Public capital and childcare capital in the two sector growth model

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Abstract

Although numerous developed countries have implemented policies to raise the fertility rate, low fertility remains. This raises the question: What kind of impact will rapid population decline have on economic growth? This study aims to answer this question with a two-sector labor-augmented growth model using a dual period overlapping-generations model. We analyze public capital by classifying it into two types: labor-augmented general public capital in the final goods sector, as studied by Futagami et al. (1993); and public capital in the childcare sector, such as nursery schools. This study clearly points up the relationship between optimal policies combatting declining birthrates and increasing economic growth.

Keywords: Public capital - Childcare capital - Income tax - Economic growth

JEL classification: D91 - E62 - O41

1. Introduction

Why are developed countries' fertility rates steadily declining across the board? Various measures against declining birthrates are already being implemented. However, the rate of decline remains rapid. First, the basic research model for endogenous birth rate is Becker and Lewis (1973), in which having children is considered the same as having a private good. They treated children as ordinary goods, except that price rises with income. This theory is based on empirical data published by Seiver (1978) and Yamaguchi and Ferguson (1995) showing that the higher the income, the greater the human capital investment. Furthermore, a study that regarded having children as a capital good was also published by Groezen et al. (2003) for a society in which the social security system was actually mutual aid, for example, a pay-as-you-go public pension system. They pointed out that it would alter the optimal social allocation through the positive external effects of treating children as capital goods.

2. The model

We use a model that has an infinite-horizon and discrete time three-period overlapping generations

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model, as indicated by Diamond (1965). We assume that there are two types of production sectors: normal private goods and public childcare goods. Childcare goods can nullify the opportunity cost of raising children. Furthermore, we assume an international open market for childcare goods, private goods, labor, and capital. We assume childcare goods are in a domestically closed market with internally determined prices .

2.1 Individuals

We assume an infinite horizon discrete-time economy that starts at t=1. We use an overlapping generation model with two lifetime periods: work and retirement. We assume that there are L individuals. Each individual has one unit of time during the first lifetime period: time used for labor or raising children. In the first period, the individual earns wage income, consumes a part of that income, and saves the rest for consumption in the second period, when there is no wage earning. There is no social security, such as public pension or long-term care. Each individual's utility derives from consumption in both periods and having children. We assume that utility is derived based on the following formula:

$$u_t = log c_t + \rho log d_{t+1} + \varepsilon log n_t, \tag{1}$$

where $\rho \in (0,1)$ indicates the subjective discount factor for consumption in the second period, and $\varepsilon > 0$ indicates the preference for having children. Furthermore, c and d represent the consumption levels in the first and second periods, respectively; and n is the number of children. Individuals face the following first-period budget constraints:

$$w_t (1-\tau) = c_t + (p^c - h_t)n_t + s_t, \tag{2}$$

where w_t indicates the wage rate, and $\tau \in (0,1)$ is the income tax rate. Individuals bear the cost $z > 0$ for raising a child. The government provides assistance, represented by $h_t \in (0,1)$. Individuals face the following second-period budget constraints:

$$d_{t+1} = s_t r_{t+1}(1-\tau), \tag{3}$$

where d indicates second-period consumption and r is the rental rate for private capital. It is assumed that labor and capital income are taxed at the same rate, τ. By combining (2) and (3), the budget constraint for the two periods is :

$$w_t\,(1-\tau) = c_t + n_t(p^c - h_t) + \frac{d_{t+1}}{r_{t+1}(1-\tau)}, \tag{4}$$

We derive the following conditions from the utility maximization problem:

$$\frac{1}{c_t} = \frac{\rho r_{t+1}(1-\tau)}{d_{t+1}} \tag{5}$$

$$\frac{1}{c_t} = \frac{\varepsilon}{n_t(p^c - h_t)} \tag{6}$$

$$\frac{\rho r_{t+1}(1-\tau)}{d_t} = \frac{\varepsilon}{n_t(p^c - h_t)} \tag{7}$$

From these conditions, and the intertemporal budget constraint, we derive the following:

$$c_t^* = \frac{w_t(1-\tau)}{(1+\varepsilon+\rho)} \tag{8}$$

$$d_{t+1}^* = \frac{\rho w_t r_{t+1}(1-\tau)^2}{(1+\varepsilon+\rho)} \tag{9}$$

$$n_t^* = \frac{\varepsilon(1-\tau)w_t}{(p^c - h_t)(1+\varepsilon+\rho)} \tag{10}$$

$$s_t^* = \frac{\rho(1-\tau)w_t}{(1+\varepsilon+\rho)} \tag{11}$$

2.2 Productions

2.2.1 Firms in a final goods sector

The economy has two production sectors: one produces final goods and the other produces nursery goods; see Chang (1970). The first sector is the normal goods production sector. The second sector is the childcare production sector, which includes nursery schools, kindergartens, and the like. The production function in the normal sector uses a labor augmented model, as in Romer (1986); labor productivity, that is, the technology of production, is indicated by public capital stock, as per Futagami, Morita, and Shibata (1993). We assume that labor productivity will be augmented by the existence of public capital stock, that is, public capital boosts the efficiency of labor. In the childcare sector, the input factor for obtaining the time related to declining opportunity cost for child rearing is only childcare labor. The production technology of each firm is obtained using the increasing-returns-to-scale function and we assume that private goods will be numeraire. The production function of the final goods sectors is indicated as follows:

$$Y_{i,t}^{p} = f\left(K_{it}^{p}, A_{t}^{p} L_{it}^{p}\right) = \left(K_{it}^{p}\right)^{\alpha} (A_{t}^{p} L_{it}^{p})^{1-\alpha}, \tag{12}$$

where, because of a competitive labor market, the wage rate of both sectors will be the same value in equilibrium. Here, technology A is indicated by public capital stock per capita as follows:

$$A_{t}^{p} = \frac{G_{t}^{p}}{L_{t}^{p}} \tag{13}$$

Under the profit-maximizing condition, the wage rate becomes equal to the productivity of labor in each sector. First, we show the wage rate in the final goods sector.

