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Abstract

Central to this study is the concept of disjointly productive players. Two players are disjointly productive if there is no synergy effect if one of these players joins a coalition containing the other. Our first new axiom states that the payoff to a player does not change when another player, disjointly productive with that player, is removed from the game. The second new axiom means that if we merge two disjointly productive players into a new player, the payoff to a third player in a game with the merged player does not change. These two axioms, along with efficiency, characterize the Shapley value and can lead to improved run times for computing the Shapley value in games with some disjointly productive players.

Keywords Cooperative game · Shapley value · Disjointly productive players · Merged (disjointly productive) players game

1 Introduction

An area of study within cooperative game theory is how payoffs of TU-values change when two or more players are merged into a single one (see, e.g., Derks and Tijs (2000)). Ideally, this also leads to axiomatizations as for the Banzhaf value (Banzhaf, 1965) in Lehrer (1988). Lehrer defines an amalgamated game where two players are merged into one who has the same effect in the new game as the two merging players had in the old game. Together with an axiom, known as standardness (Hart and Mas-Colell, 1989), Lehrer axiomatizes the Banzhaf value with a reduction axiom for these games, called 2-efficiency. A TU-value satisfies this axiom if the payoff to the merged player in the new game is equal to the sum of the payoffs to the two merging players in the old game. Since the Shapley value (Shapley 1953b) also satisfies standardness, the Shapley value is bound to fail for this axiom.

2-efficiency is also used in the axiomatizations of the Banzhaf value in Nowak (1997), Casajus(2011), and Casajus(2012). As pointed out in Alonso-Meijide et al. (2012), Casajus (2012) uses a somewhat different definition of 2-efficiency and, therefore, comes up with somewhat contradictory solutions for the Banzhaf value compared to Nowak (1997).

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Haller (1994) investigates collusion properties of TU-values. Instead of merging two players into one, one player becomes a proxy with the power of both players and the other player becomes a null player. For the Shapley value, the results can be transferred to the amalgamation of players because this value satisfies the null player out property (Derks and Haller, 1999), i.e., removing a null player does not change the payoffs to the other players.

The study of Haller (1994) was inspired by the joint-bargaining paradox in Harsanyi (1977), also known as the Harsanyi paradox (see Vidal-Puga (2012)). Harsanyi observes that in simple bargaining processes, when two or more players join together to form an acting bargaining unit, their bargaining position worsens relative to the remaining players. Moreover, Harsanyi notes that this holds for all solution concepts that satisfy efficiency and the symmetry axiom, hence also for the Shapley value.

Chae and Heidhues (2004) explain this paradox by arguing that, by merging, players trade their multiple "rights to talk" for a single one, thereby weakening their power position.

To axiomatize the class of weighted Shapley values (Shapley 1953a), Nowak and Ratzik (1995) presented the concept of mutually dependent players. These are players who are only jointly productive. Any coalition of mutually dependent players forms a partnership, introduced in Kalai and Samet (1987). For so-called weighted games, which consist of a classical TU-game and a weight vector λ, Ratzik (2012) formulates an amalgamating payoffs axiom. Here, not arbitrary players are merged into a new one, as in 2-efficiency, but only those that form a partnership, i.e., only jointly productive players.

Besner (2019) introduces a splitting axiom to axiomatize the proportional Shapley value (Besner, 2016; Béal et al., 2018). This axiom can also be interpreted as a merging axiom for weakly dependent players who are jointly productive and still have a stand-alone worth.

Unlike all of the studies above, Besner (2019) is not concerned with the ratio of the payoff for the merged player versus the sum of the payoffs for the merging players in the original game, but rather that any merger or split has no effect on the players not affected by it.

This view is also central to this study. For this, as a contrast to mutually dependent players, i.e. jointly productive players, we use the concept of disjointly productive players. Two players are disjointly productive if their marginal contribution to any coalition that does not include the other player is the same as if that coalition had previously been joined by the other one. Therefore, this can be considered as the special case of "interaction of cooperation" in Grabisch and Roubens (1999) without any interaction.

Our first new axiom then states that the payoff to a player does not change when a player who is disjointly productive to that player leaves the game. For our second axiom, we introduce a merged (disjointly productive players) game, corresponding to the merged game[1] in Lehrer (1988), but only for disjointly productive players. Our axiom then states that the payoff does not change for players who are not affected by the merger. As the main result, we show that the Shapley value, along with efficiency, is axiomatized by our two axioms. In addition to our merger axiom, if a TU-value also satisfies efficiency, as does the Shapley value, then the payoff to the merged player is equal to the payoff to the merging players from the original game. Thus, the Harsanyi paradox does not apply to disjointly productive players and they do not lose their "right to talk."

The remainder of this paper is organized as follows. Section 2 contains some preliminaries. In Section 3, we introduce the concept of disjointly productive players, related axioms, our theorem and a compact corollary. Section 4 gives a brief conclusion and some hints on how to reduce the complexity of computing the Shapley value in some situations.

[1] Lehrer (1988) introduces another merged players game that uses as a new worth for certain coalitions the maximum worth of certain subcoalitions from the old game..

