hat{n} \implies \hat{k}a - \hat{\ell}b \equiv 0 \mod \hat{n}.$$

The above two conditions together are nothing but the statement that $(a, b) \in \mathcal{A}(k, \ell, n)$ implies $(b, a) \in \mathcal{A}(k, \ell, n)$, which is equivalent to (a) by Proposition 5.38. $\square$

Proposition 5.40. *Let* $(\alpha, \beta) = \Phi(k, \ell, n)$.

(i) $\Sigma_0(\alpha, \beta) = \Sigma_0(\beta, \alpha) \iff \beta = 0$ *or* $\alpha = \beta$.

(ii) $\Sigma_0(\alpha, \beta) = \Sigma_0(\beta, \alpha) \iff$ **GCD-div** *in* (5.70).

Proof. (i) is obvious, and (ii) follows from Propositions 5.37 and 5.39. Note that $\Sigma_0(\alpha, \beta)$ is the subgroup generated by r and $p_1^a p_2^b$ for $(a, b) \in \mathcal{A}(k, \ell, n)$. $\square$

6. Bifurcating Solutions: Solving Bifurcation Equations

The $n \times n$ square lattice was introduced as a two-dimensional discretized space, and the group $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ labeling the symmetry of the $n \times n$ square lattice was presented in Chapter 2. The irreducible decomposition of the permutation representation of the group G was obtained in Chapters 3 and 4 to identify the irreducible representations. The equivariant branching lemma was presented in Chapter 5 as a pertinent and sufficient means to test the existence of a bifurcating solution, and was used to show the existence of the square patterns for each irreducible representation.

In this chapter, a bifurcation analysis by solving bifurcation equations is advanced as a more informative means to investigate the properties of bifurcating solutions for each irreducible representations. The expanded forms of bifurcation equations are derived by exploiting the symmetry of the square lattice. The stability of the bifurcating solutions is evaluated asymptotically, and stability conditions for the bifurcating solutions are presented.

This chapter is organized as follows. Fundamentals of an analysis are summarized in Section 6.1. Bifurcation points of multiplicity $M = 1$, 2, 4, and 8 are studied in Sections 6.2–6.5, respectively.

6.1. Procedure of an Analysis

Let us consider a governing equation

$$\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{0} \tag{6.1}$$

endowed with the equivariance to the group $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ formulated as

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{F}(T(g)\boldsymbol{\lambda}, \phi), \quad g \in G. \tag{6.2}$$

Recall that ϕ is a bifurcation parameter, $\boldsymbol{\lambda} \in \mathbb{R}^N$ is an $N = n^2$ dimensional independent variable vector expressing a distribution of mobile population, $\boldsymbol{F} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is a nonlinear function, and T is the N-dimensional permutation representation of the group G. Accordingly, the Jacobian matrix of $\boldsymbol{F}$ is an $N \times N$ matrix expressed as

$$J(\boldsymbol{\lambda}, \phi) = \left(\frac{\partial F_i}{\partial \lambda_j} \,\middle|\, i, j = 1, \ldots, N \right). \tag{6.3}$$

Let $(\boldsymbol{\lambda}_{\mathrm{c}}, \phi_{\mathrm{c}})$ be a critical point of multiplicity M (≥ 1), at which the Jacobian matrix of $\boldsymbol{F}$ has a rank deficiency M. The critical point $(\boldsymbol{\lambda}_{\mathrm{c}}, \phi_{\mathrm{c}})$ is assumed to be G-symmetric in the sense of

$$T(g)\boldsymbol{\lambda}_{\mathrm{c}} = \boldsymbol{\lambda}_{\mathrm{c}}, \quad g \in G. \tag{6.4}$$

Moreover, it is assumed to be group-theoretic, which means, by definition, that the M-dimensional kernel space of the Jacobian matrix at $(\boldsymbol{\lambda}_{\mathrm{c}}, \phi_{\mathrm{c}})$ is irreducible with respect to the representation T. The critical point $(\boldsymbol{\lambda}_{\mathrm{c}}, \phi_{\mathrm{c}})$ is associated with one of the irreducible representations μ of G in Table 6.1. The multiplicity M corresponds to the dimension of μ, and a matrix representation for μ is denoted by $T^{\mu}(g)$.

Table 6.1: Irreducible representations of $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ to be considered in bifurcation analysis

$n \setminus M$	1	2	4	8
$2m$	$(1;+,+,+), (1;+,+,-)$	$(2;+,+)$	$(4;k,0;+), (4;k,k;+), (4;n/2,\ell,+)$	$(8;k,\ell)$
$2m-1$	$(1;+,+,+)$		$(4;k,0;+), (4;k,k;+)$	$(8;k,\ell)$

$(4;k,0;+), (4;k,k;+)$ for k with $1 \leq k \leq \lfloor (n-1)/2 \rfloor$;
$(4;n/2,\ell;+)$ for k with $1 \leq \ell \leq \lfloor (n-1)/2 \rfloor$;
$(8;k,\ell)$ for (k,ℓ) with $1 \leq \ell \leq k-1$, $2 \leq k \leq \lfloor (n-1)/2 \rfloor$

By the Liapunov–Schmidt reduction with symmetry (Sattinger, 1979 [25]; Golubitsky et al., 1988 [26]), the full system of the governing equation (6.1) is reduced, in the neighborhood of the critical point $(\boldsymbol{\lambda}_c, \phi_c)$, to a system of bifurcation equations

$$\widetilde{\boldsymbol{F}}(\boldsymbol{w}, \widetilde{\phi}) = \boldsymbol{0} \tag{6.5}$$

in $\boldsymbol{w} \in \mathrm{Ker}(J_c)$, where $\widetilde{\boldsymbol{F}} : \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^M$ is a function, $\widetilde{\phi} = \phi - \phi_c$ denotes the increment of ϕ, $\mathrm{Ker}(J_c)$ is the kernel space of $J(\boldsymbol{\lambda}_c, \phi_c)$. Here, we define variables $\boldsymbol{w} = (w_1, \ldots, w_M)^\top$ in the bifurcation equation (6.5) by using the column vectors of $Q^\mu = [\boldsymbol{q}_1^\mu, \ldots, \boldsymbol{q}_M^\mu]$ in Section 4.3 that span $\mathrm{Ker}(J_c)$.

In this reduction process, the equivariance (6.2) of the full system is inherited by the reduced system (6.5). With the use of the matrix representation $T^\mu(g)$ for the associated irreducible representation μ, the equivariance of the bifurcation equation can be expressed as

$$T^\mu(g)\widetilde{\boldsymbol{F}}(\boldsymbol{w}, \widetilde{\phi}) = \widetilde{\boldsymbol{F}}(T^\mu(g)\boldsymbol{w}, \widetilde{\phi}), \quad g \in G. \tag{6.6}$$

The reduced equation (6.5) can possibly admit multiple solutions $\boldsymbol{w} = \boldsymbol{w}(\widetilde{\phi})$ with $\boldsymbol{w}(0) = \boldsymbol{0}$, since $(\boldsymbol{w}, \widetilde{\phi}) = (\boldsymbol{0}, 0)$ is a singular point of (6.5). This gives rise to bifurcation. Each $\boldsymbol{w}$ uniquely determines a solution $\boldsymbol{\lambda}$ to the full system (6.1).

A group-theoretic bifurcation analysis to investigate the stability of a bifurcating solution for a critical point proceeds as follows:

- Specify an irreducible representation μ of $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ in Table 6.1.
- Obtain the expanded form of the bifurcation equation by exploiting the symmetry.
- Obtain a bifurcating solution by using the equivariant branching lemma (Cicogna, 1981 [27]; Vanderbauwhede, 1982 [28]; Golubitsky et al., 1988 [26]) or solving the bifurcation equation.
- Obtain the Jacobian matrix of $\widetilde{\boldsymbol{F}}$.
- Substitute the bifurcating solution into the Jacobian matrix and evaluate the eigenvalues to determine their stability as

$$\begin{cases} \text{linearly stable:} & \text{every eigenvalue has a negative real part,} \\ \text{linearly unstable:} & \text{at least one eigenvalue has a positive real part.} \end{cases}$$

Table 6.2: Theoretically predicted bifurcating solutions for critical points with multiplicity M

M	Bifurcating solutions ($w \in \mathbb{R}$)	Existence conditions
1	w	if n is even
2	$\boldsymbol{w}_{\mathrm{sq}} = (w, w)$	if n is even
	$\boldsymbol{w}_{\mathrm{stripe}} = (w, 0)$	if n is even
4	$\boldsymbol{w}_{\mathrm{sq}} = (w, 0, w, 0)$	Always
	$\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0)$	Always
	$\boldsymbol{w}_{\mathrm{stripeII}} = (0, w, 0, 0)$	if $\check{n}$ is even
8	$\boldsymbol{w}_{\mathrm{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$	Always
	$\boldsymbol{w}_{\mathrm{sqT}} = (w, 0, w, 0, 0, 0, 0, 0)$	if $2\gcd(\hat{k}, \hat{\ell})$ is not divisible by $\gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n})$
	$\boldsymbol{w}_{\mathrm{upside-downI}} = (w, 0, 0, 0, w, 0, 0, 0)$	if $(\hat{k} + \hat{\ell})\gcd(\hat{k} + \hat{\ell}, \hat{k} - \hat{\ell})$ is not divisible by $\gcd(2\hat{k}\hat{\ell}, \hat{n})$
	$\boldsymbol{w}_{\mathrm{upside-downII}} = (0, w, 0, 0, 0, w, 0, 0)$	if $\hat{n}$ is even and $(\hat{k} + \hat{\ell})\gcd(\hat{k} + \hat{\ell}, \hat{k} - \hat{\ell})$ is not divisible by $\gcd(2\hat{k}\hat{\ell}, \hat{n})$
	$\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0, 0, 0, 0, 0)$	if $\hat{k}^2 + \hat{\ell}^2$, $2\hat{k}\hat{\ell}$, and $\hat{k}^2 - \hat{\ell}^2$ are not divisible by $\hat{n}$
	$\boldsymbol{w}_{\mathrm{stripeII}} = (0, w, 0, 0, 0, 0, 0, 0)$	if $\hat{n}$ is even and $\hat{k}^2 + \hat{\ell}^2$, $2\hat{k}\hat{\ell}$, and $\hat{k}^2 - \hat{\ell}^2$ are not divisible by $\hat{n}$

$\check{n} = n/\gcd(k, n)$ for $M = 4$ in (6.40);
$\hat{n} = n/\gcd(k, \ell, n)$, $\hat{k} = k/\gcd(k, \ell, n)$, $\hat{\ell} = \ell/\gcd(k, \ell, n)$ for $M = 8$ in (6.176)

We showed the existence of the square patterns by using the equivariant branching lemma in Chapter 5. Additionally, in this chapter, we show the existence of some other bifurcating solutions by solving bifurcation equations. Theoretically predicted bifurcating solutions are summarized in Table 6.2. A stability analysis for these solutions is also conducted in this chapter.

6.2. Bifurcation Point of Multiplicity 1

We consider a critical point associated with the one-dimensional irreducible representation $\mu = (1; +, +, -)$ of the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. The actions in $(1; +, +, -)$ on a variable $w \in \mathbb{R}$ are expressed as

$$r, s : \ w \mapsto w, \quad p_1, p_2 : \ w \mapsto -w. \tag{6.7}$$

This case is nothing but pitchfork bifurcation and is well-known.

The bifurcation equation for the critical point of multiplicity 1 is a one-dimensional equation over $\mathbb{R}$ as

$$\widetilde{F}(w, \widetilde{\phi}) = 0, \tag{6.8}$$

where $(w, \widetilde{\phi}) = (0, 0)$ is assumed to correspond to the critical point. We expand $\widetilde{F}$ into a power series as

$$\widetilde{F}(w, \widetilde{\phi}) = \sum_{a=0} A_a(\widetilde{\phi}) w^a \tag{6.9}$$

with coefficients $A_a(\widetilde{\phi}) \in \mathbb{R}$. Since $(w, \widetilde{\phi}) = (0, 0)$ corresponds to the critical point, we have

$$A_0(0) = 0, \quad A_1(0) = 0.$$

Hence, we have

$$A_1(\widetilde{\phi}) \approx A_1'(0)\widetilde{\phi}.$$

for $A_1'(0)$, which is generically nonzero.[11]

The equivariance of the bifurcation equation to the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ is identical to the equivariance to the action of the four elements r, s, p_1, and p_2 generating this group. Hence, the equivariance condition (6.6) of the bifurcation equation is written for (6.9) as

$$r, s : \widetilde{F}(w, \widetilde{\phi}) = \widetilde{F}(w, \widetilde{\phi}), \tag{6.10}$$

$$p_1, p_2 : \; -\widetilde{F}(w, \widetilde{\phi}) = \widetilde{F}(-w, \widetilde{\phi}). \tag{6.11}$$

From the equivariance condition (6.11), we have

$$\sum_{a=0} (-A_a(\widetilde{\phi}))w^a = \sum_{a=0} A_a(\widetilde{\phi})(-w)^a.$$

This condition implies $(-1)^a = -1$, that is,

$$a = 2b + 1, \quad b \in \mathbb{Z}_+,$$

where $\mathbb{Z}_+$ represents the set of nonnegative integers. Hence, (6.9) is restricted to

$$\widetilde{F}(w, \widetilde{\phi}) = w \sum_{b=0} A_{2b+1}(\widetilde{\phi})w^{2b}. \tag{6.12}$$

The form of (6.12) implies that $\widetilde{F}(w, \widetilde{\phi}) = 0$ has the trivial solution and a bifurcating solution. Note that $\widetilde{F}(w, \widetilde{\phi})$ is an odd function in w. Thus, $(w, \widetilde{\phi})$ and $(-w, \widetilde{\phi})$ are conjugate solutions for $\widetilde{F} = 0$. We hereafter call the two solutions that are conjugate as symmetric bifurcating solutions and those that are not as asymmetric ones.

We evaluate the stability of the bifurcating solution by considering the asymptotic form of the bifurcation equation. The asymptotic form of the bifurcation equation in (6.12) becomes

$$\widetilde{F}(w, \widetilde{\phi}) \approx w(A_1'(0)\widetilde{\phi} + A_3(0)w^2), \tag{6.13}$$

and the Jacobian of $\widetilde{F}$ becomes

$$\widetilde{J}(w, \widetilde{\phi}) = \frac{\partial \widetilde{F}}{\partial w} \approx A_1'(0)\widetilde{\phi} + 3A_3(0)w^2. \tag{6.14}$$

[11] Notation $A_1'(0)$ means the derivative of $A_1(\widetilde{\phi})$ with respect to $\widetilde{\phi}$, evaluated at $\widetilde{\phi} = 0$. Generically we have $A_1'(0) \neq 0$ since the group symmetry imposes no condition.

Solving $\widetilde{F} = 0$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{sq} \approx -w^2 \frac{A_3(0)}{A_1'(0)}.$$

Substituting $\widetilde{\phi}_{sq}$ into (6.14), we have

$$\widetilde{J}(w, \widetilde{\phi}_{sq}) \approx 2w^2 A_3(0). \tag{6.15}$$

Hence, the stability of the bifurcating solution in the neighborhood of the critical point depends on the sign of $A_3(0)$, that is,

$$\begin{cases} A_3(0) < 0: & \text{stable,} \\ A_3(0) > 0: & \text{unstable.} \end{cases}$$

6.3. Bifurcation Point of Multiplicity 2

We consider a critical point associated with the two-dimensional irreducible representation $\mu = (2; +, +)$ of the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. The actions in $(2; +, +)$ on a two-dimensional vector $(w_1, w_2) \in \mathbb{R}^2$ are expressed as

$$r: \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto \begin{bmatrix} w_2 \\ w_1 \end{bmatrix}, \quad s: \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \tag{6.16}$$

$$p_1: \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto \begin{bmatrix} -w_1 \\ w_2 \end{bmatrix}, \quad p_2: \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto \begin{bmatrix} w_1 \\ -w_2 \end{bmatrix}. \tag{6.17}$$

The bifurcation equation for the critical point of multiplicity 2 is a two-dimensional equation in $\boldsymbol{w} = (w_1, w_2) \in \mathbb{R}^2$ expressed as

$$\widetilde{F}_i(\boldsymbol{w}, \widetilde{\phi}) = 0, \quad i = 1, 2, \tag{6.18}$$

where $(w_1, w_2, \widetilde{\phi}) = (0, 0, 0)$ is assumed to correspond to the critical point. Accordingly, the Jacobian matrix of $\widetilde{\boldsymbol{F}}$ is an 2×2 matrix expressed as

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) = \left(\frac{\partial \widetilde{F}_i}{\partial w_j} \,\middle|\, i, j = 1, \ldots, 2 \right). \tag{6.19}$$

We expand $\widetilde{F}_1$ into a power series as

$$\widetilde{F}_1(w_1, w_2, \widetilde{\phi}) = \sum_{a=0} \sum_{b=0} A_{ab}(\widetilde{\phi}) {w_1}^a {w_2}^b \tag{6.20}$$

with coefficients $A_{ab}(\widetilde{\phi}) \in \mathbb{R}$. Since $(w_1, w_2, \widetilde{\phi}) = (0, 0, 0)$ corresponds to the critical point, we have

$$A_{00}(0) = 0, \quad A_{10}(0) = A_{01}(0) = 0.$$

Since $A_{10}'(0)$ is generically nonzero, we have

$$A_{10}(\widetilde{\phi}) \approx A_{10}'(0)\widetilde{\phi}.$$

The equivariance of the bifurcation equation to the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ is identical to the equivariance to the action of the four elements r, s, p_1, and p_2 generating this group. Hence, the equivariance condition (6.6) of the bifurcation equation is written for (6.18) as

$$\begin{aligned} r: \quad \widetilde{F}_2(w_1, w_2) &= \widetilde{F}_1(w_2, w_1), & (6.21) \\ \widetilde{F}_1(w_1, w_2) &= \widetilde{F}_2(w_2, w_1), & (6.22) \\ s: \quad \widetilde{F}_1(w_1, w_2) &= \widetilde{F}_1(w_1, w_2), & (6.23) \\ \widetilde{F}_2(w_1, w_2) &= \widetilde{F}_2(w_1, w_2), & (6.24) \\ p_1: -\widetilde{F}_1(w_1, w_2) &= \widetilde{F}_1(-w_1, w_2), & (6.25) \\ \widetilde{F}_2(w_1, w_2) &= \widetilde{F}_2(-w_1, w_2), & (6.26) \\ p_2: \quad \widetilde{F}_1(w_1, w_2) &= \widetilde{F}_1(w_1, -w_2), & (6.27) \\ -\widetilde{F}_2(w_1, w_2) &= \widetilde{F}_2(w_1, -w_2). & (6.28) \end{aligned}$$

From the equivariance condition (6.25) or (6.28), we have

$$\sum_{a=0} \sum_{b=0} (-A_{ab}(\widetilde{\phi})) {w_1}^a {w_2}^b = \sum_{a=0} \sum_{b=0} A_{ab}(\widetilde{\phi})(-w_1)^a {w_2}^b.$$

From the equivariance condition (6.26) or (6.27), we have

$$\sum_{a=0} \sum_{b=0} A_{ab}(\widetilde{\phi}) {w_1}^a {w_2}^b = \sum_{a=0} \sum_{b=0} A_{ab}(\widetilde{\phi}) {w_1}^a (-w_2)^b.$$

These conditions imply that a is odd, and b is even. Thus,

$$\begin{aligned} a &= 2c + 1, \quad c \in \mathbb{Z}_+, \\ b &= 2d, \quad d \in \mathbb{Z}_+. \end{aligned}$$

where $\mathbb{Z}_+$ represents the set of nonnegative integers. Hence, $\widetilde{F}_i$ $(i = 1, 2)$ is restricted to

$$\widetilde{F}_1(w_1, w_2, \widetilde{\phi}) = w_1 \sum_{c=0} \sum_{d=0} A_{2c+1,2d}(\widetilde{\phi}) {w_1}^{2c} {w_2}^{2d}. \tag{6.29}$$

$$\widetilde{F}_2(w_1, w_2, \widetilde{\phi}) = w_2 \sum_{c=0} \sum_{d=0} A_{2c+1,2d}(\widetilde{\phi}) {w_2}^{2c} {w_1}^{2d}. \tag{6.30}$$

Therein, $\widetilde{F}_2$ is obtained by (6.21).

We have the following propositions on the existence and the symmetry of bifurcating solutions by solving the bifurcation equation.

Proposition 6.1. *For a critical point of multiplicity* 2 *associated with* $\mu = (2; +, +)$, *we have the following bifurcating solutions:*

$$\begin{aligned} \text{Stripe pattern}: \boldsymbol{w}_{\text{stripe}} &= (w, 0) \ (w \in \mathbb{R}), \\ \text{Square pattern}: \boldsymbol{w}_{\text{sq}} &= (w, w) \ (w \in \mathbb{R}). \end{aligned}$$

Proof. Substituting $\boldsymbol{w}_{\text{stripe}} = (w, 0)$ into (6.29), we have

$$\widetilde{F}_1(w, 0, \widetilde{\phi}) = w \sum_{a=0}^{\infty} A_{2a+1,0}(\widetilde{\phi}) w^{2a} \approx w \left\{ A'_{10}(0)\widetilde{\phi} + A_{30}(0) w^2 \right\} \tag{6.31}$$

with $A'_{10}(0) = \partial A_{10}/\partial\widetilde{\phi}(0)$. Thus, $\widetilde{F}_1(w, 0, \widetilde{\phi}) = 0$ represents $\widetilde{\phi}$ versus w relation for $\boldsymbol{w}_{\text{stripe}}$. Substituting $\boldsymbol{w}_{\text{stripe}}$ into (6.30), we have $\widetilde{F}_2(w, 0, \widetilde{\phi}) = 0$. Thus, there is a bifurcating curve satisfying $\widetilde{F}_1 = \widetilde{F}_2 = 0$ for $\boldsymbol{w}_{\text{stripe}}$. Similar discussion holds for $\boldsymbol{w}_{\text{sq}}$. □

Proposition 6.2. *For a critical point of multiplicity* 2 *associated with* $\mu = (2; +, +)$*, the two bifurcating solutions* $(\boldsymbol{w}, \widetilde{\phi})$ *and* $(-\boldsymbol{w}, \widetilde{\phi})$ *are conjugate for* $\boldsymbol{w} = \boldsymbol{w}_{\text{sq}}$*,* $\boldsymbol{w}_{\text{stripe}}$*.*

Proof. Since $\boldsymbol{w}_{\text{stripe}} = (w, 0)$ and $-\boldsymbol{w}_{\text{stripe}} = (-w, 0)$ satisfy the same relation (cf., (6.31))

$$\sum_{a=0}^{\infty} A_{2a+1,0}(\widetilde{\phi}) w^{2a} = 0,$$

$\widetilde{F}_1(w, 0, \widetilde{\phi})$ is an odd function in w, that is,

$$\widetilde{F}_1(-w, 0, \widetilde{\phi}) = -\widetilde{F}_1(w, 0, \widetilde{\phi}).$$

Thus, $(\boldsymbol{w}_{\text{stripe}}, \widetilde{\phi})$ and $(-\boldsymbol{w}_{\text{stripe}}, \widetilde{\phi})$ are conjugate solutions for $\widetilde{F}_1 = 0$. Similar discussion holds for $(\boldsymbol{w}_{\text{sq}}, \widetilde{\phi})$ and $(-\boldsymbol{w}_{\text{sq}}, \widetilde{\phi})$. □

We evaluate the stability of the bifurcating solutions by considering the asymptotic form of the bifurcation equation. The asymptotic form of the bifurcation equation becomes

$$\widetilde{F}_1(w_1, w_2, \widetilde{\phi}) \approx w_1(A'_{10}(0)\widetilde{\phi} + A_{30}(0){w_1}^2 + A_{12}(0){w_2}^2), \tag{6.32}$$

$$\widetilde{F}_2(w_1, w_2, \widetilde{\phi}) \approx w_2(A'_{10}(0)\widetilde{\phi} + A_{30}(0){w_2}^2 + A_{12}(0){w_1}^2), \tag{6.33}$$

and the Jacobian matrix of $\widetilde{\boldsymbol{F}}$ in (6.19) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx \begin{bmatrix} A'_{10}(0)\widetilde{\phi} + 3A_{30}(0){w_1}^2 + A_{12}(0){w_2}^2 & 2A_{12}(0)w_1 w_2 \\ 2A_{12}(0)w_1 w_2 & A'_{10}(0)\widetilde{\phi} + 3A_{30}(0){w_2}^2 + A_{12}(0){w_1}^2 \end{bmatrix}. \tag{6.34}$$

Substituting $\boldsymbol{w}_{\text{sq}} = (w, w)$ into (6.32) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sq}} \approx -w^2 \frac{A_{30}(0) + A_{12}(0)}{A'_{10}(0)}.$$

Evaluating the Jacobian matrix (6.34) at $(\boldsymbol{w}_{\text{sq}}, \widetilde{\phi}_{\text{sq}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sq}}, \widetilde{\phi}_{\text{sq}}) \approx 2w^2 \begin{bmatrix} A_{30}(0) & A_{12}(0) \\ A_{12}(0) & A_{30}(0) \end{bmatrix}. \tag{6.35}$$

The eigenvalues of $\widetilde{J}(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}})$ are given by

$$\lambda_1, \lambda_2 \approx 2w^2(A_{30}(0) \pm A_{12}(0)).$$

Hence, the sign of the eigenvalues depends on the values of the coefficients $A_{30}(0)$ and $A_{12}(0)$.

Substituting $\boldsymbol{w}_{\mathrm{stripe}} = (w, 0)$ into (6.32) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{stripe}} \approx -w^2 \frac{A_{30}(0)}{A'_{10}(0)}.$$

Evaluating the Jacobian matrix (6.34) at $(\boldsymbol{w}_{\mathrm{stripe}}, \widetilde{\phi}_{\mathrm{stripe}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{stripe}}, \widetilde{\phi}_{\mathrm{stripe}}) \approx w^2 \begin{bmatrix} 2A_{30}(0) & 0 \\ 0 & -A_{30}(0) + A_{12}(0) \end{bmatrix}. \tag{6.36}$$

The eigenvalues of $\widetilde{J}(\boldsymbol{w}_{\mathrm{stripe}}, \widetilde{\phi}_{\mathrm{stripe}})$ are given by

$$\lambda_1 \approx 2w^2 A_{30}(0),$$
$$\lambda_2 \approx 2w^2(A_{12}(0) - A_{30}(0)).$$

Hence, the sign of the eigenvalues depends on the values of the coefficients $A_{30}(0)$ and $A_{12}(0)$.

To sum up, we have the following proposition:

Proposition 6.3. *For a critical point of multiplicity* 2 *associated with* $\mu = (2; +, +)$*, suppose that all eigenvalues of* $J(\boldsymbol{\lambda}_{\mathrm{c}}, \phi)$ *other than those for* $\mu = (2; +, +)$ *are negative. Then, we have the following statements on the stability in the neighborhood of the critical point.*

(i) *If* $A_{30}(0) < A_{12}(0) < -A_{30}(0)$ *are satisfied, the square pattern* $\boldsymbol{w}_{\mathrm{sq}}$ *is stable.*

(ii) *If* $A_{12}(0) < A_{30}(0) < 0$ *are satisfied, the stripe pattern* $\boldsymbol{w}_{\mathrm{stripe}}$ *is stable.*

(iii) *The two solutions* $\boldsymbol{w}_{\mathrm{sq}}$ *and* $\boldsymbol{w}_{\mathrm{stripe}}$ *are not stable simultaneously.*

Proof. The first and second statements are obtained by assuming that all the eigenvalues of the Jacobian matrix at each bifurcating solution are negative. The last statement are obtained by the fact that $A_{30}(0) < A_{12}(0)$ and $A_{12}(0) < A_{30}(0)$ cannot be satisfied simultaneously. □

6.4. Bifurcation Point of Multiplicity 4

We consider a critical point associated with the four-dimensional irreducible representations μ of the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$:

$$(4; k, 0, +) \text{ with } 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{6.37}$$

$$(4; k, k, +) \text{ with } 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{6.38}$$

$$(4; n/2, \ell, +) \text{ with } 1 \leq \ell \leq \frac{n}{2} - 1, \tag{6.39}$$

where $n \geq 3$ and $(4; n/2, \ell, +)$ exists when n is even. For $(4; k, 0, +)$ and $(4; k, k, +)$, we use the following notations:

$$\check{n} = \frac{n}{\gcd(k, n)}, \quad \check{k} = \frac{k}{\gcd(k, n)}. \tag{6.40}$$

For $(4; n/2, \ell, +)$, we use the following notations:

$$\tilde{n} = \frac{n}{\gcd(\ell, n)}, \quad \tilde{\ell} = \frac{\ell}{\gcd(\ell, n)}. \tag{6.41}$$

The actions in $(4; k, 0, +)$ on a four-dimensional vector $(w_1, \ldots, w_4) \in \mathbb{R}^4$ are expressed for a two-dimensional vector (z_1, z_2) with complex variables $z_j = w_{2j-1} + \mathrm{i}w_{2j}$ $(j = 1, 2)$ as (cf., (5.38))

$$r: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_2 \\ z_1 \end{bmatrix}, \quad s: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ \bar{z}_2 \end{bmatrix}, \tag{6.42}$$

$$p_1: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ z_2 \end{bmatrix}, \quad p_2: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ \omega^k z_2 \end{bmatrix} \tag{6.43}$$

with $\omega = \exp(\mathrm{i}2\pi/n)$. The actions in $(4; k, k, +)$ can be expressed as (cf., (5.39))

$$r: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_2 \\ z_1 \end{bmatrix}, \quad s: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_2 \\ \bar{z}_1 \end{bmatrix}, \tag{6.44}$$

$$p_1: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^{-k} z_2 \end{bmatrix}, \quad p_2: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^k z_2 \end{bmatrix}, \tag{6.45}$$

and the actions in $(4; n/2, \ell, +)$ can be expressed as (cf., (5.40))

$$r: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_2 \\ z_1 \end{bmatrix}, \quad s: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_1 \\ z_2 \end{bmatrix}, \tag{6.46}$$

$$p_1: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} -z_1 \\ \omega^{-\ell} z_2 \end{bmatrix}, \quad p_2: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^{\ell} z_1 \\ -z_2 \end{bmatrix}. \tag{6.47}$$

6.4.1. Derivation of Bifurcation Equation

The bifurcation equation for the critical point of multiplicity 4 is a four-dimensional equation in $\boldsymbol{w} = (w_1, \dots, w_4) \in \mathbb{R}^4$ expressed as

$$\widetilde{F}_i(\boldsymbol{w}, \widetilde{\phi}) = 0, \quad i = 1, \dots, 4, \tag{6.48}$$

where $(w_1, \dots, w_4, \widetilde{\phi}) = (0, \dots, 0, 0)$ is assumed to correspond to the critical point. Accordingly, the Jacobian matrix of $\widetilde{\boldsymbol{F}}$ is a 4×4 matrix expressed as

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) = \left(\frac{\partial \widetilde{F}_i}{\partial w_j} \,\middle|\, i, j = 1, \dots, 4 \right). \tag{6.49}$$

The bifurcation equation (6.48) can be represented as a 2-dimensional equation in complex variables $z_j = w_{2j-1} + \mathrm{i} w_{2j}$ $(j = 1, 2)$ as

$$F_i(z_1, z_2, \widetilde{\phi}) = 0, \quad i = 1, 2, \tag{6.50}$$

where $(z_1, z_2, \widetilde{\phi}) = (0, 0, 0)$ corresponds to the critical point, and there are the following relationship:

$$F_1(z_1, z_2, \widetilde{\phi}) = \widetilde{F}_1 + \mathrm{i}\widetilde{F}_2, \tag{6.51}$$

$$F_2(z_1, z_2, \widetilde{\phi}) = \widetilde{F}_3 + \mathrm{i}\widetilde{F}_4. \tag{6.52}$$

We expand F_1 into a power series as

$$F_1(z_1, z_2, \widetilde{\phi}) = \sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} A_{abcd}(\widetilde{\phi}) {z_1}^a {z_2}^b \,{\overline{z}_1}^c \,{\overline{z}_2}^d \tag{6.53}$$

with coefficients $A_{abcd}(\widetilde{\phi})$. Since $(z_1, z_2, \widetilde{\phi}) = (0, 0, 0)$ corresponds to the critical point, we have

$$A_{0000}(0) = 0, \quad A_{1000}(0) = A_{0100}(0) = A_{0010}(0) = A_{0001}(0) = 0.$$

In addition, since $a_1 = A'_{1000}(0)$ is generically nonzero, we have

$$A_{1000}(\widetilde{\phi}) \approx a_1 \widetilde{\phi}.$$

The equivariance of the bifurcation equation to the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ is identical to the equivariance to the action of the four elements r, s, p_1, and p_2 generating this group. The equivariance condition for $(4; k, 0, +)$ is written as

$$r: \quad \overline{F_2(z_1, z_2)} = F_1(\overline{z}_2, z_1), \tag{6.54}$$

$$F_1(z_1, z_2) = F_2(\overline{z}_2, z_1), \tag{6.55}$$

$$s: \quad F_1(z_1, z_2) = F_1(z_1, \overline{z}_2), \tag{6.56}$$

$$\overline{F_2(z_1, z_2)} = F_2(z_1, \overline{z}_2), \tag{6.57}$$

$$p_1: \quad \omega^k F_1(z_1, z_2) = F_1(\omega^k z_1, z_2), \tag{6.58}$$

$$F_2(z_1, z_2) = F_2(\omega^k z_1, z_2), \tag{6.59}$$

$$p_2: \quad F_1(z_1, z_2) = F_1(z_1, \omega^k z_2), \tag{6.60}$$

$$\omega^k F_2(z_1, z_2) = F_2(z_1, \omega^k z_2) \tag{6.61}$$

with $\omega = \exp(\mathrm{i}2\pi/n)$. The equivariance condition for $(4; k, k, +)$ is written as

$$r: \quad \overline{F_2(z_1, z_2)} = F_1(\bar{z}_2, z_1), \tag{6.62}$$
$$F_1(z_1, z_2) = F_2(\bar{z}_2, z_1), \tag{6.63}$$
$$s: \quad \overline{F_2(z_1, z_2)} = F_1(\bar{z}_2, \bar{z}_1), \tag{6.64}$$
$$\overline{F_1(z_1, z_2)} = F_2(\bar{z}_2, \bar{z}_1), \tag{6.65}$$
$$p_1: \quad \omega^k F_1(z_1, z_2) = F_1(\omega^k z_1, \omega^{-k} z_2), \tag{6.66}$$
$$\omega^{-k} F_2(z_1, z_2) = F_2(\omega^k z_1, \omega^{-k} z_2), \tag{6.67}$$
$$p_2: \quad \omega^k F_1(z_1, z_2) = F_1(\omega^k z_1, \omega^k z_2), \tag{6.68}$$
$$\omega^k F_2(z_1, z_2) = F_2(\omega^k z_1, \omega^k z_2). \tag{6.69}$$

The equivariance condition for $(4; n/2, \ell, +)$ is written as

$$r: \quad \overline{F_2(z_1, z_2)} = F_1(\bar{z}_2, z_1), \tag{6.70}$$
$$F_1(z_1, z_2) = F_2(\bar{z}_2, z_1), \tag{6.71}$$
$$s: \quad \overline{F_1(z_1, z_2)} = F_1(\bar{z}_1, z_2), \tag{6.72}$$
$$F_2(z_1, z_2) = F_2(\bar{z}_1, z_2), \tag{6.73}$$
$$p_1: \quad -F_1(z_1, z_2) = F_1(-z_1, \omega^{-\ell} z_2), \tag{6.74}$$
$$\omega^{-\ell} F_2(z_1, z_2) = F_2(-z_1, \omega^{-\ell} z_2), \tag{6.75}$$
$$p_2: \quad \omega^\ell F_1(z_1, z_2) = F_1(\omega^\ell z_1, -z_2), \tag{6.76}$$
$$-F_2(z_1, z_2) = F_2(\omega^\ell z_1, -z_2). \tag{6.77}$$

The equivariance condition with respect to r is equivalent to

$$F_2(z_1, z_2) = F_1(z_2, \bar{z}_1), \tag{6.78}$$
$$F_1(z_1, z_2) = \overline{F_1(\bar{z}_1, \bar{z}_2)} \tag{6.79}$$

for each irreducible representation. Hence, we can obtain F_2 from F_1 by the condition (6.78) and see that

$$A_{abcd}(\widetilde{\phi}) \in \mathbb{R} \tag{6.80}$$

by the condition (6.79).

The equivariance condition with respect to s is equivalent to $F_1(z_1, z_2) = F_1(z_1, \bar{z}_2)$ in (6.56), which gives

$$A_{abcd}(\widetilde{\phi}) = A_{adcb}(\widetilde{\phi}) \tag{6.81}$$

for each irreducible representation as explained below. For $(4; k, 0, +)$, the condition (6.56) applies. For $(4; k, k, +)$, substituting (6.62) into (6.64), we have $F_1(\bar{z}_2, z_1) = F_1(\bar{z}_2, \bar{z}_1)$. This condition is equivalent to $F_1(z_1, z_2) = F_1(z_1, \bar{z}_2)$. For $(4; n/2, \ell, +)$, the condition (6.72) gives $F_1(z_1, z_2) = \overline{F_1(\bar{z}_1, z_2)}$. Using (6.80), we have $\overline{F_1(\bar{z}_1, z_2)} = F_1(z_1, \bar{z}_2)$. Thus, we have $F_1(z_1, z_2) = F_1(z_1, \bar{z}_2)$.

For $(4; k, 0, +)$, the equivariance condition with respect to p_1 and p_2 is expressed as follows. The equivariance condition (6.58) for p_1 is expressed as

$$\sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} \omega^k A_{abcd}(\widetilde{\phi}) {z_1}^a {z_2}^b \,{\bar{z}_1}^{c}\, {\bar{z}_2}^{d} = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) (\omega^k z_1)^a {z_2}^b (\omega^{-k}\,\bar{z}_1)^c\, {\bar{z}_2}^{d},$$

which implies

$$\omega^{k(a-c-1)} = \exp\left[\frac{\mathrm{i}2\pi}{n} k(a-c-1)\right] = 1. \tag{6.82}$$

The equivariance condition (6.60) for p_2 is expressed as

$$\sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) {z_1}^a {z_2}^b \,{\bar{z}_1}^{c}\, {\bar{z}_2}^{d} = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) {z_1}^a (\omega^k z_2)^b {\bar{z}_1}^{c} (\omega^{-k}\,\bar{z}_2)^d,$$

which implies

$$\omega^{k(b-d)} = \exp\left[\frac{\mathrm{i}2\pi}{n} k(b-d)\right] = 1. \tag{6.83}$$

Using (6.78), we rewrite the remaining equivariance conditions (6.59) and (6.61) as

$$\begin{aligned} F_1(z_2, \bar{z}_1) &= F_1(z_2, \omega^{-k}\,\bar{z}_1), \\ \omega^k F_1(z_2, \bar{z}_1) &= F_1(\omega^k z_2, \bar{z}_1), \end{aligned}$$

which are expressed as

$$\begin{aligned} \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) {z_2}^a \,{\bar{z}_1}^{b}\, {\bar{z}_2}^{c} {z_1}^d &= \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) {z_2}^a (\omega^{-k}\,\bar{z}_1)^b\, {\bar{z}_2}^{c} (\omega^k z_1)^d, \\ \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} \omega^k A_{abcd}(\widetilde{\phi}) {z_2}^a \,{\bar{z}_1}^{b}\, {\bar{z}_2}^{c} {z_1}^d &= \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) (\omega^k z_2)^a \,{\bar{z}_1}^{b} (\omega^{-k}\,\bar{z}_2)^c {z_1}^d. \end{aligned}$$

Each of these conditions leads to the same result as (6.83) and (6.82), respectively. To sum up, from (6.82) and (6.83), we have the following conditions for $(4; k, 0, +)$:

$$\begin{aligned} k(a-c-1) &\equiv 0 \mod n, \\ k(b-d) &\equiv 0 \mod n. \end{aligned}$$

Using (6.40), we rewrite these conditions as

$$\begin{aligned} \check{k}(a-c-1) &\equiv 0 \mod \check{n}, \\ \check{k}(b-d) &\equiv 0 \mod \check{n}, \end{aligned}$$

which are equivalent to the following condition:

$$a = c + p\check{n} + 1, \quad b = d + q\check{n} \quad (p, q \in \mathbb{Z}). \tag{6.84}$$

Then, F_1 in (6.53) becomes

$$F_1(z_1, z_2, \widetilde{\phi}) = \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \sum_{p\in\mathbb{Z},\ c+p\check{n}+1\geq 0} \sum_{q\in\mathbb{Z},\ d+q\check{n}\geq 0} A_{c+p\check{n}+1,d+q\check{n},cd}(\widetilde{\phi}) {z_1}^{c+p\check{n}+1} {z_2}^{d+q\check{n}} {\bar{z}_1}^{c} {\bar{z}_2}^{d}. \tag{6.85}$$

Note that $a = 0$ and $c = 0$ are not satisfied simultaneously in (6.84):

$$a = 0 \Rightarrow c = -p\check{n} - 1 \neq 0, \quad c = 0 \Rightarrow a = p\check{n} + 1 \neq 0.$$

Thus, F_1 in (6.85) becomes

$$\begin{aligned} F_1(z_1, z_2, \widetilde{\phi}) = z_1 &\sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \sum_{p\in\mathbb{Z},\ c+p\check{n}+1>0} \sum_{q\in\mathbb{Z},\ d+q\check{n}\geq 0} A_{c+p\check{n}+1,d+q\check{n},cd}(\widetilde{\phi}) {z_1}^{c+p\check{n}} {z_2}^{d+q\check{n}} {\bar{z}_1}^{c} {\bar{z}_2}^{d} \\ &+ \bar{z}_1 \sum_{d=0}^{\infty} \sum_{p=1}^{\infty} \sum_{q\in\mathbb{Z},\ d+q\check{n}\geq 0} A_{0,d+q\check{n},p\check{n}-1,d}(\widetilde{\phi}) {z_2}^{d+q\check{n}} {\bar{z}_1}^{p\check{n}-2} {\bar{z}_2}^{d}. \end{aligned} \tag{6.86}$$

For $(4; k, k, +)$, the equivariance condition (6.66) is expressed as

$$\sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} \omega^k A_{abcd}(\widetilde{\phi}) {z_1}^a {z_2}^b {\bar{z}_1}^c {\bar{z}_2}^d = \sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} A_{abcd}(\widetilde{\phi}) (\omega^k z_1)^a (\omega^{-k} z_2)^b (\omega^{-k} \bar{z}_1)^c (\omega^k \bar{z}_2)^d,$$

which implies

$$\omega^{k(a-b-c+d-1)} = \exp\left[\frac{\mathrm{i}2\pi}{n} k(a - b - c + d - 1)\right] = 1. \tag{6.87}$$

The equivariance condition (6.68) is expressed as

$$\sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} \omega^k A_{abcd}(\widetilde{\phi}) {z_1}^a {z_2}^b {\bar{z}_1}^c {\bar{z}_2}^d = \sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} A_{abcd}(\widetilde{\phi}) (\omega^k z_1)^a (\omega^{k} z_2)^b (\omega^{-k} \bar{z}_1)^c (\omega^{-k} \bar{z}_2)^d,$$

which implies

$$\omega^{k(a+b-c-d-1)} = \exp\left[\frac{\mathrm{i}2\pi}{n} k(a + b - c - d - 1)\right] = 1. \tag{6.88}$$

Using (6.78), we rewrite the remaining equivariance conditions (6.67) and (6.69) as

$$\begin{aligned} \omega^{-k} F_1(z_2, \bar{z}_1) &= F_1(\omega^{-k} z_2, \omega^{-k} \bar{z}_1), \\ \omega^{k} F_1(z_2, \bar{z}_1) &= F_1(\omega^{k} z_2, \omega^{-k} \bar{z}_1), \end{aligned}$$

which are expressed as

$$\begin{aligned} &\sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} \omega^{-k} A_{abcd}(\widetilde{\phi}) {z_2}^a {\bar{z}_1}^b {\bar{z}_2}^c {z_1}^d \\ &\qquad = \sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} A_{abcd}(\widetilde{\phi}) (\omega^{-k} z_2)^a (\omega^{-k} \bar{z}_1)^b (\omega^{k} \bar{z}_2)^c (\omega^{k} z_1)^d, \\ &\sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} \omega^{k} A_{abcd}(\widetilde{\phi}) {z_2}^a {\bar{z}_1}^b {\bar{z}_2}^c {z_1}^d \\ &\qquad = \sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} A_{abcd}(\widetilde{\phi}) (\omega^{k} z_2)^a (\omega^{-k} \bar{z}_1)^b (\omega^{-k} \bar{z}_2)^c (\omega^{k} z_1)^d. \end{aligned}$$

Each of these conditions leads to the same result as (6.88) and (6.87), respectively. To sum up, from (6.87) and (6.88), we have the following conditions for $(4; k, k, +)$:

$$\begin{aligned} k(a-b-c+d-1) &\equiv 0 \quad \mathrm{mod}\ n, \\ k(a+b-c-d-1) &\equiv 0 \quad \mathrm{mod}\ n. \end{aligned}$$

We rewrite these conditions as

$$\begin{aligned} \check{k}(a-b-c+d-1) &\equiv 0 \quad \mathrm{mod}\ \check{n}, \\ \check{k}(a+b-c-d-1) &\equiv 0 \quad \mathrm{mod}\ \check{n}, \end{aligned}$$

which are equivalent to the following condition:

$$a-b-c+d-1 = v\check{n}, \quad a+b-c-d-1 = w\check{n} \quad (v, w \in \mathbb{Z}).$$

Adding and subtracting the two equations from each other, we have

$$2(a-c-1) = (v+w)\check{n}, \quad 2(b-d) = (w-v)\check{n}.$$

This condition is equivalent to

$$a = c + (v+w)\check{n}/2 + 1, \quad b = d + (w-v)\check{n}/2. \tag{6.89}$$

Since the indices a, b, c, and d are integers, we have the following condition ($p, q \in \mathbb{Z}$):

$$\begin{cases} v+w = p, \quad w-v = 2q-p & \text{for } \check{n} \text{ even}, \\ v+w = 2p, \quad w-v = 2(q-p) & \text{for } \check{n} \text{ odd}. \end{cases} \tag{6.90}$$

Note that for $\check{n}$ odd, we can replace $q-p$ as q ($q \in \mathbb{Z}$). From (6.89) and (6.90), we have the following condition:

$$\begin{cases} a = c + p\check{n}/2 + 1, \quad b = d + (2q-p)\check{n}/2 & \text{for } \check{n} \text{ even}, \\ a = c + p\check{n} + 1, \quad b = d + q\check{n} & \text{for } \check{n} \text{ odd}. \end{cases} \tag{6.91}$$

Note that for both cases in (6.91), $a = 0$ and $c = 0$ are not satisfied simultaneously:

$$\begin{cases} a = 0 \Rightarrow c = -p\check{n}/2 - 1 \neq 0, \quad c = 0 \Rightarrow a = p\check{n}/2 + 1 \neq 0 & \text{for } \check{n} \text{ even}, \\ a = 0 \Rightarrow c = -p\check{n} - 1 \neq 0, \quad c = 0 \Rightarrow a = p\check{n} + 1 \neq 0 & \text{for } \check{n} \text{ odd}. \end{cases}$$

If $\check{n}$ is even, F_1 in (6.53) becomes

$$\begin{aligned} F_1(z_1, z_2, \widetilde{\phi}) = z_1 \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \sum_{p,q \in \mathbb{Z},\ c+p\frac{\check{n}}{2}+1>0,\ d+(2q-p)\frac{\check{n}}{2} \geq 0} & A_{c+p\frac{\check{n}}{2}+1, d+(2q-p)\frac{\check{n}}{2}, cd}(\widetilde{\phi}) {z_1}^{c+p\frac{\check{n}}{2}} {z_2}^{d+(2q-p)\frac{\check{n}}{2}}\ {\overline{z}_1}^{c}\ {\overline{z}_2}^{d} \\ + \overline{z}_1 \sum_{d=0}^{\infty} \sum_{p=1}^{\infty} \sum_{q \in \mathbb{Z},\ d+(2q+p)\frac{\check{n}}{2} \geq 0} & A_{0, d+(2q+p)\frac{\check{n}}{2}, p\frac{\check{n}}{2}-1, d}(\widetilde{\phi}) {z_2}^{d+(2q+p)\frac{\check{n}}{2}}\ {\overline{z}_1}^{p\frac{\check{n}}{2}-2}\ {\overline{z}_2}^{d}. \end{aligned} \tag{6.92}$$

If $\check{n}$ is odd, F_1 in (6.53) becomes

$$F_1(z_1, z_2, \widetilde{\phi}) = z_1 \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \sum_{p\in\mathbb{Z},\ c+p\check{n}+1>0} \sum_{q\in\mathbb{Z},\ d+q\check{n}\geq 0} A_{c+p\check{n}+1,d+q\check{n},cd}(\widetilde{\phi}) {z_1}^{c+p\check{n}} {z_2}^{d+q\check{n}} \,{\bar{z}_1}^{\,c}\, {\bar{z}_2}^{\,d}$$
$$+ \bar{z}_1 \sum_{d=0}^{\infty} \sum_{p=1}^{\infty} \sum_{q\in\mathbb{Z},\ d+q\check{n}\geq 0} A_{0,d+q\check{n},p\check{n}-1,d}(\widetilde{\phi}) {z_2}^{d+q\check{n}} \,{\bar{z}_1}^{\,p\check{n}-2}\, {\bar{z}_2}^{\,d}. \tag{6.93}$$

For $(4; n/2, \ell, +)$, the equivariance condition (6.74) is expressed as

$$\sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} (-A_{abcd}(\widetilde{\phi})) {z_1}^a {z_2}^b \,{\bar{z}_1}^{\,c}\, {\bar{z}_2}^{\,d} = \sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} A_{abcd}(\widetilde{\phi})(-z_1)^a (\omega^{-\ell} z_2)^b (-\bar{z}_1)^c (\omega^{\ell}\, \bar{z}_2)^d,$$

which implies

$$-1 = (-1)^{a+c} \omega^{\ell(d-b)}.$$

We rewrite this condition as

$$\exp\left[\frac{\mathrm{i}2\pi}{n}\left\{\frac{n}{2}(a+c) + \ell(d-b)\right\}\right] = -1. \tag{6.94}$$

Therein, we used

$$(-1)^{a+c} = \exp\left[\frac{\mathrm{i}\pi}{n}(a+c)\right] \quad (a, c \in \mathbb{Z}_+),$$

where $\mathbb{Z}_+$ represents the set of nonnegative integers. The equivariance condition (6.76) is expressed as

$$\sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} (\omega^{\ell} A_{abcd}(\widetilde{\phi})) {z_1}^a {z_2}^b \,{\bar{z}_1}^{\,c}\, {\bar{z}_2}^{\,d} = \sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} A_{abcd}(\widetilde{\phi})(\omega^{\ell} z_1)^a (-z_2)^b (\omega^{-\ell}\, \bar{z}_1)^c (-\bar{z}_2)^d,$$

which implies

$$\omega^{\ell} = (-1)^{b+d} \omega^{\ell(a-c)}.$$

We rewrite this condition as

$$\exp\left[\frac{\mathrm{i}2\pi}{n}\left\{\frac{n}{2}(b+d) + \ell(a-c-1)\right\}\right] = 1. \tag{6.95}$$

Therein, we used

$$(-1)^{b+d} = \exp\left[\frac{\mathrm{i}\pi}{n}(b+d)\right] \quad (b, d \in \mathbb{Z}_+).$$

Using (6.78), we rewrite the remaining equivariance conditions (6.75) and (6.77) as

$$\omega^{-\ell} F_1(z_2, \bar{z}_1) = F_1(\omega^{-\ell} z_2, -\bar{z}_1),$$
$$-F_1(z_2, \bar{z}_1) = F_1(-z_2, \omega^{-\ell}\, \bar{z}_1),$$

which are expressed as

$$\sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0}\omega^{-\ell}A_{abcd}(\widetilde{\phi})z_2{}^a\,\bar{z}_1{}^b\,\bar{z}_2{}^c z_1{}^d = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0}A_{abcd}(\widetilde{\phi})(\omega^{-\ell}z_2)^a(-\bar{z}_1)^b(\omega^{\ell}\,\bar{z}_2)^c(-z_1)^d,$$
$$\sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0}(-A_{abcd}(\widetilde{\phi}))z_2{}^a\,\bar{z}_1{}^b\,\bar{z}_2{}^c\,z_1{}^d = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0}A_{abcd}(\widetilde{\phi})(-z_2)^a(\omega^{-\ell}\,\bar{z}_1)^b(-\bar{z}_2)^c(\omega^{\ell}z_1)^d.$$

Each of these conditions leads to the same result as (6.95) and (6.94), respectively.

