Simple Equilibria in General Contests

Bastani, Spencer and Giebe, Thomas and Gürtler, Oliver

Institute for Evaluation of Labour Market and Education Policy (IFAU), Sweden, Linnaeus University, Sweden, University of Cologne, Germany

3 December 2019

Simple Equilibria in General Contests*

Spencer Bastani† Thomas Giebe‡ Oliver Gürtler§

First version: December 3, 2019
This version: May 17, 2021

Abstract

We show how symmetric equilibria emerge in general two-player contests in which skill and effort are combined to produce output according to a general production technology and players have skills drawn from different distributions. We also show how contests with heterogeneous production technologies, cost functions and prizes can be analyzed in a surprisingly simple manner using a transformed contest that has a symmetric equilibrium. Our paper provides intuition regarding how the contest components interact to determine the incentive to exert effort, sheds new light on classic comparative statics results, and discusses the implications for the optimal composition of teams.

Keywords: contest theory, symmetric equilibrium, heterogeneity, risk, stochastic dominance

JEL classification: C72, D74, D81, J23, M51

*An earlier working paper version of this paper was circulated under the title "A General Framework for Studying Contests". We thank Peter Cramton, Qiang Fu, Stephan Lauermann, Mark Le Quement, Johannes Münster, Christoph Schottmüller, Dirk Sliwka, Lennart Struth, Zhenda Yin, seminar participants at the University of Cologne, the University of East Anglia, the Berlin-Munich Behavioral Seminar, conference participants at the EALE SOLE AASLE World Conference 2020, the CMID20 Conference on Mechanism and Institution Design in Klagenfurt, and the 2020 Annual Meeting of the Verein für Socialpolitik for helpful comments. All authors gratefully acknowledge financial support from the Jan Wallander and Tom Hedelius Foundation (grant no. P18-0208). Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2126/1 390838866.

†Institute for Evaluation of Labour Market and Education Policy (IFAU) and Research Institute of Industrial Economics (IFN); Uppsala Center for Fiscal Studies; Uppsala Center for Labor Studies; CESIfo. E-mail: spencer.bastani@ifau.uu.se.

‡Department of Economics and Statistics, School of Business and Economics, Linnaeus University, Sweden. E-mail: thomas.giebe@lnu.se.

§Department of Economics, University of Cologne, Germany. E-mail: oliver.guertler@uni-koeln.de

1 Introduction

In a contest, two or more players invest effort or other costly resources to win a prize. Many economic interactions can be modeled as a contest. Promotions, for example, represent an important incentive in many firms and organizations. Employees exert effort to perform better than their colleagues and, thus, to be considered for promotion to a more highly paid position. Litigation can also be understood as a contest, in which the different parties spend time and resources to prevail in court. Procurement is a third example, where different firms invest resources into developing a proposal or lobbying politicians, thereby increasing the odds of being selected, receiving some rent in return.

Players participating in contests are typically heterogeneous in some respect. For instance, employees differ with respect to their skills, the litigant parties differ with respect to the quality of the available evidence, and firms differ with respect to their capabilities of designing a proposal. When accounting for such heterogeneity in contest models, equilibria are often asymmetric, meaning that players choose different levels of effort. Due to this asymmetry, to keep the analyses tractable, researchers have often imposed rather strict assumptions regarding the production technology and the distributions of stochastic components of the contest.

In this paper, we consider a general contest model that allows players to be heterogeneous in terms of their skill distributions, production technologies, prizes, and cost functions. We show that, despite players being heterogeneous, there is often a simple relation between equilibrium efforts. In particular, we identify conditions, under which there is a symmetric equilibrium in which players choose the same effort. We further show that some contests that do not fulfill these conditions can be transformed in such a way that the transformed contest has a symmetric equilibrium, enabling us to establish simple relationship between the asymmetric efforts of the original contest and the symmetric efforts of the transformed contest.

The details of our model are as follows. Two players compete for a prize, deciding on their effort. The output of each player, and thereby the player's *production* or *contribution* to the contest, is determined according to a general function of individual effort and the realization of a random variable. The player with the highest output wins the prize. We refer to the random variable as the skill of the player (typically, and equivalently, referred to as noise in the contest theory literature) and the statistical distributions of possible skill realizations are allowed to be different for the competing players. The model is general in terms of the production function and the skill distributions and includes the well-known models by Tullock (1980) and Lazear and Rosen (1981) as special cases.

The skill distributions of the competing players (including the expected values) are assumed to be common knowledge, whereas the exact skill realizations are generally (symmetrically) unknown (as, e.g., in Holmström 1982). These assumptions realistically reflect that in a promotion contest, for example, the expected skill of a player may be commonly known (e.g., the education, prior work experience, or CV of a player), whereas the exact skill level for the particular job is unknown (e.g., there might be uncertainty regarding how education translates into workplace performance and job match).

The main contribution of the paper is to show the existence of a symmetric (pure-strategy equal-effort) equilibrium in a general two-player contest setting where players have heterogeneous skill distributions, and how simple equlibria can emerge also when considering heterogeneity in additional dimensions, such as production technologies, prizes, and cost functions. We also make two additional contributions.

First, we provide intuition regarding how the different components of the contest interact to determine the incentive to exert effort. More specifically, we highlight the interaction between three factors. The first factor relates to the production technology and is the ratio of the marginal product of effort and the marginal product of skill. The intuition behind this factor is that the purpose of a marginal effort increase for an individual player is to beat marginally more able rivals. The ratio describes how effective a marginal effort increase is to overcome the output advantage of marginally more skilled players. The second and third factors are represented by the product of the densities of the skill distributions of the two competing players, evaluated at the same point. The reason for the presence of this product is that a player only has a marginal incentive to exert effort in cases where the skill realizations of the two players are exactly the same, and the product describes the "likelihood" of this event to happen.

Second, we use the simple structure of the equilibrium to construct a link between our contest model and standard models of decision-making under risk (expected utility theory), allowing us to revisit important comparative statics results of contest theory. In particular, we analyze how equilibrium effort is affected by making the skill distributions of the competing players more heterogeneous, investigating both the role of differences in expected skill (conceptualized by first-order stochastic dominance) and the role of differences in the uncertainty of the skill distributions of the competing players (conceptualized by second-order stochastic dominance), and how these relationships are affected by the production technology. The general message is that making contest participants more heterogeneous can increase equilibrium effort. These findings contradict certain "standard" results known from the Tullock contest and the Lazear-Rosen tournament. Thus, the comparative statics results derived from those standard models are not representative of the conclusions derived in the more general model.

To shed further light on our results, we also provide two extensions to our main analysis. In a first extension, we study the behavior in contests when the number of players n is greater than two and show that the existence of a symmetric equilibrium, and the interpretation for the two-player case, extend to the n-player case when players have identical skill distributions. We also show that, for a specific class of skill distributions, a symmetric equilibrium exists when $n-1$ identical players compete against a player who has a higher expected skill. Moreover, we show that increasing the number of contestants can increase equilibrium effort, exploiting the fact that a contest with $n > 2$ players can be interpreted as a two-player contest in which every player competes against the strongest (i.e., the highest order statistic) of his or her opponents.[1]

In a second extension, we investigate the robustness of our results with respect to the assumption of symmetric uncertainty by analyzing the consequences of letting players be privately informed about their skills. In this case, equilibria are in general not symmetric, but focusing on symmetric players, we are able to draw interesting parallels with respect to our baseline case, highlighting the role of our general production technology in influencing the marginal incentive to exert effort.

We also discuss the implications of our findings for optimal team composition and certain real-world applications in the context of labor and personnel economics. For instance, our finding that efforts can increase if the skill distribution of one of the competing players becomes more uncertain (in the sense of second-order stochastic dominance) has several interesting managerial implications. It indicates that contest organizers might wish to increase the uncertainty regarding the skills of certain players in order to induce higher effort. In a worker-firm context, employers could achieve this by, for instance, hiring a worker for whom little prior information is available, or a minority worker with a skill level drawn from a distribution that generally tends to be more uncertain (as argued, e.g., by Bjerk 2008). This means that having diverse teams might be desirable from an employer's point of view.

[1]This result can be understood by the fact that as the number of contestants increases, the strongest opponents grow stronger in the sense of first-order stochastic dominance, allowing us to apply our results from the two-player case.

The paper is organized as follows. In Section 2 below, we discuss related literature. Section 3 introduces the contest model and discusses how our model nests the Tullock contest and the Lazear-Rosen tournament as special cases. Section 4 solves the two-player model when players have different skill distributions, whereas Section 5 addresses different production functions, prizes, and cost functions. In Section 6, we analyze the two-player case in greater detail and provide a set of comparative statics results. We also discuss implications for organizational design and optimal team composition. Section 7 studies the n-player case and takes a look at the case of privately known skills. Finally, Section 8 concludes.

2 Related Literature

There are three main approaches to the study of contests, the Tullock (or ratio-form) contest, the Lazear-Rosen tournament, and the complete-information all-pay auction.[2] In the Tullock contest, introduced by Tullock (1980), a player's winning probability is given by his/her contribution to the contest divided by the sum of the contributions of the competing players, and the contribution of each player is typically defined as a function of effort and sometimes also of skill.[3] The Lazear-Rosen tournament assumes that the player with the highest contribution wins with certainty, and contributions depend on effort, some random factors (e.g., luck), and possibly on skills. The seminal paper is by Lazear and Rosen (1981) who apply the model in a labor-market context.[4] The all-pay auction, finally, makes the same assumption as the Lazear-Rosen tournament, except that contest contributions are deterministic and do not depend on random factors.[5]

[2] The theoretical contest literature has been surveyed in a number of books and papers. See, e.g., Konrad (2009) and Vojnović (2016) for recent textbooks and Chowdhury and Gürtler (2015), Chowdhury, Esteve-Gonzalez, and Mukherjee (2019), and Fu and Wu (2019) for recent surveys.

[3] The Tullock contest has been analyzed by, e.g., Hillman and Riley (1989), Cornes and Hartley (2005), Fu and Lu (2009a,b), Corchón and Dahm (2010), Schweinzer and Segev (2012), and Chowdhury and Kim (2017). It has been axiomatized in various settings by Skaperdas (1996), Clark and Riis (1998b), and Münster (2009).

[4] It has been further analyzed by, for example, Green and Stokey (1983), Malcomson (1984, 1986), O'Keeffe, Viscusi, and Zeckhauser (1984), Lazear (1989), Schotter and Weigelt (1992), Zábojník and Bernhardt (2001), Hvide (2002), Grund and Sliwka (2005), Schöttner and Thiele (2010), Gürtler and Gürtler (2015), and Imhof and Kräkel (2016).

[5] A detailed equilibrium characterization of the all-pay auction was developed by Baye, Kovenock, and de Vries (1996). The complete-information all-pay auction (with mixed-strategy equilibria) is the most commonly used in contest theory, but a private-values version can be found as well. The all-pay auction has been further studied by, e.g., Clark and Riis (1998a), Barut and Kovenock (1998), Moldovanu and Sela (2001, 2006), Moldovanu, Sela, and Shi (2007), Cohen, Kaplan, and Sela (2008), Siegel (2009, 2010), Sela (2012), Morath and Münster (2013), Barbieri, Malueg, and Topolyan (2014), Olszewski and Siegel (2016), and Fang, Noe, and Strack (2020).

Most studies analyzing the Tullock contest and the Lazear-Rosen tournament impose assumptions that ensure that equilibria in pure strategies exist. In contrast, only mixed-strategy equilibria exist in the all-pay auction (when players are symmetrically informed about the decision situation). As we indicated in the introduction, and as we explain in more detail in Section 3, the Tullock contest and the Lazear-Rosen tournament are special cases of our model, while the all-pay auction is not. Our main contribution is to show that the simple structure of equilibria in the Tullock contest and the Lazear-Rosen tournament extends to more general production functions and skill distributions, even if players are heterogeneous along several dimensions.[6]

An additional contribution of our paper is to show that canonical results regarding how player heterogeneity affects equilibrium effort in the Tullock contest and the Lazear-Rosen tournament do not always extend to more general production functions and skill distributions.[7] For example, Schotter and Weigelt (1992) have shown that effort is higher when players have homogeneous skills relative to when they are heterogeneous, since disadvantaged players tend to give up and reduce their effort, whereas advantaged players can afford to reduce their effort. Moreover, several studies (e.g., Hvide 2002) have shown that greater uncertainty regarding the contest outcome tends to reduce effort as, intuitively, effort has a lower impact on who becomes the winner in a contest where the outcome is heavily influenced by random factors. In our setting, the above results can be overturned, as we find that greater heterogeneity in terms of the skill distributions of the competing players and more uncertainty regarding the contest outcome can in many cases result in higher equilibrium effort.[8]

3 Model

In this section, we describe the contest model. For simplicity, we start by focusing on one type of player heterogeneity (i.e., heterogeneity in the skill distributions) and thus impose symmetric production functions, prizes and cost functions. We further generalize the model by allowing those to be asymmetric in Section 5.

[6]Kirkegaard (2020) has recently proposed a contest model similar to ours. However, as his focus is on optimal contest design, we view his work as complementary to ours.

[7]Some exceptions to the standard results in the context of the Tullock contest and the Lazear-Rosen tournament have already been documented in the literature. See, e.g., the work by Drugov and Ryvkin (2017) and Fu and Wu (2020) on biases in contests, Lu, Wang, and Zhou (2021) on identity-dependent prizes, and Ryvkin and Drugov (2020) on contests with more than two players.

[8]In an n-player extension of our model, we also show that effort can increase as the number of players increases, which runs in contrast to the well-known discouragement effect in Tullock (1980).

Consider a contest between two risk-neutral players $i \in \{1,2\}$ who compete for a single prize of value $V > 0$. Both players simultaneously choose effort $e_i \geq 0$, and the cost of effort $c(e_i)$ is described by a continuously differentiable, strictly increasing and strictly convex function satisfying $c(0) = 0$. The skill (type) of player i is denoted by Θ_i. There is uncertainty about skills, which means that Θ_i is a random variable. The realization of Θ_i is denoted by θ_i and it is not known to any of the players (not even player i). It is commonly known, however, that Θ_i is independently and absolutely continuously distributed according to the pdf f_i (with cdf F_i) with finite mean μ_i. For a given density f, we will use $\text{supp}(f) = \{x \in \mathbb{R} : f(x) > 0\}$ to denote its support. We assume that the supports of f_1 and f_2 overlap on a subset of $\mathbb{R}$ with positive measure.

Symmetric uncertainty regarding skills is typically imposed in the career-concerns literature (e.g., Holmström 1982, Holmström and Ricard I Costa 1986, Dewatripont, Jewitt, and Tirole 1999, Auriol, Friebel, and Pechlivanos 2002, and Bar-Isaac and Lévy 2021) and also in the literature on promotion signaling (e.g., Waldman 1984, Bernhardt 1995, Owan 2004, Ghosh and Waldman 2010, DeVaro and Waldman 2012, and Gürtler and Gürtler 2019). This literature refers to firm-worker relationships, and the idea is that both firms and workers are uncertain about how well workers perform when they begin their working careers and that this uncertainty is reduced over time once performance information becomes available. We adopt this idea, referring to Θ_i as a player's skill, but it is also common to interpret it as noise, luck, or measurement error.

The production of player i, and hence his or her *contribution* to the contest, is given by the continuously differentiable production function $g(\theta_i, e_i)$.[9] Importantly, we assume that $\frac{\partial g}{\partial \theta_i} > 0$ for all $e_i > 0$ which means (realistically) that each player's contribution to the contest is increasing with respect to his or her skill, for a given level of effort. Player i wins the contest against the opponent player $k \in \{1,2\}$, $k \neq i$, if and only if the contribution of player i is strictly higher than the contribution of player k, namely, $g(\theta_i, e_i) > g(\theta_k, e_k)$.[10] We denote by $P_i(e_i, e_k)$ player i's probability of winning the contest (as a function of the efforts of both players) and we define the expected payoff as $\pi_i(e_i, e_k) := P_i(e_i, e_k)V - c(e_i)$. We also define $\hat{e} := c^{-1}(V)$ and $E := [0, \hat{e}]$. A player's equilibrium effort will always belong to the set E as the probability of winning is bounded above by unity.

[9]In the literature on contests, a player's contribution to the contest is also denoted as a player's *score*.

[10]Note that, as usual in such contest models, adding a common random shock to the players' outputs does not affect the event of winning and, therefore, does not have an effect on the equilibrium. Also notice that $g(\theta_i, e_i) = g(\theta_k, e_k)$ happens with probability zero. In the following, whenever we refer to two players i and k, we (implicitly) assume that $i, k \in \{1,2\}$, $i \neq k$.

We impose the following assumption:

Assumption 1. *The primitives of the model are such that: (i) $\pi_i(e_i, e_k)$ is continuously differentiable, and, (ii) any interior solution of the system of first-order conditions for the players' problems of maximizing $\pi_i(e_i, e_k)$ characterizes a pure-strategy Nash equilibrium.*

The validity of the first-order approach is typically ensured by imposing assumptions on the primitives of the model that guarantee that the objective functions π_i are quasi-concave and increasing at $e_i = 0$. Previous papers in the contest-theory literature, however, have shown that the first-order approach may be valid even when the objective functions are neither quasi-concave nor increasing at $e_i = 0$ (see, e.g., Figure 1 in Schweinzer and Segev 2012). As we do not want to rule out such cases, we assume that the Nash-equilibrium efforts are characterized by the players' first-order conditions to their maximization problems without restricting the shape of π_i too much.