2 Preliminaries

Let the countably infinite set $\mathfrak{U}$ be the universe of players. We denote by $\mathcal{N}$ the set of all non-empty and finite subsets of $\mathfrak{U}$. A (TU-)game is a pair (N, v) such that $N \in \mathcal{N}$ and v is a **coaltion function**, i.e., $v\colon 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. We call the subsets $S \subseteq N$ **coalitions** and $v(S)$ is the **worth** of the coalition S, Ω^S denotes the set of all nonempty subsets of S, (S, v) is the **restriction** of (N, v) to the player set $S \in \Omega^N$, and the set of all games (N, v) is denoted by $\mathbb{V}(N)$.

Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$. For all $S \subseteq N$, the **dividends** $\Delta_v(S)$ (Harsanyi, 1959) are defined recursively by

$$\Delta_v(S) := \begin{cases} 0, \text{ if } S = \emptyset, \text{ and} \\ v(S) - \sum_{R \subsetneq S} \Delta_v(R), \text{ if } S \in \Omega^N. \end{cases} \tag{1}$$

A TU-game $(N, u_T) \in \mathbb{V}(N)$, $T \in \Omega^N$, is called a **unanimity game** if for all $S \subseteq N$ we have $u_T(S) := 1$ if $T \subseteq S$ and $u_T(S) := 0$ otherwise. By (Shapley, 1953b), any coalition function v on N has a unique representation

$$v = \sum_{T \in \Omega^N} \Delta_v(T) u_T. \tag{2}$$

We call a coalition $S \subseteq N$ **inessential** in (N, v) if $\Delta_v(S) = 0$. The **marginal contribution** $MC_i^v(S)$ of a player $i \in N$ to a coalition $S \subseteq N \backslash \{i\}$ is given by $MC_i^v(S) := v(S \cup \{i\}) - v(S)$. A player $i \in N$ is called a **dummy player** in (N, v) if $v(S \cup \{i\}) = v(S) + v(\{i\})$, $S \subseteq N \backslash \{i\}$; we call two players $i, j \in N$, $i \neq j$, **symmetric** in (N, v) if for all $S \subseteq N \backslash \{i, j\}$, we have $v(S \cup i) = v(S \cup j)$, they are called **mutually dependent** (Nowak and Radzik, 1995) in (N, v) if $v(S \cup \{i\}) = v(S) = v(S \cup \{j\})$.

For all $N \in \mathcal{N}$, a **TU-value** φ is an operator that assigns to any $(N, v) \in \mathbb{V}(N)$ a payoff vector $\varphi(N, v) \in \mathbb{R}^N$. The **Shapley value** Sh (Shapley, 1953b) is given by

$$Sh_i(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{\Delta_v(S)}{|S|} \quad \text{for all } i \in N. \tag{3}$$

We make use of the following two standard axioms for TU-values.

Efficiency, E. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, we have $\sum_{i \in N} \varphi_i(N, v) = v(N)$.

Symmetry, S. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$ and $i, j \in N$ such that i and j are symmetric in (N, v), we have $\varphi_i(N, v) = \varphi_j(N, v)$.

3 Disjointly productive players

Nowak and Radzik (1995) introduced the concept of mutually dependent players. This means that two mutually dependent players are only jointly productive. The contribution of each of these players to any coalition that does not contain the other is zero. The following concept represents the opposite to this. Here, certain players are only productive when a certain other player is not in the group.

Definition 3.1. *For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, two players $i, j \in N$ are called **disjointly productive** in (N, v) if, for all $S \subseteq N \backslash \{i, j\}$, $S \ni j$, we have $MC_i^v(S \cup \{j\}) = MC_i^v(S)$.*

Remark 3.2. *Note that* $MC_i^v(S \cup \{j\}) = MC_i^v(S)$ *in Definition 3.1 is equivalent to* $MC_j^v(S \cup \{i\}) = MC_j^v(S)$*. Grabisch and Roubens (1999) use the quantity* $v(S \cup \{i,j\}) - v(S \cup \{i\}) - v(S \cup \{j\}) + v(S)$*, or, respectively, the average of it over all coalitions, for their study of "interaction indices." Thus, our definition is equivalent to this quantity if it is zero for all coalitions.*

If we consider the dividend as "the pure contribution of cooperation in a TU-game" (Billot and Thisse, 2005), it is consequent that any coalition containing only one of two mutually dependent players has a dividend of zero. In this sense, the contribution of cooperation made by the formation of coalitions with two disjointly productive players is also zero.

Lemma 3.3. *Let* $N \in \mathcal{N}, (N,v) \in \mathbb{V}(N)$*. Two players* $i, j \in N$ *are disjointly productive in* (N,v) *if and only if for all* $S \subseteq N$*, we have*

$$v(S) = \sum_{R \subseteq S,\, \{i,j\} \nsubseteq R} \Delta_v(R) \text{ or, equivalent by (1), } \Delta_v(S) = 0, \text{ if } \{i,j\} \subseteq S. \tag{4}$$

Proof. Let $N \in \mathcal{N}, (N,v) \in \mathbb{V}(N)$, and $i, j \in N$. By induction on the size $t := |T|$, $T \subseteq N \backslash \{i\}$, $T \ni j$, we show, for all $S \subseteq T$, $S \ni j$, that

$$v(S \cup \{i\}) = v\big((S\backslash\{j\}) \cup \{i\}\big) - v(S\backslash\{j\}) + v(S) \Leftrightarrow v(S \cup \{i\}) = \sum_{R \subseteq S,\, \{i,j\} \nsubseteq R} \Delta_v(R). \tag{5}$$