To sum up, from (6.94) and (6.95), we have the following conditions for $(4;n/2,\ell,+)$:

$$\frac{n}{2}(a+c-1)+\ell(d-b)\equiv 0 \quad \bmod n,$$
$$\frac{n}{2}(b+d)+\ell(a-c-1)\equiv 0 \quad \bmod n.$$

We rewrite these conditions as

$$\tilde{n}(a+c-1)+2\tilde{\ell}(d-b)\equiv 0 \quad \bmod 2\tilde{n},$$
$$\tilde{n}(b+d)+2\tilde{\ell}(a-c-1)\equiv 0 \quad \bmod 2\tilde{n},$$

which are equivalent to the following condition:

$$\tilde{n}(a+c-1)+2\tilde{\ell}(d-b)=2p\tilde{n}, \quad \tilde{n}(b+d)+2\tilde{\ell}(a-c-1)=2q\tilde{n} \quad (p,q\in\mathbb{Z}). \tag{6.96}$$

We investigate this condition dependent on the parity of $\tilde{n}$.

When $\tilde{n}$ is even, the condition (6.96) is equivalent to

$$(a+c-1-2p)\tilde{n}/2=(b-d)\tilde{\ell}, \quad (a-c-1)\tilde{\ell}=-(b+d-2q)\tilde{n}/2.$$

Since $\tilde{\ell}$ and $\tilde{n}$ are coprime, we have the following conditions $(v,w\in\mathbb{Z})$:

$$b-d=v\tilde{n}/2, \quad b+d-2q=w\tilde{\ell}, \tag{6.97}$$
$$a+c-1-2p=v\tilde{\ell}, \quad a-c-1=-w\tilde{n}/2. \tag{6.98}$$

Adding and subtracting the two equations in (6.97) from each other, we have

$$2(b-q)=v\tilde{n}/2+w\tilde{\ell}, \quad 2(d-q)=-v\tilde{n}/2+w\tilde{\ell}.$$

This condition is equivalent to

$$\begin{bmatrix}b\\d\end{bmatrix}=q\begin{bmatrix}1\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}v\tilde{n}/2+w\tilde{\ell}\\-v\tilde{n}/2+w\tilde{\ell}\end{bmatrix}. \tag{6.99}$$

Since the indices b and d in (6.99) are integers, we have

$$v\tilde{n}/2+w\tilde{\ell}\in 2\mathbb{Z}. \tag{6.100}$$

Note that if the condition (6.100) is satisfied, then $-v\tilde{n}/2 + w\tilde{\ell} \in 2\mathbb{Z}$ is also satisfied. Adding and subtracting the two equations in (6.98) from each other, we have

$$2(a-1-p) = v\tilde{\ell} - w\tilde{n}/2, \quad 2(c-p) = v\tilde{\ell} + w\tilde{n}/2.$$

This condition is equivalent to

$$\begin{bmatrix} a \\ c \end{bmatrix} = p\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} v\tilde{\ell} - w\tilde{n}/2 \\ v\tilde{\ell} + w\tilde{n}/2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{6.101}$$

Since the indices a and c in (6.101) are integers, we have

$$v\tilde{\ell} + w\tilde{n}/2 \in 2\mathbb{Z}. \tag{6.102}$$

Note that if the condition (6.102) is satisfied, then $v\tilde{\ell} - w\tilde{n}/2 \in 2\mathbb{Z}$ is also satisfied. Since $\tilde{\ell}$ and $\tilde{n}$ are coprime, $\tilde{\ell}$ is odd. Thus, the conditions (6.100) and (6.102) are equivalent to the following condition ($t, u, t', u' \in \mathbb{Z}$):

$$\begin{cases} (v, w) = (2t, 2u) & \text{if } \tilde{n}/2 \text{ is even,} \\ (v, w) = (2t, 2u), (2t'+1, 2u'+1) & \text{if } \tilde{n}/2 \text{ is odd.} \end{cases} \tag{6.103}$$

If $\tilde{n}/2$ is even, the indices a, b, c, and d take the form

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = p\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + q\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \\ -\tilde{n}/2 \end{bmatrix} + u\begin{bmatrix} -\tilde{n}/2 \\ \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6.104}$$

Note that $a = 0$ and $c = 0$ are not satisfied simultaneously:

$$a = 0 \Rightarrow c = u\tilde{n} - 1 \neq 0, \quad c = 0 \Rightarrow a = -u\tilde{n} + 1 \neq 0.$$

With this result, we define disjoint sets U and V as

$$\begin{aligned} U &= \{(p, q, t, u) \in \mathbb{Z}^4 \mid a > 0,\ b \geq 0,\ c \geq 0,\ d \geq 0\}, \\ V &= \{(p, q, t, u) \in \mathbb{Z}^4 \mid a = 0,\ b \geq 0,\ c > 0,\ d \geq 0\}, \end{aligned}$$

which satisfy $U \cup V = \phi$ and are rewritten as

$$U = \left\{ (p, q, t, u) \in \mathbb{Z}^4 \;\middle|\; \begin{array}{l} p + t\tilde{\ell} - u\tilde{n}/2 + 1 > 0 \\ q + t\tilde{n}/2 + u\tilde{\ell} \geq 0 \\ p + t\tilde{\ell} + u\tilde{n}/2 \geq 0 \\ q - t\tilde{n}/2 + u\tilde{\ell} \geq 0 \end{array} \right\}, \tag{6.105}$$

$$V = \left\{ (p, q, t, u) \in \mathbb{Z}^4 \;\middle|\; \begin{array}{l} p + t\tilde{\ell} - u\tilde{n}/2 + 1 = 0 \\ q + t\tilde{n}/2 + u\tilde{\ell} \geq 0 \\ u\tilde{n} - 1 > 0 \\ q - t\tilde{n}/2 + u\tilde{\ell} \geq 0 \end{array} \right\}. \tag{6.106}$$

Then, F_1 in (6.53) becomes

$$
\begin{aligned}
F_1(z_1, z_2, \widetilde{\phi}) = z_1 & \sum_{(p,q,t,u)\in U} A_{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}+1,q+t\frac{\tilde{n}}{2}+u\tilde{\ell},p+t\tilde{\ell}+u\frac{\tilde{n}}{2},q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi}) {z_1}^{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}} {z_2}^{q+t\frac{\tilde{n}}{2}+u\tilde{\ell}} {\bar{z}_1}^{p+t\tilde{\ell}+u\frac{\tilde{n}}{2}} {\bar{z}_2}^{q-t\frac{\tilde{n}}{2}+u\tilde{\ell}} \\
+ \bar{z}_1 & \sum_{(p,q,t,u)\in V} A_{0,q+t\frac{\tilde{n}}{2}+u\tilde{\ell},u\tilde{n}-1,q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi}) {z_2}^{q+t\frac{\tilde{n}}{2}+u\tilde{\ell}} {\bar{z}_1}^{u\tilde{n}-2} {\bar{z}_2}^{q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}.
\end{aligned}
\tag{6.107}
$$

If $\tilde{n}/2$ is odd, the indices a, b, c, and d take the form

$$
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = p \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + q \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \\ -\tilde{n}/2 \end{bmatrix} + u \begin{bmatrix} -\tilde{n}/2 \\ \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{6.108}
$$

$$
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = p' \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + q' \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + t' \begin{bmatrix} \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \\ -\tilde{n}/2 \end{bmatrix} + u' \begin{bmatrix} -\tilde{n}/2 \\ \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \tilde{\ell} - \tilde{n}/2 \\ \tilde{\ell} + \tilde{n}/2 \\ \tilde{\ell} + \tilde{n}/2 \\ \tilde{\ell} - \tilde{n}/2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6.109}
$$

The first relation (6.108) is nothing but (6.104). Note that (6.108) and (6.109) take different vectors. In fact, assuming (6.108) = (6.109), we have

$$
(p'-p) \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (q'-q) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + (t'-t) \begin{bmatrix} \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \\ -\tilde{n}/2 \end{bmatrix} + (u'-u) \begin{bmatrix} -\tilde{n}/2 \\ \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \tilde{\ell} - \tilde{n}/2 \\ \tilde{\ell} + \tilde{n}/2 \\ \tilde{\ell} + \tilde{n}/2 \\ \tilde{\ell} - \tilde{n}/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
$$

Substituting the first equation into the third equation, we have $(u' - u + 1/2)\tilde{n} = 0$. This is a contradiction since $u' - u \in \mathbb{Z}$. In addition, note that $a = 0$ and $c = 0$ are not satisfied simultaneously:

$$
\begin{cases}
a = 0 \Rightarrow c = u\tilde{n} - 1 \neq 0, \quad c = 0 \Rightarrow a = -u\tilde{n} + 1 \neq 0 & \text{for (6.108)}, \\
a = 0 \Rightarrow c = (2u'+1)\check{n}/2 - 1 \neq 0, \quad c = 0 \Rightarrow a = -(2u'+1)\check{n}/2 + 1 \neq 0 & \text{for (6.109)}.
\end{cases}
$$

With this result, we can define four disjoint sets U and V in (6.105) and (6.106) and U' and V' from (6.109) as

$$
U' = \left\{ (p,q,t,u) \in \mathbb{Z}^4 \;\middle|\; \begin{array}{l} p + t\tilde{\ell} - u\tilde{n}/2 + (\tilde{\ell} - \tilde{n}/2)/2 + 1 > 0 \\ q + t\tilde{n}/2 + u\tilde{\ell} + (\tilde{\ell} + \tilde{n}/2)/2 \geq 0 \\ p + t\tilde{\ell} + u\tilde{n}/2 + (\tilde{\ell} + \tilde{n}/2)/2 \geq 0 \\ q - t\tilde{n}/2 + u\tilde{\ell} + (\tilde{\ell} - \tilde{n}/2)/2 \geq 0 \end{array} \right\}, \tag{6.110}
$$

$$
V' = \left\{ (p,q,t,u) \in \mathbb{Z}^4 \;\middle|\; \begin{array}{l} p + t\tilde{\ell} - u\tilde{n}/2 + (\tilde{\ell} - \tilde{n}/2)/2 + 1 = 0 \\ q + t\tilde{n}/2 + u\tilde{\ell} + (\tilde{\ell} + \tilde{n}/2)/2 \geq 0 \\ (2u+1)\tilde{n}/2 - 1 > 0 \\ q - t\tilde{n}/2 + u\tilde{\ell} + (\tilde{\ell} - \tilde{n}/2)/2 \geq 0 \end{array} \right\}. \tag{6.111}
$$

Then, F_1 in (6.53) becomes

$$
\begin{aligned}
F_1(z_1, z_2, \widetilde{\phi}) = z_1 & \sum_{(p,q,t,u)\in U} A_{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}+1,q+t\frac{\tilde{n}}{2}+u\tilde{\ell},p+t\tilde{\ell}+u\frac{\tilde{n}}{2},q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi}) {z_1}^{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}} {z_2}^{q+t\frac{\tilde{n}}{2}+u\tilde{\ell}} {\overline{z}_1}^{p+t\tilde{\ell}+u\frac{\tilde{n}}{2}} {\overline{z}_2}^{q-t\frac{\tilde{n}}{2}+u\tilde{\ell}} \\
& + \overline{z}_1 \sum_{(p,q,t,u)\in V} A_{0,q+t\frac{\tilde{n}}{2}+u\tilde{\ell},u\tilde{n}-1,q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi}) {z_2}^{q+t\frac{\tilde{n}}{2}+u\tilde{\ell}} {\overline{z}_1}^{u\tilde{n}-2} {\overline{z}_2}^{q-t\frac{\tilde{n}}{2}+u\tilde{\ell}} \\
& + z_1 \sum_{(p,q,t,u)\in U'} A_{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})+1,q+t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2}),p+t\tilde{\ell}+u\frac{\tilde{n}}{2}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2}),q-t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})}(\widetilde{\phi}) \\
& \quad \times {z_1}^{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})} {z_2}^{q+t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2})} {\overline{z}_1}^{p+t\tilde{\ell}+u\frac{\tilde{n}}{2}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2})} {\overline{z}_2}^{q-t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})} \\
& + \overline{z}_1 \sum_{(p,q,t,u)\in V'} A_{0,q+t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2}),(2u+1)\frac{\tilde{n}}{2}-1,q-t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})}(\widetilde{\phi}) \\
& \quad \times {z_2}^{q+t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2})} {\overline{z}_1}^{(2u+1)\frac{\tilde{n}}{2}-2} {\overline{z}_2}^{q-t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})}.
\end{aligned}
\tag{6.112}
$$

When $\tilde{n}$ is odd, the condition (6.96) is rewritten as

$$(a + c - 1 - 2p)\tilde{n} = 2\tilde{\ell}(b - d), \quad 2\tilde{\ell}(a - c - 1) = -(b + d - 2q)\tilde{n}.$$

Since $2\tilde{\ell}$ and $\tilde{n}$ are coprime, we have the following conditions ($v, w \in \mathbb{Z}$):

$$b - d = v\tilde{n}, \quad b + d - 2q = 2w\tilde{\ell}, \tag{6.113}$$

$$a + c - 1 - 2p = 2v\tilde{\ell}, \quad a - c - 1 = -w\tilde{n}. \tag{6.114}$$

Adding and subtracting the two equations in (6.113) from each other, we have

$$2(b - q) = v\tilde{n} + 2w\tilde{\ell}, \quad 2(d - q) = -v\tilde{n} + 2w\tilde{\ell}.$$

This condition is equivalent to

$$\begin{bmatrix} b \\ d \end{bmatrix} = q \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} v \begin{bmatrix} \tilde{n} \\ -\tilde{n} \end{bmatrix} + w \begin{bmatrix} \tilde{\ell} \\ \tilde{\ell} \end{bmatrix}. \tag{6.115}$$

Since the indices b and d in (6.115) are integers, and $\tilde{n}$ is odd, we have $v \in 2\mathbb{Z}$. Therefore, we replace v as $2t$ ($t \in \mathbb{Z}$). Adding and subtracting the two equations in (6.114) from each other, we have

$$2(a - 1 - p) = 2v\tilde{\ell} - w\tilde{n}, \quad 2(c - p) = 2v\tilde{\ell} + w\tilde{n}.$$

This condition is equivalent to

$$\begin{bmatrix} a \\ c \end{bmatrix} = p \begin{bmatrix} 1 \\ 1 \end{bmatrix} + v \begin{bmatrix} \tilde{\ell} \\ \tilde{\ell} \end{bmatrix} + \frac{1}{2} w \begin{bmatrix} -\tilde{n} \\ \tilde{n} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{6.116}$$

Since the indices a and c in (6.116) are integers, and $\tilde{n}$ is odd, we have $w \in 2\mathbb{Z}$. Therefore, we replace w as $2u$ ($u \in \mathbb{Z}$). To sum up, we have

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = p \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + q \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2\tilde{\ell} \\ \tilde{n} \\ 2\tilde{\ell} \\ -\tilde{n} \end{bmatrix} + u \begin{bmatrix} -\tilde{n} \\ 2\tilde{\ell} \\ \tilde{n} \\ 2\tilde{\ell} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6.117}$$

Note that $a = 0$ and $c = 0$ are not satisfied simultaneously:

$$a = 0 \Rightarrow c = 2u\tilde{n} - 1 \neq 0, \quad c = 0 \Rightarrow a = -u\tilde{n} + 1 \neq 0.$$

Similarly to the case that $\tilde{n}$ is even, we define sets U and V as

$$U = \left\{ (p,q,t,u) \in \mathbb{Z}^4 \;\middle|\; \begin{array}{l} p + 2t\tilde{\ell} - u\tilde{n} + 1 > 0 \\ q + t\tilde{n} + 2u\tilde{\ell} \geq 0 \\ p + 2t\tilde{\ell} + u\tilde{n} \geq 0 \\ q - t\tilde{n} + 2u\tilde{\ell} \geq 0 \end{array} \right\}, \tag{6.118}$$

$$V = \left\{ (p,q,t,u) \in \mathbb{Z}^4 \;\middle|\; \begin{array}{l} p + 2t\tilde{\ell} - u\tilde{n} + 1 = 0 \\ q + t\tilde{n} + 2u\tilde{\ell} \geq 0 \\ 2u\tilde{n} - 1 > 0 \\ q - t\tilde{n} + 2u\tilde{\ell} \geq 0 \end{array} \right\}. \tag{6.119}$$

Then, F_1 in (6.53) becomes

$$\begin{aligned} F_1(z_1, z_2, \widetilde{\phi}) = z_1 &\sum_{(p,q,t,u)\in U} A_{p+2t\tilde{\ell}-u\tilde{n}+1, q+t\tilde{n}+2u\tilde{\ell}, p+2t\tilde{\ell}+u\tilde{n}, q-t\tilde{n}+2u\tilde{\ell}}(\widetilde{\phi}) {z_1}^{p+2t\tilde{\ell}-u\tilde{n}} {z_2}^{q+t\tilde{n}+2u\tilde{\ell}} {\overline{z}_1}^{p+2t\tilde{\ell}+u\tilde{n}} {\overline{z}_2}^{q-t\tilde{n}+2u\tilde{\ell}} \\ + \overline{z}_1 &\sum_{(p,q,t,u)\in V} A_{0, q+t\tilde{n}+2u\tilde{\ell}, 2u\tilde{n}-1, q-t\tilde{n}+2u\tilde{\ell}}(\widetilde{\phi}) {z_2}^{q+t\tilde{n}+2u\tilde{\ell}} {\overline{z}_1}^{2u\tilde{n}-2} {\overline{z}_2}^{q-t\tilde{n}+2u\tilde{\ell}}. \end{aligned} \tag{6.120}$$

6.4.2. Symmetry of Square Patterns

For the irreducible representations $\mu = (4; k, 0, +), (4; k, k, +), (4; n/2, \ell, +)$, a system of the bifurcation equations $F_1 = F_2 = 0$ has a bifurcating solution, which represent the square pattern: $(z_1, z_2) = (w, w)$ $(w \in \mathbb{R})$. In Section 5.5.2, we showed the existence of this bifurcating solution by using the equivariant branching lemma (see Propositions 5.5–5.7). In this section, we discuss the symmetry of this bifurcating solution.

Consider $\mu = (4; k, 0, +)$. Substituting the square pattern $(z_1, z_2) = (w, w)$ into (6.86), we have

$$\begin{aligned} F_1(w, w, \widetilde{\phi}) &= w \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \sum_{p\in\mathbb{Z},\ c+p\check{n}+1>0} \sum_{q\in\mathbb{Z},\ d+q\check{n}\geq 0} A_{c+p\check{n}+1, d+q\check{n}, cd}(\widetilde{\phi}) w^{2(c+d)+(p+q)\check{n}} \\ &\quad + w \sum_{d=0}^{\infty} \sum_{p=1}^{\infty} \sum_{q\in\mathbb{Z},\ d+q\check{n}\geq 0} A_{0, d+q\check{n}, p\check{n}-1, d}(\widetilde{\phi}) w^{2d+(p+q)\check{n}-2} \\ &\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + (A_{1101}(0) + A_{2010}(0)) w^2 + A_{00,\check{n}-1,0}(0) w^{\check{n}-2} \right\}. \end{aligned}$$

If $\check{n}$ is even, then $F_1(w, w, \widetilde{\phi})$ becomes an odd function in w, and hence the two bifurcating solutions $(w, w, \widetilde{\phi})$ and $(-w, -w, \widetilde{\phi})$ are conjugate. If $\check{n}$ is odd, the two solutions are not conjugate.

Consider $\mu = (4; k, k, +)$ with $\check{n}$ even. Substituting the square pattern $(z_1, z_2) = (w, w)$ into

(6.92), we have

$$
\begin{aligned}
F_1(w,w,\widetilde{\phi}) &= w\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{p,q\in\mathbb{Z},\ c+p\frac{\check{n}}{2}+1>0,\ d+(2q-p)\frac{\check{n}}{2}\geq 0} A_{c+p\frac{\check{n}}{2}+1,d+(2q-p)\frac{\check{n}}{2},cd}(\widetilde{\phi})w^{2(c+d)+q\check{n}} \\
&\quad + w\sum_{d=0}^{\infty}\sum_{p=1}^{\infty}\sum_{q\in\mathbb{Z},\ d+(2q+p)\frac{\check{n}}{2}\geq 0} A_{0,d+(2q+p)\frac{\check{n}}{2},p\frac{\check{n}}{2}-1,d}(\widetilde{\phi})w^{2d+q\check{n}-2} \\
&\approx w\{A'_{1000}(0)\widetilde{\phi} + (A_{1101}(0)+A_{2010}(0))w^2 \\
&\qquad + (A_{00,\check{n}-1,0}(0) + A_{0,\frac{\check{n}}{2},\frac{\check{n}}{2}-1,0}(0) + A_{00,\frac{\check{n}}{2}-1,\frac{\check{n}}{2}}(0))w^{\check{n}-2}\}.
\end{aligned}
$$

Since $\check{n}$ is even, $F_1(w,w,\widetilde{\phi})$ is an odd function in w. Hence, the two bifurcating solutions $(w,w,\widetilde{\phi})$ and $(-w,-w,\widetilde{\phi})$ are conjugate.

Consider $\mu = (4;k,k,+)$ with $\check{n}$ odd. Substituting the square pattern $(z_1,z_2) = (w,w)$ into (6.93), we have

$$
\begin{aligned}
F_1(w,w,\widetilde{\phi}) &= w\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{p\in\mathbb{Z},\ c+p\check{n}+1>0}\ \sum_{q\in\mathbb{Z},\ d+q\check{n}\geq 0} A_{c+p\check{n}+1,d+q\check{n},cd}(\widetilde{\phi})w^{2(c+d)+(p+q)\check{n}} \\
&\quad + w\sum_{d=0}^{\infty}\sum_{p=1}^{\infty}\sum_{q\in\mathbb{Z},\ d+q\check{n}\geq 0} A_{0,d+q\check{n},p\check{n}-1,d}(\widetilde{\phi})w^{2d+(p+q)\check{n}-2} \\
&\approx w\left\{A'_{1000}(0)\widetilde{\phi} + (A_{1101}(0)+A_{2010}(0))w^2 + A_{00,\check{n}-1,0}(0)w^{\check{n}-2}\right\}.
\end{aligned}
$$

Since $\check{n}$ is odd, $F_1(w,w,\widetilde{\phi})$ is not an odd function in w. Hence, the two bifurcating solutions $(w,w,\widetilde{\phi})$ and $(-w,-w,\widetilde{\phi})$ are not conjugate.

Consider $\mu = (4;n/2,\ell,+)$ with $\tilde{n}/2$ even. Substituting the square pattern $(z_1,z_2) = (w,w)$ into (6.107), we have

$$
\begin{aligned}
F_1(w,w,\widetilde{\phi}) &= w\sum_{(p,q,t,u)\in U} A_{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}+1,q+t\frac{\tilde{n}}{2}+u\tilde{\ell},p+t\tilde{\ell}+u\frac{\tilde{n}}{2},q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi})w^{2(p+q)+2(t+u)\tilde{\ell}} \\
&\quad + w\sum_{(p,q,t,u)\in V} A_{0,q+t\frac{\tilde{n}}{2}+u\tilde{\ell},u\tilde{n}-1,q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi})w^{2q+2u(\tilde{\ell}+\frac{\tilde{n}}{2})-2} \\
&\approx w\left\{A'_{1000}(0)\widetilde{\phi} + (A_{1101}(0)+A_{2010}(0))w^2 + A_{00,\tilde{n}-1,0}(0)w^{\tilde{n}-2}\right\}.
\end{aligned}
$$

Then, $F_1(w,w,\widetilde{\phi})$ is an odd function in w. Hence, the two bifurcating solutions $(w,w,\widetilde{\phi})$ and $(-w,-w,\widetilde{\phi})$ are conjugate.

Consider $\mu = (4;n/2,\ell,+)$ with $\tilde{n}/2$ odd. Substituting the square pattern $(z_1,z_2) = (w,w)$ into

(6.112), we have

$$
\begin{aligned}
F_1(z_1, z_2, \widetilde{\phi}) &= w \sum_{(p,q,t,u)\in U_1} A_{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}+1, q+t\frac{\tilde{n}}{2}+u\tilde{\ell}, p+t\tilde{\ell}+u\frac{\tilde{n}}{2}, q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi}) w^{2(p+q)+2(t+u)\tilde{\ell}} \\
&\quad + w \sum_{(p,q,t,u)\in V_1} A_{0, q+t\frac{\tilde{n}}{2}+u\tilde{\ell}, u\tilde{n}-1, q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi}) w^{2q+2u(\tilde{\ell}+\frac{\tilde{n}}{2})-2} \\
&\quad + w \sum_{(p,q,t,u)\in U_2} A_{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})+1, q+t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2}), p+t\tilde{\ell}+u\frac{\tilde{n}}{2}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2}), q-t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})}(\widetilde{\phi}) \\
&\qquad \times w^{2(p+q)+2(t+u+1)\tilde{\ell}} \\
&\quad + w \sum_{(p,q,t,u)\in V_2} A_{0, q+t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2}), (2u+1)\frac{\tilde{n}}{2}-1, q-t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})}(\widetilde{\phi}) w^{2q+(2u+1)(\tilde{\ell}+\frac{\tilde{n}}{2})-2} \\
&\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + (A_{1101}(0) + A_{2010}(0)) w^2 + A_{00,\tilde{n}-1,0}(0) w^{\tilde{n}-2} \right\}.
\end{aligned}
$$

Since $\tilde{\ell} + \tilde{n}/2$ is even, $F_1(w, w, \widetilde{\phi})$ is an odd function in w. Hence, the two bifurcating solutions $(w, w, \widetilde{\phi})$ and $(-w, -w, \widetilde{\phi})$ are conjugate.

Consider $\mu = (4; n/2, \ell, +)$ with $\tilde{n}$ odd. Substituting the square pattern $(z_1, z_2) = (w, w)$ into (6.120), we have

$$
\begin{aligned}
F_1(w, w, \widetilde{\phi}) &= w \sum_{(p,q,t,u)\in U} A_{p+2t\tilde{\ell}-u\tilde{n}+1, q+t\tilde{n}+2u\tilde{\ell}, p+2t\tilde{\ell}+u\tilde{n}, q-t\tilde{n}+2u\tilde{\ell}}(\widetilde{\phi}) w^{2(p+q)+4(t+u)\tilde{\ell}} \\
&\quad + w \sum_{(p,q,t,u)\in V} A_{0, q+t\tilde{n}+2u\tilde{\ell}, 2u\tilde{n}-1, q-t\tilde{n}+2u\tilde{\ell}}(\widetilde{\phi}) w^{2q+2u(2\tilde{\ell}+\tilde{n})-2} \\
&\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + (A_{1101}(0) + A_{2010}(0)) w^2 \right\}.
\end{aligned}
$$

Then, $F_1(w, w, \widetilde{\phi})$ is an odd function in w. Hence, the two bifurcating solutions $(w, w, \widetilde{\phi})$ and $(-w, -w, \widetilde{\phi})$ are conjugate.

To sum up, we have the following proposition on the symmetry of the square pattern.

Proposition 6.4. *For a critical point of multiplicity 4, the two bifurcating solutions* $(w, w, \widetilde{\phi})$ *and* $(-w, -w, \widetilde{\phi})$ $(w \in \mathbb{R})$ *are conjugate for the following cases:*

- $\mu = (4; k, 0, +)$, $(4; k, k, +)$ *with* $\check{n} = n/\gcd(n, k)$ *even,*
- $\mu = (4; n/2, \ell, +)$ *for any* $\tilde{n} = n/\gcd(n, \ell)$*,*

and are not conjugate for $\mu = (4; k, 0, +)$*,* $(4; k, k, +)$ *with* $\check{n}$ *odd.*

6.4.3. Existence and Symmetry of Stripe Patterns

In this section, we would like to show the existence and the symmetry of two types of stripe patterns, which are represented as

$$
\begin{aligned}
&\text{Type I stripe pattern} : (z_1, z_2) = (w, 0) \ (w \in \mathbb{R}), \\
&\text{Type II stripe pattern} : (z_1, z_2) = (\mathrm{i}w, 0) \ (w \in \mathbb{R}).
\end{aligned}
$$

Consider $\mu = (4; k, 0, +)$. Substituting Type I stripe pattern $(z_1, z_2) = (w, 0)$ into (6.86), we have

$$\begin{aligned} F_1(w, 0, \widetilde{\phi}) &= w \sum_{c=0}^{\infty} \sum_{p \in \mathbb{Z},\ c+p\check{n}+1>0} A_{c+p\check{n}+1,0c0}(\widetilde{\phi}) w^{2c+p\check{n}} + w \sum_{p=1}^{\infty} A_{00,p\check{n}-1,0}(\widetilde{\phi}) w^{p\check{n}-2} \\ &\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0) w^2 + A_{00,\check{n}-1,0}(0) w^{\check{n}-2} \right\}. \end{aligned}$$

Thus, $F_1(w, 0, \widetilde{\phi}) = 0$ has the trivial solution $w = 0$ and a bifurcating solution. From (6.78), we have $F_2(w, 0) = F_1(0, w)$, and hence we have $F_1 = F_2 = 0$ for $(z_1, z_2) = (w, 0)$. If $\check{n}$ is even, then $F_1(w, 0, \widetilde{\phi})$ becomes an odd function in w, and hence the two bifurcating solutions $(w, 0, \widetilde{\phi})$ and $(-w, 0, \widetilde{\phi})$ are conjugate. If $\check{n}$ is odd, the two solutions are not conjugate. Next, substituting Type II stripe pattern $(z_1, z_2) = (\mathrm{i}w, 0)$ into (6.86), we have

$$\begin{aligned} F_1(\mathrm{i}w, 0, \widetilde{\phi}) &= \mathrm{i}w \sum_{c=0}^{\infty} \sum_{p \in \mathbb{Z},\ c+p\check{n}+1>0} A_{c+p\check{n}+1,0c0}(\widetilde{\phi}) \mathrm{i}^{p\check{n}} w^{2c+p\check{n}} - \mathrm{i}w \sum_{p=1}^{\infty} A_{00,p\check{n}-1,0}(\widetilde{\phi}) (-\mathrm{i})^{p\check{n}-2} w^{p\check{n}-2} \\ &\approx \mathrm{i}w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0) w^2 - A_{00,\check{n}-1,0}(0) (-\mathrm{i})^{\check{n}-2} w^{\check{n}-2} \right\}. \end{aligned}$$

If $\check{n}$ is even ($\mathrm{i}^{p\check{n}}$ and $(-\mathrm{i})^{p\check{n}-2}$ are real), then $F_1(\mathrm{i}w, 0, \widetilde{\phi}) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution, and a discussion similar to that for Type I stripe pattern holds.

Consider $\mu = (4; k, k, +)$ with $\check{n}$ even. Substituting Type I stripe pattern $(z_1, z_2) = (w, 0)$ into (6.92), we have

$$\begin{aligned} F_1(w, 0, \widetilde{\phi}) &= w \sum_{c=0}^{\infty} \sum_{q \in \mathbb{Z},\ c+q\check{n}+1>0} A_{c+q\check{n}+1,0c0}(\widetilde{\phi}) w^{2c+q\check{n}} + w \sum_{p=1}^{\infty} A_{00,q\check{n}-1,0}(\widetilde{\phi}) w^{q\check{n}-2} \\ &\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0) w^2 + A_{00,\check{n}-1,0}(0) w^{\check{n}-2} \right\}. \end{aligned}$$

Thus, $F_1(w, 0, \widetilde{\phi}) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution. From (6.78), we have $F_2(w, 0) = F_1(0, w)$, and hence we have $F_1 = F_2 = 0$ for $(z_1, z_2) = (w, 0)$. Since $\check{n}$ is even, $F_1(w, 0, \widetilde{\phi})$ is an odd function in w, and hence the two bifurcating solutions $(w, 0, \widetilde{\phi})$ and $(-w, 0, \widetilde{\phi})$ are conjugate. Next, substituting Type II stripe pattern $(z_1, z_2) = (\mathrm{i}w, 0)$ into (6.92), we have

$$\begin{aligned} F_1(\mathrm{i}w, 0, \widetilde{\phi}) &= \mathrm{i}w \sum_{c=0}^{\infty} \sum_{q \in \mathbb{Z},\ c+q\check{n}+1>0} A_{c+q\check{n}+1,0c0}(\widetilde{\phi}) \mathrm{i}^{q\check{n}} w^{2c+q\check{n}} - \mathrm{i}w \sum_{q=1}^{\infty} A_{00,q\check{n}-1,0}(\widetilde{\phi}) (-\mathrm{i})^{q\check{n}-2} w^{q\check{n}-2} \\ &\approx \mathrm{i}w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0) w^2 - A_{00,\check{n}-1,0}(0) (-\mathrm{i})^{\check{n}-2} w^{\check{n}-2} \right\}. \end{aligned}$$

Since $\check{n}$ is even ($\mathrm{i}^{q\check{n}}$ and $(-\mathrm{i})^{q\check{n}-2}$ are real), $F_1(\mathrm{i}w, 0, \widetilde{\phi}) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution, and a discussion similar to that for Type I stripe pattern holds.

Consider $\mu = (4; k, k, +)$ with $\check{n}$ odd. Substituting Type I stripe pattern $(z_1, z_2) = (w, 0)$ into (6.93), we have

$$\begin{aligned} F_1(w, 0, \widetilde{\phi}) &= w \sum_{c=0}^{\infty} \sum_{p \in \mathbb{Z},\ c+p\check{n}+1>0} A_{c+p\check{n}+1,0c0}(\widetilde{\phi}) w^{2c+p\check{n}} + w \sum_{p=1}^{\infty} A_{00,p\check{n}-1,0}(\widetilde{\phi}) w^{p\check{n}-2} \\ &\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0) w^2 + A_{00,\check{n}-1,0}(0) w^{\check{n}-2} \right\}. \end{aligned}$$

Thus, $F_1(w,0,\widetilde{\phi}) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution. From (6.78), we have $F_2(w,0) = F_1(0,w)$, and hence we have $F_1 = F_2 = 0$ for $(z_1,z_2) = (w,0)$. Since $\check{n}$ is odd, $F_1(w,0,\widetilde{\phi})$ is not an odd function in w, and hence the two bifurcating solutions $(w,0,\widetilde{\phi})$ and $(-w,0,\widetilde{\phi})$ are not conjugate. Next, substituting Type II stripe pattern $(z_1,z_2) = (\mathrm{i}w,0)$ into (6.93), we have

$$F_1(\mathrm{i}w,0,\widetilde{\phi}) = \mathrm{i}w\sum_{c=0}^{\infty}\sum_{p\in\mathbb{Z},\ c+p\check{n}+1>0} A_{c+p\check{n}+1,0c0}(\widetilde{\phi})\mathrm{i}^{p\check{n}}w^{2c+p\check{n}} - \mathrm{i}w\sum_{p=1}^{\infty}A_{00,p\check{n}-1,0}(\widetilde{\phi})(-\mathrm{i})^{p\check{n}-2}w^{p\check{n}-2}.$$

Since $\check{n}$ is odd ($\mathrm{i}^{p\check{n}}$ and $(-\mathrm{i})^{p\check{n}-2}$ can be imaginary), $F_1(\mathrm{i}w,0,\widetilde{\phi}) = 0$ cannot be solved for $\widetilde{\phi}$.

Consider $\mu = (4; n/2, \ell, +)$ with $\tilde{n}/2$ even. In (6.104), we have

$$\begin{cases} b = q + t\tilde{n}/2 + u\tilde{\ell} = 0 \\ d = q - t\tilde{n}/2 + u\tilde{\ell} = 0 \end{cases} \Rightarrow \begin{cases} q = -u\tilde{\ell} \\ t = 0 \end{cases}.$$

Thus, we have

$$F_1(z_1,0,\widetilde{\phi}) = z_1\sum_{p,u\in\mathbb{Z},\ p-u\frac{\tilde{n}}{2}+1>0,\ p+u\frac{\tilde{n}}{2}\geq 0} A_{p-u\frac{\tilde{n}}{2}+1,0,p+u\frac{\tilde{n}}{2},0}(\widetilde{\phi}){z_1}^{p-u\frac{\tilde{n}}{2}}\,{\overline{z}_1}^{p+u\frac{\tilde{n}}{2}} + \overline{z}_1\sum_{u=1}^{\infty}A_{00,u\tilde{n}-1,0}(\widetilde{\phi}){\overline{z}_1}^{u\tilde{n}-2}. \tag{6.121}$$

Substituting Type I stripe pattern $(z_1,z_2) = (w,0)$ into (6.121), we have

$$\begin{aligned} F_1(w,0,\widetilde{\phi}) &= w\sum_{p,u\in\mathbb{Z},\ p-u\frac{\tilde{n}}{2}+1>0,\ p+u\frac{\tilde{n}}{2}\geq 0} A_{p-u\frac{\tilde{n}}{2}+1,0,p+u\frac{\tilde{n}}{2},0}(\widetilde{\phi})w^{2p} + w\sum_{u=1}^{\infty}A_{00,u\tilde{n}-1,0}(\widetilde{\phi})w^{u\tilde{n}-2} \\ &\approx w\left\{A'_{1000}(0)\widetilde{\phi} + A_{2010}(0)w^2 + A_{00,\tilde{n}-1,0}(0)w^{\tilde{n}-2}\right\}. \end{aligned}$$

Thus, $F_1(w,0,\widetilde{\phi}) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution. From (6.78), we have $F_2(w,0) = F_1(0,w)$. Thus, we have $F_1 = F_2 = 0$ for $(z_1,z_2) = (w,0)$. Since $\tilde{n}$ is even, $F_1(w,0,\widetilde{\phi})$ is an odd function in w. Hence, the two bifurcating solutions $(w,0,\widetilde{\phi})$ and $(-w,0,\widetilde{\phi})$ are conjugate. Next, substituting Type II stripe pattern $(z_1,z_2) = (\mathrm{i}w,0)$ into (6.121), we have

$$\begin{aligned} &F_1(\mathrm{i}w,0,\widetilde{\phi}) \\ &= \mathrm{i}w\sum_{p,u\in\mathbb{Z},\ p-u\frac{\tilde{n}}{2}+1>0,\ p+u\frac{\tilde{n}}{2}\geq 0} A_{p-u\frac{\tilde{n}}{2}+1,0,p+u\frac{\tilde{n}}{2},0}(\widetilde{\phi})(-1)^{p+u\frac{\tilde{n}}{2}}\mathrm{i}^{2p}w^{2p} - \mathrm{i}w\sum_{u=1}^{\infty}A_{00,u\tilde{n}-1,0}(\widetilde{\phi})\mathrm{i}^{u\tilde{n}-2}w^{u\tilde{n}-2} \\ &\approx \mathrm{i}w\left\{A'_{1000}(0)\widetilde{\phi} + A_{2010}(0)w^2 - A_{00,\tilde{n}-1,0}(0)\mathrm{i}^{\tilde{n}-2}w^{\tilde{n}-2}\right\}. \end{aligned}$$

Since $\tilde{n}$ is even ($\mathrm{i}^{u\tilde{n}-2}$ is real), $F_1(\mathrm{i}w,0,\widetilde{\phi}) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution. Then, a discussion similar to that for Type I stripe pattern holds.

Consider $\mu = (4; n/2, \ell, +)$ with $\tilde{n}/2$ odd. In (6.109), we have

$$\begin{cases} b = q + t\tilde{n}/2 + u\tilde{\ell} + (\tilde{\ell} + \tilde{n}/2)/2 = 0 \\ d = q - t\tilde{n}/2 + u\tilde{\ell} + (\tilde{\ell} - \tilde{n}/2)/2 = 0 \end{cases} \Rightarrow 2q + (2u+1)\tilde{\ell} = 0.$$

Since $\tilde{\ell}$ is odd, this relation is a contradiction. Hence, $b = 0$ and $d = 0$ are not satisfied simultaneously. In (6.108), we have

$$\left\{ \begin{array}{l} b = q + u\tilde{\ell} + t\tilde{n}/2 = 0 \\ d = q + u\tilde{\ell} - t\tilde{n}/2 = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} q = -u\tilde{\ell} \\ t = 0 \end{array} \right. .$$

To sum up, we have

$$F_1(z_1, 0, \widetilde{\phi}) = z_1 \sum_{p,u\in\mathbb{Z},\ p-u\frac{\tilde{n}}{2}+1>0,\ p+u\frac{\tilde{n}}{2}\geq 0} A_{p-u\frac{\tilde{n}}{2}+1,0,p+u\frac{\tilde{n}}{2},0}(\widetilde{\phi}) {z_1}^{p-u\frac{\tilde{n}}{2}}\ {\overline{z}_1}^{p+u\frac{\tilde{n}}{2}} + \overline{z}_1 \sum_{u=1}^{\infty} A_{00,u\tilde{n}-1,0}(\widetilde{\phi}) {\overline{z}_1}^{u\tilde{n}-2}.$$

Then, a discussion similar to that for $\mu = (4; n/2, \ell, +)$ with $(\tilde{\ell}, \tilde{n}/2) = (\mathrm{odd}, \mathrm{even})$ holds.

Consider $\mu = (4; n/2, \ell, +)$ with $\tilde{n}$ odd. In (6.117), we have

$$\left\{ \begin{array}{l} b = q + t\tilde{n} + 2u\tilde{\ell} = 0 \\ d = q - t\tilde{n} + 2u\tilde{\ell} = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} q = -2u\tilde{\ell} \\ t = 0 \end{array} \right. .$$

Thus, we have

$$F_1(z_1, 0, \widetilde{\phi}) = z_1 \sum_{p,u\in\mathbb{Z},\ p-u\tilde{n}+1>0,\ p+u\tilde{n}\geq 0} A_{p-u\tilde{n}+1,0,p+u\tilde{n},0}(\widetilde{\phi}) {z_1}^{p-u\tilde{n}}\ {\overline{z}_1}^{p+u\tilde{n}} + \overline{z}_1 \sum_{u=1}^{\infty} A_{00,2u\tilde{n}-1,0}(\widetilde{\phi}) {\overline{z}_1}^{2(u\tilde{n}-1)}. \tag{6.122}$$

Substituting Type I stripe pattern $(z_1, z_2) = (w, 0)$ into (6.122), we have

$$\begin{aligned} F_1(w, 0, \widetilde{\phi}) &= w \sum_{p,u\in\mathbb{Z},\ p-u\tilde{n}+1>0,\ p+u\tilde{n}\geq 0} A_{p-u\tilde{n}+1,0,p+u\tilde{n},0}(\widetilde{\phi}) w^{2p} + w \sum_{u=1}^{\infty} A_{00,2u\tilde{n}-1,0}(\widetilde{\phi}) w^{2(u\tilde{n}-1)} \\ &\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0) w^2 \right\}. \end{aligned}$$

Thus, $F_1(w, 0, \widetilde{\phi}) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution. From (6.78), we have $F_2(w, 0) = F_1(0, w)$. Thus, we have $F_1 = F_2 = 0$ for $(z_1, z_2) = (w, 0)$. We see that $F_1(w, 0, \widetilde{\phi})$ is an odd function in w. Hence, the two bifurcating solutions $(w, 0, \widetilde{\phi})$ and $(-w, 0, \widetilde{\phi})$ are conjugate. Next, substituting Type II stripe pattern $(z_1, z_2) = (\mathrm{i}w, 0)$ into (6.122), we have

$$\begin{aligned} &F_1(w, 0, \widetilde{\phi}) \\ &= \mathrm{i}w \sum_{p,u\in\mathbb{Z},\ p-u\tilde{n}+1>0,\ p+u\tilde{n}\geq 0} A_{p-u\tilde{n}+1,0,p+u\tilde{n},0}(\widetilde{\phi}) (-1)^{p+u\tilde{n}} \mathrm{i}^{2p} w^{2p} - \mathrm{i}w \sum_{u=1}^{\infty} A_{00,2u\tilde{n}-1,0}(\widetilde{\phi}) \mathrm{i}^{2(u\tilde{n}-1)} w^{2(u\tilde{n}-1)} \\ &\approx \mathrm{i}w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0) w^2 \right\}. \end{aligned}$$

Since the indices of i are real, $F_1(\mathrm{i}w, 0, \widetilde{\phi}) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution. Then, a discussion similar to that for Type I stripe pattern holds.

To sum up, we have the following propositions on the existence and the symmetry of the stripe patterns.

Proposition 6.5. *For a critical point of multiplicity 4, the stripe patterns $\boldsymbol{z} = (w, 0)$, $(\mathrm{i}w, 0)$ $(w \in \mathbb{R})$ exist for the following cases:*

- $\mu = (4; k, 0, +)$, $(4; k, k, +)$ *of Type I for any* $\check{n} = n/\gcd(n, k)$ *and Type II with* $\check{n}$ *even,*
- $\mu = (4; n/2, \ell, +)$ *of Type I and Type II for any* $\tilde{n} = n/\gcd(n, \ell)$*.*

Proposition 6.6. *For a critical point of multiplicity 4, the two bifurcating solutions* $(\boldsymbol{z}, \widetilde{\phi})$ *and* $(-\boldsymbol{z}, \widetilde{\phi})$ *are conjugate for* $\boldsymbol{z} = (w, 0)$, $(\mathrm{i}w, 0)$ $(w \in \mathbb{R})$ *for the following cases:*

- $\mu = (4; k, 0, +)$, $(4; k, k, +)$ *with* $\check{n} = n/\gcd(n, k)$ *even,*
- $\mu = (4; n/2, \ell, +)$ *for any* $\tilde{n} = n/\gcd(n, \ell)$*,*

and are not conjugate for $\boldsymbol{z} = (w, 0)$ *for* $\mu = (4; k, 0, +)$, $(4; k, k, +)$ *with* $\check{n}$ *odd.*

6.4.4. Stability of Bifurcating Solutions

In Section 5.5.2, we found square patterns for a critical point of multiplicity 4 by using the equivariant branching lemma. In Section 6.4.3, we showed two kinds of stripe patterns by solving the bifurcation equations. These bifurcating solutions are represented for the bifurcation equation in real variables in (6.48) as follows ($w \in \mathbb{R}$):

$$\begin{aligned} \boldsymbol{w}_{\mathrm{sq}} &= (w, 0, w, 0), \\ \boldsymbol{w}_{\mathrm{stripeI}} &= (w, 0, 0, 0), \\ \boldsymbol{w}_{\mathrm{stripeII}} &= (0, w, 0, 0). \end{aligned}$$

We would like to evaluate the asymptotic stability of these bifurcating solutions.

We denote by S the set of nonnegative indices (a, b, c, d) as

$$S = \begin{cases} \{(a, b, c, d) \in \mathbb{Z}_+^4 \mid (6.84)\} & \text{for } \mu = (4; k, 0, +), \\ \{(a, b, c, d) \in \mathbb{Z}_+^4 \mid (6.91)\} & \text{for } \mu = (4; k, k, +), \\ \{(a, b, c, d) \in \mathbb{Z}_+^4 \mid (6.96)\} & \text{for } \mu = (4; n/2, \ell, +), \end{cases} \tag{6.123}$$

where $\mathbb{Z}_+^4$ represents the set of nonnegative integers in $\mathbb{Z}^4$. Note that (a, b, c, d) must belong to S when $A_{abcd}(\widetilde{\phi}) \neq 0$. Hence, we replace the power series (6.53) with

$$F_1(z_1, z_2, \widetilde{\phi}) = \sum_S A_{abcd}(\widetilde{\phi}) {z_1}^a {z_2}^b \,{\overline{z}_1}^c \,{\overline{z}_2}^d. \tag{6.124}$$

To obtain the asymptotic form of the bifurcation equation and the Jacobian matrix, we elucidate the elements of S in (6.123) and specify the form of the power series in (6.124). In other words, we investigate which coefficient $A_{abcd}(\widetilde{\phi})$ becomes nonzero in (6.124). We focus on the coefficients of linear terms, quadratic terms, and cubic terms, which play a vital role as leading terms in (6.124). For this purpose, we take $(a, b, c, d) \in \mathbb{Z}_+^4$ with $a + b + \cdots + h \leq 3$ exhaustively and investigate whether it belongs to S or not. For $(4; k, 0, +)$, $(4; k, k, +)$, and $(4; n/2, \ell, +)$, we can see

$$(1, 0, 0, 0),\ (1, 1, 0, 1),\ (2, 0, 1, 0) \in S.$$

In addition, for some specific cases, we can see

$$\begin{array}{ll}
(0,0,2,0) \in S & \text{for } (4;k,0,+) \text{ with } \check{n} = 3, \\
(0,0,3,0) \in S & \text{for } (4;k,0,+) \text{ with } \check{n} = 4, \\
(0,0,2,0) \in S & \text{for } (4;k,k,+) \text{ with } \check{n} = 3, \\
(0,0,3,0),\ (0,2,1,0),\ (0,0,1,2) \in S & \text{for } (4;k,k,+) \text{ with } \check{n} = 4, \\
(0,0,3,0) \in S & \text{for } (4;n/2,\ell,+) \text{ with } \tilde{n} = 4.
\end{array}$$

Based on the above results, F_i ($i = 1, 2$) in (6.50) is restricted to the form of

$$F_i = a_1\widetilde{\phi} z_i + F_i^C + (\text{other terms}), \quad i = 1, 2, \tag{6.125}$$

where

$$F_1^C = a_2 z_1 z_2 \overline{z}_2 + a_3 {z_1}^2 \overline{z}_1, \tag{6.126}$$

$$F_2^C = a_2 z_2 \overline{z}_1 z_1 + a_3 {z_2}^2 \overline{z}_2 \tag{6.127}$$

with the following notations:

$$a_1 = A'_{1000}(0), \quad a_2 = A_{1101}(0), \quad a_3 = A_{2010}(0). \tag{6.128}$$

Therein, F_2 is obtained by (6.78). The form of "(other terms)" depends on the type of the irreducible representations in (6.123). Accordingly, $\widetilde{F}_i$ ($i = 1, \dots, 4$) in (6.48) is restricted to the form of

$$\widetilde{F}_i = a_1\widetilde{\phi} w_i + \widetilde{F}_i^C + (\text{other terms}), \quad i = 1, \dots, 4 \tag{6.129}$$

with

$$\widetilde{F}_1^C = a_2 w_1({w_3}^2 + {w_4}^2) + a_3 w_1({w_1}^2 + {w_2}^2), \tag{6.130}$$

$$\widetilde{F}_2^C = a_2 w_2({w_3}^2 + {w_4}^2) + a_3 w_2({w_1}^2 + {w_2}^2), \tag{6.131}$$

$$\widetilde{F}_3^C = a_2 w_3({w_1}^2 + {w_2}^2) + a_3 w_3({w_3}^2 + {w_4}^2), \tag{6.132}$$

$$\widetilde{F}_4^C = a_2 w_4({w_1}^2 + {w_2}^2) + a_3 w_4({w_3}^2 + {w_4}^2). \tag{6.133}$$

In (6.125), F_i^C corresponds to cubic terms, and the form of "(other terms)" varies with the irreducible representations. For the case $(4;k,0,+)$ with $\check{n} = 3$, we have quadratic terms as leading terms. For any other cases, we have cubic terms as leading terms that vary with the irreducible representations. From this point of view, we can classify the form of the bifurcation equation as shown in Table 6.3 for each irreducible representation.