Each of the theoretical results we present will be accompanied by at least one example for which we verify that the first-order conditions indeed characterize an equilibrium, by verifying the appropriate second-order conditions. As we permit a wide range of production functions and skill distributions, resulting in payoff functions that are not generally well-behaved, it is not feasible to pin down the exact set of parameters for which Assumption 1 is satisfied. However, in some instances, it is easy to verify that the first-order approach is valid, e.g., in the case of additive production functions and sufficiently convex cost functions. Moreover, given that the Tullock contest and the Lazear-Rosen tournament are special cases of our model, all that is known about equilibrium existence for these two models continues to hold in our setting.

Finally, we assume that there exist $\bar{e}_i, \breve{e}_i \in int\ E$ such that $\frac{\partial \pi_i(e_i,e_k)}{\partial e_i}|_{e_i=e_k=\bar{e}_i} < 0$ and $\frac{\partial \pi_i(e_i,e_k)}{\partial e_i}|_{e_i=e_k=\breve{e}_i} > 0$. This ensures that the first-order condition to player i's maximization problem can be fulfilled in a symmetric equilibrium.

Below we provide some examples of different contest models, skill distributions and production technologies that can be captured in our model.

Tullock contest The well-studied rent-seeking contest of Tullock (1980) represents a special case of our model. This is easily illustrated using the results in Jia (2008), who considers a contest with a multiplicative production technology, in which player i wins if and only if $\theta_i e_i$ is highest among all players.[11] It is shown that if Θ_i is distributed

[11] See also Clark and Riis (1996) and Fu and Lu (2012).

according to the pdf

$$f_i(x) = \gamma_i m x^{-(m+1)} \exp\left(-\gamma_i x^{-m}\right) I_{(x>0)},$$

then player i wins the contest with probability

$$P_i(e_i, e_k) = \frac{\gamma_i e_i^m}{\sum_{j=1}^2 \gamma_j e_j^m},$$

where $\gamma_i \geq 0$ for both players i and $m > 0$.[12] Hence, in our model, if $g(\theta_i, e_i) = \theta_i e_i$, and Θ_i is distributed according to the above pdf, then we obtain the above Tullock contest-success function. The literature contains a range of modifications and generalizations of this form of contest-success function, some of which cannot be micro-founded in a similar way. See the recent discussion in Kirkegaard (2020).[13]

Lazear-Rosen tournament Assuming the production technology $g(\theta_i, e_i) = \theta_i + e_i$, our model includes the standard Lazear-Rosen tournament model (in the original Lazear and Rosen 1981, it is assumed that $\mu_i = 0$).

General production technologies Our model is general with respect to the set of admissible production technologies $g(\theta_i, e_i)$. For example, feasible technologies include the CES production function $g(\theta_i, e_i) = \left(\alpha\theta_i^\rho + \beta e_i^\rho\right)^{\frac{1}{\rho}}$, with $\alpha, \beta > 0$ (except for the limiting case of perfect complements). Thus, the case of perfect substitutes, $\rho = 1$, is included as well as technologies where effort and skill are complements to different degrees, such as the standard Cobb-Douglas technology $g(\theta_i, e_i) = \theta_i^\alpha e_i^\beta$, with $\alpha, \beta > 0$ (obtained when ρ approaches zero). Another example of a feasible production technology featuring complementarities between skill and effort is given by $g(\theta_i, e_i) = \alpha\theta_i + \beta e_i + \gamma\theta_i e_i$, with $\alpha, \beta, \gamma > 0$.

Skill distributions In our model, standard continuous skill distributions can be employed with both bounded and unbounded supports. Moreover, the distributions can be different for the two players. Examples are the (truncated) Normal distribution, the Exponential distribution, Student's t-distribution, the Gamma distribution, and the Uniform distribution.

[12]This contest-success function is slightly more general than the one presented in Tullock (1980). From the contest-theory literature, it is known that m must be sufficiently small for a pure-strategy equilibrium to exist. This is covered by our Assumption 1.

[13]See Fullerton and McAfee (1999) for another example of a micro-foundation for the Tullock contest.

4 Equilibrium Characterization

We focus on pure-strategy Nash equilibria in which both players choose the same level of effort. The following lemma provides a sufficient condition for such a symmetric equilibrium to exist.

Lemma 1. *A sufficient condition for a symmetric equilibrium to exist is that* $\frac{\partial P_i(e_i,e_k)}{\partial e_i}|_{e_i=e_k=e}$ *is the same for* $i,k \in \{1,2\}, i \neq k$, *and all* $e \in int\ E$.

Proof. See Appendix A.1. □

We will make use of Lemma 1 to prove the existence of a symmetric equilibrium by checking the sufficient condition. Since this condition depends on the winning probability, we need to specify this probability first. For each $e > 0$, we define the function $g_e : \mathbb{R} \to \mathbb{R}$ by $g_e(x) = g(x,e)$. The function $g_e(x)$ is strictly increasing in x and thus invertible, and we denote the (strictly increasing) inverse by g_e^{-1}. This notation can be motivated by the fact that the event of player i winning over player k can be written as

$$\begin{aligned} & g(\theta_k, e_k) < g(\theta_i, e_i) \\ \Leftrightarrow\quad & g_{e_k}(\theta_k) < g_{e_i}(\theta_i) \\ \Leftrightarrow\quad & \theta_k < g_{e_k}^{-1}\left(g_{e_i}(\theta_i)\right). \end{aligned}$$

Considering all potential realizations of Θ_i and Θ_k, the winning probability of player i is

$$P_i(e_i,e_k) = \int_{\mathbb{R}} F_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right) f_i(x)\,dx.$$

By symmetry, the winning probability of player k is

$$P_k(e_i,e_k) = \int_{\mathbb{R}} F_i\left(g_{e_i}^{-1}\left(g_{e_k}(x)\right)\right) f_k(x)\,dx.$$

The derivative of player i's winning probability with respect to e_i is given by:[14]

$$\frac{\partial P_i(e_i,e_k)}{\partial e_i} = \int_{\mathbb{R}} f_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right) \frac{d}{de_i}\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right) f_i(x)\,dx. \tag{1}$$

The derivative of player k's winning probability with respect to e_k is given by:

$$\frac{\partial P_k(e_i,e_k)}{\partial e_k} = \int_{\mathbb{R}} f_i\left(g_{e_i}^{-1}\left(g_{e_k}(x)\right)\right) \frac{d}{de_k}\left(g_{e_i}^{-1}\left(g_{e_k}(x)\right)\right) f_k(x)\,dx. \tag{2}$$

[14]Notice that F_k is differentiable almost everywhere, since it is the cdf of the absolutely continuous random variable Θ_k with f_k as the corresponding pdf.

It can immediately be seen that expressions (1) and (2) are equal when $e_i = e_k = e \in int\ E$ since, in this case, $g_{e_k}^{-1}(g_{e_i}(x)) = g_{e_i}^{-1}(g_{e_k}(x)) = x$ and $\frac{d}{de_k} g_{e_i}^{-1}\left(g_{e_k}(x)\right) = \frac{d}{de_i} g_{e_k}^{-1}\left(g_{e_i}(x)\right)$. Thus, the sufficient condition for the existence of a symmetric equilibrium in Lemma 1 is satisfied. Hence, we have the following theorem.

Theorem 1. *There exists a symmetric equilibrium in which both players choose the same level of effort.*

Proof. See Appendix A.2. □

The theorem states that, even if the players are asymmetric (i.e., $f_1 \neq f_2$), there always exists a symmetric equilibrium of the contest game. This key result allows a tractable analysis of contests between asymmetric players in a variety of different settings. We define $a_e : \mathbb{R} \to \mathbb{R}$ by

$$a_e(x) = \frac{d}{de_i} g_{e_k}^{-1}\left(g_{e_i}(x)\right)\bigg|_{e_i = e_k = e} = \frac{\partial g(x,e)}{\partial e} \Bigg/ \frac{\partial g(x,e)}{\partial x} = MRTS(x,e), \tag{3}$$

where the equality follows from an application of the inverse function theorem and $MRTS(x,e)$ denotes the marginal rate of technical substitution between skill and effort in a symmetric equilibrium.[15] Recognizing that the two players have the same cost function $c(e)$, we can write the (identical) first-order condition for effort for the two players in a symmetric equilibrium as

$$V \int_{\mathbb{R}} a_{e^*}(x) f_k(x) f_i(x)\, dx = c'(e^*). \tag{4}$$

The key observation necessary to understand the intuition behind (4) is that a player has a positive marginal incentive to supply effort if and only if $g(\theta_k, e_k) = g(\theta_i, e_i)$. In a symmetric equilibrium where $e_k = e_i$ this implies that $\theta_k = \theta_i$. The reason a player only has a marginal incentive to exert effort when $\theta_k = \theta_i$ is that this is the only situation in which a marginal increase in output would be pivotal to winning the contest. Accordingly, equation (4) contains the "collision density" $f_k(x) f_i(x)$ that describes how likely it is that the skill realizations of the two competing players are the same. The fact that this term is the same for both players is due to our assumption of symmetric uncertainty. Furthermore, the fact that $a_e(x)$ is the same for both players follows directly from the assumption that the production function $g(\theta, e)$ is the same for both players,

[15] To see this, notice that $\frac{d}{de_i} g_{e_k}^{-1}\left(g_{e_i}(x)\right)\Big|_{e_i = e_k = e} = \frac{1}{g'_{e_k}(g_{e_k}^{-1}(g_{e_i}(x)))} \frac{d}{de_i} g_{e_i}(x)\Big|_{e_i = e_k = e} = \frac{\frac{d}{de} g_e(x)}{g'_e(x)}$.

and depends only on the level of effort e and the skill θ, both of which are the same for both players in situations where players have a marginal incentive to supply effort in symmetric equilibrium.

The function $a_e(x)$ describes how a marginal increase in effort by a player increases output relative to his or her rival and is equal to the marginal rate of technical substitution between skill and effort. The purpose of raising effort is to beat players with higher skill. The MRTS determines the range of additional types that the player can win against through a small effort increase. The lower is the sensitivity of output to skill in the production function, the smaller is the advantage of marginally more skilled rivals, and the higher is the marginal incentive to exert effort. A direct implication is that the marginal incentive to exert effort is higher in environments in which players' outputs depend to a large degree on effort than in those in which the output is mainly determined by players' skills. The reason is that the MRTS tends to be larger in the former than in the latter environments, implying a greater impact of effort on the winning probability, as just explained.

Additional intuition can be provided by considering specific functional forms. For example, if $g(\theta,e) = e+\theta$ we have that $a_e(x) = 1$ since in this case both the numerator and denominator are equal to unity. If, instead, $g(\theta,e) = \theta e$, we have that $a_e(x) = x/e$ because of the complementarity between skill and own effort in the production function. The fact that $a_e(x)$ is an increasing function of x reflects that it is in this case more valuable to increase effort the higher is the skill of the player. The fact that $a_e(x)$ is decreasing in e reflects that the marginally more able individual is harder to beat the higher is the baseline (symmetric) level of effort because of the complementarity between skill and effort.

We end this section with an illustrative example. Consider the multiplicative production technology $g(\theta_i, e_i) = \theta_i e_i$ and the cost function $c(e_i) = e_i^2/2$. Assume further that the skill distribution of player 1 follows a Uniform distribution on $[1,2]$, and the skill distribution of player 2 is given by the Student's t-distribution on support $(-\infty,\infty)$, with one degree of freedom, such that:

$$f_1(x) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}, \qquad f_2(x) = \frac{1}{\pi(1+x^2)},\ x \in \mathbb{R}.$$

The event of player 1 winning is described by $g(\theta_1, e_1) > g(\theta_2, e_2) \iff \theta_2 < g_{e_2}^{-1}(g_{e_1}(\theta_1)) =$

$\theta_1 e_1/e_2$. The probability of that event, and its first derivative with respect to e_1, are

$$P_1(e_1,e_2) = \int_{-\infty}^{\infty} F_2\left(x\frac{e_1}{e_2}\right) f_1(x)dx,$$
$$\frac{\partial P_1(e_1,e_2)}{\partial e_1} = \int_{-\infty}^{\infty} f_2\left(x\frac{e_1}{e_2}\right)\left(\frac{x}{e_2}\right) f_1(x)dx.$$

The first-order condition of player 1's maximization problem is

$$\frac{\partial P_1(e_1,e_2)}{\partial e_1} V = e_1.$$

In a symmetric equilibrium with $e_1 = e_2 = e$, this can now be written as

$$V\int_{-\infty}^{\infty} f_2(x)xf_1(x)dx = e^2.$$

For player 2 we obtain the same expression. Using our distributional assumptions, the left-hand side becomes

$$V\int_{-\infty}^{\infty} f_2(x)xf_1(x)dx = V\int_1^2 \frac{x}{\pi(1+x^2)}dx = V\frac{1}{2\pi}\log\left(\frac{5}{2}\right).$$

We thus have a symmetric equilibrium, and the corresponding effort is $e^* = \sqrt{\frac{V\log(\frac{5}{2})}{2\pi}} \approx 0.38\sqrt{V}$.

5 Generalizations

In our previous analysis, the symmetry of the equilibrium was derived under the assumption that the production functions, prizes and cost functions were the same for the competing players. We show next that these assumptions can, under certain conditions, be relaxed. This allows us to deal with heterogeneity between players beyond heterogeneity in skill distributions in a surprisingly simple manner. At the end of the subsection, we will also provide an illustrative example that combines heterogeneity in skill distributions, prizes, cost functions, and production functions.

We begin with the possibility to allow for heterogeneity in production technologies, using the observation that, in some situations, different production functions can be reinterpreted as different skill distributions. This allows us to show that a symmetric equilibrium exists using the results of Theorem 1.

Corollary 1. *Suppose that the production functions are different for the two competing* *players and can be written as $g_i(\theta_i, e_i) = \tilde{g}(h_i(\theta_i), e_i)$, $i \in \{1,2\}$. Then a symmetric equilibrium of the contest game exists with effort determined by*

$$V \int_{\mathbb{R}} a_{e^*}(x) \tilde{f}_i(x) \tilde{f}_k(x) dx = c'\left(e^*\right),$$

where h_i is a real-valued function, $\tilde{f}_i$ denotes the pdf of the random variable $\tilde{\Theta}_i := h_i(\Theta_i)$ and a_{e^*} is calculated based on the production function $\tilde{g}\left(\tilde{\theta}_i, e_i\right)$.

Proof. See Appendix A.3. □

As a specific example, consider the additive production function $g_i(\theta_i, e_i) = \eta_i(\theta_i) + \kappa(e_i)$, with η_i and κ being two strictly increasing functions. In this case, Corollary 1 can be applied since we can write $\tilde{g}(h_i(\theta_i), e_i) = h_i(\theta_i) + \kappa(e_i)$, with $h_i(\theta_i) = \eta_i(\theta_i)$. Likewise, in the case of a multiplicative production function of the form $g_i(\theta_i, e_i) = \eta_i(\theta_i)\kappa(e_i)$ (imposing the additional assumption that η_i and κ are non-negative), we can apply the corollary noting that $\tilde{g}(h_i(\theta_i), e_i) = h_i(\theta_i)\kappa(e_i)$, with $h_i(\theta_i) = \eta_i(\theta_i)$. Notice that in both examples, the component $\kappa(e_i)$ is the same for both players.

Next, we consider a situation with heterogeneous prizes given by $V_1 = sV$ and $V_2 = V$ with $s > 0$. In this case, provided that the cost functions are homogeneous, the contest can be transformed into an equivalent contest, where the players have the same prizes, but different production functions. This is shown in the following proposition.

Proposition 1. *Consider a situation with heterogeneous prizes given by $V_1 = sV$ and $V_2 = V$ with $s > 0$, and let c be homogeneous of degree $\delta > 0$. Then, the contest can be transformed into a contest where the players have the same prizes, but different production functions. This is achieved by considering the transformed effort variables $\xi_1 = e_1/s^{1/\delta}$ and $\xi_2 = e_2$. Denote the equilibrium of the transformed contest by ξ_1^* and ξ_2^*. Then, the equilibrium of the original contest is given by $e_1^* = s^{1/\delta}\xi_1^*$ and e_2^*, where ξ_1^* and e_2^* maximize*

$$\int_{\mathbb{R}} F_2\left(g_{e_2}^{-1}\left(g_{s^{1/\delta}\xi_1}(x)\right)\right) f_1(x) dx V - c(\xi_1)$$

and

$$\int_{\mathbb{R}} F_1\left(g_{s^{1/\delta}\xi_1}^{-1}\left(g_{e_2}(x)\right)\right) f_2(x) dx V - c(e_2),$$

respectively.

Proof. See Appendix A.4. □

Proposition 1 shows that, given homogeneity of the cost function, the equilibrium of a contest with heterogeneous prizes is characterized by conditions abiding a structure which is very similar to the structure of the conditions used to characterize the equilibrium in Theorem 1, although the equilibrium is in general no longer symmetric. However, as we shall see below, if the conditions of Corollary 1 are satisfied, we can derive a very simple expression for the relationship between the equilibrium effort levels. Before turning to this result, we first demonstrate that, in some situations, a contest with different cost functions is equivalent to a contest with different prizes (in terms of equilibrium effort choices) and hence can, according to Proposition 1, be transformed into a contest with different production functions. This is formalized in the following remark.[16]

Remark 1. *Suppose that players have heterogeneous cost functions that take the form $c_i(e_i) = \omega_i c(e_i)$, with $\omega_i > 0$. Then, the objective of player i can be written as:*

$$\int_{\mathbb{R}} F_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right) f_i(x)\,dx V_i - \omega_i c(e_i) = \omega_i \left(\int_{\mathbb{R}} F_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right) f_i(x)\,dx \frac{V_i}{\omega_i} - c(e_i)\right).$$

This objective is equivalent to one in which prizes are given by $\frac{V_i}{\omega_i}$, but the cost functions are the same for both players.