$$w_{t}^{p} = (1-\alpha)\left(K_{it}^{p}\right)^{\alpha}\left(A_{t}^{p} L_{it}^{p}\right)^{-\alpha} A_{t}^{p} = (1-\alpha)\left(A_{t}^{p}\right)^{1-\alpha}\left(\frac{K_{it}^{p}}{L_{it}^{p}}\right)^{\alpha} = (1-\alpha)\left(\frac{G_{t}^{p}}{L_{t}^{p}}\right)^{1-\alpha}\left(\frac{K_{it}^{p}}{L_{it}^{p}}\right)^{\alpha} \tag{14}$$

$$w_{t}^{p} = (1-\alpha)\left(\frac{G_{t}^{p}}{L_{t}^{p}}\right)\left(\frac{L_{t}^{p}}{G_{t}^{p}}\right)^{\alpha}\left(\frac{K_{t}^{p}}{L_{t}^{p}}\right)^{\alpha} = (1-\alpha)\left(\frac{G_{t}^{p}}{L_{t}^{p}}\right)\left(x_{t}^{p}\right)^{\alpha}, \tag{15}$$

where w_{t}^{p} is the wage rate in the final good sector in period t. The market is perfectly competitive, and we can say $\sum_{i=0}^{\mu} L_{it}^{p} = L_{t}^{p} = \mu L_{t}$ and $\sum_{i=0} K_{it}^{p} = K_{t}^{p}$ where L_{t} and K_{t}^{p} denote the total labor supply and the aggregate stock of private capital. The total labor force in the childcare support sector is $\sum_{i=0}^{1-\mu} L_{it}^{c} = L_{t}^{c} = (1-\mu)L_{t}$. $\mu_{t} \in [0,1]$ indicates the share of labor in a private final goods sector, and x_{t} indicates the relative value of capital, K_{t}^{p}/G_{t}^{p}, the aggregate stock of private capital to the aggregate stock of public capital Because all workers are identical and the labor market is perfectly competitive, both wage rates become the same in the two sectors. Next, the rental rate is given by the following equation:

$$r_{t} = \alpha\left(K_{t}^{p}\right)^{\alpha-1}(A_{t}^{p} L_{it}^{p})^{1-\alpha} = \alpha\left(\frac{G_{t}^{p}}{K_{t}^{p}}\right)^{1-\alpha} = \alpha\left(x_{t}^{p}\right)^{\alpha-1}, \tag{16}$$

2.2.2 Firm in a childcare sector

Next, we set a production function of public childcare support sector as follows:

$$H_{i,t}^{c} = g(A_t^c L_{it}) = A_t^c L_{it}^c, \tag{17}$$

where the assumption that the sector employs only childcare labor is made for simplicity and H^c indicates a facility such as a nursery school operated by a company or corporation that providing services reduce opportunity costs for childcare in households. An the aggregate labor force in the childcare sector is indicated as $\sum_{i=\mu}^{1-\mu} L_{it}^c = L_t^c = (1-\mu)L_t$.

$$H_t^c = g(G_t^c, L_{it}^c) = A_t^c L_t^c, \tag{18}$$

$$\pi^c = A_t^c L_t^c - w_t^c L_t^c, \tag{19}$$

where the inputs of firms show the labor like as the childminder and the public capital such as nursery school building and it is assumed that the government will bear all of these costs.

$$\frac{\partial \pi^c}{\partial L_t^c} = A_t^c - w_t^c = 0 \tag{20}$$

$$w_t^c = A_t^c = \frac{G_t^c}{L_t^c} \tag{21}$$

As we will see later, the labor market is perfectly competitive, so wage rates in both sectors are equal. Therefore, the working population in the childcare sector is shown by the following equation.

$$L_t^c = \frac{\emptyset_t L_t^p}{(1-\alpha)\left(x_t^p\right)^{\alpha}} \tag{22}$$

2.2.3 Equilibrium in the labor market in a both sector

Since the labor market assumes perfect competition, the wage rates in the two sectors will be equal. Therefore, from an equation (15) and (22), which show the wage rate of each sector, the relationship between the number of worker in each sector is shown as follows:

$$L_t^p = \frac{L_t^c (1-\alpha)\left(x_t^p\right)^{\alpha}}{\emptyset_t} \tag{23}$$

$$\mu_t = \frac{(1-\alpha)\left(x_t^p\right)^{\alpha}}{\emptyset_t + (1-\alpha)\left(x_t^p\right)^{\alpha}}, \tag{24}$$

where $\mu_t > 0$ indicates the ratio of the number of laborers in the final goods sector to the total working population, that is, $\frac{L_t^p}{L_t} > 0$. In other words, the ratio of the number of labor in the childcare sector will be represented by $(1-\mu_t) > 0$, that is $\frac{L_t^c}{L_t} > 0$. From the above, the equilibrium conditional expression in the labor market is as follows:

$$L_t^p + L_t^c = L_t \tag{25}$$

2.3 Government

Government taxes labor and capital income; its tax rates are the same as indicated by $\tau \in (0,1)$ and this tax revenue is spent on public capital investment $E_t^p > 0$ and childcare support $E_t^c > 0$, $E_t^{Lc} > 0$. Here, government expenditure is considered for two locations only. Therefore, the governmental budget constraint in period t is shown by the following equation:

$$E_t^p + E_t^c + E_t^{Lc} = \tau Y_t, \tag{26}$$

where the right-hand side of the equation is exhibited as the total income tax revenue. Next, we examine each expenditure. First, public capital investment is shown as a difference equation between public capital stock in period t+1 and period t. And the $\varphi \in [0,1]$ means the share of public capital investment on total tax revenue.