As mentioned earlier, the form of "(other terms)" in (6.129) depends on the type μ of the irreducible representations in (6.123). Therefore, we checked all the possible cases numerically and classified each case by the form of leading terms. All the possible cases and stability conditions for the bifurcating solutions are summarized in Table 6.4. The main finding of this section is as follows:

Proposition 6.7. *For a critical point of multiplicity 4, we have the following statements:*

Table 6.3: Nonzero coefficients of leading terms which belong to "other terms" in (6.125)

μ	Cases	Nonzero coefficients
$(4; k, 0, +)$	General $\check{n}$	None
	$\check{n} = 3$	$A_{0020}(0)$
	$\check{n} = 4$	$A_{0030}(0)$
$(4; k, k, +)$	General $\check{n}$	None
	$\check{n} = 3$	$A_{0020}(0)$
	$\check{n} = 4$	$A_{0030}(0),\ A_{0210}(0),\ A_{0012}(0)$
$(4; n/2, \ell, +)$	General $\tilde{n}$	None
	$\tilde{n} = 4$	$A_{0030}(0)$

$\check{n} = n/\gcd(k, n)$ in (6.40); $\tilde{n} = n/\gcd(\ell, n)$ in (6.41)

- *For* $\mu = (4; k, 0, +)$ *and* $\mu = (4; k, k, +)$ *with* $\check{n} = 3$*, the bifurcating solutions* $\boldsymbol{w}_{sq}$ *and* $\boldsymbol{w}_{stripeI}$ *are always unstable in the neighborhood of the critical point, and the bifurcating curve takes the form* $\widetilde{\phi} \approx cw$ *for some constant c.*
- *For any other cases, the stability of the bifurcating solutions* $\boldsymbol{w}_{sq}$*,* $\boldsymbol{w}_{stripeI}$*, and* $\boldsymbol{w}_{stripeII}$ *depends on the values of the coefficients of the power series expansion of the bifurcation equation in* (6.124)*, and the bifurcating curve takes the form* $\widetilde{\phi} \approx cw^2$ *for some constant c.*

To show these results, we derive the asymptotic form of the bifurcation equation for each case and conduct stability analysis for the bifurcating solutions in the remainder of this section.

Case 1: General $(4; k, 0, +)$

For general $(4; k, 0, +)$, we have

$$(1, 0, 0, 0),\ (1, 1, 0, 1),\ (2, 0, 1, 0) \in S,$$

and the other elements of S correspond to higher order terms. Then, the asymptotic form of F_i $(i = 1, 2)$ in (6.125) becomes

$$F_1(z_1, z_2, \widetilde{\phi}) \approx a_1 \widetilde{\phi} z_1 + F_1^C, \tag{6.134}$$

$$F_2(z_1, z_2, \widetilde{\phi}) \approx a_1 \widetilde{\phi} z_2 + F_2^C, \tag{6.135}$$

where F_i^C $(i = 1, 2)$ is given in (6.126) and (6.127). By (6.51) and (6.52), the asymptotic form of $\widetilde{F}_i$ $(i = 1, \ldots, 4)$ in (6.48) becomes

$$\widetilde{F}_1 \approx a_1 \widetilde{\phi} w_1 + \widetilde{F}_1^C, \tag{6.136}$$

$$\widetilde{F}_2 \approx a_1 \widetilde{\phi} w_2 + \widetilde{F}_1^C, \tag{6.137}$$

$$\widetilde{F}_3 \approx a_1 \widetilde{\phi} w_3 + \widetilde{F}_1^C, \tag{6.138}$$

$$\widetilde{F}_4 \approx a_1 \widetilde{\phi} w_4 + \widetilde{F}_1^C, \tag{6.139}$$

Table 6.4: Stability conditions of bifurcating solutions for group-theoretic critical points with multiplicity 4

μ	Cases	Solutions	Stability conditions
$(4; k, 0, +)$	$\check{n} = 3$	$\boldsymbol{w}_{\mathrm{sq}}$	Always unstable
		$\boldsymbol{w}_{\mathrm{stripeI}}$	Always unstable
		$\boldsymbol{w}_{\mathrm{stripeII}}$	Does not exist
	$\check{n} = 4$	$\boldsymbol{w}_{\mathrm{sq}}$	$a_3 < -a_5 < 0,\ a_3 + a_5 < -\lvert a_2\rvert$
		$\boldsymbol{w}_{\mathrm{stripeI}}$	$a_3 < -a_5 < 0,\ a_2 < a_3 + a_5$
		$\boldsymbol{w}_{\mathrm{stripeII}}$	$a_3 < -a_5 < 0,\ a_2 < a_3 + a_5$
$(4; k, k, +)$	$\check{n} = 3$	$\boldsymbol{w}_{\mathrm{sq}}$	Always unstable
		$\boldsymbol{w}_{\mathrm{stripeI}}$	Always unstable
		$\boldsymbol{w}_{\mathrm{stripeII}}$	Does not exist
	$\check{n} = 4$	$\boldsymbol{w}_{\mathrm{sq}}$	$a_5 + a_6 > 0,\ a_3 + a_5 < -\lvert a_2 + 2a_6\rvert$
		$\boldsymbol{w}_{\mathrm{stripeI}}$	$a_3 < -a_5 < 0,\ -2\lvert a_6\rvert < a_3 + a_5$
		$\boldsymbol{w}_{\mathrm{stripeII}}$	$a_3 < -a_5 < 0,\ -2\lvert a_6\rvert < a_3 + a_5$
$(4; n/2, \ell, +)$	$\tilde{n} = 4$	$\boldsymbol{w}_{\mathrm{sq}}$	$a_3 < -a_5 < 0,\ a_3 + a_5 < -\lvert a_2\rvert$
		$\boldsymbol{w}_{\mathrm{stripeI}}$	$a_3 < -a_5 < 0,\ a_2 < a_3 + a_5$
		$\boldsymbol{w}_{\mathrm{stripeII}}$	$a_3 < -a_5 < 0,\ a_2 < a_3 + a_5$

μ	Cases	Solutions	Stability conditions (necessary condition)
$(4; k, 0, +)$	General $\check{n}$	$\boldsymbol{w}_{\mathrm{sq}}$	$a_3 < -\lvert a_2\rvert$
		$\boldsymbol{w}_{\mathrm{stripeI}}$	$a_2 < a_3 < 0$
		$\boldsymbol{w}_{\mathrm{stripeII}}$	$a_2 < a_3 < 0$ if $\check{n}$ is even Does not exist if $\check{n}$ is odd
$(4; k, k, +)$	General $\check{n}$	$\boldsymbol{w}_{\mathrm{sq}}$	$a_3 < -\lvert a_2\rvert$
		$\boldsymbol{w}_{\mathrm{stripeI}}$	$a_2 < a_3 < 0$
		$\boldsymbol{w}_{\mathrm{stripeII}}$	$a_2 < a_3 < 0$ if $\check{n}$ is even Does not exist if $\check{n}$ is odd
$(4; n/2, \ell, +)$	General $\tilde{n}$	$\boldsymbol{w}_{\mathrm{sq}}$	$a_3 < -\lvert a_2\rvert$
		$\boldsymbol{w}_{\mathrm{stripeI}}$	$a_2 < a_3 < 0$
		$\boldsymbol{w}_{\mathrm{stripeII}}$	$a_2 < a_3 < 0$

$\check{n} = n/\gcd(k, n)$ in (6.40); $\tilde{n} = n/\gcd(\ell, n)$ in (6.41);

$a_2 = A_{1101}(0)$, $a_3 = A_{2010}(0)$, $a_4 = A_{0020}(0)$, $a_5 = A_{0030}(0)$, $a_6 = A_{0210}(0)$ in (6.124) where $\widetilde{F}_i^C$ ($i = 1, \ldots, 4$) is given in (6.130) – (6.133). Hence, the asymptotic form of the Jacobian matrix in (6.49) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1 \widetilde{\phi} I_4 + B_C \tag{6.140}$$

with

$$B_C = a_2 B_2 + a_3 B_3, \tag{6.141}$$

$$B_2 = \begin{bmatrix} {w_3}^2 + {w_4}^2 & 0 & 2w_1w_3 & 2w_1w_4 \\ 0 & {w_3}^2 + {w_4}^2 & 2w_2w_3 & 2w_2w_4 \\ 2w_1w_3 & 2w_2w_3 & {w_1}^2 + {w_2}^2 & 0 \\ 2w_1w_4 & 2w_2w_4 & 0 & {w_1}^2 + {w_2}^2 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 3{w_1}^2 + {w_2}^2 & 2w_1w_2 & 0 & 0 \\ 2w_1w_2 & {w_1}^2 + 3{w_2}^2 & 0 & 0 \\ 0 & 0 & 3{w_3}^2 + {w_4}^2 & 2w_3w_4 \\ 0 & 0 & 2w_3w_4 & {w_3}^2 + 3{w_4}^2 \end{bmatrix}.$$

Substituting $\boldsymbol{w}_{\mathrm{sq}}$ into (6.136) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{sq}} \approx -\frac{a_2 + a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.140) at $(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}}) \approx 2w^2 \begin{bmatrix} a_3 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 \\ a_2 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + O(w^3). \tag{6.142}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}})$ are given by

$$\lambda_1, \lambda_2 \approx 2(a_3 \pm a_2) w^2,$$
$$\lambda_3 \approx O(w^3) \quad \text{(repeated twice)}.$$

A necessary condition where $\boldsymbol{w}_{\mathrm{sq}}$ is stable is $a_3 < -|a_2|$. A more rigorous stability condition relies on the concrete form of the terms of $O(w^3)$ for λ_3. Thus, the stability of $\boldsymbol{w}_{\mathrm{sq}}$ depends on the values of a_2 and a_3.

Substituting $\boldsymbol{w}_{\mathrm{stripeI}}$ into (6.136) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{stripeI}} \approx -\frac{a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.140) at $(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}}) \approx w^2 \begin{bmatrix} 2a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 - a_3 & 0 \\ 0 & 0 & 0 & a_2 - a_3 \end{bmatrix} + O(w^3). \tag{6.143}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$ are given by the diagonal components, i.e.,

$$\begin{aligned} \lambda_1 &\approx 2a_3 w^2, \\ \lambda_2 &\approx O(w^3), \\ \lambda_3 &\approx (a_2 - a_3)w^2 \quad \text{(repeated twice).} \end{aligned}$$

Necessary conditions where $\boldsymbol{w}_{\text{stripeI}}$ is stable are $a_2 < a_3 < 0$. A more rigorous stability condition relies on the concrete form of the terms of $O(w^3)$ for λ_2. Thus, the stability of $\boldsymbol{w}_{\text{stripeI}}$ depends on the values of a_2 and a_3.

Substituting $\boldsymbol{w}_{\text{stripeII}}$ into (6.137) and solving $\widetilde{F}_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeII}} \approx -\frac{a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.140) at $(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}}) \approx w^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2a_3 & 0 & 0 \\ 0 & 0 & a_2 - a_3 & 0 \\ 0 & 0 & 0 & a_2 - a_3 \end{bmatrix} + O(w^3). \tag{6.144}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{stripeII}}$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$.

Case 2: $(4; k, 0, +)$ *with* $\check{n} = 3$

For the case $(4; k, 0, +)$ with $\check{n} = 3$, we have

$$(0, 0, 2, 0) \in S$$

as well as

$$(1, 0, 0, 0),\ (1, 1, 0, 1),\ (2, 0, 1, 0) \in S.$$

Then, the asymptotic form of F_i $(i = 1, 2)$ in (6.125) becomes

$$F_1(z_1, z_2, \widetilde{\phi}) \approx a_1 \widetilde{\phi} z_1 + a_4 \overline{z_1}^2 + F_1^C, \tag{6.145}$$

$$F_2(z_1, z_2, \widetilde{\phi}) \approx a_1 \widetilde{\phi} z_2 + a_4 \overline{z_2}^2 + F_2^C \tag{6.146}$$

with $a_4 = A_{0020}(0)$, where F_i^C $(i = 1, 2)$ is given in (6.126) and (6.127). By (6.51) and (6.52), the asymptotic form of $\widetilde{F}_i$ $(i = 1, \dots, 4)$ in (6.48) becomes

$$\widetilde{F}_1 \approx a_1 \widetilde{\phi} w_1 + a_4 ({w_1}^2 - {w_2}^2) + \widetilde{F}_1^C, \tag{6.147}$$

$$\widetilde{F}_2 \approx a_1 \widetilde{\phi} w_2 - 2a_4 w_1 w_2 + \widetilde{F}_2^C, \tag{6.148}$$

$$\widetilde{F}_3 \approx a_1 \widetilde{\phi} w_3 + a_4 ({w_3}^2 - {w_4}^2) + \widetilde{F}_3^C, \tag{6.149}$$

$$\widetilde{F}_4 \approx a_1 \widetilde{\phi} w_4 - 2a_4 w_3 w_4 + \widetilde{F}_4^C, \tag{6.150}$$

where $\widetilde{F}_i^C$ ($i = 1, \ldots, 4$) is given in (6.130) – (6.133). Hence, the asymptotic form of the Jacobian matrix in (6.49) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1 \widetilde{\phi} I_4 + a_4 B_4 + B_C \tag{6.151}$$

with

$$B_4 = 2 \begin{bmatrix} w_1 & -w_2 & 0 & 0 \\ -w_2 & -w_1 & 0 & 0 \\ 0 & 0 & w_3 & -w_4 \\ 0 & 0 & -w_4 & -w_3 \end{bmatrix},$$

where B_C is given in (6.141).

Substituting $\boldsymbol{w}_{\text{sq}}$ into (6.147) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sq}} \approx -\frac{a_4}{a_1} w.$$

Evaluating the Jacobian matrix (6.151) at $(\boldsymbol{w}_{\text{sq}}, \widetilde{\phi}_{\text{sq}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sq}}, \widetilde{\phi}_{\text{sq}}) \approx a_4 w \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}. \tag{6.152}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sq}}, \widetilde{\phi}_{\text{sq}})$ are given by the diagonal components, i.e., $a_4 w$ (repeated twice) and $-3a_4 w$ (repeated twice). Since the eigenvalues $a_4 w$ and $-3a_4 w$ have opposite signs, the bifurcating solution $\boldsymbol{w}_{\text{sq}}$ is always unstable.

Substituting $\boldsymbol{w}_{\text{stripeI}}$ into (6.147) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeI}} \approx -\frac{a_4}{a_1} w.$$

Evaluating the Jacobian matrix (6.151) at $(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}}) \approx a_4 w \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{6.153}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$ are given by the diagonal components, i.e., $a_4 w$, $-3a_4 w$ and $-a_4 w$ (repeated twice). Since the eigenvalues $a_4 w$ and $-3a_4 w$ have opposite signs, the bifurcating solution $\boldsymbol{w}_{\text{stripeI}}$ is always unstable.

Remark 6.1. *Since $\check{n}$ is odd, $\boldsymbol{w}_{stripeII}$ does not exist for the case $(4; k, 0, +)$ with $\check{n} = 3$. See Proposition 6.5 in Section 6.4.3.* □

Case 3: $(4; k, 0, +)$ *with* $\check{n} = 4$

For the case $(4; k, 0, +)$ with $\check{n} = 4$, we have

$$(0, 0, 3, 0) \in S$$

as well as

$$(1, 0, 0, 0),\ (1, 1, 0, 1),\ (2, 0, 1, 0) \in S.$$

Then, the asymptotic form of F_i $(i = 1, 2)$ in (6.125) becomes

$$F_1(z_1, z_2, \widetilde{\phi}) \approx a_1\widetilde{\phi}z_1 + a_5\overline{z}_1{}^3 + F_1^C, \tag{6.154}$$

$$F_2(z_1, z_2, \widetilde{\phi}) \approx a_1\widetilde{\phi}z_2 + a_5\overline{z}_2{}^3 + F_2^C \tag{6.155}$$

with $a_5 = A_{0030}(0)$, where F_i^C $(i = 1, 2)$ is given in (6.126) and (6.127). By (6.51) and (6.52), the asymptotic form of $\widetilde{F}_i$ $(i = 1, \dots, 4)$ in (6.48) becomes

$$\widetilde{F}_1 \approx a_1\widetilde{\phi}w_1 + a_5w_1(w_1{}^2 - 3w_2{}^2) + \widetilde{F}_1^C, \tag{6.156}$$

$$\widetilde{F}_2 \approx a_1\widetilde{\phi}w_2 + a_5w_2(w_2{}^2 - 3w_1{}^2) + \widetilde{F}_2^C, \tag{6.157}$$

$$\widetilde{F}_3 \approx a_1\widetilde{\phi}w_3 + a_5w_3(w_3{}^2 - 3w_4{}^2) + \widetilde{F}_3^C, \tag{6.158}$$

$$\widetilde{F}_4 \approx a_1\widetilde{\phi}w_4 + a_5w_4(w_4{}^2 - 3w_3{}^2) + \widetilde{F}_4^C, \tag{6.159}$$

where $\widetilde{F}_i^C$ $(i = 1, \dots, 4)$ is given in (6.130) – (6.133). Hence, the asymptotic form of the Jacobian matrix in (6.49) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1\widetilde{\phi}I_4 + a_5B_5 + B_C \tag{6.160}$$

with

$$B_5 = 3\begin{bmatrix} w_1{}^2 - w_2{}^2 & -2w_1w_2 & 0 & 0 \\ -2w_1w_2 & w_2{}^2 - w_1{}^2 & 0 & 0 \\ 0 & 0 & w_3{}^2 - w_4{}^2 & -2w_3w_4 \\ 0 & 0 & -2w_3w_4 & w_4{}^2 - w_3{}^2 \end{bmatrix}, \tag{6.161}$$

where B_C is given in (6.141).

Substituting $\boldsymbol{w}_{\rm sq}$ into (6.156) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\rm sq} \approx -\frac{a_5 + a_2 + a_3}{a_1}w^2.$$

Evaluating the Jacobian matrix (6.160) at $(\boldsymbol{w}_{\rm sq}, \widetilde{\phi}_{\rm sq})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\rm sq}, \widetilde{\phi}_{\rm sq}) \approx 2w^2\begin{bmatrix} a_5 + a_3 & 0 & a_2 & 0 \\ 0 & -2a_5 & 0 & 0 \\ a_2 & 0 & a_5 + a_3 & 0 \\ 0 & 0 & 0 & -2a_5 \end{bmatrix}. \tag{6.162}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}})$ are given by

$$\begin{aligned}
&\lambda_1, \lambda_2 \approx 2(a_5 + a_3 \pm a_2)w^2, \\
&\lambda_3 \approx -4a_5 w^2 \quad \text{(repeated twice).}
\end{aligned}$$

If $a_3 < -a_5 < 0$ and $a_5 + a_3 < -|a_2|$ are satisfied, $\boldsymbol{w}_{\mathrm{sq}}$ is stable. Otherwise, $\boldsymbol{w}_{\mathrm{sq}}$ is unstable. Thus, the stability of $\boldsymbol{w}_{\mathrm{sq}}$ depends on the values of a_2, a_3 and a_5.

Substituting $\boldsymbol{w}_{\mathrm{stripeI}}$ into (6.156) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{stripeI}} \approx -\frac{a_5 + a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.160) at $(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}}) \approx w^2 \begin{bmatrix} 2(a_5 + a_3) & 0 & 0 & 0 \\ 0 & -4a_5 & 0 & 0 \\ 0 & 0 & -a_5 + a_2 - a_3 & 0 \\ 0 & 0 & 0 & -a_5 + a_2 - a_3 \end{bmatrix}. \tag{6.163}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}})$ are given by the diagonal components, i.e.,

$$\begin{aligned}
&\lambda_1 \approx 2(a_5 + a_3)w^2, \\
&\lambda_2 \approx -4a_5 w^2, \\
&\lambda_3 \approx -(a_5 - a_2 + a_3)w^2 \quad \text{(repeated twice).}
\end{aligned}$$

If $a_3 < -a_5 < 0$ and $a_2 < a_5 + a_3$ are satisfied, $\boldsymbol{w}_{\mathrm{stripeI}}$ is stable. Thus, the stability of $\boldsymbol{w}_{\mathrm{stripeI}}$ depends on the values of a_2, a_3 and a_5.

Substituting $\boldsymbol{w}_{\mathrm{stripeII}}$ into (6.157) and solving $\widetilde{F}_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{stripeII}} \approx -\frac{a_5 + a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.160) at $(\boldsymbol{w}_{\mathrm{stripeII}}, \widetilde{\phi}_{\mathrm{stripeII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeII}}, \widetilde{\phi}_{\mathrm{stripeII}}) \approx w^2 \begin{bmatrix} -4a_5 & 0 & 0 & 0 \\ 0 & 2(a_5 + a_3) & 0 & 0 \\ 0 & 0 & -a_5 + a_2 - a_3 & 0 \\ 0 & 0 & 0 & -a_5 + a_2 - a_3 \end{bmatrix}. \tag{6.164}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeII}}, \widetilde{\phi}_{\mathrm{stripeII}})$ are equivalent to that for $\boldsymbol{w}_{\mathrm{stripeI}}$. Hence, stability conditions for $\boldsymbol{w}_{\mathrm{stripeII}}$ are equivalent to that for $\boldsymbol{w}_{\mathrm{stripeI}}$.

Case 4: General $(4; k, k, +)$

For general $(4; k, k, +)$, we have

$$(1, 0, 0, 0),\ (1, 1, 0, 1),\ (2, 0, 1, 0) \in S,$$

and the other elements of S correspond to higher order terms. Then, the asymptotic form of F_1 in (6.130) is equivalent to that for the case 1: General $(4; k, 0, +)$. Hence, a discussion similar to that for the case 1 holds.

Case 5: $(4;k,k,+)$ *with* $\check{n}=3$

For the case $(4;k,k,+)$ with $\check{n}=3$, we have

$$(0,0,2,0)\in S$$

as well as

$$(1,0,0,0),\ (1,1,0,1),\ (2,0,1,0)\in S.$$

Then, the asymptotic form of F_1 in (6.130) is equivalent to that for the case 2: $(4;k,0,+)$ with $\check{n}=3$. Hence, a discussion similar to that for the case 2 holds, that is, $\boldsymbol{w}_{\rm sq}$ and $\boldsymbol{w}_{\rm stripeI}$ are always unstable. Since $\check{n}$ is odd, $\boldsymbol{w}_{\rm stripeII}$ does not exist for this case (see Proposition 6.5 in Section 6.4.3).

Case 6: $(4;k,k,+)$ *with* $\check{n}=4$

For the case $(4;k,k,+)$ with $\check{n}=4$, we have

$$(0,0,3,0),\ (0,2,1,0),\ (0,0,1,2)\in S$$

as well as

$$(1,0,0,0),\ (1,1,0,1),\ (2,0,1,0)\in S.$$

From the condition (6.81), we have $A_{0210}(0)=A_{0012}(0)$. Then, the asymptotic form of F_i $(i=1,2)$ in (6.125) becomes

$$F_1(z_1,z_2,\widetilde{\phi})\approx a_1\widetilde{\phi}z_1+a_5\bar{z}_1{}^3+a_6\bar{z}_1(z_2{}^2+\bar{z}_2{}^2)+F_1^C, \tag{6.165}$$

$$F_2(z_1,z_2,\widetilde{\phi})\approx a_1\widetilde{\phi}z_2+a_5\bar{z}_2{}^3+a_6\bar{z}_2(\bar{z}_1{}^2+z_1{}^2)+F_2^C \tag{6.166}$$

with $a_6=A_{0210}(0)$, where F_i^C $(i=1,2)$ is given in (6.126) and (6.127). By (6.51) and (6.52), the asymptotic form of $\widetilde{F}_i$ $(i=1,\dots,4)$ in (6.48) becomes

$$\widetilde{F}_1\approx a_1\widetilde{\phi}w_1+a_5w_1(w_1{}^2-3w_2{}^2)+2a_6w_1(w_3{}^2-w_4{}^2)+\widetilde{F}_1^C, \tag{6.167}$$

$$\widetilde{F}_2\approx a_1\widetilde{\phi}w_2+a_5w_2(w_2{}^2-3w_1{}^2)+2a_6w_2(w_4{}^2-w_3{}^2)+\widetilde{F}_2^C, \tag{6.168}$$

$$\widetilde{F}_3\approx a_1\widetilde{\phi}w_3+a_5w_3(w_3{}^2-3w_4{}^2)+2a_6w_3(w_1{}^2-w_2{}^2)+\widetilde{F}_3^C, \tag{6.169}$$

$$\widetilde{F}_4\approx a_1\widetilde{\phi}w_4+a_5w_4(w_4{}^2-3w_3{}^2)+2a_6w_4(w_2{}^2-w_1{}^2)+\widetilde{F}_4^C, \tag{6.170}$$

where $\widetilde{F}_i^C$ $(i=1,\dots,4)$ is given in (6.130) – (6.133). Hence, the asymptotic form of the Jacobian matrix in (6.49) becomes

$$\widetilde{J}(\boldsymbol{w},\widetilde{\phi})\approx a_1\widetilde{\phi}I_4+a_5B_5+a_6B_6+B_C \tag{6.171}$$

with

$$B_5=3\begin{bmatrix} w_1{}^2-w_2{}^2 & -2w_1w_2 & 0 & 0\\ -2w_1w_2 & w_2{}^2-w_1{}^2 & 0 & 0\\ 0 & 0 & w_3{}^2-w_4{}^2 & -2w_3w_4\\ 0 & 0 & -2w_3w_4 & w_4{}^2-w_3{}^2 \end{bmatrix},$$

$$B_6 = 2\begin{bmatrix} {w_3}^2 - {w_4}^2 & 0 & 2w_1w_3 & -2w_1w_4 \\ 0 & {w_4}^2 - {w_3}^2 & -2w_2w_3 & 2w_2w_4 \\ 2w_1w_3 & -2w_2w_3 & {w_1}^2 - {w_2}^2 & 0 \\ -2w_1w_4 & 2w_2w_4 & 0 & {w_2}^2 - {w_1}^2 \end{bmatrix},$$

where B_5 and B_C are given in (6.161) and (6.141).

Substituting $\boldsymbol{w}_{\mathrm{sq}}$ into (6.167) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{sq}} \approx -\frac{a_2 + a_3 + a_5 + 2a_6}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.171) at $(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}}) \approx 2w^2 \begin{bmatrix} a_5 + a_3 & 0 & 2a_6 + a_2 & 0 \\ 0 & -2(a_5 + a_6) & 0 & 0 \\ 2a_6 + a_2 & 0 & a_5 + a_3 & 0 \\ 0 & 0 & 0 & -2(a_5 + a_6) \end{bmatrix}. \tag{6.172}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}})$ are given by

$$\begin{aligned} &\lambda_1, \lambda_2 \approx 2\{(a_5 + a_3) \pm (2a_6 + a_2)\} w^2, \\ &\lambda_3 \approx -4(a_5 + a_6) w^2 \quad \text{(repeated twice)}. \end{aligned}$$

If $a_5 + a_6 > 0$ and $a_5 + a_3 < -|2a_6 + a_2|$ are satisfied, $\boldsymbol{w}_{\mathrm{sq}}$ is stable. Otherwise, $\boldsymbol{w}_{\mathrm{sq}}$ is unstable. Thus, the stability of $\boldsymbol{w}_{\mathrm{sq}}$ depends on the values of a_2, a_3, a_5 and a_6.

Substituting $\boldsymbol{w}_{\mathrm{stripeI}}$ into (6.167) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{stripeI}} \approx -\frac{a_5 + a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.171) at $(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}}) \approx -w^2 \begin{bmatrix} -2(a_5 + a_3) & 0 & 0 & 0 \\ 0 & 4a_5 & 0 & 0 \\ 0 & 0 & a_5 - 2a_6 + a_3 & 0 \\ 0 & 0 & 0 & a_5 + 2a_6 + a_3 \end{bmatrix}. \tag{6.173}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}})$ are given by the diagonal components, i.e.,

$$\begin{aligned} &\lambda_1 \approx 2(a_5 + a_3) w^2, \\ &\lambda_2 \approx -4a_5 w^2, \\ &\lambda_3, \lambda_4 \approx -(a_5 + a_3 \pm 2a_6) w^2, \end{aligned}$$

If $a_3 < -a_5 < 0$ and $-2|a_6| < a_5 + a_3$ are satisfied, $\boldsymbol{w}_{\mathrm{stripeI}}$ is stable. Otherwise, $\boldsymbol{w}_{\mathrm{stripeI}}$ is unstable. Thus, the stability of $\boldsymbol{w}_{\mathrm{stripeI}}$ depends on the values of a_3, a_5 and a_6.

Substituting $\boldsymbol{w}_{\mathrm{stripeII}}$ into (6.168) and solving $\widetilde{F}_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{stripeII}} \approx -\frac{a_5 + a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.171) at $(\boldsymbol{w}_{\mathrm{stripeII}}, \widetilde{\phi}_{\mathrm{stripeII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeII}}, \widetilde{\phi}_{\mathrm{stripeII}}) \approx -w^2 \begin{bmatrix} 4a_5 & 0 & 0 & 0 \\ 0 & -2(a_5 + a_3) & 0 & 0 \\ 0 & 0 & a_5 - 2a_6 + a_3 & 0 \\ 0 & 0 & 0 & a_5 + 2a_6 + a_3 \end{bmatrix}. \tag{6.174}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeII}}, \widetilde{\phi}_{\mathrm{stripeII}})$ are equivalent to that for $\boldsymbol{w}_{\mathrm{stripeI}}$. Hence, stability conditions for $\boldsymbol{w}_{\mathrm{stripeII}}$ is equivalent to that for $\boldsymbol{w}_{\mathrm{stripeI}}$.

Case 7: General $(4; n/2, \ell, +)$

For general $(4; n/2, \ell, +)$, we have

$$(1, 0, 0, 0),\ (1, 1, 0, 1),\ (2, 0, 1, 0) \in S,$$

and the other elements of S correspond to higher order terms. Then, the asymptotic form of F_1 in (6.130) is equivalent to that for the case 1: General $(4; k, 0, +)$. Hence, a discussion similar to that for the case 1 holds.

Case 8: $(4; n/2, \ell, +)$ *with* $\tilde{n} = 4$

For the case $(4; n/2, \ell, +)$ with $\tilde{n} = 4$, we have

$$(0, 0, 3, 0) \in S$$

as well as

$$(1, 0, 0, 0),\ (1, 1, 0, 1),\ (2, 0, 1, 0) \in S.$$

Then, the asymptotic form of F_1 in (6.130) is equivalent to that for the case 3: $(4; k, 0, +)$ with $\check{n} = 4$. Hence, a discussion similar to that for the case 3 holds.

6.5. Bifurcation Point of Multiplicity 8

We consider a critical point associated with eight-dimensional irreducible representations μ of the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$:

$$(8; k, \ell) \ \text{ with } \ 1 \le \ell \le k-1, \ 2 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{6.175}$$

where $n \geq 5$. For $(8; k, \ell)$, we use the following notations:

$$\hat{k} = \frac{k}{\gcd(k, \ell, n)}, \quad \hat{\ell} = \frac{\ell}{\gcd(k, \ell, n)}, \quad \hat{n} = \frac{n}{\gcd(k, \ell, n)}. \tag{6.176}$$

The actions in $(8; k, \ell)$ on an eight-dimensional vector $(w_1, \ldots, w_8) \in \mathbb{R}^8$ can be expressed for a four-dimensional vector $(z_1, \ldots, z_4)$ with complex variables $z_j = w_{2j-1} + \mathrm{i}w_{2j}$ $(j = 1, \ldots, 4)$ as (cf., (5.62) and (5.63))

$$r: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z}_2 \\ z_1 \\ z_4 \\ \overline{z}_3 \end{bmatrix}, \quad s: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} z_3 \\ z_4 \\ z_1 \\ z_2 \end{bmatrix}, \tag{6.177}$$

$$p_1: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^{-\ell} z_2 \\ \omega^k z_3 \\ \omega^{-\ell} z_4 \end{bmatrix}, \quad p_2: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} \omega^\ell z_1 \\ \omega^k z_2 \\ \omega^{-\ell} z_3 \\ \omega^{-k} z_4 \end{bmatrix} \tag{6.178}$$

with $\omega = \exp(\mathrm{i}2\pi/n)$.

6.5.1. Derivation of Bifurcation Equation

The bifurcation equation for the critical point of multiplicity 8 is an eight-dimensional equation in $\boldsymbol{w} = (w_1, \ldots, w_8) \in \mathbb{R}^8$ expressed as

$$\widetilde{F}_i(\boldsymbol{w}, \widetilde{\phi}) = 0, \quad i = 1, \ldots, 8, \tag{6.179}$$

where $(w_1, \ldots, w_8, \widetilde{\phi}) = (0, \ldots, 0, 0)$ is assumed to correspond to the critical point. Accordingly, the Jacobian matrix of $\widetilde{\boldsymbol{F}}$ is an 8×8 matrix expressed as

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) = \left(\frac{\partial \widetilde{F}_i}{\partial w_j} \,\middle|\, i, j = 1, \ldots, 8 \right). \tag{6.180}$$

The bifurcation equation (6.179) can be expressed as a four-dimensional equation in complex variables $z_j = w_{2j-1} + \mathrm{i}w_{2j}$ $(j = 1, \ldots, 4)$ as

$$F_i(z_1, z_2, z_3, z_4, \widetilde{\phi}) = 0, \quad i = 1, \ldots, 4, \tag{6.181}$$

where $(z_1,\dots,z_4,\widetilde{\phi}) = (0,\dots,0,0)$ corresponds to the critical point. There are the following relationships between (6.179) and (6.181):

$$F_1(z_1,z_2,z_3,z_4,\widetilde{\phi}) = \widetilde{F}_1 + \mathrm{i}\widetilde{F}_2, \tag{6.182}$$
$$F_2(z_1,z_2,z_3,z_4,\widetilde{\phi}) = \widetilde{F}_3 + \mathrm{i}\widetilde{F}_4, \tag{6.183}$$
$$F_3(z_1,z_2,z_3,z_4,\widetilde{\phi}) = \widetilde{F}_5 + \mathrm{i}\widetilde{F}_6, \tag{6.184}$$
$$F_4(z_1,z_2,z_3,z_4,\widetilde{\phi}) = \widetilde{F}_7 + \mathrm{i}\widetilde{F}_8. \tag{6.185}$$

We expand F_1 into a power series as

$$F_1(z_1,z_2,z_3,z_4,\widetilde{\phi}) = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0}\sum_{e=0}\sum_{f=0}\sum_{g=0}\sum_{h=0} A_{abcdefgh}(\widetilde{\phi}) {z_1}^a {z_2}^b {z_3}^c {z_4}^d\, {\overline{z}_1}^e\, {\overline{z}_2}^f\, {\overline{z}_3}^g\, {\overline{z}_4}^h. \tag{6.186}$$

Since $(z_1,z_2,z_3,z_4,\widetilde{\phi}) = (0,0,0,0,0)$ corresponds to the critical point, we have

$$A_{00000000}(0) = 0, \quad A_{10000000}(0) = A_{01000000}(0) = \cdots = A_{00000001}(0) = 0.$$

Since $a_1 = A'_{10000000}(0)$ is generically nonzero, we have

$$A_{10000000}(\widetilde{\phi}) \approx a_1\widetilde{\phi}.$$

The equivariance of the bifurcation equation to the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ is identical to the equivariance to the action of the four elements r, s, p_1, and p_2 generating this group. The equivariance condition for $(8;k,\ell)$ is written as

$$r: \quad \overline{F_2(z_1,z_2,z_3,z_4)} = F_1(\overline{z}_2,z_1,z_4,\overline{z}_3), \tag{6.187}$$
$$F_1(z_1,z_2,z_3,z_4) = F_2(\overline{z}_2,z_1,z_4,\overline{z}_3), \tag{6.188}$$
$$F_4(z_1,z_2,z_3,z_4) = F_3(\overline{z}_2,z_1,z_4,\overline{z}_3), \tag{6.189}$$
$$\overline{F_3(z_1,z_2,z_3,z_4)} = F_4(\overline{z}_2,z_1,z_4,\overline{z}_3), \tag{6.190}$$
$$s: \quad F_3(z_1,z_2,z_3,z_4) = F_1(z_3,z_4,z_1,z_2), \tag{6.191}$$
$$F_4(z_1,z_2,z_3,z_4) = F_2(z_3,z_4,z_1,z_2), \tag{6.192}$$
$$F_1(z_1,z_2,z_3,z_4) = F_3(z_3,z_4,z_1,z_2), \tag{6.193}$$
$$F_2(z_1,z_2,z_3,z_4) = F_4(z_3,z_4,z_1,z_2), \tag{6.194}$$
$$p_1: \quad \omega^k F_1(z_1,z_2,z_3,z_4) = F_1(\omega^k z_1,\omega^{-\ell}z_2,\omega^k z_3,\omega^{-\ell}z_4), \tag{6.195}$$
$$\omega^{-\ell} F_2(z_1,z_2,z_3,z_4) = F_2(\omega^k z_1,\omega^{-\ell}z_2,\omega^k z_3,\omega^{-\ell}z_4), \tag{6.196}$$
$$\omega^k F_3(z_1,z_2,z_3,z_4) = F_3(\omega^k z_1,\omega^{-\ell}z_2,\omega^k z_3,\omega^{-\ell}z_4), \tag{6.197}$$
$$\omega^{-\ell} F_4(z_1,z_2,z_3,z_4) = F_4(\omega^k z_1,\omega^{-\ell}z_2,\omega^k z_3,\omega^{-\ell}z_4), \tag{6.198}$$
$$p_2: \quad \omega^\ell F_1(z_1,z_2,z_3,z_4) = F_1(\omega^\ell z_1,\omega^k z_2,\omega^{-\ell}z_3,\omega^{-k}z_4), \tag{6.199}$$
$$\omega^k F_2(z_1,z_2,z_3,z_4) = F_2(\omega^\ell z_1,\omega^k z_2,\omega^{-\ell}z_3,\omega^{-k}z_4), \tag{6.200}$$
$$\omega^{-\ell} F_3(z_1,z_2,z_3,z_4) = F_3(\omega^\ell z_1,\omega^k z_2,\omega^{-\ell}z_3,\omega^{-k}z_4), \tag{6.201}$$

$$\omega^{-k}F_4(z_1,z_2,z_3,z_4)=F_4(\omega^{\ell}z_1,\omega^{k}z_2,\omega^{-\ell}z_3,\omega^{-k}z_4). \tag{6.202}$$

The equivariance conditions with respect to r and s are expressed as follows. The equivariance condition (6.188) for r implies

$$F_2(z_1,z_2,z_3,z_4)=F_1(z_2,\bar{z}_1,\bar{z}_4,z_3). \tag{6.203}$$

The equivariance condition (6.191) and (6.192) for s implies

$$F_3(z_1,z_2,z_3,z_4)=F_1(z_3,z_4,z_1,z_2), \tag{6.204}$$
$$F_4(z_1,z_2,z_3,z_4)=F_2(z_3,z_4,z_1,z_2). \tag{6.205}$$

Combining (6.203) and (6.205), we have

$$F_4(z_1,z_2,z_3,z_4)=F_1(z_4,\bar{z}_3,\bar{z}_2,z_1). \tag{6.206}$$

Hence, we obtain F_2, F_3 and F_4 from F_1 by using (6.203), (6.204) and (6.206). Combining (6.187) and (6.203), we have

$$\overline{F_1(\bar{z}_2,z_1,\bar{z}_4,z_3)}=F_1(z_2,\bar{z}_1,z_4,\bar{z}_3). \tag{6.207}$$

Hence, we have

$$A_{ab\cdots h}(\widetilde{\phi})\in\mathbb{R}. \tag{6.208}$$

It is ensured that the equivariance conditions (6.187) – (6.194) are satisfied by (6.203), (6.204), (6.206), and (6.207).

The equivariance conditions with respect to p_1 and p_2 are expressed as follows. The equivariance condition (6.195) for p_1 is expressed as

$$\begin{aligned}&\sum_{a=0}\sum_{b=0}\cdots\sum_{h=0}\omega^{k}A_{ab\cdots h}(\widetilde{\phi}){z_1}^a{z_2}^b{z_3}^c{z_4}^d\,{\bar{z}_1}^e\,{\bar{z}_2}^f\,{\bar{z}_3}^g\,{\bar{z}_4}^h\\&=\sum_{a=0}\sum_{b=0}\cdots\sum_{h=0}\omega^{k(a+c-e-g)-\ell(b+d-f-h)}A_{ab\cdots h}(\widetilde{\phi}){z_1}^a{z_2}^b{z_3}^c{z_4}^d\,{\bar{z}_1}^e\,{\bar{z}_2}^f\,{\bar{z}_3}^g\,{\bar{z}_4}^h,\end{aligned}$$

which implies

$$\omega^{k(a+c-e-g-1)-\ell(b+d-f-h)}=\exp\left[\frac{2\pi\mathrm{i}}{n}\{k(a+c-e-g-1)-\ell(b+d-f-h)\}\right]=1. \tag{6.209}$$

The equivariance condition (6.199) for p_2 is expressed as

$$\begin{aligned}&\sum_{a=0}\sum_{b=0}\cdots\sum_{h=0}\omega^{\ell}A_{ab\cdots h}(\widetilde{\phi}){z_1}^a{z_2}^b{z_3}^c{z_4}^d\,{\bar{z}_1}^e\,{\bar{z}_2}^f\,{\bar{z}_3}^g\,{\bar{z}_4}^h\\&=\sum_{a=0}\sum_{b=0}\cdots\sum_{h=0}\omega^{k(b-d-f+h)+\ell(a-c-e+g)}A_{ab\cdots h}(\widetilde{\phi}){z_1}^a{z_2}^b{z_3}^c{z_4}^d\,{\bar{z}_1}^e\,{\bar{z}_2}^f\,{\bar{z}_3}^g\,{\bar{z}_4}^h,\end{aligned}$$

which implies

$$\omega^{k(b-d-f+h)+\ell(a-c-e+g-1)} = \exp\left[\frac{2\pi \mathrm{i}}{n}\{k(b-d-f+h)+\ell(a-c-e+g-1)\}\right] = 1. \tag{6.210}$$

Using (6.203), (6.204), or (6.206), we rewrite the equivariance conditions (6.196) – (6.198) for p_1 as follows:

$$\omega^{-\ell}F_1(z_2,\bar{z}_1,\bar{z}_4,z_3) = F_1(\omega^{-\ell}z_2,\omega^{-k}\,\bar{z}_1,\omega^{\ell}\,\bar{z}_4,\omega^{k}z_3), \tag{6.211}$$
$$\omega^{k}F_1(z_3,z_4,z_1,z_2) = F_1(\omega^{k}z_3,\omega^{-\ell}z_4,\omega^{k}z_1,\omega^{-\ell}z_2), \tag{6.212}$$
$$\omega^{-\ell}F_1(z_4,\bar{z}_3,\bar{z}_2,z_1) = F_1(\omega^{-\ell}z_4,\omega^{-k}\,\bar{z}_3,\omega^{\ell}\,\bar{z}_2,\omega^{k}z_1). \tag{6.213}$$

Similarly, we rewrite the equivariance conditions (6.200) – (6.202) for p_2 as follows:

$$\omega^{k}F_1(z_2,\bar{z}_1,\bar{z}_4,z_3) = F_1(\omega^{k}z_2,\omega^{-\ell}\,\bar{z}_1,\omega^{k}\,\bar{z}_4,\omega^{-\ell}z_3), \tag{6.214}$$
$$\omega^{-\ell}F_1(z_3,z_4,z_1,z_2) = F_1(\omega^{-\ell}z_3,\omega^{-k}z_4,\omega^{\ell}z_1,\omega^{k}z_2), \tag{6.215}$$
$$\omega^{-k}F_1(z_4,\bar{z}_3,\bar{z}_2,z_1) = F_1(\omega^{-k}z_4,\omega^{\ell}\,\bar{z}_3,\omega^{-k}\,\bar{z}_2,\omega^{\ell}z_1). \tag{6.216}$$

The equivariance conditions (6.211), (6.213), and (6.215) lead to the same result as (6.210). The equivariance conditions (6.212), (6.214), and a complex conjugate of (6.216) lead to the same result as (6.209). To sum up, we have the following conditions for $(8;k,\ell)$:

$$k(a+c-e-g-1)-\ell(b+d-f-h) \equiv 0 \mod n,$$
$$k(b-d-f+h)+\ell(a-c-e+g-1) \equiv 0 \mod n,$$

which are equivalent to

$$\hat{k}(a+c-e-g-1)-\hat{\ell}(b+d-f-h) \equiv 0 \mod \hat{n}, \tag{6.217}$$
$$\hat{k}(b-d-f+h)+\hat{\ell}(a-c-e+g-1) \equiv 0 \mod \hat{n}. \tag{6.218}$$

We rewrite the conditions (6.217) and (6.218) in a matrix form as

$$A\begin{bmatrix}\hat{k}\\ \hat{\ell}\end{bmatrix} \equiv \begin{bmatrix}0\\0\end{bmatrix} \mod \hat{n} \tag{6.219}$$

with

$$A = \begin{bmatrix} a+c-e-g-1 & -b-d+f+h \\ b-d-f+h & a-c-e+g-1 \end{bmatrix}. \tag{6.220}$$

This condition is equivalent to the following condition:

$$\exists p,q\in\mathbb{Z} \quad \text{s.t.} \quad A\begin{bmatrix}\hat{k}\\ \hat{\ell}\end{bmatrix} = \hat{n}\begin{bmatrix}p\\q\end{bmatrix}. \tag{6.221}$$

For this condition, we define a set P as

$$P = \{(a,b,\ldots,h)\in\mathbb{Z}_+^8 \mid (6.221)\text{ with }(6.220)\}, \tag{6.222}$$

where $\mathbb{Z}_+$ represents the set of nonnegative integers. Note that $(a, b, \ldots, h) \in \mathbb{Z}_+^8$ must belong to P when $A_{ab\cdots h}(\widetilde{\phi}) \neq 0$ in (6.186). Hence, we replace the power series (6.186) with

$$F_1(z_1, z_2, z_3, z_4, \widetilde{\phi}) = \sum_{P} A_{abcdefgh}(\widetilde{\phi})\, {z_1}^a {z_2}^b {z_3}^c {z_4}^d\, {\overline{z}_1}^e\, {\overline{z}_2}^f\, {\overline{z}_3}^g\, {\overline{z}_4}^h. \tag{6.223}$$

In addition, we have the following proposition:

Proposition 6.8. *If $\hat{n} = n/\gcd(n, k, \ell)$ is even, then $(a, b, \ldots, h) \in P$ satisfies $a + b + c + d + e + f + g + h \notin 2\mathbb{Z}$.*

Proof. Since $\hat{n}$ is even, $p\hat{n}$ $(p \in \mathbb{Z})$ in (6.217) and $q\hat{n}$ $(q \in \mathbb{Z})$ in (6.218) are even. Since $\hat{n}$, $\hat{k}$, and $\hat{\ell}$ do not have a common divisor, $(\hat{k}, \hat{\ell}) \neq (\text{even}, \text{even})$. To prove the statement by contradiction, assume $a + b + c + d + e + f + g + h \in 2\mathbb{Z}$.

- For the case $a + c + e + g \in 2\mathbb{Z}$ and $b + d - f - h \in 2\mathbb{Z}$, we have the following statements: If $(\hat{k}, \hat{\ell}) = (\text{odd}, \text{even})$, the left-hand side of (6.217) is odd since it takes the form:

  $$(\text{odd}) \times (\text{odd}) + (\text{even}) \times (\text{even}).$$

  If $(\hat{k}, \hat{\ell}) = (\text{even}, \text{odd})$, the left-hand side of (6.218) is odd since it takes the form:

  $$(\text{even}) \times (\text{even}) + (\text{odd}) \times (\text{odd}).$$

  Thus, the condition (6.217) and (6.218) are cannot be satisfied simultaneously.

- For the case $a + c + e + g \notin 2\mathbb{Z}$ and $b + d + f + h \notin 2\mathbb{Z}$, we have the following statements: If $(\hat{k}, \hat{\ell}) = (\text{odd}, \text{even})$, the left-hand side of (6.218) is odd since it takes the form: (odd)+(even). If $(\hat{k}, \hat{\ell}) = (\text{even}, \text{odd})$, the left-hand side of (6.217) is odd since it takes the form: (even) + (odd). Thus, the condition (6.217) and (6.218) are cannot be satisfied simultaneously.

Hence, $a + b + c + d + e + f + g + h \in 2\mathbb{Z}$ is a contradiction. □

6.5.2. Symmetry of Square Patterns

For the irreducible representation $\mu = (8; k, \ell)$, a system of the bifurcation equations $F_1 = F_2 = F_3 = F_4 = 0$ has the following bifurcating solutions:

$$\begin{aligned} &\text{Type VM square pattern} : (z_1, z_2, z_3, z_4) = (w, w, w, w)\ (w \in \mathbb{R}),\\ &\text{Type T square pattern} : (z_1, z_2, z_3, z_4) = (w, w, 0, 0)\ (w \in \mathbb{R}). \end{aligned}$$

In Section 5.6.4, we showed that the Type VM solution exists for any $(\hat{n}, \hat{k}, \hat{\ell})$, while the Type T solution exists if the values of $(\hat{n}, \hat{k}, \hat{\ell})$ satisfies

$$\overline{\textbf{GCD-div}} : 2\gcd(\hat{k}, \hat{\ell}) \text{ is not divisible by } \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}) \tag{6.224}$$

(see Proposition 5.20).

Substituting the Type VM solution $(z_1, z_2, z_3, z_4) = (w, w, w, w)$ into (6.223), we have

$$F_1(w, w, w, w, \widetilde{\phi}) = \sum_P A_{abcdefgh}(\widetilde{\phi}) w^{a+b+d+e+f+g+h}.$$

Proposition 6.8 shows that if $\hat{n}$ is even, then $F_1(w, w, w, w, \widetilde{\phi})$ becomes an odd function in w. Thus, the two bifurcating solutions $(w, w, w, w, \widetilde{\phi})$ and $(-w, -w, -w, -w, \widetilde{\phi})$ are conjugate. Substituting the Type T solution $(z_1, z_2, z_3, z_4) = (w, w, 0, 0)$ into (6.223), we have

$$F_1(w, w, 0, 0, \widetilde{\phi}) = \sum_{(a,b,0,0,e,f,0,0)\in P} A_{ab00ef00}(\widetilde{\phi}) w^{a+b+e+f}.$$

Proposition 6.8 shows that if $\hat{n}$ is even, then $a + b + e + f \notin 2\mathbb{Z}$ for $(a, b, 0, 0, e, f, 0, 0) \in P$. Thus, $F_1(w, w, 0, 0, \widetilde{\phi})$ becomes an odd function in w, and hence the two bifurcating solutions $(w, w, 0, 0, \widetilde{\phi})$ and $(-w, -w, 0, 0, \widetilde{\phi})$ are conjugate.