Initialization: Let $t = 1$ and therefore $T = \{j\}$. Then, by (1), obviously (5) is satisfied.
Induction step: Let $t \geq 2$. Assume that (5) is satisfied for all t', $t' < t$, (IH). We have

$$\begin{aligned} v\big((T\backslash\{j\}) \cup \{i\}\big) &- v(T\backslash\{j\}) + v(T) \\ &\underset{(1)}{=} \sum_{S \subseteq ((T\backslash\{j\}) \cup \{i\})} \Delta_v(S) - \sum_{S \subseteq T\backslash\{j\}} \Delta_v(S) + \sum_{S \subseteq T} \Delta_v(S) \\ &\underset{(IH)}{=} \sum_{S \subseteq T,\, \{i,j\} \nsubseteq S} \Delta_v(S), \end{aligned}$$

and (5) and, therefore, Lemma 3.3 is shown. □

Since two disjointly productive players do not mind each other's business, so to speak, they should not mind if the other player leaves the game. This is the statement of our first new axiom.

Disjointly productive players, DP. For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $i, j \in N$ such that i and j are disjointly productive in (N,v) we have

$$\varphi_i(N,v) = \varphi_i(N\backslash\{j\}, v).$$

The following definition considers games that result from the union of disjointly productive players into a single player.

Definition 3.4. *Let* $N \in \mathcal{N}$*,* $(N,v) \in \mathbb{V}(N)$*,* $i, j \in N$ *be two disjointly productive players in* (N,v)*,* $k \in \mathfrak{U}$*,* $k, \notin N$*, and* $N_{ij}^k := (N\backslash\{i,j\}) \cup \{k\}$*. The TU-game* $(N_{ij}^k, v_{ij}^k) \in \mathbb{V}(N_{ij}^k)$ *is called a* ***merged (disjointly productive) players game*** *to* (N,v) *where* v_{ij}^k *is given by*

$$v_{ij}^k(S) := \begin{cases} v(S), & k \notin S, \\ v\big((S\backslash\{k\}) \cup \{i,j\}\big), & k \in S, \end{cases} \qquad \text{for all } S \subseteq N_{ij}^k. \tag{6}$$

That is, any coalition consisting of the same players in the new and old game has the same worth in both games. Coalitions in which the merged player is a member receive the value of the corresponding coalition with all the original merging players from the old game. This definition corresponds to the definition of the merged game in Lehrer (1988) with the difference that only disjointly productive players are merged.

The following lemma states that for a player in any game, we have split games where that player is split into two disjointly productive players and the old game is a merged players game to those games. The dividends from coalitions containing some players and the split player in the original game are equal to the sum of the dividends from coalitions containing the same other players and only one each of the two split disjointly productive players in the split game.

Lemma 3.5. *Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, $k \in N$, $i, j \in \mathfrak{U}$, $i, j \notin N$, and $N_k^{ij} := (N \backslash \{k\}) \cup \{i, j\}$. Then, we have some $(N_k^{ij}, v_k^{ij}) \in \mathbb{V}(N_k^{ij})$ such that i, j are disjointly productive in (N_k^{ij}, v_k^{ij})(which we will call* ***split (in disjointly productive) players games****) and (N, v) is a merged players game to each (N_k^{ij}, v_k^{ij}) where, for all $S \subseteq N \backslash \{k\}$, v_k^{ij} is given by*

$$\Delta_{v_k^{ij}}(S) = \Delta_v(S) \text{ and } \tag{7}$$

$$\Delta_{v_k^{ij}}(S \cup \{i\}) + \Delta_{v_k^{ij}}(S \cup \{j\}) = \Delta_v(S \cup \{k\}). \tag{8}$$

Proof. Let $(N, v) \in \mathbb{V}(N)$, $k \in N$, $i, j \in \mathfrak{U}$, $i, j \notin N$, $N_k^{ij} := (N \backslash \{k\}) \cup \{i, j\}$ and be $(N_k^{ij}, v_k^{ij}) \in \mathbb{V}(N_k^{ij})$ such that (7) and (8) are satisfied. Then, we define $v_k^{ij}(S \cup \{i, j\})$ such that

$$\Delta_{v_k^{ij}}(S \cup \{i, j\}) := 0 \text{ for all } S \subseteq N_k^{ij} \backslash \{i, j\}. \tag{9}$$

This is always possible and, by Lemma 3.3, i, j are disjointly productive in (N, v). Therefore, by (6), we have the merged players game $(N, \tilde{v})$ to (N_k^{ij}, v_k^{ij}), given, for all $S \subseteq N \backslash \{k\}$, by

$$\tilde{v}(S) := v_k^{ij}(S), \text{ and} \tag{10}$$

$$\tilde{v}(S \cup \{k\}) := v_k^{ij}(S \cup \{i, j\}). \tag{11}$$

By (1), (7), and (10), we have $v(S) = \tilde{v}(S)$ for all $S \subseteq N \backslash \{k\}$. We will show, by induction on the size $s := |S|$,

$$v(S \cup \{k\}) = \tilde{v}(S \cup \{k\}) \text{ for all } S \subseteq N \backslash \{k\}. \tag{12}$$

Initialization: Let $s = 0$ and therefore $S = \emptyset$. Then, by (1), (8), (9), and (11), (12) is satisfied.

