Obviously Strategy-proof Implementation of Assignment Rules: A New Characterization

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OBVIOUSLY STRATEGY-PROOF IMPLEMENTATION OF ASSIGNMENT RULES: A NEW CHARACTERIZATION

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Abstract

We consider assignment problems where individuals are to be assigned at most one indivisible object and monetary transfers are not allowed. We provide a characterization of assignment rules that are Pareto efficient, non-bossy, and implementable in obviously strategy-proof (OSP) mechanisms. As corollaries of our result, we obtain a characterization of OSP-implementable fixed priority top trading cycles (FPTTC) rules, hierarchical exchange rules, and trading cycles rules. Troyan (2019) provides a characterization of OSP-implementable FPTTC rules when there are equal number of individuals and objects. Our result generalizes this for arbitrary values of those.

Keywords: Assignment problem; Obvious strategy-proofness; Pareto efficiency; Non-bossiness; Indivisible goods

JEL Classification: C78; D82

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1 Introduction

We consider the problem where a set of objects are to be allocated over a set of individuals based on the individuals' preferences over the objects. Each individual can receive at most one object. An assignment rule selects an allocation (of the objects over the individuals) at every collection of preferences of the individuals.

Pareto efficiency, *non-bossiness*, and *(group) strategy-proofness* are standard requirements of an assignment rule.[1] Pareto efficiency ensures that there is no other way to allocate the objects so that each individual is weakly better-off (and hence some individual is strictly better-off). Non-bossiness says that an individual cannot change the assignment of another one without changing her own assignment. Strategy-proofness ensures that no individual can be strictly better-off by misreporting her (true) preference. Group strategy-proofness ensures the same for every group of individuals, that is, no group of individuals can be better-off by misreporting their preferences. Here, we say a group of individuals is better-off if each member in it is weakly better-off and some member is strictly better-off.

Hierarchical exchange rules are introduced in Pápai (2000) where it is shown that an assignment rule is strategy-proof, non-bossy, Pareto efficient, and reallocation-proof if and only if it is a hierarchical exchange rule. A hierarchical exchange rule works in several stages. In every stage, the objects (available in that stage) are owned by certain individuals who then trade their objects by forming top trading cycles.[2] The ownership of the objects in any stage is determined by a collection of trees, called *inheritance trees* in Pápai (2000). However, as discussed in Troyan (2019), use of hierarchical exchange rules in practice is rare as participating individuals find it difficult to understand these rules, particularly the fact that these rules are indeed strategy-proof.[3]

Obvious strategy-proofness (Li, 2017) came to the literature as a remedy by strengthening strategy-proofness in a way so that it becomes clear to the participating individuals that a rule is not manipulable. The concept of obvious strategy-proofness is based on the notion of *obvious dominance* in an *extensive-form game*. A strategy s_i of an individual i in an extensive-form game is obviously dominant if, for any deviating strategy s_i', starting from any earliest information set where s_i and s_i' diverge, the best possible outcome from s_i' is no better than the worst possible outcome from s_i. An assignment rule is *obviously strategy-proof (OSP)* if one can construct an extensive-form game that has an equilibrium in obviously dominant strategies. By construction, OSP depends on the extensive-form game, so two games with the same normal form may differ on this criterion.[4]

The objective of this paper is to characterize the structure of OSP-implementable assignment rules subject to Pareto efficiency and non-bossiness. We introduce the notion of *dual ownership* for this purpose.

[1]The concept of non-bossiness is due to Satterthwaite and Sonnenschein (1981).
[2]Top trading cycle (TTC) is due to David Gale and discussed in Shapley and Scarf (1974).
[3]Similar phenomena is also observed in other settings, see Chen and Sönmez (2006), Hassidim et al. (2016), Hassidim et al. (2017), Rees-Jones (2018), and Shorrer and Sóvágó (2018) for details.
[4]This verbal description of obvious strategy-proofness is adapted from Li (2017).

A hierarchical exchange rule satisfies dual ownership if for each preference profile and each stage of the hierarchical exchange rule at that preference profile, there are at most two individuals who own all the objects available in that stage. Thus, the dual ownership property makes it very simple for the (at most two) owners in any stage to trade: they only interchange their favorite objects. In contrast, for an arbitrary hierarchical exchange rule, there might be arbitrary number of individuals trading their favorite objects in a stage, which makes it harder to asses what would happen if they do not do this truthfully.