Proposition 1 and Remark 1 highlight how, in certain cases, contests with different prizes or different cost functions can be transformed into equivalent contests with different production technologies. We now turn to showing that, if the assumptions underlying Corollary 1 are satisfied, these contests can be reinterpreted as contests with different skill distributions, allowing us to apply the equilibrium characterization from our baseline case.

Proposition 2 below provides results for two well-known production technologies that satisfy the assumptions of Corollary 1. We first consider the Cobb-Douglas production technology. In this case, we can show that the ratio of equilibrium efforts e_1^*/e_2^* depends only on the ratio of prizes and the degree of homogeneity of the cost function. We then consider the additive production technology. In this case, we obtain a similar characterization which applies to the ratio of efforts subject to an exponential transformation, $\exp\left(e_1^*\right)/\exp\left(e_2^*\right)$.

Proposition 2. *Let the two prizes be given by $V_1 = sV$ and $V_2 = V$ with $s > 0$.*

(i) If the production technology is given by $g(\theta, e) = \theta^\alpha e^\beta$, with $\alpha, \beta > 0$, and c is homogeneous of degree $\delta > 0$, an equilibrium exists with efforts given by $e_1^* = s^{1/\delta} e_2^*$ and

[16] Similar transformations between prizes and cost functions are standard in the literature.

e_2^* being determined by

$$\int_{\mathbb{R}} \tilde{f}_1(x)\tilde{f}_2(x)x\,dxV = e_2^* c'\left(e_2^*\right),$$

where $\tilde{f}_1$ and $\tilde{f}_2$ denote the pdfs of the random variables $\tilde{\Theta}_1 := s^{\frac{1}{\delta}}\Theta_1^{\frac{\alpha}{\beta}}$ and $\tilde{\Theta}_2 := \Theta_2^{\frac{\alpha}{\beta}}$, respectively.

(ii) If the production technology is given by $g(\theta,e) = \alpha\theta + \beta e$, with $\alpha,\beta > 0$, and $\tilde{c} := c \circ \ln$ is homogeneous of degree $\delta > 0$, an equilibrium exists with efforts given by $e_1^* = \ln\left(s^{\frac{1}{\delta}}\tilde{e}_2^*\right)$ and $e_2^* = \ln\left(\tilde{e}_2^*\right)$, where $\tilde{e}_2^*$ is determined by

$$\int_{\mathbb{R}} \tilde{f}_1(x)\tilde{f}_2(x)x\,dxV = \tilde{e}_2^* \tilde{c}'\left(\tilde{e}_2^*\right),$$

and $\tilde{f}_1$ and $\tilde{f}_2$ denote the pdfs of the random variables $\tilde{\Theta}_1 := \exp\left(\frac{\alpha}{\beta}\Theta_1\right)s^{1/\delta}$ and $\tilde{\Theta}_2 := \exp\left(\frac{\alpha}{\beta}\Theta_2\right)$, respectively.

Proof. See Appendix A.5. □

We conclude this section by presenting an example in which players, in addition to having different skill distributions as in our baseline case, have different production functions and cost functions, and face different prizes. This example is equivalent to a Tullock lottery contest with heterogeneous prizes and quadratic effort costs.

Example 1. *Suppose that:*

$$g_1(\theta_1,e_1) = \frac{\theta_1 e_1}{2}, \quad c_1(e_1) = e_1^2, \quad V_1 = \frac{V}{2}, \quad f_1(x) = 2\frac{\exp\left(-2x^{-1}\right)}{x^2} I_{\{x>0\}},$$
$$g_2(\theta_2,e_2) = \theta_2 e_2, \quad c_2(e_2) = \frac{e_2^2}{2}, \quad V_2 = V, \quad f_2(x) = \frac{\exp\left(-x^{-1}\right)}{x^2} I_{\{x>0\}},$$

implying that player 2 has a more efficient production technology, lower cost of exerting effort and faces a higher prize. Define $\tilde{\Theta}_1 := \frac{\Theta_1}{2}$, and denote by $\bar{f}_1(x) = \frac{\exp(-x^{-1})}{x^2} I_{\{x>0\}}$ and $\bar{F}_1(t) = \int_{-\infty}^{t} \frac{\exp(-x^{-1})}{x^2} I_{\{x>0\}}dx$ the corresponding pdf and cdf.

The objective function of player 1 can then be stated as:

$$\int_{\mathbb{R}} F_2\left(\frac{e_1 x}{e_2}\right)\bar{f}_1(x)\,dx\frac{V}{2} - e_1^2 = 2\left(\int_{\mathbb{R}} F_2\left(\frac{e_1 x}{e_2}\right)\bar{f}_1(x)\,dx\frac{V}{4} - \frac{e_1^2}{2}\right).$$

The objective function of player 2 can be stated as:

$$\int_{\mathbb{R}} \bar{F}_1\left(\frac{e_2 x}{e_1}\right) f_2(x)\,dx V - \frac{e_2^2}{2}.$$

Hence, we have transformed the original contest into a contest with different prizes $\tilde{V}_1 := \frac{V}{4}$ and $V_2 = V$, but identical production functions and cost functions. According to part (i) of Proposition 2 (noting that $s = \frac{1}{4}$ and $\delta = 2$ in the transformed contest), an equilibrium exists with efforts given by $e_1^* = \frac{e_2^*}{\sqrt{4}} = \frac{e_2^*}{2}$ where e_2^* is determined by

$$V\int_{\mathbb{R}} \tilde{f}_1(x) f_2(x) x\,dx = \left(e_2^*\right)^2 \Longleftrightarrow e_2^* = \sqrt{V\int_{\mathbb{R}} \tilde{f}_1(x) f_2(x) x\,dx},$$

with $\tilde{f}_1(x) = \frac{\exp\left(-\frac{x^{-1}}{2}\right)}{2x^2} I_{\{x>0\}}$ being the pdf corresponding to the random variable $\frac{\tilde{\Theta}_1}{2}$. Using the specific density functions, we obtain

$$e_2^* = \sqrt{V\int_0^{\infty} \frac{\exp\left(-\frac{x^{-1}}{2}\right)}{2x^2} \frac{\exp\left(-x^{-1}\right)}{x^2} x\,dx} = \sqrt{V\int_0^{\infty} \frac{\exp\left(-\frac{3}{2}x^{-1}\right)}{2x^3}\,dx} = \sqrt{\frac{2V}{9}}.$$

Notice that the same result as in Example 1 would be obtained by directly solving the Tullock contest with different prizes and quadratic costs, in which players maximize the objectives $\frac{e_1}{e_1+e_2}\frac{V}{4} - \frac{e_1^2}{2}$ and $\frac{e_2}{e_1+e_2}V - \frac{e_2^2}{2}$, respectively.

6 Comparative Statics Results

In this section, we investigate the consequences of player heterogeneity, in terms of the statistical properties of the skill distributions of the competing players, on the incentive to exert effort. To facilitate the derivation of these results, we define $r_{e,i} : \mathbb{R} \to \mathbb{R}$ given by $r_{e,i}(x) = a_e(x) f_i(x)$. Equation (4) can thus be written as:

$$V\int_{\mathbb{R}} r_{e^*,i}(x) f_k(x)\,dx = c'(e^*). \tag{5}$$

The integral now has the same structure as a decision maker's expected utility in decision theory (e.g., Levy 1992), where the function $r_{e,i}$ corresponds to the decision maker's utility function. As we will see, this link proves useful in deriving several key results. We also need one additional assumption:

Assumption 2. *The primitives of the model are such that $q: E \to \mathbb{R}$, defined by*

$$q(e) = V \int_{\mathbb{R}} r_{e,i}(x) f_k(x)\,dx - c'(e),$$

is strictly decreasing.

As c is strictly convex, Assumption 2 is not very strong and is always satisfied if $\int_{\mathbb{R}} r_{e,i}(x) f_k(x)\,dx$ is non-increasing in e. To give a specific example, consider the CES production function $g(\theta_i, e_i) = \left(\alpha\theta_i^\rho + \beta e_i^\rho\right)^{\frac{1}{\rho}}$, with $\alpha, \beta > 0$ and $\rho \leq 1$. Here $a_e(x) = \frac{\beta}{\alpha}\left(\frac{x}{e}\right)^{1-\rho}$, implying that $\int_{\mathbb{R}} a_e(x) f_1(x) f_2(x)\,dx = e^{\rho-1}\int_{\mathbb{R}} \frac{\beta}{\alpha} x^{1-\rho} f_1(x) f_2(x)\,dx$. For this specification, Assumption 2 is satisfied in all cases where players have an incentive to exert positive effort (i.e., $\int_{\mathbb{R}} \frac{\beta}{\alpha} x^{1-\rho} f_1(x) f_2(x)\,dx > 0$). Furthermore, the assumption ensures that effort is always increasing in the prize and that the considered equilibrium is unique in the class of symmetric equilibria (the latter result follows from the assumption ensuring that there is a unique e solving equation (5)).

6.1 First-Order Stochastic Dominance

A standard result in contest theory is that heterogeneity among players with respect to their skills reduces the incentive to exert effort (see, e.g., Schotter and Weigelt 1992, or Observation 1 in the survey by Chowdhury, Esteve-Gonzalez, and Mukherjee 2019). In our model, this standard result is potentially reversed, as we will now show.

Consider a contest with two players with skills drawn from two distributions with expected values μ_k and μ_i, respectively. If, from the outset, $\mu_k \geq \mu_i$ and the difference $\mu_k - \mu_i$ is increased, then the two players become more heterogeneous in terms of their expected skill. Based on this idea, we proceed by investigating the consequences of making players more heterogeneous in the sense of first-order stochastic dominance, as captured by the following definition.

Definition 1. *Let μ_k and μ_i refer to the expected values of the skill distributions (F_k, F_i) in an initial contest. Players in a contest with skill distributions $(\tilde{F}_k, F_i)$ are said to be more heterogeneous (with respect to their skills) relative to players in the initial contest with skill distributions (F_k, F_i), in a first-order sense, if either of the following conditions hold:*

(i) $\mu_k \geq \mu_i$ and $\tilde{F}_k$ dominates F_k in the sense of first-order stochastic dominance.

(ii) $\mu_k \leq \mu_i$ and $\tilde{F}_k$ is dominated by F_k in the sense of first-order stochastic dominance.

Due to Assumption 2, equilibrium effort increases if a change in the primitives of the model leads to an increase in $\int_{\mathbb{R}} r_{e,i}(x) f_k(x) dx$. As indicated before, this expression has the same structure as a decision maker's expected utility in decision theory, where the function $r_{e,i}$ is replaced by the decision maker's utility function. Since the structure of the problems is the same, we can make extensive use of results from decision theory in our analysis. We obtain the following proposition.

Proposition 3. *Consider two contests with skill distributions $(\tilde{F}_k, F_i)$ and (F_k, F_i) where $\text{supp}(\tilde{f}_k)$ and $\text{supp}(f_k)$ both are subsets of $\text{supp}(f_i)$. Let $\tilde{e}^*$ and e^* denote, respectively, the (symmetric) equilibrium efforts associated with these contests. Then, $\tilde{e}^* > e^*$ if either one of the following statements hold:*

(i) $r_{e,i}(x)$ is strictly increasing for all $x \in \text{supp}(f_i)$ and all $e \geq 0$, and $\tilde{F}_k$ dominates F_k in the sense of first-order stochastic dominance.

(ii) $r_{e,i}(x)$ is strictly decreasing for all $x \in \text{supp}(f_i)$ and all $e \geq 0$, and $\tilde{F}_k$ is dominated by F_k in the sense of first-order stochastic dominance.

Proof. See Appendix A.6. □

Note that Proposition 3 holds independently of whether $\mu_k \leq \mu_i$ or $\mu_k \geq \mu_i$. Combining Definition 1 with Proposition 3, we have the following corollary.[17]

Corollary 2. *Effort can be higher when contestants are more heterogeneous in a first-order sense.*

We illustrate the intuition behind Proposition 3 and Corollary 2 through two examples. In each example, we start from a situation of equal expected skills, and then introduce a first-order stochastic dominance shift. In the first example, which has a somewhat simpler intuition than the second, $r_{e,i}(x)$ is strictly decreasing and effort gets higher as player k becomes weaker, illustrating part (ii) of Proposition 3. In the second example, $r_{e,i}(x)$ is strictly increasing and effort gets higher as player k becomes stronger, illustrating part (i) of Proposition 3.

Example 2. *Suppose that $g(\theta, e) = \theta + e$, $\Theta_i \sim Exp\left(\frac{4}{3}\right)$, $\Theta_k \sim U\left[\frac{1}{2}, 1\right]$, $\tilde{\Theta}_k \sim U\left[\frac{7}{16}, \frac{15}{16}\right]$, $c(e) = \frac{e^2}{2}$, $V = 1$. Then $e^* = \frac{2\left(\exp\left(\frac{2}{3}\right)-1\right)}{\exp\left(\frac{4}{3}\right)} \approx 0.499$ and $\tilde{e}^* = \frac{2\left(\exp\left(\frac{2}{3}\right)-1\right)}{\exp\left(\frac{5}{4}\right)} \approx 0.543$.*

[17] There is one small caveat to Corollary 2 that we should mention. If equilibrium effort increases as contestants become more heterogeneous, then a symmetric equilibrium in which both players exert positive effort will fail to exist if the heterogeneity between players becomes too large. The reason is that the weaker player would eventually receive a negative payoff, meaning that this player would prefer to choose zero effort.

In Example 2, the first thing to notice is that the additive production technology implies that $a_e(x) = 1$. This further implies that $r_{e,i}(x)$ is strictly decreasing for all relevant x, since $f_i(x)$ is the decreasing pdf of the exponential skill distribution. The fact that $a_e(x) = 1$ also implies that the incentive to supply effort, as given by (4), only depends on the collision density $f_k(x)f_i(x)$. Since $f_i(x)$ is decreasing, and $f_k(x)$ is uniform and shifted to the left, the collision density between $\tilde{f}_k$ and f_i is everywhere larger than the collision density between f_k and f_i, see Figure 1 for an illustration. Thus, both players have a higher incentive to exert effort. The simple intuition for the example is that the marginal incentive to supply effort for both players is positive only in situations where they have equal skill, and the considered shift in distributions makes such situations unambiguously "more likely" to happen.

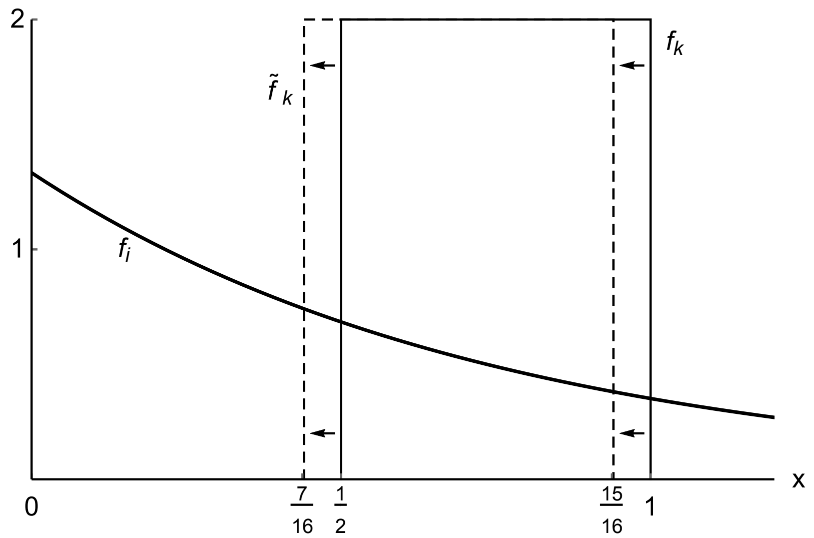

Figure 1: Illustration of Example 2

Example 3. *Suppose that* $g(\theta, e) = \theta \cdot e$, $\Theta_i \sim U[0,1]$, $\Theta_k \sim U\left[\frac{1}{4}, \frac{3}{4}\right]$, $\tilde{\Theta}_k \sim U\left[\frac{5}{16}, \frac{13}{16}\right]$, $c(e) = \frac{e^2}{2}$, $V = 1$. *Then* $e^* = \frac{1}{\sqrt{2}} \approx 0.707$ *and* $\tilde{e}^* = \frac{3}{4} = 0.75$.

In Example 3, the multiplicative production technology implies that $a_e(x) = x/e$ which is a strictly increasing function of x. This further implies that $r_{e,i}(x)$ is strictly increasing on $[0,1]$ because f_i is uniform. The shift in the skill distribution of player k from F_k to $\tilde{F}_k$ implies that the expected skill of player k increases. However, the height of the density of player k's skill distribution does not change ($f_k(x) = 2, x \in \left[\frac{1}{4}, \frac{3}{4}\right]$ and $\tilde{f}_k(x) = 2, x \in \left[\frac{5}{16}, \frac{13}{16}\right]$). Thus, since $f_i(x) = 1$, we have that $f_k(x)f_i(x) = \tilde{f}_k(x)f_i(x) = 2$ at all points where these collision densities are non-zero. However, due to the distributional shift, the subset of $\mathbb{R}$ where the two uniform distributions overlap shifts to the right. Therefore, the two distributions collide at larger values of x (see Figure 2 for an illustration). This would have no effect on the incentive to exert effort if $a_e(x)$ would be constant, as in Example 2. However, in the current example, we have that $a_e(x) = x/e$. Thus, taking into account the three terms in (4), the fact that the two distributions collide at larger values of x increases the incentive to exert effort for both players. Intuitively, given that the only relevant situations (where players have a positive marginal incentive to supply effort) now occur at larger values of skill, the fact that there is a complementarity between skill and effort in the production function implies that the incentive to supply effort is higher for both players.

