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Abstract

This study explores the evolution of income inequality in an economy featuring an endogenous transition from stagnation to growth. We incorporate heterogeneous households into a Schumpeterian model of endogenous takeoff. In the pre-industrial era, the economy is in stagnation, and income inequality is determined by an unequal distribution of land ownership and remains stationary. When takeoff occurs, the economy experiences innovation and economic growth. In this industrial era, income inequality gradually rises until the economy reaches the balanced growth path. We calibrate the model for a quantitative analysis and compare the simulation results to historical data in the UK. Extending the analysis to allow for endogenous labor supply, we find that endogenous labor supply introduces a channel through which inequality contributes to shaping the transition path of the economy and that households sort themselves into a leisure class that supply zero labor and the rest of society that supplies labor.

JEL classification: D30, O30, O40
Keywords: income inequality, innovation, economic growth, endogenous takeoff

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1 Introduction

What is the historical relationship between growth and inequality and, if any, what drives it? These questions have a long tradition in economics. Kuznets (1955) famously hypothesized that industrialization causes income inequality to rise. Williamson (1980, 1985) provides evidence for this hypothesis, showing that in Britain income inequality increases after the Industrial Revolution and keeps rising until the mid-19th century.[1] More broadly, there is abundant evidence that periods characterized by waves of innovation in technology and business organization (e.g., the period straddling the second half of the 19th century and the pre-wars 20th century) display higher and rising inequality. History thus suggests that *innovation-driven growth accelerations* cause rising inequality. The recent study by Madsen *et al.* (2021) makes this point quite forcefully. It carries out "a long-run econometric analysis for 21 OECD countries using annual data over the period 1860–2015" (p. 477) and shows that "intangibles have been a contributing factor in wealth inequality since 1860 and that the marked increase in investment in intangible assets has been a significant driver of the increasing inequality since the 1970s" (p. 477). The takeaway of the paper is that growth accelerations fueled by investment in intangibles cause rising inequality.

To illuminate analytically the mechanism that this evidence points to, we need to understand the origins of the transition from stagnation to growth and how this transition affects the evolution of the income distribution. In this study we develop a growth-theoretic framework that enables us to characterize analytically the endogenous takeoff of an economy and the evolution of the income distribution from stagnation to growth.

The framework builds on two branches of growth economics. The first is Unified Growth Theory (Galor and Weil 2000, Galor 2005 and 2011), developed to explain the transition from Malthusian stagnation to modern growth. The second is the theory of endogenous technological change (Romer 1990), developed to formalize the idea that innovation is the key driver of economic growth. Exploiting ideas from both branches, Peretto (2015) extends the Schumpeterian growth model to allow for endogenous takeoff. We incorporate in his model the approach to heterogeneous households in Chu and Cozzi (2018), to obtain a structure that allows us to characterize analytically the endogenous takeoff of the economy and its transition dynamics from stagnation to growth. The goal is to understand the evolution of the personal distribution of income throughout the process. The source of heterogeneity across households is the unequal distribution of assets. Accordingly, our framework builds on the literature, recently revived by Piketty (2014), that considers wealth inequality as the root cause of income inequality. One advantage of our analysis is that we do not impose any particular assumption on the wealth distribution except that it has well-defined moments. This property, in turn, allows us to obtain analytical solutions for popular measures of inequality, in particular the Gini coefficient.

Our first main finding is that the economy initially features a pre-industrial era, characterized by stagnation with very slow economic growth, in which income inequality is determined by the unequal distribution of land ownership and remains stationary. When the size of the market becomes sufficiently large due to population growth, the economy begins to experience innovation, and the rate of economic growth begins to rise gradually, until it converges

[1] Lindert (2000a, b) also finds a rise in income inequality in Britain in as early as the late 18th century.

to the steady state. This growth acceleration, fueled by a rising rate of return to innovation, causes income inequality to rise gradually, until it reaches a constant steady-state value. The mechanism at the heart of these dynamics is that the relative importance of asset income increases over time. The evidence supporting this interpretation is strong: Madsen (2017) shows that asset returns are an important determinant of income inequality and, as stated above, Madsen *et al.* (2021) carries out the most comprehensive study documenting the role of investment in the intangibles that drive the asset-market valuation of firms.

We also calibrate the model to current data in the UK to perform a quantitative analysis. Simulating the transitional paths of the output growth rate and the real interest rate, we find that the increase in the simulated growth rate and the simulated interest rate is consistent with historical data in the UK. We simulate the transitional path of income inequality and find that it increases sharply when the takeoff occurs. When the economy reaches the steady state, income inequality is almost twice as high as the level prior to the takeoff, and the steady-state level of income inequality is in line with the Gini coefficient of income in the UK in recent time.

We obtain the result discussed above in a baseline model with inelastic labor supply. The model has three main strengths: (i) it is analytically tractable; (ii) it identifies sharply the role of growth accelerations as a main driver of rising income inequality; (iii) it measures inequality with a well understood and widely used summary statistic of the shape of the distribution of income. The model, on the other hand, has one main weakness: inequality plays no role in shaping the transition from stagnation to growth. We thus extend the analysis allowing for labor income inequality due to endogenous labor supply. We find that endogenous labor supply introduces a channel through which inequality contributes to shaping the transition path of the economy, while it preserves the features (i)-(iii) that makes the baseline model so useful. Specifically, labor income inequality consists of two margins: an extensive margin along which households sort themselves into a leisure class that supply zero labor and the rest of society that supplies labor; an intensive margin along which households supply labor as a decreasing function of their consumption share, which in turn is an increasing function of their wealth share. The leisure class consists of households that are wealthy enough to find optimal to forgo labor income. Our model, therefore, generates endogenously the two-class structure—workers vs. capitalists—that is widely used in the literature on inequality that builds on the classical theory of the distribution of income. Models in this tradition, however, are often silent about the shape of the cross-sectional distribution of income because the imposed within-class homogeneity reduces the cross section of the whole population to two degenerate distributions. Consequently, work that uses this approach tends to measure inequality with grand ratios like the wealth share of GDP (see, e.g., Madsen *et al.* 2021 discussed above). Our structure, in contrast, allows for heterogeneity in wealth, labor supply, and thus overall income, within each class. One of our results is that a simple summary statistic of labor supply heterogeneity captures the channel through which inequality affects aggregate outcomes.

This study relates to the vast literature on innovation and economic growth. The seminal contribution by Romer (1990) features the invention of new products (i.e., horizontal innovation) as the engine of growth. Aghion and Howitt (1992) develop the creative-destruction Schumpeterian model in which economic growth is driven by the development of higher-quality products (i.e., vertical innovation) that displace existing products.[2] Subsequent studies, such as Smulders (1994), Smulders and van de Klundert (1995), Peretto (1994, 1998, 1999) and Dinopoulos and Thompson (1998), combine vertical in-house innovation by incumbent firms and horizontal innovation by entrant firms to develop the class of Schumpeterian creative-accumulation models with endogenous market structure.[3] This study contributes to this literature by introducing heterogeneous households in a tractable creative-accumulation model that features an endogenous takeoff. The goal is to explore the effects of innovation on the evolution of income inequality during the historical transition from stagnation to growth.

This study also relates to the literature on inequality and economic growth. The study most directly related to our work is Madsen *et al.* (2021) discussed above. It uses a first-generation Schumpeterian model to set up an empirical exercise guided by theory, although it restricts attention to the model's steady state. We differ chiefly in that we use a Schumpeterian model with endogenous market structure, and use our model's non-linear transitional dynamics with phase transitions to go after analytical results on the historical relationship between growth and inequality.

Early studies in this literature explore how inequality affects economic growth via capital accumulation; see for example Galor and Zeira (1993) and Aghion and Bolton (1997). Galor and Moav (2004) shows that in the early (late) stage of development, in which the accumulation of physical (human) capital is the main engine of growth, inequality stimulates (stifles) economic growth. Subsequent studies consider how inequality affects the demand and supply of resources for innovation in the Romer (1990) model; see for example, Chou and Talmain (1996), Zweimuller (2000), Foellmi and Zweimuller (2006) and Garcia-Penalosa and Wen (2008). Recent studies by Jones and Kim (2018) and Aghion *et al.* (2019) focus on the relationship between innovation and top-income inequality in the first-generation Schumpeterian model.[4] This study differs from these contributions by considering a Schumpeterian model with endogenous takeoff and analyzing the historical evolution of income inequality from stagnation to growth. The recent study by Madsen and Strulik (2020) also explores the evolution of income inequality, measured as the ratio of land rents to wages, from stagnation to growth arising from land-biased technological change driven by education. We, instead, consider other measures of income inequality, such as the Gini coefficient and the top income share, in a Schumpeterian, innovation-driven growth model.

Finally, this study relates to the literature on the Industrial Revolution and the transition to modern economic growth. As mentioned, Unified Growth Theory (Galor and Weil 2000, Galor 2005 and 2011) explores how the quality-quantity trade-off in child-rearing and the associated process of human capital accumulation allow an economy to escape the Malthusian trap and experience economic growth.[5] Galor, Moav and Vollrath (2009) explore how the inequality of land ownership in the pre-industrial era affects the transition of an economy to the industrial era via the emergence of human-capital promoting institutions. Although

[2] See also Grossman and Helpman (1991) and Segerstrom *et al.* (1990).

[3] Laincz and Peretto (2006), Ha and Howitt (2007), Madsen (2008, 2010) and Ang and Madsen (2011) provide supportive empirical evidence for this broad class of Schumpeterian models.

[4] Other studies, such as Chu (2010), Chu and Cozzi (2018) and Chu *et al.* (2019, 2021), analyze the effects of patent policy and monetary policy on innovation and income inequality.

[5] See Galor and Moav (2002), Galor and Mountford (2008) and Ashraf and Galor (2011) for other studies and empirical evidence that supports Unified Growth Theory.

the Schumpeterian model in Peretto (2015) features exogenous population growth and does not feature human capital accumulation, the innovation-driven takeoff in the model captures the Industrial Revolution, which is arguably the most important economic takeoff in human history.[6] Furthermore, this tractable growth-theoretic framework allows us to study analytically how innovation affects the rate of return on assets, and thereby the evolution of income inequality, when we incorporate heterogeneous households in the model.

The rest of this study is organized as follows. Section 2 presents the model. Section 3 analyzes the dynamics and derives the evolution of income inequality. Section 4 performs a quantitative analysis. Section 5 considers labor income inequality due to endogenous labor supply. Section 6 concludes.

2 A Schumpeterian growth model with heterogeneous households and endogenous takeoff

We introduce heterogeneous households as in Chu (2010) and Chu and Cozzi (2018) to the Schumpeterian model of endogenous takeoff in Peretto (2015). Our analysis provides a complete closed-form solution for economic growth and income inequality from stagnation to takeoff and eventually to the balanced growth path.

2.1 Heterogeneous households

There is a continuum of mass one of households indexed by $h \in [0,1]$. Household h has preferences

$$U(h) = \int_0^\infty e^{-\rho t} \ln c_t(h) dt, \tag{1}$$

where $\rho > 0$ is the subjective discount rate and $c_t(h)$ is household consumption of the final good.[7] The household maximizes (1) subject to

$$\dot{a}_t(h) = r_t a_t(h) + w_t L_t - c_t(h), \tag{2}$$

where $a_t(h)$ is household wealth and r_t is the real interest rate. The household supplies L_t units of labor inelastically to earn wage income $w_t L_t$. The household's labor endowment (the mass of identical household members) grows at rate $\lambda > 0$, i.e., $L_t = L(0) e^{\lambda t}$, $L(0) = 1$.

Standard dynamic optimization yields the familiar Euler equation

$$\frac{\dot{c}_t(h)}{c_t(h)} = r_t - \rho,$$

[6] Mokyr (2016) argues that innovations in Europe gave rise to the Industrial Revolution and sustained economic growth that subsequently spread across the world.

[7] For simplicity, we assume that flow utility is a function of the household's total consumption, rather than the mass of identical household members multiplied by the utility of consumption per household member. This allows us to abstract from differentiating between household-level consumption and individual-level consumption given that the distinction is not important to our research question.

A property of this saving rule that is quite important for our research question is that, due to the homothetic preferences (1), the welfare-maximizing growth rate of consumption is the same across households. Consequently, we can write

$$\frac{\dot{c}_t(h)}{c_t(h)} = \frac{\dot{C}_t}{C_t} = r_t - \rho. \tag{3}$$

where $C_t \equiv \int_0^1 c_t(h)dh$ is aggregate consumption.

2.2 Final good

A competitive representative firm produces a final good G_t that can be consumed, used to produce intermediate goods, invested in the improvement of the quality of existing intermediate goods, or invested in the creation of new intermediate goods. The final good is the numeraire so its price is $P_G \equiv 1$. The production technology is

$$G_t = \int_0^{N_t} X_t^{\theta}(i) \left[Z_t^{\alpha}(i) Z_t^{1-\alpha} L_t^{\gamma}(i) R_t^{1-\gamma}\right]^{1-\theta} di, \tag{4}$$

where $\{\theta, \alpha, \gamma\} \in (0, 1)$. N_t is the mass of non-durable intermediate goods, whereas $L_t(i)$ and R_t are, respectively, services of labor and land. The index i on labor says that the technology features full dilution of labor across intermediate goods, reflecting the property that both labor and intermediate goods are rival inputs. Land, instead is non-rival across intermediate goods and labor. Quality, is the good's ability to raise the productivity of the other physical factors. The contribution of good i to factor productivity downstream depends on the knowledge stock of firm i, $Z_t(i)$, and on the average knowledge of all firms, $Z_t = \int_0^{N_t} [Z_t(j)/N_t] dj$.

