Mechanization, Task Assignment, and Inequality

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May 2021

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Abstract

Mechanization (or automation) has proceeded continuously since the Industrial Revolution and seems to have accelerated recently due to the rapid advancement of information technology. This paper theoretically examines long-run trends of mechanization, shifts of tasks humans perform, and earnings levels and inequality. Specifically, the paper develops a Ricardian model of task assignment and analyzes how improvements of productivities of machines and an increase in the relative supply of skilled workers affect task assignment (which factor performs which task), earnings levels and inequality, and aggregate output. The model succeeds in capturing the great majority of the long-run trends. The paper also explores possible future trends of the variables when information technology continues to grow rapidly.

JEL Classification Numbers: J24, J31, N30, O14, O33

Keywords: mechanization, automation, task assignment, earnings inequality, information technology

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1 Introduction

Mechanization (or automation)—the replacement by machines of humans (and animals) engaged in production tasks—has proceeded continuously since the Industrial Revolution and seems to have accelerated recently due to the rapid advancement of information technology. This paper theoretically examines long-run trends of mechanization, shifts of tasks humans perform, and earnings levels and inequality. Specifically, the paper develops a Ricardian model of task assignment and analyzes how improvements of productivities of machines and an increase in the relative supply of skilled workers affect task assignment (which factor performs which task), earnings levels and inequality, and aggregate output. The model succeeds in capturing the great majority of the long-run trends. The paper also explores possible future trends of the variables when information technology continues to grow rapidly.

Facts. The long-run trends the paper focuses on are as follows.

Mechanization: During the Industrial Revolution, mechanization progressed in tasks intensive in manual labor: in manufacturing (particularly, textile and metal working), machines and factory workers replaced artisans and farmers engaged as a side job; in transportation, railroads and steamboats supplanted wagons and sailboats; in agriculture, threshing machines and reapers reduced labor input.[1] During the Second Industrial Revolution (from the second half of the 19th century to World War I), with the utilization of electric power and internal combustion engines, mechanization proceeded further in manual tasks: in manufacturing, broader sectors and production processes were automated with the introduction of mass production system; a wider range of tasks were mechanized with tractors in agriculture and with automobiles and trucks in transportation. Some analytical (cognitive) tasks too were automated: tabulating machines substituted data-processing workers at large organizations. In the post World War II era, especially since the 1970s, analytical tasks in much wider areas have been automated because of the progress of information technology: computers replaced clerical workers engaged in information processing tasks; sensors automated inspection processes in manufacturing and services; and simple troubleshooting tasks were automated with the construction of databases of known troubles.[2]

Task shifts: As a result of mechanization, humans have shifted to tasks machines cannot perform efficiently. The general trend until about the 1960s is the shift from manual tasks to analytical tasks: initially, humans shifted from manual tasks at farms, cottages, and workshops to manual tasks at factories and analytical tasks at offices and factories (generally associated with clerical, management, and technical jobs); after mechanization deepened in manufacturing, they shifted from manual tasks at factories as well as at farms to analytical tasks (Katz and Margo, 2013).[3] Since the 1970s, humans have shifted from routine analytical

[1] Works on the two revolutions by economic historians include Landes (2003) and Mokyr (1985, 1999).

[2] Case studies of effects of information technology on the workplace include Autor, Levy, and Murnane (2002) on a commercial bank and Bartel, Ichniowski, and Shaw (2007) on a bulb manufacturing factory.

[3] Although it has been widely thought that technical change during the 19th century is *unskill*-biased, Katz and Margo (2013) show that this is *not* the case for the U.S.: while the share of middle-skill workers (artisans and agricultural operators) fell and shares of low-skill workers (unskilled workers and laborers) and high-skill workers (white collar) rose in manufacturing, *for the whole economy*, shares of low-skill and middle-skill workers fell and high-skill workers rose from 1850 to 1910. (Further, the share of middle-skill workers *changed little* if clerical/sales workers are classified as middle-skilled.) They also find that the same

tasks (e.g., simple information processing tasks performed by clerks) as well as manual tasks toward non-routine analytical tasks (mainly associated with professional and technical jobs) and non-routine manual tasks in services (e.g., personal care and protective service), owing to the advancement of information technology (Autor, Levy, and Murnane, 2003; Autor, 2019).[4,5] Since the 1990s, due to the large shift from routine analytical tasks, the growth of middle-wage jobs has been weak relative to both low-wage and high-wage jobs, i.e., job polarization has been observed (Goos, Manning, and Salomons, 2014; Autor, 2019).

Earnings levels and inequality: Mechanization has affected relative demands for workers of different skill levels and thus earnings levels and inequality. In the early stage of industrialization, earnings of unskilled workers grew very moderately and the inequality between skilled and unskilled workers enlarged (Feinstein, 1998; Katz and Margo, 2013).[6] In later periods, unskilled workers have benefited more from automation, except in the 1980-early 1990s and in the mid-late 2000s of the U.S. (Autor, 2019), while, as before, the rising inequality has been the norm in economies with lightly regulated labor markets (such as the U.S.), except in periods of rapid growth of the relative supply of skilled or educated workers (such as the 1970s) and in the wartime 1940s, when the inequality fell (Goldin and Katz, 2008).[7] Since the 1990s, associated with job polarization, earnings of workers with skills for middle-wage jobs have decreased relative to earnings of those with skills for low-wage jobs and those with skills for high-wage jobs at least in the U.S. (Böhm, 2020).[8]

The model. The model economy is a static small-open competitive economy where three kinds of factors of production—skilled workers, unskilled workers, and machines—are available. Each factor is characterized by *analytical ability* and *manual ability.* Skilled workers have a higher level of analytical ability than unskilled workers, while both types

pattern is observed for the whole economy from 1920 to 1980 and main contributors of the declining share of low-skill workers were farm laborers until around 1950 and are unskilled workers and laborers (largely in manufacturing) thereafter.

[4]Similarly to Autor, Levy, and Murnane (2003), routine tasks refer to tasks whose procedures are organized so that they can be performed by machines after relevant technologies are developed.

[5]Autor, Levy, and Murnane (2003) examine changes in the composition of tasks in the U.S. from 1960 to 1998 and find that the growth of information technology is important in explaining the changes after the 1970s. Autor (2019) presents changes in occupational composition for 1970−2016.

[6]Feinstein (1998) finds that real wages of British manual workers rose very moderately from the 1770s to the 1850s (stagnated until the 1830s), implying a large increase in the disparity with skilled workers. Katz and Margo (2013) find a secular rise in the wage premium for white-collar workers for 1820−80 in the U.S..

[7]Goldin and Katz (2008) document that, after plummeting in the 1940s, the wage premium of college graduates in the U.S. kept rising except in the 1970s when the relative supply of college graduates grew rapidly. As for the wage premium of high school graduates, which is a good measure of inequality between skilled and unskilled workers until the 1940s (judging from a low elasticity of substitution between high school graduates and dropouts), it fell greatly from 1914 to 1939, when high school enrollment rates rose dramatically (from 20% to over 70%) and in the 1940s.

[8]Böhm (2020) finds that task prices (earnings per unit of skill) polarized between 1984–1992 and 2007–2009 in the U.S.: task prices of middle-wage jobs (such as clerical, sales, and production jobs) fell relative to high-wage jobs (managerial, professional, and technical jobs) and low-wage jobs (service jobs). Further, he showed that wages of those with comparative advantages in middle-wage jobs fell compared to wages of those with comparative advantages in high-wage or low-wage jobs. By contrast, *wage polarization* (the slower wage growth of middle-wage *jobs* relative to low-wage and high-wage jobs) is observed during the 1990s only in the U.S. and its evidence is weak in Europe (Autor, 2015; Naticchioni, Massari, and Ragusa, 2014).

of workers have the same level of manual ability, reflecting the fact that there is no strong correlation between the two abilities.

The final good is produced from inputs of a continuum of *tasks* that are different in *the importance of analytical ability,* a, and *the ease of codification* (*routinization*), c, using a Leontief technology.[9] In the real economy, low a and high c tasks are those involving repetitive motions such as assembling or sorting objects and typical in production jobs; low a and low c tasks are those entailing non-repetitive motions such as driving vehicles and caring for the elderly and usual in low-wage service jobs; high a and high c tasks entail simple information processing such as calculation and recording information and are typical in clerical jobs; and high a and low c tasks involve complex analysis and judgement mainly associated with management, professional, and technical jobs.

The three factors are perfectly substitutable at each task. Both abilities contribute to production at each task (except the most manual and the most analytical tasks), but the relative contribution of analytical ability is larger in tasks of the greater importance of the ability (higher a). Among tasks with given a, machines are more productive in tasks of the greater ease of codification (higher c), while workers' productivities do not depend on c.

Task assignment, factor prices, task prices, and output of a competitive equilibrium are considered. Comparative advantages of factors determine task assignment: unskilled (skilled) workers are assigned to relatively manual (analytical) tasks and machines are assigned to tasks that are easier to codify. Among tasks a given factor is employed, it is employed intensively in tasks in which its productivities are low.

Main results. Based on the model, the paper examines how task assignment, earnings levels and inequality, and output change over time, when analytical and manual abilities of machines and the relative supply of skilled workers grow exogenously over time.

Section 4 examines a simpler case (many of the results can be derived from a graphical analysis) in which the two abilities grow proportionately and machines have comparative advantages in relatively manual tasks. The analysis shows that tasks and workers strongly affected by mechanization and effects of the productivity growth on wage levels and inequality change over time. Mechanization starts from tasks that are highly manual and easy to routinize, and gradually spreads to tasks that are more analytical and difficult to routinize. Eventually, automation proceeds in highly analytical tasks previously performed by skilled workers too. Accordingly, unskilled workers shift to tasks that are more difficult to codify, so do skilled workers in later stages of mechanization, and both types shift to more analytical tasks except at the final stage. Skilled workers always benefit from the productivity growth, whereas the effect on earnings of unskilled workers is ambiguous while mechanization mainly affects them and the effect turns positive afterwards. Earnings inequality rises except in the final stage of mechanization, where it is constant. The output of the final good always increases. In contrast, an increase in the relative supply of skilled workers raises (lowers) earnings of unskilled (skilled) workers and lowers the inequality, countervailing the inequality-enhancing effect of productivity growth. (It also raises output.)

The results are consistent with the long-run trends of task shifts, wage levels, and its inequality described earlier, except the developments of earnings levels and inequality in the

[9] In this paper, the term codify/routinize means "organize procedures of tasks systematically so that tasks can be performed by machines after relevant technologies are developed".

wartime 1940s during which institutional factors are likely to be important (Goldin and Katz, 2008; Farber et al., 2021), their developments after the 1980s, and job polarization after the 1990s. However, the assumption that the two abilities of machines grow proportionately, which makes the analysis relatively simple, is rather restrictive, considering that the growth of the manual ability of machines was faster than their analytical ability for most periods of time, while the opposite seems to be true recently.

Hence, Section 5 analyzes the general case in which the two abilities may grow at different rates. Under realistic productivity growth, the model does much better jobs in explaining the developments after the 1980s, such as stagnant earnings of unskilled workers and the rising inequality in the 1980-early 1990s, than under the special case. Notably, the model shows that skilled workers shift *from non-routine analytical tasks* to manual tasks when the growth of analytical ability of machines is fast, consistent with the development after around the year 2000 in the U.S. (Beaudry, Green, and Sand, 2016).[10] Although the present model with two types of workers cannot capture the whole picture of the falling relative wage of workers with skills for middle-wage jobs after the 1990s (Böhm, 2020) (the model with three types of workers is analytically intractable), the decreasing inequality predicted by the model captures a part of the development, the falling disparity between workers with skills for low-wage jobs and those with skills for middle-wage jobs (and moderately high-wage jobs more recently).[11]

Finally, the model is used to examine possible future trends of the variables when information technology and thus the analytical ability of machines continue to grow rapidly. It is found that earnings of both types of workers increase and earnings inequality falls over time. Although the analysis based on the model with two types of workers may not capture the whole picture considering the recent widening inequality between moderately and extremely skilled workers (Alvaredo et al., 2013), the stagnant wage premium of college graduates in the 2010s (Autor, Goldin, and Katz, 2020) and episodes such as the increasing use of big data in marketing, management, and other decisions suggest that machines would replace many tasks presently performed by skilled workers in the not-distant future and thus possible effects on a great majority of the population might be captured by the model.

Related literature. The paper belongs to the literature on task (job) assignment model, which has been developed to analyze the distribution of earnings in labor economics (see Sattinger, 1993, for a review), and recently is used to examine broad issues, such as effects of technology on the labor market (Acemoglu and Autor, 2011; Acemoglu and Restrepo, 2018; Hémous and Olsen, 2020), on cross-country productivity differences (Acemoglu and Zilibotti, 2001), and on organizational structure and wages (Garicano and Rossi-Hansberg, 2006), effects of international trade and offshoring on the labor market (Grossman and Rossi-Hansberg, 2008; Costinot and Vogel, 2010; Grossman, Helpman, and Kircher, 2017), and inter-industry wage differentials and the effect of trade on wages (Sampson, 2016).

[10] Beaudry, Green, and Sand (2016) find that the employment growth of non-routine analytical jobs stalled after around 2000 despite the continuing growth of the supply of high-skill workers, suggesting a decrease in the demand for such jobs. Further, they show that the average intensity of non-routine analytical tasks for college graduates increased from the early 1980s until around 2000 but decreased thereafter.

[11] The quantitative model with three type of workers who differ in levels of analytical ability would yield the rising disparity between workers with skills for high-wage jobs and other workers as well.

The most closely related is Acemoglu and Autor (2011), who argue that the conventional non-assignment model cannot examine shifts in tasks workers with a given skill level perform and fails to capture a large part of recent trends of task shifts, earnings levels and inequality, particularly job and wage polarization and stagnant or negative earnings growth of less-educated workers in the U.S.,[12] and develop a task assignment model with three types of workers (high-skill, middle-skill, low-skill). The final good is produced from inputs of a continuum of tasks that are different in the degree of 'complexity' using a Cobb-Douglas technology. High (middle) skill workers have comparative advantages in more complex tasks against middle (low) skill workers. They analyze the situation where a part of tasks initially performed by middle-skill workers are mechanized exogenously, and show that a fraction of them shift to tasks previously performed by the other types of workers and relative earnings of high-skill workers to middle-skill workers rise and those of middle-skill workers to low-skill workers fall, reproducing job and wage polarization.[13]

The present paper builds on their work, particularly in the modeling, but there are several important differences. First, the paper is interested in the long-run trends of task shifts, earnings levels and inequality since the Industrial Revolution, while they focus on the recent development, especially job and wage polarization after the 1990s. Second, the paper examines how tasks and workers strongly affected by mechanization and its effects on earnings levels and inequality *change endogenously over time* with improvements of manual and analytical abilities of machines, whereas, because of their focus on job and wage polarization, they assume that mechanization occurs at tasks previously performed by middle-skill workers. Third, the present model assumes that tasks are different in two dimensions, the importance of analytical ability and the ease of codification (routinization), while, in their model, tasks are different in one dimension, the degree of 'complexity', which is also the case in the dynamic model of Acemoglu and Restrepo (2018).[14,15] Because of the characteriza-

[12] Limitations of the conventional model, in which workers with different skill levels are imperfect substitutes in a macro production function, pointed out by them and relevant to this paper are: (i) technical change is factor-augmenting, thus it does not model mechanization through technical change, which is also pointed out in the literature on growth models with mechanization reviewed below, (ii) the model cannot explain stagnant or negative earnings growth of particular groups in a growing economy, (iii) since all workers with a given skill level have the same 'job', shifts in jobs and tasks performed by particular groups cannot be examined, (iv) systematic changes in the composition of employment by job (task) cannot be analyzed.

[13] They also examine the situation where a part of tasks initially performed by middle-skill workers are offshored exogenously. Further, they analyze the effect of changes in factor supplies on technical change using a version of the model with endogenous factor-augmenting technical change.

[14] Acemoglu and Restrepo (2018) develop a dynamic task assignment model with two types of technological changes, the automation of tasks (the replacement of labor by capital) and the development of new tasks replacing the least 'complex' existing tasks. Their main interests are to characterize conditions for asymptotically stable balanced growth for a version of the model with directed technological changes and one type of labor (and capital and intermediates embodying technologies) and to examine the effect of shocks to technologies on factor prices and factor shares in employment and income. In an extension, they also consider a version of the model with exogenous technological changes and two types of labor (skilled labor has a comparative advantage in more 'complex' tasks) and examine the effect of technological changes on wage inequality. In particular, they show that automation raises wage inequality.

[15] Hémous and Olsen (2020) develop a dynamic model with two types of technological changes a la Acemoglu and Restrepo (2018) and with high- and low-skill workers. Unlike Acemoglu and Restrepo (2018) and the present paper, different tasks are symmetric (except whether they are automated or not), and unlike this

tion of tasks by the two variables, which is a natural extension of the analytical/manual and routine/non-routine classification of tasks standard in empirical works initiated by Autor, Levy, and Murnane (2003), types of workers displaced by machines and effects of mechanization on earnings levels and inequality change over time.

The paper is also related to the literature that examines the interaction between mechanization and economic growth, such as Zeira (1998, 2010), Givon (2006), Zuleta (2008), Acemoglu (2010), Peretto and Seater (2013), Aghion, Jones, and Jones (2019), and Ray and Mookherjee (2020). The literature is mainly interested in whether persistent growth is possible in models where economies grow through mechanization and whether the dynamics are consistent with stylized facts of growth. While the standard model assumes labor-augmenting technical change, which is labor-saving but *not* capital-using (thus does not capture mechanization), these papers (except Zeira, 2010; Ray and Mookherjee, 2020) consider technical change that is labor-saving *and* capital-using.[16] Such technical change yields a declining share of labor income or a long-run constant share, depending on production technologies. By contrast, for given technologies, Zeira (2010) examines interactions among capital accumulation, changes in factor prices, and mechanization. His model can be interpreted as a dynamic task assignment model after a slight modification of the production technology. Unlike the present model, the model assumes homogenous labor and constant productivity of machines. Ray and Mookherjee (2020) develop a general dynamic model of task assignment with physical and human capital accumulations and provide conditions for the long-run labor income share to converge to 0. They are not concerned with the transitional dynamics and the personal distribution of income

Organization of the paper. The paper is organized as follows. Section 2 presents the model and Section 3 derives equilibrium allocations for given machine abilities. Section 4 examines effects of improved machine abilities and increased relative supply of skilled workers on task assignment, earnings levels and inequality, and aggregate output, when the two abilities improve proportionately. Section 5 examines the general case in which the abilities may improve at different rates, and Section 6 concludes. Appendix A presents lemmas. Appendix B contains proofs of lemmas and propositions of Section 4, while proofs of propositions of the general case, which are very lengthy, are contained in Web Appendix.[17]

2 Model

Consider a small open economy where three types of factors of production—skilled workers, unskilled workers, and machines—are available. All markets are perfectly competitive.

Factors of production and tasks: Each factor is characterized by *analytical ability* and *manual ability*. Denote analytical abilities of a skilled worker, an unskilled worker, and

paper, the production technology is such that only low-skill workers can be displaced by machines. They show that an increase in the share of automated tasks raises skilled wage and skill premium, lowers labor income share, and has an ambiguous effect on unskilled wage. They examine quantitatively how well the model can explain evolutions of wages, wage inequality, and labor income share of the U.S. after the 1960s.