To sum up, we have the following proposition on the symmetry of the square patterns.

Proposition 6.9. *For a critical point of multiplicity 8 associated with $\mu = (8; k, \ell)$, the two bifurcating solutions $(\boldsymbol{z}, \widetilde{\phi})$ and $(-\boldsymbol{z}, \widetilde{\phi})$ are conjugate for $\boldsymbol{z} = (w, w, w, w)$, $(w, w, 0, 0)$ $(w \in \mathbb{R})$ if $\hat{n} = n/\gcd(n, k, \ell)$ is even and are not conjugate if $\hat{n}$ is odd.*

6.5.3. Existence and Symmetry of Stripe Patterns

We would like to show the existence and the symmetry of two types of stripe patterns, which are represented as

Type I stripe pattern : $(z_1, z_2, z_3, z_4) = (w, 0, 0, 0)$ $(w \in \mathbb{R})$,
Type II stripe pattern : $(z_1, z_2, z_3, z_4) = (\mathrm{i}w, 0, 0, 0)$ $(w \in \mathbb{R})$.

For both cases, we have $(a, b, \dots, h) = (a, 0, 0, 0, e, 0, 0, 0)$, and hence (6.217) and (6.218) leads to

$$\hat{k}(a - e - 1) \equiv 0, \quad \hat{\ell}(a - e - 1) \equiv 0 \mod \hat{n}, \tag{6.225}$$

which imply $a = e + p\hat{n} + 1$ $(p \in \mathbb{Z})$. Then, F_1 in (6.223) is rewritten as

$$\begin{aligned} &F_1(z_1, 0, 0, 0, \widetilde{\phi}) \\ &= \sum_{q=0}^{\infty} A_{q+1,q}(\widetilde{\phi})|z_1|^{2q} z_1 + \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} [A_{q+p\hat{n}+1,q}(\widetilde{\phi})|z_1|^{2q} {z_1}^{p\hat{n}+1} + A_{q,q+p\hat{n}-1}(\widetilde{\phi})|z_1|^{2q}\, {\overline{z}_1}^{p\hat{n}-1}] \end{aligned} \tag{6.226}$$

with $A_{ae}(\widetilde{\phi}) = A_{a000e000}(\widetilde{\phi})$.

Substituting the Type I stripe pattern $(z_1, z_2, z_3, z_4) = (w, 0, 0, 0)$ into (6.226), we have

$$\begin{aligned} F_1(w, 0, 0, 0, \widetilde{\phi}) &= w \left\{ \sum_{q=0}^{\infty} A_{q+1,q}(\widetilde{\phi}) w^{2q} + \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} [A_{q+p\hat{n}+1,q}(\widetilde{\phi}) w^{2q+p\hat{n}} + A_{q,q+p\hat{n}-1}(\widetilde{\phi}) w^{2q+p\hat{n}-2}] \right\} \\ &\approx w \left\{ A'_{10}(0)\widetilde{\phi} + A_{21}(0) w^2 + A_{0,\hat{n}-1}(0) w^{\hat{n}-2} \right\}. \end{aligned}$$

We see that $F_1(w,0,0,0,\widetilde{\phi}) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution. Note that $F_1(w,0,0,0,\widetilde{\phi})$ becomes an odd function in w if $\hat{n}$ is even. Then, the two bifurcating solutions $(w,0,0,0,\widetilde{\phi})$ and $(-w,0,0,0,\widetilde{\phi})$ are conjugate.

Substituting $(z_1,z_2,z_3,z_4) = (w,0,0,0)$ into the equivariance conditions (6.203)–(6.206), we have

$$\begin{aligned} F_2(w,0,0,0) &= F_1(0,w,0,0),\\ F_3(w,0,0,0) &= F_1(0,0,w,0),\\ F_4(w,0,0,0) &= F_1(0,0,0,w). \end{aligned} \tag{6.227}$$

With the use of P in (6.222), we have $F_i = 0$ ($i = 2,3,4$) in (6.227) if

$$\begin{aligned} (0,b,0,0,0,f,0,0) &\notin P,\\ (0,0,c,0,0,0,g,0) &\notin P,\\ (0,0,0,d,0,0,0,h) &\notin P. \end{aligned}$$

The conditions in (6.217) and (6.218) lead to

$$\begin{aligned} &\hat{k}(b-f)-\hat{\ell},\ \hat{\ell}(b-f)+\hat{k} \equiv 0 \equiv 0 \quad \text{mod } \hat{n} \quad \text{for } (a,b,c,d,e,f,g,h) = (0,b,0,0,0,f,0,0),\\ &\hat{k}(c-g)-\hat{k} \equiv 0,\ \hat{\ell}(c-g)+\hat{\ell} \equiv 0 \quad \text{mod } \hat{n} \quad \text{for } (a,b,c,d,e,f,g,h) = (0,0,c,0,0,0,g,0),\\ &\hat{k}(d-h)+\hat{\ell} \equiv 0,\ \hat{\ell}(d-h)+\hat{k} \equiv 0 \quad \text{mod } \hat{n} \quad \text{for } (a,b,c,d,e,f,g,h) = (0,0,0,d,0,0,0,h). \end{aligned}$$

These relations can be expressed in a matrix form as

$$A\boldsymbol{x} = \boldsymbol{b} \quad \text{with } A = \begin{bmatrix} \hat{k} & -\hat{n} & 0 \\ \hat{\ell} & 0 & -\hat{n} \end{bmatrix}. \tag{6.228}$$

The vectors $\boldsymbol{x}$ and $\boldsymbol{b}$ vary with (a,b,c,d,e,f,g,h) as follows:

$$\begin{aligned} \boldsymbol{x} &= \begin{bmatrix} b-f \\ p \\ q \end{bmatrix},\ \boldsymbol{b} = \begin{bmatrix} \hat{\ell} \\ -\hat{k} \end{bmatrix} \quad \text{for } (a,b,c,d,e,f,g,h) = (0,b,0,0,0,f,0,0),\\ \boldsymbol{x} &= \begin{bmatrix} c-g \\ p \\ q \end{bmatrix},\ \boldsymbol{b} = \begin{bmatrix} \hat{k} \\ -\hat{\ell} \end{bmatrix} \quad \text{for } (a,b,c,d,e,f,g,h) = (0,0,c,0,0,0,g,0),\\ \boldsymbol{x} &= \begin{bmatrix} d-h \\ p \\ q \end{bmatrix},\ \boldsymbol{b} = \begin{bmatrix} -\hat{\ell} \\ -\hat{k} \end{bmatrix} \quad \text{for } (a,b,c,d,e,f,g,h) = (0,0,0,d,0,0,0,h). \end{aligned}$$

The existence of an integer solution $\boldsymbol{x}$ of (6.228) is investigated by showing the two conditions (6.229) in Remark 6.2 below. The first condition is satisfied since we have

$$\text{rank } A = \text{rank} \begin{bmatrix} \hat{k} & -\hat{n} & 0 \\ \hat{\ell} & 0 & -\hat{n} \end{bmatrix} = 2$$

and

$$\operatorname{rank}[A \mid \boldsymbol{b}] = \operatorname{rank}\begin{bmatrix} \hat{k} & -\hat{n} & 0 & \hat{\ell} \\ \hat{\ell} & 0 & -\hat{n} & -\hat{k} \end{bmatrix} = 2 \quad \text{for } (a,b,c,d,e,f,g,h) = (0,b,0,0,0,f,0,0),$$

$$\operatorname{rank}[A \mid \boldsymbol{b}] = \operatorname{rank}\begin{bmatrix} \hat{k} & -\hat{n} & 0 & \hat{k} \\ \hat{\ell} & 0 & -\hat{n} & -\hat{\ell} \end{bmatrix} = 2 \quad \text{for } (a,b,c,d,e,f,g,h) = (0,0,c,0,0,0,g,0),$$

$$\operatorname{rank}[A \mid \boldsymbol{b}] = \operatorname{rank}\begin{bmatrix} \hat{k} & -\hat{n} & 0 & -\hat{\ell} \\ \hat{\ell} & 0 & -\hat{n} & -\hat{k} \end{bmatrix} = 2 \quad \text{for } (a,b,c,d,e,f,g,h) = (0,0,0,d,0,0,0,h).$$

For the second condition, we have

$$\begin{aligned} d_1(A) &= \gcd(\hat{\ell}, \hat{k}, \hat{n}) = 1, \\ d_1([A \mid \boldsymbol{b}]) &= \gcd(\hat{\ell}, \hat{k}, \hat{n}) = 1, \\ d_2(A) &= \gcd(\hat{k}\hat{n}, \hat{\ell}\hat{n}, \hat{n}^2) = \hat{n}. \end{aligned}$$

The value of $d_2([A \mid \boldsymbol{b}])$ varies with (a,b,c,d,e,f,g,h) as follows:

$$\begin{aligned} d_2([A \mid \boldsymbol{b}]) &= \gcd(\hat{n}, \hat{k}^2 + \hat{\ell}^2) \quad \text{for } (a,b,c,d,e,f,g,h) = (0,b,0,0,0,f,0,0), \\ d_2([A \mid \boldsymbol{b}]) &= \gcd(\hat{n}, 2\hat{k}\hat{\ell}) \quad \text{for } (a,b,c,d,e,f,g,h) = (0,0,c,0,0,0,g,0), \\ d_2([A \mid \boldsymbol{b}]) &= \gcd(\hat{n}, \hat{k}^2 - \hat{\ell}^2) \quad \text{for } (a,b,c,d,e,f,g,h) = (0,0,0,d,0,0,0,h). \end{aligned}$$

For $(a,b,c,d,e,f,g,h) = (0,b,0,0,0,f,0,0)$, we have $d_2(A) = d_2([A \mid \boldsymbol{b}])$ when $\hat{k}^2 + \hat{\ell}^2$ is divisible by $\hat{n}$. Then, the equation (6.228) has an integer solution $\boldsymbol{x}$. Hence, we have $(0,b,0,0,0,f,0,0) \in P$ and, in turn, $F_2 \neq 0$. On the contrary, we have $(0,b,0,0,0,f,0,0) \notin P$ and, in turn, $F_2 = 0$ when $\hat{k}^2 + \hat{\ell}^2$ is not divisible by $\hat{n}$. In a similar manner, we have $(0,0,c,0,0,0,g,0) \notin P$ and, in turn, $F_3 = 0$ when $2\hat{k}\hat{\ell}$ is not divisible by $\hat{n}$. We have $(0,0,0,d,0,0,0,h) \notin P$ and, in turn, $F_4 = 0$ when $\hat{k}^2 - \hat{\ell}^2$ is not divisible by $\hat{n}$. Consequently, a system of the bifurcation equations $F_1 = F_2 = F_3 = F_4 = 0$ holds for $(z_1, z_2, z_3, z_4) = (w, 0, 0, 0)$ when $\hat{k}^2 + \hat{\ell}^2$, $2\hat{k}\hat{\ell}$, and $\hat{k}^2 - \hat{\ell}^2$ are not divisible by $\hat{n}$.

Remark 6.2. *Let A be an $m \times n$ integer matrix and $\boldsymbol{b}$ an m-dimensional integer vector. A system of equations $A\boldsymbol{x} = \boldsymbol{b}$ admits an integer solution $\boldsymbol{x}$ if and only if two matrices A and $[A \mid \boldsymbol{b}]$ share the same determinantal divisors, i.e.,*

$$\operatorname{rank} A = \operatorname{rank}[A \mid \boldsymbol{b}], \quad d_k(A) = d_k([A \mid \boldsymbol{b}]) \tag{6.229}$$

for all k. Here, $d_k(A)$ is the kth determinantal divisor, which is the greatest common divisor of all $k \times k$ minors (subdeterminants) of the integer matrix A. □

Substituting Type II stripe pattern $(z_1, z_2, z_3, z_4) = (\mathrm{i}w, 0, 0, 0)$ into (6.226), we have

$$\begin{aligned} &F_1(\mathrm{i}w, 0, 0, 0, \widetilde{\phi}) \\ &= \mathrm{i}w \left\{ \sum_{q=0}^{\infty} A_{q+1,q}(\widetilde{\phi}) w^{2q} + \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} [A_{q+p\hat{n}+1,q}(\widetilde{\phi}) \mathrm{i}^{p\hat{n}} w^{2q+p\hat{n}} + A_{q,q+p\hat{n}-1}(\widetilde{\phi})(-\mathrm{i})^{p\hat{n}} w^{2q+p\hat{n}-2}] \right\} \\ &\approx \mathrm{i}w \left\{ A'_{10}(0)\widetilde{\phi} + A_{21}(0) w^2 + A_{0,\hat{n}-1}(0)(-\mathrm{i})^{\hat{n}} w^{\hat{n}-2} \right\}. \end{aligned}$$

Thus, $F_1(\mathrm{i}w, 0, 0, 0, \widetilde{\phi}) = 0$ has a bifurcating solution if $\hat{n}$ is even ($\mathrm{i}^{p\hat{n}}$ and $(-\mathrm{i})^{p\hat{n}}$ are real). Then, a discussion similar to that for the Type I stripe pattern holds.

To sum up, we have the following propositions on the existence and the symmetry of the stripe patterns.

Proposition 6.10. *For a critical point of multiplicity 8 associated with $\mu = (8; k, \ell)$, Type I stripe pattern exists if the condition*

$$\hat{k}^2 + \hat{\ell}^2,\ 2\hat{k}\hat{\ell},\ \textit{and}\ \hat{k}^2 - \hat{\ell}^2\ \textit{are not divisible by}\ \hat{n} \tag{6.230}$$

is satisfied. Therein, $\hat{k} = k/\gcd(n, k, \ell)$, $\hat{\ell} = \ell/\gcd(n, k, \ell)$, and $\hat{n} = n/\gcd(n, k, \ell)$. Type II stripe pattern exists if the condition (6.230) is satisfied and $\hat{n}$ is even.

Proposition 6.11. *For a critical point of multiplicity 8 associated with $\mu = (8; k, \ell)$, the two bifurcating solutions $(\boldsymbol{z}, \widetilde{\phi})$ and $(-\boldsymbol{z}, \widetilde{\phi})$ are conjugate for $\boldsymbol{z} = (w, 0, 0, 0)$, $(\mathrm{i}w, 0, 0, 0)$ $(w \in \mathbb{R})$ if $\hat{n} = n/\gcd(n, k, \ell)$ is even and are not conjugate for $\boldsymbol{z} = (w, 0, 0, 0)$ if $\hat{n}$ is odd.*

6.5.4. Existence and Symmetry of Upside-down Patterns

We would like to show the existence and the symmetry of two types of upside-down patterns, which are represented as

$$\begin{aligned}&\text{Type I upside-down pattern} : (z_1, z_2, z_3, z_4) = (w, 0, w, 0)\ (w \in \mathbb{R}),\\ &\text{Type II upside-down pattern} : (z_1, z_2, z_3, z_4) = (\mathrm{i}w, 0, \mathrm{i}w, 0)\ (w \in \mathbb{R}).\end{aligned}$$

For both cases, we have $(a, b, \ldots, h) = (a, 0, c, 0, e, 0, g, 0)$, and hence (6.217) and (6.218) leads to

$$\begin{aligned}\hat{k}(a - e - 1) + \hat{k}(c - g) &\equiv 0 \mod \hat{n},\\ \hat{\ell}(a - e - 1) - \hat{\ell}(c - g) &\equiv 0 \mod \hat{n},\end{aligned}$$

which imply $a = e + p\hat{n} + 1$ and $c = g + q\hat{n}$ $(p, q \in \mathbb{Z})$. Then, F_1 in (6.223) is rewritten as

$$F_1(z_1, 0, z_3, 0, \widetilde{\phi}) = \sum_{e=0}^{\infty} \sum_{g=0}^{\infty} \sum_{p \in \mathbb{Z}, e+p\hat{n}+1 \geq 0} \sum_{q \in \mathbb{Z}, g+q\hat{n} \geq 0} A_{e+p\hat{n}+1, g+q\hat{n}, e, g}(\widetilde{\phi}) {z_1}^{e+p\hat{n}+1} {z_3}^{g+q\hat{n}}\, {\overline{z}_1}^{e}\, {\overline{z}_3}^{g} \tag{6.231}$$

with $A_{aceg}(\widetilde{\phi}) = A_{a0c0e0g0}(\widetilde{\phi})$.

Substituting Type I upside-down pattern $(z_1, z_2, z_3, z_4) = (w, 0, w, 0)$ into (6.231), we have

$$\begin{aligned}F_1(w, 0, w, 0, \widetilde{\phi}) &= w \left\{ \sum_{e=0}^{\infty} \sum_{g=0}^{\infty} \sum_{p \in \mathbb{Z}, e+p\hat{n}+1 \geq 0} \sum_{q \in \mathbb{Z}, g+q\hat{n} \geq 0} A_{e+p\hat{n}+1, g+q\hat{n}, e, g}(\widetilde{\phi}) w^{2(e+g)+(p+q)\hat{n}} \right\}\\ &\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + (A_{2010}(0) + A_{1101}(0)) w^2 \right\}.\end{aligned}$$

We see that $F_1(w, 0, w, 0, \widetilde{\phi}) = 0$ has the trivial solution $(w = 0)$ and a bifurcating solution. Note that $F_1(w, 0, w, 0, \widetilde{\phi})$ becomes an odd function in w if $\hat{n}$ is even. Then, the two bifurcating solutions $(w, 0, w, 0, \widetilde{\phi})$ and $(-w, 0, -w, 0, \widetilde{\phi})$ are conjugate.

Substituting $(z_1, z_2, z_3, z_4) = (w, 0, w, 0)$ into the equivariance conditions (6.203)–(6.206), we have

$$\begin{aligned} F_2(w, 0, w, 0) &= F_4(w, 0, w, 0) = F_1(0, w, 0, w), \\ F_3(w, 0, w, 0) &= F_1(w, 0, w, 0). \end{aligned} \tag{6.232}$$

With the use of P in (6.222), we have $F_i = 0$ $(i = 2, 4)$ in (6.232) if

$$(0, b, 0, d, 0, f, 0, h) \notin P.$$

The use of $(a, b, \ldots, h) = (0, b, 0, d, 0, f, 0, h)$ in (6.217) and (6.218) leads to

$$\begin{aligned} -\hat{k} - \hat{\ell}(b + d - f - h) &\equiv 0 \mod \hat{n}, \\ \hat{k}(b - d - f + h) - \hat{\ell} &\equiv 0 \mod \hat{n}. \end{aligned}$$

This relation can be expressed in a matrix form as

$$A\boldsymbol{x} = \boldsymbol{b} \quad \text{with } A = \begin{bmatrix} -\hat{\ell} & -\hat{\ell} & -\hat{n} & 0 \\ \hat{k} & -\hat{k} & 0 & -\hat{n} \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} b - f \\ d - h \\ p \\ q \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} \hat{k} \\ \hat{\ell} \end{bmatrix}. \tag{6.233}$$

The existence of an integer solution $\boldsymbol{x}$ of (6.233) is investigated by showing the two conditions (6.229) in Remark 6.2. The first condition is satisfied since

$$\begin{aligned} \operatorname{rank} A &= \operatorname{rank} \begin{bmatrix} -\hat{\ell} & -\hat{\ell} & -\hat{n} & 0 \\ \hat{k} & -\hat{k} & 0 & -\hat{n} \end{bmatrix} = 2, \\ \operatorname{rank} [A \mid \boldsymbol{b}] &= \operatorname{rank} \begin{bmatrix} -\hat{\ell} & -\hat{\ell} & -\hat{n} & 0 & -\hat{k} \\ \hat{k} & -\hat{k} & 0 & -\hat{n} & \hat{\ell} \end{bmatrix} = 2. \end{aligned}$$

For the second condition, we have

$$\begin{aligned} d_1(A) &= \gcd(\hat{\ell}, \hat{k}, \hat{n}) = 1, \\ d_1([A \mid \boldsymbol{b}]) &= \gcd(\hat{\ell}, \hat{k}, \hat{n}) = 1, \\ d_2(A) &= \gcd(2\hat{k}\hat{\ell}, \hat{k}\hat{n}, \hat{\ell}\hat{n}, \hat{n}^2) = \gcd(2\hat{k}\hat{\ell}, \hat{n}), \\ d_2([A \mid \boldsymbol{b}]) &= \gcd(\hat{n}, 2\hat{k}\hat{\ell}, \hat{k}^2 + \hat{\ell}^2, \hat{k}^2 - \hat{\ell}^2). \end{aligned}$$

Hence, $d_2(A) = d_2([A \mid \boldsymbol{b}])$ is satisfied if

$$\gcd(\hat{k}^2 + \hat{\ell}^2, \hat{k}^2 - \hat{\ell}^2) \text{ is divisible by } \gcd(\hat{n}, 2\hat{k}\hat{\ell}),$$

Then, the equation (6.233) has an integer solution $\boldsymbol{x}$, and hence we have $(0, b, 0, d, 0, f, 0, h) \in P$ and, in turn, $F_2 = F_4 \neq 0$. On the contrary, we have $(0, b, 0, d, 0, f, 0, h) \notin P$ and, in turn, $F_2 = F_4 = 0$ if $(\hat{k} + \hat{\ell}) \gcd(\hat{k} + \hat{\ell}, \hat{k} - \hat{\ell})$ is not divisible by $\gcd(\hat{n}, 2\hat{k}\hat{\ell})$.

Substituting Type II upside-down pattern $(z_1, z_2, z_3, z_4) = (\mathrm{i}w, 0, \mathrm{i}w, 0)$ into (6.231), we have

$$
\begin{aligned}
&F_1(\mathrm{i}w, 0, \mathrm{i}w, 0, \widetilde{\phi}) \\
&= \sum_{e=0}^{\infty}\sum_{g=0}^{\infty}\sum_{p\in\mathbb{Z}, e+p\hat{n}+1\geq 0}\sum_{q\in\mathbb{Z}, g+q\hat{n}\geq 0} A_{e+p\hat{n}+1, g+q\hat{n}, e, g}(\widetilde{\phi})(\mathrm{i}w)^{e+p\hat{n}+1}(\mathrm{i}w)^{g+q\hat{n}}(-\mathrm{i}w)^{e}(-\mathrm{i}w)^{g} \\
&= \mathrm{i}w\left\{\sum_{e=0}^{\infty}\sum_{g=0}^{\infty}\sum_{p\in\mathbb{Z}, e+p\hat{n}+1\geq 0}\sum_{q\in\mathbb{Z}, g+q\hat{n}\geq 0} A_{e+p\hat{n}+1, g+q\hat{n}, e, g}(\widetilde{\phi})\mathrm{i}^{p\hat{n}}(-\mathrm{i})^{q\hat{n}}w^{2(e+g)+(p+q)\hat{n}}\right\} \\
&\approx \mathrm{i}w\left\{A'_{1000}(0)\widetilde{\phi} + (A_{2010}(0) + A_{1101}(0))w^2\right\}.
\end{aligned}
$$

Thus, $F_1(\mathrm{i}w, 0, \mathrm{i}w, 0, \widetilde{\phi}) = 0$ has a bifurcating solution if $\hat{n}$ is even ($\mathrm{i}^{p\hat{n}}$ and $(-\mathrm{i})^{q\hat{n}}$ are real). Then, a discussion similar to that for Type I stripe pattern holds.

To sum up, we have the following propositions on the existence and the symmetry of the upside-down patterns.

Proposition 6.12. *For a critical point of multiplicity 8 associated with $\mu = (8; k, \ell)$, Type I upside-down pattern exists if the condition*

$$\gcd(\hat{k}^2 + \hat{\ell}^2, \hat{k}^2 - \hat{\ell}^2) \textit{ is not divisible by } \gcd(\hat{n}, 2\hat{k}\hat{\ell}) \tag{6.234}$$

is satisfied. Therein, $\hat{k} = k/\gcd(n, k, \ell)$, $\hat{\ell} = \ell/\gcd(n, k, \ell)$, and $\hat{n} = n/\gcd(n, k, \ell)$. Type II upside-down pattern exists if the condition (6.234) is satisfied and $\hat{n}$ is even.

Proposition 6.13. *For a critical point of multiplicity 8 associated with $\mu = (8; k, \ell)$, the two bifurcating solutions $(\boldsymbol{z}, \widetilde{\phi})$ and $(-\boldsymbol{z}, \widetilde{\phi})$ are conjugate for $\boldsymbol{z} = (w, 0, w, 0)$, $(\mathrm{i}w, 0, \mathrm{i}w, 0)$ $(w \in \mathbb{R})$ if $\hat{n} = n/\gcd(n, k, \ell)$ is even and are not conjugate for $\boldsymbol{z} = (w, 0, w, 0)$ if $\hat{n}$ is odd.*

6.5.5. Stability of Bifurcating Solutions

In Section 5.6.4, we found the square patterns for a critical point of multiplicity 8 by using the equivariant branching lemma. In Section 6.5.3 and 6.5.4, we showed the stripe and upside-down patterns by solving the bifurcation equations. These bifurcating solutions are represented for the bifurcation equation in real variables in (6.179) as follows ($w \in \mathbb{R}$):

$$\begin{aligned}
\boldsymbol{w}_{\mathrm{sqVM}} &= (w, 0, w, 0, w, 0, w, 0),\\
\boldsymbol{w}_{\mathrm{sqT}} &= (w, 0, w, 0, 0, 0, 0, 0),\\
\boldsymbol{w}_{\mathrm{stripeI}} &= (w, 0, 0, 0, 0, 0, 0, 0),\\
\boldsymbol{w}_{\mathrm{stripeII}} &= (0, w, 0, 0, 0, 0, 0, 0),\\
\boldsymbol{w}_{\mathrm{upside-downI}} &= (w, 0, 0, 0, w, 0, 0, 0),\\
\boldsymbol{w}_{\mathrm{upside-downII}} &= (0, w, 0, 0, 0, w, 0, 0)
\end{aligned}$$

We would like to evaluate the asymptotic stability of these bifurcating solutions.

To obtain the the asymptotic form of the bifurcation equation and the Jacobian matrix, we first investigate which $(a, b, \ldots, h) \in \mathbb{Z}_+^8$ belongs to P in (6.222). In other words, we investigate which $A_{ab\cdots h}(\widetilde{\phi})$ becomes nonzero in (6.223). We focus on the coefficients of linear terms, quadratic terms, and cubic terms, which play a vital role as leading terms in (6.223). For this purpose, we exhaustively find $(a, b, \ldots, h) \in \mathbb{Z}_+^8$ such as

$$(a, b, \ldots, h) \in P \text{ with } a + b + \cdots + h \leq 3.$$

Let us take some $(a, b, \ldots, h) \in \mathbb{Z}_+^8$ and substitute it into the matrix A in (6.220). Then, A becomes any one of twelve possible forms as shown in Table 6.5. The condition (6.221) varies with the form of A.

For the case (i), the elements of A in (6.220) represent

$$a + c - e - g - 1 = 0, \quad a - c - e + g - 1 = 0, \tag{6.235}$$

$$-b - d + f + h = 0, \quad b - d - f + h = 0. \tag{6.236}$$

The condition in (6.221) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = 0,\ q\hat{n} = 0. \tag{6.237}$$

This condition is satisfied for any values of $(\hat{n}, \hat{k}, \hat{\ell})$ when $p = q = 0$. Recall that $a+b+\ldots+h \leq 3$. Then, $(a, b, \ldots, h)$ which satisfy (6.235) and (6.236) are enumerated as follows:

$$\begin{aligned}
&(1, 0, 0, 0, 0, 0, 0, 0),\ (2, 0, 0, 0, 1, 0, 0, 0),\ (1, 1, 0, 0, 0, 1, 0, 0),\\
&(1, 0, 1, 0, 0, 0, 1, 0),\ (1, 0, 0, 1, 0, 0, 0, 1) \in P \text{ for any } (\hat{n}, \hat{k}, \hat{\ell}).
\end{aligned} \tag{6.238}$$

For the case (ii), the elements of A in (6.220) represent

$$a + c - e - g - 1 = \alpha, \quad a - c - e + g - 1 = 0, \tag{6.239}$$

$$-b - d + f + h = 0, \quad b - d - f + h = 0. \tag{6.240}$$

Table 6.5: Possible cases for A in (6.220)

Cases		Conditions in (6.221)
(i)	$A = O$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = 0,\ q\hat{n} = 0$
(ii)	$A = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \alpha\hat{k},\ q\hat{n} = 0$
(iii)	$A = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \beta\hat{\ell},\ q\hat{n} = 0$
(iv)	$A = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = 0,\ q\hat{n} = \gamma\hat{k}$
(v)	$A = \begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = 0,\ q\hat{n} = \delta\hat{\ell}$
(vi)	$A = \begin{bmatrix} \alpha & 0 \\ \gamma & 0 \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \alpha\hat{k},\ q\hat{n} = \gamma\hat{k}$
(vii)	$A = \begin{bmatrix} 0 & \beta \\ 0 & \delta \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \beta\hat{\ell},\ q\hat{n} = \delta\hat{\ell}$
(viii)	$A = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \alpha\hat{k} + \beta\hat{\ell},\ q\hat{n} = 0$
(ix)	$A = \begin{bmatrix} 0 & 0 \\ \gamma & \delta \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = 0,\ q\hat{n} = \gamma\hat{k} + \delta\hat{\ell}$
(x)	$A = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \alpha\hat{k},\ q\hat{n} = \delta\hat{\ell}$
(xi)	$A = \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \beta\hat{\ell},\ q\hat{n} = \gamma\hat{k}$
(xii)	$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \alpha\hat{k} + \beta\hat{\ell},\ q\hat{n} = \gamma\hat{k} + \delta\hat{\ell}$

Table 6.6: Possible cases for $(a, b, c, d, e, f, g, h) \in \mathbb{Z}_+^8$

(a,b,c,d,e,f,g,h)	α	β	γ	δ	A	Existence of $p, q \in \mathbb{Z}$ in (6.220)
$(0,0,0,0,0,0,0,0)$	-1	0	0	-1	(x)	-
$(1,0,0,0,0,0,0,0)$	0	0	0	0	(i)	$p = 0,\ q = 0$ for any $(\hat{n}, \hat{k}, \hat{\ell})$
$(0,1,0,0,0,0,0,0)$	-1	-1	1	-1	(xii)	-
$(0,0,1,0,0,0,0,0)$	0	0	0	-2	(v)	-
$(0,0,0,1,0,0,0,0)$	-1	-1	-1	-1	(xii)	-
$(0,0,0,0,1,0,0,0)$	-2	0	0	-2	(x)	-
$(0,0,0,0,0,1,0,0)$	-1	1	-1	-1	(xii)	-
$(0,0,0,0,0,0,1,0)$	-2	0	0	0	(ii)	-
$(0,0,0,0,0,0,0,1)$	-1	1	1	-1	(xii)	-
$(2,0,0,0,0,0,0,0)$	1	0	0	1	(x)	-
$(0,2,0,0,0,0,0,0)$	-1	-2	2	-1	(xii)	-
$(0,0,2,0,0,0,0,0)$	1	0	0	-3	(x)	-
$(0,0,0,2,0,0,0,0)$	-1	-2	-2	-1	(xii)	-
$(0,0,0,0,2,0,0,0)$	-3	0	0	-3	(x)	-
$(0,0,0,0,0,2,0,0)$	-1	2	-2	-1	(xii)	$p = 0,\ q = -1$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (5,2,1)$
$(0,0,0,0,0,0,2,0)$	-3	0	0	1	(x)	-
$(0,0,0,0,0,0,0,2)$	-1	2	2	-1	(xii)	-
$(1,1,0,0,0,0,0,0)$	0	-1	1	0	(xi)	-
$(1,0,1,0,0,0,0,0)$	1	0	0	-1	(x)	-
$(1,0,0,1,0,0,0,0)$	0	-1	-1	0	(xi)	-
$(1,0,0,0,1,0,0,0)$	-1	0	0	-1	(x)	-
$(1,0,0,0,0,1,0,0)$	0	1	-1	0	(xi)	-
$(1,0,0,0,0,0,1,0)$	-1	0	0	1	(x)	-
$(1,0,0,0,0,0,0,1)$	0	1	1	0	(xi)	-
$(0,1,1,0,0,0,0,0)$	0	-1	1	-2	(xii)	-
$(0,1,0,1,0,0,0,0)$	-1	-2	0	-1	(xii)	-
$(0,1,0,0,1,0,0,0)$	-2	-1	1	-2	(xii)	$p = -1,\ q = 0$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (5,2,1)$
$(0,1,0,0,0,1,0,0)$	-1	0	0	-1	(x)	-
$(0,1,0,0,0,0,1,0)$	-2	-1	1	0	(xii)	-
$(0,1,0,0,0,0,0,1)$	-1	0	2	-1	(xii)	-

$\alpha = a + c - e - g - 1$; $\beta = -b - d + f + h$; $\gamma = b - d - f + h$; $\delta = a - c - e + g - 1$

Table 6.7: Possible cases for $(a, b, c, d, e, f, g, h) \in \mathbb{Z}_+^8$

(a, b, c, d, e, f, g, h)	α	β	γ	δ	A	Existence of $p, q \in \mathbb{Z}$ in (6.220)
(0, 0, 1, 1, 0, 0, 0, 0)	0	-1	-1	-2	(xii)	-
(0, 0, 1, 0, 1, 0, 0, 0)	-1	0	0	-3	(x)	-
(0, 0, 1, 0, 0, 1, 0, 0)	0	1	-1	-2	(xii)	-
(0, 0, 1, 0, 0, 0, 1, 0)	-1	0	0	-1	(x)	-
(0, 0, 1, 0, 0, 0, 0, 1)	0	1	1	-2	(xii)	-
(0, 0, 0, 1, 1, 0, 0, 0)	-2	-1	-1	-2	(xii)	-
(0, 0, 0, 1, 0, 1, 0, 0)	-1	0	-2	-1	(xii)	-
(0, 0, 0, 1, 0, 0, 1, 0)	-2	-1	-1	0	(xii)	-
(0, 0, 0, 1, 0, 0, 0, 1)	-1	0	0	-1	(x)	-
(0, 0, 0, 0, 1, 1, 0, 0)	-2	1	-1	-2	(xii)	-
(0, 0, 0, 0, 1, 0, 1, 0)	-3	0	0	-1	(x)	-
(0, 0, 0, 0, 1, 0, 0, 1)	-2	1	1	-2	(xii)	-
(0, 0, 0, 0, 0, 1, 1, 0)	-2	1	-1	0	(xii)	-
(0, 0, 0, 0, 0, 1, 0, 1)	-1	2	0	-1	(xii)	-
(0, 0, 0, 0, 0, 0, 1, 1)	-2	1	1	0	(xii)	-
(3, 0, 0, 0, 0, 0, 0, 0)	2	0	0	2	(x)	-
(0, 3, 0, 0, 0, 0, 0, 0)	-1	-3	3	-1	(xii)	$p = -1,\ q = 1$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$
(0, 0, 3, 0, 0, 0, 0, 0)	2	0	0	-4	(x)	-
(0, 0, 0, 3, 0, 0, 0, 0)	-1	-3	-3	-1	(xii)	-
(0, 0, 0, 0, 3, 0, 0, 0)	-4	0	0	-4	(x)	-
(0, 0, 0, 0, 0, 3, 0, 0)	-1	3	-3	-1	(xii)	$p = 0,\ q = 1$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (10, 3, 1)$
(0, 0, 0, 0, 0, 0, 3, 0)	-4	0	0	2	(x)	-
(0, 0, 0, 0, 0, 0, 0, 3)	-1	3	3	-1	(xii)	$p = 0,\ q = -1$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (8, 3, 1)$
(2, 1, 0, 0, 0, 0, 0, 0)	1	-1	1	1	(xii)	-
(2, 0, 1, 0, 0, 0, 0, 0)	2	0	0	0	(ii)	-
(2, 0, 0, 1, 0, 0, 0, 0)	1	-1	-1	1	(xii)	-
(2, 0, 0, 0, 1, 0, 0, 0)	0	0	0	0	(i)	$p = 0,\ q = 0$ for any $(\hat{n}, \hat{k}, \hat{\ell})$
(2, 0, 0, 0, 0, 1, 0, 0)	1	1	-1	1	(xii)	-
(2, 0, 0, 0, 0, 0, 1, 0)	0	0	0	2	(v)	-
(2, 0, 0, 0, 0, 0, 0, 1)	1	1	1	1	(xii)	-

$\alpha = a + c - e - g - 1$; $\beta = -b - d + f + h$; $\gamma = b - d - f + h$; $\delta = a - c - e + g - 1$

Table 6.8: Possible cases for $(a, b, c, d, e, f, g, h) \in \mathbb{Z}_+^8$

(a,b,c,d,e,f,g,h)	α	β	γ	δ	A	Existence of $p, q \in \mathbb{Z}$ in (6.220)
$(1,2,0,0,0,0,0,0)$	0	-2	2	0	(xi)	-
$(0,2,1,0,0,0,0,0)$	0	-2	2	-2	(xii)	-
$(0,2,0,1,0,0,0,0)$	-1	-3	1	-1	(xii)	-
$(0,2,0,0,1,0,0,0)$	-2	-2	2	-2	(xii)	-
$(0,2,0,0,0,1,0,0)$	-1	-1	1	-1	(xii)	-
$(0,2,0,0,0,0,1,0)$	-2	-2	2	0	(xii)	-
$(0,2,0,0,0,0,0,1)$	-1	-1	3	-1	(xii)	-
$(1,0,2,0,0,0,0,0)$	2	0	0	-2	(x)	-
$(0,1,2,0,0,0,0,0)$	1	-1	1	-3	(xii)	-
$(0,0,2,1,0,0,0,0)$	1	-1	-1	-3	(xii)	-
$(0,0,2,0,1,0,0,0)$	0	0	0	-4	(v)	$p = 0,\ q = -1$ for $\hat{n} = 4\hat{\ell}$
$(0,0,2,0,0,1,0,0)$	1	1	-1	-3	(xii)	-
$(0,0,2,0,0,0,1,0)$	0	0	0	-2	(v)	-
$(0,0,2,0,0,0,0,1)$	1	1	1	-3	(xii)	-
$(1,0,0,2,0,0,0,0)$	0	-2	-2	0	(xi)	-
$(0,1,0,2,0,0,0,0)$	-1	-3	-1	-1	(xii)	-
$(0,0,1,2,0,0,0,0)$	0	-2	-2	-2	(xii)	-
$(0,0,0,2,1,0,0,0)$	-2	-2	-2	-2	(xii)	$p = -1,\ q = -1$ for $\hat{n} = 2\hat{k} + 2\hat{\ell}$
$(0,0,0,2,0,1,0,0)$	-1	-1	-3	-1	(xii)	-
$(0,0,0,2,0,0,1,0)$	-2	-2	-2	0	(xii)	-
$(0,0,0,2,0,0,0,1)$	-1	-1	-1	-1	(xii)	-
$(1,0,0,0,2,0,0,0)$	-2	0	0	-2	(x)	-
$(0,1,0,0,2,0,0,0)$	-3	-1	1	-3	(xii)	$p = -1,\ q = 0$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (10, 3, 1)$
$(0,0,1,0,2,0,0,0)$	-2	0	0	-4	(x)	-
$(0,0,0,1,2,0,0,0)$	-3	-1	-1	-3	(xii)	-
$(0,0,0,0,2,1,0,0)$	-3	1	-1	-3	(xii)	$p = -1,\ q = -1$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$
$(0,0,0,0,2,0,1,0)$	-4	0	0	-2	(x)	-
$(0,0,0,0,2,0,0,1)$	-3	1	1	-3	(xii)	$p = -1,\ q = 0$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (8, 3, 1)$

$\alpha = a + c - e - g - 1;\quad \beta = -b - d + f + h;\quad \gamma = b - d - f + h;\quad \delta = a - c - e + g - 1$

Table 6.9: Possible cases for $(a, b, c, d, e, f, g, h) \in \mathbb{Z}_+^8$

(a, b, c, d, e, f, g, h)	α	β	γ	δ	A	Existence of $p, q \in \mathbb{Z}$ in (6.220)
$(1, 0, 0, 0, 0, 2, 0, 0)$	0	2	-2	0	(xi)	-
$(0, 1, 0, 0, 0, 2, 0, 0)$	-1	1	-1	-1	(xii)	-
$(0, 0, 1, 0, 0, 2, 0, 0)$	0	2	-2	-2	(xii)	-
$(0, 0, 0, 1, 0, 2, 0, 0)$	-1	1	-3	-1	(xii)	-
$(0, 0, 0, 0, 1, 2, 0, 0)$	-2	2	-2	-2	(xii)	-
$(0, 0, 0, 0, 0, 2, 1, 0)$	-2	2	-2	0	(xii)	-
$(0, 0, 0, 0, 0, 2, 0, 1)$	-1	3	-1	-1	(xii)	-
$(1, 0, 0, 0, 0, 0, 2, 0)$	-2	0	0	2	(x)	-
$(0, 1, 0, 0, 0, 0, 2, 0)$	-3	-1	1	1	(xii)	-
$(0, 0, 1, 0, 0, 0, 2, 0)$	-2	0	0	0	(ii)	-
$(0, 0, 0, 1, 0, 0, 2, 0)$	-3	-1	-1	1	(xii)	-
$(0, 0, 0, 0, 1, 0, 2, 0)$	-4	0	0	0	(ii)	$p = -1,\ q = 0$ for $\hat{n} = 4\hat{k}$
$(0, 0, 0, 0, 0, 1, 2, 0)$	-3	1	-1	1	(xii)	-
$(0, 0, 0, 0, 0, 0, 2, 1)$	-3	1	1	1	(xii)	-
$(1, 0, 0, 0, 0, 0, 0, 2)$	0	2	2	0	(xi)	-
$(0, 1, 0, 0, 0, 0, 0, 2)$	-1	1	3	-1	(xii)	-
$(0, 0, 1, 0, 0, 0, 0, 2)$	0	2	2	-2	(xii)	-
$(0, 0, 0, 1, 0, 0, 0, 2)$	-1	1	1	-1	(xii)	-
$(0, 0, 0, 0, 1, 0, 0, 2)$	-2	2	2	-2	(xii)	-
$(0, 0, 0, 0, 0, 1, 0, 2)$	-1	3	1	-1	(xii)	-
$(0, 0, 0, 0, 0, 0, 1, 2)$	-2	2	2	0	(xii)	-
$(1, 1, 1, 0, 0, 0, 0, 0)$	1	-1	1	-1	(xii)	-
$(1, 1, 0, 1, 0, 0, 0, 0)$	0	-2	0	0	(iii)	-
$(1, 1, 0, 0, 1, 0, 0, 0)$	-1	-1	1	-1	(xii)	-
$(1, 1, 0, 0, 0, 1, 0, 0)$	0	0	0	0	(i)	$p = 0,\ q = 0$ for any $(\hat{n}, \hat{k}, \hat{\ell})$
$(1, 1, 0, 0, 0, 0, 1, 0)$	-1	-1	1	1	(xii)	-
$(1, 1, 0, 0, 0, 0, 0, 1)$	0	0	2	0	(iv)	-

$\alpha = a + c - e - g - 1$; $\beta = -b - d + f + h$; $\gamma = b - d - f + h$; $\delta = a - c - e + g - 1$

Table 6.10: Possible cases for $(a, b, c, d, e, f, g, h) \in \mathbb{Z}_+^8$

(a, b, c, d, e, f, g, h)	α	β	γ	δ	A	Existence of $p, q \in \mathbb{Z}$ in (6.220)
$(1, 0, 1, 1, 0, 0, 0, 0)$	1	-1	-1	-1	(xii)	-
$(1, 0, 1, 0, 1, 0, 0, 0)$	0	0	0	-2	(v)	-
$(1, 0, 1, 0, 0, 1, 0, 0)$	1	1	-1	-1	(xii)	-
$(1, 0, 1, 0, 0, 0, 1, 0)$	0	0	0	0	(i)	$p = 0,\ q = 0$ for any $(\hat{n}, \hat{k}, \hat{\ell})$
$(1, 0, 1, 0, 0, 0, 0, 1)$	1	1	1	-1	(xii)	-
$(1, 0, 0, 1, 1, 0, 0, 0)$	-1	-1	-1	-1	(xii)	-
$(1, 0, 0, 1, 0, 1, 0, 0)$	0	0	-2	0	(iv)	-
$(1, 0, 0, 1, 0, 0, 1, 0)$	-1	-1	-1	1	(xii)	-
$(1, 0, 0, 1, 0, 0, 0, 1)$	0	0	0	0	(i)	$p = 0,\ q = 0$ for any $(\hat{n}, \hat{k}, \hat{\ell})$
$(1, 0, 0, 0, 1, 1, 0, 0)$	-1	1	-1	-1	(xii)	-
$(1, 0, 0, 0, 1, 0, 1, 0)$	-2	0	0	0	(ii)	-
$(1, 0, 0, 0, 1, 0, 0, 1)$	-1	1	1	-1	(xii)	-
$(1, 0, 0, 0, 0, 1, 1, 0)$	-1	1	-1	1	(xii)	-
$(1, 0, 0, 0, 0, 1, 0, 1)$	0	2	0	0	(iii)	-
$(1, 0, 0, 0, 0, 0, 1, 1)$	-1	1	1	1	(xii)	-
$(0, 1, 1, 1, 0, 0, 0, 0)$	0	-2	0	-2	(vii)	-
$(0, 1, 1, 0, 1, 0, 0, 0)$	-1	-1	1	-3	(xii)	-
$(0, 1, 1, 0, 0, 1, 0, 0)$	0	0	0	-2	(v)	-
$(0, 1, 1, 0, 0, 0, 1, 0)$	-1	-1	1	-1	(xii)	-
$(0, 1, 1, 0, 0, 0, 0, 1)$	0	0	2	-2	(ix)	-
$(0, 1, 0, 1, 1, 0, 0, 0)$	-2	-2	0	-2	(xii)	-
$(0, 1, 0, 1, 0, 1, 0, 0)$	-1	-1	-1	-1	(xii)	-
$(0, 1, 0, 1, 0, 0, 1, 0)$	-2	-2	0	0	(viii)	$p = -1,\ q = 0$ for $\hat{n} = 2\hat{k} + 2\hat{\ell}$
$(0, 1, 0, 1, 0, 0, 0, 1)$	-1	-1	1	-1	(xii)	-
$(0, 1, 0, 0, 1, 1, 0, 0)$	-2	0	0	-2	(x)	-
$(0, 1, 0, 0, 1, 0, 1, 0)$	-3	-1	1	-1	(xii)	-
$(0, 1, 0, 0, 1, 0, 0, 1)$	-2	0	2	-2	(xii)	-
$(0, 1, 0, 0, 0, 1, 1, 0)$	-2	0	0	0	(ii)	-
$(0, 1, 0, 0, 0, 1, 0, 1)$	-1	1	1	-1	(xii)	-
$(0, 1, 0, 0, 0, 0, 1, 1)$	-2	0	2	0	(vi)	-

$\alpha = a + c - e - g - 1;\quad \beta = -b - d + f + h;\quad \gamma = b - d - f + h;\quad \delta = a - c - e + g - 1$

Table 6.11: Possible cases for $(a, b, c, d, e, f, g, h) \in \mathbb{Z}_+^8$

(a, b, c, d, e, f, g, h)	α	β	γ	δ	A	Existence of $p, q \in \mathbb{Z}$ in (6.220)
(0, 0, 1, 1, 1, 0, 0, 0)	-1	-1	-1	-3	(xii)	-
(0, 0, 1, 1, 0, 1, 0, 0)	0	0	-2	-2	(ix)	-
(0, 0, 1, 1, 0, 0, 1, 0)	-1	-1	-1	-1	(xii)	-
(0, 0, 1, 1, 0, 0, 0, 1)	0	0	0	-2	(v)	$p = 0,\ q = -1$ for $\hat{n} = 2\hat{k} + 2\hat{\ell}$
(0, 0, 1, 0, 1, 1, 0, 0)	-1	1	-1	-3	(xii)	-
(0, 0, 1, 0, 1, 0, 1, 0)	-2	0	0	-2	(x)	-
(0, 0, 1, 0, 1, 0, 0, 1)	-1	1	1	-3	(xii)	-
(0, 0, 1, 0, 0, 1, 1, 0)	-1	1	-1	-1	(xii)	-
(0, 0, 1, 0, 0, 1, 0, 1)	0	2	0	-2	(vii)	-
(0, 0, 1, 0, 0, 0, 1, 1)	-1	1	1	-1	(xii)	-
(0, 0, 0, 1, 1, 1, 0, 0)	-2	0	-2	-2	(xii)	-
(0, 0, 0, 1, 1, 0, 1, 0)	-3	-1	-1	-1	(xii)	-
(0, 0, 0, 1, 1, 0, 0, 1)	-2	0	0	-2	(x)	-
(0, 0, 0, 1, 0, 1, 1, 0)	-2	0	-2	0	(vi)	-
(0, 0, 0, 1, 0, 1, 0, 1)	-1	1	-1	-1	(xii)	-
(0, 0, 0, 1, 0, 0, 1, 1)	-2	0	0	0	(ii)	-
(0, 0, 0, 0, 1, 1, 1, 0)	-3	1	-1	-1	(xii)	-
(0, 0, 0, 0, 1, 1, 0, 1)	-2	2	0	-2	(xii)	-
(0, 0, 0, 0, 1, 0, 1, 1)	-3	1	1	-1	(xii)	-
(0, 0, 0, 0, 0, 1, 1, 1)	-2	2	0	0	(viii)	-

$\alpha = a + c - e - g - 1$; $\beta = -b - d + f + h$; $\gamma = b - d - f + h$; $\delta = a - c - e + g - 1$

The condition in (6.221) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \alpha\hat{k},\ q\hat{n} = 0. \tag{6.241}$$

From the conditions for ℓ, k, and n in (6.175), we have necessary conditions $1 \le \ell < k < n/2$. Dividing each side of these inequalities by $\gcd(k, \ell, n)$, we have $1/\gcd(k, \ell, n) \le \hat{\ell} < \hat{k} < \hat{n}/2$. Since $\hat{\ell}$, $\hat{k}$, and $\hat{n}$ are integers, we have $1 \le \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\alpha| \le 2$, we see that $p\hat{n} = \alpha\hat{k}$ is not satisfied for any p. From this, we have $|\alpha| \ge 3$. The sum of equalities in (6.239) leads to $\alpha = 2(a - e - 1)$. From this, α is even. Recall that $a + b + \ldots + h \le 3$. From this, $\alpha = a + c - e - g - 1$ takes a value within the range of $-4 \le \alpha \le 2$. From this and $|\alpha| \ge 3$, we have $\alpha = -4$. From (6.239) and (6.240), we have $(a, b, c, d, e, f, g, h) = (0, 0, 0, 0, 1, 0, 2, 0)$. From (6.241), we have $-4\hat{k} = p\hat{n}$. This condition is satisfied for $p = -1$. Hence, we have

$$(0, 0, 0, 0, 1, 0, 2, 0) \in P \text{ for } \hat{n} = 4\hat{k}.$$

For the case (iii), the elements of A in (6.220) represent

$$a + c - e - g - 1 = 0, \quad a - c - e + g - 1 = 0, \tag{6.242}$$

$$-b - d + f + h = \beta, \quad b - d - f + h = 0. \tag{6.243}$$

The condition in (6.221) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \beta\hat{\ell},\ q\hat{n} = 0. \tag{6.244}$$

Recall that $1 \le \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\beta| \le 2$, we see that $p\hat{n} = \beta\hat{\ell}$ is not satisfied for any p. From this, we have $|\beta| \ge 3$. The sum of equalities in (6.243) leads to $\beta = -2(d - h)$. From this, β is even. Recall that $a + b + \ldots + h \le 3$. From this, $\beta = -b - d + f + h$ takes a value within the range of $-3 \le \beta \le 3$. Hence, we have $\beta = \pm 2$. This contradicts $|\beta| \ge 3$.