Let $p_t(i)$ be the price of good i and q_t be the rental price of land. Profit maximization yields the conditional demand functions:

$$R_t = \frac{(1-\gamma)(1-\theta)}{q_t} G_t; \tag{5}$$

$$L_t(i) = \left\{ \frac{\gamma(1-\theta)}{w_t} X_t^{\theta}(i) [Z_t^{\alpha}(i) Z_t^{1-\alpha} R_t^{1-\gamma}]^{1-\theta} \right\}^{1/[1-\gamma(1-\theta)]}; \tag{6}$$

$$X_t(i) = \left[\frac{\theta}{p_t(i)}\right]^{\frac{1}{1-\theta}} Z_t^{\alpha}(i) Z_t^{1-\alpha} L_t^{\gamma}(i) R_t^{1-\gamma}. \tag{7}$$

Moreover, the final producer pays total compensation to, respectively, suppliers of intermediate goods, labor and land:

$$\int_0^{N_t} p_t(i) X_t(i)\, di = \theta G_t; \tag{8}$$

$$\int_0^{N_t} w_t L_t(i)\, di = \gamma (1-\theta) G_t; \tag{9}$$

$$q_t R_t = (1-\gamma)(1-\theta) G_t. \tag{10}$$

2.3 Intermediate goods and in-house R&D

Monopolistic firm i produces with a technology that requires $X_t(i)$ units of the final good to produce $X_t(i)$ units of good i at quality $Z_t(i)$. The firm also bears a fixed operating cost $\phi Z_t^{\alpha}(i) Z_t^{1-\alpha}$ in units of the final good. The firm can allocate $I_t(i)$ units of the final good to accumulate firm-specific knowledge according to the technology

$$\dot{Z}_t(i) = I_t(i). \tag{11}$$

The firm's gross profit (i.e., profit before-R&D) is

$$\Pi_t(i) = [p_t(i) - 1] X_t(i) - \phi Z_t^{\alpha}(i) Z_t^{1-\alpha}. \tag{12}$$

The value of the firm is

$$V_t(i) = \int_t^{\infty} \exp\left(-\int_t^s r_u du\right) [\Pi_s(i) - I_s(i)] ds. \tag{13}$$

The firm maximizes (13) subject to (7) and (11). We solve this problem in the appendix; here we discuss only the elements needed to address the paper's research question.

The demand curve (7) says that an unconstrained monopolist would charge $p_t(i) = 1/\theta$. However, we assume that competitive fringe firms can produce good i at the same quality $Z_t(i)$ as the monopolist but at the higher marginal cost $\mu \in (1, 1/\theta)$.[8] The value-maximization problem then says that the monopolistic firm sets

$$p_t(i) = \min\{\mu, 1/\theta\} = \mu \tag{14}$$

to price fringe firms out of the market. The problem also delivers the firm's rate of return to quality innovation,

$$r_t^q(i) = \alpha \frac{\Pi_t(i)}{Z_t(i)} = \alpha \left[(\mu - 1)\frac{X_t(i)}{Z_t(i)} - \phi Z_t^{\alpha-1}(i) Z_t^{1-\alpha}\right],$$

which is linear in quality-adjusted firm size $x_t(i) \equiv X_t(i)/Z_t(i)$. This property is at the heart of the mechanism that we study: incentives to innovate depend on quality-adjusted firm size, which in turn depends on the size of the market.

In models of this class the equilibrium of the market for intermediate goods is symmetric: firms start with the same initial knowledge $Z_0(i) = Z_0$ for $i \in [0, N_0]$ and, facing a symmetric environment, make identical decisions. Consequently, they grow at the same rate and symmetry holds at any point in time. Using the limit price (14), we then have

$$x_t = \frac{X_t}{Z_t} = \left(\frac{\theta}{\mu}\right)^{1/(1-\theta)} \left(\frac{L_t}{N_t}\right)^{\gamma} R_t^{1-\gamma}. \tag{15}$$

This variable compresses the three variables L_t (labor input), R_t (land input) and N_t (mass of firms) into a single variable and thus makes the analysis of the model's dynamics very simple. For brevity, henceforth, we refer to x_t as "firm size". With this notation, the rate of return to quality innovation is

$$r_t^q = \alpha \frac{\Pi_t}{Z_t} = \alpha [(\mu - 1) x_t - \phi]. \tag{16}$$

[8] Specifically, we allow for diffusion of knowledge from monopolistic firms to fringe firms that enables the latter to constrain the pricing behavior of the former. This characterization disentangles markups from the technological parameter θ that in this model is a key driver of the functional distribution of income.

2.4 Entrants

A new firm pays βX_t, $\beta > 0$, units of the final good to develop a new intermediate good of average quality, Z_t, set up operations and enter the market.[9] This structure preserves the symmetry of the equilibrium of the intermediate goods market at all times. The asset-pricing equation governing the value of firms (old and new) is

$$r_t = \frac{\Pi_t - I_t}{V_t} + \frac{\dot{V}_t}{V_t}. \tag{17}$$

Entry is positive when the free-entry condition holds, i.e., when

$$V_t = \beta X_t. \tag{18}$$

Substituting (7) and (14) in (12) and then using the resulting expression, (11), (15), (17) and (18) yield the return to entry as

$$r_t^e = \frac{1}{\beta}\left(\mu - 1 - \frac{\phi + z_t}{x_t}\right) + z_t + \frac{\dot{x}_t}{x_t}, \tag{19}$$

where $z_t \equiv \dot{Z}_t / Z_t$ is the growth rate of average quality.

2.5 Value of land

Let v_t denote the value of a unit of land. The asset-pricing equation for v_t is $r_t v_t = q_t + \dot{v}_t$. This equation states that the return on land is determined by the rental price of land and the capital gain in land value.

2.6 General equilibrium

The general equilibrium of this economy is a time path of allocations $\{A_t, C_t, G_t, L_t, R_t, X_t(i), I_t(i)\}$ and a time path of prices $\{r_t, w_t, q_t, v_t, p_t(i), V_t(i)\}$ such that:

- households maximize utility taking $\{r_t, w_t, q_t\}$ as given;
- final-good firms maximize profit taking $\{p_t(i), w_t, q_t\}$ as given;
- intermediate-good firms choose $\{p_t(i), I_t(i)\}$ to maximize $V_t(i)$ taking r_t as given;
- entrants make entry decisions anticipating that when in operation they will maximize their value, i.e., they will behave as the incumbents in the previous bullet point;
- aggregate household wealth is the sum of the value of land and of the aggregate value of monopolistic firms, $A_t \equiv \int_0^1 a_t(h)dh = v_t R + V_t N_t$;

[9] Peretto and Connolly (2007) discuss alternative specifications of entry costs that yield the same qualitative results. They also show that the cost of entry scaling with market size prevents the cost from vanishing in the presence of population growth. An empirical study by Bollard *et al.* (2016) documents that entry costs do rise with the level of development, providing empirical support for our theoretical specification.

- the market for land services clears, $\int_0^1 R_t(h)dh = R$;
- the labor market clears, $\int_0^{N_t} L_t(i)di = N_t L_t(i) = L_t$;
- the market for the final good clears.

2.7 Aggregation

In symmetric equilibrium, (7) and (14) yield the reduced-form representation of final output

$$G_t = (\theta/\mu)^{\theta/(1-\theta)} N_t^{1-\gamma} Z_t L_t^{\gamma} R^{1-\gamma}, \tag{20}$$

where the land endowment R is constant. The associated growth rate of output is

$$g_t \equiv \frac{\dot{G}_t}{G_t} = (1-\gamma)n_t + z_t + \gamma\lambda. \tag{21}$$

This growth rate has three components: (i) the growth rate of the variety of intermediate goods, $n_t \equiv \dot{N}_t/N_t$; (ii) the growth rate of the average quality of intermediate goods, z_t; (iii) the growth rate of the labor force λ multiplied by the labor elasticity of final output γ.

3 Dynamics

This section analyzes the dynamics of the model. See Section 3.1 for the dynamics of the aggregate economy. See Section 3.2 for the dynamics of the wealth distribution. See Section 3.3 for the dynamics of the income distribution. Section 3.4 provides a discussion on income inequality.

3.1 Dynamics of the aggregate economy

The model identifies two eras: the pre-industrial era, where no innovation of any kind takes place, and the industrial era, where variety innovation takes place because the free-entry condition holds with equality. The industrial era consists of two phases: in phase 1, only horizontal innovation occurs; in phase 2, quality innovation also occurs.[10]

In the pre-industrial era, the demand for each intermediate product is initially so small (i.e., $x_0 < \phi/(\mu-1)$) that a would-be monopolist operating the increasing-returns technology would earn negative profit. Thus, the existing N_0 intermediate goods are produced by competitive firms that do not innovate, make zero profit at the price $p_t(i) = \mu$, and have zero stock-market value. Anticipating this, agents are not willing to pay the sunk entry cost and there is no variety innovation. Initially, therefore, all technologies exhibit constant returns to scale and the demand for each intermediate product grows only because of exogenous population growth. Eventually, the size of the market for intermediate goods is sufficiently

[10] See Bouscasse et al. (2021) for recent evidence that the historical pattern consists of a secular acceleration of economic growth that can be divided in two well-identified phases. There is some debate in the literature about the precise timing of the key events, but there is remarkable agreement on the overall time-profile of the process.

large that a would-be monopolist operating the increasing-returns technology could earn a positive profit. We assume, however, that only innovation, in this case a process innovation, allows a new firm to monopolize an existing market. The pre-industrial era, therefore, ends only when the present value of monopolistic firms is sufficiently large that the free-entry condition (18) holds.

We summarize the first important result governing the model's dynamics in the following proposition, which states that two key grand ratios are always constant.

Proposition 1 *(Grand Ratios) The equilibrium consumption-output ratio is*

$$\frac{C_t}{G_t} = \left(\frac{C}{G}\right)^* = \begin{cases} 1-\theta & n_t = 0 \\ 1-\theta+\rho\beta\theta/\mu & n_t > 0 \end{cases},$$

where $n_t = 0$ identifies the pre-industrial era (the free-entry condition does not hold) and $n_t > 0$ identifies the industrial era (the free-entry condition holds). Similarly, the consumption-wealth ratio is

$$\frac{C_t}{A_t} = \left(\frac{C}{A}\right)^* = \begin{cases} \frac{\rho}{1-\gamma} & n_t = 0 \\ \frac{\rho(1-\theta+\rho\beta\theta/\mu)}{(1-\gamma)(1-\theta)+\rho\beta\theta/\mu} & n_t > 0 \end{cases}.$$

Proof. See the Appendix. ■

The result that the consumption-output ratio is always constant implies that at all times consumption and output grow at the same rate, i.e.,

$$g_t \equiv \frac{\dot{G}_t}{G_t} = \frac{\dot{C}_t}{C_t}.$$

As the economy progresses through the three phases discussed above, the growth rate is

$$g_t = \begin{cases} \gamma\lambda & n_t = 0 \\ \gamma\lambda + (1-\gamma)n_t & n_t > 0 \text{ and } z_t = 0 \\ \gamma\lambda + (1-\gamma)n_t + z_t & n_t > 0 \text{ and } z_t > 0 \end{cases}.$$

To translate this characterization of the general equilibrium of the model into a state-space representation, in the appendix we construct the rates of variety growth (entry) n_t and of quality growth z_t as two functions of the state variable x_t that account for the non-negativity constraints $n_t \geq 0$ and $z_t \geq 0$. The end result is

$$g_t = \begin{cases} \gamma\lambda & 0 \leq x \leq x_N \\ \gamma\lambda + (1-\gamma)\left[\frac{1}{\beta}\left(\mu - 1 - \frac{\phi}{x_t}\right) - \rho\right] & x_N < x_t \leq x_Z \\ \alpha\left[(\mu-1)x_t - \phi\right] - \rho & x_t > x_Z \end{cases}, \tag{22}$$

where, x_N and x_Z are the firm-size activation thresholds of, respectively, variety and quality innovation. (The expressions for these objects as functions of the deep parameters are in the Appendix.) This piecewise function says that in each phase, growth accelerates because one form of Schumpeterian innovation starts occurring. Proposition 1, moreover, says that the consumption-output ratio jumps up when the first phase transition occurs because this event entails the costly creation of a new form of wealth—equity shares in monopolistic firms that accumulate intangible capital—that make households richer. The individual and aggregate effects of this *wealth creation event* depend on how the newly issued shares are distributed across the heterogeneous households (more on this below).

The function $n(x_t)$ constructed in the appendix yields the equilibrium law of motion of the state variable x_t. We summarize the property as follows.

Proposition 2 *(Equilibrium Dynamics) Assume*

$$\rho + \lambda < \min\left\{(1-\alpha)\phi, (1-\alpha)(\mu-1)/\beta\right\}.$$

The key properties of the model's dynamics are as follows. (i) The state variable x_t obeys the law of motion

$$\dot{x}_t = \begin{cases} \gamma\lambda x_t & 0 \le x \le x_N \\ \gamma\left[\frac{\phi}{\beta} - \left(\frac{\mu-1}{\beta} - \lambda - \rho\right)x_t\right] & x_N < x_t \le x_Z \\ \gamma\left[\lambda - \frac{[(1-\alpha)(\mu-1)-\rho\beta]x_t - (1-\alpha)\phi + \rho + \gamma\lambda}{\beta x_t - (1-\gamma)}\right]x_t & x_t > x_Z \end{cases}.$$

(ii) There exists a unique, scale-invariant, steady state

$$x^* = \frac{(1-\alpha)\phi - (\rho+\lambda)}{(1-\alpha)(\mu-1) - \beta(\rho+\lambda)} > x_Z. \tag{23}$$

(iii) Given initial condition $x_0 \in (0, x_N)$, the dynamics are globally stable and x_t converges to the steady state x^*. (iv) The steady state exhibits the scale-invariant growth rate

$$g^* = \alpha\left[(\mu-1)x^* - \phi\right] - \rho > 0. \tag{24}$$

Proof. See the Appendix. ■

We illustrate the dynamics described by Proposition 2 in two figures. Figure 1 shows that firm size x_t grows throughout the transition, following an S-shaped (i.e., logistic) path, where T_N and T_Z are the activation dates of, respectively, variety and quality innovation.[11] As x_t converges to its steady-state value x^*, we have

$$N_t = \left[\left(\frac{\theta}{\mu}\right)^{1/(1-\theta)} \frac{R^{1-\gamma}}{x^*}\right]^{1/\gamma} L_t$$

so that the mass of products/firms grows at the rate λ; see Laincz and Peretto (2006), among many others, for empirical evidence that N_t is proportional to L_t in advanced economies.