[16] Acemoglu (2010) examines whether labor scarcity encourages technological advances and shows that it does if technology is strongly labor saving. He also shows that models with mechanization-type technological change have a tendency for strongly labor-saving technology, based on the Zeira (1998) model.

[17] The address is http://www.econ.kyoto-u.ac.jp/~yuki/english.html.

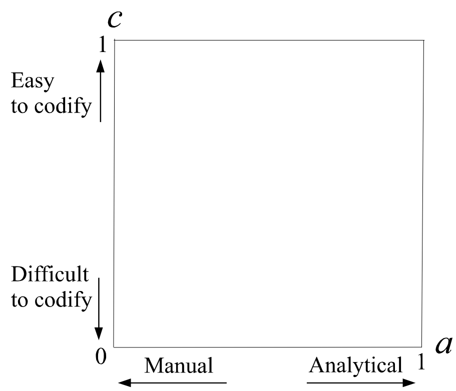


Figure 1: A continuum of tasks

a machine by h, l_a, and k_a, respectively, where $h > l_a$, and their manual abilities by l_m, l_m, and k_m, respectively. Skilled and unskilled workers have the same level of manual ability, reflecting the fact that there is no strong correlation between the two abilities.

The final good is produced from inputs of a continuum of *tasks* that are different in *the importance of analytical ability*, $a \in [0, 1]$, and *the ease of codification* (*routinization*), $c \in [0, 1]$ (Figure 1). In the real economy, low a and high c tasks are those involving repetitive motions such as assembling or sorting objects and are typical in production jobs; low a and low c tasks are those entailing non-repetitive motions such as driving vehicles and caring for the elderly and are important in low-wage service jobs; high a and high c tasks entail simple information processing such as calculation and recording information and are typical in clerical jobs; high a and low c tasks involve complex analysis and judgement mainly associated with management, professional, and technical jobs. The characterization of tasks by the two variables, a and c, is a natural extension of the analytical/manual and routine/non-routine classification of tasks standard in empirical works initiated by Autor, Levy, and Murnane (2003).

Tasks are uniformly distributed over the (a, c) space. Productivities of a skilled worker, an unskilled worker, and a machine in task (a, c) are given by:

$$A_h(a) = ah + (1 - a)l_m, \tag{1}$$

$$A_l(a) = al_a + (1 - a)l_m, \tag{2}$$

$$cA_k(a) = c[ak_a + (1 - a)k_m]. \tag{3}$$

Except the most manual tasks ($a = 0$) and the most analytical tasks ($a = 1$), both abilities contribute to the production of each task, but the relative contribution of analytical ability is greater in tasks with higher a.[18] Since $h > l_a$, skilled workers have comparative advantages in more analytical tasks relative to unskilled workers. For given a, machines are more productive in tasks with higher c, while workers are assumed to be equally productive for any c. Because of the multiplicative form of (3), irrespective of levels of k_a and k_m, humans

[18] One interpretation of the linear specification is that task (a, c) is composed of the proportion a of analytical subtasks, where only analytical ability matters, and the proportion $1 - a$ of manual ones, and the two types of subtasks requiring different abilities are perfectly substitutable in the production of the task.

are more productive than machines in tasks with very low c, ensuring that humans can always find tasks to engage in.[19]

Production: At each task, factors are perfectly substitutable as in Acemoglu and Autor (2011) and Acemoglu and Restrepo (2018), thus the production function of task (a, c) equals:

$$y(a,c) = A_h(a)n_h(a,c) + A_l(a)n_l(a,c) + cA_k(a)n_k(a,c), \tag{4}$$

where $n_i(a,c)$ $(i = h, l, k)$ is the amount of factor i engaged in the task. The output of the task, $y(a,c)$, may be interpreted as either an intermediate good or a direct input in final good production, which is produced by either final good producers or separate entities.

The final good production function is Leontief with equal weights on all tasks, that is, all tasks are equally essential in the production:

$$Y = \min_{a,c}\{y(a,c)\}. \tag{5}$$

The Leontief specification is assumed for simplicity. Similar results would be obtained as long as different types of tasks are complementary in the production, though more general specifications seem to be analytically intractable.[20]

To summarize, different tasks are complementary in final good production, but different factors are perfectly substitutable at each task. Because of this specification and the two dimensional task space, types of workers displaced by machines and effects of mechanization on earnings levels and inequality change over time.

Factor markets: A unit of each factor supplies a unit of time inelastically. Let the final good be the numeraire and let the relative price of (the output of) task (a, c) be $p(a, c)$. Then, from cost minimization problems,

$$p(a,c) = \min\left\{\frac{w_h}{A_h(a)}, \frac{w_l}{A_l(a)}, \frac{r}{cA_k(a)}\right\}, \tag{6}$$

where w_h (w_l) is earnings of a skilled (an unskilled) worker and r is exogenous interest rate.[21] That is, firms choose a factor(s) so that a unit cost of task production becomes lowest.

From (6), the basic pattern of *task assignment* can be derived (details are explained later). Since the relative productivity of skilled to unskilled workers $\frac{A_h(a)}{A_l(a)}$ increases with a, there exists unique $a^* \in (0,1)$ satisfying $\frac{A_h(a^*)}{A_l(a^*)} = \frac{w_h}{w_l}$ and skilled (unskilled) workers are chosen over unskilled (skilled) workers in tasks with $a > (<) a^*$, i.e., tasks satisfying $\frac{A_h(a)}{A_l(a)} > (<) \frac{w_h}{w_l}$. That is, skilled (unskilled) workers are assigned to relatively analytical (manual) tasks. For $a < a^*$, unskilled workers (machines) are assigned to tasks satisfying $\frac{A_l(a)}{cA_k(a)} > (<) \frac{w_l}{r}$,

[19] The maximum value of c is set to be 1 for simplicity. Qualitative results do not change if it is any other finite number. The maximum c must be finite to explain the fact that highly analytical tasks were not mechanized at least before the Second Industrial Revolution.

[20] Also, the model with a Cobb-Douglas technology seems to be very difficult to analyze. An advantage of the Leontief specification over the Cobb-Douglas is that, as shown below, it yields a realistic result that, among tasks in which a given factor is employed, it is employed intensively in tasks in which their productivities are low.

[21] The closed economy model is analytically intractable. Considering that the real interest rate has been stable in the U.K. and the U.S. over the long-run, main results would not be affected much by the assumption of the small open economy.

and for $a > a^*$, skilled workers (machines) are assigned to tasks satisfying $\frac{A_h(a)}{cA_k(a)} > (<) \frac{w_h}{r}$. Comparative advantages of factors and relative factor prices determine task assignment.

Task (intermediate good) markets: Because each task (intermediate good) is equally essential in final good production, $y(a, c) = Y$ must hold for any (a, c). Thus, the following is true for any (a, c) with $n_h(a, c) > 0$, any (a', c') with $n_l(a', c') > 0$, and any (a'', c'') with $n_k(a'', c'') > 0$, except for the set of measure 0 tasks in which multiple factors are employed:

$$A_h(a)n_h(a, c) = A_l(a')n_l(a', c') = c''A_k(a'')n_k(a'', c'') = Y. \tag{7}$$

Among tasks in which a given factor is employed, it is employed intensively in tasks in which its productivity is low, e.g., $n_h(a, c)$ is large in tasks with low $A_h(a)$.

Denote the amount of total supply of factor i $(i = h, l, k)$ by N_i, where N_k is endogenous. Then, by substituting (7) into $\iint_{n_i(a,c)>0} n_i(a, c)dadc = N_i$,

$$\frac{N_h}{\iint_{n_h(a,c)>0} \frac{1}{A_h(a)} dadc} = \frac{N_l}{\iint_{n_l(a,c)>0} \frac{1}{A_l(a)} dadc} = \frac{N_k}{\iint_{n_k(a,c)>0} \frac{1}{cA_k(a)} dadc} = Y. \tag{8}$$

The first equality of (8) is *one of the two key equations*, which holds when task assignment is determined so that market clearing conditions are satisfied for both type of workers.

Since a unit of the final good is produced from inputs of a unit of every task and the final good is the numeraire,

$$\iint p(a, c)dadc = 1 \tag{9}$$

$$\Leftrightarrow w_l \iint_{n_l(a,c)>0} \frac{1}{A_l(a)} dadc + w_h \iint_{n_h(a,c)>0} \frac{1}{A_h(a)} dadc + r \iint_{n_k(a,c)>0} \frac{1}{cA_k(a)} dadc = 1, \tag{10}$$

where the second equation is from (6). (10) is *the other key equation*, which states that task assignment must be such that the unit production cost of the final good equals 1.

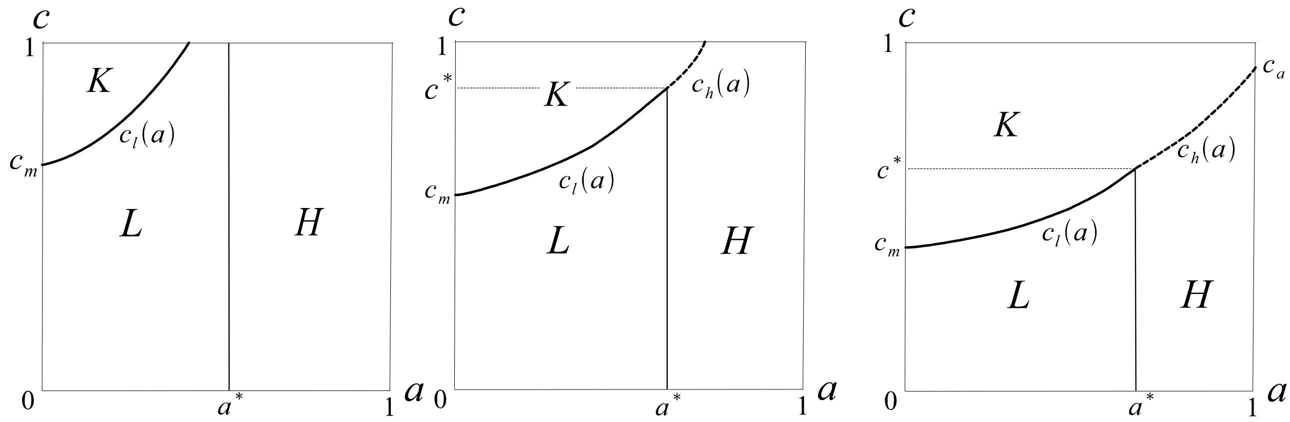
Equilibrium: A competitive equilibrium is defined by (6)$-$(8), (10), and task assignment conditions such as $\frac{A_h(a^*)}{A_l(a^*)} = \frac{w_h}{w_l}$ that are derived explicitly in the next section. As explained next, task assignment and wages w_h, w_l are determined by the first equality of (8), (10), and the task assignment conditions. Then, N_k and Y $(= y(a, c))$ are determined from the second and third equalities of (8), respectively; $n_i(a, c)$ $(i = h, l, k)$ is determined from (7); $p(a, c)$ is determined from (6).

3 Analysis

This section derives task assignment and wages explicitly for given levels of machine abilities k_a and k_m. So far, no assumptions are imposed on comparative advantages of machines to workers. Until Section 5, it is assumed that $\frac{k_a}{k_m} < \frac{l_a}{l_m} (< \frac{h}{l_m})$, that is, machines have comparative advantages in relatively manual tasks. Then, $\frac{A_l(a)}{A_k(a)}$ and $\frac{A_h(a)}{A_k(a)}$ increase with a. With this assumption, the task assignment conditions can be stated explicitly.

3.1 Task assignment conditions

Remember that, for $a < a^*$, unskilled workers (machines) perform tasks (a, c) with $\frac{A_l(a)}{cA_k(a)} > (<) \frac{w_l}{r}$, and for $a > a^*$, skilled workers (machines) perform tasks (a, c) with $\frac{A_h(a)}{cA_k(a)} > (<) \frac{w_h}{r}$, where a^* is defined by



(a) When $c^* = c_a = 1$ (b) When $c^* < c_a = 1$ (c) When $c^* < c_a < 1$

Note: $c^* \equiv min\{c_l(a^*), 1\}, c_a \equiv min\{c_h(1), 1\}$

Figure 2: Examples of task assignment when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$

$$\frac{A_h(a^*)}{A_l(a^*)} = \frac{w_h}{w_l}. \tag{11}$$

Further, since $\frac{k_a}{k_m} < \frac{l_a}{l_m} (< \frac{h}{l_m})$, machines (humans) perform tasks with relatively low (high) a and high (low) c; thus, for given c, machines perform tasks with $a > a^*$ only if they perform all tasks with $a \leq a^*$. Based on these results, critical variables and functions determining task assignment, $c_m, c^*, c_a, c_l(a)$, and $c_h(a)$, are defined next. Figure 2, which illustrates task assignment when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, is useful for understanding the following.

Unskilled workers vs. machines: From the above discussion, if $n_k(a, c) > 0$ for some (a, c), $n_k(0, 1) > 0$, i.e., whenever machines are used, they are employed in the most manual and easiest-to-codify task. Thus, there exists the level of $c \in (0, 1)$, denoted c_m, such that firms are indifferent between using machines and using unskilled workers for task $(0, c_m)$ (see Figure 2).[22] Formally, c_m is defined by

$$\frac{A_l(0)}{c_m A_k(0)} = \frac{l_m}{c_m k_m} = \frac{w_l}{r}. \tag{12}$$

From this equation, the similar condition for $a > 0$, $\frac{A_l(a)}{cA_k(a)} = \frac{w_l}{r}$, is expressed as $\frac{A_l(a)}{cA_k(a)} = \frac{l_m}{c_m k_m} \Leftrightarrow c = \frac{k_m}{l_m} \frac{A_l(a)}{A_k(a)} c_m$. Let $c_l(a) \equiv \frac{k_m}{l_m} \frac{A_l(a)}{A_k(a)} c_m$. Given a, using machines and unskilled workers are indifferent at $c = c_l(a)$ and machines (unskilled workers) are employed for $c > (<) c_l(a)$. If there exists $c < 1$ such that the two choices are indifferent at task (a^*, c), i.e., $c_l(a^*) < 1$, machines perform some tasks with $a > a^*$ (Figure 2 (b) and (c)). If $c_l(a^*) \geq 1$, machines do not perform tasks for skilled workers (Figure 2 (a)). Let $c^* \equiv \min\{c_l(a^*), 1\}$.

Skilled workers vs. machines: When $c^* < 1$, the choice between machines and skilled workers arises. From $\frac{A_h(a^*)}{A_l(a^*)} = \frac{w_h}{w_l}$ and (12), the condition $\frac{A_h(a)}{cA_k(a)} = \frac{w_h}{r}$ can be expressed as $\frac{A_h(a)}{cA_k(a)} = \frac{l_m}{k_m} \frac{A_h(a^*)}{A_l(a^*)} \frac{1}{c_m} \Leftrightarrow c = c_h(a) \equiv \frac{k_m}{l_m} \frac{A_l(a^*)}{A_h(a^*)} \frac{A_h(a)}{A_k(a)} c_m$. Given a, employing either factor is

[22] When machines are not employed in any task, c_m is set to be equal 1.

indifferent at $c = c_h(a)$. If there exists $c < 1$ such that either choice is indifferent at task $(1, c)$, i.e., $c_h(1) < 1$, machines perform some tasks with $a = 1$ (Figure 2 (c)). Let $c_a \equiv \min\{c_h(1), 1\}$.

Patterns of task assignment are clear from Figure 2. Given a, machines perform tasks with relatively high c. From the assumption that machines have comparative advantages in relatively manual tasks, given c, they perform tasks with relatively low a and the proportion of tasks performed by machines decreases with a, i.e., $c_l(a)$ and $c_h(a)$ are upward sloping.

3.2 Key equations determining equilibrium, (HL) and (P)

From their definitions, $c_l(a)$, $c_h(a)$, c^*, and c_a are functions of a^* and c_m:

$$c_l(a) = \frac{k_m}{l_m}\frac{A_l(a)}{A_k(a)}c_m, \quad c_h(a) = \frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{A_h(a)}{A_k(a)}c_m, \tag{13}$$

$$c^* \equiv \min\{c_l(a^*), 1\}, \quad c_a \equiv \min\{c_h(1), 1\}. \tag{14}$$

From (12) and (11), wages are expressed as functions of a^* and c_m:

$$w_l = \frac{l_m}{k_m}\frac{r}{c_m}, \quad w_h = \frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{r}{c_m}. \tag{15}$$

Hence, the two key equations determining equilibrium, the first equality of (8) and (10), can be expressed as (see Figure 2 for the ranges of integrations):

$$\frac{N_h}{N_l}\int_0^{a^*}\int_0^{\min\{c_l(a),1\}}\frac{1}{A_l(a)}dcda = \int_{a^*}^1\int_0^{\min\{c_h(a),1\}}\frac{1}{A_h(a)}dcda, \tag{HL}$$

$$\frac{l_m}{k_m}\frac{r}{c_m}\int_0^{a^*}\int_0^{\min\{c_l(a),1\}}\frac{1}{A_l(a)}dcda + \frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{r}{c_m}\int_{a^*}^1\int_0^{\min\{c_h(a),1\}}\frac{1}{A_h(a)}dcda$$
$$+r\left[\int_0^{a^*}\int_{\min\{c_l(a),1\}}^1\frac{1}{cA_k(a)}dcda + \int_{a^*}^1\int_{\min\{c_h(a),1\}}^1\frac{1}{cA_k(a)}dcda\right] = 1, \tag{P}$$

(HL) and (P) determine values of a^* and c_m. Then, $c_l(a), c_h(a), c^*, c_a$, and thus task assignment are determined from (13) and (14), earnings are determined from (15), and the remaining variables are determined as stated at the end of Section 2.

3.3 Illustration of the determination of equilibrium a^* and c_m

The determination of equilibrium a^* and c_m can be illustrated using a figure depicting graphs of (HL) and (P) on the (a^*, c_m) space. Since the shape of (HL) differs depending on whether c^* and c_a equal 1 or not, as shown in Figure 3, the (a^*, c_m) space is divided into three regions based on values of a^* and c_m: when $c_m \geq \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}$, $c^* = c_a = 1$ holds;[23] when $c_m \in \left[\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}, \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}\right)$, $c^* < c_a = 1$ holds;[24] when $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$, $c^* < c_a < 1$ holds.

[23] This is because $c_m \geq \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} \Leftrightarrow \frac{A_l(a^*)}{1\times A_k(a^*)} \geq \frac{l_m}{c_m k_m} = \frac{w_l}{r}$ from (15), that is, unskilled workers are weakly chosen over machines at task $(a^*, 1)$, which implies that machines are not used in any tasks with $a > a^*$.

[24] This is because $c_m < \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} \Leftrightarrow \frac{A_l(a^*)}{1\times A_k(a^*)} < \frac{w_l}{r}$ and $c_m \geq \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow \frac{h}{1\times k_a} \geq \frac{l_m}{c_m k_m}\frac{A_h(a^*)}{A_l(a^*)} = \frac{w_h}{r}$ from (15), that is, machines are strictly chosen over unskilled workers at task $(a^*, 1)$ and skilled workers are weakly chosen over machines at task $(1, 1)$, which implies that machines are employed in some tasks with $a > a^*$ but not in tasks with $a = 1$ and $c < 1$.

