Generalizing the Inequality Process' Gamma Model of Particle Wealth Statistics

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by

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[1] This paper is a revision of part of "The Inequality Process' PDF Approximation Model For Heavy-Tailed Financial Distributions", presented to a session of the Economics Statistics Section of the American Statistical Association (ASA) in the 2019 Joint Statistical Meetings and published in the JSM2019 proceedings volume (Angle, 2019). It is ASA policy that the JSM Proceedings is a preprint series and publication in it should not preclude publication in a peer-reviewed journal.

Generalizing the Inequality Process' Gamma Model of Particle Wealth Statistics

Abstract

The Inequality Process (IP) has been tested and confirmed against data on incomes that are approximately gamma distributed. The IP's gamma pdf model implies statistics of IP particle wealth expressed algebraically in terms of IP parameters but only for the subset of IP parameters that generate approximately gamma distributions of particle wealth. Many empirical distributions of income and wealth have heavier-than-gamma right tails. This paper shows that a variance-gamma (VG) model can do what the IP's gamma pdf model does, but for the full set of IP particle parameters, thus generalizing the IP's gamma pdf model without loss of parsimony because the parameters and statistics of both pdf models are re-expressed in terms of the same IP parameters.

Key Words

gamma pdf, heavier-than-gamma tails, Inequality Process, particle parameters, particle wealth, variance-gamma pdf

1.0 Introduction: The Inequality Process (IP) as a Statistical Law of Income and Wealth Distribution

1.1 The Inequality Process

The Inequality Process (IP) (Angle, 1983-2019) is a particle system, in which wealth, a positive quantity, is transferred between two particles according to the following rules:

1) All particles in a population are randomly paired.
2) Each pair flips and calls a fair coin.
3)The general pair is particle ψ and particle θ.
4) If particle ψ wins the toss, it takes an ω_θ share of particle θ's wealth.
5) If particle θ wins the toss, it takes an ω_ψ share of particle ψ's wealth.
6) Repeat.

Particle wealth changes at each encounter. In the non-evolutionary version of the process, the share of wealth a particle gives up when it loses does not change. That share is its parameter, omega, ω. Particles that lose less when they lose (i.e., with smaller omega) have a higher expectation of wealth than particles that lose more, since the probability of loss is 50% for all.

1.2 The Inequality Process' (IP's) Wide Explanandum

The verbal theory from which the Inequality Process (IP) was abstracted (Angle, 1983, 1986) asserts that, on average, workers more productive of wealth are more sheltered in the competition for wealth than less productive workers. Despite the fact that that verbal theory and the IP's parsimony are not at all intuitive for many economists, the IP has a wide empirical explanandum. See Appendix A of "***A Variance-Gamma …***" . IP papers are grouped under fifteen headings in Appendix A.1. Seven verbal maxims of economics (never supposed before the IP to be joint implications of a single, parsimonious mathematical model) are listed in Appendix A.2. Appendix A.3 lists five "stylized facts" (fuzzy invariances) of the stock market implied by the IP. Angle (2019) discusses the conjecture that the IP's apparent ubiquity arises from its evolutionary optimality.

1.3 The Inequality Process (IP) is Similar to the Kinetic Theory of Gases Model

Appendix B.1 describes the near isomorphism between the IP and the stochastic particle system of the Kinetic Theory of Gases (Whitney, 1990), the oldest and best known particle system of statistical physics. While the two particle systems have different properties, only two substitutions into the transition equations of one particle system converts it into the other. The IP's transition equations are stated in words in steps 2 through 5 above, and algebraically in Appendix B.1. Those two substitutions account for the difference in properties between the two particle systems since, apart from those substitutions, the two particle systems are isomorphic (Angle, 1990). This near isomorphism and the IP's combination of parsimony with wide empirical explanandum have perhaps facilitated the acceptance of the IP as econophysics. See Byrro Ribeiro's (2020:154-158) ***Income Distribution Dynamics of Economic Systems: An Econophysical Approach*** discussion of the IP that cites 29 IP papers. The IP is also discussed in a major review of agent-based modeling in economics (Chen, 2018).

1.4 The Inadequacy of the IP's Gamma PDF Model

The Inequality Process (IP) generates a distribution of wealth in the ω_ψ equivalence class of particles defined by its particles having equal parameters, all omegas equal to ω_ψ. The distribution of particle wealth in each ω_ψ equivalence class is approximately fitted by a gamma probability density function (pdf) ***if*** $\omega_\psi < 0.5$.

Approximations to the gamma pdf's two parameters, its shape and scale parameters, in terms of particle omegas were found by:

a) solving for the expectation of particle wealth in each equivalence class by recognizing that:

i) the Inequality Process is ergodic, $0.0 < \omega_\psi < 1.0$,
ii) particle wealth converges to a stationary distribution,
iii) each particle has an equal probability of encountering any particle,
iv) the sum of wealth over all particles is constant,
and via

b) back-substitution over the time-horizon of the process,
c) approximation to this solution by a negative binomial probability function (pf), and
d) recognition of the close relationship of the gamma probability density function (pdf) to the negative binomial probability function.

a) is exact; b) is exact; d) is exact; but c) involves guessing and numerical search. The IP's gamma pdf model of wealth statistics in the ω_ψ equivalence class, expressed in terms of ω_ψ and the harmonic mean of ω_ψ in the whole population, $\tilde{\omega}$, explains algebraically how omegas determine statistics of particle wealth. This algebraic explanation is more precise, compact, insightful, comparable, and communicable than the results of numerical search. However, this benefit of the

IP's gamma pdf model is lost when $\omega_\psi > 0.5$, because when $\omega_\psi > 0.5$ the IP generates distributions with heavier-than-gamma right tails, making the IP's gamma pdf model inapplicable. A parametric pdf model, like the IP's gamma pdf model for $\omega_\psi < 0.5$, is needed to see whether IP hypotheses, that hold when $\omega_\psi < 0.5$, also hold when $\omega_\psi > 0.5$. Murky numerics are the unsatisfactory alternative.

1.5 How the present paper generalizes the IP's gamma pdf model

So a generalization of the IP's gamma pdf model for $\omega_\psi > 0.5$, and preferably for the entire interval on which ω_ψ's are defined, $0.0 < \omega_\psi < 1.0$, is needed. A generalization should converge to the IP's gamma pdf model for $\omega_\psi < 0.5$. The present paper finds such a generalization, the variance-gamma (VG) pdf. The VG pdf has three parameters whereas the gamma pdf has two. Two of the VG's three parameters are closely related to the gamma pdf's two parameters and are employed in the IP's VG model. This paper finds a good approximation to the VG's third parameter in terms of IP parameters, the omegas, ω_ψ, and $\tilde{\boldsymbol{\omega}}$ the harmonic mean of the of ω_ψ's in the whole population of IP particles. This third VG parameter fades as ω_ψ approaches 0.5 from above, letting the VG model converge to the gamma model, thus generalizing the IP's gamma pdf model.

While the IP's VG model opens the way to testing IP hypotheses against data on income and wealth distributions with heavier-than-gamma right tails, no test of an IP hypothesis against an empirical distribution is performed in this paper. It is sufficient for one paper to show that the IP's VG model is a good approximation to IP particle wealth statistics in the ω_ψ equivalence class when $\omega_\psi > 0.5$. Because all three of the VG's parameters are re-parameterized in the same quantities used to re-parameterize the gamma's two parameters, in terms of the omegas, ω_ψ and $\tilde{\boldsymbol{\omega}}$, the IP's VG model's generalization of the gamma model incurs no loss of parsimony.

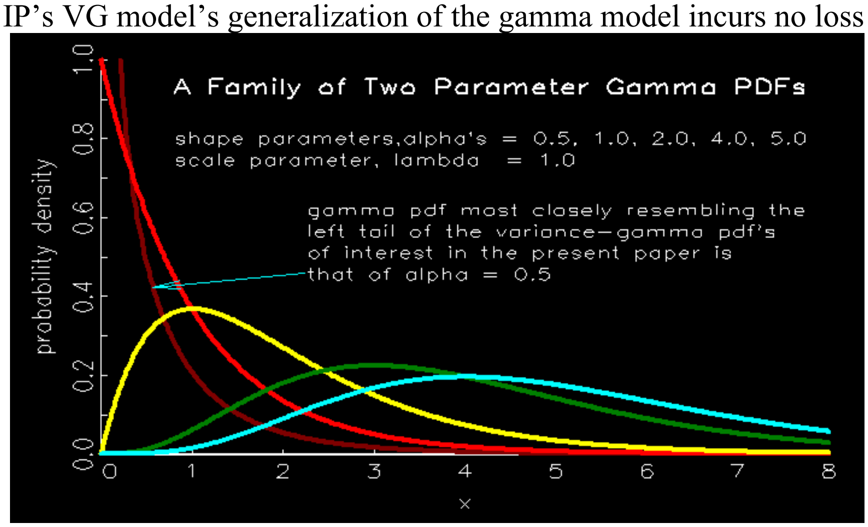

Figure 1 (This figure displays the shapes of a family of gamma pdf's with the same scale parameter, 1.0, and different shape parameters. They are all right skewed.)

