Second-order properties of quasi-concave functions

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We present proofs, some of which are very simple, of theorems on the second-order properties of quasi-concave functions.

1 Introduction

Quasi-concavity is an important concept in economics, e.g. we often assume that utility functions and production functions are quasi-concave. Arrow and Enthoven (1961) studied the problem of maximizing a quasi-concave function subject to constraints as well as gave properties of quasi-concavity in terms of bordered determinants. These properties are equivalent to the condition that the Hessian matrix be negative-definite subject to a constraint (Debreu 1952). First-order properties of quasi-concave functions are given in e.g. Mangasarian (1969, chapter 9). In this paper we give theorems on the relation between quasi-concavity and negative definiteness of the Hessian matrix and prove these theorems directly.

2 Quasi-concavity

Let f be a function from an open convex set C in $\mathbb{R}^n$ to the real line $\mathbb{R}$.

Definition 1. f is *quasi-concave* if for every x and y in C such that $f(y) \geq f(x)$, we have $f\big(\lambda x + (1-\lambda)y\big) \geq f(x)$ for every λ with $0 < \lambda < 1$. f is *strictly quasi-concave* if for every x and y in C such that $f(y) \geq f(x)$, we have $f\big(\lambda x + (1-\lambda)y\big) > f(x)$ for every λ with $0 < \lambda < 1$.

For every point x in C and for each vector h from $\mathbb{R}^n$ such that $x + h$ lies in C,we define a function g from the interval $[0, 1]$ to the real line $\mathbb{R}$ by $g(\lambda) = f(x + \lambda h)$; cf. Berge (1959, p. 217). Quasi-concavity of f can be characterized by quasi-concavity of the functions g, as shown by the following lemma.

Lemma 1. *The function f is quasi-concave if and only if for each x in C and each h in $\mathbb{R}^n$, the function g is quasi-concave on* $[0, 1]$.

Proof. Trivial. □

Because C is an open set, we can extend g to the left such that it is defined on an interval $[-\epsilon, 1]$ for some $\epsilon > 0$. Lemma 1 also holds if we replace $[0, 1]$ in its statement by $[-\epsilon, 1]$. Thus the derivatives of g, if they exist, are defined at 0.

We assume from now on that f is twice continuously differentiable. Then g is also twice continuously differentiable.

3 Necessary second-order properties

Lemma 2. *Let v be a twice continuously differentiable quasi-concave function from an open interval I in the real line $\mathbb{R}$ to the real line $\mathbb{R}$. If $v'(x) = 0$ at a point x in I, then $v''(x) \leq 0$.*

Proof. Suppose the contrary: $v''(x) > 0$. Then there exists a neighborhood N of x such that $v(y) > v(x)$ for every $y \neq x$ in N. Choose a point w in N to the left of x and a point z in N to the right of x. There exists a λ with $0 < \lambda < 1$ such that $x = \lambda w + (1 - \lambda)z$. Thus $v(\lambda w + (1 - \lambda)z) < \min\{v(w), v(z)\}$,which contradicts the quasi-concavity of v. □

Theorem 1. *If f is quasi-concave then for every point x in C there holds $h' \mathsf{D}^2 f(x)h \leq 0$ for every h such that $h' \mathsf{D} f(x) = 0$.*

Proof. Choose a point x in C and let h be a vector from $\mathbb{R}^n$ such that $h' \mathsf{D} f(x) = 0$. Define g by $g(\lambda) = f(x + \lambda h)$. By Lemma 1, g is quasi-concave. We have $g'(\lambda) = h' \mathsf{D} f(x + \lambda h)$ and $g''(\lambda) = h' \mathsf{D}^2 f(x + \lambda h)h$. Therefore $g'(0) = 0$ and thus by Lemma 2 we have $g''(0) \leq 0$. That is: $h' \mathsf{D}^2 f(x)h \leq 0$. □