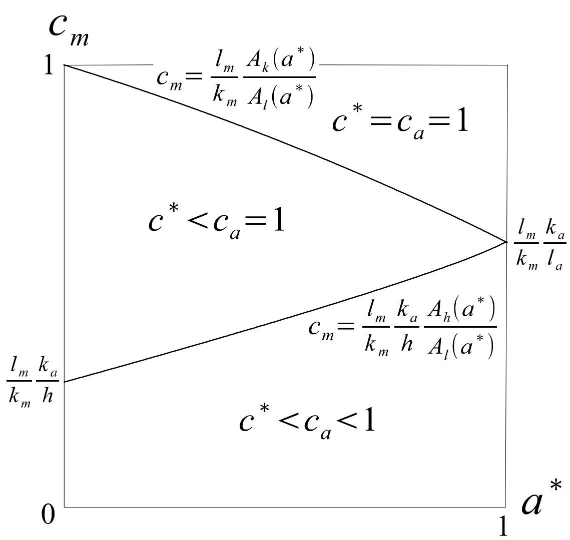
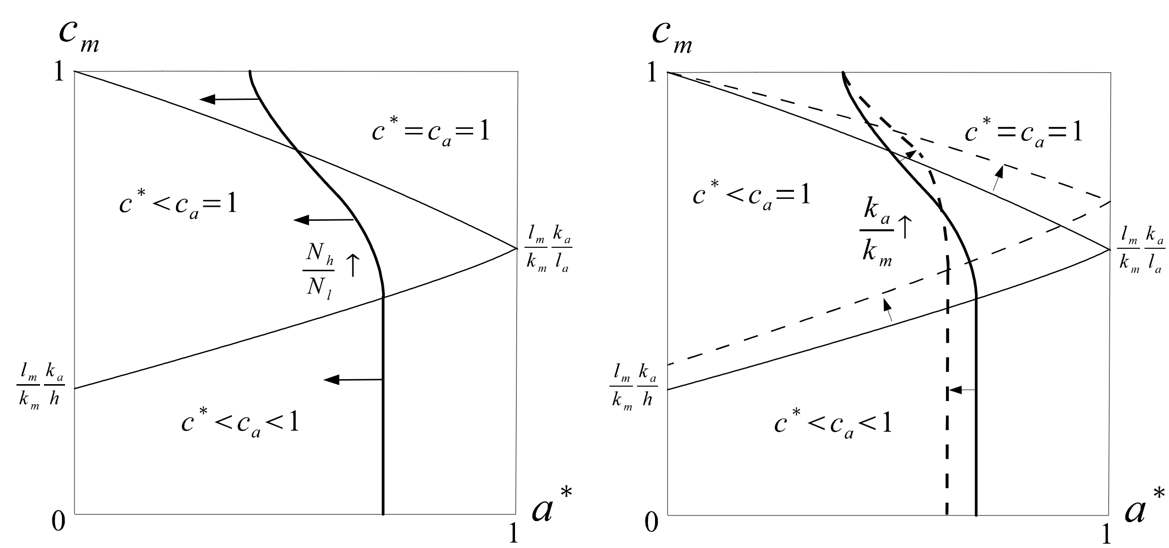


Figure 3: Values of c^* and c_a on the (a^*, c_m) space when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$



(a) Relation of (HL) with $\frac{N_h}{N_l}$ (b) Relation of (HL) with $\frac{k_a}{k_m}$

Figure 4: Shape of (HL) and its relations with $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$

3.3.1 Shape of (HL) and its relations with exogenous variables

The shape of (HL) and its relations with exogenous variables, $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$, are illustrated in Figure 4, based on Lemmas 1–3 in Appendix A.[25] The left figure shows that (HL) is

[25]The shape and the relations do *not* depend on the assumption $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, though the case $c^* = c_a = 1$ (the upper region in the figures) does not arise when $\frac{k_a}{k_m} \geq \frac{l_a}{l_m}$ and the case $c^* < c_a = 1$ (the middle region)

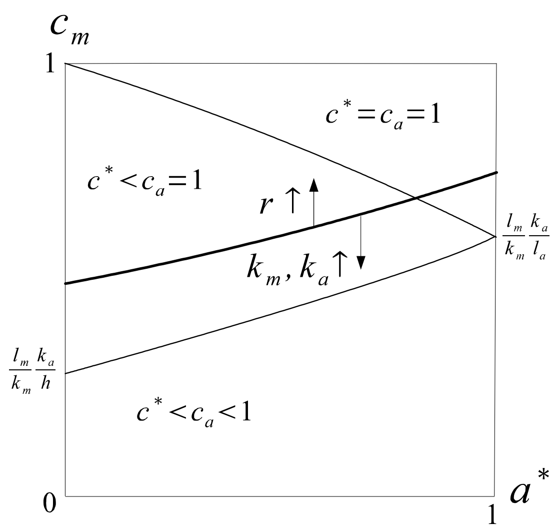


Figure 5: Shape of (P) and its relations with k_m, k_a, and r

negatively sloped when $c_a = 1$ and is vertical when $c_a < 1$ on the (a^*, c_m) space. The shape can be explained intuitively as follows. A decrease in c_m lowers $c_l(a)$ and $c_h(a)$ from (13) and raises the proportion of tasks performed by machines (see Figure 2). When $c_a = 1$, i.e., machines do not perform any tasks with $a = 1$ and $c < 1$, the mechanization mainly affects unskilled workers engaged in relatively manual tasks and thus they shift to more analytical tasks, i.e., a^* increases. By contrast, when $c_a < 1$, both types of workers are equally affected and thus a^* remains unchanged.

The left and right figures illustrate the relations of (HL) with $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$, respectively. An increase in $\frac{N_h}{N_l}$ implies that a higher portion of tasks must be performed by skilled workers and thus (HL) shifts to the left, i.e., for given c_m, a^* decreases. Less straightforward is the effect of an increase in $\frac{k_a}{k_m}$, which shifts the locus to the right (left) when c_m is high (low), definitely so when $c^* = 1$ (when $c_a < 1$). An increase in $\frac{k_a}{k_m}$ weakens comparative advantages of humans in analytical tasks and thus lowers, particularly for relatively high a, $c_l(a)$, $c_h(a)$, and the portion of tasks performed by humans (see Figure 2). When c_m (thus c^* and c_a) is high, such mechanization mainly affects unskilled workers and thus a^* must increase,[26] while the opposite is true when c_m is low.

3.3.2 Shape of (P) and its relations with exogenous variables

Figure 5 illustrates the shape of (P) and its relations with exogenous variables, k_m k_a, and r, based on Lemma 4 in Appendix A. Remember that, in order for (P) to hold, task assignment must be such that the unit production cost of the final good equals 1. c_m satisfying (P) increases with a^*, that is, (P) is upward-sloping on the (a^*, c_m) plane. This is because, if

does not arise when $\frac{k_a}{k_m} \geq \frac{h}{l_m}$.

[26] For example, when $c^* = c_a = 1$, $c_l(a)$ intersects with $c = 1$ at $a \leq a^*$ on the (a, c) plane (Figure 2 (a)). In this case, it would be clear that the mechanization mainly affects unskilled workers.

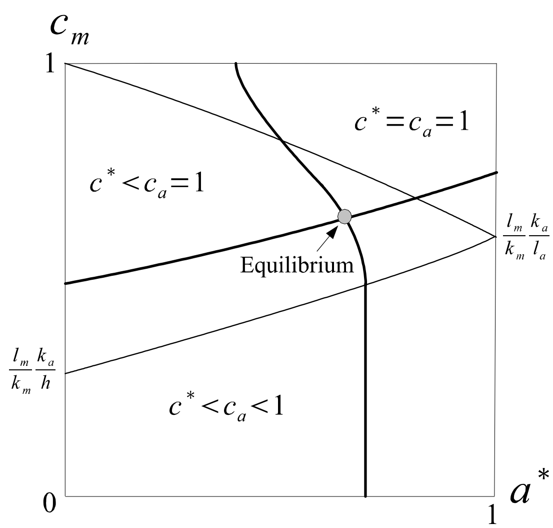


Figure 6: Determination of equilibrium a^* and c_m

an increase in a^* lowers c_m, both $w_l = \frac{l_m}{k_m}\frac{r}{c_m}$ and $w_h = \frac{A_h(a^*)}{A_l(a^*)}w_l$ increase and thus the unit production cost exceeds 1. An increase in r, which raises the cost of hiring machines, shifts the locus upward, i.e., c_m increases for given a^*. This implies that $c_l(a)$ and $c_h(a)$ increase and thus a higher portion of tasks are assigned to humans. The opposite holds when abilities of machines, k_m and k_a, increase.[27]

3.3.3 Determination of equilibrium (a^*, c_m)

As Figure 6 illustrates, equilibrium (a^*, c_m) is determined at the intersection of the two loci. Of course, the position of the intersection depends on exogenous variables such as k_m and k_a. The next two sections examine how increases in k_m, k_a, and $\frac{N_h}{N_l}$ affect the equilibrium, particularly, task assignment, earnings levels and inequality, and aggregate output.

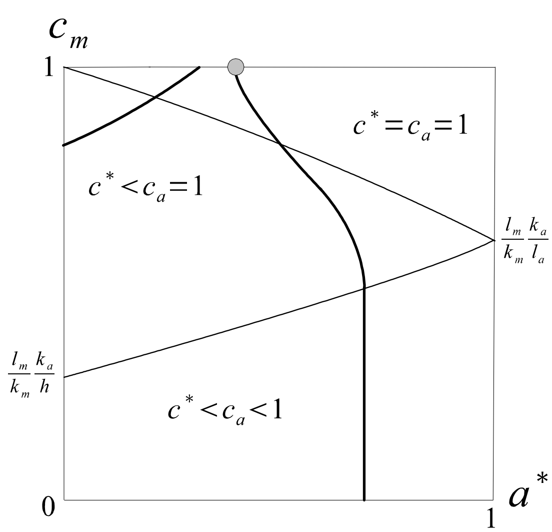
4 Mechanization with constant $\frac{k_a}{k_m}$

Suppose that abilities of machines, k_m and k_a, and thus their productivities $cA_k(a)$ increase exogenously over time. This section examines effects of the productivity growth and of an increase in $\frac{N_h}{N_l}$ on task assignment, earnings levels and inequality, and output, when k_m and k_a satisfying $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ grow proportionally. Since (HL) does not shift under constant $\frac{k_a}{k_m}$ (Figure 4 (b)), the analysis is much simpler than the general case examined in Section 5.

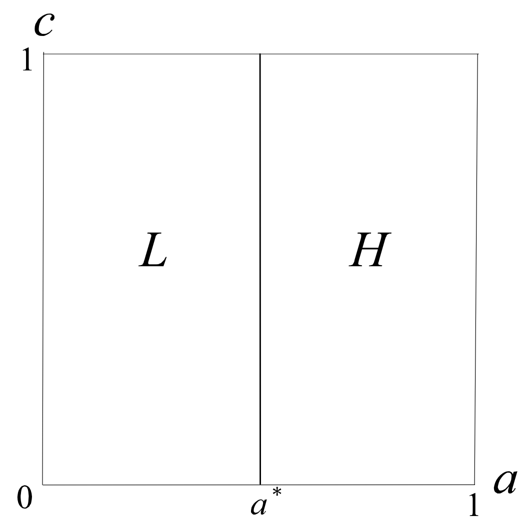
The next proposition presents the dynamics of the critical variables and functions determining task assignment. (Proofs of the propositions of this section are in Appendix B.)

Proposition 1 *Suppose that k_m and k_a satisfying $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ grow proportionally over time.*

[27] The locus never intersects with $c_m = 0$, because machines are completely useless and thus hiring machines are prohibitively expensive at the hardest-to-codify tasks.



(a) Equilibrium



(b) Task assignment

Figure 7: Equilibrium and task assignment when $c_m = c^* = c_a = 1$

(*i*) *When initial* k_m *is very low,* $c_m = c^* = c_a = 1$ *is satisfied at first;*[28] *at some point,* $c_m < c^* = c_a = 1$ *holds and thereafter* c_m *falls over time; then,* $c_m < c^* < c_a = 1$ *and* c^* *too falls; finally,* $c_m < c^* < c_a < 1$ *and* c_a *falls as well.*

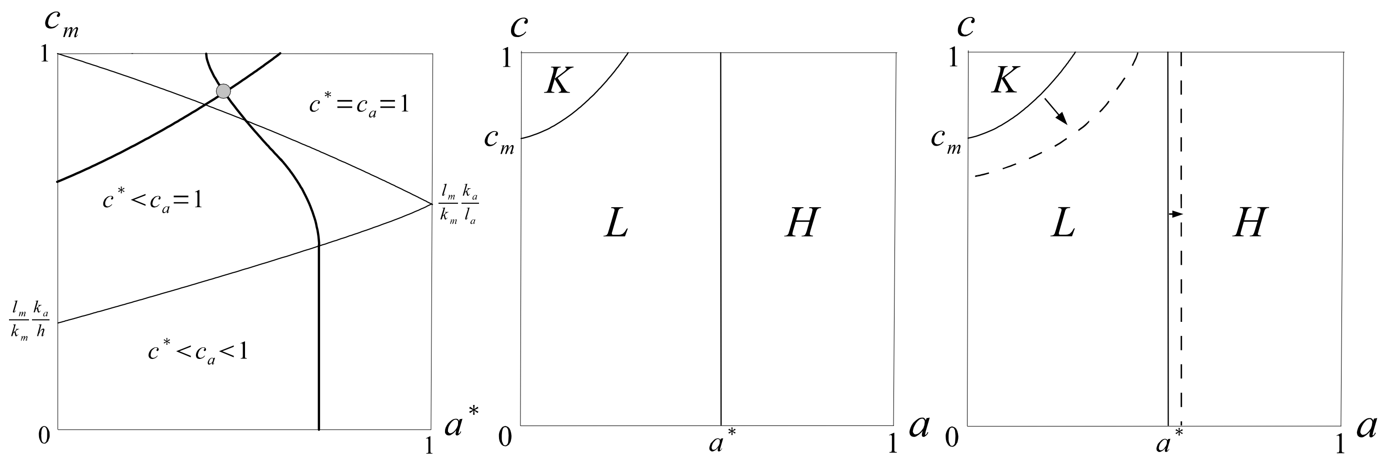
(*ii*) a^* *increases over time when* $c_m < c_a = 1$*, while* a^* *is time-invariant when* $c_m = 1$ *and when* $c_a < 1$*.*

(*iii*) $c_l(a)$ *and* $c_h(a)$ *(when* $c^* < 1$*) decrease over time when* $c_m < 1$*.*

The results of this proposition can be understood using figures similar to Figure 6. When the level of k_m is very low, there are no (a^*, c_m) satisfying (P), or (P) is located at the left side of (HL) on the (a^*, c_m) plane (Figure 7 (a)). Hence, the two loci do not intersect and an equilibrium with $c_m < 1$ does not exist. Because the manual ability of machines is very low, using machines is not rewarding and all tasks are performed by humans. Figure 7 (a) illustrates an example of the determination of equilibrium c_m and a^* in this case. Equilibrium a^* is determined at the intersection of (HL) with $c_m = 1$. Figure 7 (b) illustrates the corresponding task assignment on the (a, c) plane, which shows that unskilled (skilled) workers perform all tasks with $a < (>) a^*$.

When k_m becomes high enough that (P) is located at the right side of (HL) at $c_m = 1$, the two loci intersect and thus machines begin to be used, i.e., $c_m < 1$. Note that the level of k_a is not important for the initiation of mechanization, because mechanization starts from the most manual tasks in which analytical ability is of no use. Because of low machine productivities, they perform only highly manual and easy-to-codify tasks that were previously performed by unskilled workers, i.e., $c^* = c_a = 1$. Indeed, large-scale mechanization originated in tasks associated with simple repetitive motions in textile during the Industrial Revolution. Figure 8 (a) and (b) respectively illustrate the determination of equilibrium c_m

[28] As noted in footnote 22, the value of c_m when all tasks are performed by humans is set to be equal 1.



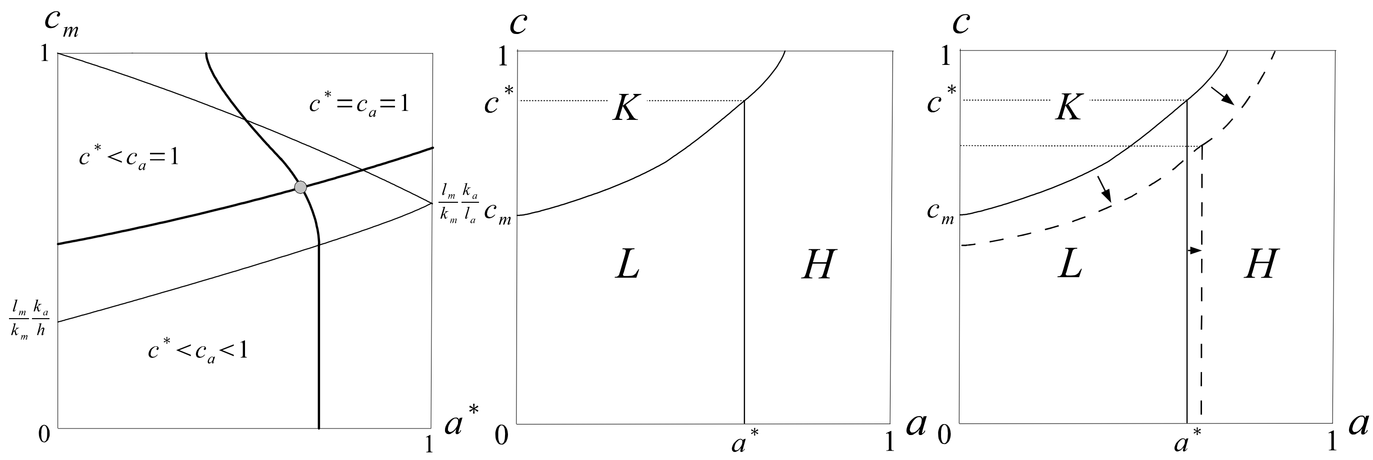
(a) Equilibrium (b) Task assignment (c) Effect of productivity growth with constant $\frac{k_a}{k_m}$

Figure 8: Equilibrium, task assignment, and the effect of productivity growth with constant $\frac{k_a}{k_m}$ when $c_m < c^* = c_a = 1$

and a^* and task assignment. Figure 8 (c) presents the effect of small increases in k_m and k_a on task assignment. Since machines perform a greater portion of highly manual and easy-to-codify tasks, a^* increases and $c_l(a)$ decreases, that is, workers shift to more analytical and, as for unskilled workers, harder-to-routinize tasks. Consistent with the model, during early stages of industrialization, humans shifted from manual tasks at farms, cottages, and workshops toward analytical tasks at offices and factories (generally associated with clerical, management, and technical jobs) as well as manual tasks at factories, and manual workers shifted to tasks involving more complex motions machines were not good at.