2.0 Empirical Validation of the Inequality Process' (IP's) Gamma Model

Figure 1 plots the gamma probability density function (pdf) with five distinct values of its shape parameter and a fixed scale parameter. If the IP's gamma pdf model can be fitted to each partial distribution the relative frequency distribution of labor income conditioned on worker education, the test of the IP's main hypothesis: wealth is transferred via competition to more productive workers, can be tested, assuming the widely accepted proposition of human capital theory that education is an investment in productivity is made here.

The test of the hypothesis is:

a) a gamma pdf fits each partial distribution of the distribution of labor income conditioned on worker level of education,

b) gamma shape parameters of these fits are positively correlated with worker level of education,

c) smaller ω_ψ is associated with a larger gamma shape parameter fitted to the distribution of particle wealth in the ω_ψ equivalence class,

if a), b), and c) then,

d) smaller ω_ψ is associated with greater worker productivity.

This hypothesis has been confirmed in U.S. data (Angle, 1990, 2002, 2003, 2006, 2007b). See Figure 2. The gamma pdf curves in Figure 2 are fitted to data by a stochastic search over a vector of IP particle parameters to minimize squared error, weighted by the number of workers at each level of education.

Figure 2 (the solid curve is based on annual March Current Population Survey data, averaged over a decade; the dotted curve is the fitted gamma pdf re-parameterized in terms of IP particle parameters estimated from the distribution of labor income conditioned on worker education in a decade.).

3.0 The Inequality Process (IP) And Heavier-Than-Gamma-Tailed Distributions

In the absence of the gamma pdf model of particle wealth, tests of IP hypotheses against wealth and income distributions with heavier-than-gamma right tails proceed using the clumsy, murky numerical methods of such as the look of scatter plots. See Angle (2018). Better insight into how IP particle parameters are related to the statistics of heavier-than-gamma-tailed distributions requires the algebra provided by a pdf model of the IP's heavier-than-gamma-tailed particle wealth distributions, i.e., $\omega_\psi > 0.5$, with that pdf model's parameters re-expressed in terms of IP particle parameters, the omegas, ω_ψ and $\tilde{\omega}$, where $\tilde{\omega}$ is the harmonic mean of the ω's of particles.

Figure 3 shows empirical distributions of income with heavier-than-gamma tails. Figure 3 is the analogue of Figure 2 for personal income from liquid assets, conditioned on recipients' level of education. Note that the relative frequency distributions of Figure 3 largely overlap, making them hard to distinguish by fitting a pdf to each relative frequency distribution. The differences between them are slight differences of density in their far right tails (the distribution over large incomes). While their relative frequency distributions are hard to distinguish regardless of what their moments, their means and variances, are. If the moments are distinctively different, the moments are preferable to the relative frequency distribution as a statistic to fit and to estimate parameters from.

3.1 A Word About Distribution Tail Labels

Right skewed distributions on the positive real line with "heavy" right tails, or, equivalently in statistical jargon, "fat" tails", are misleadingly named. Consider the extreme example of tail heaviness in a right tailed IP particle wealth distribution in a finite population of particles. In this population of n particles, there are n-1 particles with zero wealth and one particle with all wealth. That maximally rich particle is at the right limit of the relative frequency distribution of wealth on the x-axis. Up to that point, the right tail of the distribution is has zero density. The heaviness of a heavy-right-tailed distribution of wealth is <u>not</u> the presence of a large fraction of the population of particles in the right tail but rather the presence of a large proportion of that population's wealth in the right tail. See Resnick (2007) on the statistics of heavy-tailed distributions. Incidentally, this paragraph's example of a maximally heavy-tailed distribution would be replicated by the IP if all particles had an omega of 1.0. Since this situation is empirically uninteresting, particle parameters are stipulated to be less than 1.0.

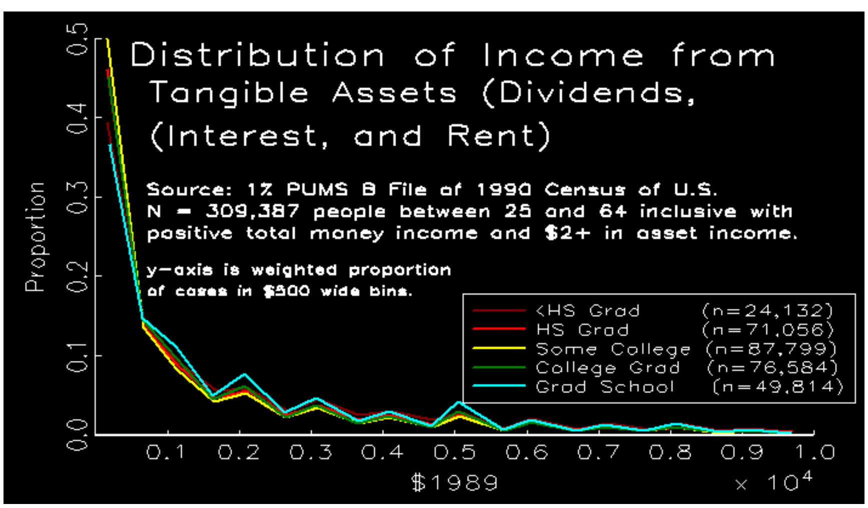

Figure 3

4.0 The Gamma PDF Becomes The IP's Gamma Model

Explaining how the Inequality Process' (IP's) variance-gamma pdf model generalizes the IP's gamma pdf model necessitates the examination of the IP's gamma pdf model, particularly how its shape and scale parameters are expressed in terms of IP particle parameters. The IP's gamma model is specific to the IP's stationary distribution of particle wealth in each ω_ψ equivalence class of particles. The stationary distribution of particle wealth in the ω_ψ equivalence class of particle is well approximated by a two parameter gamma probability density function (pdf) ***if*** $0.0 < \omega_\psi < 0.5$. The gamma pdf's two parameters are its shape parameter, α, and its scale parameter, λ. See equation (4.1a):

$$\begin{aligned} f(x) &\equiv \frac{\lambda^{\alpha}}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\lambda x} \\ x &> 0 \\ \alpha &\equiv \text{shape parameter} \\ \lambda &\equiv \text{scale parameter} \end{aligned} \tag{4.1a}$$

The two parameter gamma pdf (4.1a) becomes the IP gamma model with re-expression of the gamma pdf's shape and scale parameters in terms of IP particle parameters, the omegas, in the ω_ψ equivalence class of particle:

$$\begin{aligned} x_\psi &\equiv \text{wealth in the } \omega_\psi \text{ equivalence class in multiples of } \mu \\ x_\psi &> 0 \\ \alpha_\psi &\equiv \text{shape parameter} \approx \frac{1-\tilde{\omega}}{\omega_\psi} \\ \lambda &\equiv \text{scale parameter} \approx \frac{1-\tilde{\omega}}{\tilde{\omega}\ \ \mu} \\ \tilde{\omega} &\equiv \text{harmonic mean of } \omega_\psi\text{'s} \\ \mu &= \text{unconditional mean of wealth} \end{aligned} \tag{4.1b}$$

Where capital Ψ is the number of distinct ω_ψ equivalence classes and w_ψ the fraction of the particle population in the ω_ψ equivalence class, the harmonic mean of the ω's, is:

$$\tilde{\omega} \stackrel{\text{def}}{=} \left(\sum_{\psi=1}^{\Psi} \frac{w_\psi}{\omega_\psi} \right)^{-1} \tag{4.2}$$

Thus (4.1a) becomes the IP's gamma model of particle wealth in the ω_ψ equivalence class of particles:

$$f(x_\psi) \equiv \frac{\lambda^{\alpha_\psi}}{\Gamma(\alpha_\psi)} x_\psi^{\alpha_\psi - 1} e^{-\lambda \quad x_\psi} \tag{4.3}$$

4.1 The estimator of mean particle wealth μ_ψ, in the ω_ψ equivalence class in terms of particle parameters

(4.4) is the estimator of mean particle wealth in the ω_ψ equivalence class, μ_ψ, in terms of IP particle parameters. Given (4.1b), (4.4) follows from the expression for the gamma pdf's mean, α/λ. However (4.4)'s derivation worked the other way around, (4.1b)'s expressions for α and λ were in part derived from (4.4) which is implied by:

a) the random pairing of all particles for competition,
b) the constancy of population mean wealth,
c) the permanence of particle parameters, and,
d) the ergodicity and convergence to stationarity of the Inequality Process.