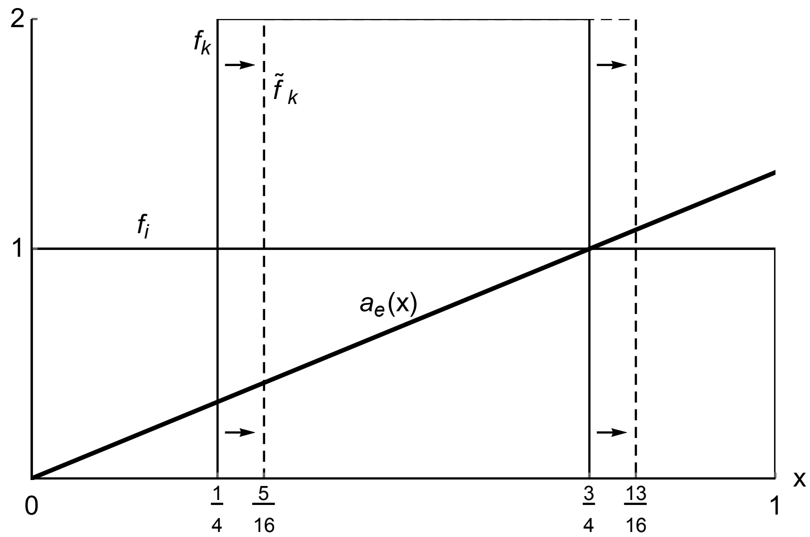

Figure 2: Illustration of Example 3

Concluding this section, we note that the conditions in Proposition 3 are sufficient, but not necessary for the result that effort can be higher when contestants are more heterogeneous. To illustrate this, we present an additional result based on normal distributions where we first determine the marginal winning probability in a situation with symmetric effort.

Proposition 4. *Suppose that* $\Theta_i \sim N(\mu_i, \sigma_i^2)$, $\Theta_k \sim N(\mu_k, \sigma_k^2)$, *and* $g(\theta, e) = \theta \cdot e$. *Then the marginal winning probability when* $e_1 = e_2 = e$ *is*

$$\frac{\partial P_i(e_i, e_k)}{\partial e_i}\Big|_{e_i = e_k = e} = \frac{(\mu_i \sigma_k^2 + \mu_k \sigma_i^2) \exp\left(-\frac{(\mu_i - \mu_k)^2}{2(\sigma_i^2 + \sigma_k^2)}\right)}{e(2\pi)^{\frac{1}{2}} (\sigma_i^2 + \sigma_k^2)^{\frac{3}{2}}}.$$

Proof. See Appendix A.7. □

In the upcoming example, it can be verified that $r_{e,i}(x) = a_e(x)f_i(x)$ is neither always increasing nor always decreasing, by virtue of the multiplicative production technology combined with the bell-shaped normal distribution. Nonetheless, equilibrium effort increases as players become more heterogeneous in the sense of increasing the distance $|\mu_i - \mu_k|$.

Example 4. *Consider Proposition 4 and assume that* $(\sigma_i, \sigma_k) = (1,1)$, $(\mu_i, \mu_k) = \left(\frac{1}{2}, \frac{1}{2}\right)$, $V = 1$, *and* $c(e) = \frac{e^2}{2}$. *Then equilibrium effort is* $e^* = \left(2\pi^{\frac{1}{4}}\right)^{-1} \approx 0.38$. *If we increase* μ_i *from* $\frac{1}{2}$ *to* $\frac{3}{2}$, *keeping* μ_k *constant, equilibrium effort increases to* $\tilde{e}^* = \left(\sqrt{2}\exp\left(\frac{1}{8}\right)\pi^{\frac{1}{4}}\right)^{-1} \approx 0.47$.

6.2 Second-Order Stochastic Dominance

The studies by Hvide (2002), Kräkel and Sliwka (2004), Kräkel (2008), Gilpatric (2009), and DeVaro and Kauhanen (2016) investigate how "risk" or "uncertainty" affects players' incentive to exert effort in contests. One result that is common to all of these analyses is that in contests between equally able players, higher risk (as measured by a higher variance of the random variables capturing the uncertainty of the contest outcome) leads to lower efforts. We revisit this result in the context of our model, and show that effort may increase as the skill distribution of one of the players becomes more uncertain.

The economic literature has identified different ways to conceptualize risk or uncertainty. We follow Rothschild and Stiglitz 1970 by using second-order stochastic dominance to measure the uncertainty regarding players' skill distributions.[18]

Definition 2. *The skill distribution* $\tilde{F}_i$ *is said to be more uncertain than the distribution* F_i *if* $\tilde{F}_i$ *is a mean-preserving spread of* F_i. *This is equivalent to* $\tilde{F}_i$ *being dominated by* F_i *in the sense of second-order stochastic dominance.*

Equipped with this definition, we can use well-known results from decision theory to obtain our next proposition:

Proposition 5. *Consider two contests with skill distributions* $(\tilde{F}_k, F_i)$ *and* (F_k, F_i) *where* $\mathrm{supp}(\tilde{f}_k)$ *and* $\mathrm{supp}(f_k)$ *both are subsets of* $\mathrm{supp}(f_i)$. *Let* $\tilde{e}^*$ *and* e^* *denote, respectively, the (symmetric) equilibrium efforts associated with these contests. Suppose that* $\tilde{F}_k$ *is more uncertain than* F_k. *Then, the following results hold:*

[18] Gerchak and He (2003) analyze how effort in two-player contests is determined by the Rényi entropy, and Drugov and Ryvkin (2020) generalize their insights to the case of more than two players. Their results, however, rely on the assumptions of an additive production function and homogeneous players (i.e., players with the same skill distributions). It is not obvious how these results transfer to a model with general production technologies and asymmetric skill distributions, which is the primary focus here.

(i) If $r_{e,i}(x)$ is strictly convex on $\mathrm{supp}(f_i)$ *for all $e \geq 0$, then $\tilde{e}^* > e^*$.*

(ii) If $r_{e,i}(x)$ is linear on $\mathrm{supp}(f_i)$ *for all $e \geq 0$, then $\tilde{e}^* = e^*$.*

(iii) If $r_{e,i}(x)$ is strictly concave on $\mathrm{supp}(f_i)$ *for all $e \geq 0$, then $\tilde{e}^* < e^*$.*

Proof. See Appendix A.8. □

The key insight needed to understand Proposition 5 is that applying a mean-preserving spread to the distribution F_k shifts probability mass from the center to the tails of the distribution, and the impact of this change on the incentive to exert effort depends on the curvature of $r_{e,i}(x)$. Notice that Proposition 5 also holds if players have the same expected skill, namely $\mu_i = \mu_k$. This means that, in a contest with two players who are expected to be equally able, higher uncertainty regarding players' skills may increase the incentive to exert effort.

Next, we illustrate and provide intuition for Proposition 5 by presenting an example set in the context of the Lazear-Rosen model with an additive production technology. The example demonstrates that increasing the uncertainty of the contest while keeping the expected skill of both players unchanged, can increase equilibrium effort.

Example 5. *Consider a contest with the additive production function $g(\theta, e) = \theta + e$, the parameter $V = 1$, and the cost function $c(e) = \frac{e^2}{2}$. Suppose $\Theta_i \sim Exp(1)$ and $\Theta_k \sim U\left[\frac{1}{2}, \frac{3}{2}\right]$ (implying $\mu_i = \mu_k = 1$). Equilibrium effort is then $e^* = \frac{\exp(1)-1}{\exp(\frac{3}{2})} \approx 0.38$. Now, consider a mean-preserving spread of the skill distribution of player k, enlarging the support of the uniform distribution, such that $\tilde{\Theta}_k \sim U[0,2]$. Then effort increases to $\tilde{e}^* = \frac{\exp(2)-1}{2\exp(2)} \approx 0.43$.*

In Example 5, we have imposed the additive production technology which implies $a_e(x) = 1$. Thus, the convexity of $r_{e,i}(x)$ referred to in part (i) of Proposition 5 is determined by the convexity of $f_i(x)$. To understand how the shift from f_k to $\tilde{f}_k$ affects the incentive to exert effort, we need to study how the integral in (4) is affected. Similar to Example 2, given that $a_e(x) = 1$, it is sufficient to compare $\int f_i(x) f_k(x) dx$ with $\int f_i(x) \tilde{f}_k(x) dx$. The shift from f_k to $\tilde{f}_k$ entails an enlargement of the support of the uniform distribution. This implies that the density decreases for intermediate values of x, but increases for low and high values of x (see Figure 3 for an illustration). Given that $f_i(x)$ is strictly decreasing, the part of the skill distribution of player k that is stretched out to the left will collide with relatively large values of f_i, whereas the the part of the skill distribution of player k that is stretched out to the right will collide with relatively small values of f_i, creating a trade-off. The fact that f_i is not only strictly decreasing, but also convex, resolves this trade-off, implying that the overall effect of the shift is to increase the value of the integral expression. Thus, both players have a higher incentive to exert effort as a result of the move from f_k to $\tilde{f}_k$. Intuitively, due to the change in the distribution of player k, situations where the competing players have the same skill become "more likely", implying an increase in equilibrium effort.

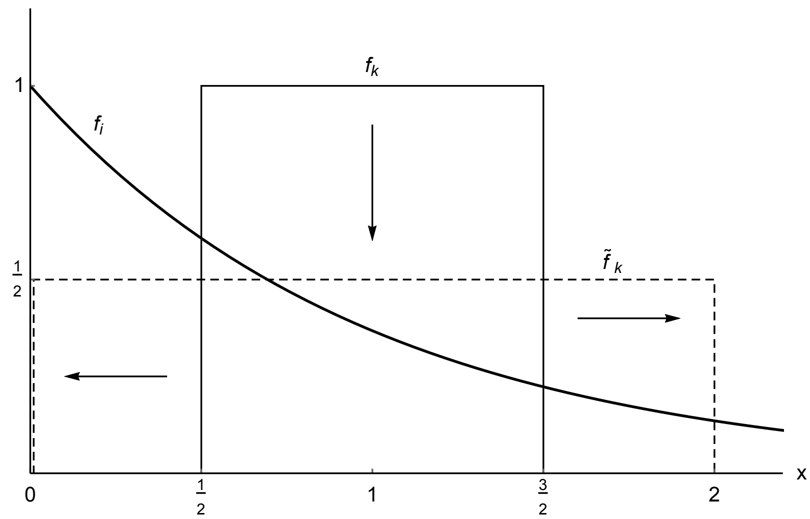

Figure 3: Illustration of Example 5

We conclude this section by defining contestant heterogeneity in a *second-order* sense and we follow the structure of the corresponding definition of heterogeneity in a first-order sense (Definition 1). In Definition 1, we used the ranking of players' mean skills to characterize the initial situation. In the new definition, we do so through the variances of the skill distributions of the competing players (restricting attention to statistical distributions with finite variance). Notice, however, that variance is not always a good measure of uncertainty or risk (see, e.g, Rothschild and Stiglitz 1970). Therefore one should keep in mind, when applying the definition below, that higher variance entails higher uncertainty only for certain skill distributions (e.g., the normal distribution).

Definition 3. *Let Var_k and Var_i refer to the variances of the skill distributions (F_k, F_i) in an initial contest. Players in a contest with skill distributions $(\tilde{F}_k, F_i)$, are said to be more heterogeneous (with respect to their skills) relative to players in the initial contest with skill distributions (F_k, F_i), in a second-order sense, if either of the following conditions hold:*

(i) $Var_k \geq Var_i$ and F_k dominates $\tilde{F}_k$ in the sense of second-order stochastic dominance.

(ii) $Var_k \leq Var_i$ *and* F_k *is dominated by* $\tilde{F}_k$ *in the sense of second-order stochastic dominance.*

Combining Proposition 5 with Definition 3, we have the following corollary.

Corollary 3. *Effort can be higher when contestants are more heterogeneous in a second-order sense.*

6.3 Implications for Optimal Team Composition

The results in the preceding two subsections have implications for optimal team composition and organizational design.[19] In particular, our results suggest that employers could find it desirable to employ a more heterogeneous workforce as an instrument to induce higher effort. In Section 6.1, we analyzed the effects of increasing the heterogeneity in players' expected skills, and showed how this can increase equilibrium effort. This means that a firm could benefit (from the perspective of inducing higher effort) by hiring some workers with a high expected skill and some with a low expected skill, based on, for example, signals such as the quality of the institution where a college graduate received his or her degree. In Section 6.2, we showed how increased uncertainty regarding skills of some players can increase equilibrium effort. Thus, a firm could benefit from hiring a mix of experienced workers (for whom the uncertainty regarding skills is relatively small) and inexperienced workers (for whom the uncertainty regarding skills is relatively large).

To see this more formally, suppose a firm already employs a worker with skill distribution F_1 and considers to hire another worker with skill distribution F_2. Moreover, assume that $r_{e,1}(x)$ is strictly decreasing and strictly convex (for example, by assuming that the production function is given by $g(\theta, e) = \theta + e$ and skills are Exponentially distributed with parameter λ).[20] Then the firm may gain from hiring another worker with a lower expected skill ($\mu_2 < \mu_1$), but where F_2 is more uncertain (meaning that worker 2's skill is drawn from a more uncertain distribution). This finding can be understood from the perspective of Proposition 3, that tells us that effort will be higher due to the lower expected skill of worker 2, combined with Proposition 5, which tells us that effort will

[19]See, e.g., Gershkov, Li, and Schweinzer (2009, 2016) and Fu, Lu, and Pan (2015).

[20]An alternative skill distribution that would also be decreasing and convex would be a normal distribution that is truncated to the left at a point to the right of the second inflection point. Such a distribution could be motivated by the observation that skills are often normally distributed and that, when employing worker 1, the firm tried to hire the most able applicant, meaning that skills in the higher end of the distribution are most relevant (see, e.g., Aguinis and O'Boyle Jr. 2014).

be higher due to the larger uncertainty regarding the skill of worker 2. In other words, hiring a worker with a lower expected skill, drawn from a more uncertain distribution, can induce higher effort. Proposition 3 and Proposition 5 also have other managerial implications as they indicate that employers may want to hire workers who have worked on different tasks in the past (or on similar tasks in a different firm or industry), to create uncertainty about workers' skills. In a similar vein, it might be desirable to implement some kind of job rotation.

7 Extensions

7.1 The Case of More Than Two Players ($n > 2$)

We now turn to the case of $n > 2$ contestants which allows us to address the interesting question of how effort depends on the number of players competing in a contest.[21] In Section 7.1.1, we show that the existence of a symmetric equilibrium generally cannot be extended to the case of $n > 2$ heterogeneous players. In Section 7.1.2, we consider the case of n homogeneous players. Section 7.1.3 examines a special case of our model where $n-1$ homogeneous players compete against a player who is more highly skilled (e.g., as in Brown 2011 and Krumer, Megidish, and Sela 2017), which serves to demonstrate that a symmetric equilibrium can exist when players are heterogeneous and the number of players is greater than two. In all these sections, we maintain the generality of the production technology.

7.1.1 The $n = 2$ Result Does Not Extend to $n > 2$

In the case of $n > 2$ players with different skill distributions, the equilibrium in our model is generally no longer symmetric. A player i will only win the contest if he or she beats all of his or her opponents. Essentially, each player is thus competing against the best of the other players, that is, the highest order statistic, and therefore faces a different "relevant rival" in the contest. This introduces an asymmetry into the model that was absent in the two-player case, and which generally leads to an asymmetric equilibrium. To see this formally, suppose, for simplicity, that $g(\theta_i, e_i) = \theta_i + e_i$, implying that $a_e(x) = 1$ (the following intuition also holds for general production technologies). Then, using a similar reasoning as in the two-player case (see Section 4), the marginal probability of winning

[21] Contests with more than two players have been studied by, e.g., Tullock (1980), Nalebuff and Stiglitz (1983), Hillman and Riley (1989), Zábojník and Bernhardt (2001), Chen (2003), Zábojník (2012), and Ryvkin and Drugov (2020).

for player i and player k in a symmetric equilibrium can be written, respectively, as:

$$\frac{\partial P_i(e_1,e_2,\dots,e_n)}{\partial e_i}|_{e_1=\dots=e_n=e} = \int_{\mathbb{R}} f_i(x)\frac{d}{dx}\left(F_k(x)\prod_{j\neq i,k}F_j(x)\right)dx$$

and

$$\frac{\partial P_k(e_1,e_2,\dots,e_2)}{\partial e_k}|_{e_1=\dots=e_n=e} = \int_{\mathbb{R}} f_k(x)\frac{d}{dx}\left(F_i(x)\prod_{j\neq i,k}F_j(x)\right)dx.$$

Applying the product differentiation rule on the RHS of the above expressions, we obtain:

$$\int_{\mathbb{R}}\left(f_i(x)f_k(x)\prod_{j\neq i,k}F_j(x)+f_i(x)F_k(x)\frac{d}{dx}\left(\prod_{j\neq i,k}F_j(x)\right)\right)dx \tag{6}$$

and

$$\int_{\mathbb{R}}\left(f_k(x)f_i(x)\prod_{j\neq i,k}F_j(x)+f_k(x)F_i(x)\frac{d}{dx}\left(\prod_{j\neq i,k}F_j(x)\right)\right)dx. \tag{7}$$

The first term in (6) and (7) corresponds to the situation in which all players $j \in \{1,\dots,n\}, j \neq i,k$ perform worse than players i and k so that the n-player contest collapses to a contest between players i and k. For this subcontest, the marginal winning probabilities are the same as shown in the analysis of the two-player contest. The second term in (6) corresponds to the situation in which player i outperforms his or her rival k, such that the contest boils down to a contest between player i and the strongest of the players $j \in \{1,\dots,n\}, j \neq i,k$. The interpretation of the second term in (7) is analogous, with the role of i and k interchanged.

