Three Remarks On Asset Pricing

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ABSTRACT

We make three remarks to the main CAPM equation presented in the well-known textbook by John Cochrane (2001). First, we believe that any economic averaging procedure implies aggregation of corresponding time series during certain time interval Δ and explain the necessity to use math expectation for both sides of the main CAPM equation. Second, the first-order condition of utility max used to derive main CAPM equation should be complemented by the second one that requires negative utility second derivative. Both define the amount of assets ξ_{max} that delivers max to utility. Expansions of the utility in a Taylor series by price and payoff variations give approximations for ξ_{max} and uncover equations on price, payoff, volatility, skewness, their covariance's and etc. We discuss why market price-volume positive correlations may prohibit existence of ξ_{max} and main CAPM equation. Third, we argue that the economic sense of the conventional frequency-based price probability may be poor. To overcome this trouble we propose new price probability measure based on widely used volume weighted average price (VWAP). To forecast price volatility one should predict evolution of squares of the value and the volume of market trades aggregated during averaging interval Δ. The forecast of the new price probability measure may be the main tough puzzle for CAPM and finance. However investors are free to chose any probability measure they prefer as ground for their investment strategies but should be ready for unexpected losses due to possible distinctions with real market trade price dynamics.

Keywords: asset pricing, volatility, price probability, market trades

JEL : C58, D4, E31, F1, G1

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1. Introduction

Asset price forecasting defines main problem and desire of participants of financial markets. Investors, traders, academic scholars and householders make their best to outrun and get ahead of others in treatment, guessing and solution of the price puzzles. Last decades give great progress in asset price valuation and setting. Starting with Hall and Hitch (1939) a lot researchers investigate price (Friedman, 1990; Heaton and Lucas, 2000) and factors those impact investors, markets (Fama, 1965), equilibrium economy (Sharpe, 1964), fluctuations (Mackey, 1989) macroeconomics (Cochrane and Hansen, 1992) and business cycles (Mills, 1946; Campbell, 1998). Muth (1961) initiated studies on dependence of asset pricing on expectations and his ideas further were developed by numerous scholars (Lucas, 1972; Malkiel and Cragg, 1980; Campbell and Shiller, 1988; Greenwood and Shleifer, 2014). Many researchers describe price dynamics and references (Goldsmith and Lipsey, 1963; Campbell, 2000; Cochrane and Culp, 2003; Borovička and Hansen, 2012; Weyl, 2019) give only a small part of them.

Asset pricing modeling cannot be separated from description of price fluctuations, volatility of the price time-series and returns. Price and returns volatility are the most important issues that impact investors expectation on current market trades and on future profits. Description of volatility is inseparable from price modeling and these problems are almost always studied together (Hall and Hitch, 1939; Fama, 1965; Stigler and Kindahl, 1970; Tauchen and Pitts, 1983; Schwert, 1988; Mankiw, Romer and Shapiro, 1991; Brock and LeBaron, 1995; Bernanke and Gertler, 1999; Andersen et.al., 2001; Poon and Granger, 2003; Andersen et.al., `2005). The list of references can be continued as hundreds and hundreds of publications describe different faces of the price-volatility puzzle.

Simple and practical advises on price modeling and forecasting among the most demanded by investors. Different price models were developed to satisfy and saturate investors desires. We refer only some pricing models (Ferson et.al., 1999; Fama and French, 2015) and studies on Capital Asset Pricing Model (CAPM) (Sharpe, 1964; Merton, 1973; Cochrane, 2001; Perold, 2004). Cochrane (2001) shows that CAPM includes different versions as ICAPM and consumption-based pricing model (Campbell, 2002) is one of CAPM variations. Further we shall consider Cochrane (2001) as clear and consistent presentation of CAPM basis, problems and achievements. His resent and unusual study (Cochrane, 2021) complements the rigorous asset price description with deep and justified general considerations of the nature, problems and possible directions for further research.

Nevertheless the asset pricing, risk, uncertainties and financial markets were studied with a great accuracy and solidity there are still "some" problems left. We assume that core economic difficulties and fundamental economic relations may still impede further significant development of the price theory. To explain the nature of the existing economic obstacles that may hamper price forecasting we simplify the main assumptions of modern asset pricing models to make the description logic more clear. Cochrane (2001) describes CAPM with great clarity and accuracy and includes almost all existing CAPM extensions. It is convenient consider asset pricing having the single source that describes different extensions and model variations from the singe approach and within the single frame. We propose that readers are sufficiently familiar with CAPM (Cochrane, 2001) and refer this monograph for any clarifications. In our paper we consider some issues concern the main CAPM equation and show why and how some simple and commonly used notions may be the origin of tough problems that prevent successful asset price forecasting.

Equation (4.5) means equation 5 in the Sec. 4 and (B.7) – notes equation 7 in Appendix B.

2. Main CAPM assumptions

The general frame that determines all CAPM versions and extensions states: "All asset pricing comes down to one central idea: the value of an asset is equal to its expected discounted payoff" (Cochrane and Culp, 2003). This formula supplemented with general equilibrium assumptions is repeated in most CAPM papers (Cochrane, 2001; Hördahl and Packer, 2007; Cochrane 2021). Below we discuss why and how this common, well-known and verified statement hides a lot of troubles that may make the asset pricing a much more tough puzzle than it seems now.

Let's follow (Cochrane, 2001) and consider the main CAPM assumptions in a more rigorous manner. The main CAMP equation has form:

$$p_t = E[m_{t+1}\, x_{t+1}] \tag{2.1}$$

In equation (2.1) p_t denotes asset price at moments t, $x_{t+1}=p_{t+1}+ d_{t+1}$ – payoff at moment $t+1$, p_{t+1} and d_{t+1} price and dividends at moment $t+1$, m_{t+1} - the stochastic discount factor and E– math expectation at moment $t+1$ made by the forecast under the information available at moment t. Cochrane (2001) considers relations (2.1) in various forms to show that almost all models of asset pricing united under the common title CAPM can be described by similar equation. We shall consider (2.1) and refer (Cochrane, 2001) for all other CAPM extensions. For readers convenience we briefly reproduce Cochrane (2001) "consumption-based" derivation of (2.1). Cochrane "models investors by utility function defined over current and future values of consumption and c_t denotes consumption at date t."

$$U(c_t;\ c_{t+1}) = u(c_t) + E[\beta u(c_{t+1})] \tag{2.2}$$

$$c_t = e_t - p_t\xi \quad ; \quad c_{t+1} = e_{t+1} + x_{t+1}\xi \tag{2.3}$$

$$x_{t+1} = p_{t+1} + d_{t+1} \tag{2.4}$$

Here (2.3) e_t and e_{t+1} "denotes original consumption level (if the investor bought none of the asset), and ξ denotes the amount of the asset he chooses to buy" (Cochrane, 2001). Payoff x_{t+1} (2.4) is determined by price p_{t+1} and dividend d_{t+1} of asset at moment $t+1$. Cochrane calls β as "subjective discount factor that captures impatience of future consumption". $E[...]$ in (2.2) denotes math expectation of random utility due to random payoff x_{t+1} (2.4) made at moment $t+1$ by forecast on base of information available at moment t. Below for brevity we shall note price at moment t as p and payoff at moment $t+1$ as x. The first-order maximum condition for (2.2) by amount of asset ξ is fulfilled by putting derivative of (2.2) by ξ equals zero (Cochrane, 2001):

$$\max_{\xi} U(c_t;\ c_{t+1}) \leftrightarrow \frac{\partial}{\partial \xi} U(c_t;\ c_{t+1}) = 0 \tag{2.5}$$

From (2.2-2.5) it is obvious that:

$$p = E\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x\right] = E[mx] \quad ; \quad m = \beta \frac{u'(c_{t+1})}{u'(c_t)} \quad ; \quad \frac{d}{dc}u(c) = u'(c) \tag{2.6}$$

and (2.6) reproduce (2.1) for m (2.6). This completes the brief derivation of the main CAPM equation (2.1) and we refer Cochrane (2001) for any further details.

3. Remarks on time scales

We start with simple remarks on math expectations and time scales. Any math expectation of price delivers price value averaged during certain interval Δ. The averaging procedure can be different but any such procedure aggregates time-series during certain interval Δ. The choice of different averaging interval Δ may define different mean values. The choice of averaging interval Δ defines the *internal* time scale of the problem under consideration. The time-horizon T: $t+1 = t+T$ defines the *external* time scale of the problem. Relations between *internal* and *external* scales determine evolution of the averaged variables, sustainability and accuracy of the model description. In simple words we underline that financial variables – price, volatility, beta – averaged during interval Δ can behave irregular or randomly on time scales T for $T \gg \Delta$. This effect is well-known and unstable dynamics of financial beta mentioned, for example, by Cochrane (2021): "Another great puzzle is how little we know about betas. In continuous-time diffusion theory, 10 seconds of millisecond data should be enough to measure betas with nearly infinite precision. In fact, betas are hard to measure and unstable over time". It's clear that if market disturbing factors have time scale d and $d > \Delta$ then averaging during interval Δ smooth only the perturbations with scales less than Δ. If market is under impact of perturbations with scales d and $\Delta < d < T$, then variables averaged during interval Δ will be disturbed over scales $d > \Delta$ and will demonstrate irregular or random properties. It is clear that dynamics of price, payoff and discount factor are under impact of factors with different time scale disturbances. Eventually, the choice of scale Δ is important for asset pricing modeling, but sadly it is not the main trouble.

