Second-order properties of quasi-concave functions

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We present proofs, some of which are very simple, of theorems on the second-order properties of quasi-concave functions.

1 Introduction

Quasi-concavity is an important concept in economics, e.g., we often assume that utility functions and production functions are quasi-concave. Arrow and Enthoven (1961) studied the problem of maximizing a quasi-concave function subject to constraints as well as gave properties of quasi-concavity in terms of bordered determinants. These properties are equivalent to the condition that the Hessian matrix be negative-definite subject to a constraint (Debreu 1952). First-order properties of quasi-concave functions are given in e.g. Mangasarian (1969, chapter 9). In this paper we give theorems on the relation between quasi-concavity and negative definiteness of the Hessian matrix and prove these theorems directly.

2 Quasi-concavity

Let $f$ be a function from an open convex set $C$ in $\mathbb{R}^n$ to the real line $\mathbb{R}$.

**Definition 1.** $f$ is **quasi-concave** if for every $x$ and $y$ in $C$ such that $f(y) \geq f(x)$, we have $f(\lambda x + (1 - \lambda)y) \geq f(x)$ for every $\lambda$ with $0 < \lambda < 1$. $f$ is **strictly quasi-concave** if for every $x$ and $y$ in $C$ such that $f(y) \geq f(x)$, we have $f(\lambda x + (1 - \lambda)y) > f(x)$ for every $\lambda$ with $0 < \lambda < 1$. 

For every point \( x \) in \( C \) and for each vector \( h \) from \( \mathbb{R}^n \) such that \( x + h \) lies in \( C \), we define a function \( g \) from the interval \([0, 1]\) to the real line \( \mathbb{R} \) by \( g(\lambda) = f(x + \lambda h) \); cf. Berge (1959, p. 217). Quasi-concavity of \( f \) can be characterized by quasi-concavity of the functions \( g \), as shown by the following lemma.

**Lemma 1.** The function \( f \) is quasi-concave if and only if for each \( x \) in \( C \) and each \( h \) in \( \mathbb{R}^n \), the function \( g \) is quasi-concave on \([0, 1]\).

**Proof.** Trivial. \( \square \)

Because \( C \) is an open set, we can extend \( g \) to the left such that it is defined on an interval \([-\epsilon, 1]\) for some \( \epsilon > 0 \). Lemma 1 also holds if we replace \([0, 1]\) in its statement by \([-\epsilon, 1]\). Thus the derivatives of \( g \), if they exist, are defined at 0.

We assume from now on that \( f \) is twice continuously differentiable. Then \( g \) is also twice continuously differentiable.

### 3 Necessary second-order properties

**Lemma 2.** Let \( v \) be a twice continuously differentiable quasi-concave function from an open interval \( I \) in the real line \( \mathbb{R} \) to the real line \( \mathbb{R} \). If \( v'(x) = 0 \) at a point \( x \) in \( I \), then \( v''(x) \leq 0 \).

**Proof.** Suppose the contrary: \( v''(x) > 0 \). Then there exists a neighborhood \( N \) of \( x \) such that \( v(y) > v(x) \) for every \( y \neq x \) in \( N \). Choose a point \( w \) in \( N \) to the left of \( x \) and a point \( z \) in \( N \) to the right of \( x \). There exists a \( \lambda \) with \( 0 < \lambda < 1 \) such that \( x = \lambda w + (1 - \lambda)z \). Thus \( v(\lambda w + (1 - \lambda)z) < \min\{v(w), v(z)\} \), which contradicts the quasi-concavity of \( v \). \( \square \)

**Theorem 1.** If \( f \) is quasi-concave then for every point \( x \) in \( C \) there holds \( h'D^2 f(x)h \leq 0 \) for every \( h \) such that \( h'D f(x) = 0 \).