$$E_t^p = G_{t+1}^p - G_t^p = \varphi\tau Y_t, \tag{27}$$

Second, the value of the public capital investment in the childcare sector E_t^c is indicated as follows:

$$E_t^c = G_{t+1}^c - G_t^c = \gamma\tau Y_t, \tag{28}$$

And finally, the labor cost in the childcare sector, E_t^{Lc} is shown as below:

$$E_t^{Lc} = w_t^c L_t^c = G_t^c = (1-\varphi-\gamma)\tau Y_t \tag{29}$$

We derive per capita childcare support h_t as follows:

$$h_t = \frac{(1-\varphi)\tau Y_t}{n_t L_t} = \frac{\mu_t \tau (1-\varphi)(p^c - h_t)(1+\varepsilon+\rho)\left[\left(x_t^p\right)^\alpha + \emptyset_t\right]}{\varepsilon(1-\tau)(1-\alpha)\left(x_t^p\right)^\alpha}\ , \tag{30}$$

In the above equation, since h_t exists on both sides, it is collectively shown by the following equation:

$$h_t = \frac{p^c \mu_t \tau (1-\varphi)(1+\varepsilon+\rho)\left[\left(x_t^p\right)^\alpha + \emptyset_t\right]}{\varepsilon\left\{(1-\tau)(1-\alpha)\left(x_t^p\right)^\alpha + \mu_t \tau (1-\varphi)(1+\varepsilon+\rho)\left[\left(x_t^p\right)^\alpha + \emptyset_t\right]\right\}}\ , \tag{31}$$

where $\emptyset_t = \frac{G_t^c}{G_t^p} > 0$ indicates the relative value between public capital in a childcare sector and final goods sector and indicates that $(1-\varphi) \in [0,1]$ indicates the share of per capita public capital in the childcare support sector. By three-side equilibrium, GDP on the production side is given as follows:

$$Y_t = Y_t^p + H_t^c = \left(K_t^p\right)^\alpha \left(G_t^p\right)^{1-\alpha} + A_t^c L_t^c = G_t^p\left[\left(x_t^p\right)^\alpha + \left(\frac{G_t^c}{L_t^c}\right) L_t^c\right] = G_t^p\left[\left(x_t^p\right)^\alpha + \emptyset_t\right], \tag{32}$$

where Y_t indicates the aggregate output in both sectors. Here, L_t^p is the aggregate labor in the final goods sector, L_t^c is the aggregate labor in the childcare support sector and $K_{t,}$ is the aggregate private capital in period t. GDP can be derived from a distribution perspective as follows:

$$Y_t = w_t^p L_t^p + K_t r_t + w_t^c L_t^c = w_t^p L_t^p + K_t r_t + \frac{G_t^c}{L_t^c} L_t^c = G_t^p\left[\left(x_t^p\right)^\alpha + \emptyset_t\right]\ , \tag{33}$$

Therefore, it is clear from Equations (32) and (33) that GDP in terms of production and distribution has the same value. We can derive the difference equation for public capital in the final good sector using Equations (27) and (32).

$$G_{t+1}^p - G_t^p = \varphi \tau G_t^p\left[\left(x_t^p\right)^\alpha + \emptyset_t\right] \tag{34}$$

3. Dynamics and Stability

By dividing both sides of Equation (34) by G_t^p, we can derive the growth of public capital.

$$g_G^p = \frac{G_{t+1}^p}{G_t^p} = \varphi\tau\left[\left(x_t^p\right)^\alpha + \emptyset_t\right] + 1 \tag{35}$$

Next, we derive a dynamic equation of the private capital stock using the equilibrium condition in the capital market as follows: There are three markets in the economy, and we consider only the capital market using Walras' law. The equilibrium condition is as follows:

$$s_t L_t = K_{t+1}^p \tag{36}$$

We substitute the optimal savings (11) for the equilibrium condition (36) and substitute the wage rate (15) for there. These allow us to rewrite condition (36) as follows:

$$K_{t+1}^p = \frac{\rho(1-\tau)w_t}{(1+\varepsilon+\rho)} L_t \tag{37}$$

Equation (38) can be obtained by dividing both sides of Equation (37) by K_t^p:

$$g_t^K = \frac{K_{t+1}^p}{K_t^p} = \frac{\rho(1-\tau)(1-\alpha)\left(x_t^p\right)^{\alpha-1}}{(1+\varepsilon+\rho)\mu_t} \tag{38}$$

The growth of x is indicated by the following equation, which combines the private capital dynamic equation (38) and the public capital dynamic equation (35).

$$g_t^x = \frac{x_{t+1}^p}{x_t^p} = \frac{\frac{K_{t+1}^p}{K_t^p}}{\frac{G_{t+1}^p}{G_t^p}} = \frac{\rho(1-\tau)(1-\alpha)\left(x_t^p\right)^{\alpha-1}}{\mu_t(1+\varepsilon+\rho)\left\{\varphi\tau\left[\left(x_t^p\right)^{\alpha}+\emptyset_t\right]+1\right\}} \tag{39}$$