As we explain the averaging interval Δ defines *internal* time the scale of the problem. In simple words: if one averages price time-series during interval Δ equals 1 hour, 1 day, 1 week then the time "meter" – "the Clocks" of the problem has minimal time scale division equals 1 hour, 1 day, 1 week. For example take initial time series with initial time scale division ε:

$$t_0, t_1, \dots t_i, .. \quad ; \quad t_i = t_0 + \varepsilon i \quad ; i = 1, \dots \tag{3.1}$$

After averaging time-series (3.1) during interval $\Delta > \varepsilon$ any variables of the problem under consideration will be presented by time series

$$t_0, t_1, \dots t_i, .. \quad ; \quad t_i = t_0 + \Delta i \quad ; i = 1, \dots \tag{3.2}$$

These simple relations cause that to model "investors by a utility function" (Cochrane, 2001) one should use the same time scale divisions "to-day" at moment t and the "next-day" at $t+1$. Time scale can't be measured "to-day" in hours and "next-day" in weeks. One should take math expectation that aggregate time-series during interval Δ for both parts of investors utility. Let's model "investors by a utility function" (2.2) without any math expectation as:

$$U(c_t;\, c_{t+1}) = u(c_t) + \beta u(c_{t+1}) \tag{3.3}$$

Relations (2.5; 2.6) for utility (3.3) take simple form

$$u'(c_t)\, p = \beta x\, u'(c_{t+1}) \tag{3.4}$$

As we mentioned above math expectation should be taken for both parts of equation (3.4):

$$E_t[p\, u'(c_t)] = E\,[\beta x\, u'(c_{t+1})] \tag{3.5}$$

Important note – math expectations *E[...]* in the left and in the right sides of (3.5) is determined by different probability measures but with identical averaging time interval Δ. In the left side math expectation $E_t[...]$ uses price p probability measure at moment t. In the right side math expectation *E[...]* uses forecast of the joint probability measure of discount factor β and payoff x at moment $t+1$. To underline that math expectation $E_t[..]$ at moment t in the left side of (3.5) is determined by probability measure that is different from joint probability in the right side we shall note it in (3.5) and further as $E_t[..]$ and the right side math expectation as *E[..]*. It is obvious that equation (3.5) can be derived starting with "investors utility function":

$$U(c_t;\ c_{t+1}) = E_t[u(c_t)] + E[\beta u(c_{t+1})] \qquad (3.6)$$

For math expectations $E_t[..]$ at moment t and *E[..]* at moment $t+1$ with identical averaging interval Δ utility function (3.6) match utility (2.2) and both moments t and $t+1$ are described with the same accuracy with time scale Δ. Below we discuss these probability measures and possible mutual relations between them. Equation (3.5) gives more general version of the main CAPM equation (2.1; 2.6). If one applies math expectation only to the right side of (3.4) obtain main CAPM equation (2.6):

$$p = E\ [\beta\ \frac{u'(c_{t+1})}{u'(c_t)}\ x] = E[mx]$$

Below we show that (3.3- 3.6) hide much more interesting relations than (2.6).

4. Remarks on main CAPM equation

Relations (2.6; 3.4; 3.5) are consequences of simple conditions (2.5). Math expectation in (2.6) establishes correlations between "next-day" utility $u(c_{t+1})$ and payoff x. However the ground for derivation of (3.5; 2.6) allows consider this equation in equivocal way. Let's mention that condition (2.5) is a simple first-order condition of max by amount of asset ξ and determines the ξ_{max} that delivers maximum to investor's utility (2.2) or (3.6). Let's show how assess the ξ_{max} from (3.5; 2.6). Let's chose the averaging interval Δ and present the price p of the asset, payoff x and subjective discount factor β as:

$$p\ =\ p_0 + \delta p\ ; \quad x = x_0 + \delta x\ \ ;\ \beta = \beta_0 + \delta\beta \qquad (4.1)$$

$$E_t[p] = p_0\ ;\ \ E[x] = x_0\ \ ;\ \ E[\beta] = \beta_0\ ;\ \ E_t[\delta p] = E[\delta x] = E[\delta\beta] = 0 \qquad (4.2)$$

We remind that we always take math expectation $E_t[..]$ as averaging during interval Δ at moment t and math expectation *E[..]* as averaging during same interval Δ at moment $t+1$ as forecast within data available at moment t. We assume that random fluctuations of price, payoff and discount factor are small to compare with their mean values and we present the utility function (3.4) by a Taylor series expansion in linear approximation by δp and δx:

$$u'(c_t) = u'(c_{t;0}) - \xi u''(c_{t;0})\delta p \quad ; \quad u'(c_{t+1}) = u'(c_{t+1;0}) + \xi u''(c_{t+1;0})\delta x \qquad (4.3)$$

$$c_{t;0} = e_t - p_0\xi \quad ; \quad c_{t+1;0} = e_{t+1} + x_0\xi$$

Take math expectation for both sides of (3.4; 4.3) obtain (3.5) or substitute Taylor series expansion of utility functions (4.4) into (3.5) and obtain equation (4.4):

$$u'(c_{t;0})p_0 - \xi u''(c_{t;0})\sigma^2(p) = u'(c_{t+1;0})E[\beta x] + \xi u''(c_{t+1;0})E[\beta x\delta x] \qquad (4.4)$$

Equation (4.4) result the first-order max conditions (2.5) and determine relations (4.5) that assess the root ξ_{max} that delivers the maximum to utility $U(c_t;c_{t+1})$ (3.6) (see A.8 - App.A):

$$\xi_{max} = \frac{u'(c_{t;0})p_0 - u'(c_{t+1;0})E[\beta x]}{u''(c_{t;0})\sigma^2(p) + u''(c_{t+1;0})E[\beta x \delta x]} \tag{4.5}$$

Let's underline that the root ξ_{max} (4.5) directly depends on volatility of price $\sigma^2(p)$ at moment t and on volatility of payoff $\sigma^2(x)$ at moment $t+1$. The expressions for $\sigma^2(p)$, $E[\beta x]$ and $E[\beta x \delta x]$ are given by (4.1-4.3; A.5-A.7):

$$E[\beta x] = \beta_0 x_0 + cov(\beta, x)$$

$$E[\beta x \delta x] = \beta_0 \sigma^2(x) + x_0 cov(\beta, x) + cov(\beta, x^2)$$

$$\sigma^2(p) = E_p\left[\delta^2 p\right] \;\; ; \;\; \sigma^2(x) = E\left[\delta^2 x\right] \;\; ; \;\; cov\left(\delta\beta, \delta^2 x\right) = E\left[\delta\beta\delta^2 x\right]$$

It is obvious that as due to (A.2) utility functions $u'(c_{t;0})$, $u'(c_{t+1;0})$, $u''(c_{t;0})$, $u''(c_{t+1;0})$ depend on ξ thus relations (4.5) give only assessment of the amount ξ_{max} that can be treated as *first approximation*. It is clear that sequential iterations may give more accurate approximations of ξ_{max}. Nevertheless our approach and (4.5) give new look on CAPM equation (2.6; 3.5). If one follows standard derivation of equation (2.6) (Cochrane, 2001) and neglects the math expectations $E_t[..]$ in the left-side of (3.5) then (2.6; 4.5) give

$$\xi_{max} = \frac{u'(c_t)p - u'(c_{t+1;0})E[\beta x]}{u''(c_{t+1;0})E[\beta x \delta x]} \tag{4.6}$$

Relations (4.6) show that even standard form of main CAPM equation (2.6) hides direct dependence of ξ_{max} on payoff volatility $\sigma^2(x)$ at moment $t+1$. If one has independent assessment of ξ_{max} then he can use it to present (4.6) in a way alike to the main CAPM equation (2.6)

$$p = \frac{u'(c_{t+1;0})}{u'(c_t)}E[\beta x] + \xi_{max}\frac{u''(c_{t+1;0})}{u'(c_t)}E[\beta x \delta x] \tag{4.7}$$

Taking into account (A.5-A.7) one can transform (4.7) alike to (2.6):

$$p = m_0 x_0 + \xi_{max} m_1 \sigma^2(x) + \varphi(\beta, x) \tag{4.8}$$

$$m_0 = \frac{u'(c_{t+1;0})}{u'(c_t)}\beta_0 \;\; ; \;\; m_1 = \frac{u''(c_{t+1;0})}{u'(c_t)}\beta_0 \tag{4.9}$$

$$\varphi(\beta, x) = [m_0 + \xi_{max} m_1 x_0]\frac{cov(\beta,x)}{\beta_0} + \xi_{max} m_1 \frac{cov(\beta,x^2)}{\beta_0} \tag{4.10}$$

For the given ξ_{max} equation (4.8) describes dependence of price p at moment t on mean discount factor m_0 (4.9) and mean payoff x_0 (4.1) and discount factor m_1 (4.9) and payoff volatility $\sigma^2(x)$ (A.6; A.7). Function $\varphi(\beta,x)$ (4.10) for the given ξ_{max} describes impact of $cov(\beta,x)$ and $cov(\beta,x^2)$ discounted by m_0 and m_1. Let's underline that while the mean discount factor $m_0>0$, the mean discount factor $m_1<0$ because utility $u'(c_t)>0$ and $u''(c_t)<0$ for all t. We underline that (4.6-4.10) have sense for the given value of ξ_{max}. It seems important that for the given value of ξ_{max} (4.8) describes negative discounted impact of payoff volatility $\sigma^2(x)$ at moment $t+1$ on asset price at moment t. If one neglects impact of function $\varphi(\beta,x)$ in (4.8) then modified main CAPM equation (4.11) takes simple form:

$$p = m_0 x_0 + \xi_{max} m_1 \sigma^2(x) \tag{4.11}$$

Equation (4.11) describes modified CAMP statement: *"the value of an asset is equal" mean payoff x_0 discounted by mean factor m_0 minus payoff volatility $\sigma^2(x)$ discounted by factor $|m_1|$ and multiplied by amount of asset ξ_{max} that delivers maximum to investors utility (2.2).*

$$\xi_{max} < \frac{m_0 x_0}{|m_1|\sigma^2(x)} = -\frac{u'(c_{t+1;0})}{u''(c_{t+1;0})} \frac{x_0}{\sigma^2(x)} \tag{4.12}$$