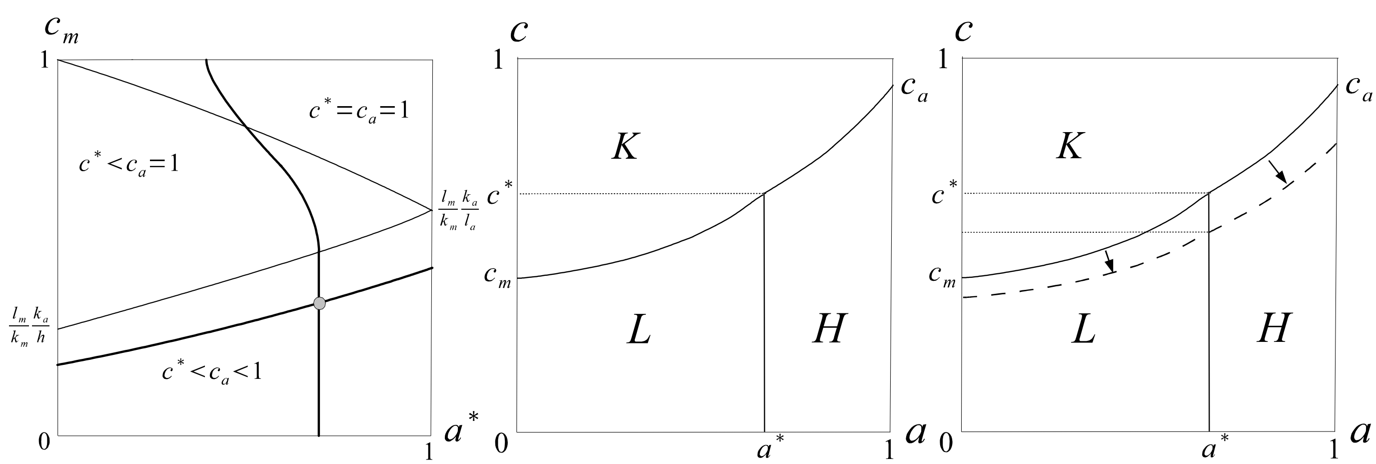
As k_m and k_a grow over time, automation spreads to relatively analytical tasks, and eventually, machines come to perform highly analytical tasks, those previously performed by skilled workers. In the real economy, the new phase of mechanization started during the Second Industrial Revolution—e.g., teleprinters replaced Morse code operators and tabulating machines substituted data-processing workers at large organizations—and has progressed on a large scale in the post World War II era, especially since the 1970s, because of the advancement of information technology. Figure 9 (a) and (b) respectively illustrate the determination of equilibrium c_m and a^* and task assignment when $c_m < c^* < c_a = 1$. Machines perform some tasks with $a > a^*$ but not the most analytical ones, i.e., $c^* < c_a = 1$. Productivity growth lowers $c_h(a)$ as well as $c_l(a)$ (and raises a^*), thus skilled workers too shift to more difficult-to-codify tasks (Figure 9 (c)). Congruent with the model, since the 1970s, humans have shifted from routine analytical tasks (such as simple information processing tasks typical in clerical jobs) as well as manual tasks toward non-routine analytical tasks mainly associated with professional and technical jobs and non-routine manual tasks in services.

Finally, the economy reaches the case $c_m < c^* < c_a < 1$, which is illustrated in Figure 10. Machines perform a portion of the most analytical tasks, i.e., $c_a < 1$. In fact, currently,



(a) Equilibrium (b) Task assignment (c) Effect of productivity growth with constant $\frac{k_a}{k_m}$

Figure 9: Equilibrium, task assignment, and the effect of productivity growth with constant $\frac{k_a}{k_m}$ when $c_m < c^* < c_a = 1$



(a) Equilibrium (b) Task assignment (c) Effect of productivity growth with constant $\frac{k_a}{k_m}$

Figure 10: Equilibrium, task assignment, and the effect of productivity growth with constant $\frac{k_a}{k_m}$ when $c_m < c^* < c_a < 1$

machines are engaged in some tasks involving analysis and decision-making, such as automated trading in financial markets. Unlike the previous cases, productivity growth affects two type of workers equally and thus a^* does not change, while $c_h(a)$ and $c_l(a)$ decrease and thus workers shift to more difficult-to-codify tasks.

In sum, when the two abilities of machines with $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ improve proportionally over time, mechanization starts from highly manual and easy-to-codify tasks and gradually spreads to more analytical and harder-to-codify tasks. Eventually, machines come to perform highly analytical tasks previously performed by skilled workers. Accordingly, unskilled workers shift to tasks that are more difficult to codify, so do skilled workers in later stages of mechanization, and both types shift to more analytical tasks except at the final stage.

The dynamics of task assignment accord with the long-run trends of mechanization and of shifts in tasks performed by humans except job polarization after the 1990s, which are detailed in the introduction and is summarized as: initially, mechanization proceeded in tasks intensive in manual labor, while automation of tasks intensive in analytical labor started during the Second Industrial Revolution and has progressed on a large scale in the post World War II era, especially since the 1970s, because of the advancement of information technology; humans shifted from manual tasks to analytical tasks until about the 1960s, whereas, thereafter, they have shifted away from routine analytical tasks as well as routine manual tasks toward non-routine analytical tasks and non-routine manual tasks in services.

Effects of the productivity growth on earnings levels and inequality, and aggregate output are examined in the next proposition.

Proposition 2 *Suppose that k_m and k_a satisfying $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ grow proportionately over time when $c_m < 1$.*

(*i*) *Earnings of skilled workers increase over time. When $c^* < c_a < 1$, earnings of unskilled workers too increase.*

(*ii*) *Earnings inequality, $\frac{w_h}{w_l}$, rises over time when $c_a = 1$ and is time-invariant when $c_a < 1$.*

(*iii*) *The output of the final good, Y, increases over time.*

The proposition shows that, while skilled workers *always* benefit from mechanization, the effect on earnings of unskilled workers is ambiguous when mechanization mainly affects them, i.e., when $c_a = 1$, and the effect turns positive when $c_a < 1$. Because different tasks are complementary in final good production, the increased productivity of machines raises the demand for tasks that are not directly affected by mechanization, shifts workers to these tasks, in which they have greater comparative advantages, and increases output. This has a positive effect on earnings. But it also leads to the substitution of workers with specific skill levels by machines and has a negative effect on their earnings. When machines replace only or mainly unskilled workers, i.e., when $c_a = 1$, *the negative substitution effect* could dominate *the positive complementarity effect* for unskilled workers and thus their earnings could decrease, while when machines replace both types of workers similarly, i.e., when $c_a < 1$, the complementarity effect dominates and their wage increases.[29] As for skilled workers, it is always the case that the complementarity effect dominates the substitution effect and thus

[29] The complementarity effect is relatively small when $c_a = 1$, because they shift not only to more difficult-to-codify tasks, i.e., tasks with greater comparative advantages relative to machines, but also to more analytical tasks, i.e., tasks with *weaker* comparative advantages relative to skilled workers.

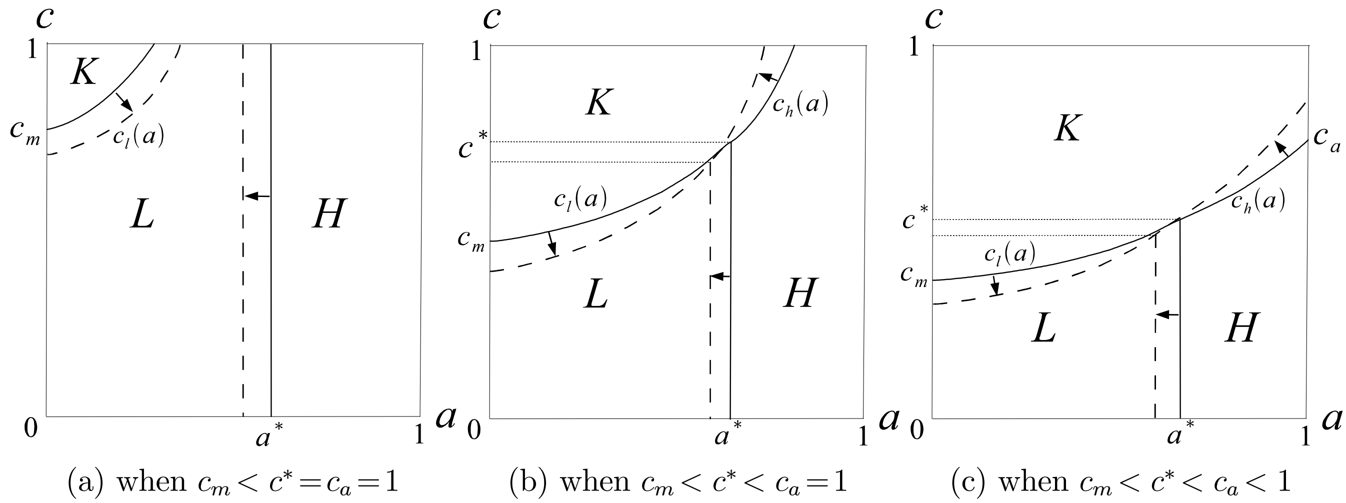


Figure 11: Effect of an increase in $\frac{N_h}{N_l}$ on task assignment when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$

their earnings increase. Mechanization worsens earnings inequality $\frac{w_h}{w_l}$ when $c_a = 1$, while it has no effect when $c_a < 1$. The output of the final good *always* increases, even if $l_a < h < l_m$ and thus workers' productivities, $A_h(a)$ and $A_l(a)$, fall as they shift to more analytical tasks.

So far, the ratio of skilled workers to unskilled workers, $\frac{N_h}{N_l}$, is held constant, though it has increased over time, particularly after the 20th century, in the real economy. Thus, the next proposition examines effects of the growth of $\frac{N_h}{N_l}$ for given machine qualities.

Proposition 3 *Suppose that $\frac{N_h}{N_l}$ grows over time when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ and $c_m < 1$.*

(i) c_m, a^*, c^* (when $c^* < 1$), and $c_l(a)$ decrease, while c_a (when $c_a < 1$) and $c_h(a)$ (when $c^* < 1$) increase over time.

(ii) w_l (w_h) rises (falls) and earnings inequality, $\frac{w_h}{w_l}$, shrinks over time.

(iii) Y increases over time under constant $N_h + N_l$.

Figure 11 illustrates the effect of an increase in $\frac{N_h}{N_l}$ on task assignment. Since skilled workers become abundant relative to unskilled workers, they take over a portion of tasks previously performed by unskilled workers, i.e., a^* decreases. Further, earnings of unskilled workers rise and those of skilled workers fall, thus some tasks previously performed by unskilled workers are *mechanized*, i.e., $c_l(a)$ decreases, while, when $c^* < 1$, skilled workers take over some tasks performed by machines before, i.e., $c_h(a)$ increases. That is, skilled workers shift to more manual tasks, and unskilled workers shift to harder-to-routinize tasks. The wage of unskilled workers increases because of the positive complementarity effect from the increased number of workers with greater abilities. The wage of skilled workers decreases because they have weaker comparative advantages in tasks they take over. Output increases mainly because skilled workers are more productive than unskilled workers.

By combining the results on effects of an increase in $\frac{N_h}{N_l}$ with those of the productivity growth, the model can explain the long-run trends of earnings levels and inequality until the

1970s,[30] except the 1940s during which institutional factors such as the policy-induced sharp increase in union membership and the wartime wage setting rules are likely to be important (Goldin and Katz, 2008; Farber et al., 2021). The trends, which are described more in detail in the introduction, are summarized as: in early stages of industrialization when mechanization directly affected unskilled workers only and the relative supply of skilled workers grew slowly, earnings of unskilled workers grew very moderately and earnings inequality rose; in later periods when skilled workers too were directly affected by automation and the relative supply of skilled workers grew faster, unskilled workers benefited more from mechanization, while, as before, the rising inequality was the norm in economies with lightly regulated labor markets (such as the U.S.), except in periods of a rapid increase in the relative supply of educated workers (such as the 1970s) and in the 1940s, when the inequality fell.

The model, however, fails to capture the trends after the 1980s, which are: earnings of unskilled workers fell or stagnated and those of skilled workers rose until the mid 1990s in the U.S. (Autor, 2019);[31] the overall inequality rose greatly after the 1980s (after the 1990s in many European economies, OECD, 2008); since the 1990s, earnings of those with skills for middle-wage jobs have fallen relative to earnings of those with skills for low-wage jobs and those with skills for high-wage jobs at least in the U.S. (Böhm, 2020). By contrast, the model predicts that earnings of unskilled workers increase and the inequality shrinks when highly analytical tasks are affected by automation, i.e., when $c_a < 1$, and the relative supply of skilled workers rises.

5 Mechanization with time-varying $\frac{k_a}{k_m}$

The previous section has examined the case in which k_m and k_a grow proportionately. This special case has been taken up first for analytical simplicity. However, the assumption of the proportionate growth is rather restrictive, because, according to the trend of mechanization described in the introduction, the growth of k_m was apparently faster than that of k_a for most periods of time (major technological developments before the 1970s increased productivities of machines to perform production and transportation tasks), while k_a seems to have been growing faster than k_m recently (because of the rapid advance of information technology).[32]

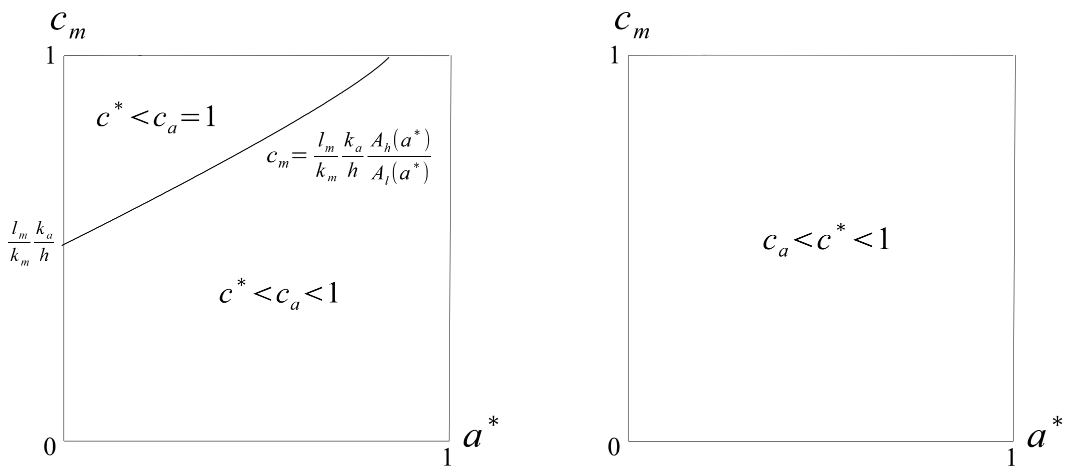
This section examines the general case in which the machine abilities may grow at different rates. This case is much more difficult to analyze because a change in $\frac{k_a}{k_m}$ shifts the graph of (HL) as well as that of (P) (see Figures 4 and 5 in Section 3). Under realistic productivity growth, the model does much better jobs in explaining the development after the 1980s than under the constant $\frac{k_a}{k_m}$ case.

Unlike the previous case, shapes of graphs in Figures 2 and 3 may change qualitatively

[30] The combined effect of an increase in $\frac{N_h}{N_l}$ and improvements of machine qualities on task assignment accords with the trend of task shifts in the real economy when $c^* = 1$. When $c^* < 1$, it is consistent with the fact, unless the negative effect of an increase in $\frac{N_h}{N_l}$ on $c_h(a)$ is very strong (see Figure 11).

[31] According to Autor (2019), composition-adjusted real wages are lower in 1995 than in 1980 for full-time male workers without graduate degrees and for full-time female workers without college degrees.

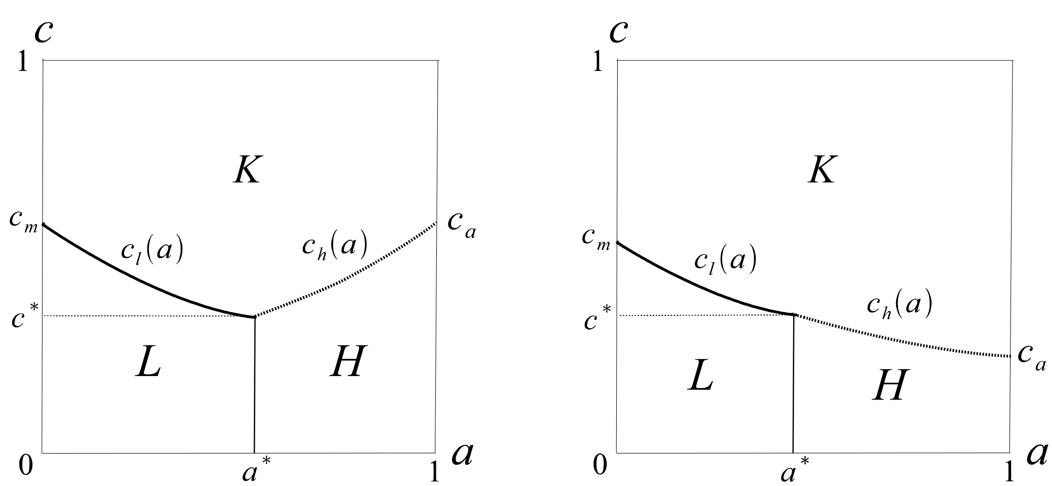
[32] Note that k_a was positive even before the Industrial Revolution. Various machines had automatic control systems whose major examples are: float valve regulators used in ancient Greece and in the medieval Arab world to control the level of water in tanks and devices such as water clocks and oil lamps; temperature regulators of furnaces invented in early 17th century Europe.



(a) when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ (b) when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$

Figure 12: c^* and c_a when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ and when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$

with productivity growth. Starting from the situation where $\frac{k_a}{k_m} < \frac{l_a}{l_m} (< \frac{h}{l_m})$ holds, if k_a keeps growing faster than k_m, i.e., the rapid growth of information technology continues, $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$, then $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$ hold eventually. That is, comparative advantages of machines to two type of workers change over time. As illustrated in Figure 12, when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$, $c^* < 1$ always holds, and when $\frac{k_a}{k_m} > \frac{h}{l_m} (> \frac{l_a}{l_m})$, $c_a < c^* < 1$ always holds.[33]



(a) when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ (b) when $\frac{k_a}{k_m} > \frac{h}{l_m}$

Figure 13: $c_l(a)$ and $c_h(a)$ when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ ($c_a < 1$ is assumed) and when $\frac{k_a}{k_m} > \frac{h}{l_m}$

Figure 13 illustrates $c_l(a)$ and $c_h(a)$ and task assignment on the (a, c) space when $\frac{k_a}{k_m} \in$

[33] This is because $c^* = \min\left\{\frac{k_m}{l_m}\frac{A_l(a^*)}{A_k(a^*)}c_m, 1\right\}$ and $c_a = \min\left\{\frac{h}{k_a}\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m, 1\right\}$.