In the steady state, by setting Equation (39) to one, x^{p*}, that is, the relative value of capital in the steady state is determined by the following equation:

$$\mu^*(1+\varepsilon+\rho)\varphi\tau(x^{p*})^{\alpha} + \mu^*(1+\varepsilon+\rho)\varphi\tau\emptyset^* + \mu^*(1+\varepsilon+\rho) = \rho(1-\tau)(1-\alpha)(x^{p*})^{\alpha-1}, \tag{40}$$

where, we see the effect of increasing the share of public capital investment on total tax revenue for the relative value of capital in a steady state, x^{p*}. Next, we turn to the growth of the ratio of public capital in the childcare and final goods sectors. The dynamics of the relative public capital values in the childcare and final goods sectors are shown in the following equation:

$$g_t^{\emptyset} = \frac{\emptyset_{t+1}}{\emptyset_t} = \frac{\frac{G_{t+1}^c}{G_{t+1}^p}}{\frac{G_t^c}{G_t^p}} = \frac{\frac{G_{t+1}^c}{G_t^c}}{\frac{G_{t+1}^p}{G_t^p}} = \frac{\frac{\gamma\tau}{\emptyset_t}\left[\left(x_t^p\right)^\alpha + \emptyset_t\right] + 1}{\varphi\tau\left[\left(x_t^p\right)^\alpha + \emptyset_t\right] + 1} = \frac{\gamma\tau\left[\left(x_t^p\right)^\alpha + \emptyset_t\right] + \emptyset_t}{\emptyset_t\left\{\varphi\tau\left[\left(x_t^p\right)^\alpha + \emptyset_t\right] + 1\right\}} \ , \quad (41)$$

where the dynamic equation in the childcare sector of the numerator is given by the following equation, using Equation (28).

$$g_G^c = \frac{G_{t+1}^c}{G_t^c} = \frac{\gamma\tau}{\emptyset_t}\left[\left(x_t^p\right)^\alpha + \emptyset_t\right] + 1 \ , \quad (42)$$

To show the relative size of capital in the two sectors in the steady state, we set equation (41) to one, obtaining the following equation:

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$$\gamma\tau[(x^{p*})^\alpha + \emptyset^*] + \emptyset^* = \emptyset^*\{\varphi\tau[(x^{p*})^\alpha + \emptyset^*] + 1\} \quad (43)$$

In the steady state, by setting an equation (41) to one, $\emptyset^*$, that is the relative value of public capital in the childcare support sector, and the public capital sector in the steady state is determined.

$$(\emptyset^*)^2 + \left[(x^{p*})^\alpha - \frac{\gamma}{\varphi}\right]\emptyset^* - \frac{\gamma}{\varphi}(x^{p*})^\alpha = 0 \quad (44)$$

$$\left(\emptyset^* - \frac{\gamma}{\varphi}\right)[\emptyset^* + (x^{p*})^\alpha] = 0 \quad (45)$$

$$\emptyset^* = \frac{\gamma}{\varphi} \ , -(x^{p*})^\alpha, \quad (46)$$

Because $\emptyset^*$ takes the value larger than zero, the only correct answer will be $\frac{\gamma}{\varphi} > 0$. Next, assuming that the ratio of the number of workers in the final sector to the total number of workers is $\mu > 0$.

$$\mu^* = \frac{(1-\alpha)(x^{p*})^\alpha}{\emptyset^* + (1-\alpha)(x^{p*})^\alpha} = \frac{\varphi(1-\alpha)(x^{p*})^\alpha}{\gamma + \varphi(1-\alpha)(x^{p*})^\alpha} \quad (47)$$

where μ^* indicates the ratio in the steady state and substituting an equation (47) and $\emptyset^* = \frac{\gamma}{\varphi}$ for an equation (40).

$$\tau\varphi^3(1-\alpha)(1+\varepsilon+\rho)(x^{p*})^{\alpha}+\tau\gamma\varphi^2(1-\alpha)(1+\varepsilon+\rho)+\varphi(1-\alpha)(1+\varepsilon+\rho)$$

$$=\gamma\rho(1-\tau)(1-\alpha)(x^{p*})^{-1}+\varphi\rho(1-\tau)(1-\alpha)^2(x^{p*})^{\alpha-1} \quad (48)$$

We will see the effect of increase the share of public capital investment on total tax revenue for the relative value of capitals in the steady state, x^{p*}as follows:

$$\frac{\partial(x^{p*})}{\partial\varphi}=-\frac{\{[\tau\varphi^2(1+\varepsilon+\rho)-\rho(1-\tau)(1-\alpha)(3x^{p*})^{-1}]3(x^{p*})^{\alpha}+2\tau\gamma\varphi(1+\varepsilon+\rho)+2\varphi(1+\varepsilon+\rho)\}}{[\alpha\varphi^3(1+\varepsilon+\rho)(x^{p*})^{\alpha-1}dx^{p*}+\gamma\rho(1-\tau)(x^{p*})^{-2}dx^{p*}+\varphi\rho(1-\tau)(1-\alpha)^2(x^{p*})^{\alpha-2}]}<0, \quad (49)$$

Next, we analyze the stability of the two dynamic equations for x^p and $\emptyset$. These dynamic equations are represented as follows:

$$x_{t+1}^p=f\left(x_t^p,\emptyset_t\right)=\frac{\rho(1-\tau)(1-\alpha)\left(x_t^p\right)^{\alpha}}{\varphi\tau\mu_t(1+\varepsilon+\rho)\left[\left(x_t^p\right)^{\alpha}+\emptyset_t\right]}\ , \quad (50)$$

$$\emptyset_{t+1}=g\left(x_t^p,\emptyset_t\right)=\frac{\gamma\tau\left[\left(x_t^p\right)^{\alpha}+\emptyset_t\right]+\emptyset_t}{\left\{\varphi\tau\left[\left(x_t^p\right)^{\alpha}+\emptyset_t\right]+1\right\}}\ , \quad (51)$$