For this approximation for positive asset price $p>0$ relations (4.12) limit the value of ξ_{max}. If one takes into account the averaging $E_t[..]$ at moment t then from (4.5) obtain equations similar to (4.7-4.12):

$$m_0 = \frac{u'(c_{t+1;0})}{u'(c_{t;0})}\beta_0 > 0\ ;\ m_1 = \frac{u''(c_{t+1;0})}{u'(c_{t;0})}\beta_0 < 0\ ;\ m_2 = \frac{u''(c_{t;0})}{u'(c_{t;0})} < 0 \tag{4.13}$$

$$p_0 = m_0 x_0 + \xi_{max}[m_1\sigma^2(x) + m_2\sigma^2(p)] + \varphi(\beta, x) \tag{4.14}$$

We use the same notions m_0, m_1 to denote discount factors taking into account replacement of $u'(c_t)$ in (4.9) by $u'(c_{t;0})$ in (4.13; 4.14). Modified main CAPM equation (4.14) for the case (3.6; 4.5; 4.13) describes dependence of mean asset price p_0 at moment t on price volatility $\sigma^2(p)$ at moment t, mean payoff x_0 and payoff volatility $\sigma^2(x)$ and function $\varphi(\beta,x)$ with coefficients (4.13) at moment $t+1$. If one neglects function $\varphi(\beta,x)$ (4.10) then obtain equation (4.15) similar to (4.11):

$$p_0 = m_0 x_0 + \xi_{max}[m_1\sigma^2(x) + m_2\sigma^2(p)] \tag{4.15}$$

Equation (4.15) describes simple relations between mean price p_0 and price volatility $\sigma^2(p)$ at moment t, mean payoff x_0 and payoff volatility $\sigma^2(x)$ at moment t+1, and discount factors m_0, m_1, m_2 (4.13). Equation (4.15) reproduces well known practice that high volatility $\sigma^2(p)$ of price at moment t and high forecast of payoff volatility $\sigma^2(x)$ at moment $t+1$ cause decline of the mean price p_0 at moment t. We leave the detailed analysis of (4.5-4.15) for the future.

4.1 The idiosyncratic risk

Here we consider the idiosyncratic risk in which case for equation (2.6) the payoff x at moment $t+1$ is not correlated with the discount factor m at moment $t+1$(Cochrane, 2001):

$$cov(m, x) = 0 \tag{4.16}$$

In this case equation (2.6) takes form:

$$p = E[mx] = E[m]E[x] + cov(m, x) = E[m]x_0 \tag{4.17}$$

The risk-free rate R^f is known ahead. Taking into account (4.1-4.4; 4.9; 4.16), from (A.16) obtain in linear approximation by δx Taylor series for utility $u'(c_{t+1})$:

$$E[m] = \beta_0 \frac{u'(c_{t+1;0})}{u'(c_t)} \left[1 - \frac{cov^2(\beta,x)}{\beta_0^2 \sigma^2(x)}\right] = m_0 \left[1 - \frac{cov^2(\beta,x)}{\beta_0^2 \sigma^2(x)}\right] = \frac{1}{R^f} \qquad (4.18)$$

Correlations $cov^2(\beta,x)$ between subjective discount factor β and payoff x and variance of the payoff $\sigma^2(x)$ impact the mean value β_0 of the subjective discount factor. We remind that due to (4.9; A.15) $cov(\beta,x)>0$ and:

$$\xi_{max} = -\frac{u'(c_{t+1;0})cov(\beta,x)}{\beta_0 u''(c_{t+1;0})\sigma^2(x)} = -\frac{m_0\, cov(\beta,x)}{m_1 \beta_0 \sigma^2(x)} \quad ; \quad cov(\beta,x) > 0$$

Otherwise the amount of asset ξ_{max} that delivers the maximum to the utility (2.5) equals zero: $\xi_{max}=0$. Relations (4.9; 4.18) define dependence of $cov(\beta,x)$ on payoff volatility $\sigma^2(x)$ and the risk-free rate R^f as:

$$cov^2(\beta,x) = \left(1 - \frac{1}{m_0 R^f}\right) \beta_0^2\, \sigma^2(x) \qquad (4.19)$$

From (4.17; 4.18; A.14; A.16)

$$p = \frac{x_0}{R^f} = x_0 \left[m_0 - \xi_{max}^2 \sigma^2(x)\, \frac{m_1^2}{m_0}\right] \qquad (4.20)$$

Now let's consider idiosyncratic risk in the case of utility (3.6) and its consequences (4.5; 4.13; 4.14). In this case one takes math expectation at moment t and moment $t+1$ and main CAPM equation (2.6) takes form of (3.5). We state that in this case condition (4.23) similar to (4.16) at moment $t+1$ between factor $[\beta u'(c_{t+1})]$ and payoff x (4.1) should be complemented by no correlations conditions (A.18-A.20) at moment t between utility $u'(c_t)$ and price p (4.1). Equations (3.5; A.19; A.20; A.21) describe p_0, x_0 and risk-free rate R^f alike to (4.17; 4.18) (see A.21):

$$p_0 = \frac{x_0}{R^f} = \frac{E[\beta u'(c_{t+1})]}{E_t[u'(c_t)]}\, x_0 \quad ; \quad \frac{1}{R^f} = \frac{E[\beta u'(c_{t+1})]}{E_t[u'(c_t)]}$$

The main interest is hidden in approximation of math expectations $E_t[u'(c_t)]$ at moment t and math expectation $E[\beta u'(c_{t+1})]$ at moment $t+1$. Indeed, relations (4.18; 4.19) are determined by Taylor series of utility $u'(c_{t+1})$ by δx in linear approximation. However it is easy to show (A.22-A.25) that substitution of linear Taylor series of utility $u'(c_t)$ by δp in (4.22) implies:

$$\sigma^2(p) = 0$$

This is the result of linear Taylor series approximation of utility $u'(c_t)$ by δp and not the affirmation of zero market price volatility at moment t. To overcome "non-market" zero-volatility case at moment t let's take the Taylor series with accuracy $\delta^2 p$ for utility $u'(c_t)$ at moment t and, to keep balance at moment $t+1$, with accuracy $\delta^2 x$ for utility $u'(c_{t+1})$ at moment $t+1$. Then relations (see App.A, A18-A.42) similar to (4.18) take form (A.37)

$$\frac{1+\frac{1}{2}\,\xi^2{}_{max} m_4 \sigma^2(p)}{R^f} = m_0 + \xi_{max} m_1\, \frac{cov(\beta,x)}{\beta_0} + \frac{1}{2}\,\xi^2{}_{max} m_5 \left[\sigma^2(x) + \frac{cov(\beta,x^2)}{\beta_0}\right]$$

No-correlations conditions (A.19; A.20) give (A.42):

$$\frac{\sigma(x)\; Sk(x)}{\sigma(p)\; Sk(p)} = -\frac{m_0 m_4^2}{2m_2^2 m_5}\frac{cov(\beta,x)}{\beta_0\sigma^2(x)} Sk(p)\sigma(p)\; - \frac{m_1 m_2 m_4}{m_2^2 m_5}[1+\frac{cov(\beta x^2)}{\beta_0\sigma^2(x)}]$$

Here $Sk(p)$ (A.32) and Sk(x) denote normalized skewness of price p at moment t and payoff x at moment $t+1$ respectively.

It is clear that approximation of utility by Taylor series with accuracy of the third order $\delta^3 p$ at moment t and $\delta^3 x$ at moment $t+1$ will take into account impact of price and payoff kurtosis. We leave these exercises for future.

4.2 The utility maximum

Relations (2.5) define the first-order conditions that may determine the amount of asset ξ_{max} that delivers the max to utility function $U(c_t;\, c_{t+1})$ (2.2; 3.6). To confirm that ξ_{max} really delivers the maximum to $U(c_t;\, c_{t+1})$ the first order conditions (2.5) must be supplemented by simple and obvious second order conditions:

$$\frac{\partial^2}{\partial \xi^2} U(c_t;\; c_{t+1}) < 0 \qquad (4.21)$$

The usage of the conditions (4.21) gives a lot of interesting consequences for asset pricing. From (2.2 – 2.4) and (4.21) obtain (see details in App.B, B.2):

$$p^2 > -E\left[\beta x^2\, \frac{u''(c_{t+1})}{u''(c_t)}\right] \qquad (4.22)$$

The same conditions (4.21) for utility function (3.6) give (B.3):

$$E_t[p^2 u''(c_t)] < -E[\beta x^2\; u''(c_{t+1})\;] \qquad (4.23)$$

Taking Taylor expansion of utilities $u''(c_t)$ by price δp at moment t and $u''(c_{t+1})$ by payoff δx at moment $t+1$ in (4.23) obtain relations on ξ_{max} (B.6-B.11). For the case (4.22) expansion in Taylor series gives

$$p^2 > -\,\frac{1}{\beta_0 m_2}\{m_1 E[\beta x^2] + m_5\; \xi E[\beta x^2 \delta x]\} \qquad (4.24)$$

Relations (4.24) should be treated as inequalities on the amount of asset ξ_{max} (B.10; B.11):

$$\xi_{max} < \frac{p^2 u''(c_t) + E[\beta x^2] u''(c_{t+1;0})}{E[\beta x^2 \delta x] u'''(c_{t+1;0})}\;\; ;\; if \quad E[\beta x^2 \delta x] u'''(c_{t+1;0}) < 0 \qquad (4.25)$$

To avoid doubling we refer to App.B for further details and similar inequalities on utility (3.6) and (4.23).