Consequently, (4.4) does not depend on ω_ψ being in the interval $0.0 < \omega_\psi < 0.5$ because (4.4) does not depend on the IP's gamma pdf model.

$$\mu_\psi = \left(\frac{\tilde{\omega} \quad \mu}{\omega_\psi} \right) \tag{4.4}$$

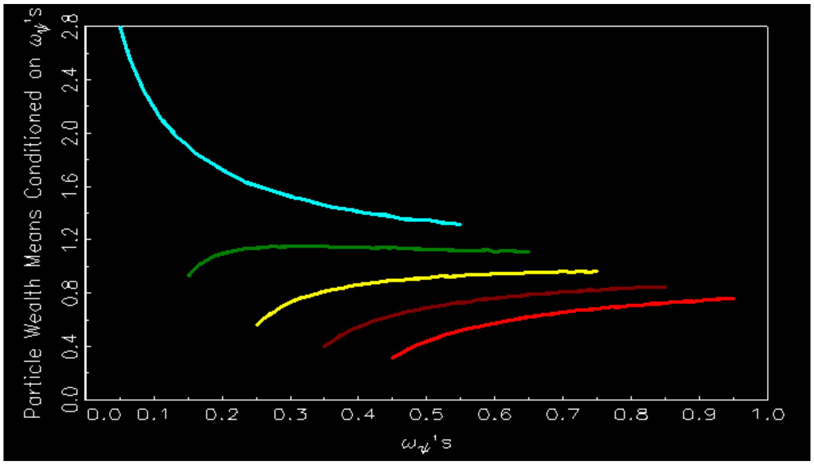

Figure 4

Figure 4 shows the estimates of the conditional means, the μ_ψ's, in each ω_ψ equivalence class, based on (4.4) (plotted as the unseen dashed piecewise linear curves), and the estimates made directly from observations on the particle wealth of the 1,000 particles in each of the five ω_ψ equivalence classes in each of the 37 IP simulations described in section 4.2 (solid piecewise linear curves). The dashed piecewise linear curves are not visible in Figure 4 because the two sets of estimates are so close that they overlap in Figure 4.

(4.4) can be deduced from an examination of Figure 5. Figure 5 plots the change in wealth of an IP particle on the y-axis against its wealth before the encounter on the x-axis. The change in wealth is due to a competitive encounter with another particle. See the IP's transition equations in Appendix B.1. Note the asymmetry of gains and losses and the approximate equality of losses with mean gains at mean particle wealth in each ω_ψ equivalence class in Figure 5.

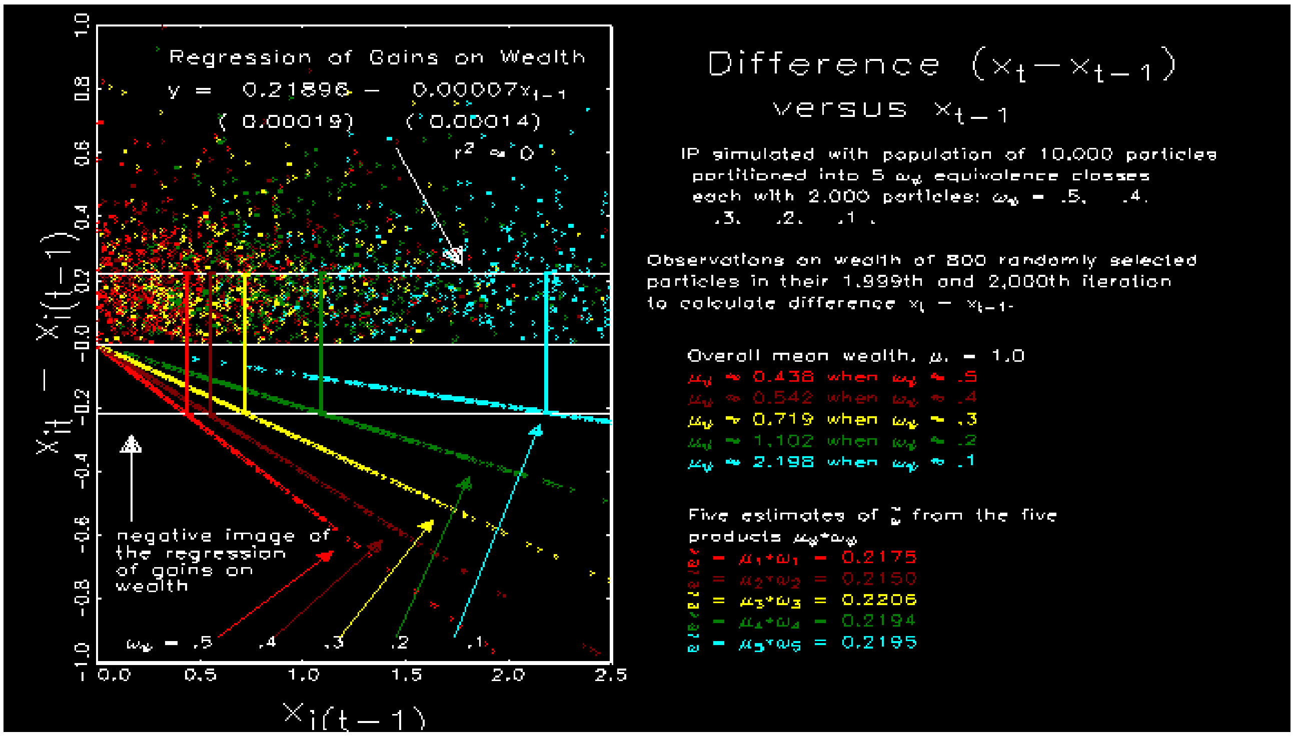

Figure 5

4.2 Solution of the Inequality Process Via Simulation

Each IP particle's parameter, ω, the number of particles in each ω_ψ equivalence class, and the whole population's harmonic mean of particle parameters, $\tilde{\omega}$ are known before each simulation run begins. Each particle's wealth is observable as the simulation proceeds. Five ω_ψ equivalence classes are distinguished in the 37 IP simulations generating estimates displayed in Figures 4, and 6 through 8. Each of the 37 simulations of the IP has a set of five particle parameters, ω_ψ's, one parameter to each ω_ψ equivalence class containing 1,000 particles. The first of the 37 IP simulations starts with { .05, .15, .25, .35, .45 } as its set of parameters. All 5,000 particles are involved in every iteration of every IP simulation. Pairing of particles for competition in each iteration of the IP is random. Without loss of generality, the grand mean of particle wealth in the population of 5,000 particles is 1.0 for interpretability and computational stability. After 2,000 iterations of the Inequality Process with a particular set of five ω_ψ's, the wealth of particles in each of the five ω_ψ equivalence classes is recorded. 2,000 iterations allow each IP simulation to converge to its stationary distribution. The mean and variance of particle wealth in each ω_ψ equivalence class are calculated from its 1,000 particles. The standard errors of estimate of these statistics are negligible.

The first IP simulation with IP particle parameters { .05, .15, .25, .35, .45 } provides the leftmost point estimates graphed in Figures 4 and 6 through 9. After each IP simulation is run, the set of five parameters is incremented by .0125 . The second IP simulation is identical to the first simulation except that its set of ω_ψ's is {.0625, .1625, .2625, .3625, .4625}, and, so on, through 35 more IP simulations, the last one of which is done with the ω_ψ set {.5, .6, .7, .8, .9 }. The 37 simulations of the IP are programmed in the GAUSS language (Aptech Systems, 2012).

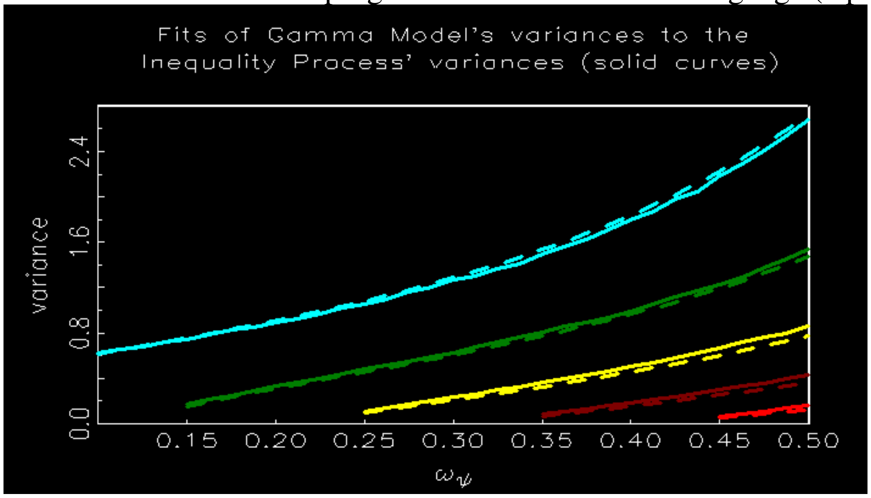

Figure 6

Figures 6 and 7 compare the IP gamma pdf model's estimates of the variance of particle wealth in each ω_ψ equivalence class with the variance estimated directly from particle wealth in that ω_ψ equivalence class. Figure 6 shows that the IP gamma model variance estimates are serviceable approximations if their ω_ψ's are in the interval $0.0 < \omega_\psi < 0.5$. Figure 7 shows that the IP gamma model variances become poorer approximations as ω_ψ's exceed .5 .

