Uncertainty Aversion and Convexity in Portfolio Choice

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Abstract

This note studies the implication of the general notion of uncertainty aversion (Schmeidler 1989) on the problem of portfolio choice, which involves allocating the proportions of fixed capital to several assets. We prove that if an investor is both risk averse and uncertainty averse, then preference in a portfolio space is convex. This result means that the convexity in a portfolio choice problem can be guaranteed without restricting preference representation to a particular functional form.

Keywords Convexity, Portfolio Choice, Ambiguity, Uncertainty Aversion, Risk Aversion

1 Introduction

The problem of portfolio choice underlines much of finance and it is commonly adopted by experimentalists to elicit risk preferences. Typically, it involves an investor choosing the proportions of fixed capital allocated to several assets (one of the assets can be a safe asset) with known return probability distributions (risk). However, the probability distributions are usually unknown or do not exist (ambiguity) in reality. Ellsberg (1969)'s seminal paper argues that people tend to be ambiguous averse, which means they prefer to bet on known probability to unknown probability. Ambiguity has since been widely studied theoretically, experimentally and its implications on financial market has been developed[1].

This paper investigates a basic theoretical question: is preference under ambiguity in a portfolio space convex? Without convexity, there is little analytical tractability and it is difficult to derive meaningful economic prediction. Take comparatives statics as an example, one can say very little about what happens to a portfolio choice if the return of an asset is increasing. Hence, similar to how risk aversion is represented by a concave Bernoulli utility function, decision theorists strive to derive concave representations for ambiguity aversion. While different functional representations require different sets of

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[1]For example, Bossaerts *et al.*(2010) have studied how ambiguity aversion affects equilibrium asset prices and asset holdings

axioms, most ambiguity theories [2] share the axiom of *uncertainty aversion*.

Definition 1 (Uncertainty aversion, Schmeidler 1989)**.** For all acts f and g, preferences $\succsim$ are *uncertainty averse* if $f \succsim g$ implies $\lambda f \oplus (1-\lambda)g \succsim g$ for any $\lambda \in [0,1]$.

where an *act* is a mapping from states to *primitive lotteries* (that is probability distributions over outcomes, henceforth lottery). The addition "$\oplus$" is called probability mixture and it operates on acts state-by-state. Let f_s and g_s denote the realizations of f and g in state s respectively. $\lambda f_s \oplus (1-\lambda)g_s$ is a compound lottery that gives rise to f_s with a probability of λ and gives rise to g_s with a probability of $1-\lambda$. Note *Uncertainty aversion* by itself directly guarantees convexity in the probability mixture space[3].

Just like risky financial assets are modelled by lotteries, ambiguous financial assets are usually modelled by acts. A portfolio consisting of two ambiguous assets f and g can be written as $\alpha f + (1-\alpha)g, \alpha \in [0,1]$. To guarantee preferences in the portfolio mixture space is convex, the following is needed.

Definition 2 (Portfolio Convexity)**.** For all acts f and g, preferences $\succsim$ are *convex* if $f \succsim g$ implies $\alpha f + (1-\alpha)g \succsim g$ for any $\alpha \in [0,1]$.

where the addition sign "$+$" refers to summing of two probability distributions state-by-state, which is called portfolio mixture hereafter.

Our main result, as stated in Proposition 1, shows that uncertainty aversion can directly imply portfolio convexity when risk aversion is assumed.

Proposition 1. *If $\succsim$ is uncertainty averse and risk averse, then $\succsim$ is portfolio convex.*

It also can be easily seen from the proof that when *strictly risk aversion* is assumed, then uncertainty aversion implies *strictly portfolio convexity*. In a similar proof, Appendix B demonstrates that portfolio convexity can also be implied by uncertainty aversion and variance aversion, which is a common assumption in financial literature.

In what follows, Section 2 formally defines probability mixture and portfolio mixture. Section 3 introduces the portfolio choice model. Section 4 provides the proof for proposition 1. Section 5 demonstrates how to extend the result to the case of multiple assets.