Almost all economic and financial notions and relations interfere with others. Main CAPM equation (2.6) and second order condition (4.21) are no exception. Accidently or not but market trade price-volume correlations are almost identical the second order conditions (4.21) that define existence of utility function maximum at point ξ_{max}. Market trade price-volume correlations are under investigation for decades (Ying, 1966; Karpoff, 1987; Gallant, Rossi and Tauchen, 1992; DeFusco, Nathanson and Zwick, 2017). Researchers report evidence as for positive as well for negative price-volume correlations for different time terms, assets and markets. For example, Ying (1966): «A large volume is usually accompanied by a rise in price". Karpoff (1987) collected data from numerous studies since 1963 till 1987 that support positive correlation between price change and volume $(\partial p/\partial\xi > 0)$ (Table 1, p.113) and data that don't support positive correlations (Table 2, p.118). Let's take that investor gains his "amount of the asset he chooses to buy" (Cochrane, 2001) within the trade with the price p and the volume ξ. Then the sign of derivative of the price p by the volume ξ determines growth or decline of the trade price p with trade volume ξ:

$$1.\ \frac{\partial p}{\partial \xi} < 0 \ \ ; \ \ 2.\ \frac{\partial p}{\partial \xi} = 0 \ \ ; \ \ 3.\ \frac{\partial p}{\partial \xi} > 0 \qquad (4.26)$$

Relations (4.26) completely determine the sign of the second order condition (4.21) and:

$$1.\frac{\partial^2}{\partial \xi^2}U(c_t;\ c_{t+1}) < 0\ ;\ \ 2.\ \frac{\partial^2}{\partial \xi^2}U(c_t;\ c_{t+1}) = 0\ \ ;\ \ 3.\frac{\partial^2}{\partial \xi^2}U(c_t;\ c_{t+1}) > 0 \quad (4.27)$$

Relations (4.27) are obvious consequences of (2.6; 3.5) and (4.21; 4.26). The negative sign of price derivative (4.26) equals negative price-volume correlations and thus negative condition (4.21; 4.27) may determine the condition for ξ_{max}. However positive sign (4.26) equals positive (4.27) price-volume correlations and hence the root ξ_{min} of (2.5) determines *minimum* of utility $U(c_t;c_{t+1})$. If so, utility (2.2; 3.6) has no maximum and main CAPM equation (2.6) describes relations at *minimum* of utility function. If the price derivative by ξ equals zero (4.26) then utility function $U(c_t;c_{t+1})$ doesn't depend on ξ (see B.22) and extremum of utility exist. Above considerations describe impact of market trade price-volume correlations studied at least for 60 years on the main CAPM equations. We assume that mutual interfere between price-volume correlations and CAPM equations deserve further investigations.

5. Remarks on price probability measure

As usual the problems that are most common and "obvious" hide most difficulties. The price probability measure is exactly the case of such hidden complexity.

The capital asset pricing model is based on assumption that it is possible to assess and forecast the price p probability and the joint probability measure that describes random payoff x and subjective discount factor β. We consider the price p probability measure only and assume that it alone delivers enough complexity to make the CAPM predictions sufficiently difficult. All asset price models are based on consideration of certain price probability measure and its forecast. We regard the choice and forecasting of the price averaging measure as most interesting, important and complex problem of financial economics and CAPM in particular.

The usual and conventional treatment of the price p probability "is based on the probabilistic approach and using A. N. Kolmogorov's axiomatic of probability theory, which is generally accepted now" (Shiryaev, 1999). Definition of price probability is based on the frequency of events - trades with price p during the averaging time interval Δ. Let's take that N trades were performed during time interval Δ and of them $n(p)$ trades with price p. Then probability $P(p)$ of price p during interval Δ is determined as $P(p) \sim n(p)/N$. The frequency of particular event is the correct, general and conventional approach to probability definition.

However we study economics, financial markets and agents expectations. These items don't accept anything standard. Let's state a simple question: does conventional frequency-based price probability definition can be applied for asset pricing? Asset pricing is the result of market trading. Economic agents take decisions on trades on base of numerous economic and financial factors and agents personal expectations. It is generally accepted that agents expectations are the driving forces for market trading and asset pricing. This duality of impact of economic factors and agent expectations on asset pricing makes the price probability problem really complex. We don't intent to prove that frequency-based price probability is incorrect. We see no space for almost any "solid proof" in economic and finance. Agents expectations and believes may refute and overturn almost any economic "solid law". However we show that the asset price probability measure different from conventional frequency based probability may be more valuable for asset pricing modeling.

Let's remind that almost 30 years ago the volume weighted average price (VWAP) was introduced and VWAP is widely used now (Berkowitz et.al 1988; Buryak and Guo, 2014; Guéant and Royer, 2014; Busseti and Boyd, 2015; Duffie and Dworczak, 2018; Padungsaksawasdi and Daigler, 2018; CME Group, 2020). The definition of VWAP during interval Δ is simple. Let's take that N market trades during interval Δ were performed with the volume $U(t_i)$, the value $C(t_i)$ and the price $p(t_i)$ at moments t_i, $i=1,\ldots N$. Then VWAP $p(1;t)$ at moment t during interval Δ equals:

$$p(1;t) = \frac{1}{U(1;t)} \sum_{i=1}^{N(t)} p(t_i)U(t_i) = \frac{C(1;t)}{U(1;t)} \quad ; \quad C(1;t) = p(1;t)U(1;t) \tag{5.1}$$

$$C(1;t) = \sum_{i=1}^{N(t)} C(t_i) \ ; \ \ U(1;t) = \sum_{i=1}^{N(t)} U(t_i) \ ; \ C(t_i) = p(t_i)U(t_i) \tag{5.2}$$

$$\Delta(t) = \left[t - \frac{\Delta}{2}, t + \frac{\Delta}{2}\right] \ ; \ t_i \in \Delta(t) \ , \ i = 1, \ldots N(t) \tag{5.3}$$

Price $p(t_i)$ in (5.2) at moment t_i defines price of the single trade with the value $C(t_i)$ and the volume $U(t_i)$. The aggregate value $C(1;t)$ and aggregate volume $U(1;t)$ of trades during interval Δ define the VWAP $p(1;t)$ (5.1). It is obvious that dividing $C(1;t)$ and $U(1;t)$ (5.1) by N one obtains value and mean volume of N trades performed during interval Δ in conventional frequency-based manner. VWAP $p(1;t)$ (5.1) plays the role of coefficient between mean value and mean volume of trades or equally the role of coefficient between aggregated value $C(1;t)$ and volume $U(1;t)$ of trades similar to the price $p(t_i)$ of the single trade (5.2). The difference between VWAP (5.1) and mean price $p(t)$ determined by frequencies $n(p_i)$ of trades with price p_i :

$$\bar{p}(t) = \frac{1}{N} \sum p_i \, n_i \quad ; \quad N = \sum n_i \tag{5.4}$$

outlines important issue. The VWAP $p(1;t)$ (5.1) match the price definition (5.2) as coefficient between the value $C(1;t)$ and the volume $U(1;t)$ aggregated during interval Δ. However frequency-based average price $\bar{p}(t)$ (5.4) corresponds to certain mean value $\bar{C}(t)$ and mean volume $\bar{U}(t)$ that may be too far from economic understanding of the average. As illustration in App.C we consider a simple example that illustrates the differences between VWAP $p(1;t)$ and relations (5.1) on one hand and frequency-based mean price $\bar{p}(t)$ (5.4) and corresponding mean value $\bar{C}(t)$ and mean volume $\bar{U}(t)$ of the market trades that match relations similar to (5.1). Certainly, investors expectations and beliefs may accept and approve any mean price definition. But we repeat – the VWAP $p(1;t)$ (5.1) has economic meaning of price as coefficient between aggregated or mean value and volume of trades. The similar relations (C.3; C.8) for usual frequency-based mean price $\bar{p}(t)$ (5.4) seems to be too exotic to have economic sense. In App.C we consider simple case of two trades with total volume 200. As we show (see App.C) for such a case usual frequency-based mean price $\bar{p}(t)$ (5.4) may match relations like (5.1) as coefficient between mean value $\bar{C}(t)$ and mean volume $\bar{U}(t)$ =1,99. It is rather difficult to find economic sense to the "mean" trade volume $\bar{U} = 1{,}99$ of two trades with total volume $U=200$. Thus in some cases conventional frequency-based mean price $\bar{p}(t)$ (5.4) may have poor economic meaning. The same troubles concern economic sense of frequency-based mean squares of price and any mean *n-th* power of the price for $n=1,2,\ldots$. To overcome such disadvantage we propose price probability measure that match price relations (5.2) and reflects the random properties of the market trades. With this in mind we introduce the VWAP-based price probability measure.

We underline that above considerations don't determine or chose correct or incorrect price probability measure. Trade decisions are mostly based on investors expectations, beliefs and social "myths & legends" and investors may adopt any definitions and measurement of average price and price probability. However the VWAP *p(1;t)* (5.1) generates the price measure that is determined by the market trade evolution. It seems reasonable that any pricing method and CAPM in particular should follow laws of market trade distributions. That can make asset pricing more justified and more market related. We consider the definition and forecasting of the price probability measure based on VWAP (5.1) in Olkhov (2020a-2021). Below we briefly introduce main notions and discuss the pricing problems.