Induction step: Let $s \geq 1$. Assume that (12) is satisfied for all $s', s' < s$, (IH). We have

$$\begin{aligned}
&\Delta_{\tilde{v}}(S \cup \{k\}) \\
&\underset{(1)}{=} \tilde{v}(S \cup \{k\}) - \sum_{R \subsetneq (S \cup \{k\})} \Delta_{\tilde{v}}(R) \\
&\underset{(6)}{=} v_k^{ij}(S \cup \{i, j\}) - \sum_{R \subsetneq (S \cup \{k\}), R \ni k} \Delta_{\tilde{v}}(R) - \sum_{R \subsetneq (S \cup \{k\}), k \notin R} \Delta_{\tilde{v}}(R) \\
&\underset{(10)}{=} v_k^{ij}(S \cup \{i, j\}) - \sum_{R \subsetneq S \cup \{k\}, R \ni k} \Delta_{\tilde{v}}(R) - \sum_{R \subseteq S} \Delta_{v_k^{ij}}(R) \\
&\underset{\substack{(IH) \\ (8)}}{=} v_k^{ij}(S \cup \{i, j\}) - \sum_{R \subsetneq S \cup \{i\}, R \ni i} \Delta_{v_k^{ij}}(R) - \sum_{R \subsetneq S \cup \{j\}, R \ni j} \Delta_{v_k^{ij}}(R) - \sum_{R \subseteq S} \Delta_{v_k^{ij}}(R) \\
&\underset{(4)}{=} \sum_{R \subseteq S \cup \{i, j\}, \{i, j\} \not\subseteq R} \Delta_{v_k^{ij}}(R) - \sum_{R \subsetneq S \cup \{i\}, R \ni i} \Delta_{v_k^{ij}}(R) - \sum_{R \subsetneq S \cup \{j\}, R \ni j} \Delta_{v_k^{ij}}(R) - \sum_{R \subseteq S} \Delta_{v_k^{ij}}(R) \\
&= \Delta_{v_k^{ij}}(S \cup \{i\}) + \Delta_{v_k^{ij}}(S \cup \{j\}) \underset{(8)}{=} \Delta_v(S \cup \{k\}).
\end{aligned}$$

□

The following new axiom states that if two disjointly productive players merge into one player who has the same impact on the new game as the two players together had previously, the payoff for the other players should not change.

Merged (disjointly productive) players game property, MP. For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, $i,j \in N$ such that i and j are disjointly productive in (N,v), $k \in \mathfrak{U}$, $k, \notin N$, and a merged players game $(N_{ij}^k, v_{ij}^k) \in \mathbb{V}(N_{ij}^k)$ to (N,v), we have

$$\varphi_\ell(N_{ij}^k, v_{ij}^k) = \varphi_\ell(N,v) \text{ for all } \ell \in N_{ij}^k \backslash \{k\}. \tag{13}$$

Our interest is also in the payoffs for the merging players versus the merged player. If the value is efficient, we get an obvious result.

Remark 3.6. *Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, $i,j \in N$ such that i and j are disjointly productive in (N,v), $k \in \mathfrak{U}$, $k, \notin N$, and $(N_{ij}^k, v_{ij}^k) \in \mathbb{V}(N_{ij}^k)$ a merged players game to (N,v). If φ is a TU-value that satisfies **E** and **MP**, we have, by (6) and (13),*

$$\varphi_k(N_{ij}^k, v_{ij}^k) = \varphi_i(N,v) + \varphi_j(N,v). \tag{14}$$

(14) corresponds to the condition for 2-efficiency in Lehrer (1988), but our merging players must be disjointly productive and the TU-value must be efficient.

Remark 3.7. *Let Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $I \subseteq N$, $|I| \geq 3$, be a coalition of players where each $i \in I$ is mutually disjointly productive to all other players $j \in I \backslash \{i\}$. If we merge two players $i,j \in I$, accordingly to Definition 3.4, into a new player $k \in \mathfrak{U}$, resulting in a merged players game $(N_{ij}^k, v_{ij}^k) \in \mathbb{V}(N_{ij}^k)$, by (4) and (8), we have $\Delta_{v_{ij}^k}(S \cup \{k\}) = 0$ for all $S \cap I \neq \emptyset$. This means, by (4), all $\ell \in (I \backslash \{i,j\}) \cup \{k\}$ are mutually disjointly productive in $\mathbb{V}(N_{ij}^k)$. Therefore, we can apply Definition 3.4 repeatedly to all $i \in I$ and have, finally, for the last merged players game, here denoted by $(\overline{N}, \overline{v})$, a player set $\overline{N} = (N \backslash I) \cup \overline{k}$ and a coalition function $\overline{v}$, given by*

$$\overline{v}(S) := \begin{cases} v(S), & \overline{k} \notin S, \\ v\big((S \backslash \{\overline{k}\}) \cup I\big), & \overline{k} \in S, \end{cases} \qquad \textit{for all } S \subseteq \overline{N}.$$

Accordingly, (8) can be adapted to

$$\Delta_{v_{ij}^{\overline{k}}}(S \cup \{\overline{k}\}) = \sum_{i \in I} \Delta_v(S \cup \{i\}),$$

and (14) can be adapted to

$$\varphi_{\overline{k}}(\overline{N}, \overline{v}) = \sum_{i \in I} \varphi_i(N,v).$$

The following lemma is similar to Lemma 2 in Besner (2019), where it is shown that a TU-value that is efficient and satisfies a player splitting axiom defined there also satisfies symmetry.