2 Portfolio Mixture "$+$" and Probability Mixture "$\oplus$"

The definition of portfolio mixture "$+$" is straightforward: it is the state-by-state sum of two probability distributions. We still define it in details so that its difference to probability mixture "$\oplus$" can be made clear. Let f and g denote two acts. The outcome of an act in a state $s \in S$, denoted by f_s, is a lottery. $f_s(z)$ should be read as the probability that f_s gives to the monetary payoff z. $(f_s + g_s)(z)$ should be read as the probability that the mixed lottery $f_s + g_s$ gives to the monetary payoff z.

[2] For example, Schmeidler's (1989)'s Choquet Expected Utility with convex capacity, Gilboa and Schmeidlerl's (1989) max-min Expected Utility, Maccheroni, Marinacci, and Rustichini's (2006) Variational Preference, Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio's (2011) penalization representation, Strzalecki (2011) Multiplier Preferences.

[3] It does not guarantee convexity in a portfolio mixture space. Appendix A provides an example that convexity in a portfolio choice problem is violated under the standard ambiguity model of Maxmin expected utility model (Gilboa, Itzhak and David Schmeidler, 1989).

2.1 Portfolio Mixture

Let f_s and g_s denote two lotteries. The portfolio mixture of the two lotteries is

$$(f_s + g_s)(z) = \int f_s(x) g_s(z - x) dx. \tag{1}$$

We can consider f_s and g_s the probability density functions for two independent real-valued random variables F and G. Let random variable H be the sum $H = F+G$. Equation (1) describes the probability density function of H.

It follows that for any $\alpha, \beta \in \mathbb{R}$

$$(\alpha f_s + \beta g_s)(z) = \int f_s(x) g_s\left(\frac{z - \alpha x}{\beta}\right) dx.$$

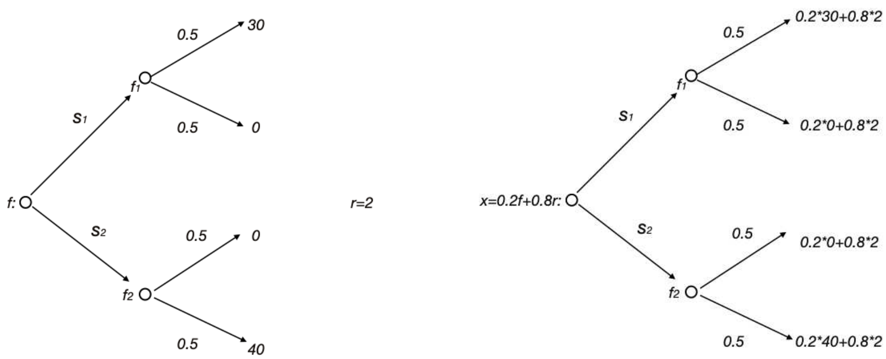
Portfolio mixture "+" operates on acts state by state, that is it results in a new act for which its outcomes in a state s is the portfolio mixture of the two resulting lotteries in that state. That is

$$(\alpha f + \beta g)_s = \alpha f_s + \beta g_s$$

for all $s \in S$.

An Example of Portfolio Mixture

Consider the special case when the act g is a constant real value $r \in \mathbb{R}$. Let $x = 0.2f + 0.8r$. The operation is rather simple: the probability distribution of x_s is the same as f_s while the original outcome z of f_s becomes $0.2z + 0.8r$. For illustrative purpose, we use discrete distributions as in Figure 1.



(a) Ambiguous Asset f and Safe Asset r

Suppose there are two states s_1 and s_2. In state 1, f_1 is a lottery that returns 30 with a probability of 0.5 and returns 0 with a probability of 0. In state 2, f_2 is a lottery that returns 0 with a probability of 0.5 and returns 40 with a probability of 0. The safe asset pays a constant r of 2 in either of the two states.

(b) Portfolio Mixture $0.2f + 0.8r$

Now suppose an investor allocates 20% to f and the remain 80% to r. Then this portfolio mix results in a new act. Since f takes four outcomes and the $r = 2$ is a constant, their weighted sum takes four outcomes. Each $(0.2p + 0.8q)_s$, $s = 1, 2$ takes two outcomes and it follows the same distribution of f_s.