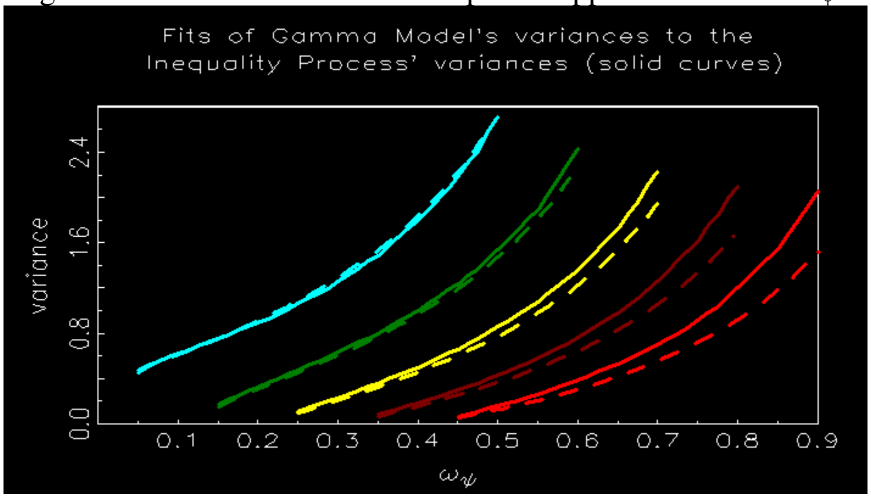

Figure 7

4.3 The Re-parameterized Gamma PDF Model's Estimator of the Variance of Particle Wealth

As previously noted, variances are a more reliable statistic to fit than the frequency distribution of IP particle wealth with heavier-than-gamma tails, i.e. for ω_ψ's exceeding .5 . So it is appropriate to review the IP gamma model's variances of particle wealth in the ω_ψ equivalence class, $\text{variance}_{\Gamma IP\ \psi}$, starting with the definition of the variance:

$$\text{variance} \stackrel{\text{def}}{=} E[(x - E[x])^2]$$
$$= E[x^2] - (E[x])^2$$

(4.5)[2]

The variance of a gamma distributed random variable, x, here particle wealth in the ω_ψ equivalence class is the ratio of the shape parameter, α, to the square of the scale parameter, λ:

$$\text{variance}_{\Gamma IP\psi} = \frac{\alpha_\psi}{\lambda^2} \qquad (4.5a)$$

So, given (4.1b), the re-parameterization of α and λ in terms of ω_ψ and $\tilde{\omega}$,
(4.5a) becomes:

$$\text{variance}_{\mathbf{\Gamma IP\psi}} \approx \frac{(\tilde{\omega}\quad \mu\ \ )}{\omega_\psi} \cdot \frac{(\tilde{\omega}\quad \mu\ \ )}{(1-\tilde{\omega}\ )} = \frac{(\tilde{\omega}\quad \mu\ \ )^2}{\omega_\psi\,(1-\tilde{\omega}\quad)} \qquad (4.5b)$$

$$= \mu_\psi \frac{(\tilde{\omega}\quad \mu\ \ )}{(1-\tilde{\omega}\quad)} \qquad (4.5c)$$

(4.5c) factors out (4.4), μ_ψ, from (4.5b). Note that the variance of particle wealth in each ω_ψ equivalence classes is the product of its mean of particle wealth (particle wealth conditioned on ω_ψ) and ***the same scaling quantity for all ω_ψ equivalence classes,*** the inverse of the re-parameterization of λ .

5.0 Features An IP Model For ω_ψ| 0.0 < ω_ψ < 1.0 Should Have

A generalization of the IP's gamma pdf model of particle wealth in the ω_ψ equivalence class for the full interval $\omega_\psi | 0.0 < \omega_\psi < 1.0$ should satisfy the following six constraints:

1) The generalization must approximate IP particle wealth statistics when $\omega_\psi > .5$.
2) The generalization should converge to the statistics of the IP's gamma model as ω_ψ decreases past 0.50, since the gamma pdf model is adequate when $\omega_\psi < .5$;
3) The generalization must have the same estimator for mean particle wealth in the ω_ψ equivalence class as equation (4.4), same as the gamma pdf model's, since (4.4) appears to be exact in expectation for all $\omega_\psi | 0.0 < \omega_\psi < 1.0$.
4) It is convenient if the generalization is so closely related to the gamma pdf that the expressions for the gamma shape and scale parameters in terms of IP particle parameters, ω_ψ and $\tilde{\omega}$, are preserved as shape and scale parameters of the generalization. $\tilde{\omega}$ is the harmonic mean of ω's in the whole population of particles.
5) The generalization must have all of its other parameters expressible in terms of IP particle parameters, ω_ψ and $\tilde{\omega}$.
6) The generalization would be particularly welcome if it were a pdf already in use as a model of heavier-than-gamma tailed distributions of income or wealth.

[2]Finding the first and second gamma pdf moments around zero is an elementary integration. To find the first moment around zero, increment the exponent on x in (4.1a) by one and integrate over x from zero to positive infinity. To find the second moment around zero, increment the exponent on x in (4.1a) by two and integrate over x from zero to positive infinity.

The variance-gamma (VG) pdf satisfies all these constraints if it can be shown that constraint #5 can be satisfied. The variance-gamma (VG) pdf has three parameters. Section 5.1, next, shows that two of the VG's three parameters are analogues of the gamma pdf's shape and scale parameters. Expressions for these two gamma parameters in terms of IP parameters exist, i.e. (4.1b). Section 6.0 introduces an expression for the VG's third parameter in terms of IP particle parameters, ω_ψ and $\widetilde{\omega}$, that works for $\omega_\psi | \ 0.0 < \omega_\psi < 1.0$, the entire interval on which ω_ψ's are defined.

Since the variance-gamma (VG) pdf is used to model right skewed, heavy-tailed financial distributions (Madan and Seneta (1990), Seneta, (2004), Finlay (2009), and Fiorani (2009)), it is conceivable that the VG's modeling of IP wealth distributions where $\omega_\psi > 0.5$ may have empirical relevance. However, there is no excursion in this paper into the attempt to demonstrate that empirical relevance.

5.1 The close relationship between the variance-gamma (VG) pdf and the gamma pdf

Compare the gamma pdf's characteristic function with the VG's pdf, i.e. (5.1) and (5.2), respectively, below. They differ only by one term. It appears in the denominator of the VG's characteristic function but not the gamma's. It is the VG's term containing σ^2. σ^2 gives the VG a heavier right tail than the gamma's.

Because (5.1) and (5.2) are otherwise equal, the re-parameterization of the gamma pdf's shape and scale parameters in terms of IP particle parameters, (4.1b), is assigned to their VG pdf counterparts. This decision is justified by the fact that σ^2 -> 0 implies that the VG pdf -> gamma pdf, because their respective characteristic functions, (5.1) and (5.2), converge as σ^2 -> 0.
Note: the presentation here of the variance-gamma (VG) pdf closely follows Kotz, Kozubowski, and Podgorski's (2001) discussion of the VG pdf. Kotz, Kozubowski, and Podgorski (2001) discuss the Laplace Distribution and its generalizations. The variance-gamma (VG) pdf is one of these. Kotz et al. (2001: 180) write: "In this book we use the terms Bessel function distribution and variance-gamma distribution interchangeably with the name generalized Laplace distribution ….". All algebra quoted from their text is in their notation, beginning with (5.2).

What is needed for the variance-gamma (VG) pdf to satisfy the above set of six constraints is an expression for the VG's σ^2 in terms of IP particle parameters, ω_ψ and $\widetilde{\omega}$. This re-parameterization is prompted by the comparison of the two characteristic functions.

Equation (4.1a) is the two parameter gamma pdf with shape and scale parameters unconstrained by re-parameterization in terms of IP particle parameters. Its characteristic function of the two parameter gamma pdf is:

$$\Psi_\Gamma(t) \equiv \left(\frac{1}{1-\left(\frac{it}{\lambda}\right)}\right)^{\alpha} \qquad (5.1)$$

where:

$\alpha \overset{\text{def}}{=} shape\ parameter > 0$
$\lambda \overset{\text{def}}{=} scale\ parameter > 0$

and $i = \sqrt{-1}$.

The characteristic function of the variance-gamma pdf (VG) equals that of the gamma pdf up to the VG's extra (third) parameter, σ^2, in the denominator of the VG's characteristic function (5.2), σ^2. Kotz et al., (2001) gives VG's characteristic function as:

$$\Psi_{VG}(t) \equiv \left(\frac{1}{1+\frac{1}{2}\sigma^2 t^2 - i\mu t}\right)^{\tau} \qquad (5.2)$$

where:

$\mu, \tau \in \mathbb{R}^+$ (positive real numbers)

(5.2) is definition 4.1.1 of Kotz et al. (2001: 180). The gamma pdf's two parameters, α and λ and their IP approximations in terms of ω_ψ and $\tilde{\omega}$ (leaving the t subscript that indicates which set of IP parameters is being simulated off) become the VG pdf parameters:

$$\tau = \alpha \approx \frac{1 - \tilde{\omega}}{\omega_\psi}$$

$$\mu = 1/\lambda \approx \left(\frac{\tilde{\omega} \quad \mu}{1-\tilde{\omega}}\right) \qquad (5.3a,b)$$

NB!: The left hand sides of (5.3a,b) are in Kotz et al. (2001) notation, while the far right hand sides are in terms of IP parameters. Kotz et al. (2001) notation for VG scale parameter μ in (5.2), appearing on the left hand side of (5.3b), should not be confused with the 'μ' of equations (4.1b), (4.4), and the far right hand side of (5.3b). In IP notation 'μ' is the unconditional mean of IP particle wealth. The decision to keep Kotz et al.'s (2001) notation in expressions quoted from Kotz et al. (2001) necessitates using 'μ' to denote these two different quantities. Context makes clear what 'μ' denotes in each instance.