**Proof.** Choose a point \( x \) in \( C \) and let \( h \) be a vector from \( \mathbb{R}^n \) such that \( h'D f(x) = 0 \). Define \( g \) by \( g(\lambda) = f(x + \lambda h) \). By Lemma 1, \( g \) is quasi-concave. We have \( g'(\lambda) = h'D f(x + \lambda h)h \) and \( g''(\lambda) = h'D^2 f(x + \lambda h)h \). Therefore \( g'(0) = 0 \) and thus by Lemma 2 we have \( g''(0) \leq 0 \). That is: \( h'D^2 f(x)h \leq 0 \). \( \square \)

### 4 Sufficient second-order properties

The converse of Theorem 1 does not hold. For example \( f(x) = x^4 \) is not quasi-concave; but for all scalars \( h \) such that \( hf'(x) = 0 \), there holds \( h^2f''(x) = 0 \). Another counterexample is \( f(x_1, x_2) = (x_1 - 5)^4(x_2 - 5)^4 \). For all vectors \( h \) such that \( h'D f(x) = 0 \) there holds \( h'D^2 f(x)h = -8h_1^4(x_1 - 5)^2(x_2 - 5)^4 \leq 0 \) if \( x_1 \neq 5 \) and \( x_2 \neq 5 \), and \( h'D^2 f(x)h = 0 \) if either \( x_1 = 5 \) or \( x_2 = 5 \); however \( f \) is not quasi-concave: \( f(2, 2) < f(2, 12) \), but \( f(7, 7) = f(\sqrt[3]{2}(2, 2) + \sqrt[3]{2}(12, 12)) < f(2, 2) \).

However, if at least one partial derivative of \( f \) is everywhere on \( C \) non-zero, then the converse of Theorem 1 holds.
Theorem 2. Let $\partial f / \partial x_n$ be non-zero on $C$ and let all the contour lines of $f$ lie in $\mathbb{R}^{n-1} \cap C$. If for every point $x$ in $C$ there holds $h' D \frac{f(x)}{h} \leq 0$ for every $h$ such that $h' D f(x) = 0$, then $f$ is quasi-concave.

**Proof.** (adapted from Arrow and Enthoven 1961, theorem 4, who prove this for the two-dimensional case). We will show that each contour line (level set) of $f$ is convex, and that this implies $f$ is quasi-concave.

Define for each point $x$ in $\mathbb{R}^n$: $\tilde{x} = (x_1, x_2, \cdots, x_{n-1})$.

We define $f_i$ to be the $i$-th element of the gradient $D f$. So $f_n \neq 0$ everywhere on $C$. Because of the continuity of $f_n$ there holds either $f_n > 0$ or $f_n < 0$. Suppose $f_n < 0$.

Choose two points $x^0$ and $y^0$ in $C$ and let $f(y^0) \geq f(x^0)$. Let $\lambda^0$ be the largest value of $\lambda$ in $[0, 1]$ such that $f(\lambda x^0 + (1 - \lambda) y^0) = f(x^0)$ and define $z^0 = \lambda^0 x^0 + (1 - \lambda^0) y^0$.

Consider the contour line $\{ x | f(x) = f(x^0) \}$. Because $f_n > 0$ there exists a twice continuously differentiable function $F : \mathbb{R}^{n-1} \cap C \to \mathbb{R}$ such that $f(\tilde{x}, F(\tilde{x})) = f(x^0)$ and $F(\tilde{x}) = x_n$ for each $\tilde{x}$ in $\mathbb{R}^{n-1} \cap C$. Let $\tilde{h} = (h_1, h_2, \cdots, h_{n-1})$ be an arbitrary vector in $\mathbb{R}^{n-1}$ and define $h_n = \sum_{i=1}^{n-1} h_i f_i(\tilde{x}, F(\tilde{x}))/f_n(\tilde{x}, F(\tilde{x}))$. Then $\partial F / \partial F(\tilde{x}) = 0$ and thus $h' D^2 f(\tilde{x}, F(\tilde{x})) h \leq 0$. But $h' D^2 f(\tilde{x}, F(\tilde{x})) h = -\tilde{h}' D^2 F(\tilde{x}) \tilde{h}/f_n(\tilde{x}, F(\tilde{x}))$ (see Note at the end of the proof) and thus $h' D^2 F(\tilde{x}) \tilde{h} \geq 0$ for every point $\tilde{x}$ in $\mathbb{R}^{n-1} \cap C$ and for each vector $\tilde{h}$ from $\mathbb{R}^{n-1}$. Therefore $F$ is a convex function.