For the case (iv), the elements of A in (6.220) represent

$$a + c - e - g - 1 = 0, \quad a - c - e + g - 1 = 0, \tag{6.245}$$

$$-b - d + f + h = 0, \quad b - d - f + h = \gamma. \tag{6.246}$$

The condition in (6.221) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = 0,\ q\hat{n} = \gamma\hat{k}. \tag{6.247}$$

Recall that $1 \le \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\gamma| \le 2$, we see that $q\hat{n} = \gamma\hat{k}$ is not satisfied for any q. From this, we have $|\gamma| \ge 3$. The sum of equalities in (6.246) leads to $\gamma = -2(d - h)$. From this, γ is even. Recall that $a + b + \ldots + h \le 3$. From this, $\gamma = b - d - f + h$ takes a value within the range of $-3 \le \gamma \le 3$. Hence, we have $\gamma = \pm 2$. This contradicts $|\gamma| \ge 3$.

For the case (v), the elements of A in (6.220) represent

$$a + c - e - g - 1 = 0, \quad a - c - e + g - 1 = \delta, \tag{6.248}$$

$$-b - d + f + h = 0, \quad b - d - f + h = 0. \tag{6.249}$$

The condition in (6.221) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = 0,\ q\hat{n} = \delta\hat{\ell}. \tag{6.250}$$

Recall that $1 \leq \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\delta| \leq 2$, we see that $q\hat{n} = \delta\hat{\ell}$ is not satisfied for any p. From this, we have $|\delta| \geq 3$. Recall that $a + b + \ldots + h \leq 3$. From this, $\delta = a - c - e + g - 1$ takes a value within the range of $-4 \leq \delta \leq 2$. From this and $|\delta| \geq 3$, we have $\delta = -4$. From (6.248) and (6.249), we have $(a, b, c, d, e, f, g, h) = (0, 0, 2, 0, 1, 0, 0, 0)$. From (6.250), we have $-4\hat{\ell} = p\hat{n}$. This condition is satisfied for $p = -1$. Hence, we have

$$(0, 0, 2, 0, 1, 0, 0, 0) \in P \text{ for } \hat{n} = 4\hat{\ell}.$$

For the case (vi), the elements of A in (6.220) represent

$$a + c - e - g - 1 = \alpha, \quad a - c - e + g - 1 = 0, \tag{6.251}$$
$$-b - d + f + h = 0, \quad b - d - f + h = \gamma. \tag{6.252}$$

The condition in (6.221) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \alpha\hat{k},\ q\hat{n} = \gamma\hat{k}.$$

Recall that $1 \leq \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\gamma| \leq 2$, we see that $q\hat{n} = \gamma\hat{k}$ is not satisfied for any q. From this, we have $|\gamma| \geq 3$. The sum of equalities in (6.252) leads to $\gamma = -2(d - h)$. From this, γ is even. Recall that $a + b + \ldots + h \leq 3$. From this, $\gamma = b - d - f + h$ takes a value within the range of $-3 \leq \gamma \leq 3$. Hence, we have $\gamma = \pm 2$. This contradicts $|\gamma| \geq 3$.

For the case (vii), the elements of A in (6.220) represent

$$a + c - e - g - 1 = 0, \quad a - c - e + g - 1 = \delta, \tag{6.253}$$
$$-b - d + f + h = \beta, \quad b - d - f + h = 0. \tag{6.254}$$

The condition in (6.221) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \beta\hat{\ell},\ q\hat{n} = \delta\hat{\ell}. \tag{6.255}$$

Recall that $1 \leq \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\beta| \leq 2$, we see that $p\hat{n} = \beta\hat{\ell}$ is not satisfied for any p. From this, we have $|\beta| \geq 3$. The sum of equalities in (6.254) leads to $\beta = -2(d - h)$. From this, β is even. Recall that $a + b + \ldots + h \leq 3$. From this, $\beta = -b - d + f + h$ takes a value within the range of $-3 \leq \beta \leq 3$. Hence, we have $\beta = \pm 2$. This contradicts $|\beta| \geq 3$.

For the case (viii), the elements of A in (6.220) represent

$$a + c - e - g - 1 = \alpha, \quad a - c - e + g - 1 = 0, \tag{6.256}$$
$$-b - d + f + h = \beta, \quad b - d - f + h = 0. \tag{6.257}$$

The condition in (6.221) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \alpha\hat{k} + \beta\hat{\ell},\ q\hat{n} = 0. \tag{6.258}$$

Recall that $a + b + \ldots + h \leq 3$. From this, $\alpha = a + c - e - g - 1$ takes a value within the range of $-4 \leq \alpha \leq 2$. The sum of equalities in (6.256) leads to $\alpha = 2(a - e - 1)$. Thus, α is even. Hence, we have $\alpha = \pm 2, -4$. In a similar manner, $\beta = -b - d + f + h$ takes a value within the range of $-3 \leq \beta \leq 3$. The sum of equalities in (6.257) leads to $\beta = -2(d - h)$. Thus, β is even. Hence, we have $\beta = \pm 2$. When we consider $\alpha = -4$, we have $(a, b, c, d, e, f, g, h) = (0, 0, 0, 0, 1, 0, 2, 0)$. Hence, we have $\beta = 0$. This contradicts $\beta \neq 0$. When we consider $\alpha = 2$, we have $(a, b, c, d, e, f, g, h) = (2, 0, 1, 0, 0, 0, 0, 0)$. Hence, we have $\beta = 0$. This contradicts $\beta \neq 0$. When we consider $\alpha = -2$ with $\beta = 2$, we have $(a, b, c, d, e, f, g, h) = (0, 0, 0, 0, 0, 1, 1, 1)$. From (6.258), we have $-2(\hat{k} - \hat{\ell}) = p\hat{n}$. Recall that $1 \leq \hat{\ell} < \hat{k} < \hat{n}/2$. From this, we have $1 \leq \hat{k} - \hat{\ell} < \hat{n}/2$. Thus, the condition $-2(\hat{k} - \hat{\ell}) = p\hat{n}$ is not satisfied for any p. When we consider $\alpha = -2$ with $\beta = -2$, we have $(a, b, c, d, e, f, g, h) = (0, 1, 0, 1, 0, 0, 1, 0)$. From (6.258), we have $-2(\hat{k} + \hat{\ell}) = p\hat{n}$. This condition is satisfied for $p = -1$. Hence, we have

$$(0, 1, 0, 1, 0, 0, 1, 0) \in P \text{ for } \hat{n} = 2\hat{k} + 2\hat{\ell}.$$

For the case (ix), the elements of A in (6.220) represent

$$a + c - e - g - 1 = 0, \quad a - c - e + g - 1 = \delta, \tag{6.259}$$

$$-b - d + f + h = 0, \quad b - d - f + h = \gamma. \tag{6.260}$$

The condition in (6.221) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = 0, \ q\hat{n} = \gamma\hat{k} + \delta\hat{\ell}. \tag{6.261}$$

Recall that $a + b + \ldots + h \leq 3$. From this, $\alpha = a - c - e + g - 1$ takes a value within the range of $-4 \leq \alpha \leq 2$. The sum of equalities in (6.259) leads to $\alpha = 2(a - e - 1)$. Thus, α is even. Hence, we have $\alpha = \pm 2, -4$. In a similar manner, $\beta = b - d - f + h$ takes a value within the range of $-3 \leq \beta \leq 3$. The sum of equalities in (6.260) leads to $\beta = -2(d - h)$. Thus, β is even. Hence, we have $\beta = \pm 2$. When we consider $\alpha = -4$, we have $(a, b, c, d, e, f, g, h) = (0, 0, 2, 0, 1, 0, 0, 0)$. Hence, we have $\beta = 0$. This contradicts $\beta \neq 0$. When we consider $\alpha = 2$, we have $(a, b, c, d, e, f, g, h) = (2, 0, 0, 0, 0, 0, 1, 0)$. Hence, we have $\beta = 0$. This contradicts $\beta \neq 0$. When we consider $\alpha = -2$ with $\beta = 2$, we have $(a, b, c, d, e, f, g, h) = (0, 1, 1, 0, 0, 0, 0, 1)$. From (6.261), we have $-2(\hat{k} - \hat{\ell}) = q\hat{n}$. Recall that $1 \leq \hat{\ell} < \hat{k} < \hat{n}/2$. From this, we have $1 \leq \hat{k} - \hat{\ell} < \hat{n}/2$. Thus, the condition $-2(\hat{k} - \hat{\ell}) = q\hat{n}$ is not satisfied for any q. When we consider $\alpha = -2$ with $\beta = -2$, we have $(a, b, c, d, e, f, g, h) = (0, 0, 1, 1, 0, 1, 0, 0)$. From (6.261), we have $-2(\hat{k} + \hat{\ell}) = q\hat{n}$. This condition is satisfied for $q = -1$. Hence, we have

$$(0, 0, 1, 1, 0, 1, 0, 0) \in P \text{ for } \hat{n} = 2\hat{k} + 2\hat{\ell}.$$

For the case (x), the elements of A in (6.220) represent

$$a + c - e - g - 1 = \alpha, \quad a - c - e + g - 1 = \delta, \tag{6.262}$$

$$-b - d + f + h = 0, \quad b - d - f + h = 0. \tag{6.263}$$

The condition in (6.221) is equivalent to

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \alpha\hat{k},\ q\hat{n} = \delta\hat{\ell}. \tag{6.264}$$

Recall that $1 \leq \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\alpha| \leq 2$, we see that $p\hat{n} = \alpha\hat{k}$ is not satisfied for any p. From this, we have $|\alpha| \geq 3$. Similarly, for the case $|\delta| \leq 2$, we see that $q\hat{n} = \delta\hat{\ell}$ is not satisfied for any q. From this, we have $|\delta| \geq 3$. According to the results in Table 6.11–6.11, only $(a, b, c, d, e, f, g, h) = (0, 0, 0, 0, 2, 0, 0, 0)$ corresponds to this case. From (6.264), we have $p\hat{n} = -3\hat{k}$ and $q\hat{n} = -3\hat{\ell}$. From $1 \leq \hat{\ell} < \hat{k} < \hat{n}/2$, we have $p = -1$ and $q = -1$. Thus, we have $\hat{n} = 3\hat{k}$ and $\hat{n} = 3\hat{\ell}$. This contradicts $\hat{k} \neq \hat{\ell}$.

For the case (xi), the elements of A in (6.220) represent

$$a + c - e - g - 1 = 0, \quad a - c - e + g - 1 = 0, \tag{6.265}$$

$$-b - d + f + h = \beta, \quad b - d - f + h = \gamma. \tag{6.266}$$

The condition in (6.221) is equivalent to

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \beta\hat{\ell},\ q\hat{n} = \gamma\hat{k}. \tag{6.267}$$

Recall that $1 \leq \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\beta| \leq 2$, we see that $p\hat{n} = \beta\hat{\ell}$ is not satisfied for any p. From this, we have $|\beta| \geq 3$. Similarly, for the case $|\gamma| \leq 2$, we see that $q\hat{n} = \gamma\hat{k}$ is not satisfied for any q. From this, we have $|\gamma| \geq 3$. According to the results in Table 6.6–6.11, no (a, b, c, d, e, f, g, h) corresponds to this case.

For the case (xii), the elements of A in (6.220) represent

$$a + c - e - g - 1 = \alpha, \quad a - c - e + g - 1 = \delta, \tag{6.268}$$

$$-b - d + f + h = \beta, \quad b - d - f + h = \gamma. \tag{6.269}$$

The condition in (6.221) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \alpha\hat{k} + \beta\hat{\ell},\ q\hat{n} = \gamma\hat{k} + \delta\hat{\ell}. \tag{6.270}$$

All (a, b, c, d, e, f, g, h) that correspond to this case are shown in Table 6.6–6.11.

Based on the above discussion, F_i $(i = 1, \ldots, 4)$ is restricted to the form of

$$F_i = a_1\widetilde{\phi}z_i + F_i^C + (\text{other terms}), \quad i = 1, \ldots, 4, \tag{6.271}$$

where

$$F_1^C = z_1(a_2|z_1|^2 + a_3|z_2|^2 + a_4|z_3|^2 + a_5|z_4|^2), \tag{6.272}$$

$$F_2^C = z_2(a_2|z_2|^2 + a_3|z_1|^2 + a_4|z_4|^2 + a_5|z_3|^2), \tag{6.273}$$

$$F_3^C = z_3(a_2|z_3|^2 + a_3|z_4|^2 + a_4|z_1|^2 + a_5|z_2|^2), \tag{6.274}$$

$$F_4^C = z_4(a_2|z_4|^2 + a_3|z_3|^2 + a_4|z_2|^2 + a_5|z_1|^2) \tag{6.275}$$

Table 6.12: Nonzero coefficients of leading terms which belong to "other terms" in (6.271)

$(\hat{n}, \hat{k}, \hat{\ell})$	Nonzero coefficients
General $(\hat{n}, \hat{k}, \hat{\ell})$	None
$(5, 2, 1)$	$A_{01001000}(0)$, $A_{00000200}(0)$, $A_{03000000}(0)$, $A_{00002100}(0)$
$(8, 3, 1)$	$A_{01010010}(0)$, $A_{00110100}(0)$, $A_{00021000}(0)$, $A_{00002001}(0)$, $A_{00000003}(0)$
$(10, 3, 1)$	$A_{01002000}(0)$, $A_{00000300}(0)$
$(4\hat{k}, \hat{k}, \hat{\ell})$	$A_{00001020}(0)$
$(4\hat{\ell}, \hat{k}, \hat{\ell})$	$A_{00201000}(0)$
$(2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$ with $(\hat{k}, \hat{\ell}) \neq (3, 1)$	$A_{01010010}(0)$, $A_{00110100}(0)$, $A_{00021000}(0)$

Table 6.13: Stability conditions of bifurcating solutions for group-theoretic critical points with multiplicity 8

$(\hat{n}, \hat{k}, \hat{\ell})$	Solutions	Stability conditions (necessary conditions)
General $(\hat{n}, \hat{k}, \hat{\ell})$	$\boldsymbol{w}_{\text{stripeI}}$, $\boldsymbol{w}_{\text{stripeII}}$	$\max(a_3, a_4, a_5) < a_2 < 0$
	$\boldsymbol{w}_{\text{upside-downI}}$, $\boldsymbol{w}_{\text{upside-downII}}$	$a_3 - a_4 + a_5 < a_2 < -\lvert a_4\rvert$
	$\boldsymbol{w}_{\text{sqT}}$	$-a_3 + a_4 + a_5 < a_2 < -\lvert a_3\rvert$
	$\boldsymbol{w}_{\text{sqVM}}$	$a_2 + a_3 < -\lvert a_4 + a_5\rvert$ $a_2 - a_3 < -\lvert a_4 - a_5\rvert$

with the following notations:[12]

$$\begin{aligned} &a_1 = A'_{10000000}(0), \quad a_2 = A_{20001000}(0), \quad a_3 = A_{11000100}(0), \\ &a_4 = A_{10100010}(0), \quad a_5 = A_{10010001}(0). \end{aligned} \tag{6.276}$$

F_2, F_3, and F_4 are obtained by (6.203), (6.204), and (6.206), respectively.

In (6.271), F_i^C corresponds to cubic terms, and the form of "(other terms)" varies with the values of $(\hat{n}, \hat{k}, \hat{\ell})$. For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$, we have quadratic terms as leading terms. For any other cases, we have cubic terms as leading terms that vary with the values of $(\hat{n}, \hat{k}, \hat{\ell})$. From this point of view, we classify the form of the bifurcation equation as shown in Table 6.12 by the values of $(\hat{n}, \hat{k}, \hat{\ell})$.

The form of "(other terms)" in (6.271) depends on the values of $(\hat{n}, \hat{k}, \hat{\ell})$ in (6.176). All the possible cases and stability conditions for the bifurcating solutions are summarized in Tables 6.13–6.15. The main finding of this section is as follows:

Proposition 6.14. *For a critical point of multiplicity 8 associated with $\mu = (8; k, \ell)$, we have the following statements:*

[12]These notations are local and should not be confused with (6.128) used in Section 6.4.1.

Table 6.14: Stability conditions of bifurcating solutions for group-theoretic critical points with multiplicity 8

$(\hat{n}, \hat{k}, \hat{\ell})$	Solutions	Stability conditions
$(5, 2, 1)$	$\boldsymbol{w}_{\text{stripeI}}$, $\boldsymbol{w}_{\text{stripeII}}$	Does not exist
	$\boldsymbol{w}_{\text{upside-downI}}$, $\boldsymbol{w}_{\text{upside-downII}}$	Does not exist
	$\boldsymbol{w}_{\text{sqT}}$	Always unstable
	$\boldsymbol{w}_{\text{sqVM}}$	$\begin{cases} a_6 + a_7 < 0 \\ 3a_6 + a_7 > 0 \quad \text{if } w > 0 \\ 2a_6 + a_7 > 0 \end{cases}$ $\begin{cases} a_6 + a_7 > 0 \\ 3a_6 + a_7 > 0 \quad \text{if } w < 0 \\ 2a_6 + a_7 > 0 \end{cases}$
$(8, 3, 1)$	$\boldsymbol{w}_{\text{stripeI}}$, $\boldsymbol{w}_{\text{stripeII}}$	Does not exist
	$\boldsymbol{w}_{\text{upside-downI}}$, $\boldsymbol{w}_{\text{upside-downII}}$	Does not exist
	$\boldsymbol{w}_{\text{sqT}}$	Does not exist
	$\boldsymbol{w}_{\text{sqVM}}$	$a_2 + a_3 + a_4 + a_5 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} < 0$ $a_2 + a_3 - a_4 - a_5 - a_{10} - a_{11} - a_{12} - 2a_{14} < 0$ $a_2 - a_3 + a_4 - a_5 - a_{10} - a_{11} - a_{12} - 2a_{14} < 0$ $a_2 - a_3 - a_4 + a_5 - a_{10} - a_{11} + a_{12} + a_{13} + a_{14} < 0$ $a_{10} + a_{11} + 2a_{12} + a_{13} - a_{14} > 0$ $a_{13} + a_{14} > 0$
$(10, 3, 1)$	$\boldsymbol{w}_{\text{stripeI}}$, $\boldsymbol{w}_{\text{stripeII}}$	Does not exist
	$\boldsymbol{w}_{\text{upside-downI}}$, $\boldsymbol{w}_{\text{upside-downII}}$	Does not exist
	$\boldsymbol{w}_{\text{sqT}}$	$a_2 + a_3 + a_{15} + a_{16} < 0$ $a_2 - a_3 - 2a_{16} < 0$ $3a_{15} + a_{16} > 0$ $a_2 + a_3 - a_4 - a_5 + a_{15} + a_{16} < 0$
	$\boldsymbol{w}_{\text{sqVM}}$	$a_2 + a_3 + a_4 + a_5 + a_{15} + a_{16} < 0$ $a_2 + a_3 - a_4 - a_5 - a_{15} - a_{16} < 0$ $a_2 - a_3 + a_4 - a_5 - 2a_{16} < 0$ $a_2 - a_3 - a_4 + a_5 - 2a_{16} < 0$ $3a_{15} + a_{16} > 0$

Table 6.15: Stability conditions of bifurcating solutions for group-theoretic critical points with multiplicity 8

$(\hat{n}, \hat{k}, \hat{\ell})$	Solutions	Stability conditions (necessary conditions)
$(4\hat{k}, \hat{k}, \hat{\ell})$	$\boldsymbol{w}_{\text{stripeI}}$, $\boldsymbol{w}_{\text{stripeII}}$	$\max(a_3, a_4 + \lvert a_{17}\rvert, a_5) < a_2 < 0$
	$\boldsymbol{w}_{\text{upside-downI}}$, $\boldsymbol{w}_{\text{upside-downII}}$	$a_3 - a_4 + a_5 - a_{17} < a_2 < -\lvert a_4 + a_{17}\rvert$
		$a_4 > 0$
	$\boldsymbol{w}_{\text{SqT}}$	Does not exist
	$\boldsymbol{w}_{\text{sqVM}}$	$a_2 + a_3 + a_4 + a_5 + a_{17} < 0$
		$a_2 + a_3 - a_4 - a_5 - a_{17} < 0$
		$a_2 - a_3 + a_4 - a_5 + a_{17} < 0$
		$a_2 - a_3 - a_4 + a_5 - a_{17} < 0$
		$a_{17} > 0$
$(4\hat{\ell}, \hat{k}, \hat{\ell})$	$\boldsymbol{w}_{\text{stripeI}}$, $\boldsymbol{w}_{\text{stripeII}}$	$\max(a_3, a_4 + \lvert a_{18}\rvert, a_5) < a_2 < 0$
	$\boldsymbol{w}_{\text{upside-downI}}$, $\boldsymbol{w}_{\text{upside-downII}}$	$a_3 - a_4 + a_5 + a_{17} < a_2 < -\lvert a_4 + a_{18}\rvert$
		$a_{18} > 0$
	$\boldsymbol{w}_{\text{SqT}}$	Does not exist
	$\boldsymbol{w}_{\text{SqVM}}$	$a_2 + a_3 + a_4 + a_5 + a_{18} < 0$
		$a_2 + a_3 - a_4 - a_5 - a_{18} < 0$
		$a_2 - a_3 + a_4 - a_5 + a_{18} < 0$
		$a_2 - a_3 - a_4 + a_5 - a_{18} < 0$
		$a_{18} > 0$
$(2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$	$\boldsymbol{w}_{\text{stripeI}}$, $\boldsymbol{w}_{\text{stripeII}}$	$\max(a_3, a_4, a_5 - \lvert a_{12}\rvert) < a_2 < 0$
with $(\hat{k}, \hat{\ell}) \neq (3, 1)$	$\boldsymbol{w}_{\text{upside-downI}}$, $\boldsymbol{w}_{\text{upside-downII}}$	$a_2 < -\lvert a_4\rvert$
		$a_2 + a_3 - a_4 - a_5 - a_{12} > -\lvert a_{10} + a_{11}\rvert$
		$a_2 + a_3 - a_4 - a_5 + a_{12} > -\lvert a_{10} - a_{11}\rvert$
	$\boldsymbol{w}_{\text{sqT}}$	Does not exist
	$\boldsymbol{w}_{\text{sqVM}}$	$a_2 + a_3 + a_4 + a_5 + a_{10} + a_{11} + a_{12} < 0$
		$a_2 + a_3 - a_4 - a_5 - a_{10} - a_{11} - a_{12} < 0$
		$a_2 - a_3 + a_4 - a_5 - a_{10} - a_{11} - a_{12} < 0$
		$a_2 - a_3 - a_4 + a_5 - a_{10} - a_{11} + a_{12} < 0$
		$a_{10} + a_{11} + 2a_{12} > 0$

- *For the case* $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$*, the bifurcating solution* $\boldsymbol{w}_{\rm sqT}$ *is always unstable in the neighborhood of the critical point, and the bifurcating curve takes the form* $\widetilde{\phi} \approx cw$ *for some constant c.*

- *For any other cases, the stability of the bifurcating solutions* $\boldsymbol{w}_{\rm stripeI}$, $\boldsymbol{w}_{\rm stripeII}$, $\boldsymbol{w}_{\rm upside-downI}$, $\boldsymbol{w}_{\rm upside-downII}$, $\boldsymbol{w}_{\rm sqT}$*, and* $\boldsymbol{w}_{\rm sqVM}$ *depends on the values of the coefficients of the power series expansion of the bifurcation equation in* (6.223)*, and the bifurcating curve takes the form* $\widetilde{\phi} \approx cw^2$ *for some constant c.*

To show these results, we focus on each case and study stability conditions for the bifurcating solutions in the remainder of this section.

Case 1: General $(\hat{n}, \hat{k}, \hat{\ell})$

For general cases, other than special cases to be treated in the sequel, the asymptotic form of F_i $(i = 1, \dots, 4)$ in (6.271) becomes

$$F_i \approx a_1 \widetilde{\phi} z_i + F_i^C, \tag{6.277}$$

where F_i^C $(i = 1, \dots, 4)$ are given in (6.272) – (6.275). Then, the asymptotic form of $\widetilde{F}_i$ $(i = 1, \dots, 8)$ in (6.182) – (6.185) becomes

$$\widetilde{F}_i \approx a_1 \widetilde{\phi} w_i + \widetilde{F}_i^C \tag{6.278}$$

with

$$\widetilde{F}_1^C = w_1\{a_2({w_1}^2 + {w_2}^2) + a_3({w_3}^2 + {w_4}^2) + a_4({w_5}^2 + {w_6}^2) + a_5({w_7}^2 + {w_8}^2)\}, \tag{6.279}$$
$$\widetilde{F}_2^C = w_2\{a_2({w_1}^2 + {w_2}^2) + a_3({w_3}^2 + {w_4}^2) + a_4({w_5}^2 + {w_6}^2) + a_5({w_7}^2 + {w_8}^2)\}, \tag{6.280}$$
$$\widetilde{F}_3^C = w_3\{a_2({w_3}^2 + {w_4}^2) + a_3({w_1}^2 + {w_2}^2) + a_4({w_7}^2 + {w_8}^2) + a_5({w_5}^2 + {w_6}^2)\}, \tag{6.281}$$
$$\widetilde{F}_4^C = w_4\{a_2({w_3}^2 + {w_4}^2) + a_3({w_1}^2 + {w_2}^2) + a_4({w_7}^2 + {w_8}^2) + a_5({w_5}^2 + {w_6}^2)\}, \tag{6.282}$$
$$\widetilde{F}_5^C = w_5\{a_2({w_5}^2 + {w_6}^2) + a_3({w_7}^2 + {w_8}^2) + a_4({w_1}^2 + {w_2}^2) + a_5({w_3}^2 + {w_4}^2)\}, \tag{6.283}$$
$$\widetilde{F}_6^C = w_6\{a_2({w_5}^2 + {w_6}^2) + a_3({w_7}^2 + {w_8}^2) + a_4({w_1}^2 + {w_2}^2) + a_5({w_3}^2 + {w_4}^2)\}, \tag{6.284}$$
$$\widetilde{F}_7^C = w_7\{a_2({w_7}^2 + {w_8}^2) + a_3({w_5}^2 + {w_6}^2) + a_4({w_3}^2 + {w_4}^2) + a_5({w_1}^2 + {w_2}^2)\}, \tag{6.285}$$
$$\widetilde{F}_8^C = w_8\{a_2({w_7}^2 + {w_8}^2) + a_3({w_5}^2 + {w_6}^2) + a_4({w_3}^2 + {w_4}^2) + a_5({w_1}^2 + {w_2}^2)\}, \tag{6.286}$$

Hence, the asymptotic form of the Jacobian matrix in (6.180) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1 \tilde{\phi} I_8 + B_C \tag{6.287}$$

with the following notations:[13]

$$B_C = a_2 B_2 + a_3 B_3 + a_4 B_4 + a_5 B_5, \tag{6.288}$$

[13]The notations here are local and should not be confused with (6.141) used in Section 6.4.1.

$$B_2 = \begin{bmatrix} B_1^2 & O \\ O & B_2^2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} B_1^3 & O \\ O & B_2^3 \end{bmatrix}, \quad B_4 = \begin{bmatrix} B_1^4 & B_3^4 \\ (B_3^4)^\top & B_2^4 \end{bmatrix}, \quad B_5 = \begin{bmatrix} B_1^5 & B_3^5 \\ (B_3^5)^\top & B_2^5 \end{bmatrix},$$

$$B_1^2 = \begin{bmatrix} 3{w_1}^2 + {w_2}^2 & 2w_1w_2 & 0 & 0 \\ 2w_1w_2 & {w_1}^2 + 3{w_2}^2 & 0 & 0 \\ 0 & 0 & 3{w_3}^2 + {w_4}^2 & 2w_3w_4 \\ 0 & 0 & 2w_3w_4 & {w_3}^2 + 3{w_4}^2 \end{bmatrix},$$

$$B_2^2 = \begin{bmatrix} 3{w_5}^2 + {w_6}^2 & 2w_5w_6 & 0 & 0 \\ 2w_5w_6 & {w_5}^2 + 3{w_6}^2 & 0 & 0 \\ 0 & 0 & 3{w_7}^2 + {w_8}^2 & 2w_7w_8 \\ 0 & 0 & 2w_7w_8 & {w_7}^2 + 3{w_8}^2 \end{bmatrix},$$

$$B_1^3 = \begin{bmatrix} {w_3}^2 + {w_4}^2 & 0 & 2w_1w_3 & 2w_1w_4 \\ 0 & {w_3}^2 + {w_4}^2 & 2w_2w_3 & 2w_2w_4 \\ 2w_1w_3 & 2w_2w_3 & {w_1}^2 + {w_2}^2 & 0 \\ 2w_1w_4 & 2w_2w_4 & 0 & {w_1}^2 + {w_2}^2 \end{bmatrix},$$

$$B_2^3 = \begin{bmatrix} {w_7}^2 + {w_8}^2 & 0 & 2w_5w_7 & 2w_5w_8 \\ 0 & {w_7}^2 + {w_8}^2 & 2w_6w_7 & 2w_6w_8 \\ 2w_5w_7 & 2w_6w_7 & {w_5}^2 + {w_6}^2 & 0 \\ 2w_5w_8 & 2w_6w_8 & 0 & {w_5}^2 + {w_6}^2 \end{bmatrix},$$

$$B_1^4 = \begin{bmatrix} ({w_5}^2 + {w_6}^2)I_2 & O \\ O & ({w_7}^2 + {w_8}^2)I_2 \end{bmatrix}, \quad B_2^4 = \begin{bmatrix} ({w_1}^2 + {w_2}^2)I_2 & O \\ O & ({w_3}^2 + {w_4}^2)I_2 \end{bmatrix},$$

$$B_3^4 = 2\begin{bmatrix} w_1w_5 & w_1w_6 & 0 & 0 \\ w_2w_5 & w_2w_6 & 0 & 0 \\ 0 & 0 & w_3w_7 & w_3w_8 \\ 0 & 0 & w_4w_7 & w_4w_8 \end{bmatrix}, \quad B_1^5 = \begin{bmatrix} ({w_7}^2 + {w_8}^2)I_2 & O \\ O & ({w_5}^2 + {w_6}^2)I_2 \end{bmatrix},$$

$$B_2^5 = \begin{bmatrix} ({w_3}^2 + {w_4}^2)I_2 & O \\ O & ({w_1}^2 + {w_2}^2)I_2 \end{bmatrix}, \quad B_3^5 = 2\begin{bmatrix} 0 & 0 & w_1w_7 & w_1w_8 \\ 0 & 0 & w_2w_7 & w_2w_8 \\ w_3w_5 & w_3w_6 & 0 & 0 \\ w_4w_5 & w_4w_6 & 0 & 0 \end{bmatrix}.$$

Substituting $\boldsymbol{w}_{\text{stripeI}} = (w, 0, 0, 0, 0, 0, 0, 0)$ into (6.278) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeI}} \approx -\frac{a_2}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.287) at $(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}}) = \widetilde{J_C}^{\text{stripeI}} \approx w^2 \begin{bmatrix} C_1 & O \\ O & C_2 \end{bmatrix} \tag{6.289}$$

with

$$C_1 = \begin{bmatrix} 2a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 + a_3 & 0 \\ 0 & 0 & 0 & -a_2 + a_3 \end{bmatrix}, \quad C_2 = \begin{bmatrix} (-a_2 + a_4)I_2 & O \\ O & (-a_2 + a_5)I_2 \end{bmatrix}. \tag{6.290}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$ are given by

$$\begin{aligned}
\lambda_1 &\approx 2a_2 w^2, \\
\lambda_2 &\approx O(w^3), \\
\lambda_3 &\approx -(a_2 - a_3)w^2 \quad \text{(repeated twice)}, \\
\lambda_4 &\approx -(a_2 - a_4)w^2 \quad \text{(repeated twice)}, \\
\lambda_5 &\approx -(a_2 - a_5)w^2 \quad \text{(repeated twice)}.
\end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$\begin{aligned}
&a_2 < 0, \\
&a_2 - a_3 > 0, \\
&a_2 - a_4 > 0, \\
&a_2 - a_5 > 0.
\end{aligned}$$

These conditions are equivalent to

$$\max(a_3, a_4, a_5) < a_2 < 0. \tag{6.291}$$

Thus, the stability of $\boldsymbol{w}_{\text{stripeI}}$ is conditional and depends on the values of $a_2, \ldots, a_5$.

Substituting $\boldsymbol{w}_{\text{stripeII}} = (0, w, 0, 0, 0, 0, 0, 0)$ into (6.278) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeII}} \approx -\frac{a_2}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.287) at $(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}}) = \widetilde{J_C}^{\text{stripeII}} \approx w^2 \begin{bmatrix} C_3 & O \\ O & C_2 \end{bmatrix} \tag{6.292}$$

with

$$C_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2a_2 & 0 & 0 \\ 0 & 0 & -a_2 + a_3 & 0 \\ 0 & 0 & 0 & -a_2 + a_3 \end{bmatrix}, \tag{6.293}$$

where C_2 is given in (6.290). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{stripeII}}$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$.

Substituting $\boldsymbol{w}_{\text{upside-downI}} = (w, 0, 0, 0, w, 0, 0, 0)$ into (6.278) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{upside-downI}} \approx -\frac{a_2 + a_4}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.287) at $(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}}) = \widetilde{J_C}^{\text{upside-downI}} \approx w^2 \begin{bmatrix} C_4 & C_5 \\ C_5 & C_4 \end{bmatrix} \tag{6.294}$$

with

$$C_4 = \begin{bmatrix} 2a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 + a_3 - a_4 + a_5 & 0 \\ 0 & 0 & 0 & -a_2 + a_3 - a_4 + a_5 \end{bmatrix}, \quad C_5 = \begin{bmatrix} 2a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.295}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$ are given by

$$\begin{aligned} &\lambda_1, \lambda_2 \approx 2(a_2 \pm a_4) w^2, \\ &\lambda_3 \approx O(w^3) \quad \text{(repeated twice)}, \\ &\lambda_4 \approx -(a_2 - a_3 + a_4 - a_5) w^2 \quad \text{(repeated 4 times)}. \end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$\begin{aligned} &a_2 < -|a_4|, \\ &a_2 - a_3 + a_4 - a_5 > 0. \end{aligned}$$

These conditions are equivalent to

$$a_3 - a_4 + a_5 < a_2 < -|a_4|. \tag{6.296}$$

Thus, the stability of $\boldsymbol{w}_{\text{upside-downI}}$ is conditional and depends on the values of $a_2, \ldots, a_5$.

Substituting $\boldsymbol{w}_{\text{upside-downII}} = (0, w, 0, 0, 0, w, 0, 0)$ into (6.278) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{upside-downII}} \approx -\frac{a_2 + a_4}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.287) at $(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}}) = \widetilde{J_C}^{\text{upside-downII}} \approx w^2 \begin{bmatrix} C_6 & C_7 \\ C_7 & C_6 \end{bmatrix} \tag{6.297}$$

with

$$C_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2a_2 & 0 & 0 \\ 0 & 0 & -a_2+a_3-a_4+a_5 & 0 \\ 0 & 0 & 0 & -a_2+a_3-a_4+a_5 \end{bmatrix}, \quad C_7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.298}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}})$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{upside-downII}}$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$.

Substituting $\boldsymbol{w}_{\text{sqT}} = (w, 0, w, 0, 0, 0, 0, 0)$ into (6.278) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqT}} \approx -\frac{a_2 + a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.287) at $(\boldsymbol{w}_{\text{sqT}}, \widetilde{\phi}_{\text{sqT}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sqT}}, \widetilde{\phi}_{\text{sqT}}) = \widetilde{J_C}^{\text{sqT}} \approx w^2 \begin{bmatrix} C_8 & O \\ O & C_9 \end{bmatrix} \tag{6.299}$$

$$C_8 = 2\begin{bmatrix} a_2 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 \\ a_3 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_9 = -(a_2 + a_3 - a_4 - a_5) I_4. \tag{6.300}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sqT}}, \widetilde{\phi}_{\text{sqT}})$ are given by

$$\begin{aligned} &\lambda_1, \lambda_2 \approx 2(a_2 \pm a_3) w^2, \\ &\lambda_3 \approx O(w^3) \quad \text{(repeated twice)}, \\ &\lambda_4 \approx -(a_2 + a_3 - a_4 - a_5) w^2 \quad \text{(repeated 4 times)}. \end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$\begin{aligned} &a_2 < -|a_3|, \\ &a_2 + a_3 - a_4 - a_5 > 0. \end{aligned}$$

These are equivalent to

$$-a_3 + a_4 + a_5 < a_2 < -|a_3|. \tag{6.301}$$

Thus, the stability of $\boldsymbol{w}_{\text{sqT}}$ is conditional and depends on the values of $a_2, \ldots, a_5$.

Substituting $\boldsymbol{w}_{\text{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$ into (6.278) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqVM}} \approx -\frac{a_2 + a_3 + a_4 + a_5}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.287) at $(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}}) = \widetilde{J_C}^{\text{sqVM}} \approx w^2 \begin{bmatrix} C_8 & C_{10} \\ C_{10} & C_8 \end{bmatrix} \tag{6.302}$$

with

$$C_{10} = 2\begin{bmatrix} a_4 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 \\ a_5 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6.303}$$

where C_8 is given in (6.300). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{sqVM}}, \widetilde{\phi}_{\mathrm{sqVM}})$ are given by

$$\begin{aligned} &\lambda_1, \lambda_2 \approx 2\{a_2 + a_3 \pm (a_4 + a_5)\}w^2, \\ &\lambda_3, \lambda_4 \approx 2\{a_2 - a_3 \pm (a_4 - a_5)\}w^2, \\ &\lambda_5 \approx O(w^3) \quad \text{(repeated 4 times)}. \end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 + a_3 < -|a_4 + a_5|, \tag{6.304}$$

$$a_2 - a_3 < -|a_4 - a_5|. \tag{6.305}$$

Thus, the stability of $\boldsymbol{w}_{\mathrm{sqVM}}$ is conditional and depends on the values of $a_2, \ldots, a_5$.

Case 2: $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$

For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$, we have

$$(0,1,0,0,1,0,0,0),\ (0,0,0,0,0,2,0,0),\ (0,3,0,0,0,0,0,0),\ (0,0,0,0,2,1,0,0) \in P$$

as well as

$$\begin{aligned} &(1,0,0,0,0,0,0,0),\ (2,0,0,0,1,0,0,0),\ (1,1,0,0,0,1,0,0), \\ &(1,0,1,0,0,0,1,0),\ (1,0,0,1,0,0,0,1) \in P \end{aligned}$$

in (6.238). Then, the asymptotic form of F_i $(i = 1, \ldots, 4)$ in (6.271) becomes

$$F_1 \approx a_1\widetilde{\phi}z_1 + a_6 z_2\bar{z}_1 + a_7\bar{z}_2{}^2 + a_8 z_2{}^3 + a_9\bar{z}_1{}^2\bar{z}_2 + F_1^C, \tag{6.306}$$

$$F_2 \approx a_1\widetilde{\phi}z_2 + a_6\bar{z}_1\bar{z}_2 + a_7 z_1{}^2 + a_8\bar{z}_1{}^3 + a_9\bar{z}_2{}^2 z_1 + F_2^C, \tag{6.307}$$

$$F_3 \approx a_1\widetilde{\phi}z_3 + a_6 z_4\bar{z}_3 + a_7\bar{z}_4{}^2 + a_8 z_4{}^3 + a_9\bar{z}_3{}^2\bar{z}_4 + F_3^C, \tag{6.308}$$

$$F_4 \approx a_1\widetilde{\phi}z_4 + a_6\bar{z}_3\bar{z}_4 + a_7 z_3{}^2 + a_8\bar{z}_3{}^3 + a_9\bar{z}_4{}^2 z_3 + F_4^C \tag{6.309}$$

with

$$a_6 = A_{01001000}(0), \quad a_7 = A_{00000200}(0), \quad a_8 = A_{03000000}(0), \quad a_9 = A_{00002100}(0),$$

where F_i^C $(i = 1, \ldots, 4)$ is given in (6.272) – (6.275). Then, the asymptotic form of $\widetilde{F}_i$ $(i = 1, \ldots, 8)$ in (6.182) – (6.185) becomes

$$\widetilde{F}_1 \approx a_1\widetilde{\phi}w_1 + a_6(w_1 w_3 + w_2 w_4) + a_7(w_3{}^2 - w_4{}^2)$$

$$+ a_8 w_3({w_3}^2 - 3{w_4}^2) + a_9\{w_3({w_1}^2 - {w_2}^2) - 2w_1 w_2 w_4\} + \widetilde{F}_1^C, \tag{6.310}$$

$$\widetilde{F}_2 \approx a_1 \widetilde{\phi} w_2 + a_6(w_1 w_4 - w_2 w_3) - 2a_7 w_3 w_4$$
$$+ a_8 w_4(3{w_3}^2 - {w_4}^2) + a_9\{-w_4({w_1}^2 - {w_2}^2) - 2w_1 w_2 w_3\} + \widetilde{F}_2^C, \tag{6.311}$$

$$\widetilde{F}_3 \approx a_1 \widetilde{\phi} w_3 + a_6(w_1 w_3 - w_2 w_4) + a_7({w_1}^2 - {w_2}^2)$$
$$+ a_8 w_1({w_1}^2 - 3{w_2}^2) + a_9\{w_1({w_3}^2 - {w_4}^2) + 2w_3 w_4 w_2\} + \widetilde{F}_3^C, \tag{6.312}$$

$$\widetilde{F}_4 \approx a_1 \widetilde{\phi} w_4 + a_6(-w_1 w_4 - w_2 w_3) + 2a_7 w_1 w_2$$
$$+ a_8 w_2(-3{w_1}^2 + {w_2}^2) + a_9\{w_2({w_3}^2 - {w_4}^2) - 2w_3 w_4 w_1\} + \widetilde{F}_4^C, \tag{6.313}$$

$$\widetilde{F}_5 \approx a_1 \widetilde{\phi} w_5 + a_6(w_5 w_7 + w_6 w_8) + a_7({w_7}^2 - {w_8}^2)$$
$$+ a_8 w_7({w_7}^2 - 3{w_8}^2) + a_9\{w_7({w_5}^2 - {w_6}^2) - 2w_5 w_6 w_8\} + \widetilde{F}_5^C, \tag{6.314}$$

$$\widetilde{F}_6 \approx a_1 \widetilde{\phi} w_6 + a_6(w_5 w_8 - w_6 w_7) - 2a_7 w_7 w_8$$
$$+ a_8 w_8(3{w_7}^2 - {w_8}^2) + a_9\{-w_8({w_5}^2 - {w_6}^2) - 2w_5 w_6 w_7\} + \widetilde{F}_6^C, \tag{6.315}$$

$$\widetilde{F}_7 \approx a_1 \widetilde{\phi} w_7 + a_6(w_5 w_7 - w_6 w_8) + a_7({w_5}^2 - {w_6}^2)$$
$$+ a_8 w_5({w_5}^2 - 3{w_6}^2) + a_9\{w_5({w_7}^2 - {w_8}^2) + 2w_7 w_8 w_6\} + \widetilde{F}_7^C, \tag{6.316}$$

$$\widetilde{F}_8 \approx a_1 \widetilde{\phi} w_8 + a_6(-w_5 w_8 - w_6 w_7) + 2a_7 w_5 w_6$$
$$+ a_8 w_6(-3{w_5}^2 + {w_6}^2) + a_9\{w_6({w_7}^2 - {w_8}^2) - 2w_7 w_8 w_5\} + \widetilde{F}_8^C, \tag{6.317}$$

where $\widetilde{F}_i^C$ $(i = 1, \ldots, 8)$ is given in (6.279) – (6.286). Hence, the asymptotic form of the Jacobian matrix in (6.180) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1 \tilde{\phi} I_8 + a_6 B_6 + a_7 B_7 + a_8 B_8 + a_9 B_9 + B_C, \tag{6.318}$$

where B_C is given in (6.288) and

$$B_6 = \begin{bmatrix} B_1^6 & O \\ O & B_2^6 \end{bmatrix}, \quad B_7 = \begin{bmatrix} B_1^7 & O \\ O & B_2^7 \end{bmatrix}, \quad B_8 = \begin{bmatrix} B_1^8 & O \\ O & B_2^8 \end{bmatrix}, \quad B_9 = \begin{bmatrix} B_1^9 & O \\ O & B_2^9 \end{bmatrix},$$

$$B_1^6 = \begin{bmatrix} w_3 & w_4 & w_1 & w_2 \\ w_4 & -w_3 & -w_2 & w_1 \\ w_3 & -w_4 & w_1 & -w_2 \\ -w_4 & -w_3 & -w_2 & -w_1 \end{bmatrix}, \quad B_2^6 = \begin{bmatrix} w_7 & w_8 & w_5 & w_6 \\ w_8 & -w_7 & -w_6 & w_5 \\ w_7 & -w_8 & w_5 & -w_6 \\ -w_8 & -w_7 & -w_6 & -w_5 \end{bmatrix},$$

$$B_1^7 = 2\begin{bmatrix} 0 & 0 & w_3 & -w_4 \\ 0 & 0 & -w_4 & -w_3 \\ w_1 & -w_2 & 0 & 0 \\ w_2 & w_1 & 0 & 0 \end{bmatrix}, \quad B_2^7 = 2\begin{bmatrix} 0 & 0 & w_7 & -w_8 \\ 0 & 0 & -w_8 & -w_7 \\ w_5 & -w_6 & 0 & 0 \\ w_6 & w_5 & 0 & 0 \end{bmatrix},$$

$$B_1^8 = 3\begin{bmatrix} 0 & 0 & {w_3}^2 - {w_4}^2 & -2w_3w_4 \\ 0 & 0 & 2w_3w_4 & {w_3}^2 - {w_4}^2 \\ {w_1}^2 - {w_2}^2 & -2w_1w_2 & 0 & 0 \\ -2w_1w_2 & -{w_1}^2 + {w_2}^2 & 0 & 0 \end{bmatrix},$$

$$B_2^8 = 3\begin{bmatrix} 0 & 0 & {w_7}^2 - {w_8}^2 & -2w_7w_8 \\ 0 & 0 & 2w_7w_8 & {w_7}^2 - {w_8}^2 \\ {w_5}^2 - {w_6}^2 & -2w_5w_6 & 0 & 0 \\ -2w_5w_6 & -{w_5}^2 + {w_6}^2 & 0 & 0 \end{bmatrix},$$

$$B_1^9 = \begin{bmatrix} 2(w_1w_3 - w_2w_4) & 2(-w_1w_4 - w_2w_3) & {w_1}^2 - {w_2}^2 & -2w_1w_2 \\ 2(-w_1w_4 - w_2w_3) & 2(-w_1w_3 + w_2w_4) & -2w_1w_2 & -{w_1}^2 + {w_2}^2 \\ {w_3}^2 - {w_4}^2 & 2w_3w_4 & 2(w_1w_3 + w_2w_4) & 2(-w_1w_4 + w_2w_3) \\ -2w_3w_4 & {w_3}^2 - {w_4}^2 & 2(-w_1w_4 + w_2w_3) & 2(-w_1w_3 - w_2w_4) \end{bmatrix},$$

$$B_2^9 = \begin{bmatrix} 2(w_5w_7 - w_6w_8) & 2(-w_5w_8 - w_6w_7) & {w_5}^2 - {w_6}^2 & -2w_5w_6 \\ 2(-w_5w_8 - w_6w_7) & 2(-w_5w_7 + w_6w_8) & -2w_5w_6 & -{w_5}^2 + {w_6}^2 \\ {w_7}^2 - {w_8}^2 & 2w_7w_8 & 2(w_5w_7 + w_6w_8) & 2(-w_5w_8 + w_6w_7) \\ -2w_7w_8 & {w_7}^2 - {w_8}^2 & 2(-w_5w_8 + w_6w_7) & 2(-w_5w_7 - w_6w_8) \end{bmatrix}.$$

Substituting $\boldsymbol{w}_{\text{sqT}} = (w, 0, w, 0, 0, 0, 0, 0)$ into (6.310) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqT}} \approx -\frac{a_6 + a_7}{a_1} w.$$

Evaluating the Jacobian matrix (6.318) at $(\boldsymbol{w}_{\text{sqT}}, \widetilde{\phi}_{\text{sqT}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sqT}}, \widetilde{\phi}_{\text{sqT}}) \approx w\begin{bmatrix} C_{11} & O \\ O & C_{12} \end{bmatrix} \tag{6.319}$$

with

$$C_{11} = \begin{bmatrix} -a_7 & 0 & a_6 + 2a_7 & 0 \\ 0 & -2a_6 - a_7 & 0 & a_6 - 2a_7 \\ a_6 + 2a_7 & 0 & -a_7 & 0 \\ 0 & -a_6 + 2a_7 & 0 & -2a_6 - a_7 \end{bmatrix}, \quad C_{12} = -(a_6 + a_7)I_4. \tag{6.320}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sqT}}, \widetilde{\phi}_{\text{sqT}})$ are given by

$$\begin{aligned} &\lambda_1 \approx (a_6 + a_7)w, \\ &\lambda_2 \approx -(a_6 + 3a_7)w, \\ &\lambda_3, \lambda_4 \approx -(2a_6 + a_7)w \pm \mathrm{i}(a_6 - 2a_7)w, \\ &\lambda_5 \approx -(a_6 + a_7)w \quad \text{(repeated 4 times)}. \end{aligned}$$

Since the eigenvalues λ_1 and λ_5 have opposite signs, there is at least one positive eigenvalue. Thus, the bifurcating solution $\boldsymbol{w}_{\rm sqT}$ is always unstable.

Substituting $\boldsymbol{w}_{\rm sqVM} = (w, 0, w, 0, w, 0, w, 0)$ into (6.310) with (6.279) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\rm sqVM} \approx -\frac{a_6 + a_7}{a_1} w.$$

Evaluating the Jacobian matrix (6.318) at $(\boldsymbol{w}_{\rm sqVM}, \widetilde{\phi}_{\rm sqVM})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\rm sqVM}, \widetilde{\phi}_{\rm sqVM}) \approx w \begin{bmatrix} C_{11} & O \\ O & C_{11} \end{bmatrix}, \tag{6.321}$$

where C_{11} is given in (6.320). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\rm sqVM}, \widetilde{\phi}_{\rm sqVM})$ are given by

$$\begin{aligned} &\lambda_1 \approx (a_6 + a_7)w, \\ &\lambda_2 \approx -(3a_6 + a_7)w, \\ &\lambda_3, \lambda_4 \approx -\{2a_6 + a_7 \pm \mathrm{i}(a_6 - 2a_7)\}w \end{aligned}$$

and are all repeated twice. Assume that all eigenvalues have negative real parts. If $w < 0$, we have the following stability conditions:

$$a_6 + a_7 < 0, \tag{6.322}$$
$$3a_6 + a_7 > 0, \tag{6.323}$$
$$2a_6 + a_7 > 0. \tag{6.324}$$

If $w < 0$, we have the following stability conditions:

$$a_6 + a_7 > 0, \tag{6.325}$$
$$3a_6 + a_7 < 0, \tag{6.326}$$
$$2a_6 + a_7 < 0. \tag{6.327}$$

Thus, the stability of $\boldsymbol{w}_{\rm sqVM}$ depends on the direction w of the bifurcating solution and the values of a_6 and a_7.