4 Sufficient second-order properties

The converse of Theorem 1 does not hold. For example $f(x) = x^4$ is not quasi-concave; but for all scalars h such that $hf'(x) = 0$, there holds $h^2 f''(x) = 0$. Another counterexample is $f(x_1, x_2) = (x_1 - 5)^4(x_2 - 5)^4$. For all vectors h such that $h' \mathsf{D} f(x) = 0$ there holds $h' \mathsf{D}^2 f(x)h = -8h_1^2(x_1 - 5)^2(x_2 - 5)^4 \leq 0$ if $x_1 \neq 5$ and $x_2 \neq 5$, and $h' \mathsf{D}^2 f(x)h = 0$ if either $x_1 = 5$ or $x_2 = 5$; however f is not quasi-concave: $f(2, 2) < f(2, 12)$, but $f(7, 7) = f({}^1\!/\!_2(2, 2) + {}^1\!/\!_2(12, 12)) < f(2, 2)$.

However, if at least one partial derivative of f is everywhere on C non-zero, then the converse of Theorem 1 holds.

Theorem 2. *Let $\partial f/\partial x_n$ be non-zero on C and let all the contour lines of f lie in $\mathbb{R}^{n-1} \cap C$. If for every point x in C there holds $h' \mathsf{D}^2 f(x)h \leq 0$ for every h such that $h' \mathsf{D} f(x) = 0$, then f is quasi-concave.*

Proof. (adapted from Arrow and Enthoven 1961, theorem 4, who prove this for the two-dimensional case). We will show that each contour line (level set) of f is convex, and that this implies f is quasi-concave.

Define for each point x in $\mathbb{R}^n$: $\tilde{x} = (x_1, x_2, \cdots, x_{n-1})$.

We define f_i to be the i-th element of the gradient $\mathsf{D} f$. So $f_n \neq 0$ everywhere on C. Because of the continuity of f_n there holds either $f_n > 0$ or $f_n < 0$. Suppose $f_n < 0$.

Choose two points x^0 and y^0 in C and let $f(y^0) \geq f(x^0)$. Let λ^0 be the largest value of λ in $[0, 1]$ such that $f(\lambda x^0 + (1 - \lambda)y^0) = f(x^0)$ and define $z^0 = \lambda^0 x^0 + (1 - \lambda^0)y^0$.

Consider the contour line $\{x | f(x) = f(x^0)\}$. Because $f_n > 0$ there exists a twice continuously differentiable function $F : \mathbb{R}^{n-1} \cap C \to \mathbb{R}$ such that $f(\tilde{x}, F(\tilde{x})) = f(x^0)$ and $F(\tilde{x}) = x_n$ for each $\tilde{x}$ in $\mathbb{R}^{n-1} \cap C$. Let $\tilde{h} = (h_1, h_{2,} \cdots, h_{n-1})$ be an arbitrary vector in $\mathbb{R}^{n-1}$ and define $h_n = -\sum_{i=1}^{n-1} h_i f_i(\tilde{x}, F(\tilde{x}))/f_n(\tilde{x}, F(\tilde{x}))$. Then $h' \mathsf{D} f(\tilde{x}, F(\tilde{x})) = 0$ and thus $h' \mathsf{D}^2 f(\tilde{x}, F(\tilde{x}))h \leq 0$. But $h' \mathsf{D}^2 f(\tilde{x}, F(\tilde{x}))h = -\tilde{h}' \mathsf{D}^2 F(\tilde{x})\tilde{h}/f_n(\tilde{x}, F(\tilde{x}))$ (see Note at the end of the proof) and thus $\tilde{h}' \mathsf{D}^2 F(\tilde{x})\tilde{h} \geq 0$ for every point $\tilde{x}$ in $\mathbb{R}^{n-1} \cap C$ and for each vector $\tilde{h}$ from $\mathbb{R}^{n-1}$. Therefore F is a convex function.