$(\frac{l_a}{l_m}, \frac{h}{l_m})$ (the figure is drawn assuming $c_a < 1$) and when $\frac{k_a}{k_m} > \frac{h}{l_m}$. Unlike the original case $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, $c_l(a)$ is downward-sloping and, when $\frac{k_a}{k_m} > \frac{h}{l_m}$, $c_h(c)$ too is downward-sloping. Hence, when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$, for given c, machines tend to perform tasks with *intermediate* a and the proportion of tasks performed by machines is *highest at* $a = a^*$. When $\frac{k_a}{k_m} > \frac{h}{l_m}$, for given c, machines tend to perform relatively *analytical* tasks and the proportion of tasks performed by machines *increases* with a.

5.1 Effects of changes in k_m, k_a, and $\frac{N_h}{N_l}$

Now, effects of changes in k_m and k_a on task assignment, earnings levels and inequality, and output are examined. Since results are different depending on whether c^* and c_a equal 1 or not (Figure 3), they are presented in three propositions.[34,35] The next proposition analyzes the effects in the first stage of mechanization, $c^* = c_a = 1$, which arises only when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$.

Proposition 4 *When* $c_m \geq \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} \Leftrightarrow c^* = c_a = 1$ *(possible only when* $\frac{k_a}{k_m} < \frac{l_a}{l_m}$*),*
(i) c_m *decreases and* a^* *increases with* k_m *and* k_a*.*
(ii) $c_l(a)$ *decreases with* k_m *and* k_a*.*
(iii) w_h*,* $\frac{w_h}{w_l}$*, and* Y *increase with* k_m *and* k_a*.* w_l *increases with* k_a*.*

The only difference from the constant $\frac{k_a}{k_m}$ case is that w_l increases when k_a rises with k_m unchanged, an unlikely situation. As before, with improved machine qualities, c_m and $c_l(a)$ decrease and a^* increases, i.e., workers shift to more analytical and, for unskilled workers, harder-to-codify tasks (see Figure 8 (c) in Section 4), and earnings of skilled workers, earnings inequality $\frac{w_h}{w_l}$, and output rise.

The next proposition examines the effects in the second stage of mechanization, $c^* < c_a = 1$, which is possible only when $\frac{k_a}{k_m} < \frac{h}{l_m}$.

Proposition 5 *When* $c_m \in \left[\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}, \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}\right) \Leftrightarrow c^* < c_a = 1$ *(possible only when* $\frac{k_a}{k_m} < \frac{h}{l_m}$*),*
(i) c_m *decreases with* k_m *and* k_a*.* a^* *increases when* $\frac{k_a}{k_m}$ *non-increases.*
(ii) $c_l(a)$ *and* $c_h(a)$ *decrease with* k_m *and* k_a*.*
(iii) w_h *and* Y *increase with* k_m *and* k_a*, while* w_l *increases with* k_a*.* $\frac{w_h}{w_l}$ *increases when* $\frac{k_a}{k_m}$ *non-increases.*

There are several differences from the constant $\frac{k_a}{k_m}$ case. First, effects of productivity growth with increasing $\frac{k_a}{k_m}$ on a^* and earnings inequality are *ambiguous*, and w_l increases with k_a. Second, although $c_l(a)$ (thus c_m) and $c_h(a)$ decrease and thus workers shift to harder-to-routinize tasks as in the original case, workers may not shift to more analytical tasks when a^* decreases (possible when $\frac{k_a}{k_m}$ increases) and when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ (Figure 13 (a)).[36] Remaining results are same as before, that is, when $\frac{k_a}{k_m}$ non-increases, a^* and earnings inequality increase; when $\frac{k_a}{k_m} \leq \frac{l_a}{l_m}$ too holds, workers shift to more analytical tasks; and earnings of skilled workers and output always increase.

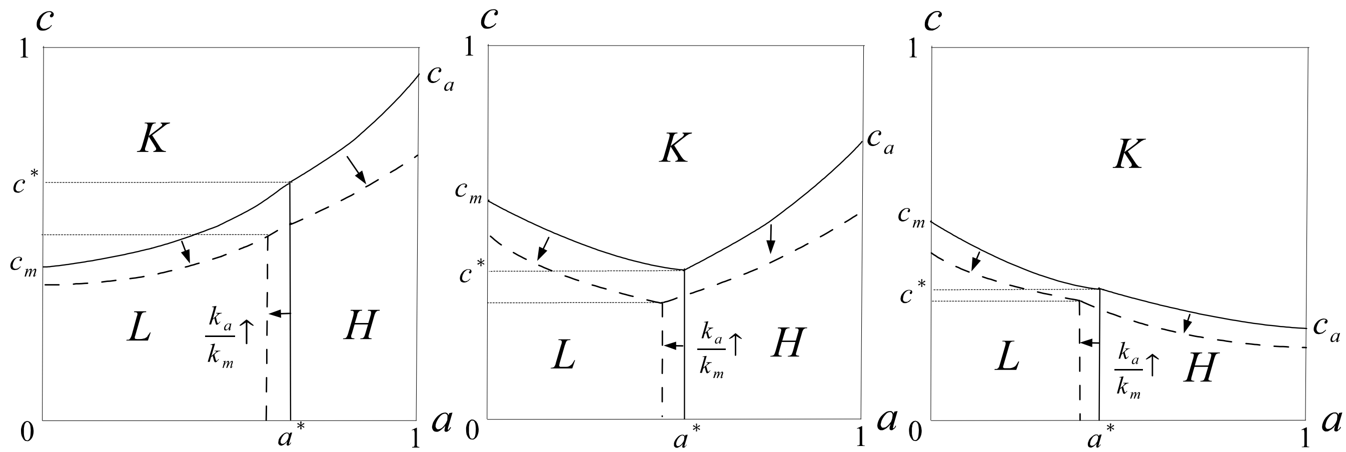
Proposition 6 examines the effects in the final stage of mechanization, $c^*, c_a < 1$.[37]

[34] When $\frac{k_a}{k_m} > \frac{l_a}{l_m}$, $c_m = 1$ is possible with c^* or $c_a < 1$. However, such situation—the most manual and easy-to-codify task is not mechanized while some of other tasks are—is unrealistic and thus is not examined.

[35] Proofs of these propositions and Proposition 7 are very lengthy and thus are relegated to Web Appendix posted on the author's web site (http://www.econ.kyoto-u.ac.jp/~yuki/english.html).

[36] For relatively high c, unskilled workers shift to more manual tasks when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$.

[37] $c^* < (>) c_a$ when $\frac{k_a}{k_m} < (>) \frac{h}{l_m}$.



(a) when $\frac{k_a}{k_m} < \frac{l_a}{l_m} (< \frac{h}{l_m})$ (b) when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ (c) when $\frac{k_a}{k_m} > \frac{h}{l_m}$

Figure 14: Effect of productivity growth with increasing $\frac{k_a}{k_m}$ when $c^*, c_a < 1$

Proposition 6 *When* $c_m < \frac{l_m}{k_m} \frac{k_a}{h} \frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow c^*, c_a < 1$,

(i) c_m *and* c_a *decrease with* k_m *and* k_a*, and* a^* *decreases with* $\frac{k_a}{k_m}$.

(ii) $c_l(a)$ *and* $c_h(a)$ *decrease with* k_m *and* k_a.

(iii) w_h *and* Y *increase with* k_m *and* k_a*, while* w_l *increases when* $\frac{k_a}{k_m}$ *non-decreases.* $\frac{w_h}{w_l}$ *decreases with* $\frac{k_a}{k_m}$.

Unlike the constant $\frac{k_a}{k_m}$ case, in which a^* and thus $\frac{w_h}{w_l}$ are constant and w_l increases over time, a^* and $\frac{w_h}{w_l}$ *decrease* with $\frac{k_a}{k_m}$ and the effect on w_l is *ambiguous* when $\frac{k_a}{k_m}$ decreases. As for task assignment, while $c_l(a)$ (thus c_m) and $c_h(a)$ decrease as in the original case (thus workers shift to harder-to-routinize tasks), tasks performed by humans change in the skill dimension as well. In particular, when $\frac{k_a}{k_m}$ rises (falls), that is, when productivity growth is such that comparative advantages of machines to humans in analytical (manual) tasks rise, unskilled workers shift to more *manual* (analytical) tasks under $\frac{k_a}{k_m} > (<) \frac{l_a}{l_m}$, and skilled workers too shift to such tasks under $\frac{k_a}{k_m} > (<) \frac{h}{l_m}$.[38] Figure 14 illustrates the effect of productivity growth with increasing $\frac{k_a}{k_m}$ on task assignment for this case. Earnings of skilled workers and output rise as before.

Finally, Proposition 7 examines effects of an increase in $\frac{N_h}{N_l}$ when $\frac{k_a}{k_m} \geq \frac{l_a}{l_m}$ is allowed.

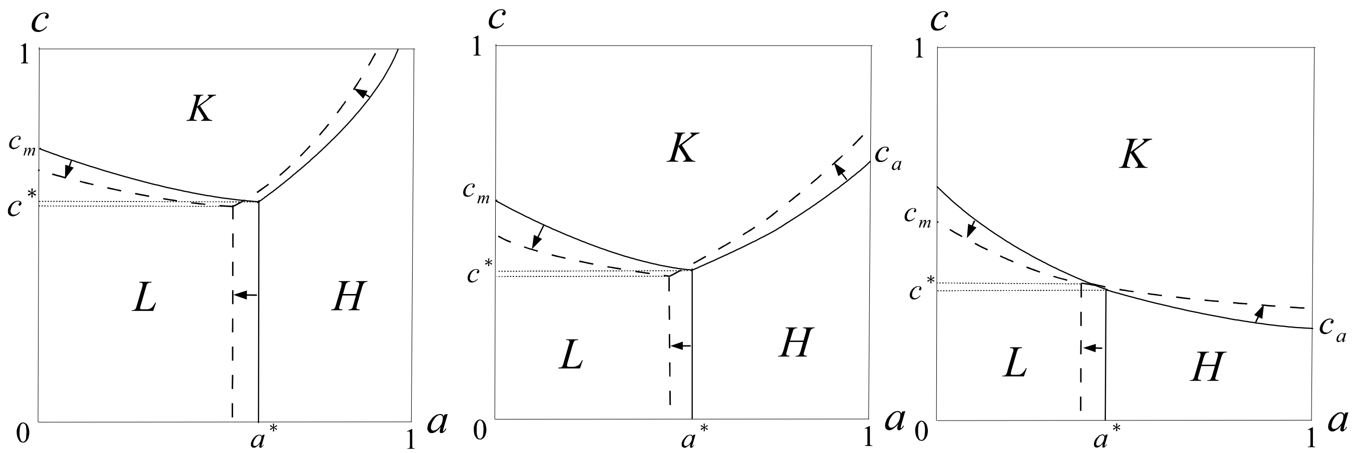
Proposition 7 *Suppose that* $\frac{N_h}{N_l}$ *grows over time when* $c_m < 1$.

(i) c_m*,* a^**, and* $c_l(a)$ *decrease, while* c_a *(when* $c_a < 1$*) and* $c_h(a)$ *(when* $c^* < 1$*) increase over time.* c^* *(when* $c^* < 1$*) falls (rises) when* $\frac{k_a}{k_m} \leq \frac{l_a}{l_m}$ $(\frac{k_a}{k_m} \geq \frac{h}{l_m})$.

(ii) w_l *(*w_h*) rises (falls) and* $\frac{w_h}{w_l}$ *shrinks over time.*

(iii) Y *increases over time under constant* $N_h + N_l$.

[38] When $\frac{k_a}{k_m}$ rises (falls) under $\frac{k_a}{k_m} < (>) \frac{l_a}{l_m}$, unskilled workers shift to more manual (analytical) tasks at low c. The same is true for skilled workers under $\frac{k_a}{k_m} < (>) \frac{h}{l_m}$. (See Figure 14.) Hence, at low c, both types of workers *always* shift to more manual (analytical) tasks when $\frac{k_a}{k_m}$ rises (falls).



(a) when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ and $c_a = 1$ (b) when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ and $c_a < 1$ (c) when $\frac{k_a}{k_m} > \frac{h}{l_m}$

Figure 15: Effect of an increase in $\frac{N_h}{N_l}$ when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ and when $\frac{k_a}{k_m} > \frac{h}{l_m}$

Figure 15 illustrates the effect of an increase in $\frac{N_h}{N_l}$ on task assignment when $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$ and when $\frac{k_a}{k_m} > \frac{h}{l_m}$. (Note that $c^* = 1$ does not occur in these cases and $c_a = 1$ does not occur when $\frac{k_a}{k_m} > \frac{h}{l_m}$.) As in the original case of $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, skilled workers take over some tasks previously performed by unskilled workers, i.e., a^* decreases, and machines (skilled workers) take over a portion of tasks performed by unskilled workers (machines) before, i.e., $c_l(a)$ decreases ($c_h(a)$ increases). However, unlike before, $c_l(a)$ is downward-sloping on the (a, c) plane, and, when $\frac{k_a}{k_m} > \frac{h}{l_m}$, $c_h(a)$ too is downward-sloping. Thus, unskilled workers shift to harder-to-routinize *and* more manual tasks, and skilled workers may not shift to more manual tasks when $\frac{k_a}{k_m} > \frac{h}{l_m}$ (see Figure 15 (c)). As in the original case, earnings of unskilled (skilled) workers rise (fall), earnings inequality shrinks, and output increases.

5.2 Contrasting the model with facts

Based on the propositions, it is examined whether the model with realistic productivity growth can explain the long-run trends of task shifts, earnings, and earnings inequality in the real economy.

Two assumptions are imposed on comparative advantage of machines against humans and the relative growth of the two abilities of machines. First, it would be plausible to suppose that $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ has continued to hold until now (thus $c_l(a)$ and $c_h(a)$ are downward-sloping on the (a, c) plane), because the proportion of tasks performed by machines seems to be still higher in more manual tasks: consider the fact that the large majority of non-routine analytical tasks generally associated with management, professional, and technical jobs and of non-routine "middle a" tasks typical in occupations such as mechanics and nurses are yet to be automated.

Second, the history of mechanization and task shifts described in the introduction suggests that k_m seems to have grown faster than k_a until sometime in the 1990s, after which the growth of k_a appears to be faster because of the ever-increasing application of information technology in wide areas.[39] The supposed turning point would be not be far off the mark considering that a decrease in the employment share of production occupations, which are intensive in manual tasks, was greatest in the 1980s and slowed down considerably after the 1990s, while a decrease in the share of clerical occupations intensive in routine analytical tasks started in the 1980s and accelerated in the 1990s (Autor, 2019). Note also that information technology seems to have contributed to the growth of k_m more than the growth of k_a initially: CNC [Computer Numerical Control] machines and industrial robots, widely used since the 1970s and the 1980s respectively, raised productivities of machines to perform manual and relatively non-routine tasks considerably. Hence, suppose that $\frac{k_a}{k_m}$ falls over time when $c_a = 1$, while when $c_a < 1$, i.e., in the final stage of mechanization, $\frac{k_a}{k_m}$ *falls initially, then rises.*

Now, the evolutions of earnings levels and inequality are examined. The result when $c^* = c_a = 1$ is almost same as the constant $\frac{k_a}{k_m}$ case (Proposition 4), thus the model is consistent with the actual trends in the early stage of mechanization. The model accords with the trends in the intermediate stage too (except a decline of the inequality in the 1940s), because the result of the case $c^* < c_a = 1$ is same as before when $\frac{k_a}{k_m}$ falls (Proposition 5).

It is in the final stage of mechanization, i.e, when $c^* < c_a < 1$, that the model with time-varying $\frac{k_a}{k_m}$ explains the trends much better than the model with constant $\frac{k_a}{k_m}$. First, the present model could be congruent with falling or stagnant earnings of U.S. unskilled workers in the 1980s-early 1990s and a large increase in the overall inequality after the 1980s (after the 1990s in many European nations). This is because the effect of productivity growth with decreasing $\frac{k_a}{k_m}$ on their earnings is ambiguous and the effect on the inequality is positive when $c^* < c_a < 1$ (Proposition 6), and the growth of $\frac{N_h}{N_l}$, which contributes to raising their earnings and lowering the inequality (Proposition 7), greatly slowed down during the period. When the growth of manual ability of machines is higher than the growth of their analytical ability, the negative substitution effect of mechanization on earnings is relatively strong and could dominate the positive complementarity effect for unskilled workers, who are engaged in relatively manual tasks; thus, their earnings could decrease and earnings inequality rises even in the final stage of mechanization. Second, it is also consistent with the sound growth of earnings of unskilled workers in the late 1990s-early 2000s and in the 2010s (Autor, 2019), because their earnings increase when $\frac{k_a}{k_m}$ rises under $c^* < c_a < 1$.[40] Third, although the present model with two types of workers cannot capture the whole picture of the falling relative wage of workers with skills for middle-wage jobs after the 1990s (Böhm, 2020) (the model with three types of workers is analytically intractable), it yields the decreasing inequality when $\frac{k_a}{k_m}$ rises under $c^* < c_a < 1$ and thus captures a part of the development, the falling disparity between workers with skills for low-wage jobs and those with skills for middle-wage jobs

[39] It is true that several components of the composite analytical ability k_a, such as numeric ability, seems to have been growing faster than the composite manual ability k_m for much longer periods. But remaining components, such as analysis and decision-making abilities, seem to have grown slowly until recently.

[40] According to Autor (2019), composition-adjusted real wages of full-time workers of all education groups exhibited sound growth in the late 1990s-early 2000s in the U.S. Thereafter, however, real wages fell or stagnated, except for workers with post-college education, whose earnings also dropped after the Great Recession. In the 2010s, all groups, particularly high school dropouts, have enjoyed strong earnings growth.

(and *moderately high-wage* jobs more recently).[41,42] When machines improve mainly in their analytical ability, the substitution effect is stronger for skilled than for unskilled workers and thus the inequality decreases.

As for the dynamics of task shifts, the result under $c^* = c_a = 1$ is same as the constant $\frac{k_a}{k_m}$ case, and so is the result under $c^* < c_a = 1$ when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ holds and $\frac{k_a}{k_m}$ falls (Propositions 4 and 5): $c_l(a)$ and $c_h(a)$ decrease and a^* increase over time, unless $\frac{N_h}{N_l}$ grows rapidly. Hence, the dynamics accord with the long-run trend until recently, i.e., workers shift to harder-to-routinize and more analytical tasks over time. By contrast, when $c^* < c_a < 1$, while $c_l(a)$ and $c_h(a)$ decrease over time (unless $\frac{N_h}{N_l}$ grows rapidly) as before, unlike the constant $\frac{k_a}{k_m}$ case, a^* increases (decreases) when $\frac{k_a}{k_m}$ falls (rises) (Proposition 6). Hence, workers shift to harder-to-codify and more analytical tasks while $\frac{k_a}{k_m}$ falls, whereas after $\frac{k_a}{k_m}$ starts to rise, they shift to harder-to-codify tasks overall and shift to more *manual tasks* at low c (Figure 14 (a)). This is consistent with the shift *from non-routine analytical* tasks as well as routine tasks to non-routine manual tasks after around the year 2000 in the U.S. (Beaudry, Green, and Sand, 2016; see footnote 10 in the introduction for details).