Remark 6.3. For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$, we have the following statements:

- The solutions $\boldsymbol{w}_{\rm stripeI}$ and $\boldsymbol{w}_{\rm stripeII}$ do not exist. See Proposition 6.10 in Section 6.5.3. In fact, $\hat{k}^2 + \hat{\ell} = 5$. This is divisible by $\hat{n} = 5$. Hence, the condition (6.230) is not satisfied.

- The solutions $\boldsymbol{w}_{\rm upside\text{-}downI}$ and $\boldsymbol{w}_{\rm upside\text{-}downII}$ do not exist. See Proposition 6.12 in Section 6.5.4. In fact, $\gcd(\hat{k}^2 + \hat{\ell}, \hat{k}^2 - \hat{\ell}) = \gcd(5, 3) = 1$. This is divisible by $\gcd(\hat{n}, 2\hat{k}\hat{\ell}) = \gcd(5, 4) = 1$. Hence, the condition (6.234) is not satisfied.

□

Case 3: $(\hat{n}, \hat{k}, \hat{\ell}) = (8, 3, 1)$

For the case of $(\hat{n}, \hat{k}, \hat{\ell}) = (8, 3, 1)$, we have

$$(0,1,0,1,0,0,1,0),\ (0,0,1,1,0,1,0,0),\ (0,0,0,2,1,0,0,0),$$
$$(0,0,0,0,2,0,0,1),\ (0,0,0,0,0,0,0,3) \in P$$

as well as

$$(1,0,0,0,0,0,0,0),\ (2,0,0,0,1,0,0,0),\ (1,1,0,0,0,1,0,0),$$
$$(1,0,1,0,0,0,1,0),\ (1,0,0,1,0,0,0,1) \in P$$

in (6.238). Then, the asymptotic form of F_i $(i = 1, \ldots, 4)$ in (6.271) becomes

$$F_1 \approx a_1\widetilde{\phi} z_1 + a_{10} z_2 z_4 \bar{z}_3 + a_{11} z_3 z_4 \bar{z}_2 + a_{12} {z_4}^2 \bar{z}_1 + a_{13} {\bar{z}_1}^2 \bar{z}_4 + a_{14} {\bar{z}_4}^3 + F_1^C, \tag{6.328}$$
$$F_2 \approx a_1\widetilde{\phi} z_2 + a_{10} \bar{z}_1 z_3 z_4 + a_{11} \bar{z}_4 z_3 z_1 + a_{12} {z_3}^2 \bar{z}_2 + a_{13} {\bar{z}_2}^2 \bar{z}_3 + a_{14} {\bar{z}_3}^3 + F_2^C, \tag{6.329}$$
$$F_3 \approx a_1\widetilde{\phi} z_3 + a_{10} z_4 z_2 \bar{z}_1 + a_{11} z_1 z_2 \bar{z}_4 + a_{12} {z_2}^2 \bar{z}_3 + a_{13} {\bar{z}_3}^2 \bar{z}_2 + a_{14} {\bar{z}_2}^3 + F_3^C, \tag{6.330}$$
$$F_4 \approx a_1\widetilde{\phi} z_4 + a_{10} \bar{z}_3 z_1 z_2 + a_{11} \bar{z}_2 z_1 z_3 + a_{12} {z_1}^2 \bar{z}_4 + a_{13} {\bar{z}_4}^2 \bar{z}_1 + a_{14} {\bar{z}_1}^3 + F_4^C \tag{6.331}$$

with

$$\begin{aligned} &a_{10} = A_{01010010}(0), \quad a_{11} = A_{00110100}(0), \quad a_{12} = A_{00021000}(0), \\ &a_{13} = A_{00002001}(0), \quad a_{14} = A_{00000003}(0), \end{aligned} \tag{6.332}$$

where F_i^C $(i = 1, \ldots, 4)$ is given in (6.272) – (6.275). Then, the asymptotic form of $\widetilde{F}_i$ $(i = 1, \ldots, 8)$ in (6.182) – (6.185) becomes

$$\begin{aligned} \widetilde{F}_1 \approx\ & a_1\widetilde{\phi} w_1 + a_{10}\{w_5(w_3 w_7 - w_4 w_8) + w_6(w_3 w_8 + w_4 w_7)\} \\ & + a_{11}\{w_3(w_5 w_7 - w_6 w_8) + w_4(w_5 w_8 + w_6 w_7)\} \\ & + a_{12}\{w_1({w_7}^2 - {w_8}^2) + 2 w_2 w_7 w_8\} + a_{13}\{w_7({w_1}^2 - {w_2}^2) - 2 w_8 w_1 w_2\} \\ & + a_{14} w_7({w_7}^2 - 3{w_8}^2) + \widetilde{F}_1^C, \end{aligned} \tag{6.333}$$
$$\begin{aligned} \widetilde{F}_2 \approx\ & a_1\widetilde{\phi} w_2 + a_{10}\{w_5(w_3 w_8 + w_4 w_7) - w_6(w_3 w_7 - w_4 w_8)\} \\ & + a_{11}\{w_3(w_5 w_8 + w_6 w_7) - w_4(w_5 w_7 - w_6 w_8)\} \\ & + a_{12}\{-w_2({w_7}^2 - {w_8}^2) + 2 w_1 w_7 w_8\} + a_{13}\{-w_8({w_1}^2 - {w_2}^2) - 2 w_7 w_1 w_2\} \\ & + a_{14} w_8(-3{w_7}^2 + {w_8}^2) + \widetilde{F}_2^C, \end{aligned} \tag{6.334}$$
$$\begin{aligned} \widetilde{F}_3 \approx\ & a_1\widetilde{\phi} w_3 + a_{10}\{w_1(w_5 w_7 - w_6 w_8) + w_2(w_5 w_8 + w_6 w_7)\} \\ & + a_{11}\{w_7(w_1 w_5 - w_2 w_6) + w_8(w_1 w_6 + w_2 w_5)\} \\ & + a_{12}\{w_3({w_5}^2 - {w_6}^2) + 2 w_4 w_5 w_6\} + a_{13}\{w_5({w_3}^2 - {w_4}^2) - 2 w_6 w_3 w_4\} \\ & + a_{14} w_5({w_5}^2 - 3{w_6}^2) + \widetilde{F}_3^C, \end{aligned} \tag{6.335}$$
$$\widetilde{F}_4 \approx a_1\widetilde{\phi} w_4 + a_{10}\{w_1(w_5 w_8 + w_6 w_7) - w_2(w_5 w_7 - w_6 w_8)\}$$

$$
\begin{aligned}
&+ a_{11}\{w_7(w_1w_6 + w_2w_5) - w_8(w_1w_5 - w_2w_6)\} \\
&+ a_{12}\{-w_4({w_5}^2 - {w_6}^2) + 2w_3w_5w_6\} + a_{13}\{-w_6({w_3}^2 - {w_4}^2) - 2w_5w_3w_4\} \\
&+ a_{14}w_6(-3{w_5}^2 + {w_6}^2) + \widetilde{F}_4^C, \qquad (6.336) \\
\widetilde{F}_5 \approx\ & a_1\widetilde{\phi}w_5 + a_{10}\{w_1(w_3w_7 - w_4w_8) + w_2(w_3w_8 + w_4w_7)\} \\
&+ a_{11}\{w_7(w_1w_3 - w_2w_4) + w_8(w_1w_4 + w_2w_3)\} \\
&+ a_{12}\{w_5({w_3}^2 - {w_4}^2) + 2w_3w_4w_6\} + a_{13}\{w_3({w_5}^2 - {w_6}^2) - 2w_4w_5w_6\} \\
&+ a_{14}w_3({w_3}^2 - 3{w_4}^2) + \widetilde{F}_5^C, \qquad (6.337) \\
\widetilde{F}_6 \approx\ & a_1\widetilde{\phi}w_6 + a_{10}\{w_1(w_3w_8 + w_4w_7) - w_2(w_3w_7 - w_4w_8)\} \\
&+ a_{11}\{w_7(w_1w_4 + w_2w_3) - w_8(w_1w_3 - w_2w_4)\} \\
&+ a_{12}\{-w_6({w_3}^2 - {w_4}^2) + 2w_3w_4w_5\} + a_{13}\{-w_4({w_5}^2 - {w_6}^2) - 2w_3w_5w_6\} \\
&+ a_{14}w_4(-3{w_3}^2 + {w_4}^2) + \widetilde{F}_6^C, \qquad (6.338) \\
\widetilde{F}_7 \approx\ & a_1\widetilde{\phi}w_7 + a_{10}\{w_5(w_1w_3 - w_2w_4) + w_6(w_1w_4 + w_2w_3)\} \\
&+ a_{11}\{w_3(w_1w_5 - w_2w_6) + w_4(w_1w_6 + w_2w_5)\} \\
&+ a_{12}\{w_7({w_1}^2 - {w_2}^2) + 2w_8w_1w_2\} + a_{13}\{w_1({w_7}^2 - {w_8}^2) - 2w_2w_7w_8\} \\
&+ a_{14}w_1({w_1}^2 - 3{w_2}^2) + \widetilde{F}_7^C, \qquad (6.339) \\
\widetilde{F}_8 \approx\ & a_1\widetilde{\phi}w_8 + a_{10}\{w_5(w_1w_4 + w_2w_3) - w_6(w_1w_3 - w_2w_4)\} \\
&+ a_{11}\{w_3(w_1w_6 + w_2w_5) - w_4(w_1w_5 - w_2w_6)\} \\
&+ a_{12}\{-w_8({w_1}^2 - {w_2}^2) + 2w_7w_1w_2\} + a_{13}\{-w_2({w_7}^2 - {w_8}^2) - 2w_1w_7w_8\} \\
&+ a_{14}w_2(-3{w_1}^2 + {w_2}^2) + \widetilde{F}_8^C, \qquad (6.340)
\end{aligned}
$$

where $\widetilde{F}_i^C$ $(i = 1, \dots, 8)$ is given in (6.279) – (6.286). Hence, the asymptotic form of the Jacobian matrix in (6.180) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1\tilde{\phi}I_8 + a_{10}B_{10} + a_{11}B_{11} + a_{12}B_{12} + a_{13}B_{13} + a_{14}B_{14} + B_C, \qquad (6.341)$$

where B_C is given in (6.288) and

$$B_{10} = \begin{bmatrix} B_1^{10} & B_3^{10} \\ B_4^{10} & B_2^{10} \end{bmatrix}, \quad B_{11} = \begin{bmatrix} B_1^{11} & B_3^{11} \\ B_4^{11} & B_2^{11} \end{bmatrix}, \quad B_{12} = \begin{bmatrix} B_1^{12} & B_3^{12} \\ (B_3^{12})^\top & B_2^{12} \end{bmatrix}, \qquad (6.342)$$

$$B_{13} = \begin{bmatrix} B_1^{13} & B_3^{13} \\ B_4^{13} & B_2^{13} \end{bmatrix}, \quad B_{14} = \begin{bmatrix} O & B_1^{14} \\ B_2^{14} & O \end{bmatrix},$$

$$B_1^{10} = \begin{bmatrix} 0 & 0 & w_5w_7 + w_6w_8 & -w_5w_8 + w_6w_7 \\ 0 & 0 & w_5w_8 - w_6w_7 & w_5w_7 + w_6w_8 \\ w_5w_7 - w_6w_8 & w_5w_8 + w_6w_7 & 0 & 0 \\ w_5w_8 + w_6w_7 & -w_5w_7 + w_6w_8 & 0 & 0 \end{bmatrix},$$

$$B_2^{10} = \begin{bmatrix} 0 & 0 & w_1w_3 + w_2w_4 & -w_1w_4 + w_2w_3 \\ 0 & 0 & w_1w_4 - w_2w_3 & w_1w_3 + w_2w_4 \\ w_1w_3 - w_2w_4 & w_1w_4 + w_2w_3 & 0 & 0 \\ w_1w_4 + w_2w_3 & -w_1w_3 + w_2w_4 & 0 & 0 \end{bmatrix},$$

$$B_3^{10} = \begin{bmatrix} w_3w_7 - w_4w_8 & w_3w_8 + w_4w_7 & w_3w_5 + w_4w_6 & w_3w_6 - w_4w_5 \\ w_3w_8 + w_4w_7 & -w_3w_7 + w_4w_8 & -w_3w_6 + w_4w_5 & w_3w_5 + w_4w_6 \\ w_1w_7 + w_2w_8 & -w_1w_8 + w_2w_7 & w_1w_5 + w_2w_6 & -w_1w_6 + w_2w_5 \\ w_1w_8 - w_2w_7 & w_1w_7 + w_2w_8 & w_1w_6 - w_2w_5 & w_1w_5 + w_2w_6 \end{bmatrix},$$

$$B_4^{10} = \begin{bmatrix} w_3w_7 - w_4w_8 & w_3w_8 + w_4w_7 & w_1w_7 + w_2w_8 & -w_1w_8 + w_2w_7 \\ w_3w_8 + w_4w_7 & -w_3w_7 + w_4w_8 & w_1w_8 - w_2w_7 & w_1w_7 + w_2w_8 \\ w_3w_5 + w_4w_6 & w_3w_6 - w_4w_5 & w_1w_5 + w_2w_6 & w_1w_6 - w_2w_5 \\ -w_3w_6 + w_4w_5 & w_3w_5 + w_4w_6 & -w_1w_6 + w_2w_5 & w_1w_5 + w_2w_6 \end{bmatrix},$$

$$B_1^{11} = \begin{bmatrix} 0 & 0 & w_5w_7 - w_6w_8 & w_5w_8 + w_6w_7 \\ 0 & 0 & w_5w_8 + w_6w_7 & -w_5w_7 + w_6w_8 \\ w_5w_7 + w_6w_8 & w_5w_8 - w_6w_7 & 0 & 0 \\ -w_5w_8 + w_6w_7 & w_5w_7 + w_6w_8 & 0 & 0 \end{bmatrix},$$

$$B_2^{11} = \begin{bmatrix} 0 & 0 & w_1w_3 - w_2w_4 & w_1w_4 + w_2w_3 \\ 0 & 0 & w_1w_4 + w_2w_3 & -w_1w_3 + w_2w_4 \\ w_1w_3 + w_2w_4 & w_1w_4 - w_2w_3 & 0 & 0 \\ -w_1w_4 + w_2w_3 & w_1w_3 + w_2w_4 & 0 & 0 \end{bmatrix},$$

$$B_3^{11} = \begin{bmatrix} w_3w_7 + w_4w_8 & -w_3w_8 + w_4w_7 & w_3w_5 + w_4w_6 & -w_3w_6 + w_4w_5 \\ w_3w_8 - w_4w_7 & w_3w_7 + w_4w_8 & w_3w_6 - w_4w_5 & w_3w_5 + w_4w_6 \\ w_1w_7 + w_2w_8 & w_1w_8 - w_2w_7 & w_1w_5 - w_2w_6 & w_1w_6 + w_2w_5 \\ -w_1w_8 + w_2w_7 & w_1w_7 + w_2w_8 & w_1w_6 + w_2w_5 & -w_1w_5 + w_2w_6 \end{bmatrix},$$

$$B_4^{11} = \begin{bmatrix} w_3w_7 + w_4w_8 & w_3w_8 - w_4w_7 & w_1w_7 + w_2w_8 & w_1w_8 - w_2w_7 \\ -w_3w_8 + w_4w_7 & w_3w_7 + w_4w_8 & -w_1w_8 + w_2w_7 & w_1w_7 + w_2w_8 \\ w_3w_5 + w_4w_6 & -w_3w_6 + w_4w_5 & w_1w_5 - w_2w_6 & w_1w_6 + w_2w_5 \\ w_3w_6 - w_4w_5 & w_3w_5 + w_4w_6 & w_1w_6 + w_2w_5 & -w_1w_5 + w_2w_6 \end{bmatrix},$$

$$B_1^{12} = \begin{bmatrix} {w_7}^2 - {w_8}^2 & 2w_7w_8 & 0 & 0 \\ 2w_7w_8 & -{w_7}^2 + {w_8}^2 & 0 & 0 \\ 0 & 0 & {w_5}^2 - {w_6}^2 & 2w_5w_6 \\ 0 & 0 & 2w_5w_6 & -{w_5}^2 + {w_6}^2 \end{bmatrix},$$

$$
B_2^{12} = \begin{bmatrix} {w_3}^2 - {w_4}^2 & 2w_3w_4 & 0 & 0 \\ 2w_3w_4 & -{w_3}^2 + {w_4}^2 & 0 & 0 \\ 0 & 0 & {w_1}^2 - {w_2}^2 & 2w_1w_2 \\ 0 & 0 & 2w_1w_2 & -{w_1}^2 + {w_2}^2 \end{bmatrix},
$$

$$
B_3^{12} = 2\begin{bmatrix} 0 & 0 & w_1w_7 + w_2w_8 & -w_1w_8 + w_2w_7 \\ 0 & 0 & w_1w_8 - w_2w_7 & w_1w_7 + w_2w_8 \\ w_3w_5 + w_4w_6 & -w_3w_6 + w_4w_5 & 0 & 0 \\ w_3w_6 - w_4w_5 & w_3w_5 + w_4w_6 & 0 & 0 \end{bmatrix},
$$

$$
B_1^{13} = 2\begin{bmatrix} w_1w_7 - w_2w_8 & -w_1w_8 - w_2w_7 & 0 & 0 \\ -w_1w_8 - w_2w_7 & -w_1w_7 + w_2w_8 & 0 & 0 \\ 0 & 0 & w_3w_5 - w_4w_6 & -w_3w_6 - w_4w_5 \\ 0 & 0 & -w_3w_6 - w_4w_5 & -w_3w_5 + w_4w_6 \end{bmatrix},
$$

$$
B_2^{13} = 2\begin{bmatrix} w_3w_5 - w_4w_6 & -w_3w_6 - w_4w_5 & 0 & 0 \\ -w_3w_6 - w_4w_5 & -w_3w_5 + w_4w_6 & 0 & 0 \\ 0 & 0 & w_1w_7 - w_2w_8 & -w_1w_8 - w_2w_7 \\ 0 & 0 & -w_1w_8 - w_2w_7 & -w_1w_7 + w_2w_8 \end{bmatrix},
$$

$$
B_3^{13} = \begin{bmatrix} 0 & 0 & {w_1}^2 - {w_2}^2 & -2w_1w_2 \\ 0 & 0 & -2w_1w_2 & -{w_1}^2 + {w_2}^2 \\ {w_3}^2 - {w_4}^2 & -2w_3w_4 & 0 & 0 \\ -2w_3w_4 & -{w_3}^2 + {w_4}^2 & 0 & 0 \end{bmatrix},
$$

$$
B_4^{13} = \begin{bmatrix} 0 & 0 & {w_5}^2 - {w_6}^2 & -2w_5w_6 \\ 0 & 0 & -2w_5w_6 & -{w_5}^2 + {w_6}^2 \\ {w_7}^2 - {w_8}^2 & -2w_7w_8 & 0 & 0 \\ -2w_7w_8 & -{w_7}^2 + {w_8}^2 & 0 & 0 \end{bmatrix},
$$

$$
B_1^{14} = 3\begin{bmatrix} 0 & 0 & {w_7}^2 - {w_8}^2 & -2w_7w_8 \\ 0 & 0 & -2w_7w_8 & -{w_7}^2 + {w_8}^2 \\ {w_5}^2 - {w_6}^2 & -2w_5w_6 & 0 & 0 \\ -2w_5w_6 & -{w_5}^2 + {w_6}^2 & 0 & 0 \end{bmatrix},
$$

$$
B_2^{14} = 3\begin{bmatrix} 0 & 0 & {w_3}^2 - {w_4}^2 & -2w_3w_4 \\ 0 & 0 & -2w_3w_4 & -{w_3}^2 + {w_4}^2 \\ {w_1}^2 - {w_2}^2 & -2w_1w_2 & 0 & 0 \\ -2w_1w_2 & -{w_1}^2 + {w_2}^2 & 0 & 0 \end{bmatrix}.
$$

Substituting $\boldsymbol{w}_{\text{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$ into (6.333) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqVM}} \approx -\frac{a_2 + a_3 + a_4 + a_5 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14}}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.341) at $(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}}) \approx w^2 \begin{bmatrix} C_{12} & C_{13} \\ C_{13} & C_{12} \end{bmatrix} + \widetilde{J_C}^{\text{sqVM}}, \tag{6.343}$$

where $\widetilde{J_C}^{\text{sqVM}}$ is given in (6.302) and

$$C_{12} = \begin{bmatrix} c_1 & 0 & c_3 & 0 \\ 0 & c_2 & 0 & c_4 \\ c_3 & 0 & c_1 & 0 \\ 0 & -c_4 & 0 & c_2 \end{bmatrix}, \quad C_{13} = \begin{bmatrix} c_3 & 0 & c_5 & 0 \\ 0 & -c_4 & 0 & c_6 \\ c_5 & 0 & c_3 & 0 \\ 0 & c_6 & 0 & c_4 \end{bmatrix},$$

$$\begin{aligned}
&c_1 = -a_{10} - a_{11} + a_{13} - a_{14}, \quad c_2 = -a_{10} - a_{11} - 2a_{12} - 3a_{13} - a_{14}, \\
&c_3 = a_{10} + a_{11}, \quad c_4 = a_{10} - a_{11}, \\
&c_5 = a_{10} + a_{11} + 2a_{12} + a_{13} + 3a_{14}, \quad c_6 = a_{10} + a_{11} + 2a_{12} - a_{13} - 3a_{14}.
\end{aligned}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$ are given by

$$\begin{aligned}
&\lambda_1, \lambda_2 \approx \{(c_1 + c_3) \pm (c_5 + c_6)\} w^2, \\
&\lambda_3, \lambda_4 \approx \{(c_1 - c_3) \pm (c_5 - c_6)\} w^2, \\
&\lambda_5, \lambda_6 \approx (c_2 \pm c_7) w^2 \quad \text{(repeated twice)},
\end{aligned}$$

which are rewritten as

$$\begin{aligned}
&\lambda_1 \approx 2(a_2 + a_3 + a_4 + a_5 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14}) w^2, \\
&\lambda_2 \approx 2(a_2 + a_3 - a_4 - a_5 - a_{10} - a_{11} - a_{12} - 2a_{14}) w^2, \\
&\lambda_3 \approx 2(a_2 - a_3 + a_4 - a_5 - a_{10} - a_{11} - a_{12} - 2a_{14}) w^2, \\
&\lambda_4 \approx 2(a_2 - a_3 - a_4 + a_5 - a_{10} - a_{11} + a_{12} + a_{13} + a_{14}) w^2, \\
&\lambda_5 \approx -2(a_{10} + a_{11} + 2a_{12} + a_{13} - a_{14}) w^2 \quad \text{(repeated twice)}, \\
&\lambda_6 \approx -4(a_{13} + a_{14}) w^2 \quad \text{(repeated twice)}.
\end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions:

$$a_2 + a_3 + a_4 + a_5 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} < 0, \tag{6.344}$$

$$a_2 + a_3 - a_4 - a_5 - a_{10} - a_{11} - a_{12} - 2a_{14} < 0, \tag{6.345}$$

$$a_2 - a_3 + a_4 - a_5 - a_{10} - a_{11} - a_{12} - 2a_{14} < 0, \tag{6.346}$$

$$a_2 - a_3 - a_4 + a_5 - a_{10} - a_{11} + a_{12} + a_{13} + a_{14} < 0, \tag{6.347}$$

$$a_{10} + a_{11} + 2a_{12} + a_{13} - a_{14} > 0, \tag{6.348}$$

$$a_{13} + a_{14} > 0. \tag{6.349}$$

Thus, the stability of $\boldsymbol{w}_{\text{sqVM}}$ depends on the values of $a_2, \ldots, a_5$ and $a_{10}, \ldots, a_{14}$.

Remark 6.4. For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (8, 3, 1)$, we have the following statements:

- The solutions $\boldsymbol{w}_{\text{stripeI}}$ and $\boldsymbol{w}_{\text{stripeII}}$ do not exist. See Proposition 6.10 in Section 6.5.3. In fact, $\hat{k}^2 - \hat{\ell} = 8$. This is divisible by $\hat{n} = 8$. Hence, the condition (6.230) is not satisfied.

- The solutions $\boldsymbol{w}_{\text{upside-downI}}$ and $\boldsymbol{w}_{\text{upside-downII}}$ do not exist. See Proposition 6.12 in Section 6.5.4. In fact, $\gcd(\hat{k}^2 + \hat{\ell}, \hat{k}^2 - \hat{\ell}) = 2\gcd(10, 8) = 2$. This is divisible by $\gcd(\hat{n}, 2\hat{k}\hat{\ell}) = \gcd(8, 6) = 2$. Hence, the condition (6.234) is not satisfied.

- The solution $\boldsymbol{w}_{\text{sqT}}$ does not exist. See Proposition 5.28 in Section 5.6.7. This case corresponds to the case $(\hat{n}, \hat{k}, \hat{\ell}) = (2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$. In fact, $\underline{2\gcd(\hat{k}, \hat{\ell})} = 2\gcd(3, 1) = 2$. This is divisible by $\gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}) = \gcd(10, 8) = 2$. Hence, $\overline{\textbf{GCD-div}}$ in (5.97) is not satisfied.

□

Case 4: $(\hat{n}, \hat{k}, \hat{\ell}) = (10, 3, 1)$

For the case of $(\hat{n}, \hat{k}, \hat{\ell}) = (10, 3, 1)$, we have

$$(0,1,0,0,2,0,0,0),\ (0,0,0,0,0,3,0,0) \in P$$

as well as

$$\begin{gathered}(1,0,0,0,0,0,0,0),\ (2,0,0,0,1,0,0,0),\ (1,1,0,0,0,1,0,0),\\ (1,0,1,0,0,0,1,0),\ (1,0,0,1,0,0,0,1) \in P\end{gathered}$$

in (6.238). Then, the asymptotic form of F_i $(i = 1, \dots, 4)$ in (6.271) becomes

$$F_1 \approx a_1\widetilde{\phi}z_1 + a_{15}z_2\overline{z}_1{}^2 + a_{16}\overline{z}_2{}^3 + F_1^C, \tag{6.350}$$
$$F_2 \approx a_1\widetilde{\phi}z_2 + a_{15}\overline{z}_1\overline{z}_2{}^2 + a_{16}z_1{}^3 + F_2^C, \tag{6.351}$$
$$F_3 \approx a_1\widetilde{\phi}z_3 + a_{15}z_4\overline{z}_3{}^2 + a_{16}\overline{z}_4{}^3 + F_3^C, \tag{6.352}$$
$$F_4 \approx a_1\widetilde{\phi}z_4 + a_{15}\overline{z}_3\overline{z}_4{}^2 + a_{16}z_3{}^3 + F_4^C \tag{6.353}$$

with

$$a_{15} = A_{01002000}(0), \quad a_{16} = A_{00000300}(0),$$

where F_i^C $(i = 1, \dots, 4)$ is given in (6.272) – (6.275). Then, the asymptotic form of $\widetilde{F}_i$ $(i = 1, \dots, 8)$ in (6.182) – (6.185) becomes

$$\widetilde{F}_1 \approx a_1\widetilde{\phi}w_1 + a_{15}\{w_3(w_1{}^2 - w_2{}^2) + 2w_4w_1w_2\} + a_{16}w_3(w_3{}^2 - 3w_4{}^2) + \widetilde{F}_1^C, \tag{6.354}$$
$$\widetilde{F}_2 \approx a_1\widetilde{\phi}w_2 + a_{15}\{w_4(w_1{}^2 - w_2{}^2) - 2w_3w_1w_2\} + a_{16}w_4(-3w_3{}^2 + w_4{}^2) + \widetilde{F}_2^C, \tag{6.355}$$
$$\widetilde{F}_3 \approx a_1\widetilde{\phi}w_3 + a_{15}\{w_1(w_3{}^2 - w_4{}^2) - 2w_2w_3w_4\} + a_{16}w_1(w_1{}^2 - 3w_2{}^2) + \widetilde{F}_3^C, \tag{6.356}$$
$$\widetilde{F}_4 \approx a_1\widetilde{\phi}w_4 + a_{15}\{-w_2(w_3{}^2 - w_4{}^2) - 2w_1w_3w_4\} + a_{16}w_2(3w_1{}^2 - w_2{}^2) + \widetilde{F}_4^C, \tag{6.357}$$
$$\widetilde{F}_5 \approx a_1\widetilde{\phi}w_5 + a_{15}\{w_7(w_5{}^2 - w_6{}^2) + 2w_8w_5w_6\} + a_{16}w_7(w_7{}^2 - 3w_8{}^2) + \widetilde{F}_5^C, \tag{6.358}$$
$$\widetilde{F}_6 \approx a_1\widetilde{\phi}w_6 + a_{15}\{w_8(w_5{}^2 - w_6{}^2) - 2w_7w_5w_6\} + a_{16}w_8(-3w_7{}^2 + w_8{}^2) + \widetilde{F}_6^C, \tag{6.359}$$

$$\widetilde{F}_7 \approx a_1\widetilde{\phi}w_7 + a_{15}\{w_5({w_7}^2 - {w_8}^2) - 2w_6w_7w_8\} + a_{16}w_5({w_5}^2 - 3{w_6}^2) + \widetilde{F}_7^C, \tag{6.360}$$
$$\widetilde{F}_8 \approx a_1\widetilde{\phi}w_8 + a_{15}\{-w_6({w_7}^2 - {w_8}^2) - 2w_5w_7w_8\} + a_{16}w_6(3{w_5}^2 - {w_6}^2) + \widetilde{F}_8^C, \tag{6.361}$$

where $\widetilde{F}_i^C$ $(i = 1, \ldots, 8)$ is given in (6.279) – (6.286). Hence, the asymptotic form of the Jacobian matrix in (6.180) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1\tilde{\phi}I_8 + a_{15}B_{15} + a_{16}B_{16} + B_C, \tag{6.362}$$

where B_C is given in (6.288) and

$$B_{15} = \begin{bmatrix} B_1^{15} & O \\ O & B_2^{15} \end{bmatrix}, \quad B_{16} = \begin{bmatrix} B_1^{16} & O \\ O & B_2^{16} \end{bmatrix},$$

$$B_1^{15} = \begin{bmatrix} 2(w_1w_3 + w_2w_4) & 2(w_1w_4 - w_2w_3) & {w_1}^2 - {w_2}^2 & 2w_1w_2 \\ 2(w_1w_4 - w_2w_3) & 2(-w_1w_3 - w_2w_4) & -2w_1w_2 & {w_1}^2 - {w_2}^2 \\ {w_3}^2 - {w_4}^2 & -2w_3w_4 & 2(w_1w_3 - w_2w_4) & 2(-w_1w_4 - w_2w_3) \\ -2w_3w_4 & -{w_3}^2 + {w_4}^2 & 2(-w_1w_4 - w_2w_3) & 2(-w_1w_3 + w_2w_4) \end{bmatrix},$$

$$B_2^{15} = \begin{bmatrix} 2(w_5w_7 + w_6w_8) & 2(w_5w_8 - w_6w_7) & {w_5}^2 - {w_6}^2 & 2w_5w_6 \\ 2(w_5w_8 - w_6w_7) & 2(-w_5w_7 - w_6w_8) & -2w_5w_6 & {w_5}^2 - {w_6}^2 \\ {w_7}^2 - {w_8}^2 & -2w_7w_8 & 2(w_5w_7 - w_6w_8) & 2(-w_5w_8 - w_6w_7) \\ -2w_7w_8 & -{w_7}^2 + {w_8}^2 & 2(-w_5w_8 - w_6w_7) & 2(-w_5w_7 + w_6w_8) \end{bmatrix},$$

$$B_1^{16} = 3\begin{bmatrix} 0 & 0 & {w_3}^2 - {w_4}^2 & -2w_3w_4 \\ 0 & 0 & -2w_3w_4 & -{w_3}^2 + {w_4}^2 \\ {w_1}^2 - {w_2}^2 & -2w_1w_2 & 0 & 0 \\ 2w_1w_2 & {w_1}^2 - {w_2}^2 & 0 & 0 \end{bmatrix},$$

$$B_2^{16} = 3\begin{bmatrix} 0 & 0 & {w_7}^2 - {w_8}^2 & -2w_7w_8 \\ 0 & 0 & -2w_7w_8 & -{w_7}^2 + {w_8}^2 \\ {w_5}^2 - {w_6}^2 & -2w_5w_6 & 0 & 0 \\ 2w_5w_6 & {w_5}^2 - {w_6}^2 & 0 & 0 \end{bmatrix}.$$

Substituting $\boldsymbol{w}_{\mathrm{sqT}} = (w, 0, w, 0, 0, 0, 0, 0)$ into (6.354) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{sqT}} \approx -\frac{a_2 + a_3 + a_{15} + a_{16}}{a_1}w^2.$$

Evaluating the Jacobian matrix (6.362) at $(\boldsymbol{w}_{\mathrm{sqT}}, \widetilde{\phi}_{\mathrm{sqT}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{sqT}}, \widetilde{\phi}_{\mathrm{sqT}}) \approx w^2 \begin{bmatrix} C_{14} & O \\ O & C_{15} \end{bmatrix} + \widetilde{J_C}^{\mathrm{sqT}}, \tag{6.363}$$

where $\widetilde{J_C}^{\text{sqT}}$ is given in (6.289) and

$$C_{14} = \begin{bmatrix} a_{15} - a_{16} & 0 & a_{15} + 3a_{16} & 0 \\ 0 & -3a_{15} - a_{16} & 0 & a_{15} - 3a_{16} \\ a_{15} + 3a_{16} & 0 & a_{15} - a_{16} & 0 \\ 0 & -a_{15} + 3a_{16} & 0 & -3a_{15} - a_{16} \end{bmatrix}, \tag{6.364}$$

$$C_{15} = -(a_{15} + a_{16})I_4. \tag{6.365}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sqT}}, \widetilde{\phi}_{\text{sqT}})$ are given by

$$\begin{aligned}
&\lambda_1 \approx 2(a_2 + a_3 + a_{15} + a_{16})w^2, \\
&\lambda_2 \approx 2(a_2 - a_3 - 2a_{16})w^2, \\
&\lambda_3, \lambda_4 \approx -\{3a_{15} + a_{16} \pm \mathrm{i}(a_{15} - 3a_{16})\}w^2, \\
&\lambda_5 \approx -(a_2 + a_3 - a_4 - a_5 + a_{15} + a_{16})w^2 \quad \text{(repeated 4 times)}.
\end{aligned}$$

Assuming that all eigenvalues have negative real parts, we have the following stability conditions:

$$a_2 + a_3 + a_{15} + a_{16} < 0, \tag{6.366}$$

$$a_2 - a_3 - 2a_{16} < 0, \tag{6.367}$$

$$3a_{15} + a_{16} > 0, \tag{6.368}$$

$$a_2 + a_3 - a_4 - a_5 + a_{15} + a_{16} < 0. \tag{6.369}$$

Thus, the stability of $\boldsymbol{w}_{\text{sqT}}$ depends on the values of $a_2, \ldots, a_5, a_{15}$ and a_{16}.

Substituting $\boldsymbol{w}_{\text{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$ into (6.354) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqVM}} \approx -\frac{a_2 + a_3 + a_4 + a_5 + a_{15} + a_{16}}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.362) at $(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}}) \approx w^2 \begin{bmatrix} C_{14} & O \\ O & C_{14} \end{bmatrix} + \widetilde{J_C}^{\text{sqVM}}, \tag{6.370}$$

where C_{14} is given in (6.364), and $\widetilde{J_C}^{\text{sqVM}}$ is given in (6.302). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$ are given by

$$\begin{aligned}
&\lambda_1 \approx 2(a_2 + a_3 + a_4 + a_5 + a_{15} + a_{16})w^2, \\
&\lambda_2 \approx 2(a_2 + a_3 - a_4 - a_5 - a_{15} - a_{16})w^2, \\
&\lambda_3 \approx 2(a_2 - a_3 + a_4 - a_5 - 2a_{16})w^2, \\
&\lambda_4 \approx 2(a_2 - a_3 - a_4 + a_5 - 2a_{16})w^2, \\
&\lambda_5, \lambda_6 \approx -\{3a_{15} + a_{16} \pm \mathrm{i}(a_{15} - 3a_{16})\}w^2 \quad \text{(repeated twice)}.
\end{aligned}$$

Assuming that all eigenvalues have negative real parts, we have the following stability conditions:

$$a_2 + a_3 + a_4 + a_5 + a_{15} + a_{16} < 0, \tag{6.371}$$
$$a_2 + a_3 - a_4 - a_5 - a_{15} - a_{16} < 0, \tag{6.372}$$
$$a_2 - a_3 + a_4 - a_5 - 2a_{16} < 0, \tag{6.373}$$
$$a_2 - a_3 - a_4 + a_5 - 2a_{16} < 0, \tag{6.374}$$
$$3a_{15} + a_{16} > 0. \tag{6.375}$$

Thus, the stability of $\boldsymbol{w}_{\mathrm{sqVM}}$ depends on the values of $a_2, \dots, a_5, a_{15}$ and a_{16}.

Remark 6.5. For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (10, 3, 1)$, we have the following statements:

- The solutions $\boldsymbol{w}_{\mathrm{stripeI}}$ and $\boldsymbol{w}_{\mathrm{stripeII}}$ do not exist. See Proposition 6.10 in Section 6.5.3. In fact, $\hat{k}^2 + \hat{\ell} = 10$. This is divisible by $\hat{n} = 10$. Hence, the condition (6.230) is not satisfied.

- The solutions $\boldsymbol{w}_{\mathrm{upside\text{-}downI}}$ and $\boldsymbol{w}_{\mathrm{upside\text{-}downII}}$ do not exist. See Proposition 6.12 in Section 6.5.4. In fact, $\gcd(\hat{k}^2 + \hat{\ell}, \hat{k}^2 - \hat{\ell}) = 2\gcd(10, 8) = 2$. This is divisible by $\gcd(\hat{n}, 2\hat{k}\hat{\ell}) = \gcd(10, 6) = 2$. Hence, the condition (6.234) is not satisfied.

□

Case 5: $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{k}, \hat{k}, \hat{\ell})$

For the case of $(\hat{n}, \hat{k}, \hat{\ell})$ with $\hat{n} = 4\hat{k}$, we have

$$(0, 0, 0, 0, 1, 0, 2, 0) \in P$$

as well as

$$(1, 0, 0, 0, 0, 0, 0, 0),\ (2, 0, 0, 0, 1, 0, 0, 0),\ (1, 1, 0, 0, 0, 1, 0, 0),$$
$$(1, 0, 1, 0, 0, 0, 1, 0),\ (1, 0, 0, 1, 0, 0, 0, 1) \in P$$

in (6.238). Then, the asymptotic form of F_i $(i = 1, \dots, 4)$ in (6.271) becomes

$$F_1 \approx a_1\widetilde{\phi}z_1 + a_{17}\bar{z}_1\bar{z}_3{}^2 + F_1^C, \tag{6.376}$$
$$F_2 \approx a_1\widetilde{\phi}z_2 + a_{17}\bar{z}_2 z_4{}^2 + F_2^C, \tag{6.377}$$
$$F_3 \approx a_1\widetilde{\phi}z_3 + a_{17}\bar{z}_3\bar{z}_1{}^2 + F_3^C, \tag{6.378}$$
$$F_4 \approx a_1\widetilde{\phi}z_4 + a_{17}\bar{z}_4 z_2{}^2 + F_4^C \tag{6.379}$$

with

$$a_{17} = A_{00001020}(0),$$

where F_i^C $(i = 1, \dots, 4)$ is given in (6.272) – (6.275). Then, the asymptotic form of $\widetilde{F}_i$ $(i = 1, \dots, 8)$ in (6.182) – (6.185) becomes

$$\widetilde{F}_1 \approx a_1\widetilde{\phi}w_1 + a_{17}\{\ \ w_1(w_5{}^2 - w_6{}^2) - 2w_2w_5w_6)\} + \widetilde{F}_1^C, \tag{6.380}$$

$$
\begin{aligned}
\widetilde{F}_2 &\approx a_1\widetilde{\phi}w_2 + a_{17}\{-w_2({w_5}^2 - {w_6}^2) - 2w_1w_5w_6\} + \widetilde{F}_2^C, && (6.381)\\
\widetilde{F}_3 &\approx a_1\widetilde{\phi}w_3 + a_{17}\{\ \ w_3({w_7}^2 - {w_8}^2) + 2w_4w_7w_8\} + \widetilde{F}_3^C, && (6.382)\\
\widetilde{F}_4 &\approx a_1\widetilde{\phi}w_4 + a_{17}\{-w_4({w_7}^2 - {w_8}^2) + 2w_3w_7w_8\} + \widetilde{F}_4^C, && (6.383)\\
\widetilde{F}_5 &\approx a_1\widetilde{\phi}w_5 + a_{17}\{\ \ w_5({w_1}^2 - {w_2}^2) - 2w_6w_1w_2\} + \widetilde{F}_5^C, && (6.384)\\
\widetilde{F}_6 &\approx a_1\widetilde{\phi}w_6 + a_{17}\{-w_6({w_1}^2 - {w_2}^2) - 2w_5w_1w_2\} + \widetilde{F}_6^C, && (6.385)\\
\widetilde{F}_7 &\approx a_1\widetilde{\phi}w_7 + a_{17}\{\ \ w_7({w_3}^2 - {w_4}^2) + 2w_8w_3w_4\} + \widetilde{F}_7^C, && (6.386)\\
\widetilde{F}_8 &\approx a_1\widetilde{\phi}w_8 + a_{17}\{-w_8({w_3}^2 - {w_4}^2) + 2w_7w_3w_4\} + \widetilde{F}_8^C, && (6.387)
\end{aligned}
$$

where $\widetilde{F}_i^C$ $(i = 1, \ldots, 8)$ is given in (6.279) – (6.286). Hence, the asymptotic form of the Jacobian matrix in (6.180) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1\tilde{\phi}I_8 + a_{17}B_{17} + B_C, \tag{6.388}$$

where B_C is given in (6.288) and

$$B_{17} = \begin{bmatrix} B_1^{17} & B_3^{17} \\ (B_3^{17})^\top & B_2^{17} \end{bmatrix},$$

$$B_1^{17} = \begin{bmatrix} {w_5}^2 - {w_6}^2 & -2w_5w_6 & 0 & 0 \\ -2w_5w_6 & -{w_5}^2 + {w_6}^2 & 0 & 0 \\ 0 & 0 & {w_7}^2 - {w_8}^2 & 2w_7w_8 \\ 0 & 0 & 2w_7w_8 & -{w_7}^2 + {w_8}^2 \end{bmatrix},$$

$$B_2^{17} = \begin{bmatrix} {w_1}^2 - {w_2}^2 & -2w_1w_2 & 0 & 0 \\ -2w_1w_2 & -{w_1}^2 + {w_2}^2 & 0 & 0 \\ 0 & 0 & {w_3}^2 - {w_4}^2 & 2w_3w_4 \\ 0 & 0 & 2w_3w_4 & -{w_3}^2 + {w_4}^2 \end{bmatrix},$$

$$B_3^{17} = 2\begin{bmatrix} w_1w_5 - w_2w_6 & -w_1w_6 - w_2w_5 & 0 & 0 \\ -w_1w_6 - w_2w_5 & -w_1w_5 + w_2w_6 & 0 & 0 \\ 0 & 0 & w_3w_7 + w_4w_8 & -w_3w_8 + w_4w_7 \\ 0 & 0 & w_3w_8 - w_4w_7 & w_3w_7 + w_4w_8 \end{bmatrix}.$$

Substituting $\boldsymbol{w}_{\text{stripeI}} = (w, 0, 0, 0, 0, 0, 0, 0)$ into (6.380) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeI}} \approx -\frac{a_2}{a_1}w^2.$$

Evaluating the Jacobian matrix (6.388) at $(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}}) \approx w^2\begin{bmatrix} O & O \\ O & C_{16} \end{bmatrix} + \widetilde{J}_C^{\text{stripeI}}, \tag{6.389}$$

where $\widetilde{J_C}^{\text{stripeI}}$ is given in (6.289) and

$$C_{16} = \begin{bmatrix} a_{17} & 0 & 0 & 0 \\ 0 & -a_{17} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$ are given by

$$\begin{aligned}
&\lambda_1 \approx 2a_2 w^2, \\
&\lambda_2 \approx O(w^3), \\
&\lambda_3, \lambda_4 \approx -(a_2 - a_4 \pm a_{17})w^2, \\
&\lambda_5 \approx -(a_2 - a_3)w^2 \quad \text{(repeated twice)}, \\
&\lambda_6 \approx -(a_2 - a_5)w^2 \quad \text{(repeated twice)}.
\end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$\begin{aligned}
&a_2 < 0, \\
&a_2 - a_4 \pm a_{17} > 0, \\
&a_2 - a_3 > 0, \\
&a_2 - a_5 > 0.
\end{aligned}$$

These are equivalen to

$$\max(a_3, a_4 + |a_{17}|, a_5) < a_2 < 0. \tag{6.390}$$

Thus, the stability of $\boldsymbol{w}_{\text{stripeI}}$ depends on the values of $a_2, \ldots, a_5$ and a_{17}.

Substituting $\boldsymbol{w}_{\text{stripeII}} = (0, w, 0, 0, 0, 0, 0, 0)$ into (6.380) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeII}} \approx -\frac{a_2}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.388) at $(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}}) \approx w^2 \begin{bmatrix} O & O \\ O & -C_{16} \end{bmatrix} + \widetilde{J_C}^{\text{stripeII}}, \tag{6.391}$$

where C_{16} is given in (6.5.5), and $\widetilde{J_C}^{\text{stripeII}}$ is given in (6.292). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{stripeII}}$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$.

Substituting $\boldsymbol{w}_{\text{upside-downI}} = (w, 0, 0, 0, w, 0, 0, 0)$ into (6.278) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{upside-downI}} \approx -\frac{a_2 + a_4 + a_{17}}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.287) at $(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}}) \approx w^2 \begin{bmatrix} C_{17} & C_{18} \\ C_{18} & C_{17} \end{bmatrix} + \widetilde{J_C}^{\text{upside-downI}} \tag{6.392}$$

with

$$C_{17} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2a_{17} & 0 & 0 \\ 0 & 0 & -a_{17} & 0 \\ 0 & 0 & 0 & -a_{17} \end{bmatrix}, \quad C_{18} = \begin{bmatrix} 2a_{17} & 0 & 0 & 0 \\ 0 & -2a_{17} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.393}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$ are given by

$$\begin{aligned} &\lambda_1, \lambda_2 \approx 2a_2 \pm (a_4 + a_{17})w^2, \\ &\lambda_3 \approx -4a_{17}w^2, \\ &\lambda_4 \approx O(w^3), \\ &\lambda_5 \approx -(a_2 - a_3 + a_4 - a_5 + a_{17})w^2 \quad \text{(repeated 4 times).} \end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$\begin{aligned} &a_2 < -|a_4 + a_{17}|, \\ &a_{17} > 0, \\ &a_2 - a_3 + a_4 - a_5 + a_{17} > 0. \end{aligned}$$

These conditions are equivalent to

$$\begin{aligned} &a_3 - a_4 + a_5 - a_{17} < a_2 < -|a_4 + a_{17}| \\ &a_4 > 0. \end{aligned}$$

Thus, the stability of $\boldsymbol{w}_{\text{upside-downI}}$ is conditional and depends on the values of $a_2, \ldots, a_5$ and a_{17}.

Substituting $\boldsymbol{w}_{\text{upside-downII}} = (0, w, 0, 0, 0, w, 0, 0)$ into (6.278) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{upside-downII}} \approx -\frac{a_2 + a_4 + a_{17}}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.287) at $(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}}) = \widetilde{J_C}^{\text{upside-downII}} \approx w^2 \begin{bmatrix} C_{19} & -C_{18} \\ -C_{18} & C_{19} \end{bmatrix} \tag{6.394}$$

with

$$C_{19} = \begin{bmatrix} -2a_{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_{17} & 0 \\ 0 & 0 & 0 & -a_{17} \end{bmatrix}, \tag{6.395}$$

where C_{18} is given in (6.393). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}})$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{upside-downII}}$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$.

Substituting $\boldsymbol{w}_{\text{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$ into (6.380) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqVM}} \approx -\frac{a_2 + a_3 + a_4 + a_5 + a_{17}}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.388) at $(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}}) \approx w^2 \begin{bmatrix} C_{20} & C_{21} \\ C_{21} & C_{20} \end{bmatrix} + \widetilde{J_C}^{\text{sqVM}}, \tag{6.396}$$

where $\widetilde{J_C}^{\text{sqVM}}$ is given in (6.302) and

$$C_{20} = 2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -a_{17} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_{17} \end{bmatrix}, \quad C_{21} = 2 \begin{bmatrix} a_{17} & 0 & 0 & 0 \\ 0 & -a_{17} & 0 & 0 \\ 0 & 0 & a_{17} & 0 \\ 0 & 0 & 0 & a_{17} \end{bmatrix}.$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$ are given by

$$\begin{aligned}
\lambda_1 &\approx 2(a_2 + a_3 + a_4 + a_5 + a_{17})w^2, \\
\lambda_2 &\approx 2(a_2 + a_3 - a_4 - a_5 - a_{17})w^2, \\
\lambda_3 &\approx 2(a_2 - a_3 + a_4 - a_5 + a_{17})w^2, \\
\lambda_4 &\approx 2(a_2 - a_3 - a_4 + a_5 - a_{17})w^2, \\
\lambda_5 &\approx -4a_{17}w^2, \quad \text{(repeated twice)} \\
\lambda_6 &\approx O(w^3) \quad \text{(repeated twice)}.
\end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 + a_3 + a_4 + a_5 + a_{17} < 0, \tag{6.397}$$
$$a_2 + a_3 - a_4 - a_5 - a_{17} < 0, \tag{6.398}$$
$$a_2 - a_3 + a_4 - a_5 + a_{17} < 0, \tag{6.399}$$
$$a_2 - a_3 - a_4 + a_5 - a_{17} < 0, \tag{6.400}$$
$$a_{17} > 0. \tag{6.401}$$

Thus, the stability of $\boldsymbol{w}_{\text{sqVM}}$ depends on the values of $a_2, \ldots, a_5$ and a_{17}.