Figure 1: Portfolio Mixture of One Ambiguous Asset and One Safe Asset

2.2 Probability Mixture

Previous axiomatizing decision models under ambiguity often take advantage of the following probability mixture operation $\oplus$ of two lotteries:

$$(\lambda f_s \oplus (1-\lambda)g)(z) = \lambda f_s(z) + (1-\lambda)g_s(z)$$

where the latter "+" is the addition of real numbers and $\lambda \in (0,1)$. The probability mixture of two lotteries f_s and g_s gives a two-stage compound lottery $(\lambda f_s \oplus (1-\lambda)g_s)$ where at the first stage f_s realizes with a probability of λ and g_s realizes with a probability of $1-\lambda$.

Similar to portfolio mix, probability mixture operates state by state on acts:

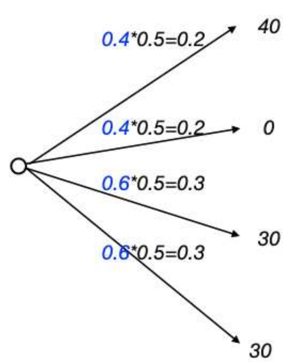
$$(\lambda f \oplus (1-\lambda)g)_s = \lambda f_s \oplus (1-\lambda)g_s \qquad for\ all\ s \in S.$$

An Example of how the two mixtures differ

The following example illustrates how the two mixtures differ. Figure 2 compares how they yield different lotteries in a typical state s.

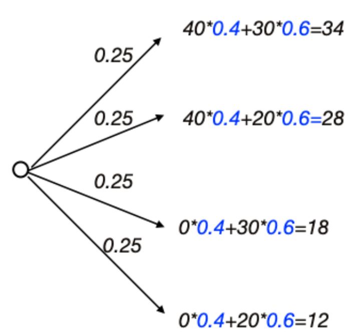
(a) Two Independent Lotteries f_s and g_s

Suppose for some state s, f gives rise to a lottery f_s and g gives rise to a lottery g_s.



(b) Probability Mixture $0.4p \oplus 0.6q$

This is the compounded lottery. The coefficients 0.4 and 0.6 are used for calculating the probabilities.



(c) Portfolio Mixture $0.4p + 0.6q$

This is the weighted sum of the outcomes.The coefficients 0.4 and 0.6 are used for weighting the outcomes.

Figure 2: Probability Mixture and Portfolio Mixture in a typical state s

Further examples of probability mixture can be found in Figure 7.2 of Kreps (1988).

3 The Portfolio Choice Model

Recent decision models under ambiguity are often (see Marchina and Siniscalchi 2014 for a survey) built on a type of Anscombe-Aumann (AA) framework, where $f : S \to \Delta(Z)$ is called an AA act or a two-stage horse-roulette act (hereafter act) that maps states into the linear space $X = \Delta(Z)$. $Z = \mathbb{R}_+$ is the monetary outcome space and Δ is a probability simplex. The classic Expected utility model is maintained for preferences over primitive lotteries. This objective-subjective approach provides a framework for representing uncertain prospects that involve both objective and subjective uncertainty. In this set-up, ambiguity aversion attitudes featured in the Ellsberg paradox can be incorporated.

Applying this AA framework to portfolio choice, then ambiguous assets would be represented by acts[4]. Consider the typical two-assets portfolio choice problem: an investor decides the proportions $(\alpha, 1-\alpha) \in \mathbb{R}^2_+$of fixed capital to allocate between two ambiguous assets. Denote the set of states by S that the outcome of the ambiguous asset will depend on. Suppose there is a finite number of states that is also denoted by S. The gross return (hereafter return) of investing in an uncertain asset is $f : S \to X$. The return of the uncertain asset in state s is denoted by f_s, which is a lottery. And hence, ambiguity is expressed in this way: the subjective uncertainty (states) will solve and, depend on how it resolves the return of the uncertain asset is a lottery. While the information about the probability of the subjective uncertainty is not available, the specification and the parameters of the lottery can be estimated using statistics [5].