$$x_{t+1}^p=\left[\frac{\rho(1-\tau)(1-\alpha)(x^{p*})^{\alpha-1}-\varphi\tau\mu_t(1+\varepsilon+\rho)\emptyset^*}{\varphi\tau\mu_t(1+\varepsilon+\rho)}\right]^{\frac{1}{\alpha}} \quad (52)$$

$$+\left\{\left[\frac{\alpha\rho(1-\tau)(1-\alpha)}{\varphi\tau\mu_t(1+\varepsilon+\rho)}\right]\frac{[(x_t^p)^{\alpha}+\emptyset_t]\alpha(x_t^p)^{\alpha-1}-\alpha(x_t^p)^{2\alpha}}{\left[\left(x_t^p\right)^{\alpha}+\emptyset_t\right]^2\left(x_t^p\right)}\right\}(x_t^p-x^{p*})+\left\{\left[\frac{\alpha\rho(1-\tau)(1-\alpha)}{\varphi\tau\mu_t(1+\varepsilon+\rho)}\right]\frac{(x_t^p)^{\alpha}}{\left[\left(x_t^p\right)^{\alpha}+\emptyset_t\right]}\right\}(\emptyset_t-\emptyset^*)$$

$$\emptyset_{t+1}=\frac{\gamma}{\varphi}+\frac{\{\varphi\tau[(x_t^p)^{\alpha}+\emptyset_t]+1\}(\gamma\tau+1)-\{\gamma\tau[(x_t^p)^{\alpha}+\emptyset_t]+\emptyset_t\}\varphi\tau}{\left\{\varphi\tau\left[\left(x_t^p\right)^{\alpha}+\emptyset_t\right]+1\right\}^2}(x_t^p-x^{p*}) \quad (53)$$

$$+\left\{\frac{\{\varphi\tau[(x_t^p)^{\alpha}+\emptyset_t]+1\}(\gamma\tau+1)-\varphi\tau\{\gamma\tau[(x_t^p)^{\alpha}+\emptyset_t]+\emptyset_t\}}{\left\{\varphi\tau\left[\left(x_t^p\right)^{\alpha}+\emptyset_t\right]+1\right\}^2}\right\}(\emptyset_t-\emptyset^*)$$

$$\begin{bmatrix} x_{t+1}^{p} - x^{p*} \\ \emptyset_{t+1} - \emptyset^{*} \end{bmatrix}$$

$$= \begin{bmatrix} \left\{ \left[\frac{\alpha\rho(1-\tau)(1-\alpha)}{\varphi\tau\mu_t(1+\varepsilon+\rho)} \right] \frac{[(x_t^p)^\alpha + \emptyset_t]\alpha(x_t^p)^{\alpha-1} - \alpha(x_t^p)^{2\alpha}}{[(x_t^p)^\alpha + \emptyset_t]^2 (x_t^p)} \right\} & \left\{ \left[\frac{\alpha\rho(1-\tau)(1-\alpha)}{\varphi\tau\mu_t(1+\varepsilon+\rho)} \right] \frac{(x_t^p)^\alpha}{[(x_t^p)^\alpha + \emptyset_t]} \right\} \\ \left\{ \frac{\{\varphi\tau[(x_t^p)^\alpha + \emptyset_t] + 1\}(\gamma\tau + 1) - \{\gamma\tau[(x_t^p)^\alpha + \emptyset_t] + \emptyset_t\}\varphi\tau}{\{\varphi\tau[(x_t^p)^\alpha + \emptyset_t] + 1\}^2} \right\} & \left\{ \frac{\{\varphi\tau[(x_t^p)^\alpha + \emptyset_t] + 1\}(\gamma\tau + 1) - \varphi\tau\{\gamma\tau[(x_t^p)^\alpha + \emptyset_t] + \emptyset_t\}}{\{\varphi\tau[(x_t^p)^\alpha + \emptyset_t] + 1\}^2} \right\} \end{bmatrix} \begin{bmatrix} (x_t^p - x^{p*}) \\ (\emptyset_t - \emptyset^{*}) \end{bmatrix} \tag{54}$$

To simplify the notation, we rewrite the above equation to the first order, with a two variable difference, as shown below:

$$\begin{bmatrix} \widehat{x_{t+1}^{p}} \\ \widehat{\emptyset_{t+1}} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \widehat{x_{t}^{p}} \\ \widehat{\emptyset_{t}} \end{bmatrix} = A \begin{bmatrix} \widehat{x_{t}^{p}} \\ \widehat{\emptyset_{t}} \end{bmatrix} \tag{55}$$

We assume that the eigenvalues of the determinant of Equation (54) are λ_i. Next, we find the solution to the characteristic equation.

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \ , \tag{56}$$

Then, by expanding the above equation, the following equation can be obtained.

$$F(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc) = 0, \tag{57}$$

where the above equation shows the characteristic equation. If the trace and determinant in Equation (56) are defined as T=$a+d$ and D $= ad - bc$, then Equation (55) can be rewritten as follows:

$$F(\lambda) = \lambda^2 - T\lambda + D = 0, \tag{58}$$

$$A = \begin{bmatrix} \left\{ \left[\frac{\alpha\rho(1-\tau)(1-\alpha)}{\varphi\tau\mu_t E} \right] \frac{[(x_t^p)^\alpha + \emptyset_t]\alpha(x_t^p)^{\alpha-1} - \alpha(x_t^p)^{2\alpha}}{[(x_t^p)^\alpha + \emptyset_t]^2 (x_t^p)} \right\} & \left\{ \left[\frac{\alpha\rho(1-\tau)(1-\alpha)}{\varphi\tau\mu_t E} \right] \frac{(x_t^p)^\alpha}{[(x_t^p)^\alpha + \emptyset_t]} \right\} \\ \left\{ \frac{\{\varphi\tau[(x_t^p)^\alpha + \emptyset_t] + 1\}(\gamma\tau + 1) - \{\gamma\tau[(x_t^p)^\alpha + \emptyset_t] + \emptyset_t\}\varphi\tau}{\{\varphi\tau[(x_t^p)^\alpha + \emptyset_t] + 1\}^2} \right\} & \left\{ \frac{\{\varphi\tau[(x_t^p)^\alpha + \emptyset_t] + 1\}(\gamma\tau + 1) - \varphi\tau\{\gamma\tau[(x_t^p)^\alpha + \emptyset_t] + \emptyset_t\}}{\{\varphi\tau[(x_t^p)^\alpha + \emptyset_t] + 1\}^2} \right\} \end{bmatrix}, \tag{59}$$

where E and A indicate $(1+\varepsilon+\rho)$ and $\left[\left(x_t^p\right)^\alpha+\emptyset_t\right]$ for simplicity.

$$\mathrm{T}=\frac{\alpha^2\rho(1-\tau)(1-\alpha)(x_t^p)^{\alpha-1}\{(x_t^p)^\alpha[(x_t^p)^\alpha+\emptyset_t]^{-1}-(x_t^p)^{-1}\}\{\varphi\tau[(x_t^p)^\alpha+\emptyset_t]+1\}}{\varphi\tau\mu_t E\left[\left(x_t^p\right)^\alpha+\emptyset_t\right]\left\{\varphi\tau\left[\left(x_t^p\right)^\alpha+\emptyset_t\right]+1\right\}}$$

$$-\frac{[(x_t^p)^\alpha+\emptyset_t]\{1+\gamma\tau-\{\varphi\gamma\tau^2(x_t^p)^\alpha+\varphi\gamma\tau^2\emptyset_t+\varphi\tau\emptyset_t\}\{\varphi\tau[(x_t^p)^\alpha+\emptyset_t]+1\}^{-1}\}}{\left[\left(x_t^p\right)^\alpha+\emptyset_t\right]\left\{\varphi\tau\left[\left(x_t^p\right)^\alpha+\emptyset_t\right]+1\right\}} \tag{60}$$

$$D=\frac{\langle[\alpha\rho(1-\tau)(1-\alpha)]\{\alpha[(x_t^p)^\alpha+\emptyset_t](x_t^p)^{\alpha-2}-\alpha(x_t^p)^{2\alpha-1}\}\rangle\langle\{\varphi\tau[(x_t^p)^\alpha+\emptyset_t]+1\}(\gamma\tau+1)-\varphi\tau\{\gamma\tau[(x_t^p)^\alpha+\emptyset_t]+\emptyset_t\}\rangle}{\varphi\tau\mu_t E\left[\left(x_t^p\right)^\alpha+\emptyset_t\right]^2\left\{\varphi\tau\left[\left(x_t^p\right)^\alpha+\emptyset_t\right]+1\right\}^2}$$

$$+\left\{\frac{\alpha\rho(1-\tau)(1-\alpha)(x_t^p)^\alpha[(x_t^p)^\alpha+\emptyset_t]\{\emptyset_t-\gamma\tau-\varphi\tau(x_t^p)^\alpha-\varphi\tau\emptyset_t-1\}}{\varphi\tau\mu_t E\left[\left(x_t^p\right)^\alpha+\emptyset_t\right]^2\left\{\varphi\tau\left[\left(x_t^p\right)^\alpha+\emptyset_t\right]+1\right\}^2}\right\} \tag{61}$$

$$\mathrm{T}=-\left\{\frac{[E\alpha(x_t^p)^{\alpha-1}-\alpha(x_t^p)^{2\alpha}]\alpha\rho(1-\tau)(1-\alpha)}{\varphi\tau\mu_t AE^2\left(x_t^p\right)}\right\}-\left\{\frac{\{\varphi\tau E+1\}(\gamma\tau+1)-\varphi\tau\{\gamma\tau E+\emptyset_t\}}{\{\varphi\tau E+1\}^2}\right\} \tag{62}$$

$$\mathrm{D}=\left\{-\frac{\left[E\alpha\left(x_t^p\right)^{\alpha-1}-\alpha\left(x_t^p\right)^{2\alpha}\right]\alpha\rho(1-\tau)(1-\alpha)}{\varphi\tau\mu_t AE^2\left(x_t^p\right)}\right\}\left\{-\frac{\{\varphi\tau E+1\}(\gamma\tau+1)-\{\gamma\tau E+\emptyset_t\}\varphi\tau}{\{\varphi\tau E+1\}^2}\right\} \tag{63}$$

Therefore, we can derive the two eigenvalues, λ_1 and λ_2.

$$\lambda_1=-\left\{\frac{[E\alpha(x_t^p)^{\alpha-1}-\alpha(x_t^p)^{2\alpha}]\alpha\rho(1-\tau)(1-\alpha)}{\varphi\tau\mu_t AE^2\left(x_t^p\right)}\right\} \tag{64}$$

$$\lambda_2=-\left\{\frac{\{\varphi\tau E+1\}(\gamma\tau+1)-\varphi\tau\{\gamma\tau E+\emptyset_t\}}{\{\varphi\tau E+1\}^2}\right\} \tag{65}$$

The parameters in Equations (63) and (64) are quantified concretely as follows:

$$(\alpha,\varepsilon,\rho,\tau,x,z,\varphi,\gamma)=(0.5,0.7,0.7,0.3,3,0.68,0.17)\ , \tag{66}$$

$$\lambda_1=0.54,\qquad \lambda_2=0.029\ , \tag{67}$$

where the result means that there are innumerable paths that converge to the steady state $x^{p*},\emptyset^*$ for any given x_t^p, $\emptyset_t$. Thus, there is a sink-point equilibrium.