Remark 6.6. For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{k}, \hat{k}, \hat{\ell})$, $\boldsymbol{w}_{\text{sqT}}$ does not exist. See Proposition 5.28 in Section 5.6.7. □

Case 6: $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{\ell}, \hat{k}, \hat{\ell})$

For the case of $(\hat{n}, \hat{k}, \hat{\ell})$ with $\hat{n} = 4\hat{\ell}$, we have

$$(0, 0, 2, 0, 1, 0, 0, 0) \in P$$

as well as

$$\begin{aligned}&(1, 0, 0, 0, 0, 0, 0, 0),\ (2, 0, 0, 0, 1, 0, 0, 0),\ (1, 1, 0, 0, 0, 1, 0, 0),\\ &(1, 0, 1, 0, 0, 0, 1, 0),\ (1, 0, 0, 1, 0, 0, 0, 1) \in P\end{aligned}$$

in (6.238). Then, the asymptotic form of F_i $(i = 1, \ldots, 4)$ in (6.271) becomes

$$F_1 \approx a_1\widetilde{\phi}z_1 + a_{18}{z_3}^2\overline{z}_1 + F_1^C, \tag{6.402}$$
$$F_2 \approx a_1\widetilde{\phi}z_2 + a_{18}{\overline{z}_4}^2\overline{z}_2 + F_2^C, \tag{6.403}$$
$$F_3 \approx a_1\widetilde{\phi}z_3 + a_{18}{z_1}^2\overline{z}_3 + F_3^C, \tag{6.404}$$
$$F_4 \approx a_1\widetilde{\phi}z_4 + a_{18}{\overline{z}_2}^2\overline{z}_4 + F_4^C \tag{6.405}$$

with

$$a_{18} = A_{00201000}(0).$$

where F_i^C $(i = 1, \ldots, 4)$ is given in (6.272) – (6.275). Then, the asymptotic form of $\widetilde{F}_i$ $(i = 1, \ldots, 8)$ in (6.182) – (6.185) becomes

$$\widetilde{F}_1 \approx a_1\widetilde{\phi}w_1 + a_{18}\{\ \ w_1({w_5}^2 - {w_6}^2) + 2w_2w_5w_6)\} + \widetilde{F}_1^C, \tag{6.406}$$
$$\widetilde{F}_2 \approx a_1\widetilde{\phi}w_2 + a_{18}\{-w_2({w_5}^2 - {w_6}^2) + 2w_1w_5w_6)\} + \widetilde{F}_2^C, \tag{6.407}$$
$$\widetilde{F}_3 \approx a_1\widetilde{\phi}w_3 + a_{18}\{\ \ w_3({w_7}^2 - {w_8}^2) - 2w_4w_7w_8)\} + \widetilde{F}_3^C, \tag{6.408}$$
$$\widetilde{F}_4 \approx a_1\widetilde{\phi}w_4 + a_{18}\{-w_4({w_7}^2 - {w_8}^2) - 2w_3w_7w_8)\} + \widetilde{F}_4^C, \tag{6.409}$$
$$\widetilde{F}_5 \approx a_1\widetilde{\phi}w_5 + a_{18}\{\ \ w_5({w_1}^2 - {w_2}^2) + 2w_6w_1w_2)\} + \widetilde{F}_5^C, \tag{6.410}$$
$$\widetilde{F}_6 \approx a_1\widetilde{\phi}w_6 + a_{18}\{-w_6({w_1}^2 - {w_2}^2) + 2w_5w_1w_2)\} + \widetilde{F}_6^C, \tag{6.411}$$
$$\widetilde{F}_7 \approx a_1\widetilde{\phi}w_7 + a_{18}\{\ \ w_7({w_3}^2 - {w_4}^2) - 2w_8w_3w_4)\} + \widetilde{F}_7^C, \tag{6.412}$$
$$\widetilde{F}_8 \approx a_1\widetilde{\phi}w_8 + a_{18}\{-w_8({w_3}^2 - {w_4}^2) - 2w_7w_3w_4)\} + \widetilde{F}_8^C, \tag{6.413}$$

where $\widetilde{F}_i^C$ $(i = 1, \ldots, 8)$ is given in (6.279) – (6.286). Hence, the asymptotic form of the Jacobian matrix in (6.180) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1\tilde{\phi}I_8 + a_{18}B_{18} + B_C, \tag{6.414}$$

where B_C is given in (6.288) and

$$B_{18} = \begin{bmatrix} B_1^{18} & B_3^{18} \\ (B_3^{18})^\top & B_2^{18} \end{bmatrix},$$

$$B_1^{18} = \begin{bmatrix} {w_5}^2 - {w_6}^2 & 2w_5w_6 & 0 & 0 \\ 2w_5w_6 & -{w_5}^2 + {w_6}^2 & 0 & 0 \\ 0 & 0 & {w_7}^2 - {w_8}^2 & -2w_7w_8 \\ 0 & 0 & -2w_7w_8 & -{w_7}^2 + {w_8}^2 \end{bmatrix},$$

$$B_2^{18} = \begin{bmatrix} {w_1}^2 - {w_2}^2 & 2w_1w_2 & 0 & 0 \\ 2w_1w_2 & -{w_1}^2 + {w_2}^2 & 0 & 0 \\ 0 & 0 & {w_3}^2 - {w_4}^2 & -2w_3w_4 \\ 0 & 0 & -2w_3w_4 & -{w_3}^2 + {w_4}^2 \end{bmatrix},$$

$$B_3^{18} = 2\begin{bmatrix} w_1w_5 + w_2w_6 & -w_1w_6 + w_2w_5 & 0 & 0 \\ w_1w_6 - w_2w_5 & w_1w_5 + w_2w_6 & 0 & 0 \\ 0 & 0 & w_3w_7 - w_4w_8 & -w_3w_8 - w_4w_7 \\ 0 & 0 & -w_3w_8 - w_4w_7 & -w_3w_7 + w_4w_8 \end{bmatrix}.$$

Substituting $\boldsymbol{w}_{\text{stripeI}}$ into (6.406) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeI}} \approx -\frac{a_2}{a_1}w^2.$$

Evaluating the Jacobian matrix (6.414) at $(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}}) \approx w^2 \begin{bmatrix} O & O \\ O & C_{22} \end{bmatrix} + \widetilde{J_C}^{\text{stripeI}}, \tag{6.415}$$

where $\widetilde{J_C}^{\text{stripeI}}$ is given in (6.289) and

$$C_{22} = \begin{bmatrix} a_{18} & 0 & 0 & 0 \\ 0 & -a_{18} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$ are given by

$$\begin{aligned} &\lambda_1 \approx 2a_2w^2, \\ &\lambda_2 \approx O(w^3), \\ &\lambda_3, \lambda_4 \approx -(a_2 - a_4 \pm a_{18})w^2, \\ &\lambda_5 \approx -(a_2 - a_3)w^2 \quad \text{(repeated twice)}, \\ &\lambda_6 \approx -(a_2 - a_5)w^2 \quad \text{(repeated twice)}. \end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 < 0,$$

$$
\begin{aligned}
&a_2 - a_4 \pm a_{18} > 0,\\
&a_2 - a_3 > 0,\\
&a_2 - a_5 > 0.
\end{aligned}
$$

These are equivalen to

$$\max(a_3, a_4 + |a_{18}|, a_5) < a_2 < 0. \tag{6.416}$$

Thus, the stability of $\boldsymbol{w}_{\text{stripeI}}$ depends on the values of $a_2, \ldots, a_5$ and a_{18}.

Substituting $\boldsymbol{w}_{\text{stripeII}} = (0, w, 0, 0, 0, 0, 0, 0)$ into (6.380) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeII}} \approx -\frac{a_2}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.414) at $(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}}) \approx w^2 \begin{bmatrix} O & O \\ O & -C_{22} \end{bmatrix} + \widetilde{J_C}^{\text{stripeII}}, \tag{6.417}$$

where C_{22} is given in (6.5.5), and $\widetilde{J_C}^{\text{stripeII}}$ is given in (6.292). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{stripeII}}$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$.

Substituting $\boldsymbol{w}_{\text{upside-downI}} = (w, 0, 0, 0, w, 0, 0, 0)$ into (6.278) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{upside-downI}} \approx -\frac{a_2 + a_4 + a_{18}}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.287) at $(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}}) \approx w^2 \begin{bmatrix} C_{23} & C_{24} \\ C_{24} & C_{23} \end{bmatrix} + \widetilde{J_C}^{\text{upside-downI}} \tag{6.418}$$

with

$$C_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2a_{18} & 0 & 0 \\ 0 & 0 & -a_{18} & 0 \\ 0 & 0 & 0 & -a_{18} \end{bmatrix}, \quad C_{24} = \begin{bmatrix} 2a_{18} & 0 & 0 & 0 \\ 0 & 2a_{18} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.419}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$ are given by

$$
\begin{aligned}
&\lambda_1, \lambda_2 \approx 2a_2 \pm (a_4 + a_{18}) w^2,\\
&\lambda_3 \approx -a_{18} w^2,\\
&\lambda_4 \approx O(w^3),\\
&\lambda_5 \approx -(a_2 - a_3 + a_4 - a_5 + a_{18}) w^2 \quad \text{(repeated 4 times)}.
\end{aligned}
$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 < -|a_4 + a_{18}|,$$

$$a_{18} > 0,$$
$$a_2 - a_3 + a_4 - a_5 + a_{18} > 0.$$

These conditions are equivalent to

$$a_3 - a_4 + a_5 + a_{17} < a_2 < -|a_4 + a_{18}|$$
$$a_{18} > 0.$$

Thus, the stability of $\boldsymbol{w}_{\text{upside-downI}}$ is conditional and depends on the values of $a_2, \ldots, a_5$ and a_{18}.

Substituting $\boldsymbol{w}_{\text{upside-downII}} = (0, w, 0, 0, 0, w, 0, 0)$ into (6.278) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{upside-downII}} \approx -\frac{a_2 + a_4 + a_{18}}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.287) at $(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}}) = \widetilde{J_C}^{\text{upside-downII}} \approx w^2 \begin{bmatrix} C_{25} & C_{24} \\ C_{24} & C_{25} \end{bmatrix} \tag{6.420}$$

with

$$C_{25} = \begin{bmatrix} -2a_{18} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_{18} & 0 \\ 0 & 0 & 0 & -a_{18} \end{bmatrix}, \tag{6.421}$$

where C_{18} is given in (6.393). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}})$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{upside-downII}}$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$.

Substituting $\boldsymbol{w}_{\text{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$ into (6.406) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqVM}} \approx -\frac{a_2 + a_3 + a_4 + a_5 + a_{18}}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.414) at $(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}}) \approx w^2 \begin{bmatrix} C_{21} & C_{22} \\ C_{22} & C_{21} \end{bmatrix} + \widetilde{J_C}^{\text{sqVM}}, \tag{6.422}$$

where $\widetilde{J_C}^{\text{sqVM}}$ is given in (6.302) and

$$C_{21} = 2\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -a_{18} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_{18} \end{bmatrix}, \quad C_{22} = 2\begin{bmatrix} a_{18} & 0 & 0 & 0 \\ 0 & a_{18} & 0 & 0 \\ 0 & 0 & a_{18} & 0 \\ 0 & 0 & 0 & -a_{18} \end{bmatrix}.$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$ are given by

$$\lambda_1 \approx 2(a_2 + a_3 + a_4 + a_5 + a_{18})w^2,$$

$$
\begin{aligned}
\lambda_2 &\approx 2(a_2 + a_3 - a_4 - a_5 - a_{18})w^2,\\
\lambda_3 &\approx 2(a_2 - a_3 + a_4 - a_5 + a_{18})w^2,\\
\lambda_4 &\approx 2(a_2 - a_3 - a_4 + a_5 - a_{18})w^2,\\
\lambda_5 &\approx -4a_{18}w^2 \quad \text{(repeated twice)},\\
\lambda_6 &\approx O(w^3) \quad \text{(repeated twice)}.
\end{aligned}
$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 + a_3 + a_4 + a_5 + a_{18} < 0, \tag{6.423}$$
$$a_2 + a_3 - a_4 - a_5 - a_{18} < 0, \tag{6.424}$$
$$a_2 - a_3 + a_4 - a_5 + a_{18} < 0, \tag{6.425}$$
$$a_2 - a_3 - a_4 + a_5 - a_{18} < 0, \tag{6.426}$$
$$a_{18} > 0. \tag{6.427}$$

Thus, the stability of $\boldsymbol{w}_{\mathrm{sqVM}}$ depends on the values of $a_2, \dots, a_5$ and a_{18}.

Remark 6.7. For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{\ell}, \hat{k}, \hat{\ell})$, $\boldsymbol{w}_{\mathrm{sqT}}$ does not exist. See Proposition 5.28 in Section 5.6.7. □

Case 7: $(\hat{n}, \hat{k}, \hat{\ell}) = (2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$, $(\hat{k}, \hat{\ell}) \neq (3, 1)$

For the case of $(\hat{n}, \hat{k}, \hat{\ell})$ with $\hat{n} = 2(\hat{k} + \hat{\ell})$ and $(\hat{k}, \hat{\ell}) \neq (3, 1)$, we have

$$(0,1,0,1,0,0,1,0),\ (0,0,1,1,0,1,0,0),\ (0,0,0,2,1,0,0,0) \in P.$$

as well as

$$
\begin{aligned}
&(1,0,0,0,0,0,0,0),\ (2,0,0,0,1,0,0,0),\ (1,1,0,0,0,1,0,0),\\
&(1,0,1,0,0,0,1,0),\ (1,0,0,1,0,0,0,1) \in P
\end{aligned}
$$

in (6.238). Then, the asymptotic form of F_i $(i = 1, \dots, 4)$ in (6.271) becomes

$$F_1 \approx a_1\widetilde{\phi}z_1 + a_{10}z_2z_4\bar{z}_3 + a_{11}z_3z_4\bar{z}_2 + a_{12}{z_4}^2\bar{z}_1 + F_1^C, \tag{6.428}$$
$$F_2 \approx a_1\widetilde{\phi}z_2 + a_{10}\bar{z}_1z_3z_4 + a_{11}\bar{z}_4z_3z_1 + a_{12}{z_3}^2\bar{z}_2 + F_2^C, \tag{6.429}$$
$$F_3 \approx a_1\widetilde{\phi}z_3 + a_{10}z_4z_2\bar{z}_1 + a_{11}z_1z_2\bar{z}_4 + a_{12}{z_2}^2\bar{z}_3 + F_3^C, \tag{6.430}$$
$$F_4 \approx a_1\widetilde{\phi}z_4 + a_{10}\bar{z}_3z_1z_2 + a_{11}\bar{z}_2z_1z_3 + a_{12}{z_1}^2\bar{z}_4 + F_4^C \tag{6.431}$$

with a_{10}, a_{11}, a_{12} given in (6.332), and F_i^C $(i = 1, \dots, 4)$ given in (6.272) – (6.275). Then, the asymptotic form of $\widetilde{F}_i$ $(i = 1, \dots, 8)$ in (6.182) – (6.185) becomes

$$
\begin{aligned}
\widetilde{F}_1 \approx{}& a_1\widetilde{\phi}w_1 + a_{10}\{w_5(w_3w_7 - w_4w_8) + w_6(w_3w_8 + w_4w_7)\}\\
&+ a_{11}\{w_3(w_5w_7 - w_6w_8) + w_4(w_5w_8 + w_6w_7)\}
\end{aligned}
$$

$$
\begin{aligned}
&+ a_{12}\{w_1({w_7}^2 - {w_8}^2) + 2w_2w_7w_8\} + \widetilde{F}_1^C, && (6.432)\\
\widetilde{F}_2 \approx\ & a_1\widetilde{\phi}w_2 + a_{10}\{w_5(w_3w_8 + w_4w_7) - w_6(w_3w_7 - w_4w_8)\}\\
&+ a_{11}\{w_3(w_5w_8 + w_6w_7) - w_4(w_5w_7 - w_6w_8)\}\\
&+ a_{12}\{-w_2({w_7}^2 - {w_8}^2) + 2w_1w_7w_8\} + \widetilde{F}_2^C, && (6.433)\\
\widetilde{F}_3 \approx\ & a_1\widetilde{\phi}w_3 + a_{10}\{w_1(w_5w_7 - w_6w_8) + w_2(w_5w_8 + w_6w_7)\}\\
&+ a_{11}\{w_7(w_1w_5 - w_2w_6) + w_8(w_1w_6 + w_2w_5)\}\\
&+ a_{12}\{w_3({w_5}^2 - {w_6}^2) + 2w_4w_5w_6\} + \widetilde{F}_3^C, && (6.434)\\
\widetilde{F}_4 \approx\ & a_1\widetilde{\phi}w_4 + a_{10}\{w_1(w_5w_8 + w_6w_7) - w_2(w_5w_7 - w_6w_8)\}\\
&+ a_{11}\{w_7(w_1w_6 + w_2w_5) - w_8(w_1w_5 - w_2w_6)\}\\
&+ a_{12}\{-w_4({w_5}^2 - {w_6}^2) + 2w_3w_5w_6\} + \widetilde{F}_4^C, && (6.435)\\
\widetilde{F}_5 \approx\ & a_1\widetilde{\phi}w_5 + a_{10}\{w_1(w_3w_7 - w_4w_8) + w_2(w_3w_8 + w_4w_7)\}\\
&+ a_{11}\{w_7(w_1w_3 - w_2w_4) + w_8(w_1w_4 + w_2w_3)\}\\
&+ a_{12}\{w_5({w_3}^2 - {w_4}^2) + 2w_3w_4w_6\} + \widetilde{F}_5^C, && (6.436)\\
\widetilde{F}_6 \approx\ & a_1\widetilde{\phi}w_6 + a_{10}\{w_1(w_3w_8 + w_4w_7) - w_2(w_3w_7 - w_4w_8)\}\\
&+ a_{11}\{w_7(w_1w_4 + w_2w_3) - w_8(w_1w_3 - w_2w_4)\}\\
&+ a_{12}\{-w_6({w_3}^2 - {w_4}^2) + 2w_3w_4w_5\} + \widetilde{F}_6^C, && (6.437)\\
\widetilde{F}_7 \approx\ & a_1\widetilde{\phi}w_7 + a_{10}\{w_5(w_1w_3 - w_2w_4) + w_6(w_1w_4 + w_2w_3)\}\\
&+ a_{11}\{w_3(w_1w_5 - w_2w_6) + w_4(w_1w_6 + w_2w_5)\}\\
&+ a_{12}\{w_7({w_1}^2 - {w_2}^2) + 2w_8w_1w_2\} + \widetilde{F}_7^C, && (6.438)\\
\widetilde{F}_8 \approx\ & a_1\widetilde{\phi}w_8 + a_{10}\{w_5(w_1w_4 + w_2w_3) - w_6(w_1w_3 - w_2w_4)\}\\
&+ a_{11}\{w_3(w_1w_6 + w_2w_5) - w_4(w_1w_5 - w_2w_6)\}\\
&+ a_{12}\{-w_8({w_1}^2 - {w_2}^2) + 2w_7w_1w_2\} + \widetilde{F}_8^C, && (6.439)
\end{aligned}
$$

where $\widetilde{F}_i^C$ $(i = 1, \ldots, 8)$ is given in (6.279) – (6.286). Hence, the asymptotic form of the Jacobian matrix in (6.180) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1\tilde{\phi}I_8 + a_{10}B_{10} + a_{11}B_{11} + a_{12}B_{12} + B_C, \tag{6.440}$$

with B_C given in (6.288), B_{10}, B_{11} and B_{12} given in (6.342).

Substituting $\boldsymbol{w}_{\text{stripeI}} = (w, 0, 0, 0, 0, 0, 0, 0)$ into (6.432) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeI}} \approx -\frac{a_2}{a_1}w^2.$$

Evaluating the Jacobian matrix (6.440) at $(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}}) \approx w^2 \begin{bmatrix} O & O \\ O & C_{23} \end{bmatrix} + \widetilde{J}_C^{\ \text{stripeI}}, \tag{6.441}$$

where $\widetilde{J_C}^{\text{stripeI}}$ is given in (6.289) and

$$C_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{12} & 0 \\ 0 & 0 & 0 & -a_{12} \end{bmatrix}.$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$ are given by

$$\begin{aligned} \lambda_1 &\approx 2a_2w^2, \\ \lambda_2, \lambda_3 &\approx -(a_2 - a_5 \pm a_{12})w^2, \\ \lambda_4 &\approx O(w^3), \\ \lambda_5 &\approx -(a_2 - a_3)w^2, \quad \text{(repeated twice)} \\ \lambda_6 &\approx -(a_2 - a_4)w^2, \quad \text{(repeated twice).} \end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$\begin{aligned} &a_2 < 0, \\ &a_2 - a_5 > -|a_{12}|, \\ &a_2 - a_3 > 0, \\ &a_2 - a_4 > 0. \end{aligned}$$

These are equivalent to

$$\max(a_3, a_4, a_5 - |a_{12}|) < a_2 < 0. \tag{6.442}$$

Thus, the stability of $\boldsymbol{w}_{\text{stripeI}}$ depends on the values of $a_2, \ldots, a_5$ and a_{12}.

Substituting $\boldsymbol{w}_{\text{stripeII}} = (0, w, 0, 0, 0, 0, 0, 0)$ into (6.432) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeII}} \approx -\frac{a_2}{a_1}w^2.$$

Evaluating the Jacobian matrix (6.440) at $(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}}) \approx w^2 \begin{bmatrix} O & O \\ O & -C_{23} \end{bmatrix} + \widetilde{J_C}^{\text{stripeII}}, \tag{6.443}$$

where C_{23} is given in (6.5.5), and $\widetilde{J_C}^{\text{stripeII}}$ is given in (6.292). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{stripeII}}$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$.

Substituting $\boldsymbol{w}_{\text{upside-downI}} = (w, 0, 0, 0, w, 0, 0, 0)$ into (6.432) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{upside-downI}} \approx -\frac{a_2 + a_4}{a_1}w^2.$$

Evaluating the Jacobian matrix (6.440) at $(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}}) \approx w^2 \begin{bmatrix} C_{24} & C_{25} \\ C_{25} & C_{24} \end{bmatrix} + \widetilde{J_C}^{\text{upside-downI}} \tag{6.444}$$

with

$$C_{24} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{12} & 0 \\ 0 & 0 & 0 & -a_{12} \end{bmatrix}, \quad C_{25} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{10} + a_{11} & 0 \\ 0 & 0 & 0 & a_{10} - a_{11} \end{bmatrix}. \tag{6.445}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$ are given by

$$\begin{aligned}
&\lambda_1, \lambda_2 \approx 2(a_2 \pm a_4)w^2, \\
&\lambda_3, \lambda_4 \approx \{-(a_2 + a_3 - a_4 - a_5 - a_{12}) \pm (a_{10} + a_{11})\}w^2, \\
&\lambda_5, \lambda_6 \approx \{-(a_2 + a_3 - a_4 - a_5 + a_{12}) \pm (a_{10} - a_{11})\}w^2, \\
&\lambda_7 \approx O(w^3) \quad \text{(repeated twice)}.
\end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$\begin{aligned}
&a_2 < -|a_4|, \\
&a_2 + a_3 - a_4 - a_5 - a_{12} > -|a_{10} + a_{11}|, \\
&a_2 + a_3 - a_4 - a_5 + a_{12} > -|a_{10} - a_{11}|.
\end{aligned}$$

Thus, the stability of $\boldsymbol{w}_{\text{upside-downI}}$ depends on the values of $a_2, \ldots, a_5$ and $a_{10}, \ldots, a_{12}$.

Substituting $\boldsymbol{w}_{\text{upside-downII}} = (0, w, 0, 0, 0, w, 0, 0)$ into (6.278) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{upside-downII}} \approx -\frac{a_2 + a_4}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.287) at $(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}}) \approx w^2 \begin{bmatrix} -C_{24} & C_{26} \\ C_{26} & -C_{24} \end{bmatrix} + \widetilde{J_C}^{\text{upside-downII}} \tag{6.446}$$

with

$$C_{26} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{10} - a_{11} & 0 \\ 0 & 0 & 0 & a_{10} + a_{11} \end{bmatrix}. \tag{6.447}$$

where C_{24} is given in (6.445). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}})$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{upside-downII}}$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$.

Substituting $\boldsymbol{w}_{\text{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$ into (6.432) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqVM}} \approx -\frac{a_2 + a_3 + a_4 + a_5 + a_{10} + a_{11} + a_{12}}{a_1} w^2.$$

Evaluating the Jacobian matrix (6.440) at $(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}}) \approx w^2 \begin{bmatrix} C_{27} & C_{28} \\ C_{28} & C_{27} \end{bmatrix} + \widetilde{J_C}^{\text{sqVM}}, \tag{6.448}$$

where $\widetilde{J_C}^{\text{sqVM}}$ is given in (6.302) and

$$C_{27} = \begin{bmatrix} -a_{10} - a_{11} & 0 & a_{10} + a_{11} & 0 \\ 0 & -a_{10} - a_{11} - 2a_{12} & 0 & a_{10} - a_{11} \\ a_{10} + a_{11} & 0 & -a_{10} - a_{11} & 0 \\ 0 & -a_{10} + a_{11} & 0 & -a_{10} - a_{11} - 2a_{12} \end{bmatrix},$$

$$C_{28} = \begin{bmatrix} a_{10} + a_{11} & 0 & a_{10} + a_{11} + 2a_{12} & 0 \\ 0 & -a_{10} + a_{11} & 0 & a_{10} + a_{11} + 2a_{12} \\ a_{10} + a_{11} + 2a_{12} & 0 & a_{10} + a_{11} & 0 \\ 0 & a_{10} + a_{11} + 2a_{12} & 0 & a_{10} - a_{11} \end{bmatrix}.$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$ are given by

$$\begin{aligned}
\lambda_1 &\approx 2(a_2 + a_3 + a_4 + a_5 + a_{10} + a_{11} + a_{12})w^2, \\
\lambda_2 &\approx 2(a_2 + a_3 - a_4 - a_5 - a_{10} - a_{11} - a_{12})w^2, \\
\lambda_3 &\approx 2(a_2 - a_3 + a_4 - a_5 - a_{10} - a_{11} - a_{12})w^2, \\
\lambda_4 &\approx 2(a_2 - a_3 - a_4 + a_5 - a_{10} - a_{11} + a_{12})w^2, \\
\lambda_5 &\approx -2(a_{10} + a_{11} + 2a_{12})w^2 \quad \text{(repeated twice)}, \\
\lambda_6 &\approx O(w^3) \quad \text{(repeated twice)}.
\end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 + a_3 + a_4 + a_5 + a_{10} + a_{11} + a_{12} < 0, \tag{6.449}$$

$$a_2 + a_3 - a_4 - a_5 - a_{10} - a_{11} - a_{12} < 0, \tag{6.450}$$

$$a_2 - a_3 + a_4 - a_5 - a_{10} - a_{11} - a_{12} < 0, \tag{6.451}$$

$$a_2 - a_3 - a_4 + a_5 - a_{10} - a_{11} + a_{12} < 0, \tag{6.452}$$

$$a_{10} + a_{11} + 2a_{12} > 0. \tag{6.453}$$

Thus, the stability of $\boldsymbol{w}_{\text{sqVM}}$ depends on the values of $a_2, \ldots, a_5$ and $a_{10}, \ldots, a_{12}$.

Remark 6.8. For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$, $\boldsymbol{w}_{\text{sqT}}$ does not exist. See Proposition 5.28 in Section 5.6.7. □

7. Bifurcating Solutions and Invariant Patterns for the Replicator Dynamic

In this chapter, we study bifurcation mechanisms for a spatial economic model with the replicator dynamics on the square lattice. Direct and further bifurcating solutions from the uniform state form a complicated network of solution curves. We aim to elucidate the mechanism of this complicated network in the point of view of the symmetry of the square lattice.

We conduct a numerical bifurcation analysis for a spatial economic model and demonstrate the emergence of theoretically predicted bifurcating solutions in Chapter 6. We search for these bifurcating solutions and investigate their stability by using comparative static analysis with respect to the trade freeness, which is one of the major parameter of a spatial economic model and is employed here as a bifurcation parameter.

We futhermore focus on invariant patterns that retain their spatial distribution when the value of a bifurcation parameter changes. The existence of invariant patterns is the special feature of the replicator dynamics that hitherto has not been given much attention in nonlinear mathematics. In fact, solution curves of invariant patterns are connected by those of non-invariant ones, thereby forming a complicated mesh-like structure. From this point of view, we propose a bifurcation analysis procedure using invariant patterns efficiently to find stable equilibria.

This chapter is organized as follows. A general framework of spatial economic models with the replicator dynamics is represented in Section 7.1. The equivariance of the governing equation for spatial economic models on the square lattice is expressed in Section 7.2. A theory of invariant patterns for the replicator dynamics is introduced in Section 7.3. Numerical bifurcation analysis are conducted in Section 7.4.

7.1. Spatial Economic Model with the Replicator Dynamics

We introduce a general framework of spatial economic models (e.g., see Fujita et al., 1999 [30]).

7.1.1. General Framework

Consider an economy composed of K regions. Each mobile agent, who can migrate between regions, selects one region. A spatial distribution of mobile agents is denoted by $\boldsymbol{\lambda} = \{\lambda_i \geq 0 \mid i = 1, \dots, K\}$. The indirect utility function of mobile agents is denoted by $\boldsymbol{v} \in \mathbb{R}_+^K$, which is defined as a smooth function of $\boldsymbol{\lambda}$ and the trade freeness ϕ. A spatial equilibrium is denoted by $\boldsymbol{\lambda}^*$, which is defined as a spatial distribution that satisfies the following conditions:

$$\begin{cases} v^* - v_i(\boldsymbol{\lambda}, \phi) = 0 & \text{if } \lambda_i > 0, \\ v^* - v_i(\boldsymbol{\lambda}, \phi) \geq 0 & \text{if } \lambda_i = 0, \end{cases} \tag{7.1}$$

and $\sum_{i=1}^{K} \lambda_i = 1$, where v^* denotes the equilibrium utility level. This condition means that there is no incentive for mobile agents to change the location choice.

Spatial equilibria and their stability depend on the definition of the utility function of each model. We focus on a multi-regional version of the FO model (Forslid and Ottaviano, 2003 [31]) as a specific example in the numerical bifurcation analysis The basic assumptions of this model is introduced briefly.

There are two factors of production: skilled and unskilled workers. The endowments of skilled and unskilled workers are H (= 1) and L (= 1), respectively. Skilled workers are mobile and move between regions. Unskilled workers are immobile and are evenly distributed across the K regions with the population L/K in each region.

There are two sectors of production: A-sector (agriculture) and M-sector (manufacturing). A-sector produces horizontally homogeneous goods, which require one unit of unskilled workers to produce one unit of goods under constant returns to scale and perfect competition. M-sector produces horizontally differentiated goods, which require a fixed requirement of α units of skilled workers and a marginal input requirement of β units of unskilled workers under increasing returns to scale and Dixit-Stiglitz monopolistic competition.

There are two major micro economic parameters for the FO model: σ (> 1) expresses the constant elasticity of substitution between any two M-sector goods, and $\mu \in (0, 1)$ denotes the constant expenditure share on M-sector goods.

Goods of both sectors are transported between regions and consumed in each region. The transportation of A-sector goods is cost free, while the transportation of M-sector goods demands the iceberg costs. In other words, for each unit of M-sector goods transported from a region i to j ($\neq i$), only a fraction $1/\tau_{ij} < 1$ arrives. We assume that $\tau_{ii} = 1$ for all $i \in \{1, \cdots, K\}$ and that $\tau_{ij} = \tau_{ij}(\tau)$ is a function in the transportation cost parameter $\tau > 0$ as

$$\tau_{ij} = \exp[\tau\, m(i, j)], \tag{7.2}$$

where $m(i, j)$ represents the shortest distance between region i and j. The trade freeness is defined by

$$\phi = \exp[-(\sigma - 1)d\tau], \quad \phi \in (0, 1), \tag{7.3}$$

where d denotes the nominal distance. The trade freeness ϕ is inversely proportional to the transportation cost parameter τ.

7.1.2. Replicator Dynamics

Mobile agents migrate to a region where they achieve the highest indirect utility. In order to describe the process of the migration of mobile agents, we consider the replicator dynamics:

$$\frac{d\boldsymbol{\lambda}}{dt} = \boldsymbol{F}(\boldsymbol{\lambda}, \phi), \tag{7.4}$$

where

$$\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = (F_i(\boldsymbol{\lambda}, \phi) \mid i = 1, \ldots, K) \tag{7.5}$$

with

$$F_i(\boldsymbol{\lambda}, \phi) = (v_i(\boldsymbol{\lambda}, \phi) - \bar{v}(\boldsymbol{\lambda}, \phi))\lambda_i, \tag{7.6}$$

and $\bar{v}$ is the average utility defined by

$$\bar{v} = \sum_{i=1}^{K} \lambda_i v_i. \tag{7.7}$$

The population of mobile agents in a region is determined by the indirect utility v_i, the average utility $\bar{v}$, and the number λ_i of mobile agents in the region.

We can convert a problem to find a set of stable equilibria of spatial economic models into another problem to find a set of stable stationary points of the replicator dynamics (see Sandholm, 2010 [32]). Stationary points $\boldsymbol{\lambda}^*(\phi)$ of the replicator dynamics are defined as points which satisfy a static governing equation

$$\boldsymbol{F}(\boldsymbol{\lambda}^*, \phi) = \mathbf{0}. \tag{7.8}$$

Using the eigenvalues of the Jacobian matrix $J(\boldsymbol{\lambda}^*, \phi) = \partial \boldsymbol{F}/\partial \boldsymbol{\lambda}(\boldsymbol{\lambda}^*, \phi)$, we evaluate the stability of a stationary point as

$$\begin{cases} \text{linearly stable:} & \text{every eigenvalue has a negative real part,} \\ \text{linearly unstable:} & \text{at least one eigenvalue has a positive real part.} \end{cases}$$

A stationary point is asymptotically stable or unstable according to whether it is linearly stable or unstable.

7.2. Equivariance of the Governing Equation on the Square Lattice

The $n \times n$ square lattice provides uniformly distributed $n \times n$ discrete regions ($K = n^2$), which are connected by links of the same length d forming a square mesh. As represented in the following proposition, the FO model with the replicator dynamics to which applied the $n \times n$ square lattice satisfies the equivariance.

Proposition 7.1. *The FO model with* (7.6) *on the square lattice is equivariant to* $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_\mathrm{n} \times \mathbb{Z}_\mathrm{n})$*, that is,*

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{F}(T(g)\boldsymbol{\lambda}, \phi), \quad g \in G \tag{7.9}$$

for the N-dimensional permutation representation $T(g)$ *of* G.[14]

Proof. Each element g of G acts as a permutation of place numbers $(1, \ldots, K)$, and the action of $g \in G$ is expressed as $g : i \mapsto i^*$. For the FO model, we have $v_i(T(g)\boldsymbol{\lambda}, \phi) = v_{i^*}(\boldsymbol{\lambda}, \phi)$ because of the form of the transport cost parameter in (7.2). We also have $\bar{v}(T(g)\boldsymbol{\lambda}, \phi) = \bar{v}(\boldsymbol{\lambda}, \phi)$ by (7.7). Therefore, we have

$$F_i(T(g)\boldsymbol{\lambda}, \phi) = (v_{i^*}(\boldsymbol{\lambda}, \phi) - \bar{v}(\boldsymbol{\lambda}, \phi))\lambda_{i^*} = F_{i^*}(\boldsymbol{\lambda}, \phi)$$

for the function F_i in (7.6). This proves the equivariance (7.9). □

Note that the uniform state

$$\boldsymbol{\lambda}_{\text{uniform}} = (1/K, \ldots, 1/K)^\top \tag{7.10}$$

satisfies the governing equation (7.8) for any ϕ, and hence this solution is one of the invariant patterns (see Proposition 7.4). The symmetry of the uniform state is represented as

$$T(g)\boldsymbol{\lambda}_{\text{uniform}} = \boldsymbol{\lambda}_{\text{uniform}}, \quad g \in G, \tag{7.11}$$

and hence this solution is G-symmetric. Critical points on the uniform state are investigated numerically in Section 7.4.3.

[14] The concrete form of $T(g)$ was given in Section 4.1.

Stationary points form solution curves $(\boldsymbol{\lambda}^*(\phi), \phi)$. In general, a spatial distribution $\boldsymbol{\lambda}^*(\phi)$ changes as the value of ϕ along a solution curve. In contrast, there can be a special solution curve $(\boldsymbol{\lambda}^*(\phi), \phi) = (\bar{\boldsymbol{\lambda}}, \phi)$ that has a constant spatial distribution $\bar{\boldsymbol{\lambda}}$ along a solution curve.[15] Such a distribution $\bar{\boldsymbol{\lambda}}$ is called an invariant pattern, and $(\bar{\boldsymbol{\lambda}}, \phi)$ is a solution for any ϕ. In contrast, a solution curve with distribution $\boldsymbol{\lambda}^*(\phi)$ that varies with ϕ is called a non-invariant pattern. Thus, the spatial patterns for stationary points are classified into

$$\begin{cases} \text{invariant pattern:} & \boldsymbol{\lambda}^* = \bar{\boldsymbol{\lambda}}, \\ \text{non-invariant pattern:} & \boldsymbol{\lambda}^* = \boldsymbol{\lambda}^*(\phi). \end{cases}$$

Rearranging the order of the components of $\boldsymbol{\lambda}^*$, we introduce $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0)$ with $\boldsymbol{\lambda}_+ = \{\lambda_i > 0 \mid i = 1, \dots, m\}$ and $\boldsymbol{\lambda}_0 = \mathbf{0}$ for later discussion of invariant patterns. As a candidate of invariant patterns, we consider a spatial distribution of a special form

$$(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0) = \left(\frac{1}{m}\mathbf{1}, \mathbf{0}\right), \quad 1 \le m \le K \tag{7.12}$$

with an m-dimensional vector $\mathbf{1} = (1, \dots, 1)^\top$. This distribution expresses equal complete agglomeration to m places and can be an invariant pattern under some symmetry conditions in the following proposition:

Proposition 7.2. *A spatial distribution* $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0) = (\frac{1}{m}\mathbf{1}, \mathbf{0})$ *is an invariant pattern if this distribution satisfies*

(i) $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0) = (\frac{1}{m}\mathbf{1}, \mathbf{0})$ *is invariant to some subgroup G' of G.*

(ii) *The set of points for $\boldsymbol{\lambda}_+$ belongs to an orbit of G'.*

Proof. Since the m places of $\boldsymbol{\lambda}_+$ belong to an orbit, we have $v_1 = \cdots = v_m$. Then, we have $\bar{v} = \sum_{i=1}^m \lambda_i v_i = v_i$ and $v_i - \bar{v} = 0$ $(i = 1, \dots, m)$. Hence, we have $F_i(\frac{1}{m}\mathbf{1}, \mathbf{0}, \phi) = \mathbf{0}$ $(i = 1, \dots, m)$. For $K - m$ places with no population, we have $\lambda_j = 0$ $(j = m + 1, \dots, K)$. Hence, we have $F_i(\frac{1}{m}\mathbf{1}, \mathbf{0}, \phi) = \mathbf{0}$ $(i = m + 1, \dots, K)$. This shows that $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \phi) = (\frac{1}{m}\mathbf{1}, \mathbf{0}, \phi)$ is a solution for any ϕ. Hence, $(\frac{1}{m}\mathbf{1}, \mathbf{0})$ is an invariant pattern. □

Spatial distributions for $m = 1$, 2, and K in (7.12) are called mono-centric, duo-centric, and uniform distribution, respectively.[16] We have the following propositions for these distributions.

Proposition 7.3. *A mono-centric distribution at any place is an invariant pattern for any economy.*

Proof. Consider $\lambda_1 = 1$ and $\lambda_i = 0$ $(i = 2, \dots, K)$. Then, we have $\bar{v} = \sum_{i=1}^m \lambda_i v_i = v_1$. Thus, we have $v_1 - \bar{v} = 0$. Hence, we have $F_1(1, \mathbf{0}, \phi) = \mathbf{0}$. For $K - 1$ places with no population, we have $\lambda_i = 0$. Hence, we have $F_i(1, \mathbf{0}, \phi) = \mathbf{0}$ $(i = 2, \dots, K)$. This shows that $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \phi) = (1, \mathbf{0}, \phi)$ serves as a solution for any ϕ. Hence, a mono-center at one place is an invariant pattern. □

[15] Such a solution curve observed in the two-place economy (Fujita et al., 1999 [30]).

[16] These three distributions are proved to be invariant patterns for the hexagonal lattice (Ikeda et al., 2019 [10]).

Proposition 7.4. *The uniform and a duo-centric distribution are invariant patterns for an $n \times n$ square lattice.*

Proof. Consider two nodes (n_1, n_2) and (n_1', n_2'). Then, we have

$$r^2 {p_1}^i {p_2}^j \cdot (n_1, n_2) \equiv (-n_1 - i, -n_2 - j) \mod n.$$

Hence, for any pair of (n_1, n_2) and (n_1', n_2'), we see that

$$g \cdot (n_1, n_2) \equiv (n_1', n_2'), \quad g \cdot (n_1', n_2') \equiv (n_1, n_2) \mod n$$

by $g = r^2 {p_1}^i {p_2}^j$ with $i = -n_1 - n_1'$ and $j = -n_2 - n_2'$. By choosing $G' = \langle r^3 {p_1}^i {p_2}^j \rangle$, we see that a duo-center ($m = 2$) at any places is an invariant pattern by Proposition 7.2. The uniform distribution can be shown as an invariant pattern by extending the proof for the duo-center. □

We search for invariant patterns on the $n \times n$ square lattice by finding a set of m nodal points and a subgroup G' that satisfy Proposition 7.2 for the group $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_\mathrm{n} \times \mathbb{Z}_\mathrm{n})$. We propose the following procedure to obtain all invariant patterns.

- Choose a set of m nodal points among a total of n^2 nodal points.
- Find elements of G that retain the set of points invariant.
- If these elements form a group and permute any two of the m nodal points, this group is chosen as G' in Proposition 7.2 to ensure that the set of points gives an invariant pattern.

In this procedure, it is convenient to note that the number m of agglomerated places is not arbitrary but depends on the lattice size n as explained in the following proposition:

Proposition 7.5. *If a spatial distribution $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0) = (\frac{1}{m}\mathbf{1}, \mathbf{0})$ is an invariant pattern on an $n \times n$ square lattice, then the number m $(1 \leq m \leq n^2)$ divides $8n^2$.*

Proof. Since G' is a subgroup of G with $|G| = |\langle r, s, p_1, p_2 \rangle| = 8n^2$, $|G'|$ divides $8n^2$ by Lagrange's theorem. The number m of elements of an orbit divides $|G'|$ (Kochendörfer, 1970 [33, §3.1.2]). Hence, $8n^2$ is divisible by m. □

For example, list of all invariant patterns for $n = 6$ are depicted in Figures. 7.1 and 7.2.

7.4. Bifurcation Analysis of the 6×6 *Square Lattice*

We focus on the 6×6 square lattice that accommodates various kinds of bifurcating solutions. We conduct a numerical bifurcation analysis for the FO model and investigate the connectivity of bifurcating solutions to invariant patterns.

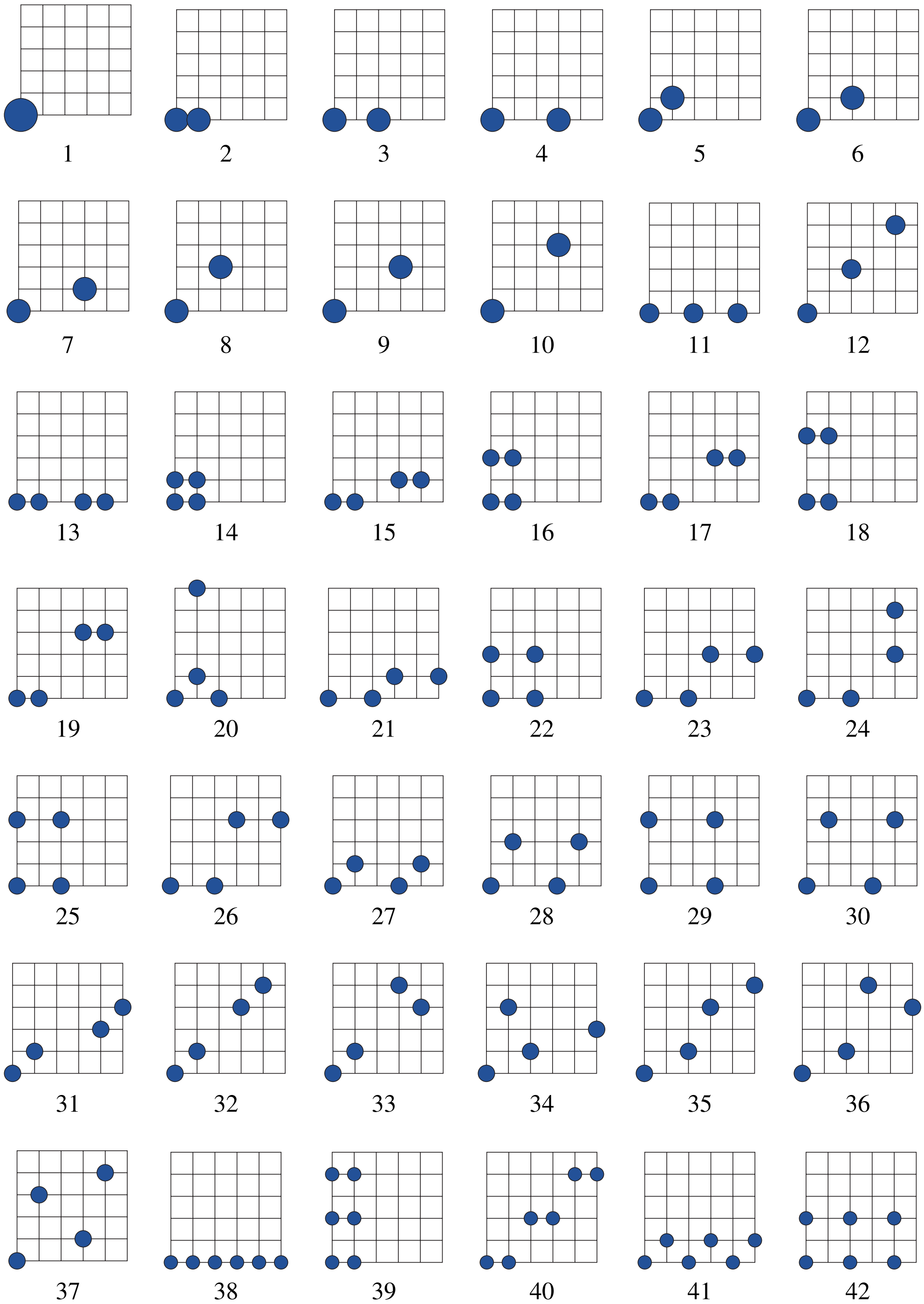


Figure 7.1: List of invariant patterns for the 6×6 square lattice. The size of a circle represents the mass of population in each place.

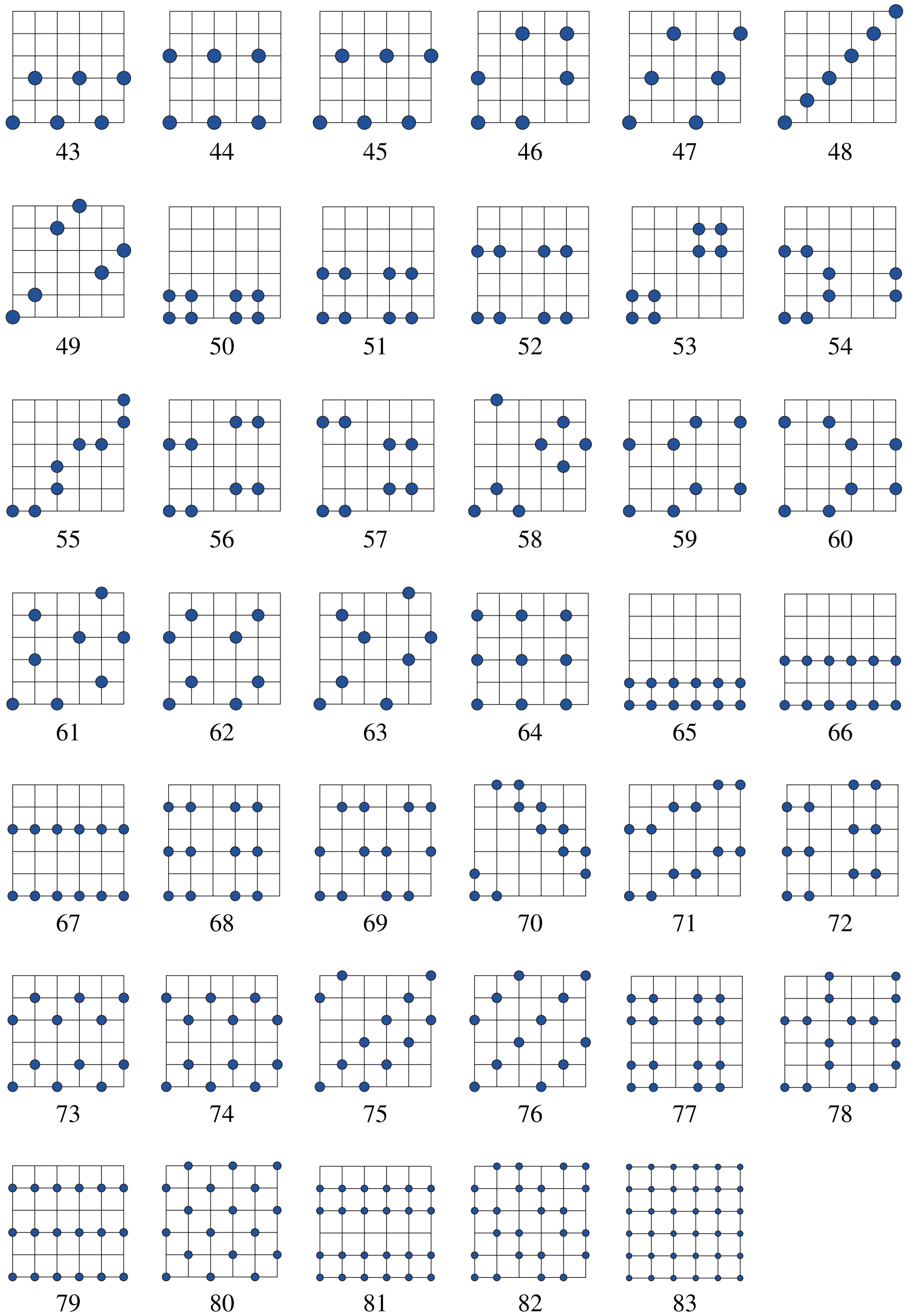


Figure 7.2: List of invariant patterns for the 6×6 square lattice. The size of a circle represents the mass of population in each place.