[11] The model yields analytical solutions for these dates; see the Appendix.

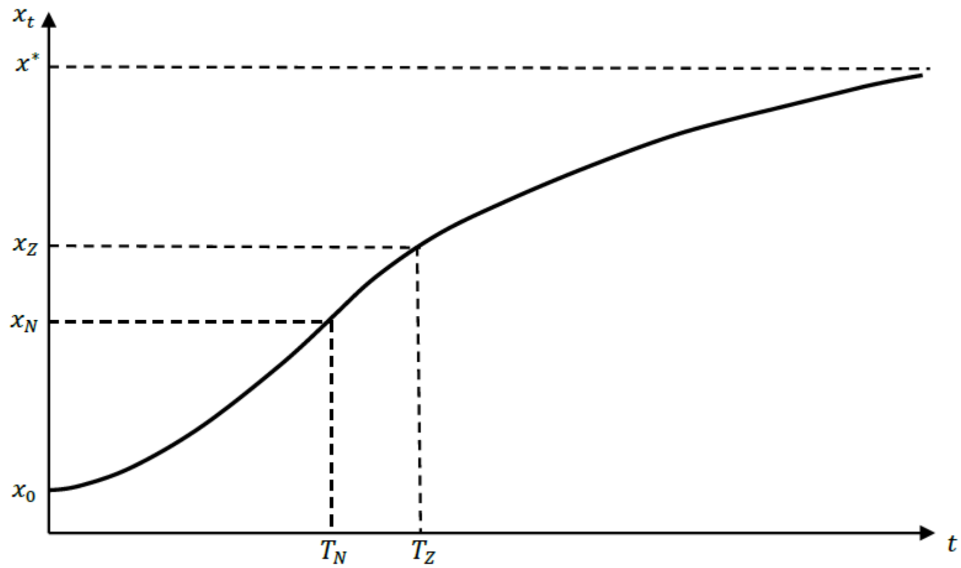

Figure 1: Transition path of the firm size

Figure 2 shows the dynamics of economic growth that we obtain by feeding the path of x_t to the growth rate equation (22). In the pre-industrial era, the growth rate of output is simply $g_t = \gamma\lambda$ due to the absence of innovation. In the industrial era, the growth rate accelerates, initially fueled only by variety innovation and then also by quality innovation. As x_t converges to x^*, the growth rate converges to the steady-state value g^* in (24).

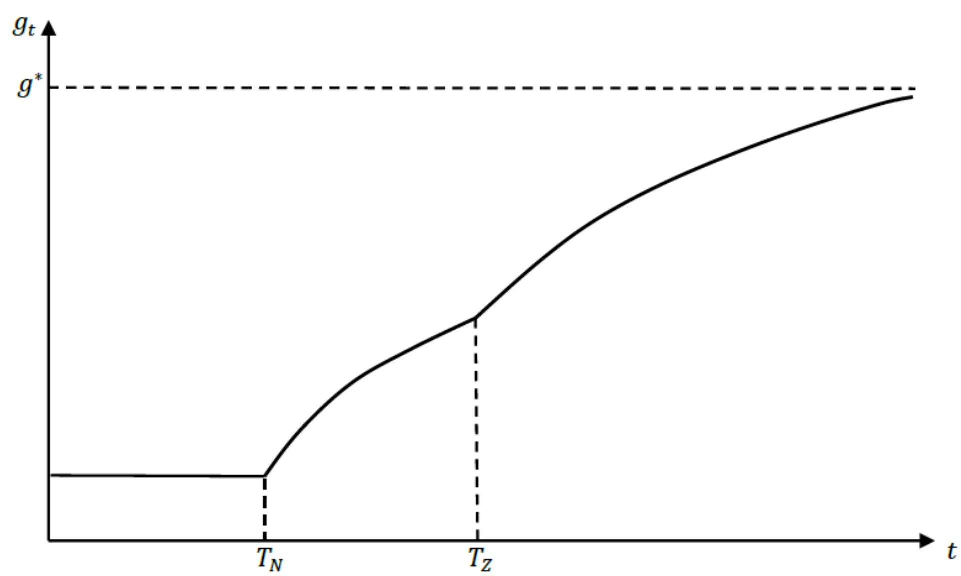

Figure 2: Transition path of the growth rate

This gradual acceleration of economic growth is consistent with historical data for the

UK. Figure 3 plots the log of the UK real GDP from 1700 to 2016.[12] The slope of the plot is the growth rate. According to the data, the average growth rate in the UK is 0.71% in the first half of the 18th century, 1.24% in the second half of the 18th century, 1.86% in the first half of the 19th century, 2.23% in the second half of the 19th century, 1.50% in the first half of the 20th century and 2.55% from the second half of the 20th century onwards. Except for the wartime periods in the first half of the 20th century, the UK experiences a gradually rising growth rate as in our Schumpeterian model of endogenous takeoff. We are interested in the implications of these dynamics for inequality.

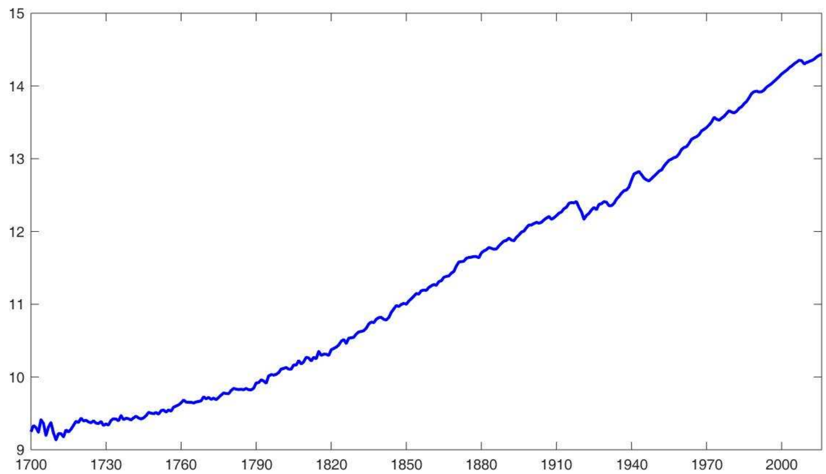

Figure 3: Log of real GDP in the UK from 1700 to 2016

3.2 Dynamics of the wealth distribution

Let $s_{c,t}(h) \equiv c_t(h)/C_t$ and $s_{a,t}(h) \equiv a_t(h)/A_t$ be, respectively, the share of consumption and of wealth of household h at time t.[13] The most important implication of the saving behavior characterized by equation (3) is that households want to achieve the same consumption growth rate, since they face the common interest rate r and they have the same discount rate ρ. It follows that the household consumption share $s_{c,t}(h)$ is always constant. The aggregate transition dynamics, however, suggest that the consumption share is phase specific. Moreover, in general equilibrium the household's consumption-saving behavior yields that the wealth share is also always constant and phase specific.

The following proposition summarizes our main formal result on the dynamics of the household consumption and wealth shares.

[12] Data source: Federal Reserve Bank of St. Louis.

[13] Note that we do not treat the household's land holding $R_t(h)$ as a constant but, consistently with how we wrote the budget constraint (2), we treat it as an asset that the household can trade at price v_t.

Proposition 3 *(Household Shares) Consider household h for $h \in [0,1]$. Let $s_{R,0}(h) \geq 0$ be the household's land share at time $t = 0$ and let $s_{a,T_N}(h)$ be the household's wealth share at time $t = T_N$. Define also the composite parameter $\Theta \equiv \beta\theta/(1-\theta)$. In equilibrium, the household's consumption and wealth shares are, respectively:*

$$s_{a,t}(h) = s_a^*(h) = \begin{cases} s_{R,0}(h) & 0 \leq x_t \leq x_N \\ s_{a,T_N}(h) & x_t > x_N \end{cases};$$

$$s_{c,t}(h) = s_c^*(h) = \begin{cases} \gamma + (1-\gamma)\, s_{R,0}(h) & 0 \leq x_t \leq x_N \\ \frac{\gamma}{\rho\Theta/\mu+1} + \frac{\rho\Theta/\mu+1-\gamma}{\rho\Theta/\mu+1} s_{a,T_N}(h) & x_t > x_N \end{cases}.$$

Proof. See the Appendix. ■

The proposition says that agents have perfect foresight and incorporate in their decisions all available information at $t = 0$. Consequently, the consumption share must jump at $t = 0$. The implication for the dynamics of the shares is twofold. First, even though our scheme allows for trade of land, in equilibrium households do not trade it and $s_{R,t}(h) = s_{R,0}(h)$ for all $t \in [0, T_N]$. Second, both shares are constant at all $t \neq T_N$, with a discrete adjustment of the consumption share at $t = T_N$ that gives the optimal intertemporal response to the arrival of new wealth. Note that to complete the second argument we need to take a stand on the distribution of industrial wealth at $t = T_N$. To get started, we assume that it tracks the distribution of land. Thus, $s_{a,T_N}(h) = R_0(h)/R \equiv s_R(h)$. Alternative assumptions on the initial distribution of industrial wealth are feasible but do not change the qualitative results of our analysis.

The following narrative describing the dynamics of the wealth distribution emerges from this characterization. In the pre-industrial era the distribution of wealth is stationary and determined by the initial (exogenous) distribution of land. When the economy enters the industrial era, a new form of wealth appears—equity shares in industrial firms that develop and apply new technology. We find that the distribution of wealth, now consisting of both land and industrial shares, continues to be stationary and jointly determined by the distribution of land and by the initial distribution of industrial shares at the time of the takeoff. To simplify we assume that such initial distribution tracks the distribution of land.[14] It is important to note that the economy features transition dynamics determined by the evolution of firm size. However, despite these dynamics the wealth distribution remains stationary because we work with a structure where households want parallel log-consumption paths. Since this implies constant consumption shares and households face a common rate of return to assets, it follows that the household wealth shares must be constant as well.

To close this subsection, we sort households in ascending order of wealth and define the Gini coefficient of wealth at time t,

$$\sigma_{a,t} \equiv 1 - 2\int_0^1 \mathcal{L}_{a,t}(h)dh.$$

[14] According to Clark (2008), the landed aristocracy and their descendants were still among the wealthiest group at least by the 1860s.

The Lorenz curve of wealth inside the integral is

$$\mathcal{L}_{a,t}(h) \equiv \frac{\int_0^h a_t(\chi)d\chi}{\int_0^1 a_t(\chi)d\chi} = \frac{\int_0^h a_t(\chi)d\chi}{A_t} = \int_0^h s_{a,t}(\chi)d\chi = \int_0^h s_{a,0}(\chi)d\chi = \int_0^h s_R(\chi)d\chi.$$

In light of Proposition 3, the Gini coefficient of wealth is stationary. Moreover, we have $\sigma_{a,t} = \sigma_{a,0} = \sigma_R$ for all t, where σ_R is the Gini coefficient of land ownership at time 0. The model produces any other summary statistics that we might wish to consider; for example, the share of wealth of the top ε households is

$$S_{a,t}^{\varepsilon} \equiv \int_{1-\varepsilon}^1 s_{a,t}(h)dh = \int_{1-\varepsilon}^1 s_R(h)dh.$$

We focus on the Gini coefficient because of its prominent role in the literature.

3.3 Dynamics of the income distribution

Household h earns income $y_t(h) \equiv r_t a_t(h) + w_t L_t$. Aggregating across households yields

$$Y_t \equiv \int_0^1 y_t(h)dh = r_t A_t + w_t L_t.$$

Note that Y_t is *not* G_t because in this model there are intermediate goods. Specifically, Y_t is GDP, while G_t is aggregate production of the final good.

Let $s_{y,t}(h) \equiv y_t(h)/Y_t$ denote the income share of household h and $s_{L,t} \equiv wL_t/Y_t$ be the aggregate labor share. We have

$$s_{y,t}(h) = \frac{r_t a_t(h) + w_t L_t}{r_t A_t + w_t L_t} = (1 - s_{L,t})\, s_R(h) + s_{L,t}, \tag{25}$$

where we have used $s_{a,t}(h) = s_R(h)$. This equation says that the household's income share has two determinants: the household's land share $s_R(h)$ and the aggregate labor share $s_{L,t}$. An increase in the aggregate labor share raises the income share of household h if the household's wealth share $s_R(h)$ is less than one (the average wealth share), it lowers it if the reverse is true. This simple property constitutes the main transmission channel of macro events to household income and thus to the cross-sectional distribution of income.

Equation (25) allows us to derive any summary statistic of the income distribution. The Gini coefficient of income is[15]

$$\sigma_{y,t} = (1 - s_{L,t})\, \sigma_R. \tag{26}$$

This expression says that income inequality is decreasing in the labor share. Accounting for the dynamics of the labor share, we then have (see the Appendix for derivations):

$$\sigma_{y,t} = \begin{cases} \left(1 + \frac{\rho}{\rho+\gamma\lambda}\frac{\gamma}{1-\gamma}\right)^{-1} \sigma_R & 0 \leq x_t \leq x_N \\ \left(1 + \frac{\rho}{\rho+g_t}\frac{\gamma}{1-\gamma+\rho\Theta/\mu}\right)^{-1} \sigma_R & x_t > x_N \end{cases}. \tag{27}$$

[15] The coefficient of variation of income is still given by $\sigma_{y,t} = (1 - s_L)\, \sigma_R$ if we instead define $\sigma_{y,t} \equiv \sqrt{\int_0^1 [s_{y,t}(h) - 1]^2 dh}$ and $\sigma_R \equiv \sqrt{\int_0^1 [s_R(h) - 1]^2 dh}$ as the coefficients of variation of income and wealth.