6. The VWAP-based price probability measure

The main idea of our approach: any mean *n-th* power of the price should be coefficient between aggregated (or mean) *n-th* power of the value and the volume. It is obvious that for any *n* for any particular single trade at moment t_i valid:

$$C^n(t_i) = p^n(t_i)U^n(t_i) \ ; \quad n = 1,2,\ldots \tag{6.1}$$

In (6.1) $C^n(t_i)$, $U^n(t_i)$ and $p^n(t_i)$ denote the *n-th* power of the value, the volume and the price of a single trade at moment t_i. We state that similar relations should match the mean *n-th* power of the price. Let's aggregate *n-th* power of value $C^n(t_i)$ and volume $U^n(t_i)$ of $N(t)$ trades performed during interval Δ:

$$C(n;t) = \sum_{i=1}^{N(t)} C^n(t_i) = \sum_{i=1}^{N(t)} p^n(t_i)U^n(t_i) \ ; \ U(n;t) = \sum_{i=1}^{N(t)} U^n(t_i) \tag{6.2}$$

It is obvious that $C(n;t)$ and $U(n;t)$ have meaning of mean *n-th* power of the value and the volume accurate to the factor $1/N$ in conventional frequency-based manner. Let's take the coefficient $p(n;t)$ between them as mean *n-th* power of the price (6.3):

$$C(n;t) = p(n;t)U(n;t) \tag{6.3}$$

Mean *n-th* power of the price $p(n;t)$ can be presented in a form alike to VWAP – as *n-th* power of the volume weighted average *n-th* power of the price (6.2-6.4):

$$p(n;t) = \frac{1}{U(n;t)} \sum_{i=1}^{N(t)} p^n(t_i)U^n(t_i) = \frac{C(n;t)}{U(n;t)} \tag{6.4}$$

Relations (6.2-6.4) for any $n=1,2\ldots$ define mean *n-th* power of the price $p(n;t)$ averaged during interval Δ in a form similar to price definition (6.1) as coefficient between mean *n-th* power of the value $C(n;t)$ and the volume $U(n;t)$ defined by (6.2) with accuracy to factor $1/N$.

If one assumes that price p is a random value then one can define the price probability measure $\eta(p;t)$ that delivers the mean n-th power price $p(n;t)$ (6.2-6.4) as:

$$p(n;t) = \int dp\, \eta(p;t) p^n \tag{6.5}$$

Using (6.2-6.4) one can determine price probability measure $\eta(p;t)$ through it characteristic function $F_p(x;t)$ (we omit (2π) factors for simplicity):

$$F_p(x;t) = \int dp\, \eta(p;t) \exp(ixp) \quad ; \quad \eta(p;t) = \int dx\, F_p(x;t) \exp(-ixp) \tag{6.6}$$

$$F_p(x;t) = \sum_{i=0}^{\infty} \frac{i^n}{n!} p(n;t)\, x^n \tag{6.7}$$

$$\frac{d^n}{(i)^n dx^n} F_p(x;t)|_{x=0} = \int dp\, \eta(p;t) p^n = p(n;t) \tag{6.8}$$

Characteristic function $F_p(x;t)$ (6.3;6.6-6.8) completely describes the price probability measure $\eta(p;t)$ and all mean n-th power of the price $p(n;t)$.

If one considers the price p as a random process then one should describe the random price process by characteristic functional similar to (6.7; 6.8). For brevity we omit here description of the random price process and refer (Klyatskin, 2005; 2015) for clear description of methods and usage of the characteristic functions and characteristic functionals and functional derivatives as a helpful tool for modeling stochastic systems.

Description and forecasting of the price probability measure $\eta(p;t)$ (6.5-6.8) is a really tough problem. Indeed, (6.5-6.8) shows that prediction of the price probability measure $\eta(p;t)$ requires forecasting the mean n-th power of the price $p(n;t)$ (6.3; 6.4) for all $n=1,2..$. This requires forecasting of all aggregated n-th power of the market trades value $C(n;t)$ and volume $U(n;t)$ (6.2) and that is definitely not a simple problem.

Let's mention here only one issue. As we show in Sec. 2-4 the CAPM equations (2.6) or (3.5) describe math expectations at moment t and moment $t+1$. This requires take into account same averaging time scales Δ at moment t and at moment $t+1$. Taylor series expansion of utility function (4.4) by price fluctuations uncovers impact of price volatility $\sigma^2(p)$ and payoff volatility $\sigma^2(x)$ (A.7; A.8). Relations (6.3; 6.4) define the mean price $p(1;t)$ and the mean square price $p(2;t)$ and hence define price volatility $\sigma^2(p)$:

$$\sigma^2(p) = E_t\left[\delta^2 p\right] = p(2;t) - p^2(1;t) \tag{6.9}$$

Relations (6.9) indicates that price volatility forecast requires prediction of the mean price $p(1;t)$ and mean square price $p(2;t)$ and that implies description of $C(1;t)$, $C(2;t)$, $U(1;t)$ $U(2;t)$ (6.2-6.4). Forecasting of the mean price $p(1;t)$ averaged during interval Δ requires prediction of evolution of the aggregated market trade values $C(1;t)$ and volumes $U(1;t)$. This is the problem of the so-called first-order economic theory that describes dynamics and mutual relations between aggregated macroeconomic variables and trade values and volumes of the first order. Basically to some extend this problem is roughly described by the current macroeconomic theory. However, price volatility is determined by evolution of the squares of trade value *C(2;t)* and volume *U(2;t)* aggregated during interval *Δ* and that can't be described by the first-order trades. To describe aggregated squares of market trades value *C(2;t)* and volume *U(2;t)* one should develop independent theory and we note it as the second-order economic theory. Moreover, brief consideration of the idiosyncratic risk case in Sec.4.1 and App.A shows that CAMP equations depend on price *Sk(p)* and payoff *Sk(x)* skewness (A.32; A.39-42). Skewness forecasting requires prediction of the mean *3-d* power of price *p(3;t)* at moment *t* determined by relations:

$$C(3;t) = p(3;t)U(3;t)$$

Forecasting of price *p(3;t)* and price skewness *Sk(p)* and payoff skewness *Sk(x)* from moment *t* to moment *t+1* requires development of the third-order economic theory that predicts evolution of aggregated market trade values *C(3;t)* and volumes *U(3;t)*. We mentioned in Sec.4.1 that approximation of utilities up to *3-d* order of Taylor series introduces price and payoff kurtosis determined by the *4-th* statistical moments. Forecasts of kurtosis require the forth-order economic theory and so on. For brevity we don't consider here the problems of description of the second-order economic theory and refer for details to (Olkhov, 2020b).

7. Conclusion

Our consideration of the main CAPM equation (2.1) (Cochrane, 2001) outlines three critical remarks that may be taken into account by investors and researchers.

1. Any consideration of the market time-series should be performed for certain time interval *Δ* responsible for averaging of time-series. Relations between averaging interval *Δ* and time horizon *T* of the problem define smoothness or irregular properties of the averaged variables. If *Δ<<T* then averaged variables may show irregular dynamics on horizon *T*. Long averaging interval *Δ* may hide market information important for taking investment decisions on scales less then *Δ*. The choice of *Δ* and the change of averaged variables while change of interval *Δ* are critically important for any long term investing.

2. The main CAPM equation (2.6) may have the meaning different from the standard one: "price should be the expected discounted payoff, using the investor's marginal utility to discount the payoff" (Cochrane, 2001). On one hand derivation of equations (2.6) should take into account above remark on averaging time-scales *Δ*. Any averaging procedure implies aggregating variables during certain interval *Δ*. One should make averaging at moment *t* and

at moment $t+1$ using same averaging interval Δ. Moreover, the simple Taylor series expansion of the utility functions (4.4) by price and payoff variations near the mean values of price and payoff uncovers the initial meaning of the main CAPM equation (2.6; 3.6). First-order Taylor series show that relation (2.5) treated as justification of the main CAPM equation (2.6) should be treated as equations (4.5; 4.6) on the asset amount ξ_{max} that delivers the maximum to investor's utility function under condition (2.5). In case of idiosyncratic risk these equations determine relations (4.19) between covariance $cov^2(\beta,x)$ of subjective discount factor β and payoff x on one hand and payoff volatility $\sigma^2(x)$ and risk-free rate R^f on the other hand. Equation (3.6) and utility Taylor series introduce impact of price $Sk(p)$ (A.32) and payoff $Sk(x)$ (A.39-A.42) skewness. Further expansion of utility into Taylor series up to $\delta^3 p$ and $\delta^3 x$ will introduce impact of price and payoff kurtosis.

Further in Sec. 4.2 and App.B we discuss the meaning of the first-order condition of utility max (2.5) and indicate that to obtain the max of utility condition (2.5) should be complemented by the second-order condition (4.21). Both conditions (2.5; 4.21) determine equation and inequality on asset amount ξ_{max} that should deliver max to utility function. We argue that price-volume correlations studied for decades may prevent CAPM desires to get utility max. Mutual impact of the CAPM max conditions (2.5; 4.21) and price-volume correlations should be studied further with more accuracy.

3. All asset pricing models and CAPM in particular forecast price within certain price probability measure and corresponding math expectation $E[..]$. The choice of probability measure and its forecasting are the critical issues for any asset pricing models. We conclude that there are certain doubts in correctness of the conventional frequency-based price probability measure (5.4). Indeed, economic meaning of the price of the single trade and any mean *n-th* power of the price implies that these price factors should match simple relations (5.1; 5.2; 6.1; 6.3) as coefficients between corresponding power of the value and the volume of market trades. If one regards price of the single trade then relations (6.1) must be valid. The math expectation of *n-th* power of the price $E_t[p^n]$ within any price probability measure must follow similar relations (5.1; 6.3) for the *n-th* power of the value and the volume aggregated during certain time interval Δ. Violation of these relations cause the run out of economic sense of price.

We propose the VWAP-based price probability measure $\eta(p;t)$ (6.5-6.8) determined by (6.3; 6.4) for all $n=1,2,\ldots$. The mean price $p(1;t)$ for $n=1$ coincides with VWAP (5.1; 5.2). Price probability measure $\eta(p;t)$ (6.5-6.8) completely depends on the market trade volume and value time-series. For $n=1,2..$ any mean *n-th* power of the price $p(n;t)$ match relations (6.3) and any $p(n;t)$ has meaning of mean *n-th* power of the price as coefficient between *n-th* power of the value and the volume of trades aggregated during interval Δ.