We show that an assignment rule is OSP-implementable, Pareto efficient, and non-bossy if and only if it is a hierarchical exchange rule satisfying dual ownership (Theorem 4.1). Since strategy-proofness and non-bossiness together are equivalent to group strategy-proofness (see Pápai (2000) for details), Theorem 4.1 can be reformulated in terms of group strategy-proofness (Corollary 4.1). We also show that a hierarchical exchange rule is OSP-implementable if and only if it satisfies dual ownership, and a *trading cycles rule* is OSP-implementable if and only if it is a hierarchical exchange rule satisfying dual ownership.[5]

Troyan (2019) introduces the notion of *dual dictatorship* in the context of fixed priority top trading cycles (FPTTC) rules.[6] It follows from Theorem 1 and Theorem 2 of his paper that dual dictatorship is both necessary and sufficient condition for an FPTTC rule to be OSP-implementable. However, there is a mistake in his characterization–although dual dictatorship is a sufficient condition for OSP-implementability of an FPTTC rule, it is *not* necessary.[7] Since FPTTC rules are special cases of hierarchical exchange rules (see Pápai (2000) for details), we obtain as a corollary (Corollary 5.2) of our result that dual ownership is a necessary and sufficient condition for OSP-implementability of an FPTTC rule. It is worth mentioning that Troyan (2019) assumes that the number of individuals is the same as the number of objects, whereas we derive our results for arbitrary values of those.

As we have mentioned earlier, Pápai (2000) characterizes hierarchical exchange rules as the only assignment rules satisfying strategy-proofness, non-bossiness, Pareto efficiency and reallocation-proofness. Our results complement hers in two ways. Firstly, whereas strategy-proofness, non-bossiness, and Pareto efficiency are desirable, reallocation-proofness is not that desirable. So, replacing strategy-proofness and reallocation-proofness by OSP-implementability, and characterizing the relevant class of hierarchical exchange rules is a significant contribution in our opinion. Secondly, hierarchical exchange rules are somewhat complicated for participants to understand. So, finding the class of such rules that can be implemented by obviously strategy-proof mechanisms is important for their application. Nevertheless, OSP-implementability is a desirable criteria on its own.

[5]Trading cycles rules are introduced in Pycia and Ünver (2017) as generalization of hierarchical exchange rules. They show that an assignment rule is strategy-proof, non-bossy, and Pareto efficient if and only if it is a trading cycles rule.

[6]Troyan (2019) uses the term "TTC rule" to refer to an FPTTC rule in his paper.

[7]Theorem 2 in Troyan (2019) states that "weak acyclicity" and dual dictatorship are equivalent properties of an FPTTC rule. This result is correct on its own, however, because of the mistake in Theorem 1, it is not correct that an FPTTC rule is OSP-implementable if and only if it satisfies dual dictatorship.

1.1 Related literature

Obvious strategy-proofness was introduced by Li (2017), who studies this property extensively for both the scenarios where monetary transfers are allowed and not allowed. When monetary transfers are not allowed, he analyses the implementability of serial dictatorship and top trading cycles rules under obvious strategy-proofness. Bade and Gonczarowski (2017) *constructively* characterize Pareto-efficient social choice rules that admit obviously strategy-proof implementations in popular domains (object assignment, single-peaked preferences, and combinatorial auctions). Pycia and Troyan (2019) characterize the full class of obviously strategy-proof mechanisms in environments without transfers. They also introduce a natural strengthening of obvious strategy-proofness called *strong obvious strategy-proofness* to characterize the well-known *random priority mechanism* as the unique mechanism that is efficient and fair. Ashlagi and Gonczarowski (2018) consider two-sided matching with one strategic side and show that for general preferences, no mechanism that implements the men-optimal stable matching (or any other stable matching) is obviously strategy-proof for men. They also provide a sufficient condition for a deferred acceptance rule to be OSP-implementable. Later, Thomas (2020) provides a necessary and sufficient condition for the same.

1.2 Organization of the paper

The organization of this paper is as follows. In Section 2, we introduce basic notions and notations that we use throughout the paper, define assignment rules and discuss their standard properties, and introduce the notion of obvious strategy-proofness. Section 3 introduces the notion of hierarchical exchange rules. In Section 4, we introduce the dual ownership property of a hierarchical exchange rule and present our main result (characterization of all OSP-implementable, Pareto efficient, and non-bossy assignment rules). In Section 5, we present a characterization of OSP-implementable hierarchical exchange rules, a characterization of OSP-implementable trading cycles rules, and a characterization of OSP-implementable FPTTC rules. We further discuss the relation between our result regarding FPTTC rules and that of Troyan (2019). All the proofs are collected in the Appendix.

2 Preliminaries

2.1 Basic notions and notations

Let $N = \{1, \dots, n\}$ be a (finite) set of individuals and A be a (non-empty and finite) set of objects. An ***allocation*** is a function $\mu : N \to A \cup \{\emptyset\}$ such that $|\mu^{-1}(x)| \leq 1$ for all $x \in A$. Here, $\mu(i) = x$ means individual i is assigned object x under μ, and $\mu(i) = \emptyset$ means individual i is not assigned any object under μ. We denote by $\mathcal{M}$ the set of all allocations. For $N' \subseteq N$, $A' \subseteq A$ such that $|N'| = |A'| \neq 0$, let $\mathcal{M}(N', A')$ denote the set of all bijections from N' to A'.