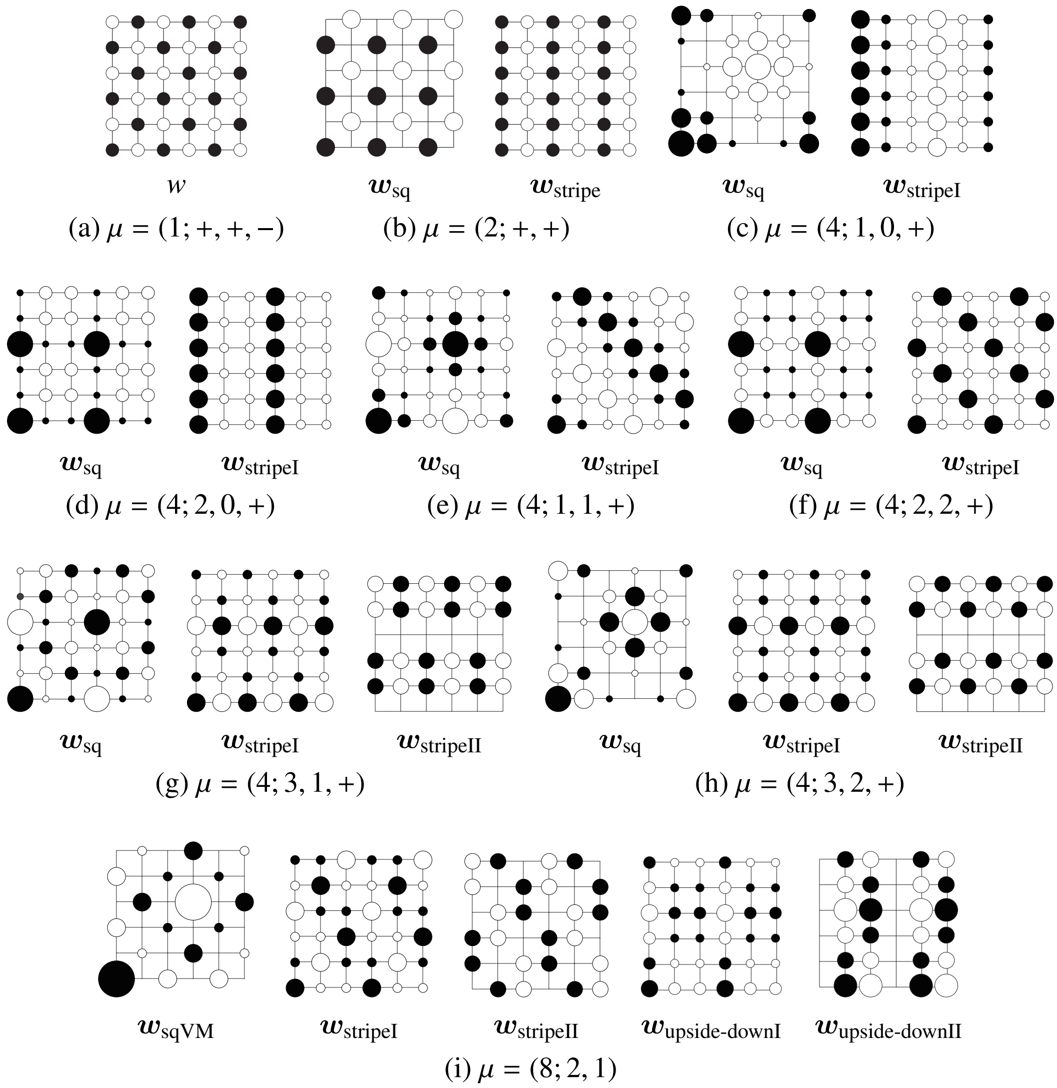


Figure 7.3: Bifurcating solutions for the 6×6 square lattice. A black circle denotes a positive component, and a white circle denotes a negative component. The size of a circle represents the magnitude of the associated component.

Table 7.1: Bifurcating solutions for the 6×6 square lattice

μ	Bifurcating solutions ($w \in \mathbb{R}$)
$(1; +, +, -)$	w
$(2; +, +)$	$\boldsymbol{w}_{\mathrm{sq}} = (w, w)$, $\boldsymbol{w}_{\mathrm{stripe}} = (w, 0)$
$(4; 1, 0, +)$	$\boldsymbol{w}_{\mathrm{sq}} = (w, 0, w, 0)$, $\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0)$
$(4; 2, 0, +)$	$\boldsymbol{w}_{\mathrm{sq}} = (w, 0, w, 0)$, $\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0)$
$(4; 1, 1, +)$	$\boldsymbol{w}_{\mathrm{sq}} = (w, 0, w, 0)$, $\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0)$
$(4; 2, 2, +)$	$\boldsymbol{w}_{\mathrm{sq}} = (w, 0, w, 0)$, $\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0)$
$(4; 3, 1, +)$	$\boldsymbol{w}_{\mathrm{sq}} = (w, 0, w, 0)$, $\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0)$, $\boldsymbol{w}_{\mathrm{stripeII}} = (0, w, 0, 0)$
$(4; 3, 2, +)$	$\boldsymbol{w}_{\mathrm{sq}} = (w, 0, w, 0)$, $\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0)$, $\boldsymbol{w}_{\mathrm{stripeII}} = (0, w, 0, 0)$
$(8; 2, 1)$	$\boldsymbol{w}_{\mathrm{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$,
	$\boldsymbol{w}_{\text{upside-downI}} = (w, 0, 0, 0, w, 0, 0, 0)$, $\boldsymbol{w}_{\text{upside-downII}} = (0, w, 0, 0, 0, w, 0, 0)$,
	$\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0, 0, 0, 0, 0)$, $\boldsymbol{w}_{\mathrm{stripeII}} = (0, w, 0, 0, 0, 0, 0, 0)$,

7.4.1. Bifurcating Solutions from the Uniform distribution

As a consequence of the irreducible decomposition (4.11) of the permutation representation T for this lattice, the irreducible representation μ of the group $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_6 \times \mathbb{Z}_6)$ to be considered in bifurcation analysis is restricted to

$$\begin{aligned}\mu = {} & (1; +, +, +),\ (1; +, +, -),\ (2; +, +),\ (4; 1, 0, +),\ (4; 2, 0, +), \\ & (4; 1, 1, +),\ (4; 2, 2, +),\ (4; 3, 1, +),\ (4; 3, 2, +),\ (8; 2, 1).\end{aligned} \tag{7.13}$$

Theoretically possible bifurcating solutions associated with μ in (7.13) are listed in Table 7.1 and depicted in Figure 7.3. Note that for $\mu = (4; 2, 0, +)$ and $(4; 2, 2, +)$, the two solutions $\boldsymbol{w}_{\mathrm{sq}}$ and $-\boldsymbol{w}_{\mathrm{sq}}$, which have opposite signs, represent different physical behaviour. The same holds for the solutions $\boldsymbol{w}_{\mathrm{stripeI}}$ and $-\boldsymbol{w}_{\mathrm{stripeI}}$. Other bifurcating solutions with opposite signs represent the same physical behaviour.

Remark 7.1. For the 6×6 square lattice, we have the following statements:

- For $\mu = (4; 1, 0, +),\ (4; 2, 0, +),\ (4; 1, 1, +),\ (4; 2, 2, +)$, the solution $\boldsymbol{w}_{\mathrm{stripeII}} = (0, w, 0, 0)$ does not exist. See Proposition 6.5 in Section 6.4.3. Note that the condition in Proposition 6.5 is not satisfied since $\check{n}$ is odd for these cases.

- For $\mu = (8; 2, 1)$, the solution $\boldsymbol{w}_{\mathrm{sqT}} = (w, 0, w, 0, 0, 0, 0, 0)$ does not exist. See Proposition 5.28 in Section 5.6.7. This case corresponds to the case $(\hat{n}, \hat{k}, \hat{\ell}) = (2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$. In fact, $\underline{2\gcd(\hat{k}, \hat{\ell})} = 2\gcd(2, 1) = 2$. This is divisible by $\gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}) = \gcd(6, 6) = 1$. Hence, **GCD-div** in (5.97) is not satisfied.

□

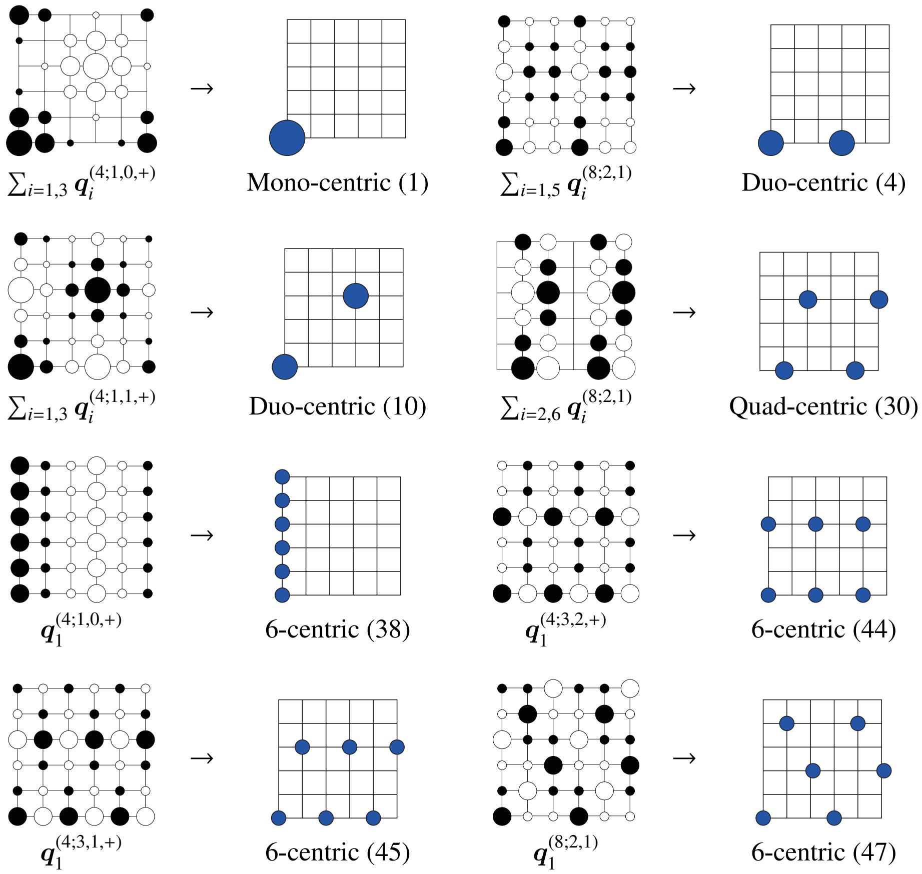


Figure 7.4: Invariant patterns that are engendered through asymmetric bifurcating solutions from the uniform state for the 6×6 square lattice. The figures to the left represent bifurcating solutions, and the ones to the right represent corresponding invariant patterns. The number in the label of each invariant pattern corresponds to Figures 7.1 and 7.2.

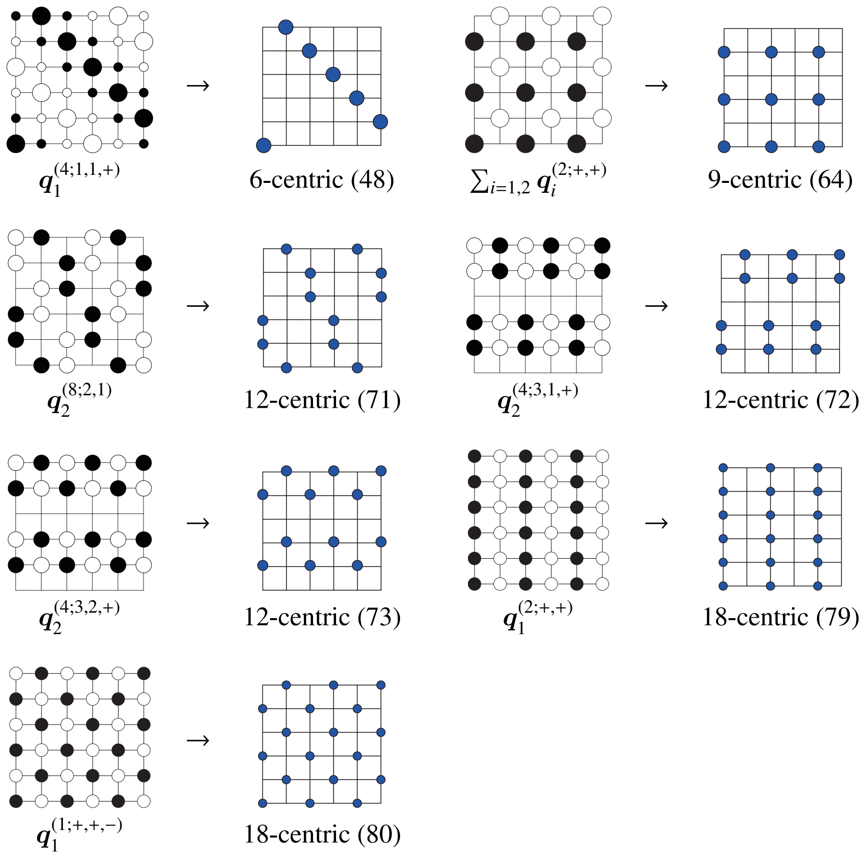


Figure 7.5: Invariant patterns that are engendered through asymmetric bifurcating solutions from the uniform state for the 6×6 square lattice. The figures to the left represent bifurcating solutions, and the ones to the right represent corresponding invariant patterns. The number in the label of each invariant pattern corresponds to Figures 7.1 and 7.2.

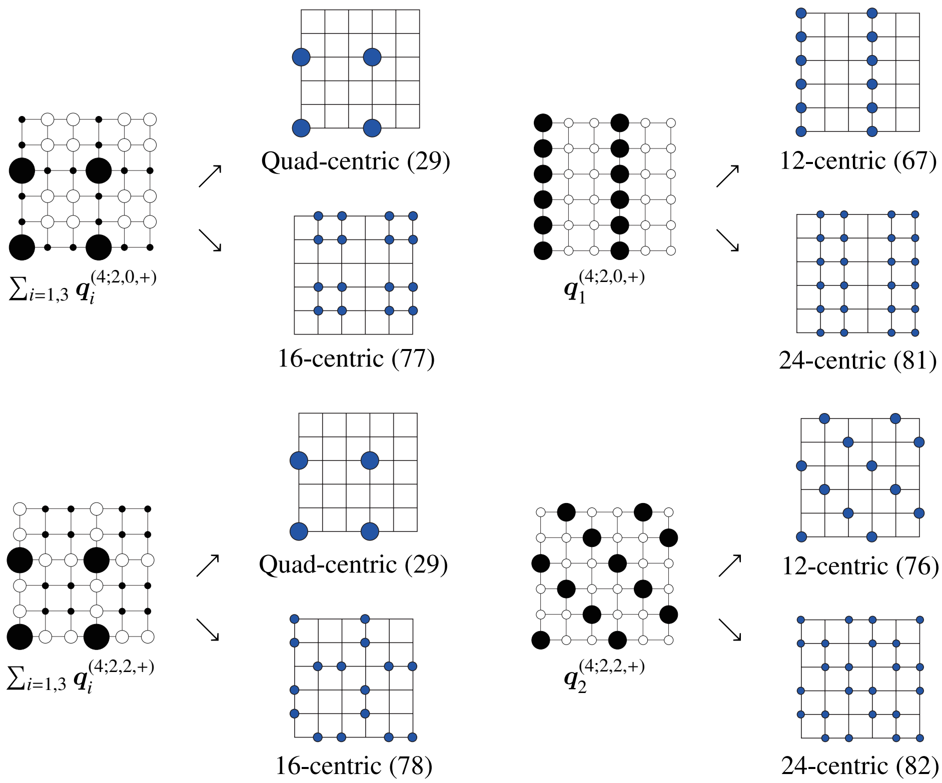


Figure 7.6: Invariant patterns that are engendered through asymmetric bifurcating solutions from the uniform state for the 6×6 square lattice. The figures to the left represent bifurcating solutions, and the ones to the right represent corresponding invariant patterns. The number in the label of each invariant pattern corresponds to Figures 7.1 and 7.2.

7.4.2. Connectivity of Bifurcating Solutions to Invariant Patterns

We investigate the connectivity of the uniform state to invariant patterns via bifurcating solutions presented in Figure 7.3. Figures 7.4–7.6 present several pairs of the eigenvector of a bifurcating solution in the left and the associated invariant pattern in the right connected by an arrow $\rightarrow$. Each pair displays similar geometrical patterns. Moreover, we explain below such similarity arises from bifurcation mechanism. In the numerical bifurcation analysis of the FO model presented in Section 7.1, population in places with the positive components of bifurcating solutions tended to increase, while population in places with the negative components of bifurcating solutions tended to decrease along all bifurcating curves from the uniform state. Based on this tendency, we propose the following conjecture that explains the connectivity between the pairs in Figures 7.4–7.6.

Conjecture 1. *Population is agglomerated completely to places with the largest positive components of the eigenvalues for the bifurcating solutions.*

Under this conjecture, we predict that invariant patterns shown in Figures 7.4–7.6 can be engendered from the uniform state as consequence of direct bifurcations. For example, a mono-center can be engendered from a critical point associated with $\boldsymbol{q}_1^{(4;1,0)} + \boldsymbol{q}_3^{(4;1,0)}$ (see the top-left of Figure 7.4). Such connectivity is also observed for the other pairs connected by the arrow $\rightarrow$. This conjecture is fairly in line with the bifurcation and the agglomeration behaviour of the FO model to be investigated in Section 7.4.3 and is insightful in the understanding of spatial economic agglomerations.

A remark is on the symmetry/asymmetry of the bifurcating solutions. When the solutions in the positive and the negative directions from the bifurcation point are conjugate, these solutions can arrive at the same invariant pattern (see Figures 7.4 and 7.5). When the two solutions are not conjugate, these solutions can arrive at two different patterns (see Figure 7.6).

7.4.3. Stability of Bifurcating Solutions and Invariant Patterns

We conducted the comparative static analysis with respect to the trade freeness ϕ of the FO model. Parameter values for the FO model were chosen as $(\sigma, \mu) = (6.0, 0.4)$, following Fujita et al., 1999 [30]. The nominal distance of the square lattice was chosen as $d = 1/n = 1/6$.

We found stable invariant patterns engendered by direct bifurcation from the uniform state and computed solution curves for non-invariant patterns that connect the invariant patterns with the uniform state. Figure 7.7 shows that 9 invariant patterns are engendered by direct bifurcations from the uniform state and have become stable. When the trade freeness ϕ is increased from a small value, the uniform state loses stability at the critical point A associated with $\mu = (1; +, +, -)$. Then, the bifurcating solution (Figure 7.3 (a)) is engendered from A. The bifurcating solution curve is connected to an invariant pattern of 18 places and regains stability. As we expected in the previous section, population is agglomerated completely to places with the positive components of the bifurcating solution. The same holds for the solution curves from the critical points B, C, D, E, F, and G associated with $\mu = (4; 2, 2, +)$, $(4; 3, 1, +)$, $(8; 2, 1)$, $(4; 2, 0, +)$, $(4; 1, 1, +)$, and $(4; 1, 0, +)$, respectively. For these solution curves, we see that almost all the non-invariant patterns are unstable. Accordingly, we find out a mesh-like structure of the solution curves for stable invariant patterns and unstable non-invariant ones. Hence, a knowledge of invariant patterns is useful in the understanding of the mechanism of such a bifurcation behaviour.

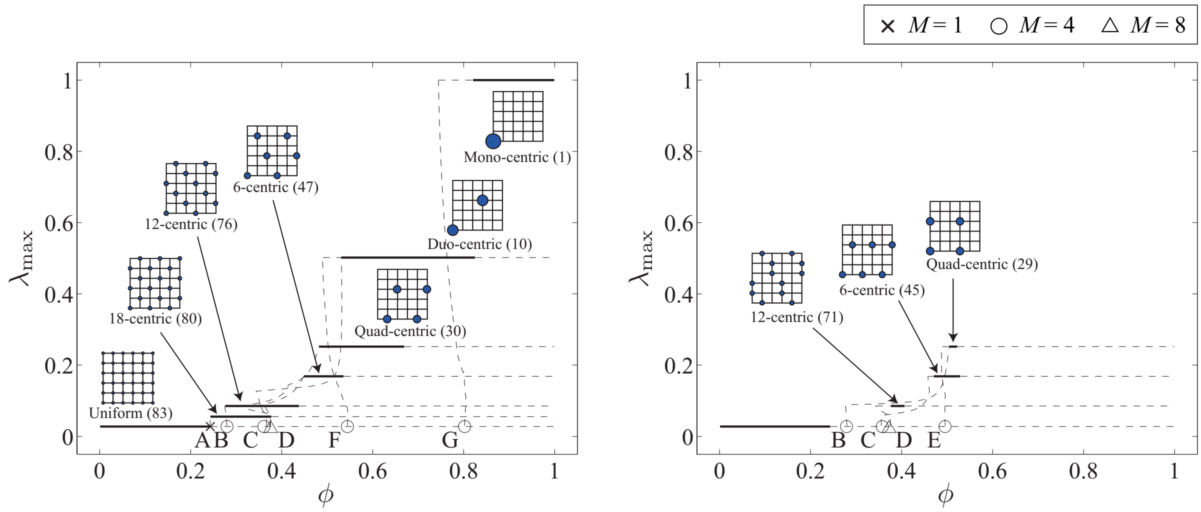


Figure 7.7: Stable invariant patterns engendered by direct bifurcation for the 6×6 square lattice. The vertical axis shows $\lambda_{\max} = \max(\lambda_1, \ldots, \lambda_K)$. M represents the multiplicity of critical points. Solid curves represent stable equilibria, and dashed ones represent unstable ones. The number in the label of each invariant pattern corresponds to Figures 7.1 and 7.2.

We investigate the stability of all invariant patterns for the 6×6 square lattice. Figure 7.8 shows invariant patterns that become stable for some ϕ. As shown in this figure, as many as 22 patterns are stable for some ϕ including other patterns than that are connected to the uniform state. We see a tendency that when the trade freeness ϕ is increased from a small value, the number of places that have population is decreased.

We identified group-theoretic critical points on the uniform state $\boldsymbol{\lambda}_{\text{uniform}} = (1/36, \ldots, 1/36)^\top$ for all the irreducible representations μ in (7.13) and computed bifurcating solution curves from these points. Figures 7.9–7.17 shows bifurcating solution curves for each μ. From these results, we have demonstrated the emergence of each bifurcating solution that was theoretically predicted in Chapters 5 and 6. We see that all the bifurcating solutions are unstable just after bifurcation although stable ones are theoretically possible. For almost all the bifurcating solution curves, population tend to be agglomerated completely to places with the largest positive or negative components of the bifurcating solution after the bifurcation. Note that $\boldsymbol{w}_{\text{sq}}$ with $\mu = (4; 3, 2, +)$ in Figure 7.16 and $\boldsymbol{w}_{\text{sqVM}}$ with $\mu = (8; 2, 1)$ in Figure 7.17 are exceptions to this tendency. These solutions have a common property that some places have a zero component. For solutions with such a property, computing the bifurcating solution curves is troublesome since we cannot predict increase and decrease in population in places with a zero component.

The main contribution of this chapter is not only demonstrating the emergence of bifurcating solutions for the FO model but also proposing a general framework to understand bifurcation behaviour for any spatial economic model with symmetry. Through the same procedure conducted in this chapter, we can completely figure out bifurcation behaviour for any model.

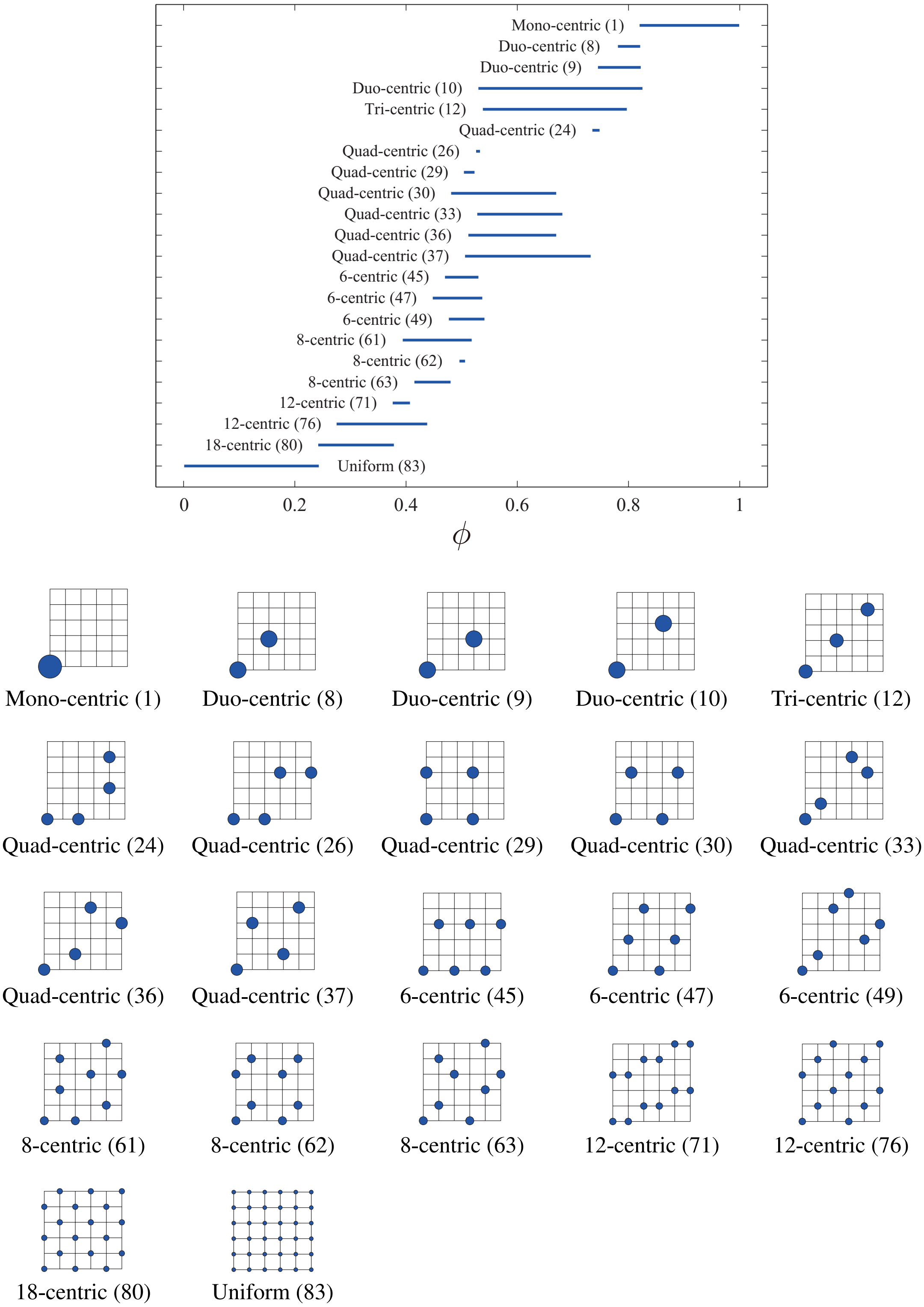


Figure 7.8: The ranges of ϕ for stable invariant patterns for the 6×6 square lattice. The number in the label of each invariant pattern corresponds to Figures 7.1 and 7.2.

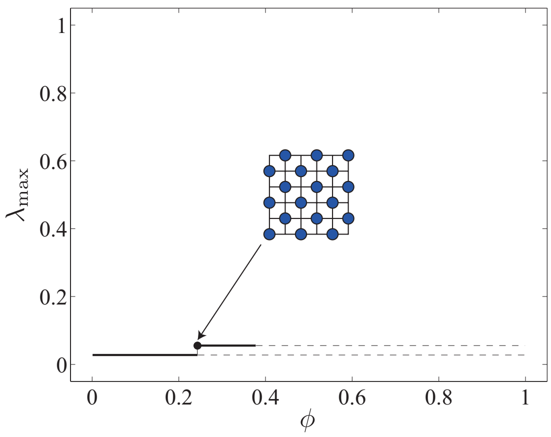
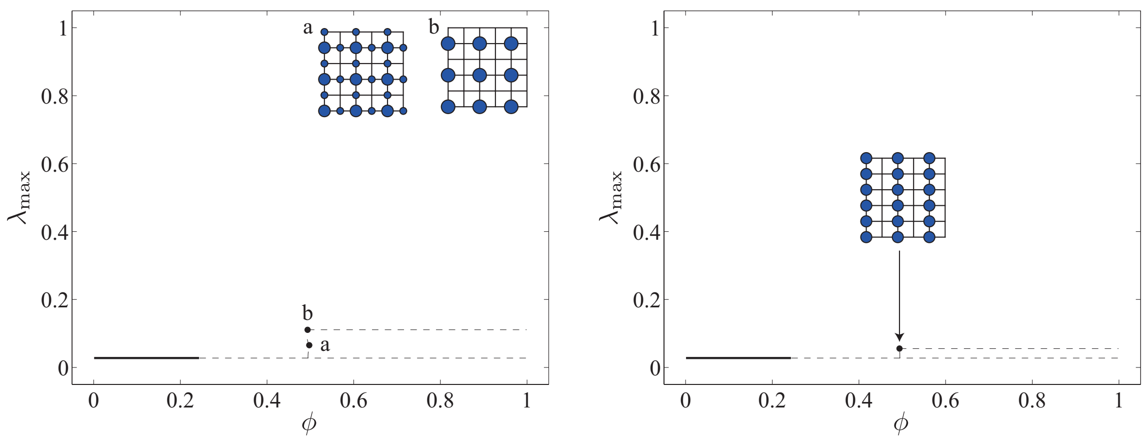
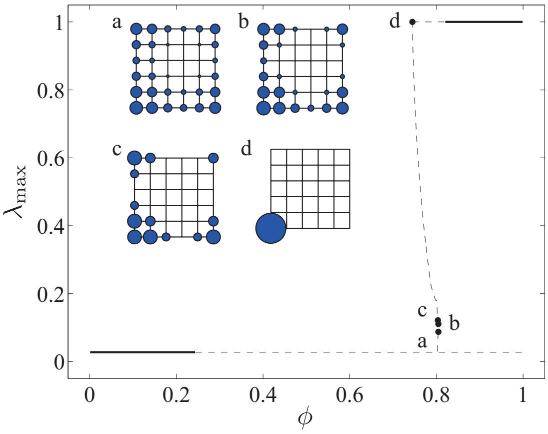


Figure 7.9: Bifurcating curves for $\mu = (1; +, +, -)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.

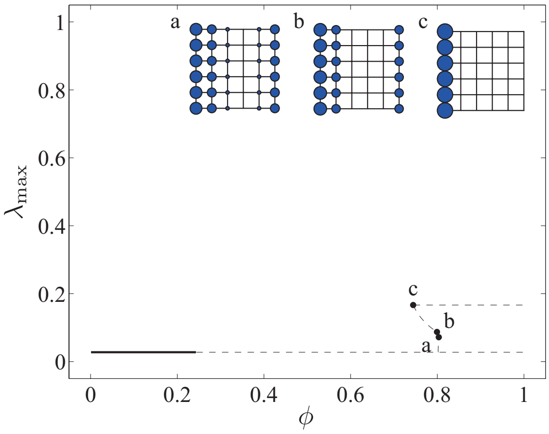


$\boldsymbol{w}_{\text{sq}}$ $\boldsymbol{w}_{\text{stripe}}$

Figure 7.10: Bifurcating curves for $\mu = (2; +, +)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.



$\boldsymbol{w}_{\mathrm{sq}}$



$\boldsymbol{w}_{\mathrm{stripeI}}$

Figure 7.11: Bifurcating curves for $\mu = (4; 1, 0, +)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.

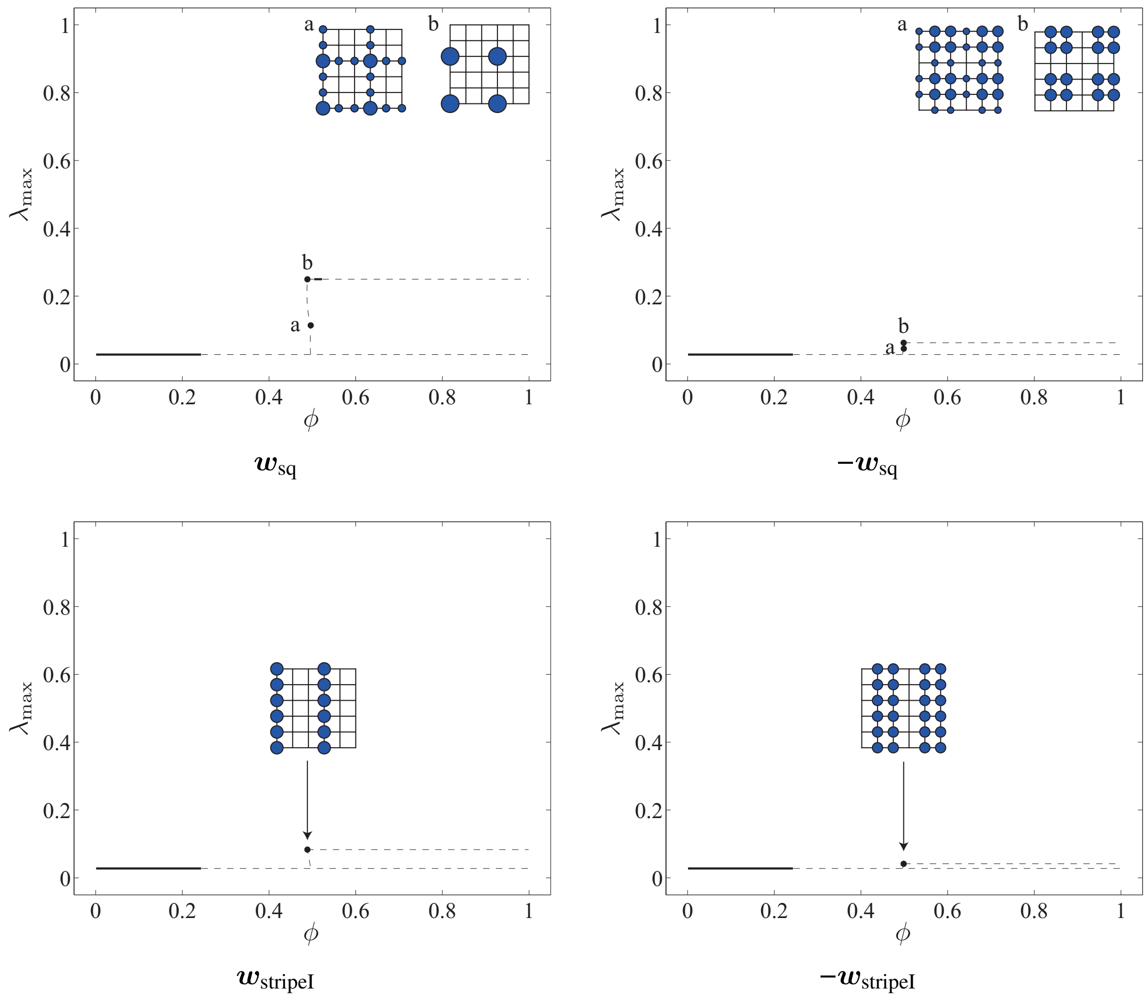


Figure 7.12: Bifurcating curves for $\mu = (4; 2, 0, +)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.

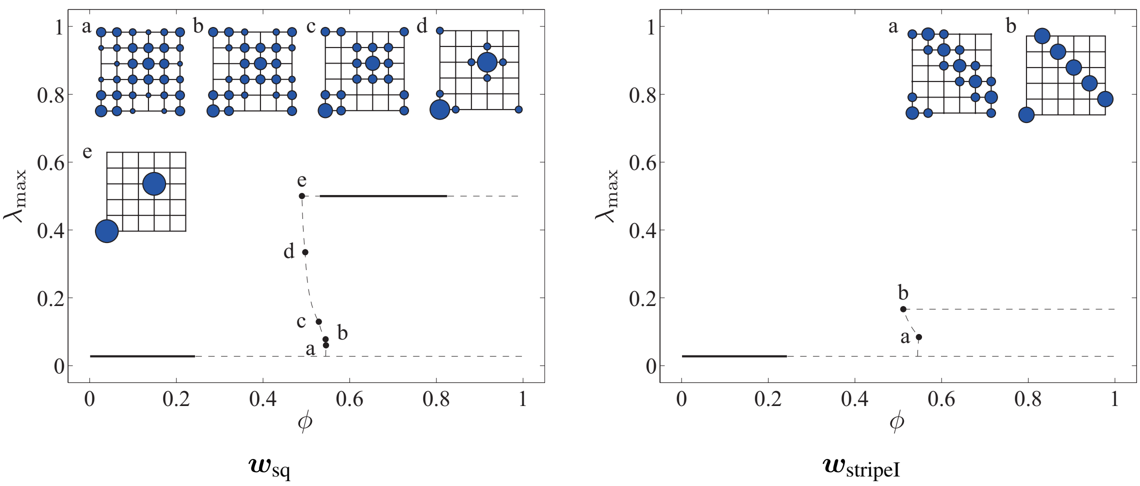


Figure 7.13: Bifurcating curves for $\mu = (4; 1, 1, +)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.

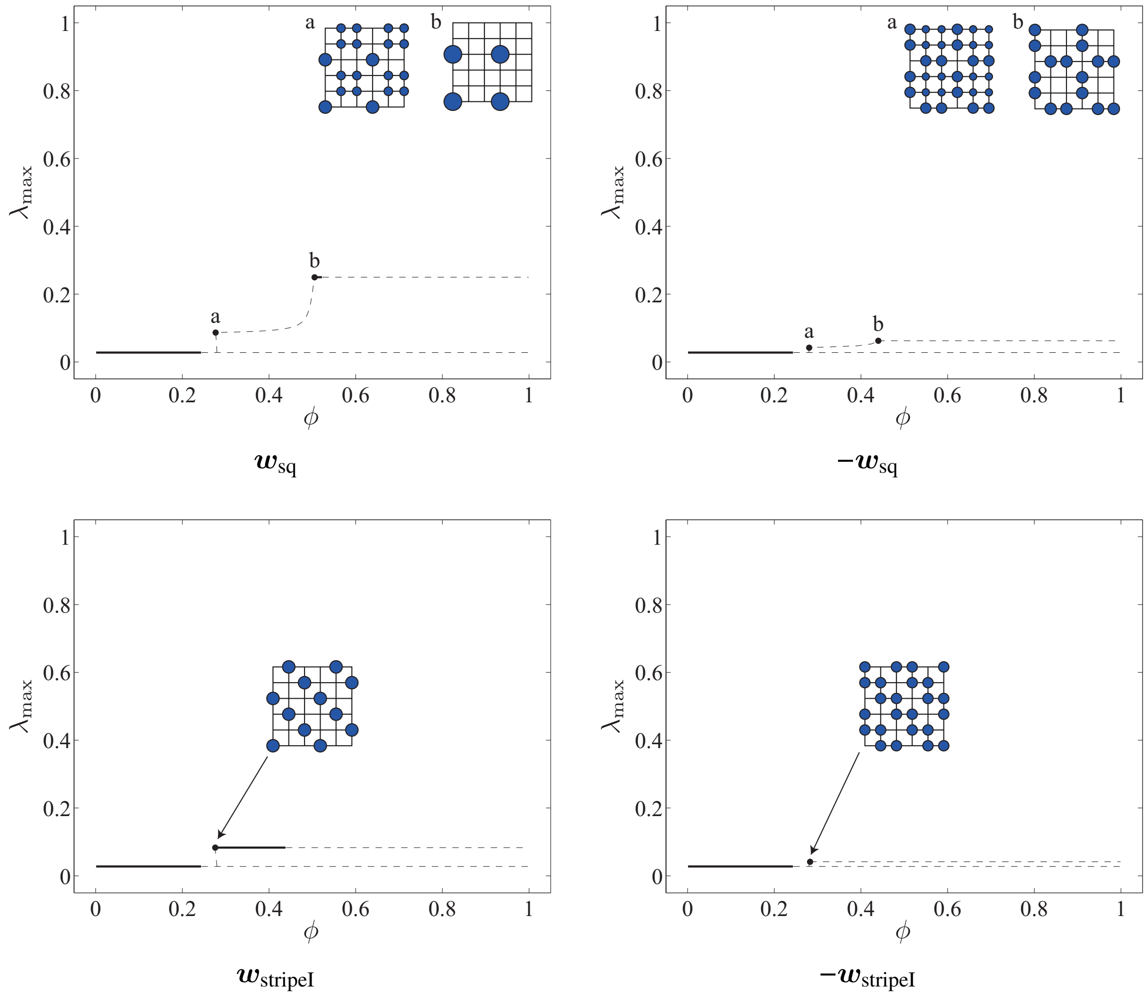


Figure 7.14: Bifurcating curves for $\mu = (4; 2, 2, +)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.

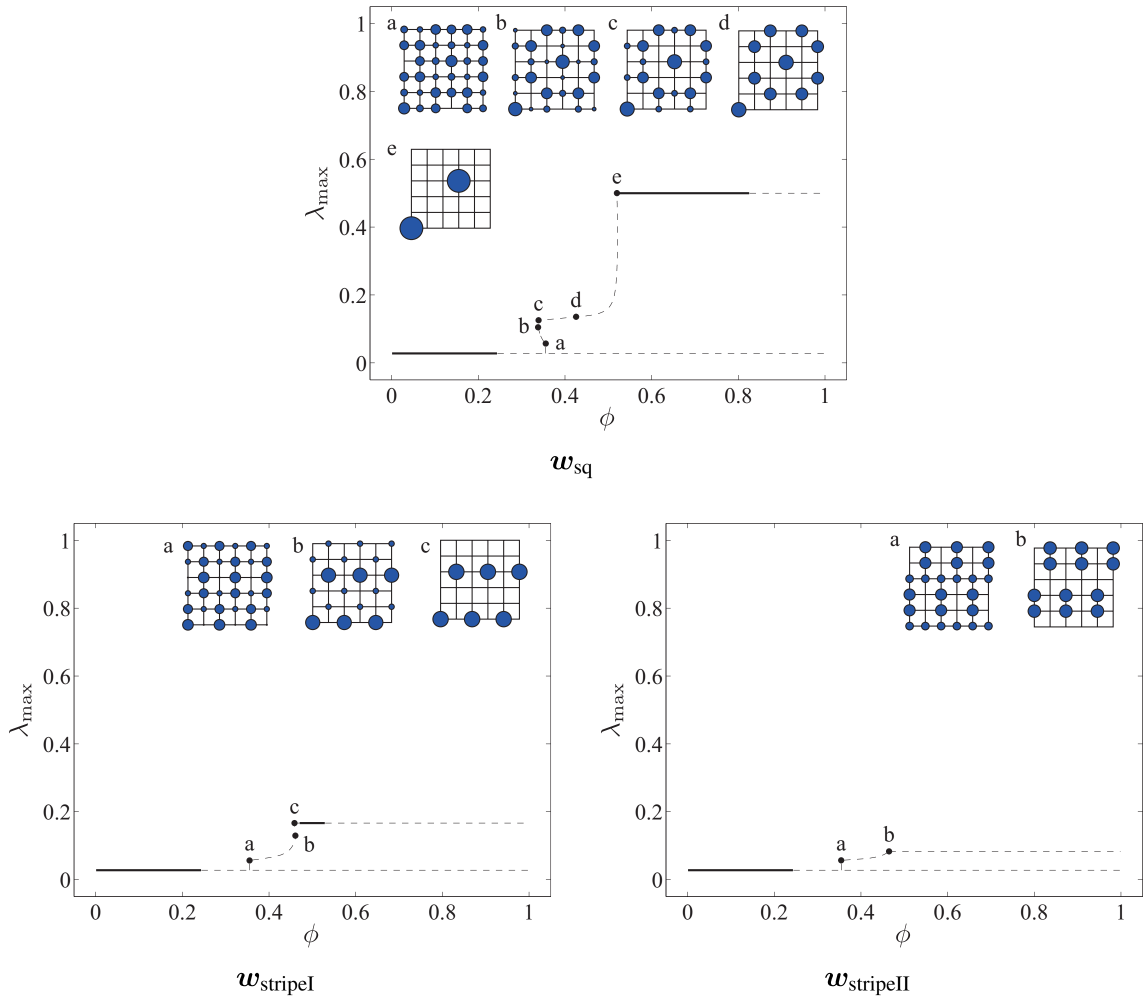
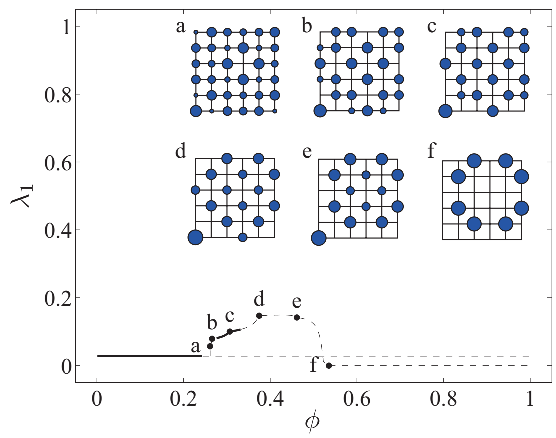
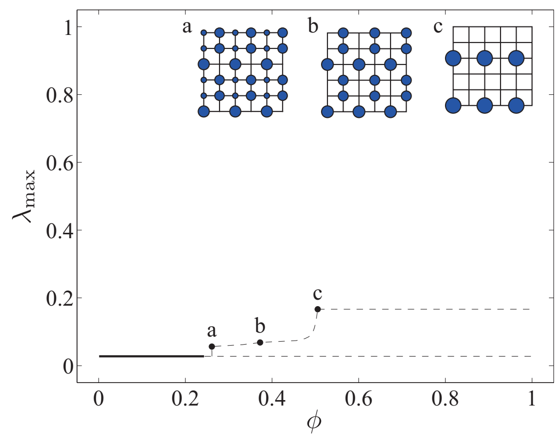


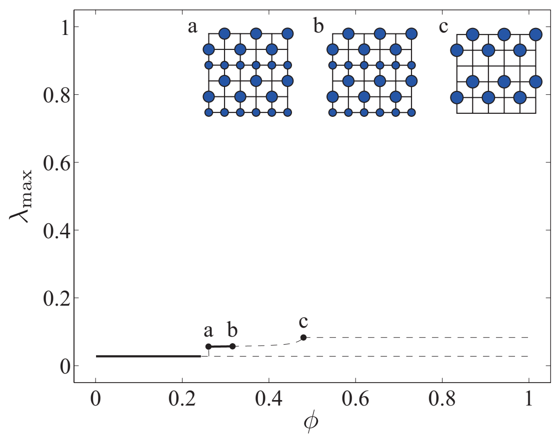
Figure 7.15: Bifurcating curves for $\mu = (4; 3, 1, +)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.



$\boldsymbol{w}_{\mathrm{sq}}$



$\boldsymbol{w}_{\mathrm{stripeI}}$



$\boldsymbol{w}_{\mathrm{stripeII}}$

Figure 7.16: Bifurcating curves for $\mu = (4; 3, 2, +)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.

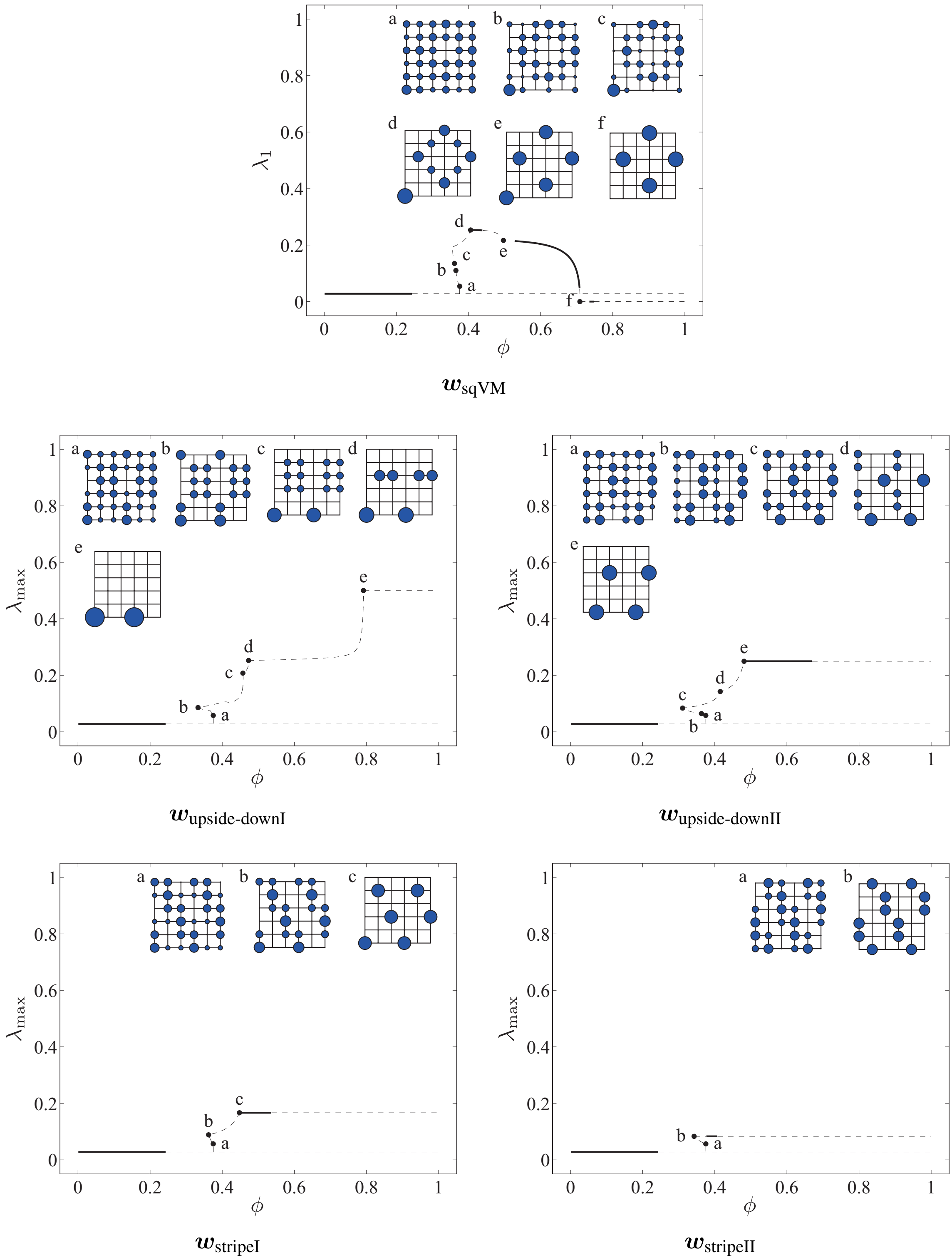


Figure 7.17: Bifurcating curves for $\mu = (8; 2, 1)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.

8. Concluding Remarks

This paper developed a group-theoretic methodology for analyzing spatial economic models on a square lattice in collaboration with nonlinear mathematics and new economic geography. Such a methodology provides an effective approach to elucidate the complicated agglomeration behaviour of spatial economic models systematically.

Chapters 2–4 provided preparation of fundamental issues for the group-theoretic bifurcation analysis. Chapter 2 introduced an $n \times n$ square lattice and a group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ expressing the symmetry of this lattice. Chapter 3 gave a series of irreducible representations of the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. Chapter 4 presented matrix representations of this group.

Chapters 5 and 6 revealed the mechanism of the self-organization of square patterns as bifurcation phenomena in a system of equations modeled on the square lattice. Two different approaches, using the equivariant branching lemma and solving the bifurcation equation, proceeded.

Chapter 7 applied the group-theoretic methodology to a prototype of spatial economic models. Square agglomeration patterns, which are consistent with bifurcating solutions revealed in the previous chapters, emerged from the uniform population distribution. Under the replicator dynamics, invariant patterns, which are connected with the bifurcating solutions via bifurcating curves, played a vital role in the elucidation of agglomeration behaviour as a solutions to the governing equation irrespective of the trade freeness.

As the theoretical contribution of this paper, we presented a complete list of typical bifurcating solutions from the uniform distribution on the square lattice for an arbitrary lattice size n (see Figure 7.3 for $n = 6$). As another kind of possible agglomeration patterns, we obtained invariant patterns on the square lattice. Invariant patterns display characteristic population distribution depicted in Figures 7.4–7.6.

In the numerical analysis of a prototype spatial economic model, the Forslid and Ottaviano model, we showed the connectivity between bifurcating solutions and invariant patterns via bifurcating solutions from the uniform state (Conjecture 1). We found a mesh-like structure of the solution curves for stable invariant patterns and unstable non-invariant ones. A knowledge of such a bifurcation mechanism would make a substantial contribution to the understanding of two-dimensional economic agglomerations. It is emphasized that this paper not only demonstrated the emergence of bifurcating solutions for the Forslid and Ottaviano model but also proposed a general framework to understand bifurcation behaviour applicable to any spatial economic model.

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