Take an arbitrary λ in $[0, \lambda^0]$ and define $w = x^0 + (1 - \lambda)y^0$. There holds $w = \mu x^0 + (1 - \mu)z^0$ with $\mu = (\lambda - \lambda^0)/(1 - \lambda^0)$. Then by the convexity of F we have $F(\tilde{w}) = F(\mu\tilde{x}^0 + (1 - \mu)z^0) \leq \mu F(\tilde{x}^0) + (1 - \mu)F(\tilde{z}^0) = \mu x_n^0 + (1 - \mu)z_n^0 = w_n$. Because $f_n > 0$, it then follows that $f(w) \geq f(\tilde{w}, F(\tilde{w})) = f(x^0)$.

By definition of λ^0 we have $f(\lambda x^0 + (1 - \lambda)y^0) > f(x^0)$ for every λ in $(\lambda^0, 1]$. Thus for each λ in $[0, 1]$ there holds $f(\lambda x^0 + (1 - \lambda)y^0) \geq f(x^0)$. Therefore f is quasi-concave if $f_n > 0$.

In the same way we can prove that f is quasi-concave if $f_n < 0$.

Note: Proof of $h' \mathsf{D}^2 f(\tilde{x}, F(\tilde{x}))h = -\tilde{h}' \mathsf{D}^2 F(\tilde{x})\tilde{h}/f_n(\tilde{x}, F(\tilde{x}))\tilde{h}$

Define f_i and F_i to be the i-th element of the gradient $\mathsf{D} f(\tilde{x}, F(\tilde{x}))$ respectively $\mathsf{D} F(\tilde{x})$, and f_{ij} and F_{ij} to be the (i, j)-th element of the Hessian matrix $\mathsf{D}^2 f(\tilde{x}, F(\tilde{x}))$ respectively $\mathsf{D}^2 F(\tilde{x})$. We have $F_i = -f_i/f_n$ and thus $F_{ij} = [f_i(f_{nj} + f_{nn}F_j) - f_n(f_{ij} + f_{in}F_j)]/f_n^2 = (f_i f_{nj} + f_j f_{in} - f_n f_{ij} - f_i f_j f_{nn}/f_n)/f_n^2$. The proof now follows by substituting this in $\tilde{h}' \mathsf{D}^2 F(\tilde{x})\tilde{h} = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} h_i h_j F_{ij}$ and noting that $-\sum_{i=1}^{n-1} h_i f_i/f_n = h_n$. □

As shown by the following theorem, the condition that at least one partial derivative is non-zero, is not required for a function to be *strictly* quasi-concave.

Theorem 3. *If for every point x in C there holds $h' \mathsf{D}^2 f(x)h < 0$ for each $h \neq 0$ such that $h' \mathsf{D} f(x) = 0$, then f is strictly quasi-concave.*

Proof. Suppose f is not strictly quasi-concave. Then there exist two points x and y such that $f(y) \geq f(x)$ and $f(\lambda^0 x + (1 - \lambda^0)y) \leq f(x)$ for a λ^0 with $0 < \lambda^0 < 1$. Define $h = y - x$, and g by $g(\lambda) = g(x + \lambda h)$. Then we have $g(1) \geq g(0)$ and $g(\lambda^0) \leq g(0)$. So there exists an α in $(0, 1)$ such that $g(\alpha) \leq g(\lambda)$ for every λ in $[0, 1]$. Therefore $g'(\alpha) = 0$ and $g''(\alpha) \geq 0$. That is $h' \mathsf{D} f(x + \alpha h) = 0$, but $h' \mathsf{D}^2 f(x + \alpha h)h \geq 0$, which gives a contradiction. □

The converse of Theorem 3 does not hold. For example $f(x) = x^4$ is strictly concave and thus strictly quasi-concave, but for each $h \neq 0$ we have $hf'(0) = 0$, but $h^2 f''(0) = 0$. The converse of Theorem 3 does not hold even if a partial derivative is everywhere non-zero. For example $f(x_1, x_2) = 3x_2 - (x_1 - 1)^4 - 3x_1$ is strictly concave and thus strictly quasi-concave, $\partial f / \partial x_2 = 3$ is everywhere non-zero, but if $x_1 = 1$, then for all $h \neq 0$ such that $h' \mathsf{D} f(x) = 0$ there holds $h' \mathsf{D}^2 f(x) h = 0$.

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