Lemma 3.8. *If a TU-value φ satisfies **E** and **MP**, then φ also satisfies **S**.*

Proof. The proof is similar to the proof of Lemma 2 in Besner (2019).

Let $N=\{1,2,...,n\}$, $n\geq 2$, $(N,v)\in\mathbb{V}^N$, φ be a TU-value that satisfies **E** and **MP**, and, w.l.o.g., player 1 and player 2 be symmetric in (N,v). If we split player 1, in accordance to **MP** and Lemma 3.5, into two new disjointly productive players, player $n+1$ and player $n+2$, and define $N_1^{n+1,n+2}:=\{2,3,...,n,n+1,n+2\}$, we have

$$\varphi_2(N_1^{n+1,n+2},v_1^{n+1,n+2})=\varphi_2(N,v). \tag{15}$$

If we split player 2, in accordance to **MP** and Lemma 3.5, into the same players as before, player $n+1$ and player $n+2$, instead, and define $N_2^{n+1,n+2}:=\{1,3,4,...,n,n+1,n+2\}$, we have

$$\varphi_1(N_2^{n+1,n+2},v_2^{n+1,n+2})=\varphi_1(N,v), \tag{16}$$

where we choose, for all $S\subseteq N\backslash\{1,2\}$,

$$\begin{aligned}
v_2^{n+1,n+2}(S\cup\{n+1\}) &:= v_1^{n+1,n+2}(S\cup\{n+1\}),\\
v_2^{n+1,n+2}(S\cup\{n+2\}) &:= v_1^{n+1,n+2}(S\cup\{n+2\}),\\
v_2^{n+1,n+2}(S\cup\{1\}\cup\{n+1\}) &:= v_1^{n+1,n+2}(S\cup\{2\}\cup\{n+1\}),\text{ and}\\
v_2^{n+1,n+2}(S\cup\{1\}\cup\{n+1\}) &:= v_1^{n+1,n+2}(S\cup\{2\}\cup\{n+1\}).
\end{aligned}$$

This is possible because players 1 and 2 are symmetric in (N,v).

In the same way, now in the game $(N_1^{n+1,n+2},v_1^{n+1,n+2})$, we split player 2 into two new disjointly productive players, player $n+3$ and player $n+4$, and, analogously, in the game $(N_2^{n+1,n+2},v_2^{n+1,n+2})$ player 1 into the same players as before, player $n+3$ and player $n+4$. Note that we have $N_{12}^{(n+1,n+2)^{n+3,n+4}}=N_{21}^{(n+1,n+2)^{n+3,n+4}}=\{3,4,...,n,n+1,n+2,n+3,n+4\}$, and, since players 1 and 2 are symmetric in (N,v), we can choose

$$v_{21}^{(n+1,n+2)^{n+3,n+4}}(S)=v_{12}^{(n+1,n+2)^{n+3,n+4}}(S)\text{ for all }S\subsetneq N_{12}^{(n+1,n+2)^{n+3,n+4}}=N_{21}^{(n+1,n+2)^{n+3,n+4}}.$$

By **E**, we obtain

$$\begin{aligned}
&\varphi_{n+3}\Big(N_{12}^{(n+1,n+2)^{n+3,n+4}},v_{12}^{(n+1,n+2)^{n+3,n+4}}\Big)+\varphi_{n+4}\Big(N_{12}^{(n+1,n+2)^{n+3,n+4}},v_{12}^{(n+1,n+2)^{n+3,n+4}}\Big)\\
&\qquad\underset{Rem.\,3.6}{=}\varphi_2(N_1^{n+1,n+2},v_1^{n+1,n+2})\underset{(15)}{=}\varphi_2(N,v),\\
&\varphi_{n+3}\Big(N_{21}^{(n+1,n+2)^{n+3,n+4}},v_{21}^{(n+1,n+2)^{n+3,n+4}}\Big)+\varphi_{n+4}\Big(N_{21}^{(n+1,n+2)^{n+3,n+4}},v_{21}^{(n+1,n+2)^{n+3,n+4}}\Big)\\
&\qquad\underset{Rem.\,3.6}{=}\varphi_1(N_2^{n+1,n+2},v_2^{n+1,n+2})\underset{(16)}{=}\varphi_1(N,v).
\end{aligned}$$

It follows, $\varphi_1(N,v)=\varphi_2(N,v)$, and **S** is shown. □

We come to our main result.

Theorem 3.9. *Sh is the unique TU-value that satisfies **E**, **DP**, and **MP**.*

Proof. Let $N\in\mathcal{N}$, $(N,v)\in\mathbb{V}(N)$.