The income share of the top ε households is

$$S_{y,t}^{\varepsilon} \equiv \int_{1-\varepsilon}^{1} s_{y,t}(h)dh = (1 - s_{L,t}) S_{a,t}^{\varepsilon} + s_{L,t}\varepsilon.$$

We can use (26) to eliminate the labor share and write

$$S_{y,t}^{\varepsilon} = \frac{\sigma_{y,t}}{\sigma_R} \left(S_{a,t}^{\varepsilon} - \varepsilon \right) + \varepsilon. \tag{28}$$

This expression says that the top ε income share is increasing in the ratio of the Gini indices $\sigma_{y,t}/\sigma_R$ if and only if $S_{a,t}^{\varepsilon} > \varepsilon$. In other words, a rising Gini coefficient of income does not necessarily yield a rising top ε income share. The condition for this to happen is that the share of wealth of the top ε household be larger than ε. This condition clearly holds in the data since wealth is highly concentrated.

We summarize our result on the evolution of income inequality as follows. In the pre-industrial era, income inequality is a constant multiple of land ownership inequality. At the beginning of the industrial era, income inequality jumps up because of the unequal distribution of industrial wealth that we assume tracks the distribution of land. Thereafter income inequality is a multiple of land inequality, with a multiplier that is an increasing function of the growth rate g_t, whose dynamics are described by (22). In phase 1 the growth rate is fueled only by variety innovation and rises gradually. Eventually, phase 2 starts and quality innovation adds its contribution, providing a new acceleration of the growth rate with final convergence to the steady state g^* described in (24). Figure 4 summarizes the dynamics of the Gini index $\sigma_{y,t}$ from stagnation to takeoff and eventually to the steady state.

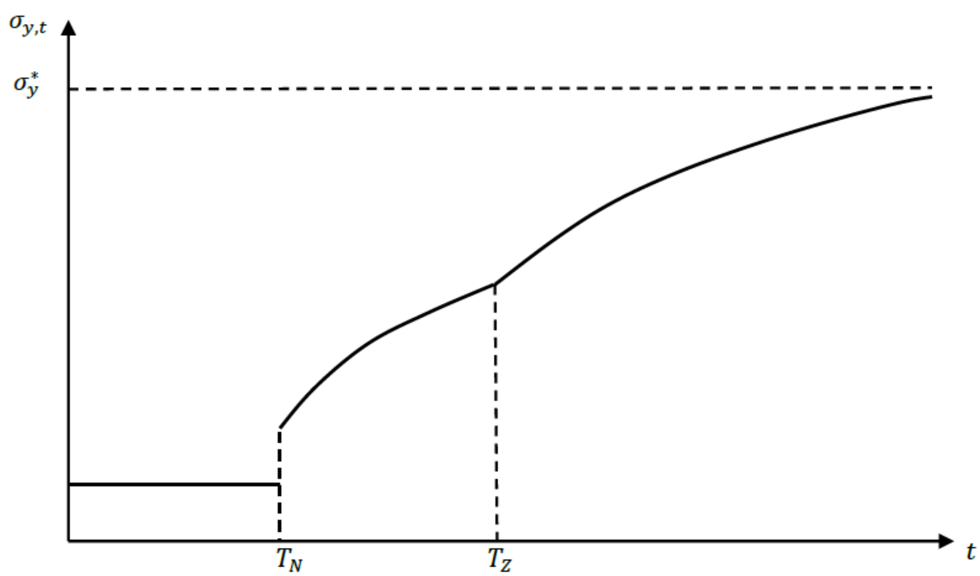

Figure 4: Transition path of income inequality

3.4 Discussion and interpretation

To extract the key insight provided by the model's dynamics, we ask why the labor share is decreasing in the growth rate of final output g_t. We note first that the consumption-output ratio is always constant, see Proposition 1, and thus the Euler equation (3) yields $r_t = \rho + g_t$. We then split wealth in its two separate components—land $v_t R_t(h)$ and industrial shares $a_t^N(h)$—and write the household budget (2) as

$$\dot{a}_t^N(h) + \dot{v}_t R_t(h) + v_t \dot{R}_t(h) = r_t a_t^N(h) + r_t v_t R_t(h) + w_t L_t - c_t(h).$$

Aggregating and noting that the land market clears, $\int_0^1 R_t(h)dh = R$, we obtain

$$\dot{A}_t^N + \dot{v}_t R = \underbrace{r_t A_t^N + r_t v_t R + w_t L_t}_{\text{income}} - C_t.$$

Therefore, we write

$$s_{L,t} = \frac{w_t L_t}{r_t A_t^N + r_t v_t R + w_t L_t} = \frac{w_t L_t}{(\rho + g_t)\left(A_t^N + v_t R\right) + w_t L_t}.$$

It thus seems that all else equal the labor share is decreasing in the growth rate simply because the real interest rate is increasing in it. However, and more interestingly, the equation says that the labor share is decreasing in the growth rate because faster growth raises the real interest rate and thereby raises aggregate income via the asset income channel. This is not a variant of the now popular $r > g$ story (see, e.g., Piketty 2014): here the labor share does not fall over time because the real interest rate is higher than the growth rate, it falls over time because the growth rate *accelerates* throughout the industrial era. Indeed, when the growth rate is constant, as in the pre-industrial era and in the steady state of the industrial era, the labor share is constant. As discussed, Madsen *et al.* (2021) provide the most recent and comprehensive evidence supporting this mechanism.

To see the mechanism in finer detail note that in the pre-industrial era $g_t = \gamma\lambda$, a constant, while $A_t^N = 0$. We thus have

$$s_{L,t} = \frac{\frac{\rho}{\rho+\gamma\lambda}\frac{\gamma}{1-\gamma}}{1 + \frac{\rho}{\rho+\gamma\lambda}\frac{\gamma}{1-\gamma}}.$$

In the industrial era, instead, we rewrite the expression for the labor share as

$$s_{L,t} = \frac{\frac{\rho}{\rho+g_t}\frac{\gamma}{1-\gamma+\rho\Theta/\mu}}{1 + \frac{\rho}{\rho+g_t}\frac{\gamma}{1-\gamma+\rho\Theta/\mu}}.$$

The model's mechanism, therefore, is that inequality is initially constant and then rises along the transition because the growth acceleration that takes place throughout the industrial era, which is a manifestation of the rising rate of return to innovation, causes asset income to rise faster than wage income. This differential growth propagates through the households, continuously spreading out the distribution of their incomes because of the heterogeneity in their assets holdings. This dynamic mechanism emphasizes the property mentioned above that in this model wealth inequality is the root cause of income inequality. This is consistent with the thrust of the recent literature based on Piketty (2014), which sees fundamental differences across households in their sources of income—capital vs. labor—as the root cause of income inequality. Our main mechanism differs in that the dynamics of income inequality are driven by growth accelerations, not by the mere fact that the interest rate is larger than the growth rate.

To summarize, in this model the income distribution is non-degenerate, endogenous, and non-stationary but analytically tractable. The dynamics produce a clear insight: the secular acceleration of the growth rate in the aftermath of the Industrial Revolution produced a secular rise of income inequality because asset income grew faster than labor income. The tractability of the model is predicated on the property that the wealth distribution is stationary. This property follows from the assumption that our households have identical homothetic preferences over consumption and thus want parallel log-consumption paths with intercepts that differ because of the unequal distribution of land. Admittedly, this feature makes the model silent on the evolution of the wealth distribution. We offer in its defense three arguments. First, our wealth distribution is stationary, not exogenous. That is, the only thing that we take as exogenous to the model is the initial distribution of land. Once the industrial era begins, we have a mechanism that produces endogenously a new stationary distribution of wealth. Second, there is evidence that wealth inequality evolves around a stationary value in the very long run.[16] Thus, working with a model that treats the stationary distribution of wealth as the root cause of income inequality is not only interesting and fruitful but also empirically legitimate. Third, models of heterogeneous agents that produce a fully endogenous wealth distribution tend to be so complex that more often than not one is forced to impose stationarity anyway. Since our goal is to study the secular dynamics of income inequality, such an approach is not useful. In contrast, our approach is so tractable that we obtain a complete analytical characterization of the dynamics of economic growth and a complete analytical characterization of the distribution of income at any point in time.

4 Quantitative analysis

In this section, we calibrate the model to UK data in order to perform a quantitative analysis. The model features the following parameters: $\{\rho, \alpha, \lambda, \theta, \beta, \gamma, \mu, \phi\}$. We set the discount rate ρ to 0.04. We follow Iacopetta *et al.* (2019) to set the degree of technology spillovers $1-\alpha$ to 0.833. In the UK, the long-run population growth rate λ is 0.6%.[17] Then, we calibrate the remaining parameters $\{\theta, \beta, \gamma, \mu, \phi\}$ by matching the following moments for the UK economy: 52.6% for labor income as a share of output,[18] 74.4% for consumption as a share of output,[19] 12.3% for housing rents as a share of output,[20] 2.5% for the growth rate of output,[21] and

[16] Madsen (2019) provides evidence that wealth inequality remains at a stationary value in the long run with short-run deviations that can last over decades. We are essentially treating these short-run deviations as exogenous.

[17] Data source: Maddison Project Database.

[18] Data source: Office for National Statistics.

[19] Data source: Office for National Statistics.

[20] Data source: New Economics Foundation.

[21] Data source: Federal Reserve Bank of St. Louis.

18.4% for investment as a share of output.[22] Table 1 summarizes the calibrated parameter values.[23] These parameter values imply a rate of asset returns of 6.5% and R&D as a share of output of 2.0%, which are in line with UK data.

Table 1: Calibrated parameter values							
ρ	α	λ	θ	β	γ	μ	ϕ
0.040	0.167	0.006	0.351	14.468	0.810	2.138	0.245

Figure 5 presents the simulated paths of the output growth rate and the real interest rate along with the HP-filter trends of the GDP growth rate and the rate of return on non-residential fixed capital in the UK.[24] We choose an initial value x_0 such that the takeoff occurs in the late 18th century.[25] This figure shows that the output growth rate increases from about 0.5% in the late 18th century to 2.5% in recent time. This gradual increase in the growth rate and the magnitude of the increase are in line with historical data in the UK. Figure 5 also shows that the real interest rate increases from 4.5% in the late 18th century to an average of 5.9% in the 19th century and reaches an average of 6.4% in the 20th century. The average rates of return on non-residential fixed capital in the UK were 5.1% in the 18th century, 6.0% in the 19th century, and 7.0% from the 20th century onwards.[26] Therefore, the increase in the rate of return on assets and the magnitude of the increase in asset returns predicted by our model are also in line with historical data.

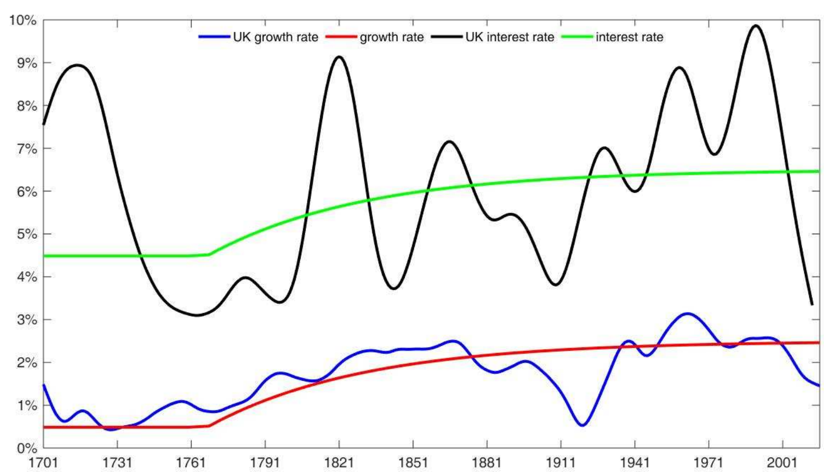

Figure 5: Simulated paths of the growth rate and the interest rate

[22] Data source: Office for National Statistics. To compute this moment from the model, we add up expenses on intermediate goods and horizontal/vertical R&D. One can think of the intermediate goods in our model as investment in capital that depreciates rapidly.

[23] The calibrated value of μ seems high but implies a reasonable profit share of output of 11.5%.

[24] Here we use a smoothing parameter of 1000 on the annual data in order to extract a smoother trend.

[25] According to Ashton (1998), the Industrial Revolution started in as early as 1760.

[26] See Madsen (2017). The authors are grateful to Jakob Madsen for sharing this data series.

The increase in the real interest rate in Figure 5 implies an increase in income inequality in our model. Figure 6 presents the simulated path of income inequality in terms of percent changes from its initial value prior to the takeoff. This figure shows that income inequality increases sharply by about 50% when the takeoff occurs. When the economy reaches the balanced growth path, income inequality would have almost doubled. Our model takes the degree of wealth inequality as given. If we consider a Gini coefficient of wealth of 0.732 in recent time,[27] then we can also simulate the Gini coefficient of income. Figure 7 reports the simulated path of income inequality along with the Gini coefficient of income in the UK from 1961 to 2017.[28] It shows that the simulated Gini coefficient of income increases from 0.15 before the takeoff to 0.29 in the steady state.