Denote f and g the two assets. Then the final wealth of the investor's portfolio in state s is

$$x_s = \alpha f_s + (1-\alpha) g_s.$$

and the generic form of the final wealth of the investor' portfolio is written as

$$x = \alpha f + (1-\alpha) g. \quad (2)$$

where the addition operation "+" in (2) is the *portfolio mixture* defined in Section 2.1. Hence, it is different from the algebraic addition in a classic portfolio choice model within Arrow-Debreu framework. It is similar to the one in Gollier (2013)'s portfolio model.

Preferences $\succsim$ are defined on final wealth x. Since S is finite, we let the vectors denote acts, for example, $\mathbf{f} = (f_1, ..., f_S)$ represents the act f. Let $\mathbf{F}$ denote the return matrix $(\mathbf{f}, \mathbf{g})$. Then the budget set is

$$B(\mathbf{F}) = \{\mathbf{x} \in X^S : \mathbf{x} = \alpha \mathbf{f} + (1-\alpha)\mathbf{g}, 0 \leq \alpha \leq 1.\} \quad (3)$$

If preferences on $B(\mathbf{F})$ are convex, then there exists a demand

$$\mathbf{x}^* \in \{\mathbf{x} \in B : \forall \mathbf{y} \in B, u(\mathbf{x}^*) \geq u(\mathbf{y})\}$$

where $u(x)$ is a quasiconcave function. The corresponding *portfolio choice* is $\alpha^* = (\mathbf{x}^* - \mathbf{g})/(\mathbf{f} - \mathbf{g})$. The proof of Proposition 1 demonstrates preference on $B(\mathbf{F})$ is convex.

[4]In an Arrow Debreu framework under risk, assets are usually represented by $f : S \to R$.

[5]For example, consider how investors may formulate the effect of international travel restrictions on an airline company' return: in state 1 (with a travel restriction), the return is uniformly distributed on the region of two times the standard deviation of 0.2 around the mean of 0.4 ; state 2 (without travel restriction), the region is then two times the standard deviation of 0.2 around a higher mean of 1.4. While the mean and standard deviations can be calculated based on statistical data, there is not enough information on the probability of the event of travel restrictions being imposed.

4 Proof for Proposition 1

Risk aversion is defined formally as follows.

Definition 3. Preference over lotteries are *risk averse* if any lottery is evaluated by its probability cumulative distribution $F(\cdot)$ by the von-Neumann-Vorgenstern (vNM) utility function $U(F) = \int u(t)f(t)dt$ where $u(\cdot)$ is a concave Bernoulli utility function.

Proof. The key of the proof is Lemma 1. It proves that the portfolio mixture of two lotteries is preferred to the probability mixture of the two lotteries for all risk averse investors. Essentially it shows the former *second order stochastic dominates* the latter.

Consider any arbitrary acts f and g such that $f \succsim g$. By uncertainty aversion, for any $\lambda \in [0,1]$ we have $\lambda f \oplus (1-\lambda)f \succsim g$.

Now consider an arbitrary state s. Let $L_1 = \lambda f_s \oplus (1-\lambda)g_s$ and $L_2 = \lambda f_s + (1-\lambda)g_s$. By Lemma 1, we have $L_2 \succsim L_1$. This holds for all states. Therefore $\lambda f + (1-\lambda)g \succsim \lambda f \oplus (1-\lambda)g$. By transitivity, it follows that $\lambda f + (1-\lambda)g \succsim g$. □

Lemma 1. If preferences over lotteries are risk averse, then the portfolio mixture of two lotteries is preferred to the probability mixture of two lotteries.

Proof. Let f_s and g_s denote two lotteries. Let P and Q denote two independent random variables for which f_s and g_s describe their probability density distributions respectively.