Take an arbitrary $\lambda$ in $[0, A^0]$ and define $w = x^0 + (1 - \lambda) y^0$. There holds $w = \mu x^0 + (1 - \mu) z^0$ with $\mu = (\lambda - A^0)/(1 - \lambda^0)$. Then by the convexity of $F$ we have $F(w) = F(\mu x^0 + (1 - \mu) z^0) \leq \mu F(x^0) + (1 - \mu) F(z^0) = \mu x_n^0 + (1 - \mu) z_n^0 = w_n$. Because $f_n > 0$, it then follows that $f(w) \geq f(w, F(w)) = f(x^0)$.

By definition of $\lambda^0$ we have $f(\lambda x^0 + (1 - \lambda) y^0) > f(x^0)$ for every $\lambda$ in $(A^0, 1]$. Thus for each $\lambda$ in $[0, 1]$ there holds $f(\lambda x^0 + (1 - \lambda) y^0) \geq f(x^0)$. Therefore $f$ is quasi-concave if $f_n > 0$.

In the same way we can prove that $f$ is quasi-concave if $f_n < 0$.

**Note:** Proof of $h' D^2 f(\tilde{x}, F(\tilde{x})) h = -\tilde{h}' D^2 F(\tilde{x}) \tilde{h}/f_n(\tilde{x}, F(\tilde{x})) \tilde{h}$

Define $f_i$ and $F_i$ to be the $i$-th element of the gradient $D f(\tilde{x}, F(\tilde{x}))$ respectively $D F(\tilde{x})$, and $f_{ij}$ and $F_{ij}$ to be the $(i, j)$-th element of the Hessian matrix $D^2 f(\tilde{x}, F(\tilde{x}))$ respectively $D^2 F(\tilde{x})$. We have $F_i = -f_i/f_n$ and thus $F_{ij} = [f_i(f_{nj} + f_{mn} F_j) - f_n(f_{ij} + f_{jn} F_j)]/f_n^2 = (f_i f_{nj} + f_{ij} f_{jn} - f_n f_{ij})/f_n^2$. The proof now follows by substituting this in $\tilde{h}' D^2 F(\tilde{x}) \tilde{h} = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} h_i h_j f_{ij}$ and noting that $-\sum_{i=1}^{n-1} h_i f_i/f_n = h_n$.

As shown by the following theorem, the condition that at least one partial derivative is non-zero, is not required for a function to be strictly quasi-concave.

**Theorem 3.** If for every point $x$ in $C$ there holds $h' D^2 f(x) h < 0$ for each $h \neq 0$ such that $h' D f(x) = 0$, then $f$ is strictly quasi-concave.

**Proof.** Suppose $f$ is not strictly quasi-concave. Then there exist two points $x$ and $y$ such that $f(y) \geq f(x)$ and $f(\lambda x + (1 - \lambda) y) \leq f(x)$ for a $\lambda^0$ with $0 < \lambda^0 < 1$. Define $h = y - x$, and $g$ by $g(\lambda) = g(x + \lambda h)$. Then we have $g(1) \geq g(0)$ and $g(\lambda^0) \leq g(0)$. So there exists an $\alpha$ in $(0, 1)$ such that $g(\alpha) \leq g(\lambda)$ for every $\lambda$ in $[0, 1]$. Therefore $g'(\alpha) = 0$ and $g''(\alpha) \geq 0$. That is $h' D f(x + \alpha h) = 0$, but $h' D^2 f(x + \alpha h) h \geq 0$, which gives a contradiction.
The converse of Theorem 3 does not hold. For example $f(x) = x^4$ is strictly concave and thus strictly quasi-concave, but for each $h \neq 0$ we have $hf''(0) = 0$, but $h^2f'''(0) = 0$. The converse of Theorem 3 does not hold even if a partial derivative is everywhere non-zero. For example $f(x_1, x_2) = 3x_2 - (x_1 - 1)^4 - 3x_1$ is strictly concave and thus strictly quasi-concave, $\partial f / \partial x_2 = 3$ is everywhere non-zero, but if $x_1 = 1$, then for all $h \neq 0$ such that $h'Df(x) = 0$ there holds $h'D^2f(x)h = 0$.

References