Some authors write the gamma pdf's scale parameter as $\beta = 1/\lambda$ with β as the gamma scale parameter. Expressing the gamma pdf's scale parameter this way is consistent with Kotz, et al. (2001)'s notation for the VG scale parameter, μ. So one could re-write (5.3b) as:

$$\mu = \beta \approx \left(\frac{\tilde{\omega} \quad \mu}{1-\tilde{\omega}}\right) \qquad (5.3c)$$

5.2 Three Reasons to Fit the VG's Variance Rather than its PDF

5.21 Reason 1

It is not practical to fit a VG pdf directly to frequency distributions of IP particle wealth with heavier-than-gamma right tails. The reason why is illustrated by Figure 3. Its income distributions with heavier-than-gamma right tails largely overlap. So do distributions of IP particle wealth when particles are in ω_ψ equivalence classes with their ω_ψ's > 0.5, those with heavier-than-gamma right tails. Such distributions differ subtly in their small densities over large amounts of particle wealth, i.e., they do not have distinctly different relative frequency distributions. However, they do have distinctly different variances. Hence the need to fit variances.

5.22 Reason 2

The second reason not to fit a variance-gamma (VG) pdf directly to a relative frequency distribution of IP particle wealth in the ω_ψ equivalence class is that the VG parameter tau, τ, the analogue of the gamma pdf's parameter alpha, α, in (5.1) and (5.2), the respective characteristic functions, has to be an integer for the VG's density function to exist in closed algebraic form (Kotz et al., 2001: 192). As you can see from eqs. (4.1b) and (5.3a), tau, τ, expressed in terms of IP particle parameters, will not in general be an integer.[3]

5.23 Reason 3

The third reason the VG pdf should be fitted to the variance of IP particle wealth in the is that, as Kotz et al., (2001) state, the VG's moment generating function, (5.4), is not limited to integer tau, τ. Higher moments than the variance might be fitted, but these are noisier than the variance. The variance is preferable to the second moment around zero because the variance of particle wealth in the ω_ψ equivalence class incorporates the reliably estimated mean of particle wealth in that particle equivalence class. Kotz et al. (2001:192) state the VG's moment generating function as[4]:

$$E(Y^n) = \frac{1}{\sqrt{\pi}\,\Gamma(\tau)} \sum_{k=0}^{[[n/2]]} \binom{n}{2k} \sigma^{2k} \mu^{n-2k} 2^k \Gamma(1/2 + k)\,\Gamma(\tau + n - k) \tag{5.4}$$

Eq. (5.4) generates moments around zero. The first two VG moments around zero in Kotz et al. (2001) notation, instantiated from (5.4), are:

$$\begin{aligned} \mathrm{E}[\mathrm{X}] &= \tau\mu \\ \mathrm{E}[\mathrm{X}^2] &= (\tau+1)\tau\mu^2 + \tau\sigma^2 \end{aligned} \tag{5.5}$$

Note that since the gamma pdf's mean is α/λ, or equivalently, $\alpha\beta$ from (5.3b,c), (5.5) implies that the gamma's mean equals the VG's mean, E[x], where x is particle wealth. See (4.5) for the calculation of the variance from the first moment (the mean) and second moment around zero. Note that the VG's parameter, σ^2, the parameter not shared with the gamma pdf, appears in the VG's second moment around zero, but not its mean. Consequently, because the second moment around zero appears in the variance (4.5), an estimator for σ^2 must be found in terms of ω_ψ and $\tilde{\omega}$ if the ω_ψ's are to be estimated from the VG's variance fitted to IP particle wealth variances without foreknowledge of their ω_ψ's or from empirical income or wealth distributions.

[3] The utility of the Inequality Process' stationary distribution to model the VG's pdf between integer values of tau, τ is not pursued here.

[4] Kotz et al., (2001) define '[[n/2]]' as the greatest integer less than or equal to n/2. Γ( ) is the gamma function (not to be confused with the gamma pdf). The gamma function is defined as $\Gamma(\alpha) \stackrel{\text{def}}{=} \int_0^\infty x^{\alpha-1} e^{-x}\, dx$.

6.0 Re-Expressing VG Parameter σ^2 in Terms of ω_ψ and $\tilde{\omega}$

6.1 The Inequality Process' (IP's) Variance-Gamma (VG) Model of the variance of particle wealth in the ω_ψ equivalence class

The VG pdf's variance of its distributed quantity x, calculated from (5.5) according to (4.5), and stated in Kotz et al. (2001), is:

$$\begin{aligned} \text{variance}_{\mathbf{VG}} &= \tau(\mu^2 + \sigma^2) \\ &= \tau\mu^2 + \tau\sigma^2 \end{aligned} \qquad (6.1)$$

(6.1) is in Kotz et al. (2001) notation. The first term of (6.1), $\tau\mu^2$, equals the IP gamma model's variance in the ω_ψ equivalence class of particles, given (4.1b) and (5.3):

$$\text{variance}_{\boldsymbol{\Gamma}\mathbf{IP}\psi} = \frac{\alpha}{\lambda^2} \approx \left(\frac{1-\tilde{\omega}}{\omega_\psi}\right)\cdot\left(\frac{\tilde{\omega}\quad\mu}{1-\tilde{\omega}}\right)^2 = \frac{(\tilde{\omega}\quad\mu\ )^2}{\omega_\psi\,(1-\tilde{\omega}\ \ )}$$

$$= \frac{(\tilde{\omega}\quad\mu\ \ )}{\omega_\psi}\cdot\frac{(\tilde{\omega}\quad\mu\ \ )}{(1-\tilde{\omega}\ \ )} \qquad (6.2)$$

The μ in (6.2) is the unconditional mean of particle wealth, its meaning in the Inequality Process.

The second term in (6.1), $\tau\sigma^2$, is a non-negative add-on to the gamma pdf's variance in the ω_ψ equivalence class, guaranteeing that the VG's variance is equal to or greater than IP gamma pdf model's variance, and converges to it as σ^2 -> zero. An expression for σ^2 in the ω_ψ equivalence class, σ^2_ψ, in terms of IP parameters would make the IP's VG model as algebraically interpretable and useful for $\omega_\psi| 0.0 < \omega_\psi < 1.0$ as the IP's gamma pdf model is for $\omega_\psi| 0.0 < \omega_\psi < 0.5$.

6.2 Finding an expression for the parameter the VG does not share with the gamma pdf, σ^2_ψ, in terms of IP particle parameters in the ω_ψ equivalence class

Figure 6 and Figure 7 show that an expression for σ^2_ψ in terms of IP particle parameters should act like a dimmer-switch (rheostat) making σ^2_ψ smaller as ω_ψ -> 0.5 and less. (5.2) implies the dimmer-switch interpretation of σ^2_ψ by showing that the VG's characteristic function, and consequently all its statistics, converge to those of the gamma pdf as σ^2_ψ approaches zero. See Figure 8, in which the IP's VG model's variance (6.4) incorporates (6.3) as σ^2_ψ. Figure 8 shows (6.3), an expression for σ^2_ψ, yields good fits for IP parameters $\omega_\psi| 0.0 < \omega_\psi < 1.0$. Compare Figure 8b to Figure 8a or to Figure 7, with ω_ψ's greater than 0.5.

$$\sigma^2_\psi \approx \omega^4_\psi\left(\frac{\tilde{\omega}\quad\mu}{1-\tilde{\omega}}\right) \qquad (6.3)$$

No claim is made for (6.3) other than it fulfills the "dimmer-switch" constraint ω_ψ -> 0.5 implied by (5.2) and (6.3) provides a good approximation as demonstrated in Figure 8. The IP's VG estimator of the variance of particle wealth in the ω_ψ equivalence class becomes with (6.3):

$$\text{variance}_{\mathbf{VGIP}\psi} = \tau(\mu^2 + \sigma^2) \qquad \text{Kotz, et al. (2001) notation}$$

and in IP notation:

$$\approx \left(\frac{1-\tilde{\omega}}{\omega_\psi}\right)\left[\left(\frac{\tilde{\omega}\quad\mu}{1-\tilde{\omega}}\right)^2 + \omega_\psi^4\,\frac{(\tilde{\omega}\quad\mu\ \ )}{(1-\tilde{\omega}\ \ )}\right]$$

$$\approx \mu_\psi\,\frac{(\tilde{\omega}\quad\mu\ \ )}{(1-\tilde{\omega}\ \ )} + \mu_\psi\,\omega_\psi^4 \tag{6.4}$$

ω_ψ^4 is the "dimmer-switch" term forcing the VG's variance to converge to the gamma's as ω_ψ decreases. (6.3) and (6.4) imply that $\omega_\psi = 0.5$ is just approximately where σ_ψ^2 became small enough to be ignored.