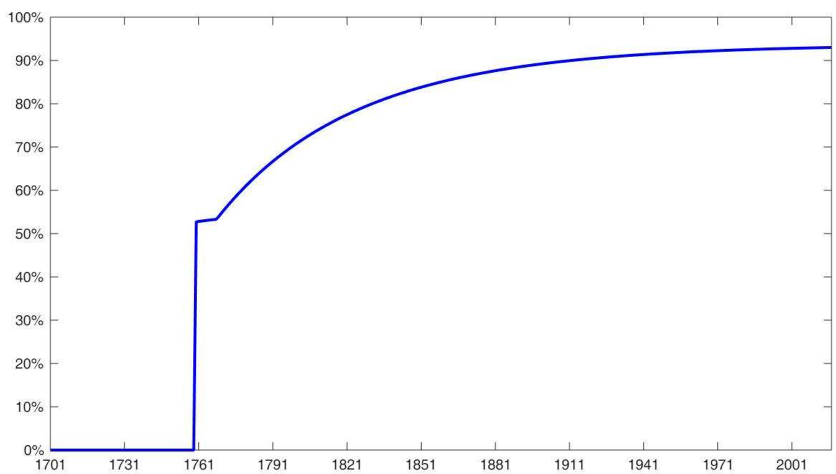

Figure 6: Simulated path of income inequality (percent change)

[27]Data source: Credit Suisse Global Wealth Databook.

[28]Data source: Institute for Fiscal Studies. Data available from 1961.

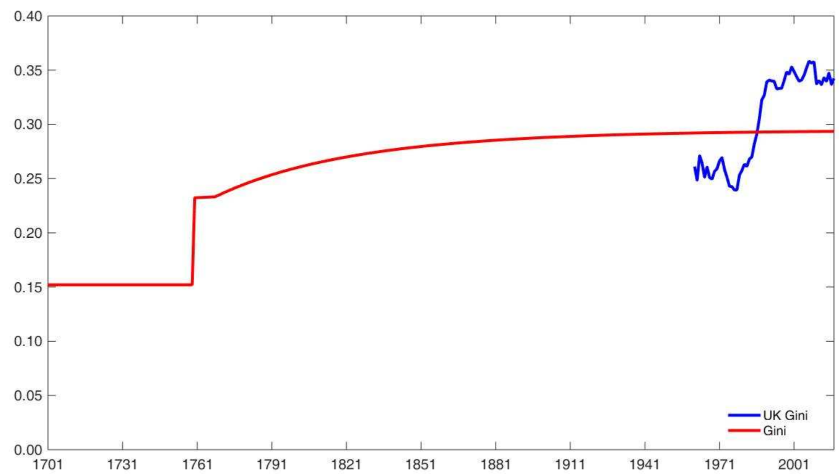

Figure 7: Simulated path of income inequality (Gini coefficient)

Williamson (1980, 1985) and Lindert (2000a, 2000b) examine historical data in Britain and document that income inequality, based on different measures, increases in the late 18th century/early 19th century and levels off after the mid-19th century. Then, income inequality, measured by the top 1% income share, decreases from the early 20th century to the late 1970's.[29] As for the Gini coefficient of income, it decreases from 0.27 in the early 1960's to 0.24 in the late 1970's before rising again to as high as 0.36 in recent time with an average value of 0.30 from 1961 to 2017 in the UK. Therefore, the long-run level of income inequality predicted by our model is in line with recent data in the UK. Furthermore, our model is able to deliver the pattern of rising income inequality in the late 18th century/early 19th century and its leveling off in the late 19th century. However, our model is unable to explain the decrease in income inequality from the early 20th century to the late 1970's. The reason is that this decrease in income equality is driven by a decrease in wealth inequality,[30] whereas our model takes wealth inequality as given.

To address this issue, we consider historical data on the income and wealth shares owned by the top households, which have longer time series than the Gini coefficient. Therefore, we now use historical data on the top 10% wealth share in the UK along with the asset-wage income ratio $r_t A_t/(w_t L_t)$ computed from our model to simulate the top 10% income share. Figure 8 presents the simulated path of the top 10% income share along with data in the UK from 1900 to 2010.[31] Given the data on wealth inequality, our model now predicts that income inequality rises in the 19th century and falls from the early 20th century to the 1970's. After that, income inequality becomes rising again. This pattern matches the data.

[29] World Inequality Database documents a decrease in the top 1% income share from 20% in the early 20th century to 5% in the late 1970's.

[30] World Inequality Database documents a decrease in the top 1% wealth share from 70% in the early 20th century to less than 20% in the early 1980's.

[31] Data source: Piketty (2014). Data on the top 10% wealth (income) share is available from 1810 (1900).

Furthermore, the average value of the top 10% income share in the UK from 1900 to 2010 is 0.37, whereas our model predicts an average value of 0.36 in this period.

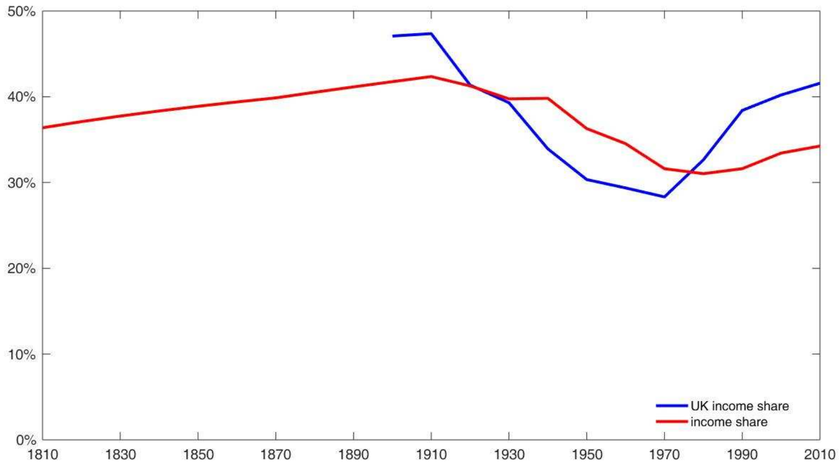

Figure 8: Simulated path of the top 10% income share

5 Labor income inequality

Our baseline model has three main strengths: (i) it is eminently tractable; (ii) it identifies sharply the role of growth accelerations as a main driver of rising income inequality; (iii) it measures inequality with a well understood and widely used summary statistic of the shape of the distribution of income. The baseline model, on the other hand, has one main weakness: inequality plays no role in shaping the transition from stagnation to growth. In this section, we extend the analysis allowing for labor income inequality due to endogenous labor supply. The advantage of doing this is twofold. First, endogenous labor supply is interesting per se and including it makes the analysis more empirically relevant. Second, endogenous labor supply introduces a channel through which inequality contributes to shaping the transition path of the economy.

5.1 The model with endogenous labor supply

We generalize the utility function of household $h \in [0, 1]$ to

$$U(h) = \int_0^\infty e^{-\rho t} \left\{ \ln c_t(h) + \frac{\eta}{1 - 1/\omega} [1 - l_t(h)/L_t]^{1-1/\omega} \right\} dt, \tag{29}$$

where $l_t(h)$ is the household's labor supply and $1 - l_t(h)/L_t$ is leisure per member of the household. The parameter $\eta > 0$ determines the importance of leisure, whereas the parameter $\omega > 0$ determines the elasticity of intertemporal substitution for leisure. The budget constraint is

$$\dot{a}_t(h) = r_t a_t(h) + w_t l_t(h) - c_t(h). \tag{30}$$

The novel element is the household's endogenous supply of labor

$$\frac{l_t(h)}{L_t} = 1 - \left[\frac{\eta c_t(h)}{w_t L_t}\right]^{\omega}. \tag{31}$$

The rest of the model is the same as before.

5.2 Special case: $\omega = 1$

We begin our analysis with the special case $\omega = 1$, which gives log-log utility. Aggregating the labor supply (31) yields

$$\frac{l_t}{L_t} = \int_0^1 \frac{l_t\left(h\right)}{L_t} dh = \int_0^1 \left(1 - \left[\frac{s_c^*(h)}{\frac{w_t L_t}{\eta C_t}}\right]\right) dh = 1 - \int_0^1 \left[\frac{s_c^*(h)}{\frac{w_t L_t}{\eta C_t}}\right] dh = 1 - \frac{\eta C_t}{w_t L_t}.$$

We stress that because labor supply is linear in consumption, the heterogeneity across households washes out. Next, we use the labor demand (9), note that Proposition 1 holds in this extension as well, and rearrange terms to write

$$\frac{\eta C_t}{w_t L_t} = \frac{\eta C_t}{w_t l_t}\frac{l_t}{L_t} = \frac{\eta C_t}{\gamma\left(1-\theta\right) G_t}\frac{l_t}{L_t} = \frac{\eta\left(\frac{C}{G}\right)^*}{\gamma\left(1-\theta\right)}\frac{l_t}{L_t}.$$

The resulting equilibrium employment ratio is

$$\frac{l_t}{L_t} = \left(\frac{l}{L}\right)^* \equiv \begin{cases} \left(1+\frac{\eta}{\gamma}\right)^{-1} & 0 \le x_t \le x_N \\ \left[1+\frac{\eta}{\gamma}\left(1+\frac{\rho\Theta}{\mu}\right)\right]^{-1} & x_t > x_N \end{cases}. \tag{32}$$

Since it depends on the consumption-output ratio, the employment ratio jumps when the consumption ratio jumps.

The simplicity of this special case allows us to show immediately the implications of endogenous labor supply for the model's dynamics. To express the dynamics in terms of a pre-determined state variable that does not jump, we note that in this extension

$$\frac{X_t}{Z_t} = \left(\frac{\theta}{\mu}\right)^{1/(1-\theta)}\left(\frac{l_t}{L_t}\right)^{\gamma}\left(\frac{L_t}{N_t}\right)^{\gamma} R^{1-\gamma}.$$

We thus modify the definition of x_t to

$$x_t \equiv \left(\frac{L_t}{l_t}\right)^{\gamma}\frac{X_t}{Z_t} = \left(\frac{\theta}{\mu}\right)^{1/(1-\theta)}\left(\frac{L_t}{N_t}\right)^{\gamma} R^{1-\gamma}. \tag{33}$$

The interpretation of x_t is no longer firm size, but the mapping to that concept is transparent.

Employment growth is $\dot{l}_t/l_t = \lambda$ and the growth rate of output is

$$g_t = \begin{cases} \gamma\lambda & 0 \le x_t \le x_N \\ \gamma\lambda + (1-\gamma)\left\{\frac{1}{\beta}\left[\mu - 1 - \left[1 + \frac{\eta}{\gamma}\left(1 + \frac{\rho\Theta}{\mu}\right)\right]^{\gamma}\frac{\phi}{x_t}\right] - \rho\right\} & x_N < x_t \le x_Z \\ \alpha\left\{(\mu-1)\left[1 + \frac{\eta}{\gamma}\left(1 + \frac{\rho\Theta}{\mu}\right)\right]^{-\gamma} x_t - \phi\right\} - \rho & x_t > x_Z \end{cases}. \tag{34}$$

In this extension as well the state variable x_t grows from an initial value x_0 and gradually converges to the steady-state value x^* following an S-shaped path. The value $[(l/L)^*]^{\gamma}x^*$ is the same as x^* in (23) in the baseline case due to the model's scale-invariance property. The growth rate g_t is constant in the pre-industrial era and then gradually increases throughout the industrial era, converging to the same value g^* in (24) as in the baseline case due to scale-invariance. Moreover, we show in the appendix that the differential equation governing the dynamics of the household wealth share still yields the result that the wealth share is constant at all times. To summarize, the dynamics of the economy are qualitatively the same as those discussed in Propositions 2-3.

We now derive the implications of this structure for the income distribution. Since the wealth share is constant, we rewrite the budget constraint (30) as $c_t(h) = (r_t - g_t)a_t(h) + w_t l_t(h)$ since $\dot{a}_t(h)/a_t(h) = \dot{A}_t/A_t = g_t$. Using this result and (31), we write labor income as

$$w_t l_t(h) = \frac{1}{1+\eta}\left[w_t L_t - \eta(r_t - g_t)a_t(h)\right].$$

This result says that, since $r_t > g_t$, wealthier households supply less labor and earn lower labor income. Accounting for this labor income inequality, we show in the Appendix that the household's income share is

$$s_{y,t}(h) = \frac{r_t a_t(h) + w_t l_t(h)}{r_t A_t + w_t l_t} = \frac{(r_t + \eta g_t)A_t}{(r_t + \eta g_t)A_t + w_t L_t} s_R(h) + \frac{w_t L_t}{(r_t + \eta g_t)A_t + w_t L_t}. \tag{35}$$

We define the term

$$\tilde{s}_{L,t} \equiv \frac{w_t L_t}{(r_t + \eta g_t)A_t + w_t L_t} = \frac{s_{L,t}}{(1+\eta)\frac{l_t}{L_t}} \tag{36}$$

and rewrite (35) as in the baseline case,

$$s_{y,t}(h) = (1 - \tilde{s}_{L,t})\, s_R(h) + \tilde{s}_{L,t}. \tag{37}$$

Our extended result therefore is that the evolution of the household income share is determined by the evolution of the ratio $(r_t + \eta g_t)A_t/(w_t L_t)$, rather than $r_t A_t/(w_t L_t)$. The additional term $\eta g_t A_t$ captures the effect on labor income of standard labor supply behavior.