In sum, unlike the proportionate growth case, the model with realistic productivity growth is consistent with a large part of the developments after the 1980s. The result suggests that mechanization driven by the rising productivity of machines and the increased proportion of skilled workers are important in understanding the long-term evolution of task shifts, earnings levels and inequality from the era of the Industrial Revolution until the present. Of course, other factors, such as changes in union density (Farber et al., 2021) and increased trade with and increased offshoring to developing countries after the 1990s (Firpo, Fortin, and Lemieux, 2013; Ebenstein et al., 2014), too have significant effects,[43] but only the two changes considered in the paper seem to have influenced the evolution continuously.

If the rapid progress of information technology continues and $\frac{k_a}{k_m}$ keeps rising, comparative advantages of machines to two type of workers could change over time, i.e., first, from $\frac{k_a}{k_m} < \frac{l_a}{l_m}$ to $\frac{k_a}{k_m} \in (\frac{l_a}{l_m}, \frac{h}{l_m})$, then to $\frac{k_a}{k_m} > \frac{h}{l_m}$. The model predicts what will happen to task assignment, earnings, and earnings inequality under such situations. As before, both types of workers shift to tasks that are more difficult to routinize (unless $\frac{N_h}{N_l}$ rises greatly, which is very unlikely). By contrast, unlike before, unskilled workers shift to more *manual* tasks (even at high c), and, when $\frac{k_a}{k_m} > \frac{h}{l_m}$, skilled workers too shift to such tasks (Figure 14 (c)). That is, workers shift to relatively manual and difficult-to-codify tasks: the recent shift to low-wage service occupations such as personal care and protective service may continue into

[41] As mentioned below, the wage premium of college graduates (a weighted average of the college *and* post-college wage premium relative to high school graduates) is stagnant in the 2010s (Autor, Goldin, and Katz, 2020), which suggests that the earnings disparity between workers with skills for low-wage jobs and those with skills for moderately high-wage jobs is *falling* recently.

[42] The quantitative model with three type of workers, who differ in levels of analytical ability or ability to perform non-routine tasks, would yield the rising inequality between workers with skills for high-wage jobs and other workers as well.

[43] Farber et al. (2021) find negative effects of union density on various measures of income inequality for the U.S. economy, using data from 1936 to 2014. Firpo, Fortin, and Lemieux (2013) find that the effect of trade and offshoring on wage inequality is important after the 1990s and strong in the 2000s for the U.S. economy. Ebenstein et al. (2014) find that the effect of trade and offshoring on real wages and employment is large after 1997 (until the end of the sample period, 2002).

the future. However, the model predicts that earnings of unskilled workers as well as those of skilled workers rise and earnings inequality shrinks over time. The analysis based on the model with two types of workers would not capture the whole picture, considering the recent widening inequality between moderately and extremely high-skill workers (Alvaredo et al., 2013). And, the extended model with more than two types of workers, which is not analytically tractable, may not be sufficient to understand the evolution of the right tail of the distribution at which, Alvaredo et al. (2013), based on international evidence, argue that institutional and policy changes play important roles. However, the stagnant wage premium of college graduates (a weighted average of the college *and* post-college wage premium relative to high school graduates) in the 2010s (Autor, Goldin, and Katz, 2020) and episodes such as declining newspaper industry, burgeoning online education, and the increasing use of big data in marketing, trading, management and other decisions (such as the diagnosis of diseases) suggest that machines would replace a large number of tasks presently performed by highly skilled workers in the not-distant future and thus possible effects on a great majority of the population might be captured by the present model.[44]

6 Conclusion

Since the Industrial Revolution, mechanization (or automation) has affected types of tasks humans perform, relative demands for workers of different skill levels, earnings levels and inequality, and aggregate output. This paper has developed a Ricardian model of task assignment and examined how improvements of qualities of machines and an increase in the relative supply of skilled workers affect these variables. The analysis has shown that tasks and workers strongly affected by the productivity growth and the effects on earnings levels and inequality change over time. The model is consistent with long-run trends of these variables in the real economy, except a decline of the inequality in the wartime 1940s and job polarization and the fall of the relative wage of workers with skills for middle-wage jobs after the 1990s, though the model does capture an important part of the latter development. The model has also been employed to examine possible future trends of these variables when the rapid growth of information technology continues. It is found that earnings of both skilled and unskilled workers increase and earnings inequality falls over time.

Several extensions of the model would be fruitful for analyzing the recent evolutions of the labor market quantitatively. First, in order to understand job polarization and the related development of earnings more accurately, the model with more than two type of workers, who differ in levels of analytical ability or ability to perform non-routine tasks, could be developed. Second, empirical works find that international trade and offshoring have important effects on earnings inequality after the 1990s, thus it may be interesting to examine effects of these factors and productivity growth jointly.

[44] Autor, Goldin, and Katz (2020) find that the growth of the wage premium of college graduates slowed down considerably after 2000 and the premium has stopped increasing in the 2010s, while the growth of the wage premium of post-college graduates remains strong.

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Appendix A: Lemmas

This appendix presents lemmas examining the shape of (HL) and its relations with exogenous variables illustrated in Figure 4 of Section 3, and a lemma examining the shape of (P) and its relations with exogenous variables illustrated in Figure 5. Proofs are in Appendix B.

The next lemma presents the result when $c^*, c_a < 1$ ($c^* < (>) c_a$ when $\frac{k_a}{k_m} < (>) \frac{h}{l_m}$), the area below $c_m = \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$ of Figure 3. Note that *no* assumptions are imposed on relations of analytical abilities to manual abilities, although presentations in the lemmas might appear to suppose $h > l_m$, $l_m > l_a$, and $k_m > k_a$.

Lemma 1 *When $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow c^*, c_a < 1$, (HL) is expressed as*

$$\frac{N_h}{N_l}\ln\left(\frac{k_m}{A_k(a^*)}\right) = \frac{A_l(a^*)}{A_h(a^*)}\ln\left(\frac{A_k(a^*)}{k_a}\right), \quad \text{when } \frac{k_a}{k_m} \neq 1, \tag{16}$$

$$\frac{N_h}{N_l}a^* = \frac{A_l(a^*)}{A_h(a^*)}(1-a^*), \quad \text{when } \frac{k_a}{k_m} = 1. \tag{17}$$

a^* satisfying the equation decreases with $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$.

Unlike the cases below, (HL) is independent of c_m. a^* satisfying the equation decreases with $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$. The next lemma presents the result when $c^* < c_a = 1$, the area below $c_m = \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}$ and on or above $c_m = \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$ of Figure 3. This case arises only when $\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} > \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow \frac{k_a}{k_m} < \frac{h}{l_m}$.

Lemma 2 *When $c_m \in \left[\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}, \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}\right) \Leftrightarrow c^* < c_a = 1$, which arises only when $\frac{k_a}{k_m} < \frac{h}{l_m}$, (HL) is expressed as*

$$\text{when } \frac{k_a}{k_m} \neq 1, \quad \frac{N_h}{N_l}\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\ln\left(\frac{k_m}{A_k(a^*)}\right)$$
$$= \frac{1}{h-l_m}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)} + (h-l_m)c_m}{\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}h\right] + \frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m-k_a}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)} + (h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right], \tag{18}$$

$$\text{when } \frac{k_a}{k_m} = 1, \quad \frac{N_h}{N_l}\frac{c_m a^*}{l_m} = \frac{1}{h-l_m}\left\{\ln\left[\frac{h}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\right] - \frac{A_l(a^*)}{l_m}c_m + 1\right\}. \tag{19}$$

a^* satisfying the equation decreases with c_m and $\frac{N_h}{N_l}$ ($\frac{\partial a^*}{\partial c_m} = 0$ at $c_m = \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$), and decreases (increases) with $\frac{k_a}{k_m}$ for small (large) c_m.

Unlike the previous case, a^* satisfying (HL) decreases with c_m (except at $c_m = \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$, where $\frac{\partial a^*}{\partial c_m} = 0$), and it increases with $\frac{k_a}{k_m}$ when c_m is large. Finally, the next lemma presents the result when $c^* = c_a = 1$, the area on or above $c_m = \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}$ of Figure 3. This case arises only when $\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} < 1 \Leftrightarrow \frac{k_a}{k_m} < \frac{l_a}{l_m}$.

Lemma 3 *When $c_m \geq \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} \Leftrightarrow c^* = c_a = 1$, which arises only when $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, (HL) is expressed as*

$$\frac{N_h}{N_l}\left\{\frac{1}{l_m - l_a}\ln\left[\frac{l_a k_m - l_m k_a}{(k_m - k_a)l_m - (l_m - l_a)k_m c_m}\frac{l_m}{A_l(a^*)}\right] + \frac{k_m c_m}{(k_m - k_a)l_m}\ln\left[\frac{(k_m - k_a)l_m - (l_m - l_a)k_m c_m}{(l_a k_m - l_m k_a)c_m}\right]\right\}$$
$$= \frac{1}{h - l_m}\ln\left(\frac{h}{A_h(a^*)}\right), \quad \textit{when } \frac{k_a}{k_m} \neq 1, \tag{20}$$

$$\frac{N_h}{N_l}\frac{1}{l_a - l_m}\left\{\ln\left[\frac{c_m A_l(a^*)}{l_m}\right] + 1 - c_m\right\} = \frac{1}{h - l_m}\ln\left(\frac{h}{A_h(a^*)}\right), \quad \textit{when } \frac{k_a}{k_m} = 1, \tag{21}$$

where $a^* \in (0,1)$ holds for any c_m. a^* satisfying the equation decreases with c_m and $\frac{N_h}{N_l}$, and it increases with $\frac{k_a}{k_m}$ ($\lim_{c_m \to 1}\frac{\partial a^*}{\partial c_m} = \lim_{c_m \to 1}\frac{\partial a^*}{\partial \frac{k_a}{k_m}} = 0$).

a^* satisfying (HL) decreases with c_m as in the previous case, while it increases with $\frac{k_a}{k_m}$ ($\lim_{c_m \to 1}\frac{\partial a^*}{\partial c_m} = \lim_{c_m \to 1}\frac{\partial a^*}{\partial \frac{k_a}{k_m}} = 0$, though).

Finally, the next lemma presents the shape of (P) and its relations with k_m, k_a, and r.

Lemma 4 *c_m satisfying (P), which is positive, increases with a^* and r, and decreases with k_m and k_a.*

7 Appendix B: Proofs of Lemmas and Propositions 1-3

Proof of Lemma 1. [**Derivation of the LHS of the equation**]**:** When $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$ and thus $c_m < \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} \Leftrightarrow c^* = c_l(a^*) < 1$, the LHS of (HL) equals $\frac{N_h}{N_l}$ times

$$\int_0^{a^*}\int_0^{c_l(a)}\frac{1}{A_l(a)}dcda = \int_0^{a^*}\frac{c_l(a)}{A_l(a)}da = \frac{k_m}{l_m}c_m\int_0^{a^*}\frac{da}{A_k(a)}. \tag{22}$$

Hence, when $\frac{k_a}{k_m} \neq 1$, the LHS of (HL) equals

$$\frac{N_h}{N_l}\frac{k_m}{l_m}\frac{c_m}{k_m - k_a}\ln\left(\frac{k_m}{A_k(a^*)}\right). \tag{23}$$

Applying l'Hôpital's rule to the above equation, the LHS of (HL) when $\frac{k_a}{k_m} = 1$ equals

$$-\frac{N_h}{N_l}\frac{1}{l_m}\frac{c_m}{\lim_{\frac{k_a}{k_m}\to 1}(1 - \frac{k_a}{k_m})}\lim_{\frac{k_a}{k_m}\to 1}\ln\left(a^*\frac{k_a}{k_m} + 1 - a^*\right) = \frac{N_h}{N_l}\frac{c_m}{l_m}\lim_{\frac{k_a}{k_m}\to 1}\left(\frac{a^*}{a^*\frac{k_a}{k_m} + 1 - a^*}\right)$$
$$= \frac{N_h}{N_l}\frac{c_m a^*}{l_m}. \tag{24}$$

[Derivation of the RHS of the equation]: When $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)} \Leftrightarrow c_a = c_h(1) < 1$, the RHS of (HL) is expressed as

$$\int_{a^*}^{1}\int_0^{c_h(a)} \frac{1}{A_h(a)} dcda = \int_{a^*}^{1} \frac{c_h(a)}{A_h(a)} da = \frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)} c_m \int_{a^*}^{1} \frac{da}{A_k(a)}. \tag{25}$$

Hence, when $\frac{k_a}{k_m} \neq 1$, the RHS of (HL) equals

$$\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m - k_a} \ln\left(\frac{A_k(a^*)}{k_a}\right). \tag{26}$$

By applying l'Hôpital's rule to the above equation, the LHS of (HL) when $\frac{k_a}{k_m} = 1$ equals

$$\begin{aligned}\frac{A_l(a^*)}{A_h(a^*)}\frac{1}{l_m}\frac{c_m}{\lim_{\frac{k_a}{k_m}\to 1}(1-\frac{k_a}{k_m})} \lim_{\frac{k_a}{k_m}\to 1} \ln\left[a^* + (1-a^*)\frac{k_m}{k_a}\right] &= -\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{l_m} \lim_{\frac{k_a}{k_m}\to 1}\left(\frac{-(1-a^*)(\frac{k_a}{k_m})^{-2}}{a^* + (1-a^*)\frac{k_m}{k_a}}\right) \\ &= \frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{l_m}(1-a^*). \end{aligned} \tag{27}$$

[Relations of a^* satisfying the equation with $\frac{N_h}{N_l}$ and $\frac{k_a}{k_m}$]: Clearly, a^* satisfying the equation decreases with $\frac{N_h}{N_l}$. Noting that, from (23) and (26), (HL) when $\frac{k_a}{k_m} \neq 1$ can be expressed as

$$\frac{k_m}{l_m}\frac{c_m}{k_m - k_a}\left[-\frac{N_h}{N_l}\ln\left(a^*\frac{k_a}{k_m} + 1 - a^*\right) - \frac{A_l(a^*)}{A_h(a^*)}\ln\left(a^* + (1-a^*)\frac{k_m}{k_a}\right)\right] = 0, \tag{28}$$

the derivative of the above equation with respect to $\frac{k_a}{k_m}$ equals

$$\begin{aligned}&\frac{k_m}{l_m}\frac{c_m}{k_m - k_a}\left(-\frac{N_h}{N_l}\frac{a^*}{a^*\frac{k_a}{k_m} + 1 - a^*} - \frac{A_l(a^*)}{A_h(a^*)}\frac{-(1-a^*)(\frac{k_a}{k_m})^{-2}}{a^* + (1-a^*)\frac{k_m}{k_a}}\right) \\ &= \frac{k_m}{l_m}\frac{c_m}{k_m - k_a}\frac{k_m}{A_k(a^*)}\left[-\frac{N_h}{N_l}a^* + \frac{A_l(a^*)}{A_h(a^*)}(1-a^*)\frac{k_m}{k_a}\right],\end{aligned} \tag{29}$$

where the expression inside the large bracket can be rewritten as

$$\begin{aligned}&-\frac{N_h}{N_l}a^* + \frac{A_l(a^*)}{A_h(a^*)}(1-a^*)\frac{k_m}{k_a} = \left[\ln\left(\frac{A_k(a^*)}{k_a}\right)\right]^{-1}\frac{N_h}{N_l}\left[-a^*\ln\left(a^* + (1-a^*)\frac{k_m}{k_a}\right) - (1-a^*)\frac{k_m}{k_a}\ln\left(a^*\frac{k_a}{k_m} + 1 - a^*\right)\right] \\ &= \left[\ln\left(\frac{A_k(a^*)}{k_a}\right)\right]^{-1}\frac{N_h}{N_l}\frac{k_m}{k_a}\left[a^*\frac{k_a}{k_m}\ln\left(\frac{k_a}{k_m}\right) - \left(a^*\frac{k_a}{k_m} + 1 - a^*\right)\ln\left(a^*\frac{k_a}{k_m} + 1 - a^*\right)\right].\end{aligned} \tag{30}$$

The expression inside the large bracket of the above equation is positive, because the expression equals 0 at $\frac{k_a}{k_m} = 1$ and its derivative with respect to $\frac{k_a}{k_m}$ equals

$$a^*\left[\ln\left(\frac{k_a}{k_m}\right) - \ln\left(a^*\frac{k_a}{k_m} + 1 - a^*\right)\right], \tag{31}$$

which is negative (positive) for $\frac{k_a}{k_m} < (>)1$. Thus, noting that $\ln\left(\frac{A_k(a^*)}{k_a}\right) > (<)0$ for $\frac{k_a}{k_m} < (>)1$, (29) is positive. The derivative of (28) with respect to a^* is positive from $\partial\frac{A_l(a^*)}{A_h(a^*)}/\partial a^* < 0$. Hence, a^* satisfying (16) decreases with $\frac{k_a}{k_m}$ when $\frac{k_a}{k_m} \neq 1$. When $\frac{k_a}{k_m} \to 1$, (29) equals

$$\lim_{\frac{k_a}{k_m}\to 1}\left\{\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\frac{1}{a^*\frac{k_a}{k_m}+1-a^*}\left[-\frac{N_h}{N_l}a^*+\frac{A_l(a^*)}{A_h(a^*)}(1-a^*)\frac{k_m}{k_a}\right]\right\}$$

$$=-\frac{c_m}{l_m}\lim_{\frac{k_a}{k_m}\to 1}\left\{\frac{-\left(a^*\frac{k_a}{k_m}+1-a^*\right)\frac{A_l(a^*)}{A_h(a^*)}(1-a^*)(\frac{k_a}{k_m})^{-2}-\left(-\frac{N_h}{N_l}a^*+\frac{A_l(a^*)}{A_h(a^*)}(1-a^*)\frac{k_m}{k_a}\right)a^*}{\left(a^*\frac{k_a}{k_m}+1-a^*\right)^2}\right\}$$

$$=\frac{c_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}(1-a^*)>0. \tag{32}$$

where (17) is used to derive the last equality. Thus, the same result holds when $\frac{k_a}{k_m}=1$ too. ■

Proof of Lemma 2. [Derivation of the equation]: Since $c^*<1$, the LHS of (HL) equals (23) (when $\frac{k_a}{k_m}\neq 1$) and (24) (when $\frac{k_a}{k_m}=1$) in the proof of Lemma 1.