I. Existence: It is well-known that Sh satisfies **E**. By (3) and Lemma 3.3, it is obvious that Sh satisfies **DP**.

• **MP**: Let $i, j \in N$ be such that i and j are disjointly productive in (N, v) and $(N^k_{ij}, v^k_{ij}) \in \mathbb{V}(N^k_{ij})$ be a merged players game to (N, v). We have

$$\begin{aligned} Sh_\ell(N^k_{ij}, v^k_{ij}) &\underset{(3)}{=} \sum_{\substack{S \subseteq N^k_{ij},\\ S \ni \ell}} \frac{\Delta_{v^k_{ij}}(S)}{|S|} \\ &\underset{(8)}{\underset{(7)}{=}} \sum_{\substack{S \subseteq (N\backslash\{i,j\}),\\ S \ni \ell}} \frac{\Delta_v(S)}{|S|} + \sum_{\substack{S \subseteq (N\backslash\{j\}),\\ \{i,\ell\} \subseteq S}} \frac{\Delta_v(S)}{|S|} + \sum_{\substack{S \subseteq (N\backslash\{i\}),\\ \{j,\ell\} \subseteq S}} \frac{\Delta_v(S)}{|S|} \\ &= \sum_{\substack{S \subseteq N, \{i,j\} \not\subseteq S,\\ S \ni \ell}} \frac{\Delta_v(S)}{|S|} \underset{(4)}{=} Sh_\ell(N, v) \text{ for all } \ell \in N^k_{ij}\backslash\{k\}, \end{aligned}$$

and **MP** is shown.

II. Uniqueness: Let φ be a TU-value that satisfies all axioms of Theorem 3.9 and, therefore, by Lemma 3.8, also **S**. By **MP**, applying Lemma 3.5 and Remark 3.7, we split succesively each player $i \in N$ of the $n = |N|$ players in 2^{n-1} disjointly productive players i_S, $S \subseteq N$, $S \ni i$. In the final split game, we call it $(\overline{N}, \overline{v})$, we have $n \cdot 2^{n-1}$ players. Each of the coalitions containing all players with the same coalition $S \subseteq N$ as a subscript get a worth of the dividend $\Delta_v(S)$ and all other coalitions are defined as inessential in $(\overline{N}, \overline{v})$, i.e., we have, for all $T \in \Omega^{\overline{N}}$,

$$\Delta_{\overline{v}}(T) := \begin{cases} \Delta_v(S), & T = \bigcup_{i \in S}\{i_S\},\ S \in \Omega^N, \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

We illustrate our procedure with a small example: Let $(N', w) \in \mathbb{V}(N')$, $N' = \{1, 2, 3\}$. At first, we split for a new game (N'_1, w_1) player 1, using Lemma 3.5 and Remark 3.7, into four players $1_{\{1\}}, 1_{\{1,2\}}, 1_{\{1,3\}}, 1_{\{1,2,3\}}$ with the player set $N'_1 := (N'\backslash\{1\}) \cup \{1_{\{1\}}, 1_{\{1,2\}}, 1_{\{1,3\}}, 1_{\{1,2,3\}}\}$. By Remark 3.7, we define w_1 by

$$\begin{aligned} \Delta_{w_1}(\{1_{\{1\}}\}) &:= \Delta_w(1), & \Delta_{w_1}(\{1_{\{1,2\}}, 2\}) &:= \Delta_w(\{1, 2\}), \\ \Delta_{w_1}(\{1_{\{1,3\}}, 3\}) &:= \Delta_w(\{1, 3\}), & \Delta_{w_1}(\{1_{\{1,2,3\}}, 2, 3\}) &:= \Delta_w(\{1, 2, 3\}), \end{aligned}$$

$w_1(S) := w(S)$ for all $S \subseteq N'\backslash\{1\}$, and all other coalitions are defined as inessential in (N'_1, w_1).

In the next step, we split for a new game (N'_{12}, w_{12}) player 2, using Lemma 3.5 and Remark 3.7, into four players $2_{\{2\}}, 2_{\{1,2\}}, 2_{\{2,3\}}, 2_{\{1,2,3\}}$ with the player set $N'_{12} := (N'_1\backslash\{2\}) \cup \{2_{\{2\}}, 2_{\{1,2\}}, 2_{\{2,3\}}, 2_{\{1,2,3\}}\}$. By Remark 3.7, we define w_{12} by

$$\begin{aligned} \Delta_{w_{12}}(\{2_{\{2\}}\}) &:= \Delta_w(2), & \Delta_{w_{12}}(\{1_{\{1,2\}}, 2_{\{1,2\}}\}) &:= \Delta_w(\{1, 2\}), \\ \Delta_{w_{12}}(\{2_{\{2,3\}}, 3\}) &:= \Delta_w(\{2, 3\}), & \Delta_{w_{12}}(\{1_{\{1,2,3\}}, 2_{\{1,2,3\}}, 3\}) &:= \Delta_w(\{1, 2, 3\}), \end{aligned}$$

$w_{12}(S) := w_1(S)$ for all $S \subseteq N'_1\backslash\{2\}$, and all other coalitions are defined as inessential in (N'_{12}, w_{12}).

Finally, we split for a new game (N'_{123}, w_{123}) player 3, using Lemma 3.5 and Remark 3.7, into four players $3_{\{3\}}, 3_{\{1,3\}}, 3_{\{2,3\}}, 3_{\{1,2,3\}}$ with the player set $N'_{123} := 1_{\{1\}}, 1_{\{1,2\}}, 1_{\{1,3\}}, 1_{\{1,2,3\}}, 2_{\{2\}}, 2_{\{1,2\}}, 2_{\{2,3\}}, 2_{\{1,2,3\}}, 3_{\{3\}}, 3_{\{1,3\}}, 3_{\{2,3\}}, 3_{\{1,2,3\}}$. By Remark 3.7, we define w_{123} by