Setting expression (6) equal to expression (7), we obtain

$$\int_{\mathbb{R}} f_i(x)F_k(x)\frac{d}{dx}\left(\prod_{j\neq i,k}F_j(x)\right)dx = \int_{\mathbb{R}} f_k(x)F_i(x)\frac{d}{dx}\left(\prod_{j\neq i,k}F_j(x)\right)dx$$
$$\Leftrightarrow \quad \int_{\mathbb{R}}\left(\frac{f_i(x)}{F_i(x)}-\frac{f_k(x)}{F_k(x)}\right)F_i(x)F_k(x)\frac{d}{dx}\left(\prod_{j\neq i,k}F_j(x)\right)dx = 0.$$

Notice that i and k were arbitrarily selected. Hence, in order for a symmetric equilibrium to exist, it must be the case that the above condition holds for all $i,k \in \{1,\dots,n\}, i \neq k$. We conclude that the condition above is generally violated when the skill distributions of the competing players are distinct, which implies that a symmetric equilibrium generally does not exist in the case of $n > 2$ players.[22]

[22]A sufficient condition for a symmetric equilibrium to exist is that $\frac{f_i(x)}{F_i(x)} = \frac{f_k(x)}{F_k(x)}$ for all $x \in \mathbb{R}$ and all $i,k \in \{1,\dots,n\}, i \neq k$. However, since $\frac{f_i(x)}{F_i(x)} = \frac{d\log F_i(x)}{dx}$ is the reversed hazard rate, which completely charac-

7.1.2 The Case of Homogeneous Players

Suppose that all players share the same skill distribution, i.e., $f_1 = f_2 = ... = f_n =: f$, and define $r_e(x) := a_e(x) f(x)$.

Proposition 6. *In an n-player contest with homogeneous skill distributions, a symmetric Nash equilibrium with $e_1 = e_2 = \cdots = e_n = e^*$ exists and is characterized by*

$$V \int_{\mathbb{R}} r_{e^*}(x)(n-1)(F(x))^{n-2} f(x)\,dx = V \int_{\mathbb{R}} r_{e^*}(x) \frac{d}{dx}\left((F(x))^{n-1}\right) dx = c'\left(e^*\right). \tag{8}$$

Proof. See Appendix A.9. □

Notice that $(F(x))^{n-1}$ describes the cdf of the highest order statistic out of a group of $n-1$ players. The condition from the proposition therefore illustrates what was mentioned before: the n-player contest boils down to a two-player contest, in which every player competes against the strongest of the other players.

A particular focus in the literature has been on the relation between effort and the number of competitors. Early studies of the n-player Tullock contest with $\gamma_1 = ... = \gamma_n$, $m = 1$, and linear effort costs found that equilibrium effort is given by $e^* = \frac{n-1}{n^2} V$, so that effort is decreasing in n (e.g., Tullock 1980, Hillman and Riley 1989). With a convex cost function (as in our setting), the condition would change to $e^* c'(e^*) = \frac{n-1}{n^2} V$, but effort would still be decreasing in n. The result can be explained by a discouragement effect. If a player competes against many rivals, his or her chance of winning is relatively low and the player reduces effort in turn.

In what follows, we study the relationship between effort and the number of competitors in our model. To do so, we need to extend Assumption 2 to the n-player case.

Assumption 3. *The primitives of the model are such that $q_n : E \to \mathbb{R}$, defined by*

$$q_n(e) = V \int_{\mathbb{R}} r_e(x) \frac{d}{dx}\left((F(x))^{n-1}\right) dx - c'(e),$$

is strictly decreasing.

We observe that, in addition to the discouragement effect mentioned before, there is also an encouragement effect, inducing players to increase their effort as they compete against more players. This is reflected by the factor $(n-1)$ in $\int_{\mathbb{R}} r_e(x)(n-1)(F(x))^{n-2} f(x)\,dx$

terizes a statistical distribution, this condition would only hold for all $x \in \mathbb{R}$ if F_i and F_k refer to identical distributions.

in Proposition 6 above. As we will show, the encouragement effect might dominate, opening up for the possibility that effort increases in the number of competitors (for a related result, see also Ryvkin and Drugov 2020). In our proof, we make use of the fact that increasing n leads to a distribution of the highest order statistic that first-order stochastically dominates the original distribution. We can thus invoke Proposition 3 to study the effects of an increase in n on equilibrium effort.[23]

Proposition 7. *Consider the n-player contest with homogeneous skill distributions and let e^* denote the symmetric Nash equilibrium effort. Then the following statements hold:*

i) If $r_e(x)$ is strictly increasing for all $x \in \mathrm{supp}(f)$ and all $e \geq 0$, then e^* increases in n.

ii) If $r_e(x)$ is strictly decreasing for all $x \in \mathrm{supp}(f)$ and all $e \geq 0$, then e^* decreases in n.

iii) If $r_e(x)$ is constant for all $x \in \mathrm{supp}(f)$ and all $e \geq 0$, then e^* does not depend on n.

Proof. See Appendix A.10. □

We conclude this subsection with an example to illustrate the potentially positive relationship between effort and the number of players in the context of the well-known Lazear-Rosen model.

Example 6. *Consider a contest with an additive production function $g(\theta, e) = \theta + e$, $V = 1$, and cost function $c(e) = \frac{e^2}{2}$. Suppose each Θ_i is distributed according to the modified reflected exponential distribution with mean $\mu = 1$ and pdf $f(x) = \frac{1}{2}\exp\left(\frac{1}{2}(x-3)\right)$ for $x \leq 3$ and zero otherwise (see, e.g., Rinne 2014). With two players, equilibrium effort is $e^* = \frac{1}{4}$. With three players, equilibrium effort increases to $\tilde{e}^* = \frac{1}{3}$.*

7.1.3 A Contest With One Player Who Is More Highly Skilled

We now turn to a special case of our contest model with $n > 2$ players where we obtain a symmetric equilibrium even when players are asymmetric in the sense of having different expected skills. For this purpose, suppose that $\Theta_i = t_i + \mathscr{E}_i$ for $i = 1, \ldots, n$ where $t_1 > t_2 = \cdots = t_n = t$ and the $\mathscr{E}_i$, $i = 1, \ldots, n$, are i.i.d. according to the *reflected exponential distribution* with cdf $H(x) = \exp(\lambda x)$ defined on $(-\infty, 0]$ with $\lambda > 0$ (Rinne 2014). In this case, we have:

$$f_i(x) = \begin{cases} \lambda \exp(\lambda(x - t_i)), & \text{for } x \leq t_i \\ 0, & \text{for } x > t_i \end{cases}$$

[23] Notice that similar to what was mentioned in connection to Corollary 2, there is a small caveat to part (i) of Proposition 7. If e^* is increasing in n, a symmetric equilibrium in which all players exert positive effort will fail to exist if n becomes so large that $V/n < c(e^*)$, as (some) players would prefer to choose an effort of zero.

and

$$F_i(x) = \begin{cases} \exp(\lambda(x - t_i)), & \text{for } x \le t_i \\ 1, & \text{for } x > t_i, \end{cases}$$

implying that $\frac{f_i(x)}{F_i(x)} = \lambda$ on the support of f_i which is $(-\infty, t_i]$. Consider the condition

$$\int_{\mathbb{R}} \left(\frac{f_i(x)}{F_i(x)} - \frac{f_k(x)}{F_k(x)} \right) F_i(x) F_k(x) \frac{d}{dx} \left(\prod_{j \neq i,k} F_j(x) \right) dx = 0,$$

that we derived in Subsection 7.1.1.[24] It is satisfied for all $i, k \in \{2, \dots, n\}$ since in this case, $\frac{f_i(x)}{F_i(x)} = \frac{f_k(x)}{F_k(x)} = \lambda$ on the common support $(\infty, t]$ of f_i and f_k (since we assumed from the outset that $t_2 = t_3 = \dots = t_n = t$). Consider now the case where $i = 1$ and $k \in \{2, \dots, n\}$. In this case, we have that $\frac{f_1(x)}{F_1(x)} = \frac{f_k(x)}{F_k(x)} = \lambda$ for $x \le t$. For $x > t$, we have $\prod_{j \neq 1,k} F_j(x) = 1 \Rightarrow \frac{d}{dx} \left(\prod_{j \neq 1,k} F_j(x) \right) = 0$. Hence, we conclude that the condition is satisfied for all $i, k \in \{1, \dots, n\}$, $i \neq k$ and all $x \in \mathbb{R}$, and that the marginal winning probabilities are the same. This implies that a symmetric equilibrium exists in which all players choose the same equilibrium effort e^*.

Next, we compute an example with a multiplicative production technology and show that the marginal winning probability is increasing in the number of players.

Proposition 8. *Consider the contest described above. Suppose the production technology takes the form $g(\theta, e) = \theta e$ and assume that t is chosen sufficiently large so that $n\lambda t - 1 > 0$. Then, the marginal winning probability given equal effort e is equal to:*

$$\Psi(n) = \frac{(n-1)}{e} \exp(-\lambda(t_1 - t)) \frac{(n\lambda t - 1)}{n^2}, \quad \textit{with} \quad \Psi'(n) > 0.$$

Proof. See Appendix A.11. □

7.2 Privately Known Skills

We now turn to examine how our analysis is affected by assuming that players have private information regarding their own skill. For this purpose, we assume that each player $i \in \{1, \dots, n\}$ observes his or her own skill realization θ_i before choosing effort e_i. This means that each player chooses a strategy consisting of a function $e_i(\theta_i)$ that specifies the effort level for each value of θ_i. Everything else in our model remains unchanged.

[24]This condition was derived under the assumption of an additive production technology. In Appendix B.1, we provide a proof for the existence of a symmetric equilibrium in the general case.

In particular, all the opponents' skills Θ_k, $k \in \{1,\ldots,n\}, k \neq i$ remain uncertain, as in the main model, and their distributions are common knowledge.

This private-information assumption effectively implies that player i can, in a deterministic manner, choose output $g(\theta_i, e_i)$ by making the appropriate effort choice e_i. The decision problem of player i can therefore, equivalently, be expressed as the specification of optimal effort $e_i(\theta_i)$ or the choice of optimal output $z_i(\theta_i) := g(\theta_i, e_i(\theta_i))$, as a best response to the opponents' choice of effort or output. Assuming that optimal output is strictly increasing in skill (this will be confirmed in our examples), z_i is invertible with inverse z_i^{-1}.

In the two-player case, where player i competes against another player k, player i wins the contest for given realizations of Θ_i and Θ_k if and only if the following condition holds

$$g(\theta_k, e_k) < g(\theta_i, e_i) \iff z_k(\theta_k) < z_i(\theta_i) \iff \theta_k < z_k^{-1}(z_i(\theta_i)).$$

Taking into account that, from the perspective of player i, the uncertainty of the contest only concerns the skill realization of player k, we have that equilibrium efforts $e_i(\theta_i)$ and $e_k(\theta_k)$ satisfy:

$$e_i(\theta_i) \in \underset{e_i}{\arg\max} \left\{ F_k(z_k^{-1}(z_i(\theta_i)))V - c(e_i) \right\},$$
$$e_k(\theta_k) \in \underset{e_k}{\arg\max} \left\{ F_i(z_i^{-1}(z_k(\theta_k)))V - c(e_k) \right\}.$$

It can thus immediately be seen that the first-order condition for player i only involves the skill distribution of the opposing player k, whereas the first-order condition for player k only involves the skill distribution of the opposing player i. Hence, the symmetry that was present in the main model, where the first-order condition for each player involved the product of f_i and f_k (see equation (4)), vanishes when skills are privately known. We thus conclude that the equilibrium effort functions $e_i(\theta_i)$ and $e_k(\theta_k)$ generally are not symmetric.

The n-player case with privately known skills is handled in an almost identical fashion. Instead of competing against player k, player i can be viewed as competing against the strongest of the opponents $j \in \{1,\ldots,n\}$, $j \neq i$, in the sense of the highest order statistic. We analyze the n-player case with symmetric skill distributions below. Our analysis generalizes existing contest models with privately informed players with respect to the production technology. For instance, when imposing a multiplicative production function, our model is equivalent to the single-prize version of Moldovanu and Sela (2001). Likewise, in the case of an additive production function, our model matches the nondiscriminatory contest in Pérez-Castrillo and Wettstein (2016). We note, however, that these models are more general in other respects. Whereas Moldovanu and Sela (2001) allow for multiple prizes, Pérez-Castrillo and Wettstein (2016) allow for the single prize to differ between contestants.

The case of n homogeneous players. We revisit the setting with n homogeneous players considered in Section 7.1.2 and introduce the private-information assumption. The exposition also serves to illustrate the two-player case with symmetric players and private information.

In a symmetric setting, we naturally expect symmetric equilibria in which players employ the same effort function, $e(\theta_i)$. Thus, equal types imply equal effort and output even when individuals are privately informed about their own type. As we did in Section 7.1.2, we analyze the n-player case by analyzing how player i competes against the highest order statistic of his or her opponents. We denote the distribution function of this order statistic by $F^{(n-1)}$ with the associated probability density function $f^{(n-1)}$. Given that players are assumed to have independent skill distributions, $F^{(n-1)}(x) = F(x)^{n-1}$ and $f^{(n-1)}(x) = (n-1)F(x)^{n-2}f(x)$.

To solve for a symmetric equilibrium, we consider the problem of player i maximizing his or her expected payoff when all his or her rivals adopt the common effort function $e(\theta_k)$, or equivalently, the common output function $z(\theta_k) := g(\theta_k, e(\theta_k)), i \neq k$ with corresponding inverse z^{-1}. The equilibrium effort of player i is thus given by:

$$e_i(\theta_i) \in \arg\max_{e_i} \left\{ F^{(n-1)}(z^{-1}(g(\theta_i, e_i)))V - c(e_i) \right\}.$$

For each value of θ_i, there is an associated first-order condition:

$$f^{(n-1)}(z^{-1}(z_i(\theta_i))) \frac{1}{z'(z^{-1}(z_i(\theta_i)))} \frac{\partial g(\theta_i, e_i(\theta_i))}{\partial e_i} V = c'(e_i(\theta_i)),$$

where $\frac{\partial g(\theta_i, e_i(\theta_i))}{\partial e_i}$ is the partial derivative of $g(\theta_i, e_i(\theta_i))$ with respect to the second argument. In a symmetric equilibrium, we can drop the index i, thus the first-order condition in equilibrium can be written as

$$f^{(n-1)}(\theta) \frac{\partial g/\partial e}{z'(\theta)} V = c'(e(\theta)).$$

The above condition implicitly defines the symmetric equilibrium effort function $e(\theta)$. Note that since

$$z'(\theta) = \frac{dg(\theta, e(\theta))}{d\theta} = \frac{\partial g}{\partial \theta} + \frac{\partial g}{\partial e}\frac{de(\theta)}{d\theta},$$

we have that the first-order condition can be written as:

$$f^{(n-1)}(\theta)\frac{\partial g/\partial e}{\frac{\partial g}{\partial \theta} + \frac{\partial g}{\partial e}e'(\theta)}V = c'(e(\theta)). \tag{9}$$

Condition (9) has an intuitive interpretation. The LHS is the marginal probability of winning times the prize V in a symmetric equilibrium from the perspective of a player who knows that his or her skill is θ. Given that a player only has a marginal incentive to exert effort when the strongest opponent (the highest order statistic) has the same skill, $f^{(n-1)}(\theta)$ is the "likelihood" of this situation. There are two main differences with respect to the corresponding condition for the case of symmetric uncertainty (equation (4)). First, because players know their own skill level, there is no need to integrate over all possible realizations of a considered player's own skill. Second, instead of $a_e(x) = \frac{\partial g(x,e)}{\partial e} \Big/ \frac{\partial g(x,e)}{\partial x}$ (which appeared inside the integral of (4)), we now have the factor $\frac{\partial g/\partial e}{\frac{\partial g}{\partial \theta} + \frac{\partial g}{\partial e}e'(\theta)}$ which includes the new term $\frac{\partial g}{\partial e}e'(\theta)$ in the denominator. This new term arises because effort is a function of skill in the private information case.

Recall that when we discussed the intuition behind $a_e(x)$ in equation (4), we explained that the purpose of a marginal effort increase is to beat rivals who have marginally higher skill. In the current setting, the output advantage of marginally more able rivals is not only determined by $\frac{\partial g}{\partial \theta}$ (which is positive) but also by the additional term $\frac{\partial g}{\partial e}e'(\theta)$ which generally has an ambiguous sign. If $\frac{\partial g}{\partial e}$ and $e'(\theta)$ are both strictly positive, more highly skilled rivals are harder to beat not only because of their skill advantage, but also because they exert higher effort, reducing the marginal incentive to exert effort by any player.

In the following example, we compute the equilibrium effort for a specific skill distribution and production function.[25]

Example 7. *Consider a contest with n symmetric players with privately known skills independently drawn from the uniform distribution on* $[0,1]$. *The production function is given by* $g(\theta, e) = \theta e$*, and the cost function is* $c(e) = \frac{e^2}{2}$. *Then, the symmetric equilibrium*

[25]The detailed derivations for this example are provided in Appendix B.2.

effort is:

$$e(\theta) = \sqrt{\frac{2(n-1)}{n+1} V\theta^{n-1}}.$$

Notice that for the contest in the above example, $r_e(x)$ is strictly increasing on $[0,1]$ for all $e \geq 0$. Hence, equilibrium effort in the symmetric uncertainty case would be *increasing* in n according to Proposition 7. To obtain an analogue of this result in the case of private information, we can compute the expectation of the equilibrium effort in Example 7 to obtain:

$$E[e(\theta)] = \sqrt{(n-1)\left(\frac{2}{n+1}\right)^3 V}. \tag{10}$$

We immediately see that the expected effort in (10) is *decreasing* in n. Hence, Example 7 serves to demonstrate that the comparative statics results from the baseline case with symmetric uncertainty do not necessarily carry over to the private-information case.