The RHS of (HL) when $c_a=1 \Leftrightarrow c_h(1)\geq 1$, $c^*<1 \Leftrightarrow c_h(a^*)<1$, and $\frac{k_a}{k_m}\neq 1$ is expressed as

$$\int_{a^*}^{c_h^{-1}(1)}\int_0^{c_h(a)}\frac{dcda}{A_h(a)}+\int_{c_h^{-1}(1)}^1\int_0^1\frac{dcda}{A_h(a)}=\int_{a^*}^{c_h^{-1}(1)}\frac{c_h(a)}{A_h(a)}da+\int_{c_h^{-1}(1)}^1\frac{da}{A_h(a)}$$

$$=\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\int_{a^*}^{c_h^{-1}(1)}\frac{da}{A_k(a)}+\int_{c_h^{-1}(1)}^1\frac{da}{A_h(a)}$$

$$=\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m-k_a}\ln\left(\frac{A_k(a^*)}{A_k(c_h^{-1}(1))}\right)+\frac{1}{h-l_m}\ln\left(\frac{h}{A_h(c_h^{-1}(1))}\right), \tag{33}$$

where $c_h^{-1}(1)$, i.e., the value of a when $c_h(a)=1$, equals, from (1) and (3),

$$\frac{A_h(a)}{A_k(a)}=\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{1}{c_m}\Leftrightarrow a(h-l_m)+l_m=\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{1}{c_m}[-a(k_m-k_a)+k_m]$$

$$\Leftrightarrow a=\frac{l_m\left(\frac{A_h(a^*)}{A_l(a^*)}-c_m\right)}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}. \tag{34}$$

Hence, from (33) and

$$A_k(c_h^{-1}(1))=\frac{-l_m\left(\frac{A_h(a^*)}{A_l(a^*)}-c_m\right)(k_m-k_a)+k_m\left[(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m\right]}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}$$

$$=\frac{(hk_m-l_mk_a)c_m}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}, \tag{35}$$

$$A_h(c_h^{-1}(1))=\frac{l_m\left(\frac{A_h(a^*)}{A_l(a^*)}-c_m\right)(h-l_m)+l_m\left[(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m\right]}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}$$

$$=\frac{\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}, \tag{36}$$

the RHS of (HL) when $\frac{k_a}{k_m}\neq 1$, equals

$$\frac{1}{h-l_m}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}h\right]+\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m-k_a}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]. \tag{37}$$

By applying l'Hôpital's rule to the above equation, the RHS when $\frac{k_a}{k_m}=1$ equals

$$\begin{aligned}
&\frac{1}{h-l_m}\lim_{\frac{k_a}{k_m}\to 1}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(h-l_m\frac{k_a}{k_m})}h\right]+\frac{\frac{1}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m}{\lim_{\frac{k_a}{k_m}\to 1}(1-\frac{k_a}{k_m})}\lim_{\frac{k_a}{k_m}\to 1}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(h-l_m\frac{k_a}{k_m})c_m}{a^*\frac{k_a}{k_m}+(1-a^*)}}\right]\\
&=\frac{1}{h-l_m}\ln\left[\frac{h}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\right]-\frac{1}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\lim_{\frac{k_a}{k_m}\to 1}\left[\frac{a^*}{a^*\frac{k_a}{k_m}+(1-a^*)}+\frac{l_m}{h-l_m\frac{k_a}{k_m}}-\frac{l_m\frac{A_h(a^*)}{A_l(a^*)}}{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\right]\\
&=\frac{1}{h-l_m}\ln\left[\frac{h}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\right]-\frac{1}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\left[\frac{A_h(a^*)}{h-l_m}-\frac{l_m\frac{A_h(a^*)}{A_l(a^*)}}{(h-l_m)c_m}\right]\\
&=\frac{1}{h-l_m}\left\{\ln\left[\frac{h}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m\right]-\frac{A_l(a^*)}{l_m}c_m+1\right\}.
\end{aligned}\tag{38}$$

[Relations of a^* satisfying the equation with $\frac{N_h}{N_l}$ and c_m]: When $\frac{k_a}{k_m}\neq 1$, the derivative of the LHS$-$RHS of (18) with respect to a^* equals

$$\begin{aligned}
&\frac{N_h}{N_l}\frac{k_m}{l_m}c_m\frac{1}{A_k(a^*)}+\frac{1}{h-l_m}\left[\frac{1}{\frac{A_h(a^*)}{A_l(a^*)}}-\frac{(k_m-k_a)\frac{l_m}{k_m}}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\right]\frac{\partial\frac{A_h(a^*)}{A_l(a^*)}}{\partial a^*}+\frac{k_m}{l_m}c_m\frac{A_l(a^*)}{A_h(a^*)}\frac{1}{A_k(a^*)}\\
&\qquad-\frac{c_m\frac{A_l(a^*)}{A_h(a^*)}}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\frac{\partial\frac{A_h(a^*)}{A_l(a^*)}}{\partial a^*}-\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]\\
&=c_m\left[\begin{array}{c}\frac{N_h}{N_l}\frac{k_m}{l_m}\frac{1}{A_k(a^*)}+\frac{\frac{A_l(a^*)}{A_h(a^*)}}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\frac{\partial\frac{A_h(a^*)}{A_l(a^*)}}{\partial a^*}+\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{1}{A_k(a^*)}\\
-\frac{\frac{A_l(a^*)}{A_h(a^*)}}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\frac{\partial\frac{A_h(a^*)}{A_l(a^*)}}{\partial a^*}-\frac{k_m}{l_m}\frac{1}{k_m-k_a}\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]\end{array}\right]
\end{aligned}$$

$$=\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\left\{\left(\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right)\frac{k_m-k_a}{A_k(a^*)}-\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\ln\left[1+\frac{(k_m-k_a)\frac{A_h(a^*)}{A_k(a^*)}\left[\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}-c_m\right]}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]\right\}>0, \tag{39}$$

where the last equality is derived by using

$$\begin{aligned}
\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}&=\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m-\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}+\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\\
&=1+\frac{(k_m-k_a)\frac{A_h(a^*)}{A_k(a^*)}\left[\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}-c_m\right]}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}>(<)1\text{ when }\frac{k_a}{k_m}<(>)1\quad\left(\because c_m<\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}\right).
\end{aligned}\tag{40}$$

The derivative of the LHS-RHS of (18) with respect to c_m when $\frac{k_a}{k_m}\neq 1$ equals

$$\frac{1}{(h-l_m)c_m}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}h\right]-\frac{1+\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\frac{A_l(a^*)}{A_h(a^*)}(h-l_m)}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}+\frac{k_m}{l_m}\frac{1}{k_m-k_a}\frac{A_l(a^*)}{A_h(a^*)}$$

$$=\frac{1}{(h-l_m)c_m}\ln\left[1+\frac{(h-l_m)hk_m\left[c_m-\frac{l_m}{h}\frac{k_a}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\right]}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}\right]\geq 0\quad\left(\because c_m\geq\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}\right),\tag{41}$$

where the last equality is derived by using

$$\frac{(k_m-k_a)l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)k_mc_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}h=\frac{\left[(k_m-k_a)l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)k_mc_m\right]h-l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)+l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}$$

$$=1+\frac{(h-l_m)hk_m\left[c_m-\frac{l_m}{h}\frac{k_a}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\right]}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}.\tag{42}$$

Hence, when $\frac{k_a}{k_m}\neq 1$, a^* satisfying (18) decreases with $\frac{N_h}{N_l}$ and c_m ($\frac{\partial a^*}{\partial c_m}=0$ at $c_m=\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$).

The corresponding derivatives when $\frac{k_a}{k_m}\to 1$ are

$$a^*:\lim_{\frac{k_a}{k_m}\to 1}\left(\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\left\{\left(\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right)\frac{1-\frac{k_a}{k_m}}{a^*\frac{k_a}{k_m}+1-a^*}-\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\ln\left[1+\frac{(1-\frac{k_a}{k_m})A_h(a^*)\left[(a^*\frac{k_a}{k_m}+1-a^*)\frac{l_m}{A_l(a^*)}-c_m\right]}{(h-l_m\frac{k_a}{k_m})c_m}\right]\right\}\right)$$

$$=-\frac{c_m}{l_m}\lim_{\frac{k_a}{k_m}\to 1}\left\{\begin{gathered}\left(\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right)\frac{-(a^*\frac{k_a}{k_m}+1-a^*)-(1-\frac{k_a}{k_m})(1-a^*)}{(a^*\frac{k_a}{k_m}+1-a^*)^2}-\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\left[1+\frac{(1-\frac{k_a}{k_m})A_h(a^*)\left[(a^*\frac{k_a}{k_m}+1-a^*)\frac{l_m}{A_l(a^*)}-c_m\right]}{(h-l_m\frac{k_a}{k_m})c_m}\right]^{-1}\\ \times\frac{(h-l_m\frac{k_a}{k_m})A_h(a^*)\left[-\left((a^*\frac{k_a}{k_m}+1-a^*)\frac{l_m}{A_l(a^*)}-c_m\right)+(1-\frac{k_a}{k_m})\frac{(1-a^*)l_m}{A_l(a^*)}\right]+l_m(1-\frac{k_a}{k_m})A_h(a^*)\left[(a^*\frac{k_a}{k_m}+1-a^*)\frac{l_m}{A_l(a^*)}-c_m\right]}{(h-l_m\frac{k_a}{k_m})^2c_m}\end{gathered}\right\}$$

$$=\frac{c_m}{l_m}\left\{\left(\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right)-\frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\frac{A_h(a^*)\left(\frac{l_m}{A_l(a^*)}-c_m\right)}{(h-l_m)c_m}\right\}>0,\tag{43}$$

$$c_m:\frac{1}{(h-l_m)c_m}\ln\left[1+\frac{(h-l_m)h\left[c_m-\frac{l_m}{h}\frac{A_h(a^*)}{A_l(a^*)}\right]}{l_m\frac{A_h(a^*)}{A_l(a^*)}(h-l_m)}\right]\geq 0.\tag{44}$$

Therefore, the same results hold when $\frac{k_a}{k_m}=1$ as well.

[Relations of a^* satisfying the equation with $\frac{k_a}{k_m}$]: Since (18) can be expressed as

$$-\frac{N_h}{N_l}\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right)\tag{45}$$

$$=\frac{1}{h-l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(h-l_m\frac{k_a}{k_m})}h\right]+\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\frac{A_l(a^*)}{A_h(a^*)}\ln\left[\frac{a^*\frac{k_a}{k_m}+1-a^*}{h-l_m\frac{k_a}{k_m}}\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{c_m}\right],$$

the derivative of the LHS−RHS of (18) with respect to $\frac{k_a}{k_m}$ when $\frac{k_a}{k_m}\neq 1$ equals

$$-\frac{N_h}{N_l}\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\left[\frac{\ln\left(a^*\frac{k_a}{k_m}+1-a^*\right)}{1-\frac{k_a}{k_m}}+\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}\right]-\frac{1}{l_m}\frac{c_m}{(1-\frac{k_a}{k_m})^2}\frac{A_l(a^*)}{A_h(a^*)}\ln\left[\frac{a^*\frac{k_a}{k_m}+1-a^*}{h-l_m\frac{k_a}{k_m}}\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{c_m}\right]$$

$$+\frac{l_m}{h-l_m}\left[\frac{\frac{A_h(a^*)}{A_l(a^*)}}{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}-\frac{1}{h-l_m\frac{k_a}{k_m}}\right]-\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\frac{A_l(a^*)}{A_h(a^*)}\left[\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}+\frac{l_m}{h-l_m\frac{k_a}{k_m}}-\frac{l_m\frac{A_h(a^*)}{A_l(a^*)}}{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\right]$$

$$
\begin{aligned}
&= \frac{1}{(h-l_m)(1-\frac{k_a}{k_m})} \ln\left[\frac{(1-\frac{k_a}{k_m})l_m \frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{l_m \frac{A_h(a^*)}{A_l(a^*)}(h-l_m\frac{k_a}{k_m})} h\right] - \frac{N_h}{N_l}\frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*} + \frac{l_m\left(\frac{A_h(a^*)}{A_l(a^*)}-c_m\right)}{(h-l_m\frac{k_a}{k_m})\left[(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m\right]} \\
&\qquad - \frac{1}{l_m}\frac{c_m}{1-\frac{k_a}{k_m}}\frac{A_l(a^*)}{A_h(a^*)}\left[\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*} - \frac{l_m(h-l_m)\left(\frac{A_h(a^*)}{A_l(a^*)}-c_m\right)}{(h-l_m\frac{k_a}{k_m})\left[(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m\right]}\right] \\
&= \frac{k_m}{k_m-k_a}\left\{-\left[\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right]\frac{k_m}{l_m}\frac{a^*}{A_k(a^*)}c_m + \frac{k_m\left(1-c_m\frac{A_l(a^*)}{A_h(a^*)}\right)}{hk_m-l_mk_a} + \frac{1}{h-l_m}\ln\left[\frac{(k_m-k_a)l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)k_mc_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m-l_mk_a)}h\right]\right\}. \qquad (46)
\end{aligned}
$$

Since the derivative on (HL) is examined, by substituting (18) into the above equation

$$
\frac{k_m}{k_m-k_a}\left\{\begin{array}{c} -\left[\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right]\frac{k_m}{l_m}\frac{a^*}{A_k(a^*)}c_m + \frac{k_m\left(1-c_m\frac{A_l(a^*)}{A_h(a^*)}\right)}{hk_m-l_mk_a} + \frac{N_h}{N_l}\frac{k_m}{l_m}\frac{c_m}{k_m-k_a}\ln\left(\frac{k_m}{A_k(a^*)}\right) \\ -\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m-k_a}\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]\end{array}\right\} \qquad (47)
$$

$$
= \frac{k_mc_m}{(k_m-k_a)^2}\frac{k_m}{l_m}\left\{\begin{array}{c}\frac{N_h}{N_l}\left[\ln\left(\frac{k_m}{A_k(a^*)}\right)+1-\frac{k_m}{A_k(a^*)}\right] \\ -\frac{A_l(a^*)}{A_h(a^*)}\left[(k_m-k_a)\frac{A_h(a^*)}{A_l(a^*)}\frac{1}{c_m}\frac{A_l(a^*)k_mc_m-l_mA_k(a^*)}{A_k(a^*)(hk_m-l_mk_a)}+\ln\left[\frac{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{\frac{(hk_m-l_mk_a)c_m}{A_k(a^*)}}\right]\right]\end{array}\right\}. \qquad (48)
$$

The above expression is positive at $c_m=\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$ from (29) in the proof of Lemma 1 and is negative at $c_m=\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}$ from (59) in the proof of Lemma 3. Further, the derivative of the expression inside the big bracket of the above equation with respect to c_m equals

$$
\begin{aligned}
&-(k_m-k_a)\frac{1}{c_m^2}\frac{l_m}{hk_m-l_mk_a}-\frac{A_l(a^*)}{A_h(a^*)}\left[\frac{h-l_m}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}-\frac{1}{c_m}\right] \\
&= \frac{l_m}{k_m}\frac{k_m-k_a}{c_m}\left[-\frac{1}{c_m}\frac{k_m}{hk_m-l_mk_a}+\frac{1}{(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}\right] = -\frac{l_m^2}{k_m}\frac{1}{c_m^2}\frac{(k_m-k_a)^2\left[\frac{A_h(a^*)}{A_l(a^*)}-c_m\right]}{(hk_m-l_mk_a)\left[(k_m-k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m\right]}, \qquad (49)
\end{aligned}
$$

which is negative for $c_m\in\left[\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)},\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}\right]$ from $\frac{A_h(a^*)}{A_l(a^*)}-c_m\geq\frac{A_h(a^*)k_m-l_mA_k(a^*)}{A_l(a^*)k_m}=\frac{(hk_m-l_mk_a)a^*}{A_l(a^*)k_m}>0$ ($\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}>\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}\Leftrightarrow\frac{k_a}{k_m}<\frac{h}{l_m}$). Hence, there exists a unique $c_m\in(\frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)},\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)})$ such that (46) is positive (negative) for smaller (greater) c_m.

When $\frac{k_a}{k_m}\to 1$, (46) equals

$$
\begin{aligned}
&\lim_{\frac{k_a}{k_m}\to 1}\frac{1}{1-\frac{k_a}{k_m}}\left\{-\left[\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right]\frac{1}{l_m}\frac{a^*}{a^*\frac{k_a}{k_m}+1-a^*}c_m+\frac{1-c_m\frac{A_l(a^*)}{A_h(a^*)}}{h-l_m\frac{k_a}{k_m}}+\frac{1}{h-l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}{l_m\frac{A_h(a^*)}{A_l(a^*)}(h-l_m\frac{k_a}{k_m})}h\right]\right\} \\
&= -\lim_{\frac{k_a}{k_m}\to 1}\left\{\left[\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right]\frac{1}{l_m}\frac{a^{*2}c_m}{(a^*\frac{k_a}{k_m}+1-a^*)^2}+\frac{l_m\left(1-c_m\frac{A_l(a^*)}{A_h(a^*)}\right)}{(h-l_m\frac{k_a}{k_m})^2}+\frac{1}{h-l_m}\left[\frac{-l_m\frac{A_h(a^*)}{A_l(a^*)}}{(1-\frac{k_a}{k_m})l_m\frac{A_h(a^*)}{A_l(a^*)}+(h-l_m)c_m}+\frac{l_m}{h-l_m\frac{k_a}{k_m}}\right]\right\} \\
&= -\left\{\left[\frac{N_h}{N_l}+\frac{A_l(a^*)}{A_h(a^*)}\right]\frac{a^{*2}c_m}{l_m}-\frac{l_m\left(1-c_m\frac{A_l(a^*)}{A_h(a^*)}\right)\left(\frac{1}{c_m}\frac{A_h(a^*)}{A_l(a^*)}-1\right)}{(h-l_m)^2}\right\}. \qquad (50)
\end{aligned}
$$

The expression is positive at $c_m=\frac{l_m}{h}\frac{A_h(a^*)}{A_l(a^*)}$ from (32) in the proof of Lemma 1 and negative at $c_m=\frac{l_m}{A_l(a^*)}$ from (61) in the proof of Lemma 3. Further, the derivative of the expression with respect to c_m is negative. Hence, the same result holds when $\frac{k_a}{k_m}=1$ as well. ■

Proof of Lemma 3. [Derivation of the equation]: The LHS of (HL) when $c^* = 1 \Leftrightarrow c_l(a^*) \geq 1$ and $\frac{k_a}{k_m} \neq 1$ equals $\frac{N_h}{N_l}$ times

$$\int_0^{c_l^{-1}(1)}\int_0^{c_l(a)} \frac{dcda}{A_l(a)} + \int_{c_l^{-1}(1)}^{a^*}\int_0^1 \frac{dcda}{A_l(a)} = \int_0^{c_l^{-1}(1)} \frac{c_l(a)}{A_l(a)}da + \int_{c_l^{-1}(1)}^{a^*} \frac{da}{A_l(a)}$$
$$= \frac{k_m}{l_m} c_m \int_0^{c_l^{-1}(1)} \frac{da}{A_k(a)} + \int_{c_l^{-1}(1)}^{a^*} \frac{da}{A_l(a)}$$
$$= \frac{k_m}{l_m}\frac{c_m}{k_m - k_a} \ln\left(\frac{k_m}{A_k(c_l^{-1}(1))}\right) + \frac{1}{l_m - l_a} \ln\left(\frac{A_l(c_l^{-1}(1))}{A_l(a^*)}\right), \tag{51}$$

where the value of $c_l^{-1}(1)$, i.e., a when $c_l(a) = 1$, equals, from (2) and (3),

$$\frac{A_l(a)}{A_k(a)} = \frac{l_m}{k_m}\frac{1}{c_m} \Leftrightarrow -a(l_m - l_a) + l_m = \frac{l_m}{k_m}\frac{1}{c_m}[-a(k_m - k_a) + k_m]$$
$$\Leftrightarrow a = \frac{l_m(1 - c_m)}{(k_m - k_a)\frac{l_m}{k_m} - (l_m - l_a)c_m}. \tag{52}$$