$$\begin{aligned} \Delta_{w_{123}}(\{1_{\{1\}}\}) &:= \Delta_w(1), & \Delta_{w_{123}}(\{2_{\{2\}}\}) &:= \Delta_w(\{2\}), \\ \Delta_{w_{123}}(\{3_{\{3\}}\}) &:= \Delta_w(3), & \Delta_{w_{123}}(\{1_{\{1,2\}}, 2_{\{1,2\}}\}) &:= \Delta_w(\{1, 2\}), \\ \Delta_{w_{123}}(\{1_{\{1,3\}}, 3_{\{1,3\}}\}) &:= \Delta_w(\{1, 3\}) & \Delta_{w_{123}}(\{2_{\{2,3\}}, 3_{\{2,3\}}\}) &:= \Delta_w(\{2, 3\}), \end{aligned}$$

$$\Delta_{w_{123}}(\{1_{\{1,2,3\}}, 2_{\{1,2,3\}}, 3_{\{1,2,3\}}\}) := \Delta_w(\{1,2,3\}),$$

and all other coalitions are defined as inessential in (N'_{123}, w_{123}).

Back to our original split game $(\overline{N}, \overline{v})$, by **E**, **MP**, Lemma 3.5, and Remark 3.7, we have that the sum of the payoffs to all players who are split from the same player in the original game equals the payoff to that player in the original game, i.e.,

$$\sum_{i_S,\, S\in\Omega^N,\, S\ni i} \varphi_{i_S}(\overline{N}, \overline{v}) = \varphi_i(N, v) \text{ for all } i \in N. \tag{18}$$

By (17), for each $T \in \Omega^{\overline{N}}, T = \bigcup_{i\in S}\{i_S\}$, $S \in \Omega^N$, all $j \in \overline{N}\backslash T$ are disjointly productive to any $k \in T$. Therefore, repeatedly using **DP**, we have, for each $T \in \Omega^{\overline{N}}, T = \bigcup_{i\in S}\{i_S\}$, $S \in \Omega^N$,

$$\varphi_k(\overline{N}, \overline{v}) = \varphi_k(T, \overline{v}) \text{ for all } k \in T.$$

All $k \in T$ are symmetric in $(T, \overline{v})$. Thus, by **E** and **S**, φ is unique for all $j \in \overline{N}$ in $(\overline{N}, \overline{v})$, and therefore, by (18), for all $i \in N$ in (N, v), and Theorem 3.9 is shown. □

In the context of coalition structures (Aumann and Drèze, 1974; Owen, 1977), Hart and Kurz (1983) presented an axiom, called dummy coalition, which can be seen as a generalization of the dummy player property for games with a coalition structure. We adapt this axiom for TU-games and call a coalition $S \in \Omega^N$ a **dummy coalition** in (N, v) if $v(T \cup R) = v(T) + v(R)$ for all $T \subseteq N\backslash S$ and $R \subseteq S$.

Dummy coalition, DC. For all $(N, v) \in \mathbb{V}(N)$ and $S \in \Omega^N$ such that S is a dummy coalition in (N, v), we have $\sum_{i\in S} \varphi_i(N, v) = v(S)$.

By this axiom, all players of a dummy coalition receive together as a payoff what the coalition alone generates for itself.

Remark 3.10. *Note that the grand coalition is always a dummy coalition. Therefore,* ***DC*** *implies* ***E****.*

Remark 3.11. *It is well-known and easy to show that $i \in N$ is a dummy player in (N, v) if and only if we have $\Delta_v(S) = 0$ for all $S \subseteq N, \{i\} \subsetneq S$.*

A similar result holds for a dummy coalition.

Lemma 3.12. *Let $N \in \mathcal{N}, (N, v) \in \mathbb{V}(N)$. $S \in \Omega^N$ is a* ***dummy coalition*** *in (N, v) if and only if we have $\Delta_v(T) = 0$ for all $T \subseteq N$, $T \nsubseteq S, (T \cap S) \neq \emptyset$.*

Proof. Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and $S \in \Omega^N$. We have to show, for all $R \in \Omega^S$ and all $T \in \Omega^{N\backslash S}$,

$$v(R \cup T) = v(R) + v(T) \Leftrightarrow \Delta_v(R \cup T) = 0; \tag{19}$$

We use a first induction I_1 on the size $s := |S|$.

Initialization I_1: Let $s = 1$. Then, $R = S$ and (19) follows by Remark 3.11.

Induction step I_1: Let $s \geq 2$. Assume that (19) is satisfied for all s', $s' < s$, (IH_1). We use a second induction I_2 on the size $t := |T|$.

Initialization I_2: Let $t = 1$ and, therefore, $T = \{i\}$, $i \in N \backslash S$. We have

$$0 = v(R \cup \{i\}) - v(R) - v(\{i\}) \underset{(1)}{=} \sum_{Q \subseteq (R \cup \{i\})} \Delta_v(Q) - \sum_{Q \subseteq R} \Delta_v(Q) - \Delta_v(\{i\})$$
$$= \sum_{\substack{Q \subseteq (R \cup \{i\}) \\ Q \ni i}} \Delta_v(Q) + \sum_{Q \subseteq R} \Delta_v(Q) - \sum_{Q \subseteq R} \Delta_v(Q) - \Delta_v(\{i\}) \underset{(IH_1)}{=} \Delta_v(R \cup \{i\}),$$

and (19) is shown.