8 Concluding Remarks

We have explored simple equilibria in contests between heterogeneous players. Under general assumptions about the production technology and the skill distributions of the competing players, we have shown that the contest has a symmetric equilibrium in which all players exert the same effort. We have also provided intuition regarding how the different components of the contest interact to determine the incentive to exert effort and revisited several important comparative statics results of contest theory, showing that standard results in the literature are not necessarily robust to generalizations of the production technology or skill distributions. In particular, we have found that making players more heterogeneous can increase the incentive to exert effort. We have also investigated the robustness of our results with respect to the assumption of symmetric uncertainty and the number of players.

We would like to mention a few broader implications of our analysis. First, our main result regarding the emergence of symmetric equilibria in the presence of heterogeneous players is quite surprising, and an important message of our paper is that differences between people do not necessarily translate into different behavior in contest situations. Second, our finding that making skill distributions more heterogeneous can increase equilibrium effort, has implications for optimal team composition, as employers could find it desirable to increase the diversity of the workforce by hiring a worker drawn from a more uncertain skill distribution, such as a minority worker, or a worker for whom less prior information is available.

There are several possible extensions to our analysis. For instance, prior work has investigated strategic information revelation by the tournament designer (e.g., Aoyagi 2010). If the tournament designer possesses some private information about the players' skills, he or she may decide to reveal some or all of this information to trigger higher effort. Another extension would be to consider an endogenous prize structure. Finally, the implications of our model for promotion tournaments and hiring decisions could be further explored. We leave these interesting topics as avenues for future research.

Appendix

A Proofs

A.1 Proof of Lemma 1

Suppose that $\frac{\partial P_i(e_i,e_k)}{\partial e_i}|_{e_i=e_k=e}$ is the same for both $i \in \{1,2\}$ and all $e \in int\, E$. Then we have $\frac{\partial P_1(e_1,e_2)}{\partial e_1}|_{e_1=e_2=e} V - c'(e) = \frac{\partial P_2(e_2,e_1)}{\partial e_2}|_{e_1=e_2=e} V - c'(e)$ for all $e \in int\, E$. Since $\pi_i(e_i,e_k)$ is continuously differentiable, $\frac{\partial P_i(e_i,e_k)}{\partial e_i}|_{e_i=e_k=e} V - c'(e)$ is a continuous function of e. Furthermore, recall that there exist $\bar{e}_i, \breve{e}_i \in int\, E$ such that $\frac{\partial \pi_i(e_i,e_k)}{\partial e_i}|_{e_i=e_k=\bar{e}_i} < 0$ and $\frac{\partial \pi_i(e_i,e_k)}{\partial e_i}|_{e_i=e_k=\breve{e}_i} > 0$. Hence, by the Intermediate Value Theorem, there is some $e^* \in int\ E$ such that $\frac{\partial P_i(e_i,e_k)}{\partial e_i}|_{e_i=e_k=e^*} V - c'(e^*) = 0$. By Assumption 1, $e_1 = e_2 = e^*$ is a Nash equilibrium.

A.2 Proof of Theorem 1

Since we wish to apply the sufficient condition from Lemma 1, we restrict attention to $e_i > 0$. Then, the function $g_e : \mathbb{R} \to \mathbb{R}$ defined by $g_e(x) = g(x,e)$ is strictly increasing and, thus, invertible. The inverse, g_e^{-1}, is strictly increasing as well. For the two (different) players $i,k \in \{1,2\}$, we observe

$$\begin{aligned} & g(\theta_i, e_i) < g(\theta_k, e_k) \\ \Leftrightarrow\ & g_{e_i}(\theta_i) < g_{e_k}(\theta_k) \\ \Leftrightarrow\ & \theta_i < g_{e_i}^{-1}\left(g_{e_k}(\theta_k)\right). \end{aligned}$$

Player k thus wins with probability

$$\int F_i\left(g_{e_i}^{-1}\left(g_{e_k}(x)\right)\right) f_k(x)\,dx.$$

Differentiating with respect to e_k, we obtain

$$\int f_i\left(g_{e_i}^{-1}\left(g_{e_k}(x)\right)\right)\left(\frac{d}{de_k} g_{e_i}^{-1}\left(g_{e_k}(x)\right)\right) f_k(x)\,dx.$$

According to Lemma 1, and noting that $g_{e_k}^{-1}(g_{e_i}(x)) = g_{e_i}^{-1}(g_{e_k}(x)) = x$ if $e_i = e_k$, a sufficient condition for a symmetric equilibrium to exist is that

$$\begin{aligned} & \int \left(\frac{d}{de_1} g_{e_2}^{-1}\left(g_{e_1}(x)\right)\Big|_{e_1=e_2=e}\right) f_1(x) f_2(x)\,dx \\ = & \int \left(\frac{d}{de_2} g_{e_1}^{-1}\left(g_{e_2}(x)\right)\Big|_{e_1=e_2=e}\right) f_1(x) f_2(x)\,dx, \end{aligned}$$

for all $e \in int\ E$. Since $\frac{d}{de_1} g_{e_2}^{-1}\left(g_{e_1}(x)\right)\Big|_{e_1=e_2=e} = \frac{d}{de_2} g_{e_1}^{-1}\left(g_{e_2}(x)\right)\Big|_{e_1=e_2=e}$, this condition is always fulfilled.

A.3 Proof of Corollary 1

Define $\tilde{\Theta}_i := h_i(\Theta_i)$. The considered contest is then equivalent (in terms of equilibrium effort choices) to a contest in which skills are given by $\tilde{\Theta}_i$ and the production function $\tilde{g}\left(\tilde{\theta}_i, e_i\right)$ is the same for both players. Hence, a symmetric equilibrium exists with effort determined by the condition provided in the corollary.

A.4 Proof of Proposition 1

Player 1's objective function is given by

$$\int_{\mathbb{R}} F_2\left(g_{e_2}^{-1}\left(g_{e_1}(x)\right)\right) f_1(x)\, dx sV - c(e_1).$$

Using $\xi_1 = e_1/s^{1/\delta}$, the preceding expression becomes

$$\begin{aligned} & \int_{\mathbb{R}} F_2\left(g_{e_2}^{-1}\left(g_{s^{1/\delta}\xi_1}(x)\right)\right) f_1(x)\, dx sV - c\left(s^{1/\delta}\xi_1\right) \\ = & \int_{\mathbb{R}} F_2\left(g_{e_2}^{-1}\left(g_{s^{1/\delta}\xi_1}(x)\right)\right) f_1(x)\, dx sV - sc(\xi_1) \\ = & \ s\left(\int_{\mathbb{R}} F_2\left(g_{e_2}^{-1}\left(g_{s^{1/\delta}\xi_1}(x)\right)\right) f_1(x)\, dx V - c(\xi_1)\right). \end{aligned}$$

Maximizing this function is equivalent to maximizing

$$\int_{\mathbb{R}} F_2\left(g_{e_2}^{-1}\left(g_{s^{1/\delta}\xi_1}(x)\right)\right) f_1(x)\, dx V - c(\xi_1).$$

Player 2's objective function can be stated as

$$\begin{aligned} & \int_{\mathbb{R}} F_1\left(g_{e_1}^{-1}\left(g_{e_2}(x)\right)\right) f_2(x)\, dx V - c(e_2) \\ = & \int_{\mathbb{R}} F_1\left(g_{s^{1/\delta}\xi_1}^{-1}\left(g_{e_2}(x)\right)\right) f_2(x)\, dx V - c(e_2). \end{aligned}$$

A.5 Proof of Proposition 2

(i) Player 1 wins if and only if

$$\begin{aligned} & \theta_1^{\alpha} e_1^{\beta} > \theta_2^{\alpha} e_2^{\beta} \\ \Leftrightarrow \quad & \theta_1^{\frac{\alpha}{\beta}} e_1 > \theta_2^{\frac{\alpha}{\beta}} e_2. \end{aligned}$$

Substituting $\xi_1 = e_1/s^{1/\delta}$, the condition becomes

$$\theta_1^{\frac{\alpha}{\beta}} s^{\frac{1}{\delta}} \xi_1 > \theta_2^{\frac{\alpha}{\beta}} e_2.$$

Now define $\tilde{\Theta}_1 := s^{\frac{1}{\delta}} \Theta_1^{\frac{\alpha}{\beta}}$ and $\tilde{\Theta}_2 := \Theta_2^{\frac{\alpha}{\beta}}$, and denote the corresponding pdfs and cdfs by $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{F}_1$, and $\tilde{F}_2$, respectively.

Player 1's objective function can be stated as

$$\begin{aligned} & \int_{\mathbb{R}} \tilde{F}_2\left(\frac{\xi_1 x}{e_2}\right) \tilde{f}_1(x)\, dx s V - c\left(s^{1/\delta} \xi_1\right) \\ = \quad & s\left(\int_{\mathbb{R}} \tilde{F}_2\left(\frac{\xi_1 x}{e_2}\right) \tilde{f}_1(x)\, dx V - c(\xi_1)\right). \end{aligned}$$

Maximization of the objective function is equivalent to maximization of

$$\int_{\mathbb{R}} \tilde{F}_2\left(\frac{\xi_1 x}{e_2}\right) \tilde{f}_1(x)\, dx V - c(\xi_1),$$

so we consider this latter problem. Since player 2's objective function can be stated as

$$\int_{\mathbb{R}} \tilde{F}_1\left(\frac{e_2 x}{\xi_1}\right) \tilde{f}_2(x)\, dx V - c(e_2),$$

we have transformed the contest into the form of our main model, meaning that an equilibrium $\left(e_1^*, e_2^*\right)$, where $e_2^* = \xi_1^* = \frac{e_1^*}{s^{\frac{1}{\delta}}}$ is characterized by

$$\int_{\mathbb{R}} \tilde{f}_1(x) \tilde{f}_2(x) x\, dx V = e_2^* c'\left(e_2^*\right),$$

exists.

(ii) Notice that player 1 wins if and only if

$$\begin{aligned} & \alpha\theta_1 + \beta e_1 > \alpha\theta_2 + \beta e_2 \\ \Leftrightarrow \quad & \frac{\alpha}{\beta}\theta_1 + e_1 > \frac{\alpha}{\beta}\theta_2 + e_2 \\ \Leftrightarrow \quad & \exp\left(\frac{\alpha}{\beta}\theta_1 + e_1\right) > \exp\left(\frac{\alpha}{\beta}\theta_2 + e_2\right) \\ \Leftrightarrow \quad & \exp\left(\frac{\alpha}{\beta}\theta_1\right) \exp(e_1) > \exp\left(\frac{\alpha}{\beta}\theta_2\right) \exp(e_2). \end{aligned}$$

Define $\tilde{e}_i = \exp(e_i)$ for $i \in \{1,2\}$. From part (i) of the proposition, we know that, in equilibrium, $\tilde{e}_1 = s^{1/\delta}\tilde{e}_2$. (Note that this is equivalent to $e_i = \ln(\tilde{e}_i)$ and $\exp(e_1) = s^{1/\delta}\exp(e_2) \Leftrightarrow$

$e_1 = \frac{1}{\delta}\ln s + e_2$.) Hence, defining $\xi_1 = \frac{\tilde{e}_1}{s^{1/\delta}}$, the event of player 1 winning can be restated as

$$\exp\left(\frac{\alpha}{\beta}\theta_1\right)s^{1/\delta}\xi_1 > \exp\left(\frac{\alpha}{\beta}\theta_2\right)\tilde{e}_2.$$

Now define $\tilde{\Theta}_1 := \exp\left(\frac{\alpha}{\beta}\Theta_1\right)s^{1/\delta}$ and $\tilde{\Theta}_2 := \exp\left(\frac{\alpha}{\beta}\Theta_2\right)$, and denote the corresponding pdfs and cdfs by $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{F}_1$, and $\tilde{F}_2$, respectively.

Player 1's objective function can then be stated as

$$\begin{aligned}
& \int_{\mathbb{R}} \tilde{F}_2\left(\frac{\xi_1 x}{\tilde{e}_2}\right)\tilde{f}_1(x)\,dx\,sV - c\left(\ln\left(s^{\frac{1}{\delta}}\xi_1\right)\right) \\
= & \int_{\mathbb{R}} \tilde{F}_2\left(\frac{\xi_1 x}{\tilde{e}_2}\right)\tilde{f}_1(x)\,dx\,sV - sc(\ln(\xi_1)) \\
= & \; s\left(\int_{\mathbb{R}} \tilde{F}_2\left(\frac{\xi_1 x}{\tilde{e}_2}\right)\tilde{f}_1(x)\,dx\,V - \tilde{c}(\xi_1)\right),
\end{aligned}$$

where the first transformation used the homogeneity of the function $c \circ \ln(e)$. Maximizing the objective is equivalent to maximizing

$$\int_{\mathbb{R}} \tilde{F}_2\left(\frac{\xi_1 x}{\tilde{e}_2}\right)\tilde{f}_1(x)\,dx\,V - \tilde{c}(\xi_1)$$

and we consider this latter problem in what follows. Since player 2's objective function can be stated as

$$\begin{aligned}
& \int_{\mathbb{R}} \tilde{F}_1\left(\frac{\tilde{e}_2 x}{\xi_1}\right)\tilde{f}_2(x)\,dx\,V - c(\ln(\tilde{e}_2)) \\
= & \int_{\mathbb{R}} \tilde{F}_1\left(\frac{\tilde{e}_2 x}{\xi_1}\right)\tilde{f}_2(x)\,dx\,V - \tilde{c}(\tilde{e}_2),
\end{aligned}$$

we have transformed the contest into the form of our main model, meaning that an equilibrium with $\xi_1^* = \tilde{e}_2^*$ characterized by

$$\int_{\mathbb{R}} \tilde{f}_1(x)\tilde{f}_2(x)x\,dx\,V = \tilde{e}_2^*\tilde{c}'\left(\tilde{e}_2^*\right)$$

exists. The optimal efforts of the original contest are given by $e_1^* = \ln\left(s^{\frac{1}{\delta}}\xi_1^*\right)$ and $e_2^* = \ln\left(\tilde{e}_2^*\right)$.

A.6 Proof of Proposition 3

Suppose that Assumption 2 holds, and consider case (i), i.e., $r_{e,i}(x)$ is monotonically increasing in x, and $\tilde{F}_k$ first-order stochastically dominates F_k. Denote the equilibrium effort levels for the two contests by $\tilde{e}^*$ and e^*, respectively. Our goal is to show that

$\tilde{e}^* > e^*$.

The proof proceeds by contradiction, so suppose $\tilde{e}^* \leq e^*$. Now observe that

$$\begin{aligned}
&V \int r_{\tilde{e}^*,i}(x)\tilde{f}_k(x)dx - c'\left(\tilde{e}^*\right) \geq \\
&V \int r_{e^*,i}(x)\tilde{f}_k(x)dx - c'\left(e^*\right) > \\
&V \int r_{e^*,i}(x)f_k(x)dx - c'\left(e^*\right) = 0.
\end{aligned}$$

The first inequality follows from $\tilde{e}^* \leq e^*$ together with Assumption 2. The second inequality follows from $r_{e,i}(x)$ being monotonically increasing on supp(f_i), $\tilde{F}_k$ first-order stochastically dominating F_k, and the fact that we have assumed that both supp($\tilde{f}_k$) and supp(f_k) are subsets of supp(f_i).[26] The equality follows since e^* is characterized by the first-order condition $V \int r_{e^*,i}(x)f_k(x)dx - c'(e^*) = 0$. We conclude that

$$V \int r_{\tilde{e}^*,i}(x)\tilde{f}_k(x)dx - c'\left(\tilde{e}^*\right) > 0.$$

This shows that the first-order condition for equilibrium effort cannot be fulfilled in the case of the distribution $\tilde{F}_k$, giving us the desired contradiction.

By an analogous argument, we can show that $\tilde{e}^* > e^*$ also in case (ii) where $r_{e,i}$ is monotonically decreasing in x for all $e > 0$ and F_k first-order stochastically dominates $\tilde{F}_k$. In this case, $\int r_{e,i}(x)\tilde{f}_k(x)dx > \int r_{e,i}(x)f_k(x)dx$ for all $e > 0$ (see, e.g., Levy 1992, p.557).

A.7 Proof of Proposition 4

Suppose that $g(\theta, e) = \theta e$. This means that

$$a_e(x) = \frac{d}{de_i}\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right)\bigg|_{e_i=e_k=e} = \frac{d}{de_i}\left(\frac{xe_i}{e_k}\right)\bigg|_{e_i=e_k=e} = \frac{x}{e}.$$

In the considered situation, the marginal winning probability is

$$\frac{1}{2\pi\sigma_1\sigma_2}\int \frac{x}{e}\exp\left(-\frac{\left(x-\mu_1\right)^2}{2\sigma_1^2} - \frac{\left(x-\mu_2\right)^2}{2\sigma_2^2}\right)dx.$$

[26] See, e.g., Levy 1992, p.557. Notice that, in decision theory, the utility function is defined for all possible payoffs and therefore no additional constraints regarding the statistical supports need to be imposed.