It could be said that replacement of usual and common frequency-based price probability (5.6) by the VWAP-based price probability measure $\eta(p;t)$ (6.5-6.8) may deliver economic meaning to mean price $p(1;t)$, mean square price $p(2;t)$ (6.3; 6.4) for $n=1,2$ and may help forecast the price volatility $\sigma^2(p)$ (6.9). But there's no such thing as a free lunch. Introduction of the new VWAP-based price probability measure $\eta(p;t)$ (6.5-6.8) undoubtedly delivers economic meaning to any mean *n-th* power of the price $p(n;t)$ but the same time uncovers the hidden complexity of price forecasting. Due to (6.2-6.4) each mean *n-th* power of the price $p(n;t)$ is determined by corresponding *n-th* power of the value and the volume of market trades aggregated during interval Δ. Thus forecast of $p(n;t)$ requires forecast of $C(n;t)$ and $U(n;t)$ (6.2). Model that forecast the mean price $p(1;t)$ and the value $C(1;t)$ and the volume $U(1;t)$ doesn't allow forecast $C(2;t)$, $U(2;t)$ and hence $p(2;t)$ and $\sigma^2(p)$. Description of $C(n;t)$ and $U(n;t)$ for each $n=1,2,..$ requires development of separate additional economic theory of *n-th* order. Thus forecast of price volatility $\sigma^2(p)$ (6.9) with respect to probability measure $\eta(p;t)$ (6.5-6.8) need prediction of the squares of the value $C(2;t)$ and the volume $U(2;t)$ of trades aggregated during interval Δ. This makes forecasting of the price probability measure $\eta(p;t)$ (6.5-6.8) a really tough puzzle.

Illusion of simplicity of the main CAPM equation (2.6) is balanced by hidden complexity of the price probability forecasts.

However investors may choose any definition of price probability they prefer. Investors may chose conventional well-known frequency-based price measure (5.4) as ground for their investment decisions and adopt any available price forecast that has no relations with complex description of markets via $C(n;t), U(n;t)$. That may be very beneficial for investors and may be not. There's no such thing as a free lunch.

Asset pricing problem remains attractive and complex subject for researchers, unsearchable for investors and will remain so for many years or forever.

Appendix A

Taylor Series of Utility

Let's take linear Taylor series expansion (A.3) of utilities (3.4) and (4.1-4.4) near (A.2):

$$c_t = c_{t;0} - \xi\delta p \ ; \ c_{t+1} = c_{t+1;0} + \xi\delta x \qquad \text{(A.1)}$$

$$c_{t;0} = e_t - p_0\xi \ ; \quad c_{t+1;0} = e_{t+1} + x_0\xi \qquad \text{(A.2)}$$

$$p = p_0 + \delta p \ ; \ E_t[p] = p_0 \ ; \ x = x_0 + \delta x \ ; \ E[x] = x_0 \ ; \ \beta = \beta_0 + \delta\beta \ ; \ E[\beta] = \beta_0$$

$$u'(c_t) = u'(c_{t;0}) - \xi u''(c_{t;0})\delta \mathrm{p} \ ; u'(c_{t+1}) = u'(c_{t+1;0}) + \xi u''(c_{t+1;0})\delta \mathrm{x} \qquad \text{(A.3)}$$

Utility functions (A.3) at points (A.2) are regular and random properties are represented by variations $\delta p, \delta x, \delta\beta$. Due to linear Taylor series (A.3) equation (3.4) takes form:

$$[u'(c_{t;0}) - \xi u''(c_{t;0})\delta \mathrm{p}][p_0 + \delta p] = (\beta_0 + \delta\beta)[u'(c_{t+1;0}) + \xi u''(c_{t+1;0})\delta \mathrm{x}][x_0 + \delta x] \qquad \text{(A.4)}$$

Taking math expectation for both sides of (A.4) obtain

$$u'(c_{t;0})p_0 - u'(c_{t+1;0})E[\beta x] = \xi\left[u''(c_{t;0})\sigma^2(p) + u''(c_{t+1;0})E[\beta x\delta \mathrm{x}]\right] \qquad \text{(A.5)}$$

$$E[\beta x] = \beta_0 x_0 + cov(\beta, x) \quad ; \quad cov(\beta, x) = E[\delta\beta\delta x] \qquad \text{(A.6)}$$

$$E[\beta x\delta x] = \beta_0\sigma^2(x) + x_0 cov(\beta, x) + cov(\beta, x^2)$$

$$\sigma^2(p) = E_t\left[\delta^2 p\right] \ ; \ \sigma^2(x) = E\left[\delta^2 x\right] \ ; \ cov(\beta, x^2) = E\left[\delta\beta\delta^2 x\right] \qquad \text{(A.7)}$$

Equation (A.5) by linear Taylor series (A.3) defines the amount of asset ξ_{max} that delivers the maximum of utility function $U(c_t;c_{t+1})$ (3.5):

$$\xi_{max} = \frac{u'(c_{t;0})p_0 - u'(c_{t+1;0})E[\beta x]}{u''(c_{t;0})\sigma^2(p) + u''(c_{t+1;0})E[\beta x\delta x]} \qquad \text{(A.8)}$$

It is easy to show that if one starts with basic equation (2.6) and don't take into account math expectation in the left side of (2.6) then equation (A.5) takes form

$$p = \frac{u'(c_{t+1;0})}{u'(c_t)}E[\beta x] + \xi_{max}\frac{u''(c_{t+1;0})}{u'(c_t)}E[\beta x\delta \mathrm{x}] \qquad \text{(A.9)}$$

Equation (A.9) describes same relations as equation (2.6) but in the linear approximation of utility function $u'(c_{t+1})$ by Taylor series by δx and subjective discount factor β as (4.1). Equation (A.9) describes relations between p – asset price at moment t in the left side and mean payoff x_0, mean subjective discount factor β_0, payoff volatility $\sigma^2(x)$ and cov(βx) and cov(βx^2) (A.7). Equation (A.9) has sense for amount of asset ξ_{max} that delivers the max to utility $U(c_t;c_{t+1})$ and is the root of equation (2.5). One may use (A.9) similar to (2.6) and treat it as conditions between asset price p at moment t and mean discount payoff at moment $t+1$ but should take into account the impact of payoff volatility $\sigma^2(x)$ and cov(βx) and cov(βx^2) (A.7) for asset amount ξ_{max}. The asset amount ξ_{max} should take value that match first-order maximum conditions (2.5) and hence relations on ξ_{max} (A.9) should take form

$$\xi_{max} = \frac{u'(c_t)p - u'(c_{t+1;0})E[\beta x]}{u''(c_{t+1;0})E[\beta x \delta x]} \quad \text{(A.10)}$$

One should take into account properties of utility functions:

$$u'(c) > 0 \; ; \quad u''(c) < 0 \quad \text{(A11)}$$

Hence to get positive amount of asset $\xi_{max} > 0$ for (A.10; A.11):

$$u'(c_{t;0})p_0 < u'(c_{t+1;0})E[\beta x]$$

We remind that in this paper we consider price p at moment t and payoff x at moment $t+1$.

Now let's consider the case of idiosyncratic risk (Cochrane, 2001). To derive the expression for $E[m]$ (4.17) let's take Taylor series of utility $u'(c_{t+1})$ by δx at moment $t+1$ and $\delta\beta$ of m (2.6) taking into account (4.13):

$$m_0 = \frac{u'(c_{t+1;0})}{u'(c_{t;0})}\beta_0 > 0 \; ; \; m_1 = \frac{u''(c_{t+1;0})}{u'(c_{t;0})}\beta_0 < 0 \; ; \; m_2 = \frac{u''(c_{t;0})}{u'(c_{t;0})} < 0$$

$$m = \beta\frac{u'(c_{t+1})}{u'(c_t)} = E[m] + \delta m$$

$$E[m] = [m_0 + \xi_{max}\, m_1 \frac{cov(\beta,x)}{\beta_0}] \quad \text{(A.12)}$$

$$\delta m = \left[m_0 \frac{\delta\beta}{\beta_0} + \xi_{max}\, m_1 \delta x \right] \quad \text{(A.13)}$$

Now let's use condition of no-correlations (4.16):

$$cov(m,x) = E[\delta m \delta x] = 0$$

From (A.6; A.7; A.13) obtain

$$cov(m,x) = \left[m_0 \frac{cov(\beta,x)}{\beta_0} + \xi_{max} m_1 \sigma^2(x) \right] = 0 \quad \text{(A.14)}$$

Hence (4.9) ξ_{max} takes form:

$$\xi_{max} = -\frac{u'(c_{t+1;0})cov(\beta,x)}{\beta_0 u''(c_{t+1;0})\sigma^2(x)} = -\frac{m_0\, cov(\beta,x)}{m_1 \beta_0 \sigma^2(x)} \quad ; \quad cov(\beta,x) > 0 \quad \text{(A.15)}$$

In order to be positive ξ_{max} and due to (A.5) $cov(\beta,x)$ should be positive too. Now substitute (A.15) into (A.12) and obtain:

$$E[m] = m_0\left[1 - \frac{cov^2(\beta,x)}{\beta_0^2 \sigma^2(x)}\right] = 1/R^f \quad \text{(A.16)}$$

From (A.15; A.16) obtain:

$$\xi^2{}_{max}\sigma^2(x) = \left(1 - \frac{1}{m_0 R^f}\right)\frac{m_0^2}{m_1^2} \quad ; \quad m_0 R^f > 1 \quad \text{(A.17)}$$

As we mentioned above the risk-free rate R^f is known ahead hence (A.17) can be treated as condition on m_0: $m_0 > 1/R^f$. Equation (A.17) establishes relations between ξ_{max}, $\sigma^2(x)$, discount factors m_0, m_1 (4.9) and risk-free rate R^f. Underline that (A.17) for idiosyncratic risk and main

CAPM equation (2.6) describe clear market rule: the growth of payoff volatility $\sigma^2(x)$ leads to decline of ξ_{max}.