Proposition 1.

The economy in this model always converges to a steady state no matter where the initial point is assumed to be. In other words, there is an equilibrium solution of the sink point.

4. Discussion

First, consider the case in which the share of public capital investment is increased in the steady state. The number of children in the steady state is indicated as follows:

$$n_t^* = \frac{C}{D} = \frac{\varepsilon^2(1-\tau)(1-\alpha)A_t^p(x^{p*})^\alpha\{(1-\tau)\gamma+\varphi+\tau(1-\varphi)(1+\varepsilon+\rho)[\varphi(x^{p*})^\alpha+\gamma]\}}{p^c(1+\varepsilon+\rho)\{\varepsilon(1-\tau)\gamma+\varphi-\tau(1-\varepsilon)(1-\varphi)(1+\varepsilon+\rho)[\varphi(x^{p*})^\alpha+\gamma]\}} \quad (68)$$

To analyze the effect of increasing the share of public capital investment on the number of children, we first perform total differentiation on x^{p*} and φ that satisfy Equation (34) as follows:

$$\frac{\partial(x^{p*})}{\partial\varphi} = -\frac{\{[\tau\varphi^2(1+\varepsilon+\rho)-\rho(1-\tau)(1-\alpha)(3x^{p*})^{-1}]3(x^{p*})^\alpha+2\tau\gamma\varphi(1+\varepsilon+\rho)+2\varphi(1+\varepsilon+\rho)\}}{[\alpha\varphi^3(1+\varepsilon+\rho)(x^{p*})^{\alpha-1}+\gamma\rho(1-\tau)(x^{p*})^{-2}+\varphi\rho(1-\tau)(1-\alpha)^2(x^{p*})^{\alpha-2}]} \quad (69)$$

$$\frac{\partial n_t^*}{\partial\varphi} = \frac{C'D-CD'}{D^2} < 0 \quad (70)$$

$$C' = \alpha\varepsilon^2(1-\tau)(1-\alpha)A_t^p(x^{p*})^{\alpha-1}\frac{\partial(x^{p*})}{\partial\varphi}\left\{(1-\tau)\gamma+\varphi+\varphi\tau(1-\varphi)(1+\varepsilon+\rho)\left[(x^{p*})^\alpha+\frac{\gamma}{\varphi}\right]\right\}+ \quad (71)$$

$$\varepsilon^2(1-\tau)(1-\alpha)A_t^p(x^{p*})^\alpha\left\{1+(\tau-2\tau\varphi)(1+\varepsilon+\rho)\left[(x^{p*})^\alpha+\frac{\gamma}{\varphi}\right]+\varphi\tau(1-\varphi)(1+\varepsilon+\rho)\left[\alpha(x^{p*})^{\alpha-1}\frac{\partial(x^{p*})}{\partial\varphi}-\frac{\gamma}{\varphi^2}\right]\right\}$$

$$C' = -\left\{\frac{\{[\tau\varphi^2(1+\varepsilon+\rho)-\rho(1-\tau)(1-\alpha)(3x^{p*})^{-1}]3(x^{p*})^\alpha+2\tau\gamma\varphi(1+\varepsilon+\rho)+2\varphi(1+\varepsilon+\rho)\}}{[\alpha\varphi^3(1+\varepsilon+\rho)(x^{p*})^{\alpha-1}+\gamma\rho(1-\tau)(x^{p*})^{-2}+\varphi\rho(1-\tau)(1-\alpha)^2(x^{p*})^{\alpha-2}]}\right\}$$

$$\times\left\{[\alpha\varepsilon^2(1-\tau)(1-\alpha)A_t^p(x^{p*})^{\alpha-1}]\left\{(1-\tau)\gamma+\varphi+\varphi\tau(1-\varphi)(1+\varepsilon+\rho)\left[(x^{p*})^\alpha+\frac{\gamma}{\varphi}\right]\right\}\right.$$

$$\left.+\varphi\tau(1-\varphi)(1+\varepsilon+\rho)\alpha(x^{p*})^{\alpha-1}\right\} < 0 \quad (72)$$

$$D' = p^c(1+\varepsilon+\rho)\left\{1-\tau(1-2\varphi)(1-\varepsilon)(1+\varepsilon+\rho)\left[(x^{p*})^\alpha+\frac{\gamma}{\varphi}\right]-\tau\varphi\alpha(x^{p*})^{\alpha-1}(1-\varepsilon)(1-\varphi)(1+\varepsilon+\rho)\frac{\partial(x^{p*})}{\partial\varphi}\right.$$

$$\left.+\frac{1}{\varphi^2}\tau\gamma(1-\varepsilon)(1-\varphi)(1+\varepsilon+\rho)\right\} > 0 \quad (73)$$

Proposition 2.

An increase in public capital investment reduces the number of children in a steady state.