Hence, from (51) and

$$A_k(c_l^{-1}(1)) = \frac{-l_m(1-c_m)(k_m - k_a) + k_m\left[(k_m - k_a)\frac{l_m}{k_m} - (l_m - l_a)c_m\right]}{(k_m - k_a)\frac{l_m}{k_m} - (l_m - l_a)c_m}$$
$$= \frac{(l_a k_m - l_m k_a)c_m}{(k_m - k_a)\frac{l_m}{k_m} - (l_m - l_a)c_m}, \tag{53}$$
$$A_l(c_l^{-1}(1)) = \frac{-l_m(1-c_m)(l_m - l_a) + l_m\left[(k_m - k_a)\frac{l_m}{k_m} - (l_m - l_a)c_m\right]}{(k_m - k_a)\frac{l_m}{k_m} - (l_m - l_a)c_m}$$
$$= \frac{\frac{l_m}{k_m}(l_a k_m - l_m k_a)}{(k_m - k_a)\frac{l_m}{k_m} - (l_m - l_a)c_m}, \tag{54}$$

the LHS of (HL) when $\frac{k_a}{k_m} \neq 1$ equals

$$\frac{N_h}{N_l}\left\{\frac{1}{l_m - l_a}\ln\left[\frac{l_a k_m - l_m k_a}{(k_m - k_a)l_m - (l_m - l_a)k_m c_m}\frac{l_m}{A_l(a^*)}\right] + \frac{k_m c_m}{(k_m - k_a)l_m}\ln\left[\frac{(k_m - k_a)l_m - (l_m - l_a)k_m c_m}{(l_a k_m - l_m k_a)c_m}\right]\right\}. \tag{55}$$

Applying l'Hôpital's rule to the above equation, the LHS of (HL) when $\frac{k_a}{k_m} = 1$ equals

$$\frac{N_h}{N_l}\left\{\frac{1}{l_m - l_a}\lim_{\frac{k_a}{k_m}\to 1}\ln\left[\frac{l_a - l_m\frac{k_a}{k_m}}{(1 - \frac{k_a}{k_m})l_m - (l_m - l_a)c_m}\frac{l_m}{A_l(a^*)}\right] + \frac{c_m}{\lim_{\frac{k_a}{k_m}\to 1}(1 - \frac{k_a}{k_m})l_m}\lim_{\frac{k_a}{k_m}\to 1}\ln\left[\frac{(1 - \frac{k_a}{k_m})l_m - (l_m - l_a)c_m}{(l_a - l_m\frac{k_a}{k_m})c_m}\right]\right\}$$
$$= \frac{N_h}{N_l}\left\{\frac{1}{l_m - l_a}\ln\left[\frac{l_m}{c_m A_l(a^*)}\right] + c_m \lim_{\frac{k_a}{k_m}\to 1}\left(\frac{1}{(1 - \frac{k_a}{k_m})l_m - (l_m - l_a)c_m} - \frac{1}{l_a - l_m\frac{k_a}{k_m}}\right)\right\}$$
$$= \frac{N_h}{N_l}\frac{1}{l_a - l_m}\left\{\ln\left[\frac{c_m A_l(a^*)}{l_m}\right] + 1 - c_m\right\}. \tag{56}$$

[$a^* \in (0,1)$ **for any** c_m]**:** $a^* < 1$ is obvious from the equation. Since $c_m \geq \frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}$, $a^*=0$ is possible only at $c_m = 1$. However, at $c_m = 1$, the equation becomes $\frac{N_h}{N_l}\frac{1}{l_m-l_a}\ln\left(\frac{l_m}{A_l(a^*)}\right) = \frac{1}{h-l_m}\ln\left(\frac{h}{A_h(a^*)}\right)$ and thus $a^*>0$.

[**Relations of** a^* **satisfying the equation with** $\frac{N_h}{N_l}$, c_m, **and** $\frac{k_a}{k_m}$]**:** Since the derivative of the LHS$-$RHS of (20) and (21) with respect to a^* equals $\frac{N_h}{N_l}\frac{1}{A_l(a^*)}+\frac{1}{A_h(a^*)} > 0$, a^* satisfying the equation decreases with $\frac{N_h}{N_l}$.

When $\frac{k_a}{k_m}\neq 1$, a^* satisfying (20) decreases with c_m, because the derivative of the expression inside the large curly bracket of (20) with respect to c_m equals

$$\left(1-\frac{(l_m-l_a)k_m c_m}{(k_m-k_a)l_m}\right)\frac{k_m}{(k_m-k_a)l_m-(l_m-l_a)k_m c_m}-\frac{k_m}{(k_m-k_a)l_m}+\frac{k_m}{(k_m-k_a)l_m}\ln\left[\frac{(k_m-k_a)l_m-(l_m-l_a)k_m c_m}{(l_a k_m-l_m k_a)c_m}\right]$$
$$=\frac{1}{(1-\frac{k_a}{k_m})l_m}\ln\left[1+\frac{1-c_m}{c_m}\frac{(1-\frac{k_a}{k_m})l_m}{l_a-l_m\frac{k_a}{k_m}}\right] > 0. \tag{57}$$

$\lim_{c_m\to 1}\frac{\partial a^*}{\partial c_m}=0$ is clear from the above equation.

Since (20) can be expressed as

$$\frac{N_h}{N_l}\left\{\frac{1}{l_m-l_a}\ln\left[\frac{l_a-l_m\frac{k_a}{k_m}}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}\frac{l_m}{A_l(a^*)}\right]+\frac{c_m}{(1-\frac{k_a}{k_m})l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]\right\}$$
$$=\frac{1}{h-l_m}\ln\left(\frac{h}{A_h(a^*)}\right), \tag{58}$$

when $\frac{k_a}{k_m}\neq 1$, the derivative of the expression inside the large curly bracket of (20) with respect to $\frac{k_a}{k_m}$ equals

$$\frac{\frac{l_m}{l_m-l_a}-\frac{c_m}{1-\frac{k_a}{k_m}}}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}-\frac{\frac{l_m}{l_m-l_a}-\frac{c_m}{1-\frac{k_a}{k_m}}}{l_a-l_m\frac{k_a}{k_m}}+\frac{c_m}{(1-\frac{k_a}{k_m})^2 l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]$$
$$=-\left(\frac{l_m}{l_m-l_a}-\frac{c_m}{1-\frac{k_a}{k_m}}\right)\frac{(l_m-l_a)(1-c_m)}{[(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m](l_a-l_m\frac{k_a}{k_m})}+\frac{c_m}{(1-\frac{k_a}{k_m})^2 l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]$$
$$=-\frac{c_m}{(1-\frac{k_a}{k_m})^2 l_m}\left(\frac{1-c_m}{c_m}\frac{(1-\frac{k_a}{k_m})l_m}{l_a-l_m\frac{k_a}{k_m}}-\ln\left[1+\frac{1-c_m}{c_m}\frac{(1-\frac{k_a}{k_m})l_m}{l_a-l_m\frac{k_a}{k_m}}\right]\right) < 0. \tag{59}$$

The derivative is negative because the expression inside the large parenthesis of (59) equals 0 at $c_m = 1$ and, when $\frac{k_a}{k_m} < (>)1$, it increases (decreases) with $\frac{1-c_m}{c_m}\frac{(1-\frac{k_a}{k_m})l_m}{l_a-l_m\frac{k_a}{k_m}}$ and thus decreases with c_m. Hence, a^* satisfying (20) increases with $\frac{k_a}{k_m}$ when $\frac{k_a}{k_m}\neq 1$. $\lim_{c_m\to 1}\frac{\partial a^*}{\partial \frac{k_a}{k_m}}=0$ is clear from the above equation.

The corresponding derivatives when $\frac{k_a}{k_m}\to 1$ are

$$c_m:\lim_{\frac{k_a}{k_m}\to 1}\left\{\frac{1}{(1-\frac{k_a}{k_m})l_m}\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]\right\}=\frac{-1}{l_m}\lim_{\frac{k_a}{k_m}\to 1}\left[\frac{-l_m}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}+\frac{l_m}{l_a-l_m\frac{k_a}{k_m}}\right]$$
$$=\frac{1}{l_a-l_m}\frac{1-c_m}{c_m} > 0. \tag{60}$$

$$\frac{k_a}{k_m}:\lim_{\frac{k_a}{k_m}\to 1}\left\{-\frac{c_m}{(1-\frac{k_a}{k_m})^2 l_m}\left(\frac{1-c_m}{c_m}\frac{(1-\frac{k_a}{k_m})l_m}{l_a-l_m\frac{k_a}{k_m}}-\ln\left[\frac{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}{(l_a-l_m\frac{k_a}{k_m})c_m}\right]\right)\right\}$$
$$=\lim_{\frac{k_a}{k_m}\to 1}\left\{\frac{c_m}{2(1-\frac{k_a}{k_m})l_m}\left(\frac{1-c_m}{c_m}\frac{-(l_a-l_m\frac{k_a}{k_m})l_m+(1-\frac{k_a}{k_m})l_m^2}{(l_a-l_m\frac{k_a}{k_m})^2}-\left[\frac{-l_m}{(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m}+\frac{l_m}{l_a-l_m\frac{k_a}{k_m}}\right]\right)\right\}$$
$$=\frac{c_m}{2l_m}\left(\frac{1-c_m}{c_m}\lim_{\frac{k_a}{k_m}\to 1}\frac{2l_m^2(l_a-l_m)}{(l_a-l_m\frac{k_a}{k_m})^3}+\lim_{\frac{k_a}{k_m}\to 1}\left[\frac{-l_m^2}{[(1-\frac{k_a}{k_m})l_m-(l_m-l_a)c_m]^2}+\frac{l_m^2}{(l_a-l_m\frac{k_a}{k_m})^2}\right]\right)$$
$$=\frac{c_m}{2l_m}\frac{l_m^2}{(l_a-l_m)^2}\left[2\frac{1-c_m}{c_m}+\left(1-\frac{1}{c_m^2}\right)\right]=-\frac{1}{2}\frac{l_m}{(l_a-l_m)^2}\frac{(1-c_m)^2}{c_m}<0, \tag{61}$$

where $l_a-l_m>0$ from $\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)}<1\Leftrightarrow 1<\frac{l_a}{l_m}$. Thus, the same results hold when $\frac{k_a}{k_m}=1$ too. ■

Proof of Lemma 4. [Relations of c_m satisfying (P) **with** a^*, k_m, k_a, **and** r]: Derivatives of the LHS of (P) with respect to a^*, c_m, k_m, and k_a equal

$$a^*:\frac{\partial\frac{A_h(a^*)}{A_l(a^*)}}{\partial a^*}\frac{l_m}{k_m}\frac{r}{c_m}\int_{a^*}^1\int_0^{\min\{c_h(a),1\}}\frac{dcda}{A_h(a)}>0, \tag{62}$$
$$c_m:-\frac{l_m}{k_m}\frac{r}{c_m^2}\left\{\int_0^{a^*}\int_0^{\min\{c_l(a),1\}}\frac{dcda}{A_l(a)}+\frac{A_h(a^*)}{A_l(a^*)}\int_{a^*}^1\int_0^{\min\{c_h(a),1\}}\frac{dcda}{A_h(a)}\right\}<0, \tag{63}$$
$$k_m:\ -\frac{1}{k_m}\left\{1-r\left[\int_0^{a^*}\int_{\min\{c_l(a),1\}}^1\frac{dcda}{cA_k(a)}+\int_{a^*}^1\int_{\min\{c_h(a),1\}}^1\frac{dadc}{cA_k(a)}\right]\right\}$$
$$-r\left[\int_0^{a^*}\int_{\min\{c_l(a),1\}}^1\frac{(1-a)dcda}{c(A_k(a))^2}+\int_0^{a^*}\int_{\min\{c_l(a),1\}}^1\frac{(1-a)dcda}{c(A_k(a))^2}\right]<0, \tag{64}$$
$$k_a:\ -r\left[\int_0^{a^*}\int_{\min\{c_l(a),1\}}^1\frac{adcda}{c(A_k(a))^2}+\int_0^{a^*}\int_{\min\{c_l(a),1\}}^1\frac{adcda}{c(A_k(a))^2}\right]<0, \tag{65}$$

where $c_l(a^*)=c_h(a^*)=c^*$, $\frac{1}{c_l(a)A_k(a)}=\frac{l_m}{k_m}\frac{1}{c_m}\frac{1}{A_l(a)}$, and $\frac{1}{c_h(a)A_k(a)}=\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\frac{1}{c_m}\frac{1}{A_h(a)}$ are used to derive the equations. The results are straightforward from the equations.

[(**P**) **does not hold at** $c_m=0$]**:** Noting that $c_l(a)=\frac{k_m}{l_m}\frac{A_l(a)}{A_k(a)}c_m$ and $c_h(a)=\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{A_h(a)}{A_k(a)}c_m$, when $c_m\to 0$, the LHS of (P) becomes

$$r\int_0^{a^*}\frac{da}{A_k(a)}+r\int_{a^*}^1\frac{da}{A_k(a)}+r\int_0^1\int_0^1\frac{dadc}{cA_k(a)}=r\int_0^1\frac{da}{A_k(a)}-\frac{r}{k_m-k_a}\ln(\frac{k_m}{k_a})\lim_{c\to 0}\ln c=+\infty>1. \tag{66}$$

Hence, (P) does not hold at $c_m=0$. ■

Proof of Proposition 1. At $c_m=1$, $c_l(a)$, $c_h(a)>1$ from (13), thus (P) equals

$$\frac{l_m}{k_m}r\int_0^{a^*}\frac{da}{A_l(a)}+\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)}r\int_{a^*}^1\frac{da}{A_h(a)}=1. \tag{67}$$

When k_m is very small, the LHS of the above equation is strictly greater than 1 for any $a^*\in[0,1]$ (thus, (P) does not hold for any c_m and a^* from Lemma 4), or a^* satisfying the equation is weakly smaller than $a^*\in(0,1)$ satisfying (HL) at $c_m=1$ ($a^*\in(0,1)$ holds on

(HL) from Lemma 3). In such case, there is no $a^* \in (0,1)$ and $c_m < 1$ satisfying both (HL) and (P), and thus machines are not employed, i.e., $c_m = 1$, in equilibrium, where equilibrium a^* is determined from (HL) with $c_m = 1$.

When k_m becomes large enough that a^* satisfying (67) is greater than $a^* \in (0,1)$ satisfying (HL) at $c_m = 1$, an equilibrium with $c_m < 1$ exists from shapes of (HL) and (P). The dynamics of c_m and a^* are straightforward from shapes of the two loci. The dynamics of c^* and c_a are from $c^* = \min\left\{\frac{k_m}{l_m}\frac{A_l(a^*)}{A_k(a^*)}c_m, 1\right\}$, $c_a = \min\left\{\frac{h}{k_a}\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}c_m, 1\right\}$, and the assumptions that $\frac{k_a}{k_m}$ is time-invariant and satisfies $\frac{k_a}{k_m} < \frac{l_a}{l_m}$. The dynamics of $c_l(a)$ and $c_h(a)$ are from those of the other variables. ■

Proof of Proposition 2. (i) When $c_m \geq \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$, earnings of skilled workers increase over time from Propositions 4 (iii) and 5 (iii) in Web Appendix. Earnings of both types of workers increase when $c_m < \frac{l_m}{k_m}\frac{k_a}{h}\frac{A_h(a^*)}{A_l(a^*)}$ from Proposition 6 (iii) in Web Appendix. (ii) is straightforward from Proposition 1 and the earnings equations (eq. 15).

(iii) Y decreases with the LHS and RHS of (HL) from (8). When $c^* = c_a = 1$ and $\frac{k_a}{k_m} \neq 1$, the RHS of (HL) equals $\frac{1}{h - l_m}\ln\left(\frac{h}{A_h(a^*)}\right)$ from Lemma 3, which decreases with the growth of k_m and k_a with constant $\frac{k_a}{k_m}$ from Proposition 1. When $c^* < c_a < 1$ and $\frac{k_a}{k_m} \neq 1$, the RHS equals $\frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{c_m}{k_m - k_a}\ln\left(\frac{A_k(a^*)}{k_a}\right)$ from (26) in the proof of Lemma 1, which decreases with the productivity growth from Proposition 1. When $c^* < c_a = 1$ and $\frac{k_a}{k_m} \neq 1$, the derivative of the RHS with respect to c_m equals, from (41) in the proof of Lemma 2 and (18),

$$
\begin{aligned}
&-\frac{1}{(h - l_m)c_m}\ln\left[1 + \frac{(h - l_m)hk_m\left[c_m - \frac{l_m}{h}\frac{k_a}{k_m}\frac{A_h(a^*)}{A_l(a^*)}\right]}{l_m\frac{A_h(a^*)}{A_l(a^*)}(hk_m - l_m k_a)}\right] + \frac{N_h}{N_l}\frac{k_m}{l_m}\frac{1}{k_m - k_a}\ln\left(\frac{k_m}{A_k(a^*)}\right) \\
&= \frac{k_m}{l_m}\frac{A_l(a^*)}{A_h(a^*)}\frac{1}{k_m - k_a}\ln\left[\frac{(k_m - k_a)\frac{l_m}{k_m}\frac{A_h(a^*)}{A_l(a^*)} + (h - l_m)c_m}{\frac{(hk_m - l_m k_a)c_m}{A_k(a^*)}}\right] > 0,
\end{aligned}
\tag{68}
$$

and the derivative with respect to a^* equals, from (39) in the proof of Lemma 2 ,

$$
-\frac{k_m}{l_m}\frac{c_m}{k_m - k_a}\left\{\frac{A_l(a^*)}{A_h(a^*)}\frac{k_m - k_a}{A_k(a^*)} - \frac{\partial\frac{A_l(a^*)}{A_h(a^*)}}{\partial a^*}\ln\left[1 + \frac{(k_m - k_a)\frac{A_h(a^*)}{A_k(a^*)}\left[\frac{l_m}{k_m}\frac{A_k(a^*)}{A_l(a^*)} - c_m\right]}{\frac{(hk_m - l_m k_a)c_m}{A_k(a^*)}}\right]\right\} < 0. \tag{69}
$$

From signs of the derivatives and Proposition 1, the RHS of (HL) decreases with the productivity growth. Hence, Y increases over time when $\frac{k_a}{k_m} \neq 1$. The result when $\frac{k_a}{k_m} = 1$ can be proved similarly. ■

Proof of Proposition 3. Since an increase in $\frac{N_h}{N_l}$ shifts (HL) to the left on the (a^*, c_m) space from Lemmas 1–3, the result that c_m and a^* decrease is straightforward from Figures 8–10. Then, $w_l = \frac{l_m}{k_m}\frac{r}{c_m}$ rises and $\frac{w_h}{w_l} = \frac{A_h(a^*)}{A_l(a^*)}$ falls. Since $c^* \equiv \min\left\{\frac{k_m}{l_m}\frac{A_l(a^*)}{A_k(a^*)}c_m, 1\right\}$, c^* falls when $c^* < 1$ from $\frac{k_a}{k_m} < \frac{l_a}{l_m}$, $\frac{da^*}{d\frac{N_h}{N_l}} < 0$, and $\frac{dc_m}{d\frac{N_h}{N_l}} < 0$. $c_l(a)$ decreases from $\frac{dc_m}{d\frac{N_h}{N_l}} < 0$. Proofs of the results for $c_h(a)$, c_a, w_h, and Y are in the proof of Proposition 7 in Web Appendix. ■