Induction step I_2: Let $t \geq 2$. Assume that (19) is satisfied for all t', $t' < t$, (IH_2). We have

$$0 = v(R \cup T) - v(R) - v(T) \underset{(1)}{=} \sum_{Q \subseteq (R \cup T)} \Delta_v(Q) - \sum_{Q \subseteq R} \Delta_v(Q) - \sum_{Q \subseteq T} \Delta_v(T)$$
$$= \sum_{\substack{Q \subseteq (R \cup T) \\ R \cap Q \neq \emptyset, T \cap Q \neq \emptyset}} \Delta_v(Q) + \sum_{Q \subseteq T} \Delta_v(Q) + \sum_{Q \subseteq R} \Delta_v(Q) - \sum_{Q \subseteq R} \Delta_v(Q) - \sum_{Q \subseteq T} \Delta_v(T)$$
$$\underset{\substack{(IH_1) \\ (IH_2)}}{=} \Delta_v(R \cup T),$$

and (19), and therefore, Lemma 3.12 is shown. □

Remark 3.13. *Let $N \in \mathcal{N}$ and $(N, v) \in \mathbb{V}(N)$. By Lemma 3.12, each coalition $S \in \Omega^N$ in a unanimity game u_S is a dummy coalition. Since Sh satisfies additivity*[2] *and by (2), it is obvious that Sh also satisfies **DC**.*

By Lemma 3.12, all coalitions $T \in \Omega^{\overline{N}}, T = \bigcup_{i \in S}\{i_S\}$, $S \in \Omega^N$ in the proof of Theorem 3.9 are dummy coalitions in $(\overline{N}, \overline{v})$. Therefore, by **DC**, also (18) is satisfied. By Remark 3.10 and the proof of Theorem 3.9, the following compact axiomatization is obvious.

Corollary 3.14. *Sh is the unique TU-value that satisfies **DC** and **MP**.*

We complete this section by showing the logically independence of the axioms in our main result.

Remark 3.15. *The axioms in Theorem 3.9 are logically independent:*

- ***E**: The TU-value $\varphi := 2Sh$ satisfies **DP** and **MP** but not **E**.*
- ***DP**: The TU-value ϕ, defined for all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}^N$, by*

$$\phi_i(N, v) := \begin{cases} \dfrac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N), & \sum_{j \in N} v(\{j\}) \neq 0, \\ Sh_i(N, v), & \text{otherwise}, \end{cases} \quad \text{for all } i \in N.$$

  *satisfies **E** and **MP** but not **DP**.*

- ***MP**: The TU-values ϕ^c, defined in Besner (2020) for all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}^N$, and all $c > 0$, by*

$$\phi_i^c(N, v) := \sum_{S \subseteq N, S \ni i} \frac{|v(\{i\})| + c}{\sum_{j \in S}(|v(\{j\})| + c)} \Delta_v(S) \quad \text{for all } i \in N.$$

  *satisfy **E** and **DP** but not **MP**.*

[2]**Additivity**: For all $N \in \mathcal{N}$, $(N, v), (N, w) \in \mathbb{V}(N)$, we have $\varphi(N, v) + \varphi(N, w) = \varphi(N, v + w)$.

4 Conclusion and a note on the complexity of computing the Shapley value

The merged players game property, when interpreted as a **split into disjointly productive players property**, means that if a player splits into several other disjointly productive players, the payoff to the players non-involved should not change.

In this age of increasing online activity, it is often impossible for participants to know whether different user accounts always trace back to different users. If different people in different groups cooperate to different degrees and the resulting cooperation gains are to be distributed, it should not matter under a fair solution concept whether a participant participates with only one account or under multiple accounts, as long as he or she makes the same total contribution overall.

However, these multiple accounts of a single individual may just be considered disjointly productive, since this participant does not generate coalitional gains only with him/herself. Therefore, satisfying the disjointly productive players property seems desirable for a fair solution concept in this regard.

Due to the steadily increasing use of artificial intelligence and machine learning, cooperative game theory is gaining importance, this is especially true for the Shapley value (see, e.g., Štrumbelj and Kononenko (2014), Takeishi (2019), Rodríguez-Pérez and Bajorath (2020)). Often, different (input) features are used as players and the payoffs are calculated via the different interactions or effects of the features among each other using the Shapley value.

For reasons of complexity, approximation methods for the Shapley value must already be used for relatively small sets of players. Nevertheless, if the players have certain structures or properties among them, it may be possible to use exact algorithms. It is well-known that null players and dummy players can be easily removed from the game.

If we know that certain features or players have no influence on each other, i.e., they are disjointly productive, both of the new introduced axioms come into play. Due to the disjointly productive players property, the payoff to the other disjointly productive players does not change when we remove a disjointly productive player. Especially, dummy coalitions can be removed from the game, reducing the complexity of the calculation for both the remaining players and the players within the dummy coalition. Due to the merged players property, we can simply merge all mutually disjointly productive players into one player and then compute the payoffs to the others with less complexity.

With a simple trick, mutually dependent players can also be valuable for calculating the Shapley value. If we have a group of such only jointly productive players, we combine them into a single player. For the payoff calculation, we then have to apply a weighted Shapley value, where the merged player, e.g., gets as weight the number of merged players and all others keep a weight of one (see Ratzik (2012)). If we reduce the number of players needed for a calculation, the accuracy of the results obtained by approximation methods also improves.

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