Next, we analyze the effect of an increasing share of public capital investment on growth using Equation (42).

$$g_G^c = \frac{G_{t+1}^c}{G_t^c} = \frac{\gamma\tau}{\emptyset_t}[(x^{p*})^\alpha + \emptyset_t] + 1 = \tau[\varphi(x^{p*})^\alpha + \gamma] + 1 \tag{74}$$

$$\frac{\partial g_G^c}{\partial\varphi} = \tau\left[(x^{p*})^\alpha + \frac{\gamma}{\varphi}\right] + \tau\left[\alpha(x^{p*})^{\alpha-1}\frac{\partial(x^{p*})}{\partial\varphi} - \frac{\gamma}{\varphi^2}\right] = -\tau\left\{\frac{\gamma(1-\varphi) + \varphi^2(x^{p*})^\alpha}{\varphi^2}\right\} \tag{75}$$

$$-\tau\left\{\frac{\alpha(x^{p*})^{\alpha-1}\{[\tau\varphi^2(1+\varepsilon+\rho) - \rho(1-\tau)(1-\alpha)(3x^{p*})^{-1}]3(x^{p*})^\alpha + 2\tau\gamma\varphi(1+\varepsilon+\rho) + 2\varphi(1+\varepsilon+\rho)\}}{[\alpha\varphi^3(1+\varepsilon+\rho)(x^{p*})^{\alpha-1} + \gamma\rho(1-\tau)(x^{p*})^{-2} + \varphi\rho(1-\tau)(1-\alpha)^2(x^{p*})^{\alpha-2}]}\right\} < 0$$

Proposition 3.

The phenomenon of rising share of public capital investment results in a declining growth rate.

Next, the per capita growth of private capital, public capital, and GDP is indicated as follows:

$$g = \frac{K_{t+1}^p/L_{t+1}^p}{K_t^p/L_t^p} = \frac{G_{t+1}^p/L_{t+1}^p}{G_t^p/L_t^p} = \frac{G_{t+1}^c/L_{t+1}^c}{G_t^c/L_t^c} = \frac{Y_{t+1}/L_{t+1}}{Y_t/L_t} \tag{76}$$

We can rewrite above equation as follows:

$$g_{per}^* = \frac{K_{t+1}^p/K_t^p}{L_{t+1}^p/L_t^p} = \frac{G_{t+1}^p/G_t^p}{L_{t+1}^p/L_t^p} = \frac{G_{t+1}^c/G_t^c}{L_{t+1}^c/L_t^c} = \frac{Y_{t+1}/Y_t}{L_{t+1}/L_t} \tag{77}$$

Therefore, the growth of an aggregate variable is shown by the next equation.

$$g^* = \frac{G_{t+1}^c}{G_t^c} = n^* g_{per}^* \tag{78}$$

$$g_{per}^* = \frac{G_{t+1}^c}{G_t^c} = \frac{g^*}{n^*} \tag{79}$$

$$\frac{\partial g_{per}^*}{\partial \varphi} = \frac{\partial \left(\frac{g^*}{n^*}\right)}{\partial \varphi} = \frac{\partial g_{per}^*}{\partial \varphi} n^* + \frac{\partial n^*}{\partial \varphi} g^* = \frac{n^* \left(\frac{\partial g^*}{\partial \varphi}\right) - g^* \left(\frac{\partial n^*}{\partial \varphi}\right)}{n^{*2}} =$$

$$-\tau \left\{ \left\{ \frac{\gamma(1-\varphi) + \varphi^2 (x^{p*})^\alpha}{\varphi^2} \right\} + \left\{ \frac{\alpha (x^{p*})^{\alpha-1} \{[\tau\varphi^2(1+\varepsilon+\rho) - \rho(1-\tau)(1-\alpha)(3x^{p*})^{-1}] 3(x^{p*})^\alpha + 2\tau\gamma\varphi(1+\varepsilon+\rho) + 2\varphi(1+\varepsilon+\rho)\}}{[\alpha\varphi^3(1+\varepsilon+\rho)(x^{p*})^{\alpha-1} + \gamma\rho(1-\tau)(x^{p*})^{-2} + \varphi\rho(1-\tau)(1-\alpha)^2 (x^{p*})^{\alpha-2}]} \right\} \right\}$$

$$\times \left\{ \frac{\varepsilon^2(1-\tau)(1-\alpha)A_t^p (x^{p*})^\alpha \left\{(1-\tau)\gamma + \varphi + \varphi\tau(1-\varphi)(1+\varepsilon+\rho)\left[(x^{p*})^\alpha + \frac{\gamma}{\varphi}\right]\right\}}{p^c(1+\varepsilon+\rho)\left\{\varepsilon(1-\tau)\gamma + \varphi - \tau\varphi(1-\varepsilon)(1-\varphi)(1+\varepsilon+\rho)\left[(x^{p*})^\alpha + \frac{\gamma}{\varphi}\right]\right\}} \right\} \quad (80)$$

$$\times \left\{ \frac{\varepsilon^2(1-\tau)(1-\alpha)A_t^p (x^{p*})^\alpha \{(1-\tau)\gamma + \varphi + \tau(1-\varphi)(1+\varepsilon+\rho)[\varphi(x^{p*})^\alpha + \gamma]\}}{p^c(1+\varepsilon+\rho)\{\varepsilon(1-\tau)\gamma + \varphi - \tau(1-\varepsilon)(1-\varphi)(1+\varepsilon+\rho)[\varphi(x^{p*})^\alpha + \gamma]\}} \right\} + \{\tau[\varphi(x^{p*})^\alpha + \gamma] + 1\} \frac{\partial n_t^*}{\partial \varphi} < 0$$

Proposition 3.

Decreased childcare support in the national budget clearly reduces both per capita and general growth rates.

5. Concluding Remarks

I wrote this paper to answer the question: What is the best policy to counteract a declining birthrate? An original feature of our work is that the model of labor-augmented public capital stock is divided into two types of public capital: capital in the final goods sector and the public childcare sector. Next, using a two-sector model, we analyzed how birth and growth rates would be affected by a change in government budget constraints on the distribution rate of public capital in the final goods sector . As a first result, predictably, an increase in the allocation rate to public capital in the final goods sector reduces the birthrate. These analyses also show that the increase in public capital in the final goods sector decreases, both from the per-capita perspective and from the overall perspective.

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