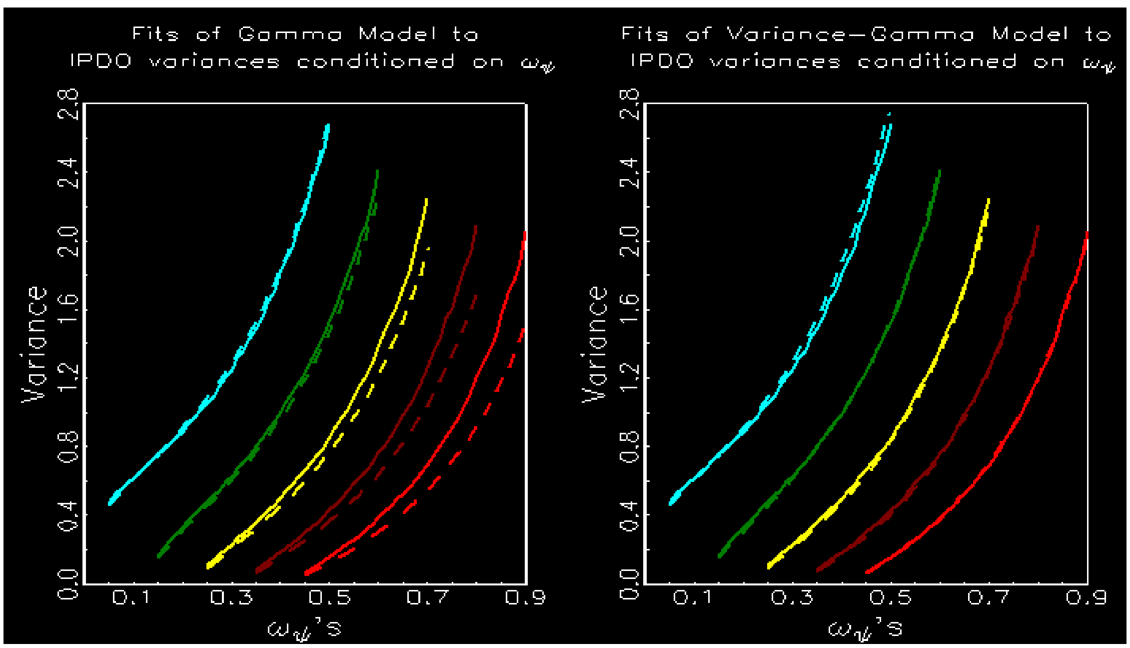

Figure 8

7.0 Conclusions

The present paper finds an expression for σ_ψ^2, the parameter of the variance-gamma (VG) pdf not shared with the gamma pdf, in terms of ω_ψ and $\tilde{\omega}$, in the ω_ψ equivalence class of Inequality Process (IP) particles. ω_ψ is the parameter of an IP particle, the amount of its wealth it loses to another particle if it loses its encounter with that particle. $\tilde{\omega}$ is the harmonic mean of all the ω's in the population of particles.

The IP's variance-gamma (VG) pdf model like the IP's gamma pdf model is a parametric model of the distribution of particle wealth in the ω_ψ equivalence class of IP particles, and all other statistics of particle wealth in that equivalence class. A parametric model of wealth statistics in the ω_ψ equivalence class is needed to test whether statistics of IP wealth are consistent with the verbal theory from which the IP was abstracted (Angle, 1983, 1986), and whether the IP implies statistical invariances in empirical distributions of income and wealth. The hypotheses tested are in terms of IP particle parameters. It is possible to do these tests numerically but numerics are less conclusive than the clarity, precision, compactness, insightfulness, comparability, communicability, and clues for further research that statistics expressed in an algebra of IP particle parameters provide ***if*** the parametric model fits statistics of IP particle wealth.

The IP's gamma pdf model is satisfactory as a parametric model of IP particle wealth in the ω_ψ equivalence class ***if*** the ω_ψ in question is less than 0.5, i.e., particles in its equivalence class lose less than half their wealth in any one encounter with another particle. But the IP's gamma pdf model leaves the top half of the interval on which the ω's are defined, $\omega_\psi | \; 0.50 < \omega_\psi < 1.0$, terra incognita because ω's in this interval generate IP particle wealth distributions with heavier-than-gamma pdf right tails, i.e., the gamma pdf does not fit. But the variance-gamma (VG) pdf, with its three parameters expressed in terms of IP particle parameters fits the variance of particle wealth in the ω_ψ equivalence class for $\omega_\psi | \; 0.50 < \omega_\psi < 1.0$ and converges to the IP's gamma pdf model as ω_ψ -> 0.50 from above and continues downward. Thus the IP's VG pdf model generalizes the IP's gamma pdf model and enables the testing of IP hypotheses expressed in terms of its particle parameters for particle wealth distributions with heavier-than-gamma right tails, and, potentially empirical distributions of income and wealth with heavier-than-gamma tails. The reason why the variances of distributions with heavier-than-gamma tails are fitted rather than their relative frequency distributions of wealth or income is explained in the text. Testing IP hypotheses against empirical distributions with heavier-than-gamma-tails with the IP's VG model is a project for further research.

8.0 Appendices

Appendix A.1: Confirmed Inequality Process (IP) Hypotheses

1. The universal pairing (all times, all places, all cultures, all races) of the appearance of extreme social inequality in the chiefdom, the society of the god-king, after egalitarian hunter/gatherers acquire a storeable food surplus (Angle, 1983, 1986).
2. The pattern of the Gini concentration ratio of personal wealth and income over the course of techno-cultural evolution beyond the chiefdom (Angle, 1983, 1986).
3. The right skew and gently tapering right tail of all distributions of income and wealth, a broad statement of the Pareto Law of income and wealth distribution. (Angle, 1983, 1986).
4. a) The sequence of shapes of the distribution of labor income by level of worker education, b) why this sequence of shapes changes little over decades, and c) why a gamma pdf model works well for fitting the distribution of labor income at each level of worker education (Angle, 1990, 2002, 2003, 2006, 2007b);
5. How the unconditional distribution of personal income appears to be gamma regardless of level of geographic aggregation although the gamma distribution is not closed under mixture (Angle, 996);
6. Why sequences of Gini concentration ratios of labor income by level of education from low to high recapitulates the sequence of Gini concentration ratios of labor income over the course of techno-cultural

evolution (a social science analogue of "ontogeny repeats phylogeny" (Angle, 1983, 1986, 2002, 2003, 2006, 2007b);
7. Why the sequence of shapes of the distribution of labor income by level of education from low to high recapitulates the sequence of shapes of the distribution of labor income over the course techno-cultural evolution, a social science analogue of "ontogeny repeats phylogeny" (Angle, 1983, 1986, 2002, 2003, 2006, 2007b);
8. The dynamics of the distribution of labor income conditioned on education as a function of the unconditional mean of labor income and the distribution of education in the labor force (Angle, 2003a, 2006, 2007b);
9. The pattern of correlations of the relative frequency of an income smaller than the mean with relative frequencies of other income amounts (Angle, 2005; 2007a).
10. The surge in the relative frequency of large incomes in a business expansion (Angle, 2007b);
11. The "heaviness" of the right tail of income being heavy enough to account for total annual wage and salary income in the U.S. National Income and Product Accounts (Angle, 2002c; 2003a).
12. Why and how the distribution of labor income is different from the distribution of income from tangible assets; (Angle, 1997)
13. Why the IP's parameters estimated from a time-series of the labor incomes of individual workers are ordered as predicted by the IP's meta-theory and approximate estimates of the same parameters from cross-sectional data on the distribution of wage income conditioned on education; (Angle, 2002)
14. The Kuznets Curve in the Gini concentration ratio of labor income during the industrialization of an agrarian economy; (Angle, Nielsen, and Scalas, 2009)
15. An elaboration of the basic Inequality Process in which all particles have an equal probability of winning a competitive encounter for wealth. This elaboration allows a majority group of particles to rig the probability of one of its members winning a competitive encounter with a member particle at $.5 + \varepsilon$, which equals probability of the minority group particle losing that encounter. This elaboration of the IP yields the following features of the joint distribution of personal income to African-Americans and 'other Americans' (i.e., non-African-Americans): a) the smaller median personal income of African-Americans than 'other Americans'; b) the difference in shapes between the African-American distribution of personal income and that of 'other Americans'; this difference corresponds to a larger Gini concentration of the African American distribution; c) the % minority effect on discrimination (the larger the minority, the more severe discrimination on a per capita basis, as reflected in a bigger difference between the median personal incomes of African-Americans and 'other Americans' in areas with a larger % African-American); d) the relatively high ratio of median African-American personal income to the median of 'other Americans' in areas where the Gini concentration ratio of the personal income of 'other Americans' is low; e) the relatively high ratio of median African-American to that of 'other Americans' in areas where the median income of 'other Americans' is high; f) the fact that relationships in d) and e) can be reduced in magnitude by controlling for a measure of economic development of an area or % African-American; g) the greater hostility of poorer 'other Americans' to African-Americans than wealthier 'other Americans' (Angle, 1992).