Define a new random variable $R_1 := BP + (1-B)Q$, where B is a binary, independent random variable for which the probability that $B = 1$ is α and the probability that $B = 0$ is $1-\alpha$. Define another random variable $R_2 := \alpha P + (1-\alpha)Q$. Thus, the probability distribution for R_1 is the probability mixture of f_s and g_s, denoted by L_1. And the probability distribution for R_2 is the portfolio mixture of f_s and g_s, denoted by L_2.

Let $F_p(\cdot), F_q(\cdot), F_{L_1}(\cdot), F_{L_2}(\cdot) : \mathbb{R} \to [0,1]$ denote the cumulative probability distribution functions for P, Q, R_1 and R_2 respectively. Let $u(\cdot)$ denote an arbitrary concave Bernoulli function. Then we have the expected utility of L_1 as

$$U(L_1) = \int u(z)dF_{L_1}(z) = \int \Big(\alpha f_s + (1-\alpha)g_s\Big)dz = \alpha \int u(z)dF_p(z) + (1-\alpha)\int u(z)dF_q(z)$$

and the expected utility of the L_2 as

$$U(L_2) = \int u(z)dF_{L_2}(z) = \int u(z)\int f_s(y)g_s(\frac{z-\alpha y}{\beta})dy = \int\int u\Big(\alpha y + (1-\alpha)z\Big)dF_p(y)dF_q(z).$$

Since $u(\cdot)$ is concave, it has the property that for any $\alpha \in [0,1]$ and any y, z

$$u(\alpha y + (1-\alpha)z) \geq \alpha u(y) + (1-\alpha)u(z).$$

Hence

$$\begin{aligned}
U(L_2) &\geq \int\int\Big(\alpha u(y)+(1-\alpha)u(z)\Big)dF_p(y)dF_q(z)\\
&= \alpha\int\int u(y)dF_p(y)dF_q(z)+(1-\alpha)\int\int u(z)dF_p(y)dF_q(z)\\
&= \alpha\int\Big(\int u(y)dF_p(y)\Big)dF_q(z)+(1-\alpha)\int\Big(\int u(z)dF_q(z)\Big)dF_p(y)\\
&= \alpha\int E(u(P))dF_q(z)+(1-\alpha)\int E(u(Q))dF_p(y)\\
&= \alpha E(u(P))\int dF_q(z)+(1-\alpha)E(u(Q))\int dF_p(y)\\
&= \alpha E(u(P))+(1-\alpha)E(u(Q))=U(L_1)
\end{aligned}$$

By the definition of risk aversion, we have $L_2 \succsim L_1$. □

5 The Case of Multiple Assets

We have demonstrated that portfolio convexity, which involves only two assets by definition, can be obtained from uncertainty aversion intuitively. This result can also be leveraged to obtain the convexity when there are $N \geq 2$ assets. Since we can arbitrarily choose two assets $f, g \in X^S, f \succsim g$ that satisfy $\lambda f + (1-\lambda)g \succsim g$ for any $\lambda \in [0,1]$, this means preference $\succsim$ is convex everywhere in X^S. Consider a return matrix of $N \geq 2$ assets

$$\mathbf{F}=\begin{pmatrix}\mathbf{f}_1^1 & \dots & \mathbf{f}_1^N\\ \vdots & \mathbf{f}_s^n & \vdots\\ \mathbf{f}_S^1 & \dots & \mathbf{f}_S^N\end{pmatrix}$$

where its n-th column $\mathbf{f}^n, n=1,...,N$ denote asset n. Denote $\boldsymbol{\alpha}=(\boldsymbol{\alpha}_1,..,\boldsymbol{\alpha}_N)\in\mathbb{R}^N$ the portfolio vector. Then the budget set can be written as

$$B(\mathbf{F})=\{\mathbf{x}\in X^S:\mathbf{x}=\mathbf{F}\boldsymbol{\alpha}\}.$$

Since $B(\mathbf{F})$ is a convex subset of X^S, preference is convex on $B(\mathbf{F})$.