To prove the proposition, it is sufficient to show that

$$\frac{1}{2\pi\sigma_1\sigma_2}\int x\exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}-\frac{(x-\mu_2)^2}{2\sigma_2^2}\right)dx$$
$$=\frac{\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)\exp\left(-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right)}{(2\pi)^{\frac{1}{2}}\left(\sigma_1^2+\sigma_2^2\right)^{\frac{3}{2}}}.$$

Define

$$Z:=\frac{1}{2\pi\sigma_1\sigma_2}\int x\exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}-\frac{(x-\mu_2)^2}{2\sigma_2^2}\right)dx$$

and notice that:

$$\begin{aligned}
&\frac{(x-\mu_1)^2}{2\sigma_1^2}+\frac{(x-\mu_2)^2}{2\sigma_2^2}\\
=\;&\frac{\sigma_2^2(x-\mu_1)^2\left(\sigma_1^2+\sigma_2^2\right)+\sigma_1^2(x-\mu_2)^2\left(\sigma_1^2+\sigma_2^2\right)}{2\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)}\\
=\;&\frac{\sigma_2^2\left(x^2-2x\mu_1+\mu_1^2\right)\left(\sigma_1^2+\sigma_2^2\right)+\sigma_1^2\left(x^2-2x\mu_2+\mu_2^2\right)\left(\sigma_1^2+\sigma_2^2\right)}{2\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)}\\
=\;&\frac{x^2\left(\sigma_1^2+\sigma_2^2\right)^2-2x\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)\left(\sigma_1^2+\sigma_2^2\right)+\left(\mu_1^2\sigma_2^2+\mu_2^2\sigma_1^2\right)\left(\sigma_1^2+\sigma_2^2\right)}{2\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)}\\
=\;&\frac{x^2\left(\sigma_1^2+\sigma_2^2\right)^2-2x\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)\left(\sigma_1^2+\sigma_2^2\right)+\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)^2}{2\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)}\\
&-\frac{\left(\mu_1^2\sigma_2^4+2\mu_1\sigma_2^2\mu_2\sigma_1^2+\mu_2^2\sigma_1^4-\mu_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)-\mu_2^2\sigma_1^2\left(\sigma_1^2+\sigma_2^2\right)\right)}{2\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)}\\
=\;&\frac{\left(x\left(\sigma_1^2+\sigma_2^2\right)-\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)\right)^2+\left(\mu_1^2\sigma_1^2\sigma_2^2-2\mu_1\sigma_2^2\mu_2\sigma_1^2+\mu_2^2\sigma_1^2\sigma_2^2\right)}{2\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)}\\
=\;&\frac{\left(x\left(\sigma_1^2+\sigma_2^2\right)-\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)\right)^2}{2\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)}+\frac{(\mu_1-\mu_2)^2}{2\left(\sigma_1^2+\sigma_2^2\right)}.
\end{aligned}$$

Using this, we obtain

$$Z=\frac{\exp\left(-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right)}{2\pi\sigma_1\sigma_2}\int x\exp\left(-\frac{\left(x\left(\sigma_1^2+\sigma_2^2\right)-\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)\right)^2}{2\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)}\right)dx$$
$$=\frac{\exp\left(-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right)}{(2\pi)^{\frac{1}{2}}\left(\sigma_1^2+\sigma_2^2\right)^{\frac{3}{2}}}\left(\sigma_1^2+\sigma_2^2\right)\frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}}\int x\exp\left(-\frac{\left(x-\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2}\right)dx.$$

Now notice that

$$\frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}}\int x\exp\left(-\frac{\left(x-\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2}\right)dx$$

describes the mean of a normally distributed random variable with variance $\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2$ and mean $\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}$, hence

$$\frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}}\int x\exp\left(-\frac{\left(x-\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2}\right)dx=\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}.$$

We obtain

$$\begin{aligned}Z &= \frac{\exp\left(-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right)}{(2\pi)^{0.5}\left(\sigma_1^2+\sigma_2^2\right)^{1.5}}\left(\sigma_1^2+\sigma_2^2\right)\frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}\\ &= \frac{\left(\mu_1\sigma_2^2+\mu_2\sigma_1^2\right)\exp\left(-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right)}{(2\pi)^{\frac{1}{2}}\left(\sigma_1^2+\sigma_2^2\right)^{\frac{3}{2}}}.\end{aligned}$$

A.8 Proof of Proposition 5

Because of Assumption 2, and the condition characterizing equilibrium effort, we need to show that $\int r_{e,i}(x)\tilde{f}_k(x)dx > (=,<) \int r_{e,i}(x)f_k(x)dx$ if $r_{e,i}$ is convex (linear, concave). The proof is very similar to part a) of the proof of Theorem 2 in Rothschild and Stiglitz (1970, p.237). In the case of convex $r_{e,i}$, the inequality in their proof is reversed, while it is replaced by an equality if $r_{e,i}$ is linear.

A.9 Proof of Proposition 6

Player i wins the contest with probability

$$\int\prod_{k\neq i}F_k\left(g_{e_k}^{-1}\left(g_{e_i}(x)\right)\right)f_i(x)dx.$$

Differentiating with respect to e_i, we obtain

$$\int \left(\prod_{k \neq i} F_k \left(g_{e_k}^{-1} \left(g_{e_i}(x) \right) \right) \right) \left(\sum_{k \neq i} \frac{f_k \left(g_{e_k}^{-1} \left(g_{e_i}(x) \right) \right) \left(\frac{d}{de_i} g_{e_k}^{-1} \left(g_{e_i}(x) \right) \right)}{F_k \left(g_{e_k}^{-1} \left(g_{e_i}(x) \right) \right)} \right) f_i(x) dx.$$

In a symmetric equilibrium with $e_1^* = ... = e_n^* =: e^*$, and symmetric skill distributions, this marginal effect of effort on the probability of winning simplifies to

$$\int \left(\prod_{k \neq i} F(x) \right) \left(\sum_{k \neq i} \left(\frac{d}{de_i} g_{e_k}^{-1} \left(g_{e_i}(x) \right) \right) \bigg|_{e_1^* = ... = e_n^* = e^*} \frac{f(x)}{F(x)} \right) f(x) dx,$$

and must be identical for all i. We can restate the above expression as

$$\int r_{e^*}(x)(n-1)(F(x))^{n-2} f(x) dx = \int r_{e^*}(x) \left(\frac{d}{dx} (F(x))^{n-1} \right) dx,$$

which is identical for all i.

A.10 Proof of Proposition 7

Part i) As explained in the main body of the paper, the equilibrium first-order condition for an n-player contest is equivalent to that of a two-player contest in which the second player's skill distribution is replaced by the strongest rival's skill distribution (the highest order statistic) of the n-player contest. We show that $\int r_e(x) \left(\frac{d}{dx} (F(x))^{n-1} \right) dx$ is increasing in n. If $n_1, n_2 \in \mathbb{N}$, with $n_1 > n_2$, then $(F(x))^{n_1 - 1}$ first-order stochastically dominates $(F(x))^{n_2 - 1}$, and the result follows from Proposition 3.

Part ii) Suppose that $r_e(x)$ is monotonically decreasing in x for all $e \geq 0$, and let $n_1, n_2 \in \mathbb{N}$, with $n_1 > n_2$. It follows that $(F(x))^{n_1 - 1}$ first-order stochastically dominates $(F(x))^{n_2 - 1}$, as just mentioned, implying that

$$\int r_e(x) \left(\frac{d}{dx} (F(x))^{n_1 - 1} \right) dx < \int r_e(x) \left(\frac{d}{dx} (F(x))^{n_2 - 1} \right) dx.$$

Part iii) If $r_e(x) = r_e$ is constant in x for all $e \geq 0$, we have

$$\int r_e(x) \left(\frac{d}{dx} (F(x))^{n-1} \right) dx = r_e \int \left(\frac{d}{dx} (F(x))^{n-1} \right) dx = r_e,$$

which is independent of n.

A.11 Proof of Proposition 8

As shown before, if $g(\theta_i, e_i) = \theta_i e_i$, we have

$$\left(\frac{d}{de_i} g_{e_k}^{-1}\left(g_{e_i}(t_i + x)\right)\right)\Big|_{e_1 = \ldots = e_n = e} = \frac{(t_i + x)}{e}.$$

Thus, making use of expression (11), derived in Section B.1, and denoting $\Delta t = t_1 - t > 0$,

$$\begin{aligned}
& \lambda^2 \int^{-\Delta t} \left(H(x) \prod_{k \neq 1} H(\Delta t + x)\right) \left(\sum_{k \neq i} \left(\frac{d}{de_i} g_{e_k}^{-1}\left(g_{e_i}(t_1 + x)\right)\right)\Big|_{e_1 = \ldots = e_n = e}\right) dx \\
= \; & \lambda^2 \int^{-\Delta t} (\exp(\lambda x) \cdot \exp((n-1)\lambda(\Delta t + x)))(n-1) \frac{(t_1 + x)}{e} dx \\
= \; & \frac{\lambda^2 (n-1)}{e} \int^{-\Delta t} \exp(n\lambda y + (n-1)\lambda \Delta t)(t_1 + y) dy.
\end{aligned}$$

The map $\phi_2 : \mathbb{R}_x \to \mathbb{R}_y$ given by $x \to y = -\Delta t + x$ is a smooth diffeomorphism with $\det |\phi_2'(x)| = 1$. Applying the associated change of variables to the integral, we obtain

$$\begin{aligned}
& \frac{\lambda^2 (n-1)}{e} \int^{0} \exp(n\lambda(x - \Delta t) + (n-1)\lambda \Delta t)(t + x) dx \\
= \; & \frac{(n-1)}{e} \exp(-\lambda \Delta t) \lambda^2 \left(\int^0 x \exp(n\lambda x) dx + t \int^0 \exp(n\lambda x) dx\right).
\end{aligned}$$

Notice that

$$n\lambda \int^0 x \exp(n\lambda x) dx$$

is the mean of a random variable that is distributed according to the reflected exponential distribution with parameter $n\lambda$, hence

$$\begin{aligned}
& n\lambda \int^0 x \exp(n\lambda x) dx = -\frac{1}{n\lambda} \\
\Leftrightarrow \; & \int^0 x \exp(n\lambda x) dx = -\frac{1}{n^2 \lambda^2}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& n\lambda \int^0 \exp(n\lambda x) dx = 1 \\
\Leftrightarrow \; & \int^0 \exp(n\lambda x) dx = \frac{1}{n\lambda}.
\end{aligned}$$

It follows that

$$\begin{aligned}
& \frac{(n-1)}{e}\exp(-\lambda\Delta t)\lambda^2\left(\int^0 x\exp(n\lambda x)\,dx + t\int^0 \exp(n\lambda x)\,dx\right) \\
= \; & \frac{(n-1)}{e}\exp(-\lambda\Delta t)\lambda^2\frac{(-1+n\lambda t)}{n^2\lambda^2} \\
= \; & \frac{(n-1)}{e}\exp(-\lambda\Delta t)\frac{(-1+n\lambda t)}{n^2}.
\end{aligned}$$

Notice that the last expression is positive if and only if $n\lambda t - 1 > 0$. Taking the derivative of the expression w.r.t. n results in an expression that is positive if $t > 0$, which is implied by $n\lambda t - 1 > 0$.

B Other Computations and Derivations

B.1 Additional Derivations for Section 7.1.3.

Player i outperforms player k iff

$$\begin{aligned}
& g_{e_i}(t_i+\varepsilon_i) > g_{e_k}(t_k+\varepsilon_k) \\
\Leftrightarrow \quad & \varepsilon_k < g_{e_k}^{-1}\left(g_{e_i}(t_i+\varepsilon_i)\right) - t_k.
\end{aligned}$$

Recall that the $\mathscr{E}_i$ are i.i.d., following the reflected exponential distribution on $(-\infty, 0]$. The cdf is denoted by H and the pdf by h. Hence, player i wins the contest with probability

$$\int \prod_{k\neq i} H\left(g_{e_k}^{-1}\left(g_{e_i}(t_i+x)\right) - t_k\right) h(x)\,dx.$$

In a symmetric equilibrium with $e_1^* = ... = e_n^* =: e^*$, the marginal effect of effort on the probability of winning,

$$\int \left(\prod_{k\neq i} H(t_i+x-t_k)\right)\left(\sum_{k\neq i}\left(\frac{d}{de_i}g_{e_k}^{-1}\left(g_{e_i}(t_i+x)\right)\right)\bigg|_{e_1^*=...=e_n^*=e^*}\frac{h(t_i+x-t_k)}{H(t_i+x-t_k)}\right)h(x)\,dx,$$

must be the same for all i. Denote $\Delta t = t_1 - t > 0$. For player 1, we have,

$$\int \left(\prod_{k\neq 1} H(\Delta t+x)\right)\left(\sum_{k\neq 1}\left(\frac{d}{de_1}g_{e_k}^{-1}\left(g_{e_1}(t_1+x)\right)\right)\bigg|_{e_1^*=...=e_n^*=e^*}\frac{h(\Delta t+x)}{H(\Delta t+x)}\right)h(x)\,dx.$$

For any other player $i \in \{2, \ldots, n\}$, we have

$$\int \left(H(-\Delta t + y) \prod_{k \neq 1,i} H(y) \right) \left(\left(\frac{d}{de_i} g_{e_1}^{-1} \left(g_{e_i}(t+y) \right) \right) \Big|_{e_1^* = \ldots = e_n^* = e^*} \frac{h(-\Delta t + y)}{H(-\Delta t + y)} \right.$$
$$\left. + \sum_{k \neq 1,i} \left(\frac{d}{de_i} g_{e_k}^{-1} \left(g_{e_i}(t+y) \right) \right) \Big|_{e_1^* = \ldots = e_n^* = e^*} \frac{h(y)}{H(y)} \right) h(y) dy.$$

The map $\phi_1 : \mathbb{R}_x \to \mathbb{R}_y$ given by $x \to y = \Delta t + x$ is a smooth diffeomorphism with $\det |\phi_1'(x)| = 1$. Applying the associated change of variables to the preceding expression, we obtain

$$\int \left(H(x) \prod_{k \neq 1,i} H(\Delta t + x) \right) \left(\left(\frac{d}{de_i} g_{e_1}^{-1} \left(g_{e_i}(t_1 + x) \right) \right) \Big|_{e_1^* = \ldots = e_n^* = e^*} \frac{h(x)}{H(x)} \right.$$
$$\left. + \sum_{k \neq 1,i} \left(\frac{d}{de_i} g_{e_k}^{-1} \left(g_{e_i}(t_1 + x) \right) \right) \Big|_{e_1^* = \ldots = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} \right) h(\Delta t + x) dx.$$

The expressions for the two types of players can be restated as

$$\int \left(H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left(\sum_{k \neq 1} \left(\frac{d}{de_1} g_{e_k}^{-1} \left(g_{e_1}(t_1 + x) \right) \right) \Big|_{e_1^* = \ldots = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} \right) \frac{h(x)}{H(x)} dx,$$
$$\int \left(H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left(\left(\frac{d}{de_i} g_{e_1}^{-1} \left(g_{e_i}(t_1 + x) \right) \right) \Big|_{e_1^* = \ldots = e_n^* = e^*} \frac{h(x)}{H(x)} \right.$$
$$\left. + \sum_{k \neq 1,i} \left(\frac{d}{de_i} g_{e_k}^{-1} \left(g_{e_i}(t_1 + x) \right) \right) \Big|_{e_1^* = \ldots = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} \right) \frac{h(\Delta t + x)}{H(\Delta t + x)} dx.$$

Notice that both expressions are equal to zero for $x \geq -\Delta t$. Hence, they can be restated as

$$\int^{-\Delta t} \left(H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left(\sum_{k \neq 1} \left(\frac{d}{de_1} g_{e_k}^{-1} \left(g_{e_1}(t_1 + x) \right) \right) \Big|_{e_1^* = \ldots = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} \right) \frac{h(x)}{H(x)} dx,$$
$$\int^{-\Delta t} \left(H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left(\left(\frac{d}{de_i} g_{e_1}^{-1} \left(g_{e_i}(t_1 + x) \right) \right) \Big|_{e_1^* = \ldots = e_n^* = e^*} \frac{h(x)}{H(x)} \right.$$
$$\left. + \sum_{k \neq 1,i} \left(\frac{d}{de_i} g_{e_k}^{-1} \left(g_{e_i}(t_1 + x) \right) \right) \Big|_{e_1^* = \ldots = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} \right) \frac{h(\Delta t + x)}{H(\Delta t + x)} dx.$$

For $x < -\Delta t$, we observe $\frac{h(x)}{H(x)} = \frac{h(\Delta t + x)}{H(\Delta t + x)} = \lambda$, and the expressions become

$$\lambda^2 \int^{-\Delta t} \left(H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left(\sum_{k \neq 1} \left(\frac{d}{de_1} g_{e_k}^{-1} \left(g_{e_1}(t_1 + x) \right) \right) \Big|_{e_1^* = \ldots = e_n^* = e^*} \right) dx,$$
$$\lambda^2 \int^{-\Delta t} \left(H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left(\sum_{k \neq i} \left(\frac{d}{de_i} g_{e_k}^{-1} \left(g_{e_i}(t_1 + x) \right) \right) \Big|_{e_1^* = \ldots = e_n^* = e^*} \right) dx \tag{11}$$

which are identical.

B.2 Computations for Example 7

The first-order condition (9) is equivalent to (we ease notation by writing e instead of $e(\theta)$)

$$c'(e)\frac{\partial g/\partial\theta}{\partial g/\partial e} - f^{(n-1)}(\theta)V + c'(e)\frac{de}{d\theta} = 0, \tag{12}$$

which can be restated as

$$P(\theta,e) + Q(\theta,e)\frac{de}{d\theta} = 0,$$

with $P(\theta,e) := c'(e)\frac{\partial g/\partial\theta}{\partial g/\partial e} - f^{(n-1)}(\theta)V$ and $Q(\theta,e) := c'(e)$.