The Gini coefficient of income is

$$\sigma_{y,t} = (1 - \tilde{s}_{L,t})\, \sigma_R. \tag{38}$$

In the pre-industrial era, the ratio $(r_t + \eta g_t)A_t/(w_t L_t)$ is

$$\frac{(r_t + \eta g_t)A_t}{w_t L_t} = \frac{\rho + (1+\eta)\gamma\lambda}{\rho}\left(\frac{1-\gamma}{\gamma}\right)\frac{l_t}{L_t} = \frac{\rho + (1+\eta)\gamma\lambda}{\rho}\left(\frac{1-\gamma}{\gamma+\eta}\right).$$

Substituting this expression in (38) yields an expression similar to (27), except for the addition of the parameter η. Finally, in the industrial era the ratio $(r_t + \eta g_t)A_t/(w_t l_t)$ becomes

$$\frac{(r_t + \eta g_t)A_t}{w_t L_t} = \frac{\rho + (1+\eta)g_t}{\rho}\left(\frac{1-\gamma+\rho\Theta/\mu}{\gamma}\right)\frac{l_t}{L_t} = \frac{\rho + (1+\eta)g_t}{\rho}\left[\frac{1-\gamma+\rho\Theta/\mu}{\gamma+\eta(1+\rho\Theta/\mu)}\right].$$

Substituting this expression in (38) yields an expression similar to (27). In the industrial era, income inequality $\sigma_{y,t}$ gradually increases until g_t converges to the same steady-state value g^* as in our baseline case.

We stress that in this structure the employment ratio l_t/L_t is the only channel through which the equilibrium of the labor market affects the economy's dynamics. This property is important to understand the transmission mechanism of the heterogeneity in labor income that we discuss next.

5.3 General case: $\omega \neq 1$

We now consider the general case $\omega \neq 1$. With the preferences in (29) that yield labor supply (31), we have two new properties: (i) the curvature of $l_t(h)/L_t$ with respect to the consumption per capita-to-wage ratio, $[c_t(h)/L_t]/w_t$, yields that household heterogeneity in labor supply, and therefore in labor income, matters for aggregate outcomes; (ii) household behavior allows for $l_t(h)/L_t = 0$ for some h. To make the exposition as clear as possible, we focus first on property (i) and then discuss property (ii).

5.3.1 The role of labor income inequality

In this subsection we shut down property (ii), that is, we consider the case $l_t(h)/L_t > 0$ for all $h \in [0,1]$. Aggregating the labor supply (31) yields

$$\frac{l_t}{L_t} = \int_0^1 \frac{l_t(h)}{L_t}dh = \int_0^1 \left(1 - \left[\frac{s_c^*(h)}{\frac{w_t L_t}{\eta C_t}}\right]^\omega\right)dh = 1 - \int_0^1 \left[\frac{s_c^*(h)}{\frac{w_t L_t}{\eta C_t}}\right]^\omega dh = 1 - \left(\frac{\eta C_t}{w_t L_t}\Delta_c^*\right)^\omega,$$

where the operator

$$\Delta_c^* \equiv \left(\int_0^1 [s_c^*(h)]^\omega dh\right)^{\frac{1}{\omega}}$$

accounts for the heterogeneity in labor supply behavior, which in this specification does not wash out. Proceeding as in the previous case, we obtain the labor market clearing condition

$$\frac{l_t}{L_t} = 1 - \left[\left(\frac{C}{G}\right)^* \frac{\eta\Delta_c^*}{\gamma(1-\theta)}\right]^\omega \left(\frac{l_t}{L_t}\right)^\omega.$$

This is an equation in the endogenous variable l_t/L_t and two objects, Δ_c^* and $(C/G)^*$, that are functions of the model's parameters. The left-hand side is increasing; the right hand side is decreasing. Therefore, we have the unique solution

$$\left(\frac{l}{L}\right)^* \equiv \arg\text{solve}\left\{\frac{l_t}{L_t} = 1 - \left[\left(\frac{C}{G}\right)^* \frac{\eta\Delta_c^*}{\gamma(1-\theta)}\right]^\omega \left(\frac{l_t}{L_t}\right)^\omega\right\}, \tag{39}$$

with the important comparative statics property that

$$\frac{\partial\left(\frac{l}{L}\right)^*}{\partial\Delta_c^*} < 0.$$

Substituting this result in the household labor supply yields

$$\frac{l_t\left(h\right)}{L_t} = 1 - \left[\left(\frac{C}{G}\right)^* \frac{\eta s_c^*(h)}{\gamma\left(1-\theta\right)}\left(\frac{l}{L}\right)^*\right]^\omega > 0 \quad \text{with } s_c^*(h) < \frac{\gamma\left(1-\theta\right)}{\eta\left(\frac{C}{G}\right)^*\left(\frac{l}{L}\right)^*}.$$

As stated, we rule out the corner solution $l_t(h)/L_t = 0$ so that all households work.

Since the employment ratio is always constant, in this case as well, and the case that we study in the next subsection, the differential equation governing the dynamics of the household wealth share yields that the wealth share is constant at all times. The formal proof is identical to that developed in the previous subsection.

The presence of the operator Δ_c^* in the solution (39) is our property (i), namely, equilibrium aggregate labor supply, and therefore equilibrium employment, depends on the heterogeneity across household in their individual labor supply. The question then is, what is Δ_c^*, the term accounting for such heterogeneity? The answer is that Δ_c^* is the power mean of the consumption shares, where, because of the unit continuum of households, the consumption share is also consumption relative to mean consumption. The parameter ω drives how the operator "penalizes" or "rewards" the dispersion of consumption relative to the mean. Specifically, for $\omega = 1$ we have $\Delta_c^* = 1$ regardless of the dispersion of consumption relative to the mean. For $\omega \neq 1$, instead, Δ_c^* deviates from unity unless $s_c^*(h) = 1$ for all $h \in [0,1]$ (i.e., a completely equal society). In particular, an unequal society has $\Delta_c^* > 1$ for $\omega > 1$ and $\Delta_c^* < 1$ for $\omega < 1$. Moreover, Δ_c^* is increasing in consumption inequality for $\omega > 1$ and decreasing in it for $\omega < 1$. Finally, given that the employment ratio l_t/L_t is decreasing in the dispersion index Δ_c^*, the unequal society features higher employment than an equal one under $\omega < 1$ and lower employment under $\omega > 1$. Similarly, the employment ratio l_t/L_t is increasing in consumption inequality under $\omega < 1$ and decreasing under $\omega > 1$.

To understand why the value of ω determines how inequality affects employment, we plot (31) in Figure 9. This figure shows that when ω is greater (less) than 1, the decrease in labor supply by rich households, which have above average consumption, is greater (less) than the increase in labor supply by poor households, which have below average consumption; as a result, inequality that gives rise to rich and poor households reduces (raises) aggregate employment.

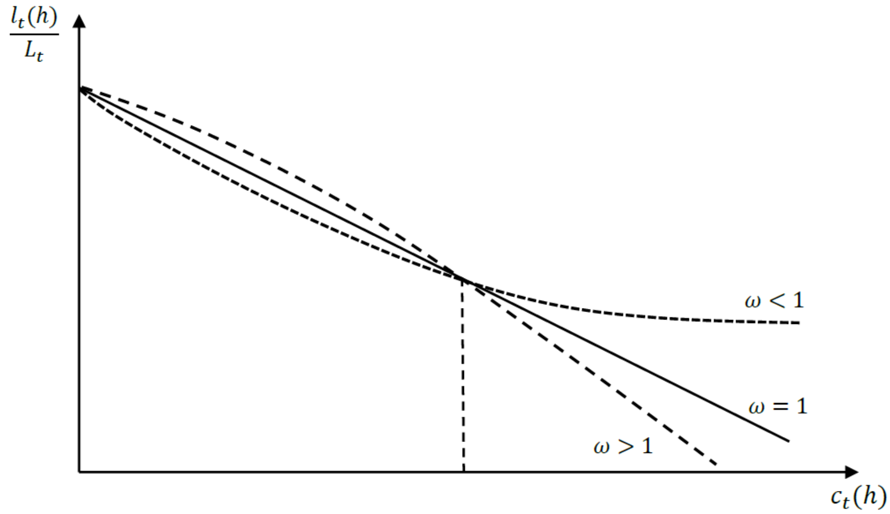

Figure 9: Equation (31)

The use of power means of variables expressed in deviation from the mean as summary statistics for inequality is quite common. In fact, our operator Δ_c^* is related to well-known measures of inequality widely discussed in the literature. Specifically, it is a transformation of the Generalized Entropy index with parameter ω, which in our notation reads:

$$GE(\omega) = \frac{1}{\omega(1-\omega)} \int_0^1 [1 - [s_c^*(h)]^\omega]\, dh, \quad \omega \neq 0, 1.$$

Therefore,

$$\Delta_c^* = [1 - \omega(1-\omega) \cdot GE(\omega)]^{\frac{1}{\omega}},$$

where, for $\omega < 1$, $A_\omega = \omega(1-\omega) \cdot GE(\omega)$ is the Atkinson index of inequality, a popular and well understood measure. In other words, for $\omega < 1$ our Δ_c^* is proportional to $A_\omega^{1/\omega}$.

A simple example illuminates the mechanism. Let initial household wealth $a_0(h)$ be a draw from a Pareto II distribution with CDF

$$F(a) = 1 - \left(1 + \frac{a - a_{\min}}{\kappa}\right)^{-\xi}, \quad a \geq a_{\min}, \quad \kappa > 0, \quad \xi > 1,$$

where the location parameter $a_{\min}$ is the lowest value of wealth. To allow for households with zero wealth, we set $a_{\min} = 0$, which gives us the Lomax distribution.[32] The scale and shape parameters, κ and ξ, have the standard interpretation popularized by the literature on Pareto distributions of wealth and income.[33] The CDF of the wealth share $s_a^*(h) = a_0(h)/A_0$ is

$$F(s_a) = 1 - \left(1 + \frac{s_a}{\xi - 1}\right)^{-\xi}, \quad \kappa > 0, \quad \xi > 1.$$

[32] We do not use the popular Pareto I distribution because it requires $s_a > \kappa > 0$ and thus does not allow for zero-wealth households.

[33] The scale parameter, however, is ultimately linked to supply of assets. In the pre-industrial era, in particular, we have $\kappa/(\xi - 1) = R$ so that we must set $\kappa = R(\xi - 1)$.

Note that this has mean 1 since $A = \int_0^1 a(h)\, dh$ is mean wealth. Note also that

$$\frac{\partial F(s_a)}{\partial \xi} = \xi \left(1 + \frac{s_a}{\xi - 1}\right)^{-\xi-1} \frac{1}{\xi - 1} + \left(1 + \frac{s_a}{\xi - 1}\right)^{-\xi} \log\left(1 + \frac{s_a}{\xi - 1}\right) > 0.$$

Thus, as ξ rises, the distribution of wealth becomes more equal.

As stated in Proposition 3, which extends to this case of endogenous labor supply, the consumption share $s_c^*(h)$ is a phase-specific linear function of the wealth share $s_a^*(h)$. To minimize notation, we write $s_c^*(h) = \varphi_0 + \varphi_1 s_a^*(h)$, where the exact expressions for the phase-specific coefficients are in the Appendix and by construction $\varphi_0 + \varphi_1 = 1$. Since we are taking a linear transformation, the distribution of the consumption shares is Pareto II with CDF

$$F_c(s_c) = 1 - \left(1 + \frac{s_c - \varphi_0}{(\xi - 1)\varphi_1}\right)^{-\xi}.$$

This too has the property that as ξ rises, it shifts up, i.e., the distribution of consumption becomes more equal. Note that while the support of the wealth distribution is $[0, \infty)$, the distribution of the consumption shares features a positive lower bound, φ_0, which corresponds to the consumption share of the zero-wealth households.[34] We then have

$$\Delta_c^* = \left(E\left[(s_c^*)^{\omega}\right]\right)^{\frac{1}{\omega}}.$$

Unfortunately, the calculation of $E\left[(s_c^*)^{\omega}\right]$ for this class of distributions is complex and rarely delivers easily interpretable closed-form solutions.

However, this analytical structure produces a clear insight. A society with more unequal land ownership, i.e., a society with smaller shape parameter ξ of the Pareto II distribution of land, has a more unequal Pareto II distribution of consumption and thus a more unequal Pareto II distribution of labor supply. Such labor supply behavior yields lower employment under $\omega > 1$ and higher employment under $\omega < 1$. This intratemporal causal chain traces how wealth inequality propagates throughout the economy, affecting its scale of operation and thereby its growth path. It also stresses the importance of the parameter ω that regulates the responsiveness of labor supply to consumption. The intertemporal part of the causal chain is that in our scale-invariant model the employment ratio determines the threshold x_N for our state variable, x_t, and thus determines the overall shape of the transition path via its effect on the timing of key events, even though it does not affect steady-state growth. Specifically, the takeoff date is $T_N = \ln(x_N/x_0)/\lambda$, with $x_N = \phi/[(\mu - 1 - \beta\rho)(l/L)^{\gamma}]$ being decreasing in the employment ratio l/L. Consequently, a more unequal society takes off later under under $\omega > 1$ and earlier under $\omega < 1$. These differences in the timing of takeoff never wash out, holding constant everything else. Thus, initial wealth inequality, which in our scheme is the root of all inequality, has effects that echo for centuries and are amplified by the growth acceleration that occurs with the takeoff.

[34] Note also that $E[s_c^*] = \frac{(\xi-1)\varphi_1}{\xi-1} + \varphi_0 = \varphi_1 + \varphi_0 = 1$.

5.3.2 Endogenous formation of the leisure class

Property (ii) occurs when the inequality

$$\left[\frac{\eta c_t(h)}{w_t L_t}\right]^{\omega} \geq 1 \Rightarrow \frac{c_t(h)}{C_t}\frac{\eta C_t}{w_t L_t} \geq 1 \Rightarrow s_c^*(h) \geq \frac{w_t L_t}{\eta C_t}$$

holds for some h. We thus write household labor supply as

$$\frac{l_t(h)}{L_t} = \begin{cases} 1 - \left[\frac{s_c^*(h)}{\frac{w_t L_t}{\eta C_t}}\right]^{\omega} & s_c^*(h) < \frac{w_t L_t}{\eta C_t} \\ 0 & s_c^*(h) \geq \frac{w_t L_t}{\eta C_t} \end{cases}.$$

Next, we note that if we sort households over the unit interval in ascending order of consumption share, the condition $s_c^*(h) \geq \frac{w_t L_t}{\eta C_t}$ defines the cutoff value

$$\bar{h}_t = \arg\text{solve}\left\{s_c^*(h) = \frac{w_t L_t}{\eta C_t}\right\}$$

such that households in the set $[0, \bar{h}_t)$ supply labor and households in the set $\left[\bar{h}_t, 1\right]$ do not. The model, therefore, generates endogenously a *leisure class*, i.e., households who do not work and live off asset income, which consists of industrial dividends, land rents and capital gains on the prices of industrial shares and land.