Appendix A.2: Seven Maxims of Economics Implied by the IP[5]

Maxim of Economics:	Inequality Process' Implication:
1) All distributions of labor income are right skewed with tapering right tails; hence the impossibility of radical egalitarianism, the ideologically motivated findings of Vilfredo Pareto's study of income and wealth distribution.	The IP generates right skewed distributions shaped like empirical distributions of labor income or personal assets (depending on the particle parameter). The IP implies that the unconditional distribution of personal money income from labor is an exponential family pdf (probability density function) shape mixture. Such a mixture has a right tail approximately as heavy as empirical right tails of money income and the Pareto pdf, the model of those right tails preferred in economics.
2) Differences of wealth and income arise easily, naturally, and inevitably via a ubiquitous stochastic process; a general statement of Gibrat's Law; hence the impossibility of radical egalitarianism. Like Pareto, Robert Gibrat's interest in income distribution was motivated by the desire to deny the possibility of a radically egalitarian income distribution.	In the IP, differences of wealth arise easily, naturally, and inevitably, via an ubiquitous stochastic process.
3) A worker's earnings are tied to that worker's productivity [i.e., a central tenet of economics since Aesop's fable of the ant and the grasshopper was all there was to economics] but there is a wide variety of dissimilar returns to similarly productive workers.	An IP particle's expected wealth is determined by the ratio of mean productivity in the population to that particular particle's productivity (the ratio of the harmonic mean of particle parameters in the population to an individual particle's parameter). The IP implies a distribution around this expectation whose shape is determined by each particle's productivity.
4) Labor incomes small and large benefit from a business expansion strong enough to increase mean labor income, i.e., there is a community of interest between all workers regardless of their earnings in a business expansion. A conclusion encapsulated in a favorite saying of mainstream economists: "A rising tide lifts all boats."	In the IP's Macro Model, an increase in the unconditional mean of wealth increases all percentiles of the stationary distribution of wealth by an equal factor. In pithy statement form: "A rising tide lifts the logarithm of all boats equally.".
5) Competition transfers wealth to the more productive of wealth via transactions without central direction, i.e., via parallel processing.	In the IP, competition between particles causes wealth to flow via transactions from particles that are by hypothesis and empirical analogue less productive of wealth to those that are more productive of wealth, enabling the more productive to create more wealth, explaining economic growth without a) requiring knowledge of how wealth is produced or b) central direction, i.e., with a minimum of information, two reasons for hypothesizing that the IP would arise to allocate wealth in every economy. These features enable the IP to operate homogeneously over the entire course of techno-cultural evolution independently of wealth level.

[5] Angle, 2006e, 2013a.

6) The ratio of mean labor income in two different occupations remains roughly constant as long as the skill levels in the two occupations remain roughly constant.	This conclusion falls out of the ratio of expected wealth in subsets of the population with two distinct values of wealth productivity (distinct values of the particle parameter).
7) Competition and transactions maximize societal gross product and over the long run drive techno-cultural evolution.	The Inequality Process operates as an evolutionary wealth maximizer in the whole population of particles, given a relaxation of the zero-sum constraint on wealth transfers within the model, by transferring wealth to the more productive and doing so more efficiently as mean wealth productivity in the population of particles increases.

Appendix A.3 : The IP Implies Empirical Invariances of Stock Market Statistics

1.	Association between greater corporate market capitalization and a lower mean absolute value of the logarithm of its daily stock returns. Volatility is defined here as the mean absolute logarithm of daily returns. Source: Malkiel (2015:124)
2.	Big stock price movements down are associated with greater volatility, while big stock price movements up are associated with lower volatility. In finance this phenomenon is terms "leverage effect". Source: Tsay (2013:177).
3.	(t+k) autocorrelations of daily log returns to stocks of a particular corporation converge to near zero for k small beyond k = 1. Sources: Georgakopoulos (2015:115), Resnick (2007:6), Tsay (2013:178).
4.	t+k autocorrelations of squared daily log returns to stocks of a particular corporation show long term memory (i.e., do not converge to zero) as k increases. Sources: Georgakopoulos (2015:115), Resnick (2007:6).
5.	Bollinger Band-like bounded volatility of particle wealth. Source: Kaufman (2005: 294).

Appendix B.1: The IP's Similarity To The Kinetic Theory Of Gases

Two substitutions into the Inequality Process' transition equations for the exchange of a positive quantity, x, between two particles, (B.1a,b), transform them into the transition equations of the interacting particle system model of the kinetic theory of gases (Angle, 1990), the best known statistical law of physics). The transition equations of the Inequality Process are:

$$x_{it} = x_{i(t-1)} + d_t \omega_\theta x_{j(t-1)} - (1-d_t)\omega_\psi x_{i(t-1)}$$
$$x_{jt} = x_{j(t-1)} - d_t \omega_\theta x_{j(t-1)} + (1-d_t)\omega_\psi \quad {}_{i(t-1)}$$

(B.1a,b)

$x_{it} \equiv$ particle i's wealth at time − step t in multiples of μ, the unconditional mean of wealth

$x_{j(t-1)} \equiv$ particle j's wealth at time − step $(t-1)$

$0 < \omega_{\theta j} < 1.0$, fraction lost in loss by particle j

$0 < \omega_{\psi i} < 1.0$, fraction lost in loss by particle i

$d_t =$ an i.i.d. 0,1 uniform discrete r.v. equal to 1 with probability .5 at time − step t (a Bernoulli variable)

$\mu =$ unconditional mean of wealth

If

1) d_t, a discrete 0,1 uniform random variable is replaced by a continuous [0,1] uniform random variable, ϵ_t , and,

2) the ω's are replaced by 1.0,

then (B.1a,b) has been transformed into the transition equations of the particle system of the kinetic theory of gases:

$$
\begin{aligned}
x_{it} &= \varepsilon_t\left(x_{i(t-1)} + x_{j(t-1)}\right) \\
x_{jt} &= (1-\varepsilon_t)\left(x_{i(t-1)} + x_{j(t-1)}\right)
\end{aligned}
\qquad \text{(B.1c,d)}
$$

where:

$x_{i(t-1)}$ = particle i's kinetic energy at time-step (t-1)
x_{jt} = particle j's kinetic energy at time-step t
ϵ_t = a [0,1] continuous uniform random variable at time-step t

(B.1c,d) is Whitney's (1990:103) statement of the transition equations for the transfer of kinetic energy between two molecules in the kinetic theory of gases. So, in this narrow sense, it is certain that the Inequality Process is like an established model of statistical physics, part of Auguste Comte's 19th century vision of what sociology should become.

The transformation from (B.1.a,b) into (B.1.c,d) is perhaps more easily recognized if (B.1.a,b) is re-written as

$$
\begin{aligned}
x_{it} &= \left(1-\omega_\psi\right) x_{i(t-1)} + d_t\left(\omega_\psi\, x_{i(t-1)} + \omega_\theta\, x_{j\,(t-1)}\right) \\
x_{jt} &= \left(1-\omega_\theta\right) x_{j\,(t-1)} + (1-d_t)\left(\omega_\psi\, x_{i(t-1)} + \omega_\theta\, x_{j\,(t-1)}\right)
\end{aligned}
\qquad \text{(B.1e,f)}
$$

with $d_t \to \epsilon_t$ and the ω's -> 1.0. Both particle systems are otherwise identical apart from the labels on the variables. In both particle systems, particles are collectively isolated. Since in both particle systems, random pairings of particles result in transfers of a positive quantity that is neither created nor destroyed, the sum of that quantity over all particles is constant.

B.2 Sketch of Proof The Gamma PDF is Not Implied By The Inequality Process

The gamma pdf is a maximum entropy distribution, i.e., the result of maximizing the entropy statistic subject to two linear constraints: a) that the sum of all x's (the distributed quantity, here particle wealth) is constant, and, b) the sum of the logarithm of all x's is constant. Boltzmann found the stationary distribution of molecular kinetic energy implied by the kinetic theory of gases by maximizing the entropy statistic subject to the first constraint. The same constraint holds true for the Inequality Process: the sum of particle wealth exiting an encounter with another particle is equals their sum going into that encounter. However, the equality of the product of the two particle wealth amounts after the encounter with that before *does not hold, and consequently the sum of the logarithms of their particle wealth amounts also does not hold.* It would have to if the Inequality Process' stationary distribution of wealth were the gamma pdf.

9.0 References

Angle, John. 2019. "The Inequality Process' PDF Approximation Model For Heavy-Tail Financial Distributions. ***Proceedings of the 2019 Joint Statistical Meetings***, (American Statistical Association, Economic and Business Statistics Section, Joint Statistical Meetings). Pp. 227-244. CD-ROM. Alexandria, VA: American Statistical Association.

_____. 2018. "The Inequality Process' Footprint On Stock Market "Stylized Facts". ***Proceedings of the 2018 Joint Statistical Meetings***, (American Statistical Association, Economic and Business Statistics Section, Joint Statistical Meetings). Pp. 759-773. CD-ROM. Alexandria, VA: American Statistical Association.

_____. 2017, "Detecting Politically Motivated Tampering With Workers' Labor Income In Survey Data". ***Proceedings of the 2017 Joint Statistical Meetings*****,** (American Statistical Association, Social Statistics Section, Joint Statistical Meetings). Pp.1681-1686. CD-ROM. Alexandria, VA: American Statistical Association.

------. 2013a. "How To Win Acceptance of The Inequality Process as Economics?". ***Management and Society Review*** 2(#2): 117-134.

-----. 2013b. "The Inequality Process is "Demonic": Conditional Entropy Maximization and Minimization and Wage Incomes". ***Proceedings of the 2013 Joint Statistical Meetings*****,** (American Statistical Association, Social Statistics Section, Joint Statistical Meetings. Pp.1375-1389. CD-ROM. Alexandria, VA: American Statistical Association.

_____. 2012. "The Inequality Process v. The Saved Wealth Model: Which is the More Likely to Imply an Analogue of Thermodynamics in Social Science". **Journal of Mathematical Sociology**. An earlier version of this paper **"** The Inequality Process vs. The Saved Wealth Model. Two Particle Systems of Income Distribution; Which Does Better Empirically?" is on RePEc/MPRA. [*http://mpra.ub.uni-muenchen.de/20835/*], a revision of "A Test of Two Similar Particle System Models of Wage Income Distribution Conditioned on Education", 2009 Joint Statistical Meetings, in its proceedings volume.