Equations (A.12-A.17) describe the idiosyncratic risk for the approximation (4.17):

$$p = E[mx] = E[m]E[x] + cov(m, x) = E[m]x_0$$

Now let's consider the idiosyncratic risk for the case (3.5) and (4.22-4.24):

$$E_t[p\, u'(c_t)] = E\, [\beta x\, u'(c_{t+1})] \tag{A.18}$$

$$E_t[u'(c_t)p] = E_t[u'(c_t)]\, p_0 \;\; ; \;\; cov(u'(c_t), p) = E_t[\delta u'(c_t)\delta p] = 0 \tag{A.19}$$

$$E[\beta u'(c_{t+1})x] = E[\beta u'(c_{t+1})]\, x_0 \;\; ; \;\; cov(\beta u'(c_{t+1}), x) = E\big[\delta\big(\beta u'(c_t)\big)\delta p\big] = 0 \tag{A.20}$$

$$p_0 = \frac{x_0}{Rf} = \frac{E[\beta u'(c_{t+1})]}{E_t[u'(c_t)]}\, x_0 \;\; ; \quad \frac{1}{Rf} = \frac{E[\beta u'(c_{t+1})]}{E_t[u'(c_t)]} \tag{A.21}$$

It is easy to show that linear expansion of $u'(c_t)$ into Taylor series by δp

$$u'(c_t) = u'\big(c_{t;0}\big) - \xi_{max} u''\big(c_{t;0}\big)\delta p \tag{A.22}$$

$$E_t[u'(c_t)] = u'\big(c_{t;0}\big) \quad ; \quad \delta u'(c_t) = -\xi_{max} u''\big(c_{t;0}\big)\delta p \tag{A.23}$$

if substituted in (A.19) gives:

$$E_t[\delta u'(c_t)\delta p] = -\xi_{max} u''\big(c_{t;0}\big)\sigma^2(p) = 0 \tag{A.24}$$

Hence

$$\sigma^2(p) = 0 \tag{A.25}$$

Zero market price volatility (A.25) is result of linear approximation of Taylor series (A.22). To overcome this zero-volatility problem let's take the Taylor series with accuracy to $\delta^2 p$ at moment t and to $\delta^2 x$ at moment $t+1$:

$$u'(c_t) = u'\big(c_{t;0}\big) - \xi_{max} u''\big(c_{t;0}\big)\delta p + \frac{1}{2}\, \xi^2{}_{max} u'''\big(c_{t;0}\big)\delta^2 p \tag{A.26}$$

$$u'(c_{t+1}) = u'\big(c_{t+1;0}\big) + \xi_{max} u''\big(c_{t+1;0}\big)\delta x + \frac{1}{2}\, \xi^2{}_{max} u'''\big(c_{t+1;0}\big)\delta^2 x \tag{A.27}$$

In the approximation (4.1; A.26; A.27) math expectations and variations of utilities at moment t take form:

$$E_t[u'(c_t)] = u'\big(c_{t;0}\big) + \frac{1}{2}\, \xi^2{}_{max} u'''\big(c_{t;0}\big)\sigma^2(p) \tag{A.28}$$

$$\delta u'(c_t) = u'(c_t) - E_t[u'(c_t)] = -\xi_{max} u''\big(c_{t;0}\big)\delta p + \frac{1}{2}\, \xi^2{}_{max} u'''\big(c_{t;0}\big)\left[\delta^2 p - \sigma^2(p)\right] \tag{A.29}$$

$$E_t[\delta u'(c_t)] = 0$$

For (A.29) no-correlation condition (A.19) at moment t takes form:

$$cov(u'(c_t), p) = -u''\big(c_{t;0}\big)\sigma^2(p) + \frac{1}{2}\, \xi_{max} u'''\big(c_{t;0}\big)\gamma^3(p) = 0 \tag{A.30}$$

$$u''\big(c_{t;0}\big)\sigma^2(p) = \frac{1}{2}\, \xi_{max} u'''\big(c_{t;0}\big)\gamma^3(p) \tag{A.31}$$

$$Sk(p) = \frac{\gamma^3(p)}{\sigma^3(p)} \quad ; \quad \gamma^3(p) = E_t[\delta^3 p] \tag{A.32}$$

$\gamma^3(p)$ in (A.30-A.32) describes price skewness that is usually normalized by price volatility $\sigma^3(p)$ (Xu, 2007, p.2540) and we denote normalized price skewness as $Sk(p)$ (A.32). Thus non-zero price skewness (A.32) is the condition for non-trivial approximation of utility (A.26) and (A.31) takes form (A.33):

$$\xi_{max}\, Sk(p)\, \sigma(p) = 2\frac{m_2}{m_4} \quad ; \quad m_2 = \frac{u''(c_{t;0})}{u'(c_{t;0})} \;\; ; \;\; m_4 = \frac{u'''(c_{t;0})}{u'(c_{t;0})} \qquad \text{(A.33)}$$

Relations (A.33) and can be treated as equation on ξ_{max}. Taking into account relations (4.13):

$$m_0 = \frac{u'(c_{t+1;0})}{u'(c_{t;0})}\beta_0 > 0 \; ; \;\; m_1 = \frac{u''(c_{t+1;0})}{u'(c_{t;0})}\beta_0 < 0 \; ; \;\; m_2 = \frac{u''(c_{t;0})}{u'(c_{t;0})} < 0 \qquad \text{(A.34)}$$

relations for (A.20) take form:

$$E[\beta u'(c_{t+1})] = u'(c_{t;0})\left\{m_0 + \xi_{max} m_1 \frac{cov(\beta,x)}{\beta_0} + \frac{1}{2}\,\xi^2{}_{max} m_5 \left[\sigma^2(x) + \frac{cov(\beta,x^2)}{\beta_0}\right]\right\} \qquad \text{(A.35)}$$

$$m_5 = \frac{u'''(c_{t+1;0})}{u'(c_{t;0})}\beta_0 \qquad \text{(A.36)}$$

From (A.28; A.35) obtain relation (A.21) for risk-free rate R^f:

$$\frac{1+\frac{1}{2}\xi^2{}_{max} m_4 \sigma^2(p)}{Rf} = m_0 + \xi_{max} m_1 \frac{cov(\beta,x)}{\beta_0} + \frac{1}{2}\,\xi^2{}_{max} m_5 \left[\sigma^2(x) + \frac{cov(\beta,x^2)}{\beta_0}\right] \qquad \text{(A.37)}$$

No-correlation condition (A.20) at moment $t+1$ takes form:

$$cov([\beta u'(c_{t+1})], x) = u'(c_{t+1;0}) cov(\beta, x) + \xi_{max} u''(c_{t+1;0})[\beta_0 \sigma^2(x) + cov(\beta x^2)] +$$
$$+\frac{1}{2}\,\xi^2{}_{max} u'''(c_{t+1;0})[\,\beta_0 \gamma^3(x) + cov(\beta x^3)] = 0 \qquad \text{(A.37)}$$

Thus no-correlations equation (A.37) can be presented as:

$$m_0 \frac{cov(\beta,x)}{\beta_0 \sigma^2(x)} + \xi_{max} m_1 \left[1 + \frac{cov(\beta x^2)}{\beta_0 \sigma^2(x)}\right] + \frac{1}{2}\,\xi^2{}_{max} m_5 \left[\, Sk(x)\sigma(x) + \frac{cov(\beta x^3)}{\beta_0 \sigma^2(x)}\right] = 0 \quad \text{(A.38)}$$

We denote normalized skewness of payoff $Sk(x)$ (A.39) at moment $t+1$ similar to (A.32):

$$Sk(x) = \frac{\gamma^3(x)}{\sigma^3(x)} \quad ; \quad \gamma^3(x) = E[\delta^3 x] \qquad \text{(A.39)}$$

If one neglects the forth-order factor $cov(\beta x^3)$ in (A.38):

$$cov(\beta x^3) = E[\delta\beta\delta^3 x] \qquad \text{(A.40)}$$

then no-correlations equation (A.38) takes form:

$$\frac{1}{2}\,\xi^2{}_{max} m_5\, \sigma(x)\, Sk(x) = -m_0 \frac{cov(\beta,x)}{\beta_0 \sigma^2(x)} - \xi_{max} m_1 [1 + \frac{cov(\beta x^2)}{\beta_0 \sigma^2(x)}] \qquad \text{(A.41)}$$

Taking into account no-correlation condition at moment t (A.33) as equation on ξ_{max} obtain:

$$\frac{\sigma(x)\, Sk(x)}{\sigma(p)\, Sk(p)} = -\frac{m_0 m_4^2}{2 m_2^2 m_5} \frac{cov(\beta,x)}{\beta_0 \sigma^2(x)} Sk(p)\sigma(p) \; - \frac{m_1 m_2 m_4}{m_2^2 m_5}[1 + \frac{cov(\beta x^2)}{\beta_0 \sigma^2(x)}] \qquad \text{(A.42)}$$