To construct the equilibrium we need to check how the individual household decision depends on the aggregate state of the labor market. That is, we need to derive the equilibrium expression for the wage by aggregating across households and then check how the individual household responds to that wage, allowing for the possibility of the corner solution.

Aggregation gives us

$$\frac{l_t}{L_t} = \int_0^1 \frac{l_t(h)}{L_t}dh = \int_0^{\bar{h}_t}\left(1 - \left[\frac{s_c^*(h)}{\frac{w_t L_t}{\eta C_t}}\right]^{\omega}\right)dh + \int_{\bar{h}_t}^1 0dh = \bar{h}_t\left[1 - \left(\frac{\eta C_t}{w_t L_t}\bar{\Delta}_c^*\right)^{\omega}\right],$$

where

$$\bar{\Delta}_c^* \equiv \left(\frac{1}{\bar{h}_t}\int_0^{\bar{h}_t}\left[s_c^*(h)\right]^{\omega} dh\right)^{\frac{1}{\omega}}.$$

Proceeding as in the previous case, we obtain the labor market clearing condition

$$\frac{l_t}{L_t} = \bar{h}_t\left[1 - \left[\left(\frac{C}{G}\right)^* \frac{\eta\bar{\Delta}_c^*}{\gamma(1-\theta)}\frac{l_t}{L_t}\right]^{\omega}\right].$$

We stress that $\bar{\Delta}_c^*$ is not a function of $\bar{h}_t$ and is constant. Next, we define the new object

$$\bar{h}_t = \bar{h}\left(\frac{l_t}{L_t}\right) \equiv \arg\text{solve}\left\{s_c^*(h) = \frac{\gamma(1-\theta)}{\eta\left(\frac{C}{G}\right)^*\frac{l_t}{L_t}}\right\},$$

with $\bar{h}'(\cdot) < 0$, and write

$$\frac{l_t}{L_t} = \bar{h}\left(\frac{l_t}{L_t}\right)\left[1 - \left[\left(\frac{C}{G}\right)^* \frac{\eta\bar{\Delta}_c^*}{\gamma(1-\theta)}\right]^{\omega}\left(\frac{l_t}{L_t}\right)^{\omega}\right].$$

This is again an equation in the endogenous variable l_t/L_t and two objects, $\bar{\Delta}_c^*$ and $(C/G)^*$, that are functions of the model's parameters. The left-hand side is increasing; the right hand side is decreasing. Therefore, we have the unique solution

$$\left(\frac{l}{L}\right)^* \equiv \arg\text{solve}\left\{\frac{l_t}{L_t} = \bar{h}\left(\frac{l_t}{L_t}\right)\left[1 - \left[\left(\frac{C}{G}\right)^* \frac{\eta\bar{\Delta}_c^*}{\gamma(1-\theta)}\right]^\omega \left(\frac{l_t}{L_t}\right)^\omega\right]\right\}. \tag{40}$$

Substituting this result in the household labor supply yields

$$\frac{l_t(h)}{L_t} = \begin{cases} 1 - \left[\left(\frac{C}{G}\right)^* \frac{\eta s_c^*(h)}{\gamma(1-\theta)}\left(\frac{l}{L}\right)^*\right]^\omega & s_c^*(h) < \frac{\gamma(1-\theta)}{\eta\left(\frac{C}{G}\right)^*\left(\frac{l}{L}\right)^*} \\ 0 & s_c^*(h) \geq \frac{\gamma(1-\theta)}{\eta\left(\frac{C}{G}\right)^*\left(\frac{l}{L}\right)^*} \end{cases}.$$

This expression identifies which households want to go to the corner solution.

In this characterization, the equilibrium of the labor market is the standard intersection of labor demand and labor supply. What differs from the standard approach is that here labor supply is the joint solution of two equations. The first says that labor supply is the integral over the set of households that supply labor $\left[0, \bar{h}\right]$, the second determines the set of such households. Consequently, the model allows for heterogeneity in labor supply over two margins: the extensive margin, where the household determines whether to supply labor or not; the intensive margin, where the household determines the fraction of time spent working, conditional on having determined that the fraction is positive.

The operator $\bar{\Delta}_c^*$ plays the same role as the operator Δ_c^* discussed in the previous case. However, we must note that in the variable $s_{c,t}(h) = c_t(h)/C_t$ the C_t at the denominator is aggregate consumption. To account for the existence of the leisure class, we define

$$C_t^l \equiv \int_0^{\bar{h}} c_t(h)\, dh$$

and for $h \in \left[0, \bar{h}\right]$ we write

$$s_{c,t}(h) = \frac{c_t(h)}{C_t^l/\bar{h}_t}\frac{C_t^l/\bar{h}_t}{C_t} \Rightarrow s_c^*(h) = \left(\frac{c(h)}{C^l/\bar{h}}\right)^*\left(\frac{C^l}{C}\right)^*\frac{1}{\bar{h}}$$

Then, we write

$$\bar{\Delta}_c^* = \left(\frac{1}{\bar{h}}\int_0^{\bar{h}}\left[s_c^*(h)\right]^\omega dh\right)^{\frac{1}{\omega}} = \frac{1}{\bar{h}}\left(\frac{C^l}{C}\right)^*\left(\frac{1}{\bar{h}}\int_0^{\bar{h}}\left[\left(\frac{c(h)}{C^l/\bar{h}}\right)^*\right]^\omega dh\right)^{\frac{1}{\omega}},$$

and note that the variable in the integral is household consumption relative to mean consumption for the set of households that supply labor. We thus have the same interpretation as before for the operator $\bar{\Delta}_c^*$, with the refinement that it is the power mean of the consumption relative to the mean of the households that supply labor, adjusted for the endogenous two-classes structure of society, the term $1/\bar{h}$ that pushes it up due to the extensive margin of heterogeneity, and for the consumption share of the households that supply labor, the term $\left(C^l/C\right)^*$.

6 Conclusion

This study explored the historical evolution of income inequality from stagnation to growth in a tractable Schumpeterian model with endogenous takeoff and heterogeneous households. Our first result can be summarized as follows. In the pre-industrial era, the economy is in stagnation and income inequality is determined solely by the unequal distribution of land ownership and remains stationary. In the industrial era, the gradually rising growth rate causes income inequality to increase over time until the economy reaches the steady state. We calibrate the model to perform a quantitative analysis and find that the simulation results are roughly in line with historical data for the UK.

The result above obtains in a baseline model with inelastic labor supply that has three main strengths: (i) it is analytically tractable; (ii) it identifies sharply the role of growth accelerations as a main driver of rising income inequality; (iii) it measures inequality with a well understood and widely used summary statistic of the shape of a non-degenerate distribution of income. The baseline model, however, says that inequality plays no role in shaping the transition from stagnation to growth. To address this weakness, we extended the analysis allowing for labor income inequality due to endogenous labor supply. The advantage of doing this is twofold. First, the aggregate dynamics of our economy remain eminently tractable and consist of the two-phase secular transition documented for the simple baseline model with inelastic labor supply. Second, allowing for endogenous labor supply extends considerably the scope of our analysis: inequality affects the employment ratio and thus, through that standard role that scale plays in Schumpeterian models, contributes to shaping the transition path of the economy. Specifically, inequality that results in higher employment produces an earlier takeoff and a higher growth rate during the transition path. However, as the equilibrium growth rate converges to the steady state, this effect disappears due to the scale-invariance of our Schumpeterian growth model.

In this scheme, labor income inequality consists of two margins: an extensive margin along which households sort themselves in a leisure class that supply zero labor and the rest of society that supplies labor; an intensive margin along which households supply labor as a decreasing function of their consumption share, which in turn is an increasing function of their wealth share. The leisure class consists of households that are wealthy enough to find optimal to forgo labor income. Because of this property our model generates endogenously the two-class structure—workers vs. capitalists—that is widely used in the literature on inequality that builds on the classical theory of the distribution of income. The practice there is to postulate the two classes with fixed size and with homogeneity within each class. The resulting models are then silent about the shape of the cross-sectional distribution of income since the imposed within-class homogeneity reduces the cross section of the whole population to two degenerate distributions. Consequently, work that uses this approach tends to measure inequality with grand ratios like the wealth share of GDP (see, e.g., Madsen *et al.* 2021, which we discussed in detail). Our structure, in contrast, allows for heterogeneity in wealth, labor supply, and thus overall income, within each class.

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A Appendix

Rate of return to quality innovation. The current-value Hamiltonian of firm i is

$$H_t(i) = \Pi_t(i) - I_t(i) + \eta_t(i)\dot{Z}_t(i) + \xi_t(i)[\mu - p_t(i)],$$

where $\eta_t(i)$ is the co-state variable on (11) and $\xi_t(i)$ is the multiplier on $p_t(i) \leq \mu$. Substituting (7), (11) and (12) into $H_t(i)$, we have:

$$\frac{\partial H_t(i)}{\partial p_t(i)} = 0 \Rightarrow \frac{\partial \Pi_t(i)}{\partial p_t(i)} = \xi_t(i); \tag{A.1}$$

$$\frac{\partial H_t(i)}{\partial I_t(i)} = 0 \Rightarrow \eta_t(i) = 1; \tag{A.2}$$

$$\frac{\partial H_t(i)}{\partial Z_t(i)} = \alpha\left\{[p_t(i) - 1]\left[\frac{\theta}{p_t(i)}\right]^{1/(1-\theta)} L_t^{\gamma}(i)R^{1-\gamma} - \phi\right\}\frac{Z_t^{1-\alpha}}{Z_t^{1-\alpha}(i)} = r_t\eta_t(i) - \dot{\eta}_t(i). \tag{A.3}$$

If $p_t(i) < \mu$, then $\xi_t(i) = 0$; in this case, $\partial\Pi_t(i)/\partial p_t(i) = 0$ yields $p_t(i) = 1/\theta$. If the constraint on $p_t(i)$ is binding, then $\xi_t(i) > 0$; in this case, $p_t(i) = \mu$. Given $\mu < 1/\theta$, we have $p_t(i) = \mu$. We use (A.2), (15) and $p_t(i) = \mu$ in (A.3) and impose symmetry for (16). ■

Proof of Proposition 1. Aggregation of the budget constraints of the heterogeneous households yields

$$\dot{A}_t = r_t A_t + w_t L_t - C_t. \tag{A.4}$$

As a result of the market structure described above, wealth in the pre-industrial era consists only of land, i.e., $A_t = Rv_t$, and (A.4) reduces to

$$R\dot{v}_t = (q_t + \dot{v}_t)R + w_t L_t - C_t \Rightarrow C_t = q_t R + w_t L_t,$$

which says that in this era consumption equals income, the sum of land income and labor income. Using the factor payments (9)-(10), the expression yields

$$\frac{C_t}{G_t} = \left(\frac{C}{G}\right)^* = 1 - \theta.$$

Using the Euler equation (3), the factor payment (9) and this result we write (A.4) as

$$\frac{\dot{C}_t}{C_t} - \frac{\dot{A}_t}{A_t} = \frac{C_t}{A_t} - \rho - \frac{\gamma(1-\theta)G_t}{A_t} = \frac{C_t}{A_t}\left[1 - \gamma(1-\theta)\left(\frac{G}{C}\right)^*\right] - \rho.$$

This unstable differential equation says that to satisfy the households' transversality condition the consumption-wealth ratio, C_t/A_t, jumps to the steady-state value

$$\frac{C_t}{A_t} = \left(\frac{C}{A}\right)^* = \frac{\rho}{1 - \gamma(1-\theta)\left(\frac{G}{C}\right)^*} = \frac{\rho}{1-\gamma}.$$

In contrast, in the industrial era the free-entry condition (18) holds, the value of monopolistic firms is $N_t V_t = \beta\theta G_t/\mu$ and wealth is $A_t = Rv_t + \beta\theta G_t/\mu$. We then write (A.4) as

$$R\dot{v}_t + \frac{\beta\theta}{\mu}\dot{G}_t = (q_t + \dot{v}_t) R + r_t \frac{\beta\theta}{\mu} G_t + w_t L_t - C_t.$$

Using the factor payments (9)-(10) and the saving rule (3), we reduce this expression to

$$\frac{\dot{C}_t}{C_t} - \frac{\dot{G}_t}{G_t} = \left(\frac{C_t}{G_t} - 1 + \theta\right) \frac{\mu}{\beta\theta} - \rho.$$

This unstable differential equation says that to satisfy the households' transversality condition, the consumption-output ratio, C_t/G_t, jumps to the steady-state value

$$\frac{C_t}{G_t} = \left(\frac{C}{G}\right)^* = 1 - \theta + \rho\beta\theta/\mu.$$

Proceeding as in the previous case, we obtain that the consumption-wealth ratio, C_t/A_t, jumps to the steady-state value

$$\frac{C_t}{A_t} = \left(\frac{C}{A}\right)^* = \frac{\rho}{1 - \gamma(1-\theta)\left(\frac{G}{C}\right)^*} = \frac{\rho(1 - \theta + \rho\beta\theta/\mu)}{(1-\gamma)(1-\theta) + \rho\beta\theta/\mu}.$$

■

Proof of Proposition 2. Given $x_t = (\theta/\mu)^{1/(1-\theta)} (L_t/N_t)^\gamma R^{1-\gamma}$, the growth rate of x_t is given by

$$\frac{\dot{x}_t}{x_t} = \gamma(\lambda - n_t). \tag{A.5}$$

In the pre-industrial era, the variety growth rate n_t is zero; in this case, the dynamics of x_t is simply given by $\dot{x}_t = \gamma\lambda x_t$.