_____. 2010. "The Inequality Process as an Evolutionary Algorithm". ***Proceedings of the 2010 Joint Statistical Meetings*****,** (American Statistical Association, Section on Statistical Learning and Data Mining), Joint Statistical Meetings. Pp. 2295-2309. CD-ROM. Alexandria, VA: American Statistical Association.

_____, François Nielsen, and Enrico Scalas. 2009. "The Kuznets Curve and the Inequality Process". Pp. 125-138. In Banasri Basu, Bikas K. Chakrabarti, Satya R. Chakravarty, Kausik Gangopadhyay, editors, ***Econophysics and Economics of Games, Social Choices and Quantitative Techniques***. (Proceedings of the Econophys-Kolkata IV Conference, March 2009, Kolkata, India, jointly sponsored by the Indian Statistical Institute and the Saha Institute of Nuclear Physics.) [http://www.isical.ac.in/cmp/econophys4.com/]. Milan: Springer.

_____. 2007a. "The Macro Model of the Inequality Process and The Surging Relative Frequency of Large Wage Incomes". Pp. 171-196 in A. Chatterjee and B.K. Chakrabarti, (eds.), ***The Econophysics of Markets and Networks*** (Proceedings of the Econophys-Kolkata III Conference, March 2007 [http://www.saha.ac.in/cmp/econophys3.cmp/]). Milan: Springer [ISBN: 978-8847006645]. [on-line at: http://arxiv.org/abs/0705.3430].

_____. 2007b. "The Inequality Process is an evolutionary process". ***The Constructal Theory of Social Dynamics***. Adrian Bejan and Gilbert Merkx, eds. (Proceedings of the Conference on the Constructal Theory of Social Dynamics, Duke University, April 2006). New York: Springer [ISBN: 978-0387476803].

_____. 2006a. "Not a Hollowing Out, A Stretching: Trends in U.S. Nonmetro Wage Income Distribution, 1961-2003". Self published as a downloadable *.pdf file on RePEc/MPRA. [on-line at: http://mpra.ub.uni-muenchen.de/10111/].

_____. 2006b (received 8/05; electronic publication: 12/05; hardcopy publication 7/06). "The Inequality Process as a wealth maximizing algorithm"'. ***Physica A: Statistical Mechanics and Its Applications*** 367:388-414 (DOI information: 10.1016/j.physa.2005.11.017). A draft version of the paper can be downloaded from the Luxembourg Income Study website: [http://www.lisproject.org/publications].

_____. 2006c. "A measure of intergroup discrimination: color and wage income in the nonmetropolitan U.S.". ***Proceedings of the 2006 Joint Statistical Meetings*****, (**American Statistical Association, Social Statistics Section), August, 2006. Pp, 1889-1894. [CD-ROM], Alexandria, VA: American Statistical Association.

_____. 2006d. "The Inequality Process as an evolutionary process". ***Proceedings (late breaking papers) of the Genetic and Evolutionary Computation Conference (GECCO), July, 2006***. Late breaking paper (lbp) #137.

_____. 2006e. "A comment on Gallegati et al.'s "Worrying Trends in Econophysics" ". Pp. 250-253 in A. Chatterjee and B.K. Chakrabarti, (eds.), ***The Econophysics of Stocks and Other Markets*** (Proceedings of the Econophys Kolkata II Conference, February, 2006
[http://www.saha.ac.in/cmp/econophys2.cmp/]). Milan: Springer. (ISBN: 978-8847005013).

_____. 2003a. "The dynamics of the distribution of wage and salary income in the nonmetropolitan U.S.". ***Estadistica***.55: 59-93.
_____. 2003b. "Inequality Process, The". An entry in T. Liao, et al., (eds.), ***The Encyclopedia of Social Science Research Methods***. Volume 2: 488-490. Thousand Oaks, CA: Sage.
_____. 2003c. "Imitating the salamander: a model of the right tail of the wage distribution truncated by topcoding". November, 2003 ***Conference of the Federal Committee on Statistical Methodology***, [http://www.fcsm.gov/events/papers2003.html].
_____. 2002a. "The statistical signature of pervasive competition on wages and salaries". ***Journal of Mathematical Sociology***. 26:217-270. [abstracted in Cambridge Science Abstracts and Sociological Abstracts].
_____. 2002c. "Contingent forecasting of bulges in the left and right tails of the nonmetro wage and salary income distribution". ***Proceedings of the 2002 Federal Forecasters' Conference***. (Awarded "Best Paper" by Organizing Committee of the 2003 Federal Forecasters' Conference, 9/18/03). Washington, DC: U.S. Government Printing Office.
_____. 1999. "Contingent forecasting of the size of a vulnerable nonmetro population". ***Proceedings of the 1999 Federal Forecasters' Conference***. Pp. 161-169. (Awarded "Best Paper" by Organizing Committee of the 2000 Federal Forecasters' Conference, 9/14/00). Washington, DC: U.S. Government Printing Office.
_____. 1997. "A theory of income distribution". ***1997 Proceedings of the American Statistical Association*****,** Social Statistics Section. Pp. 388-393. Alexandria, VA: American Statistical Association.
_____. 1996. "How the gamma law of income distribution appears invariant under aggregation". ***Journal of Mathematical Sociology***. 21:325-358.
_____, 1993a. "Deriving the size distribution of personal wealth from 'the rich get richer, the poor get poorer'". ***Journal of Mathematical Sociology*** 18:27-46.
_____, 1992. "The Inequality Process and the distribution of income to blacks and whites". ***Journal of Mathematical Sociology*** 17:77-98. [Included in the Inventory of the John Hope Franklin Papers, 1889-1998 and undated, bulk 1970-1998, Rare Book, Manuscripts, and Special Collections Library, Duke University Libraries, http://library.duke.edu/digitalcollections/rbmscl/franklinjohnhope/inv/].
_____. 1990. "A stochastic interacting particle system model of the size distribution of wealth and income." ***1990 Proceedings of the American Statistical Association*****,** Social Statistics Section. Pp. 279-284. Alexandria, VA: American Statistical Association.
_____, 1986. "The surplus theory of social stratification and the size distribution of Personal Wealth." **Social Forces** 65:293-326.
_____. 1983. "The surplus theory of social stratification and the size distribution of Personal Wealth." ***1983 Proceedings of the American Statistical Association*****,** Social Statistics Section**.** Pp. 395-400. Alexandria, VA: American Statistical Association.
Aptech Systems, Inc. (2012). GAUSS (Version 12.0) ***User Guide***. Black Diamond, WA: Aptech Systems.
Byrro Ribeiro, Marcelo. 2020. ***Income Distribution Dynamics of Economic Systems: An Econophysics Approach.*** Cambridge, UK: Cambridge University Press.
Chen, Shu-Heng. 2018. ***Agent-Based Computational Economics: How the idea originated and where it is going.*** New York: Routledge.
Finlay, Richard. 2009. ***The Variance Gamma (VG) Model With Long Run Dependence: A model for financial data incorporating long range dependence in squared returns***. Saarbrücken, Germany: VDM Verlag Dr. Müller.
Fiorani, Filo. 2009. ***Option Pricing Under The Variance Gamma Process***. Saarbrücken, Germany: VDM Verlag Dr. Müller.
Georgakoupoulos, Harry. 2015. ***Quantitative Trading With R: Understanding Mathematical and Computational Tools from a Quant's Perspective.*** New York: Palgrave Macmillan.
Kaufman, Perry. 2005. ***New Trading Systems and Methods.*** Hoboken, New Jersey: Wiley.
Kleiber, Christian, and Samuel Kotz. 2003. ***Statistical Size Distributions in Economics and Actuarial Science***. Hoboken, New Jersey: Wiley.
Kotz, Samuel, Tomasz Kozubowski, Krzysztof Podgorski. 2001. ***The Laplace Distribution and Generalizations***. Boston, Massachusetts: Birkhäuser.
Lux, Thomas. 2005. "Emergent statistical wealth distributions in simple monetary exchange models: a critical review". Pp. 51-60 in A. Chatterjee, S. Yarlagadda, and B.K. Chakrabarti, (eds.), ***Econophysics of Wealth Distributions***, (the proceedings volume of the International Workshop on the Econophysics of Wealth Distributions, March, 2005, Kolkata, India.

Madan, Dilip, and Eugene Seneta. 1990. "The Variance Gamma (V.G.) Model for Share Market Returns". ***The Journal of Business***. Vol. 63, #4, pp. 511-524.
Malkiel, Burton. 2015[1973]. ***A Random Walk Down Wall Street.*** New York: Norton.
Resnick, Sidney. 2007. ***Heavy-Tail Phenomena: Probabilistic and Statistical Modeling.*** New York: Springer.
Seneta, Eugene. 2004. "Fitting the Variance-Gamma Model to Financial Data". ***Journal of Applied Probability/Stochastic Methods and Their Application***. Vol. 41, pp. 177-187.
Tsay, Ruey. 2013. ***An Introduction to Analysis of Financial Data with R.*** New York: Wiley.
Whitney, Charles. 1990. ***Random Processes in Physical Systems***. New York: Wiley Interscience.