Appendix B

The Utility Max Second Order Condition

To determine the point ξ_{max} that delivers maximum to investor's utility function $U(c_t;c_{t+1})$ (2.2; 3.6) one should check two simple conditions. The first max condition (2.5) and the second condition that requires negative second derivative of utility $U''(c_t; c_{t+1})<0$ at ξ_{max}

$$\frac{\partial^2}{\partial\xi^2}U(c_t;\ c_{t+1})|_{\xi_{max}} < 0 \tag{B.1}$$

Taking derivatives of utility function (2.2) obtain

$$p^2 > -E\left[\beta x^2\ \frac{u''(c_{t+1})}{u''(c_t)}\right] \tag{B.2}$$

Taking second derivative of utility function (3.6) obtain

$$E_t[p^2 u''(c_t)] < -E[\beta x^2\ u''(c_{t+1})\ ] \tag{B.3}$$

Taylor series expansion of utilities functions $u''(c_t)$ and $u''(c_{t+1})$ by price disturbances δp at moment t and payoff disturbances δx (4.4) at moment $t+1$ and (B.3) give:

$$u''(c_t) = u''\left(c_{t;0}\right) - \xi u'''\left(c_{t;0}\right)\delta p\ \ ;\ u''(c_{t+1}) = u''\left(c_{t+1;0}\right) - \xi u'''\left(c_{t+1;0}\right)\delta x \tag{B.4}$$

$$E_t[p^2]u''\left(c_{t;0}\right) + E[\beta x^2]u''\left(c_{t+1;0}\right) < \xi\left\{E_t[p^2\delta p]u'''\left(c_{t;0}\right) - E[\beta x^2\delta x]u'''\left(c_{t+1;0}\right)\right\} \tag{B.5}$$

Due to (A.32; A.39)

$$E_t[p^2] = p_0^2 + \sigma^2(p)\ ;\quad E_t[p^2\delta p] = [2p_0 + Sk(p)\sigma(p)]\ \sigma^2(p)$$

$$E[\beta x^2\delta x] = \beta_0[2x_0 + Sk(x)\sigma(x)]\ \sigma^2(x) + x_0^2 cov(\beta, x) + 2x_0 covcov(\beta, x^2) + E[\delta\beta\delta^3 x]$$

If

$$E_t[p^2\delta p]u'''\left(c_{t;0}\right) > E[\beta x^2\delta x]u'''\left(c_{t+1;0}\right) \tag{B.6}$$

Then

$$\xi_{max}\ > \frac{E_t[p^2]u''(c_{t;0})+E[\beta x^2]u''(c_{t+1;0})}{E_t[p^2\delta p]u'''(c_{t;0})-E[\beta x^2\delta x]u'''(c_{t+1;0})} \tag{B.7}$$

If

$$E_t[p^2\delta p]u'''\left(c_{t;0}\right) < E[\beta x^2\delta x]u'''\left(c_{t+1;0}\right) \tag{B.8}$$

$$\xi_{max} < \frac{E_t[p^2]u''(c_{t;0})+E[\beta x^2]u''(c_{t+1;0})}{E_t[p^2\delta p]u'''(c_{t;0})-E[\beta x^2\delta x]u'''(c_{t+1;0})} \tag{B.9}$$

For the case (B.2) similar relations take form:

$$\xi_{max} < \frac{p^2 u''(c_t)+E[\beta x^2]u''(c_{t+1;0})}{E[\beta x^2\delta x]u'''(c_{t+1;0})}\ \ ; if\ 0 > E[\beta x^2\delta x]u'''\left(c_{t+1;0}\right) \tag{B.10}$$

and

$$\xi_{max} > \frac{p^2 u''(c_t)+E[\beta x^2]u''(c_{t+1;0})}{E[\beta x^2\delta x]u'''(c_{t+1;0})}\ \ ; if\ 0 < E[\beta x^2\delta x]u'''\left(c_{t+1;0}\right) \tag{B.11}$$

Relations (4.5) for (B.6-B.9) and (4.6) for (B.10; B.11) determine ξ_{max} that delivers the maximum to utility functions $U(c_t;c_{t+1})$ in the form (3.6) and (2.2) respectively.

Any economic model and CAPM in particular interfere with other economic and financial properties. Conditions (B.1) are almost equal to market trade price-volume relations. Numerous researchers study the dependence between market price and trade volume (Ying, 1966; Karpoff, 1987; Gallant, Rossi and Tauchen, 1992; DeFusco, Nathanson and Zwick, 2017). It is obvious (2.6; 3.5) that the second derivative of utility (4.1) has the same sign as derivative of price p by trade volume ξ. If the second derivative of utility (4.18) equals zero then derivative of price p by trade volume ξ also zero. One may take derivatives of (2.6; 3.5) by trade volume ξ for $n=1,2,..$ and obtain:

$$p^n = (-1)^{n+1} E\left[\beta x^n \frac{u^{(n)}(c_{t+1})}{u^{(n)}(c_t)}\right] \;\; ; \;\; n = 1,2,.. \qquad \text{(B.12)}$$

$$E_t\left[p^n u^{(n)}(c_t)\right] = (-1)^{n+1} E\left[\beta x^n \, u^{(n)}(c_{t+1})\right] \qquad \text{(B.13)}$$

Equations (B.13; B.14) equal the condition:

$$\frac{\partial^n}{\partial \xi^n} U(c_t;\, c_{t+1}) = 0 \;\; ; n = 1,2, \ldots \qquad \text{(B.14)}$$

and hence utility (2.2; 3.5) should be constant without maximum by ξ_{max}.

Appendix C

The frequency-based mean price and mean volume

To illustrate the problems associated with frequency-based mean price definition (5.4) we take trivial example that outlines troubles of conventional definition of mean price (5.4).

At first let's take two market trades with same asset performed during interval *Δ*. Let's take the first trade with volume 100 and price 10$ and the second trade with same volume 100 but with price *20$*. Due to (5.4) the probability *P(10)* of price 10$ equals probability *P(20)* of price *20$* and equals ½. Due to (5.4) the frequency-based mean price $\bar{p}(t)=15\$$. The VWAP for this case gives the same $p(1;t) = 15\$$.

$$\bar{p}(t) = 15\$ \quad ; \quad p(1;t) = 15\$ \tag{C.1}$$

The total value U_T of two trades during interval *Δ* equals U_T =200 and the mean trade value equals 100. The total value C_T equals

$$C_T = 3000\$ \quad ; \quad U_T = 200 \ ; \ C_T = \bar{p}(t)U_T \ ; \quad \bar{p}(t) = 15\$ \tag{C.2}$$

If one take the mean value per trade $\bar{C}$ = 1500$ and mean value per trade $\bar{U}$ = 100 then mean frequency-based price $\bar{p}(t)$ equals 15$

$$\bar{C} = \bar{p}(t)\bar{U} \ ; \quad \bar{p}(t) = 15\$ \tag{C.3}$$

Now let's consider two trades during same interval *Δ* with the same price 10$ and 20$. But now the first trade with price 10$ is performed with the volume equals 1 and the second trade with price 20$ is performed with volume 199. The total volume of two trades remains the same and equals 200. The price frequencies are the same and equal ½. Thus the frequency-based mean price doesn't change and $\bar{p}(t)$=15$. Let's ask a question – what are the mean values and mean volumes of these two trades that match the same frequency-based mean price $\bar{p}(t)$. Let's take the mean volume $\bar{U}$ as

$$\bar{U} = P(1)U(1) + P(2)U(2)$$

Let's take the share the first trade volume *U(1)* to be *P(1)=a*. Then the share of the second trade volume *U(2)* equals *P(2)=1-a*. Then the mean volume $\bar{U}$ equals

$$\bar{U} = P(1)U(1) + P(2)U(2) = 1 * a + 199 * (1 - a) = 199 - 198 * a \tag{C.4}$$

The mean value $\bar{C}$ of two trades in this case equals

$$\bar{C} = 10 * a + 20 * 199 * (1 - a) = 3980 - 3970 * a \tag{C.5}$$

Now (C.3) determines equation on share *a* for $\bar{p}(t)$=15$:

$$\bar{C} = \bar{p}(t)\bar{U} = 3980 - 3970 * a = 15 * [199 - 198 * a] \tag{C.6}$$

From (C.6) obtain

$$P(1) = a = 0{,}995 \quad \text{(C.7)}$$

Hence "mean" volume $\bar{U}$ and value $\bar{C}$ equal

$$\bar{U} = 199 - 198 * a = 1{,}99 \quad ; \quad \bar{C} = 3980 - 3970 * a = 29{,}85\$ \quad \text{(C.8)}$$

For these "mean" volume $\bar{U}$ and value $\bar{C}$ valid:

$$\bar{C} = \bar{p}(t)\bar{U} = 29{,}85\$ = 15\$ * 1{,}99$$

VWAP for this case equals

$$p(1;t) = \frac{1}{200}[10 + 20 * 199] = 19{,}95\$ \quad \text{(C.9)}$$

Mean volume $<U>$ and value $<C>$ of two trades during interval Δ in terms of VWAP remain the same and equal ½ of the total volume and value:

$$< U >= \frac{1}{2}\, U(1;t) = 100 \quad ; \quad < C >= \frac{1}{2}\, C(1;t) = 1995$$

This trivial case shows well know fact that frequency-based mean price $\bar{p}(t)$ doesn't change for different distributions of trade volume. In both cases the frequency-based mean price $\bar{p}(t)$=*15$* and total volume of two trades $U_T = 200$ remain const. To deliver the economic meaning to the frequency-based mean price $\bar{p}(t)$ and match (C.3) the "mean" trade volume $\bar{U}$ of two trades in the second case must be

$$\bar{U} = 1{,}99 \quad \text{(C.10)}$$

Such "mean" volume $\bar{U}$ (C.10) of two trades with total volume U=200 seems strange and amazing. This mean volume anomaly emphasizes the troubles with the frequency-based price probability. Above relations are trivial and for sure well known. Nevertheless it is useful time-by-time remind that for certain cases the standard conventional frequency-based price probability measure (5.4) derives the mean price $\bar{p}(t)$=*15$* that corresponds to the "mean" trade volume $\bar{U} = 1{,}99$ of two trades with total volume *U=200* that match (C.3). It is difficult to find economic sense to such "mean" trade volume $\bar{U} = 1{,}99$ of two trades with total volume *U=200*. That makes us believe that frequency-based mean price $\bar{p}(t)$=*15$* may have poor economic meaning.

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