In the first industrial era ($x_t > x_N$), the growth rate of variety is given by

$$n_t = \frac{1}{\beta}\left(\mu - 1 - \frac{\phi + z_t}{x_t}\right) - \rho, \tag{A.6}$$

which is obtained by substituting (21), (A.5) and $r_t = \rho + g_t$ into (19). In this era, the quality growth rate z_t is zero, so the variety growth rate n_t is positive if and only if $x_t > \phi/(\mu - 1 - \beta\rho) \equiv x_N$, where x_N is a threshold for the firm size x_t above which variety innovation starts to occur at time $T_N = \ln(x_N/x_0)/\lambda$. Equation (A.6) shows that when $x_t > x_N$, variety innovation occurs (i.e., $n_t > 0$); in this case, we substitute (A.6) into (A.5) to derive the dynamics of x_t in the first industrial era.

In the second industrial era ($x_t > x_Z > x_N$), quality innovation also occurs (i.e., $z_t > 0$). Substituting (16) and (21) into $r_t = \rho + g_t$ yields

$$g_t = (1-\gamma)n_t + z_t + \gamma\lambda = \alpha\left[(\mu - 1)x_t - \phi\right] - \rho. \tag{A.7}$$

Then, we combine (A.6) and (A.7) to solve for $n(x_t)$:

$$n_t = \frac{[(1-\alpha)(\mu-1)-\rho\beta]\, x_t - (1-\alpha)\phi + \rho + \gamma\lambda}{\beta x_t - (1-\gamma)}. \tag{A.8}$$

Substituting (A.8) into (A.5) yields

$$\dot{x}_t = \gamma \frac{(1-\alpha)\phi - \lambda - \rho - [(1-\alpha)(\mu-1) - \beta(\lambda+\rho)]\, x_t}{\beta - (1-\gamma)/x_t}. \tag{A.9}$$

We assume $\beta - (1-\gamma)/x_t > 0$ for all $x_t > \phi/(\mu-1)$. Then, this equation has a unique steady state that is stable if $(1-\alpha)\phi - \lambda - \rho > 0$ and $(1-\alpha)(\mu-1) - \beta(\lambda+\rho) > 0$ from which we obtain $\rho + \lambda < \min\{(1-\alpha)\phi, (1-\alpha)(\mu-1)/\beta\}$. Then, $\dot{x}_t = 0$ yields x^* in (24). We impose parameter restrictions to ensure $x^* > x_Z$, where

$$x_Z \equiv \underset{x}{\arg\,\mathrm{solve}} \left\{ [(\mu-1)\, x - \phi] \left(\alpha - \frac{1-\gamma}{\beta x} \right) = \gamma(\rho+\lambda) \right\},$$

which is obtained by combining (A.6) and (A.7) to solve for $z(x_t)$ and then setting $z_t = 0$. ■

Proof of Proposition 3. We know that the household consumption share is constant. We thus denote its value $s_c^*(h)$. The growth rate of the household wealth share is

$$\frac{\dot{s}_{a,t}(h)}{s_{a,t}(h)} = \frac{\dot{a}_t(h)}{a_t(h)} - \frac{\dot{A}_t}{A_t} = \frac{w_t L_t - s_c^*(h)\, C_t}{a_t(h)} - \frac{w_t L_t - C_t}{A_t}.$$

Collecting the consumption-wealth ratio C_t/A_t and using the factor payments (9)-(10), we obtain

$$\begin{aligned} \frac{\dot{s}_{a,t}(h)}{s_{a,t}(h)} &= \frac{C_t}{A_t} \frac{\frac{w_t L_t}{G_t}\frac{G_t}{C_t} - s_c^*(h)}{a_t(h)/A_t} - \frac{w_t L_t - C_t}{A_t} \\ &= \frac{C_t}{A_t} \left[\frac{\gamma(1-\theta)\frac{G_t}{C_t} - s_c^*(h)}{s_{a,t}(h)} - \gamma(1-\theta)\frac{G_t}{C_t} + 1 \right]. \end{aligned}$$

We then have the following two cases.

In the pre-industrial era land is the only available asset and we have $s_{a,t}(h) = s_{R,t}(h) \equiv R_t(h)/R$. Proposition 1 then yields

$$\dot{s}_{a,t}(h) = \frac{\rho}{1-\gamma} [\gamma - s_c^*(h) + (1-\gamma)\, s_{a,t}(h)].$$

This differential equation in the pre-determined state variable $s_{a,t}(h)$ holds for all $t \in [0, T_N]$ with $s_{a,t}(h) \in (0,1)$ if and only if the household consumption share $s_c^*(h)$ is always at the value that makes the bracket zero so that $\dot{s}_{a,t}(h) = 0$ for all $t \in [0, T_N]$. Moreover, since agents have perfect foresight and incorporate in their decisions all available information at $t = 0$, the consumption share must jump at $t = 0$. This yields

$$s_c^*(h) = \gamma + s_{R,0}(h)(1-\gamma)$$

for all $t \in [0, T_N]$. Thus, even though our scheme allows for trade of land, in equilibrium households do not trade it and $s_{R,t}(h) = s_{R,0}(h)$ for all $t \in [0, T_N]$. Consequently, our first result is that the pre-industrial wealth distribution is stationary and equal to the initial distribution of land.

In the industrial era the factor payments (9)-(10) and Proposition 1 yield

$$\dot{s}_{a,t}(h) = \frac{\rho(1+\rho\Theta/\mu)}{1-\gamma+\rho\Theta/\mu}\left[\frac{\gamma}{\rho\Theta/\mu+1} - s_c^*(h) + \frac{\rho\Theta/\mu+1-\gamma}{\rho\Theta/\mu+1}s_{a,t}(h)\right],$$

where $\Theta \equiv \beta\theta/(1-\theta)$. The differential equation then holds for all $t > T_N$ with $s_{a,t}(h) \in (0,1)$ if and only if the household's consumption share $s_c^*(h)$ jumps immediately to the value that makes the bracket zero so that $\dot{s}_{a,t}(h) = 0$ for all $t > T_N$. We then have

$$s_c^*(h) = \frac{\gamma}{\rho\Theta/\mu+1} + \frac{\rho\Theta/\mu+1-\gamma}{\rho\Theta/\mu+1}s_{a,T_N}(h).$$

To complete the argument, we need to take a stand on the distribution of industrial wealth at $t = T_N$. To get started, we assume that such initial distribution tracks the distribution of land. Thus, $s_{a,T_N}(h) = R_0(h)/R \equiv s_R(h)$. ■

Gini coefficient of income. The income received by household h is given by

$$y_t(h) = r_t a_t(h) + w_t L_t = r_t A_t s_{a,t}(h) + w_t L_t = r_t A_t s_R(h) + w_t L_t.$$

We order households in ascending order of wealth and of income. The Lorenz curves of, respectively, wealth and income are:

$$\mathcal{L}_a(h) = \int_0^h s_R(\chi)d\chi;$$

$$\mathcal{L}_{y,t}(h) \equiv \frac{\int_0^h y_t(\chi)d\chi}{\int_0^1 y_t(\chi)d\chi} = \frac{r_t A_t \int_0^h s_R(\chi)d\chi + w_t L_t \int_0^h 1d\chi}{Y_t}.$$

The Gini coefficients of respectively, wealth and income are:

$$\sigma_a \equiv 1 - 2\int_0^1 \mathcal{L}_a(h)dh; \tag{A.10}$$

$$\sigma_{y,t} = 1 - 2\int_0^1 \mathcal{L}_{y,t}(h)dh, \tag{A.11}$$

Substituting (A.10) into (A.11), and noting that $\int_0^h 1d\chi = h$, yields

$$\sigma_{y,t} = 1 - \frac{2r_t A_t}{Y_t}\left[\int_0^1 \mathcal{L}_a(h)dh + \frac{w_t L_t}{r_t A_t}\int_0^1 hdh\right],$$

where $\int_0^1 hdh = 0.5$. Substituting the Gini coefficient of wealth into this expression yields the Gini coefficient of income as

$$\sigma_{y,t} = \frac{r_t A_t}{Y_t}\sigma_R = \left(1 - \frac{w_t L_t}{Y_t}\right)\sigma_R,$$

which is equation (26) in the main text. ■

Labor share of income. Using the Euler equation (3) and the aggregate labor income equation (9), we obtain

$$\frac{r_t A_t}{w_t L_t} = \frac{\rho + g_t}{\gamma(1-\theta)} \frac{A_t}{G_t} = \frac{\rho + g_t}{\gamma(1-\theta)} \frac{A_t}{C_t} \frac{C_t}{G_t}.$$

Using Proposition 1 we rewrite this expression as

$$\frac{r_t A_t}{w_t L_t} = \begin{cases} \frac{\rho+\gamma\lambda}{\gamma\rho}(1-\gamma) & 0 \leq x_t \leq x_N \\ \frac{\rho+g_t}{\gamma\rho}(1-\gamma+\rho\Theta/\mu) & x_t > x_N \end{cases}.$$

Finally, we substitute this result into $s_{L,t} = 1/\left(1 + \frac{r_t A_t}{w_t L_t}\right)$ to derive (27) in the main text. ■

Stationarity of the household wealth share with endogenous labor supply. The growth rate of the wealth share is

$$\frac{\dot{s}_{a,t}(h)}{s_{a,t}(h)} = \frac{\dot{a}_t(h)}{a_t(h)} - \frac{\dot{A}_t}{A_t} = \frac{w_t l_t(h) - s_c^*(h) C_t}{a_t(h)} - \frac{w_t l_t - C_t}{A_t}.$$

Collecting the consumption-wealth ratio C_t/A_t, and using the factor payments and Proposition 1, we obtain

$$\begin{aligned} \frac{\dot{s}_{a,t}(h)}{s_{a,t}(h)} &= \left(\frac{C}{A}\right)^* \frac{\frac{w_t l_t(h)}{G_t}\left(\frac{G}{C}\right)^* - s_c^*(h)}{a_t(h)/A_t} - \frac{w_t l_t - C_t}{A_t} \\ &= \left(\frac{C}{A}\right)^* \left[\frac{\gamma(1-\theta)\left(\frac{G}{C}\right)^* \frac{l_t(h)}{l_t} - s_c^*(h)}{s_{a,t}(h)} - \gamma(1-\theta)\left(\frac{G}{C}\right)^* + 1\right]. \end{aligned}$$

The new term here is the household's relative labor supply, $s_l(h) \equiv l_t(h)/l_t$, which is constant. We thus have again that the bracket must be zero at all times, i.e., the proof of stationarity of the wealth shares is mathematically the same as in the baseline model with inelastic labor supply. The resulting phase-specific consumption share is

$$s_c^*(h) = \gamma(1-\theta)\left(\frac{G}{C}\right)^* \left(\frac{l(h)}{l}\right)^* + \left[\gamma(1-\theta)\left(\frac{G}{C}\right)^* - 1\right] s_a^*(h).$$

As argued in the text, we can write this expression as

$$\begin{aligned} s_c^*(h) &= \gamma(1-\theta)\left(\frac{G}{C}\right)^* s_l^*(h) + \left[\gamma(1-\theta)\left(\frac{G}{C}\right)^* - 1\right] s_a^*(h) \\ &= \varphi_0 + \varphi_1 s_a^*(h), \end{aligned}$$

where the consumption-output ratio is given by Proposition 1. ■

Derivation of equations (35)-(36). Recall that household labor income is

$$w_t l_t(h) = \frac{1}{1+\eta}\left[w_t L_t - \eta(r_t - g_t) a_t(h)\right] = \frac{1}{1+\eta}\left[w_t L_t - \eta(r_t - g_t) A_t s_R(h)\right].$$

Using this expression, we write the household income share as

$$\begin{aligned}
s_{y,t}(h) &= \frac{r_t a_t(h) + w_t l_t(h)}{r_t A_t + w_t l_t} \\
&= \frac{r_t A_t s_R(h) + \frac{1}{1+\eta}\left[w_t L_t - \eta(r_t - g_t) A_t s_R(h)\right]}{r_t A_t + \frac{1}{1+\eta}\left[w_t L_t - \eta(r_t - g_t) A_t\right]} \\
&= \frac{\left[r_t - \frac{\eta}{1+\eta}(r_t - g_t)\right] A_t s_R(h) + \frac{1}{1+\eta} w_t L_t}{\left[r_t - \frac{\eta}{1+\eta}(r_t - g_t)\right] A_t + \frac{1}{1+\eta} w_t L_t} \\
&= \frac{\frac{1}{1+\eta}(r_t + \eta g_t) A_t s_R(h) + \frac{1}{1+\eta} w_t L_t}{\frac{1}{1+\eta}(r_t + \eta g_t) A_t + \frac{1}{1+\eta} w_t L_t} \\
&= \frac{(r_t + \eta g_t) A_t}{(r_t + \eta g_t) A_t + w_t L_t} s_R(h) + \frac{w_t L_t}{(r_t + \eta g_t) A_t + w_t L_t}.
\end{aligned}$$

We define

$$\tilde{s}_{L,t} \equiv \frac{w_t L_t}{(r_t + \eta g_t) A_t + w_t L_t}$$

and note that

$$\tilde{s}_{L,t} = \frac{\frac{1}{1+\eta} w_t L_t}{\frac{1}{1+\eta}\left[(r_t + \eta g_t) A_t + w_t L_t\right]} = \frac{\frac{1}{1+\eta} w_t l_t \frac{L_t}{l_t}}{Y_t} = \frac{s_{L,t}}{(1+\eta)\frac{l_t}{L_t}}.$$

■