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Appendix A An Counter Example of Convexity under Uncertainty Aversion

Suppose there are two states of world with two set of priors $p_1 = [0.2, 0.8]$ and $p_2 = [0.6, 0.4]$. Denote $\mathbf{f}$ and $\mathbf{g}$ two assets that map states to monetary return where $f = [0, 4]$ and $g = [4, 0]$. Let $[\alpha, 1-\alpha]$ denote the portfolio choice, which are the proportions of wealth invested in f and g. Let x denote the final wealth of a portfolio, then $\mathbf{x} = \alpha\mathbf{f} + (1-\alpha)\mathbf{g}$. Let the utility function of monetary outcomes be convex $u(x) = x^2$. The Maxmin Expected Utility model postulates that an agent evaluates the portfolio x according two

$$MEU(\mathbf{x}) = min\Big(p_1 u(\mathbf{x}), p_2 u(\mathbf{x})\Big)$$

Consider the following $\mathbf{x}_1 = [2, 2], \mathbf{x}_2 = [4, 0], \mathbf{x}_3 = [3, 1]$ It can be easily verified that $MEU(\mathbf{x}_1) > MEU(\mathbf{x}_2)$ and $MEU(\mathbf{x}_2) > MEU(\mathbf{x}_3)$. This means $\mathbf{x}_1 \succ \mathbf{x}_2$ and $\mathbf{x}_2 \succ \mathbf{x}_3$ while $\mathbf{x}_3 = 0.5\mathbf{x}_1 + 0.5 * \mathbf{x}_2$. A contradiction of convexity.

Appendix B How Uncertainty Averse and Variance Averse implies Portfolio Convexity

Formally, Variance Aversion is defined as follows.

Definition 4. $\succsim$ on lotteries are *variance averse* if for two lotteries with the same mean, the lottery with a smaller variance is preferred.

Proposition 2. *If $\succsim$ is uncertainty averse and variance averse, then $\succsim$ is convex.*

Proof. Lemma 2 proves that the portfolio mix of two lotteries is preferred to the probability mix for all variance averse investor. Following similarly arguments in the proof of Proposition 1, we have that for $f \succsim g$, there is $\lambda f + (1-\lambda)g \succsim g$.

Lemma 2. If preferences over objective lotteries are variance averse, then portfolio mix of two lotteries is preferred to the probability mix of two lotteries.

Proof. Let f_s and g_s denote two lotteries. Let P and Q denote two independent random variables for which f_s and g_s describe their probability density distributions respectively.

Define a new random variable $R_1 := BP + (1-B)Q$, where B is a binary, independent random variable for which the probability that $B = 1$ is α and the probability that $B = 0$ is $1-\alpha$. Define another random variable $R_2 := \alpha P + (1-\alpha)Q$. Thus, the probability distribution for R_1 is the probability mixture of f_s and g_s, denoted by L_1. And the probability distribution for R_2 is the portfolio mixture of f_s and g_s, denoted by L_2.

Using Law of Total Variance , we have $Var(R_1) = E_B(Var_B(R_1|B)) + Var_B(E_B(R_1|B))$. Since $Var_B(E_B(R_1)) \geq 0$, we have $Var(R_1) \geq E(Var(R_1|B))$. Recall that $R_1 = P$ if $B = 1$and $R_1 = Q$ if $B = 0$, so $Var(R_1) \geq E_B(Var_B(R_1)) = \alpha Var(P) + (1-\alpha)Var(Q)$. Since $\alpha \in [0, 1]$, we have $\alpha \geq \alpha^2$ and $(1-\alpha) \geq (1-\alpha)^2$.

$$Var(R_1) > \alpha^2 Var(P) + (1-\alpha)^2 Var(Q)).$$

Since P and Q are independent, we have

$$E(R_2) = \alpha E(P) + (1-\alpha)E(Q)$$

and

$$Var(R_2) = \alpha^2 Var(P) + (1-\alpha)^2 Var(Q).$$

In summary, $E(R_1) = E(R_2)$ and $Var(R_1) \geq Var(R_2)$. By variance averse we have

$$\alpha f_s + (1-\alpha) g_s \succsim \alpha f_s \oplus (1-\alpha) g_s.$$

□

□