Is there an integrating factor $\mu(\theta,e)$ such that $\frac{\partial(\mu P)}{\partial e} = \frac{\partial(\mu Q)}{\partial\theta}$? In other words, is there $\mu(\theta,e)$ such that

$$\frac{\partial\mu}{\partial e}P + \mu\frac{\partial P}{\partial e} = \frac{\partial\mu}{\partial\theta}Q + \mu\frac{\partial Q}{\partial\theta}?$$

The latter equation can be stated as

$$\begin{aligned}&\frac{\partial\mu}{\partial e}\left(c'(e)\frac{\partial g/\partial\theta}{\partial g/\partial e} - f^{(n-1)}(\theta)V\right)\\ &+\mu\left(c''(e)\frac{\partial g/\partial\theta}{\partial g/\partial e} + c'(e)\frac{\partial^2 g/\partial\theta\partial e\cdot\partial g/\partial e - \partial g/\partial\theta\cdot\partial^2 g/\partial e^2}{(\partial g/\partial e)^2}\right) = \frac{\partial\mu}{\partial\theta}c'(e).\end{aligned}$$

Now, for our example, assume $g(\theta,e) = \theta e$ and $c(e) = 0.5e^2$, and ignore the argument θ in $e(\theta)$. Then the equation simplifies to

$$\frac{\partial\mu}{\partial e}\left(\frac{e^2}{\theta} - f^{(n-1)}(\theta)V\right) + \mu\left(2\frac{e}{\theta}\right) = \frac{\partial\mu}{\partial\theta}e.$$

Suppose that $\frac{\partial\mu}{\partial e} = 0$. Then μ needs to satisfy

$$\mu\frac{2}{\theta} = \frac{\partial\mu}{\partial\theta}$$

and a solution is $\mu(\theta,e) = \theta^2$ (confirming $\frac{\partial\mu}{\partial e} = 0$).

Using $g(\theta,e) = \theta e$ and $c(e) = 0.5e^2$, our differential equation (12) can be stated as

$$\frac{e^2}{\theta} - f^{(n-1)}(\theta)V + e\frac{de}{d\theta} = 0,$$

and multiplication with $\mu(\theta,e)=\theta^2$ leads to

$$\theta e^2-\theta^2 f^{(n-1)}(\theta)V+e\theta^2\frac{de}{d\theta}=0.$$

An integral is

$$L(\theta,e(\theta))=\frac{e(\theta)^2\theta^2}{2}-V\int_0^\theta x^2 f^{(n-1)}(x)\,dx,$$

which can easily be verified by computing $\frac{dL(\theta,e(\theta))}{d\theta}$.

With a general distribution, effort is given by the solution to

$$\begin{aligned}\frac{e(\theta)^2\theta^2}{2}-V\int_0^\theta x^2 f^{(n-1)}(x)\,dx&=\hat{c}\\ \Leftrightarrow\quad e(\theta)&=\sqrt{\frac{2V}{\theta^2}\int_0^\theta x^2 f^{(n-1)}(x)\,dx+\frac{2\hat{c}}{\theta^2}},\end{aligned}$$

where $\hat{c}$ is some constant.

Using the assumption that skills are uniformly distributed on $[0,1]$ (implying $f(x)=1$ and $F^{(n-1)}(t)=t^{n-1}\Rightarrow f^{(n-1)}(t)=(n-1)t^{n-2}$), we can compute effort and expected effort. In particular,

$$\int_0^\theta x^2 f^{(n-1)}(x)\,dx=(n-1)\int_0^\theta x^n\,dx=\frac{n-1}{n+1}\theta^{n+1},$$

meaning that the integral becomes

$$L(\theta,e)=\frac{e^2\theta^2}{2}-V\frac{n-1}{n+1}\theta^{n+1}.$$

Hence, the solution to the differential equation is given by

$$\frac{1}{2}e^2\theta^2-V\frac{n-1}{n+1}\theta^{n+1}=\hat{c},$$

where $\hat{c}$ is some constant. Solving for e, we obtain

$$e(\theta)=\sqrt{2V\frac{n-1}{n+1}\theta^{n-1}+\frac{2\hat{c}}{\theta^2}}.$$

Conjecturing $e(0)=0$, we have $\hat{c}=0$ and

$$e(\theta)=\sqrt{2V\frac{n-1}{n+1}\theta^{n-1}}.$$

It follows that expected effort is

$$E[e(\theta)] = \sqrt{2V\frac{n-1}{n+1}\int_0^1 x^{\frac{n-1}{2}}dx} = \sqrt{8V\frac{n-1}{(n+1)^3}},$$

which is strictly decreasing in n.

It is straightforward to verify that the equilibrium effort function satisfies $e(0) = 0$ and is strictly increasing in the skill θ. This implies that for any given skill θ, output $g(\theta, e(\theta)) = \theta e(\theta)$ is increasing in skill as well, and the inverse z^{-1} exists.

References

AGUINIS, H., AND E. O'BOYLE JR. (2014): "Star Performers in Twenty-First Century Organizations," *Personnel Psychology*, 67(2), 313–350.

AOYAGI, M. (2010): "Information feedback in a dynamic tournament," *Games and Economic Behavior*, 70(2), 242–260.

AURIOL, E., G. FRIEBEL, AND L. PECHLIVANOS (2002): "Career concerns in teams," *Journal of Labor Economics*, 20(2), 289–307.

BAR-ISAAC, H., AND R. LÉVY (2021): "Motivating employees through career paths," *Journal of Labor Economics (accepted)*.

BARBIERI, S., D. A. MALUEG, AND I. TOPOLYAN (2014): "The best-shot all-pay (group) auction with complete information," *Economic Theory*, 57(3), 603–640.

BARUT, Y., AND D. KOVENOCK (1998): "The symmetric multiple prize all-pay auction with complete information," *European Journal of Political Economy*, 14(4), 627–644.

BAYE, M. R., D. KOVENOCK, AND C. G. DE VRIES (1996): "The all-pay auction with complete information," *Economic Theory*, 8(2), 291–305.

BERNHARDT, D. (1995): "Strategic Promotion and Compensation," *The Review of Economic Studies*, 62(2), 315–339.

BJERK, D. (2008): "Glass Ceilings or Sticky Floors? Statistical Discrimination in a Dynamic Model of Hiring and Promotion," *The Economic Journal*, 118(530), 961–982.

BROWN, J. (2011): "Quitters never win: The (adverse) incentive effects of competing with superstars," *Journal of Political Economy*, 119(5), 982–1013.

CHEN, K.-P. (2003): "Sabotage in promotion tournaments," *Journal of Law, Economics, and Organization*, 19(1), 119–140.

CHOWDHURY, S., P. ESTEVE-GONZALEZ, AND A. MUKHERJEE (2019): "Heterogeneity, Leveling the Playing Field, and Affirmative Action in Contests," *Working Paper*.

CHOWDHURY, S. M., AND O. GÜRTLER (2015): "Sabotage in contests: A survey," *Public Choice*, 164(1), 135–155.

CHOWDHURY, S. M., AND S.-H. KIM (2017): "“Small, yet Beautiful”: Reconsidering the optimal design of multi-winner contests," *Games and Economic Behavior*, 104, 486 – 493.

CLARK, D., AND C. RIIS (1998a): "Competition over More than One Prize," *American Economic Review*, 88(1), 276–89.

CLARK, D. J., AND C. RIIS (1996): "On the Win Probability in Rent-Seeking Games," *Discussion Paper in Economics E4/96, University of Tromsø, Norway*.

——— (1998b): "Contest success functions: An extension," *Economic Theory*, 11(1), 201–204.

COHEN, C., T. R. KAPLAN, AND A. SELA (2008): "Optimal rewards in contests," *The RAND Journal of Economics*, 39(2), 434–451.

CORCHÓN, L., AND M. DAHM (2010): "Foundations for contest success functions," *Economic Theory*, 43(1), 81–98.

CORNES, R., AND R. HARTLEY (2005): "Asymmetric contests with general technologies," *Economic Theory*, 26(4), 923–946.

DEVARO, J., AND A. KAUHANEN (2016): "An “Opposing Responses” Test of Classic versus Market-Based Promotion Tournaments," *Journal of Labor Economics*, 34(3), 747–779.

DEVARO, J., AND M. WALDMAN (2012): "The Signaling Role of Promotions: Further Theory and Empirical Evidence," *Journal of Labor Economics*, 30(1), 91–147.

DEWATRIPONT, M., I. JEWITT, AND J. TIROLE (1999): "The economics of career concerns, Part I: Comparing information structures," *The Review of Economic Studies*, 66(226), 183–198.

DRUGOV, M., AND D. RYVKIN (2017): "Biased contests for symmetric players," *Games and Economic Behavior*, 103, 116–144.

——— (2020): "How noise affects effort in tournaments," *Journal of Economic Theory*, 188, 105065.

FANG, D., T. NOE, AND P. STRACK (2020): "Turning up the heat: The discouraging effect of competition in contests," *Journal of Political Economy*, 128(5), 1940–1975.

FU, Q., AND J. LU (2009a): "The beauty of "bigness": On optimal design of multi-winner contests," *Games and Economic Behavior*, 66(1), 146 – 161.

——— (2009b): "The optimal multi-stage contest," *Economic Theory*, 51(2), 351–382.

——— (2012): "Micro foundations of multi-prize lottery contests: a perspective of noisy performance ranking," *Social Choice and Welfare*, 38(3), 497–517.

FU, Q., J. LU, AND Y. PAN (2015): "Team Contests with Multiple Pairwise Battles," *American Economic Review*, 105(7), 2120–40.

FU, Q., AND Z. WU (2019): "Contests: Theory and Topics," *Oxford Research Encyclopedia of Economics and Finance*.

——— (2020): "On the optimal design of biased contests," *Theoretical Economics*, 15(4), 1435–1470.

FULLERTON, R. L., AND R. P. MCAFEE (1999): "Auctioning Entry into Tournaments," *Journal of Political Economy*, 107(3), 573–605.

GERCHAK, Y., AND Q.-M. HE (2003): "When will the Range of Prizes in Tournaments Increase in the Noise or in the Number of Players?," *International Game Theory Review*, 05(02), 151–165.

GERSHKOV, A., J. LI, AND P. SCHWEINZER (2009): "Efficient tournaments within teams," *The RAND Journal of Economics*, 40(1), 103–119.

——— (2016): "How to share it out: The value of information in teams," *Journal of Economic Theory*, 162, 261–304.

GHOSH, S., AND M. WALDMAN (2010): "Standard promotion practices versus up-or-out contracts," *The RAND Journal of Economics*, 41(2), 301–325.

GILPATRIC, S. M. (2009): "RISK TAKING IN CONTESTS AND THE ROLE OF CARROTS AND STICKS," *Economic Inquiry*, 47(2), 266–277.

GREEN, J. R., AND N. L. STOKEY (1983): "A Comparison of Tournaments and Contracts," *Journal of Political Economy*, 91(3), 349–364.

GRUND, C., AND D. SLIWKA (2005): "Envy and compassion in tournaments," *Journal of Economics & Management Strategy*, 14(1), 187–207.

GÜRTLER, M., AND O. GÜRTLER (2015): "The optimality of heterogeneous tournaments," *Journal of Labor Economics*, 33(4), 1007–1042.

——— (2019): "Promotion signaling, discrimination, and positive discrimination policies," *The RAND Journal of Economics*, 50(4), 1004–1027.

HILLMAN, A. L., AND J. G. RILEY (1989): "Politically Contestable Rents and Transfers," *Economics & Politics*, 1(1), 17–39.

HOLMSTRÖM, B. (1982): "Managerial Incentive Problems: A Dynamic Perspective," in *Essays in Economics and Management in Honor of Lars Wahlbeck, Helsinki 1982.*, pp. 169–182. Reprinted in *The Review of Economic Studies* (1999), 66(1).

HOLMSTRÖM, B., AND J. RICARD I COSTA (1986): "Managerial Incentives and Capital Management," *Quarterly Journal of Economics*, 101(4), 835–860.

HVIDE, H. K. (2002): "Tournament Rewards and Risk Taking," *Journal of Labor Economics*, 20(4), 877–898.

IMHOF, L., AND M. KRÄKEL (2016): "Ex post unbalanced tournaments," *The RAND Journal of Economics*, 47(1), 73–98.

JIA, H. (2008): "A stochastic derivation of the ratio form of contest success functions," *Public Choice*, 135(3-4), 125–130.

KIRKEGAARD, R. (2020): "Contest Design with Stochastic Performance," *Working Paper*.

KONRAD, K. A. (2009): *Strategy and Dynamics in Contests*. Oxford University Press.

KRÄKEL, M. (2008): "Optimal risk taking in an uneven tournament game with risk averse players," *Journal of Mathematical Economics*, 44(11), 1219–1231.

KRÄKEL, M., AND D. SLIWKA (2004): "Risk taking in asymmetric tournaments," *German Economic Review*, 5(1), 103–116.

KRUMER, A., R. MEGIDISH, AND A. SELA (2017): "Round-Robin Tournaments with a Dominant Player," *The Scandinavian Journal of Economics*, 119(4), 1167–1200.

LAZEAR, E. P. (1989): "Pay Equality and Industrial Politics," *Journal of Political Economy*, 97(3), 561–580.

LAZEAR, E. P., AND S. ROSEN (1981): “Rank-Order Tournaments as Optimum Labor Contracts,” *Journal of Political Economy*, 89(5), 841–864.

LEVY, H. (1992): “Stochastic Dominance and Expected Utility: Survey and Analysis,” *Management Science*, 38(4), 555–593.

LU, J., Z. WANG, AND L. ZHOU (2021): “Optimal Favoritism in Contests with Identity-Contingent Prizes,” *Available at SSRN 3807738*.

MALCOMSON, J. M. (1984): “Work incentives, hierarchy, and internal labor markets,” *Journal of Political Economy*, 92(3), 486–507.

MALCOMSON, J. M. (1986): “Rank-Order Contracts for a Principal with Many Agents,” *The Review of Economic Studies*, 53(5), 807–817.

MOLDOVANU, B., AND A. SELA (2001): “The Optimal Allocation of Prizes in Contests,” *American Economic Review*, 91(3), 542–558.

——— (2006): “Contest architecture,” *Journal of Economic Theory*, 126(1), 70–96.

MOLDOVANU, B., A. SELA, AND X. SHI (2007): “Contests for Status,” *Journal of Political Economy*, 115(2), 338–363.

MORATH, F., AND J. MÜNSTER (2013): “Information acquisition in conflicts,” *Economic Theory*, 54(1), 99–129.

MÜNSTER, J. (2009): “Group contest success functions,” *Economic Theory*, 41(2), 345–357.

NALEBUFF, B. J., AND J. E. STIGLITZ (1983): “Prizes and incentives: Towards a general theory of compensation and competition,” *The Bell Journal of Economics*, 14(1), 21–43.

O’KEEFFE, M., W. K. VISCUSI, AND R. J. ZECKHAUSER (1984): “Economic contests: Comparative reward schemes,” *Journal of Labor Economics*, 2(1), 27–56.

OLSZEWSKI, W., AND R. SIEGEL (2016): “Large Contests,” *Econometrica*, 84(2), 835–854.

OWAN, H. (2004): “Promotion, Turnover, Earnings, and Firm-Sponsored Training,” *Journal of Labor Economics*, 22(4), 955–978.

PÉREZ-CASTRILLO, D., AND D. WETTSTEIN (2016): "Discrimination in a model of contests with incomplete informatio about ability," *International Economic Review*, 57(3), 881–914.

RINNE, H. (2014): "The hazard rate: Theory and inference - With supplementary MATLAB Programs," Discussion paper, Justus-Liebig-University, Giessen, Germany.

ROTHSCHILD, M., AND J. E. STIGLITZ (1970): "Increasing risk I: A definition," *Journal of Economic Theory*, 2, 225–243.

RYVKIN, D., AND M. DRUGOV (2020): "The shape of luck and competition in winner-take-all tournaments," *Theoretical Economics*, 15(4), 1587–1626.

SCHOTTER, A., AND K. WEIGELT (1992): "Asymmetric Tournaments, Equal Opportunity Laws, and Affirmative Action: Some Experimental Results," *Quarterly Journal of Economics*, 107(2), 511–539.

SCHÖTTNER, A., AND V. THIELE (2010): "Promotion Tournaments and Individual Performance Pay," *Journal of Economics & Management Strategy*, 19(3), 699–731.

SCHWEINZER, P., AND E. SEGEV (2012): "The optimal prize structure of symmetric Tullock contests," *Public Choice*, 153(1-2), 69–82.

SELA, A. (2012): "Sequential two-prize contests," *Economic Theory*, 51(2), 383–395.

SIEGEL, R. (2009): "All-Pay Contests," *Econometrica*, 77(1), 71–92.

——— (2010): "Asymmetric contests with conditional investments," *The American Economic Review*, 100(5), 2230–2260.

SKAPERDAS, S. (1996): "Contest success functions," *Economic Theory*, 7(2), 283–290.

TULLOCK, G. (1980): "Efficient Rent Seeking," in *Towards a Theory of the Rent-Seeking Society*, ed. by J. Buchanan, R. Tollison, and G. Tullock, pp. 97–112. Texas A&M University Press, College Station, Tx.

VOJNOVIĆ, M. (2016): *Contest theory: Incentive mechanisms and ranking methods*. Cambridge University Press.

WALDMAN, M. (1984): "Job Assignments, Signalling, and Efficiency," *The RAND Journal of Economics*, 15(2), 255–267.

ZÁBOJNÍK, J. (2012): “Promotion tournaments in market equilibrium,” *Economic Theory*, 51(1), 213–240.

ZÁBOJNÍK, J., AND D. BERNHARDT (2001): “Corporate Tournaments, Human Capital Acquisition, and the Firm Size—Wage Relation,” *The Review of Economic Studies*, 68(3), 693–716.