Let $\mathbb{L}(A)$ denote the set of all strict linear orders over A.[8] An element of $\mathbb{L}(A)$ is called a ***preference*** over A. For a preference P, let R denote the weak part of P, that is, for all $x, y \in A$, xRy if and only if $[xPy$ or $x = y]$. We assume that the set of admissible preferences of each individual is $\mathbb{L}(A)$. An element $P_N = (P_1, \ldots, P_n)$ of $\mathbb{L}^n(A)$ is called a ***preference profile***. Given a preference profile P_N, we denote by (P'_i, P_{-i}) the preference profile obtained from P_N by changing the preference of individual i from P_i to P'_i and keeping all other preferences unchanged. For $P \in \mathbb{L}(A)$ and non-empty $A' \subseteq A$, let $\tau(P, A')$ denote the most-preferred object in A' according to P, that is, $\tau(P, A') = x$ if and only if $[x \in A'$ and xPy for all $y \in A' \setminus \{x\}]$. For ease of presentation, we denote $\tau(P, A)$ by $\tau(P)$.

For ease of presentation we use the following convention throughout the paper: for a set $\{1, \ldots, g\}$ of integers, whenever we refer to the number $g + 1$, we mean 1. For instance, if we write $s_t \geq r_{t+1}$ for all $t = 1, \ldots, g$, we mean $s_1 \geq r_2, \ldots, s_{g-1} \geq r_g$, and $s_g \geq r_1$.

2.2 Assignment rules and their standard properties

An ***assignment rule*** is a function $f : \mathbb{L}^n(A) \to \mathcal{M}$. For an assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ and a preference profile $P_N \in \mathbb{L}^n(A)$, let $f_i(P_N)$ denote the assignment of individual i by f at P_N.

An allocation μ ***Pareto dominates*** another allocation ν at a preference profile P_N if $\mu(i)R_i\nu(i)$ for all $i \in N$ and $\mu(j)P_j\nu(j)$ for some $j \in N$. An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is called ***Pareto efficient*** at a preference profile $P_N \in \mathbb{L}^n(A)$ if there is no allocation that Pareto dominates $f(P_N)$ at P_N, and it is called ***Pareto efficient*** if it is Pareto efficient at every preference profile in $\mathbb{L}^n(A)$.

Non-bossiness is a standard notion in matching theory which says that if an individual misreports her preference and her assignment does not change by the same, then the assignment of any other individual cannot change. Formally, an assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is ***non-bossy*** if for all $P_N \in \mathbb{L}^n(A)$, all $i \in N$, and all $\tilde{P}_i \in \mathbb{L}(A)$, $f_i(P_N) = f_i(\tilde{P}_i, P_{-i})$ implies $f(P_N) = f(\tilde{P}_i, P_{-i})$.

An individual i *manipulates* an assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ at a preference profile $P_N \in \mathbb{L}^n(A)$ via a preference $\tilde{P}_i \in \mathbb{L}(A)$ if $f_i(\tilde{P}_i, P_{-i})P_if_i(P_N)$. An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is ***strategy-proof*** if no individual can manipulate it at any preference profile.

Group strategy-proofness says that no group of individuals will have an incentive to misreport their preferences. More formally, a group of individuals $N' \subseteq N$ *manipulates* an assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ at a preference profile $P_N \in \mathbb{L}^n(A)$ via a collection of preferences $\tilde{P}_{N'} \in \mathbb{L}^{|N'|}(A)$ if $f_i(\tilde{P}_{N'}, P_{-N'})R_if_i(P_N)$ for all $i \in N'$ and $f_j(\tilde{P}_{N'}, P_{-N'})P_jf_j(P_N)$ for some $j \in N'$. An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is ***group strategy-proof*** if no group of individuals can manipulate it at any preference profile.

2.3 Obviously strategy-proof assignment rules

Li (2017) introduces the notion of *obviously strategy-proof implementation*. We use the following notions and

[8]A ***strict linear order*** is a semiconnex, asymmetric, and transitive binary relation.

notations to present it.

We denote a rooted (directed) tree by T. For a tree T, we denote its set of nodes by $V(T)$, set of all edges by $E(T)$, root by $r(T)$, and set of leaves (terminal nodes) by $L(T)$. For a node $v \in V(T)$, we denote the set of all outgoing edges from v by $E^{out}(v)$. For an edge $e \in E(T)$, we denote its source node by $s(e)$. A *path* in a tree is a sequence of nodes such that every two consecutive nodes form an edge.

A *leaves-to-allocations* function $\eta^{LA} : L(T) \to \mathcal{M}$ assigns an allocation to each leaf of T, and a *nodes-to-individuals* function $\eta^{NI} : V(T) \setminus L(T) \to N$ assigns an individual to each internal node of T. An *edges-to-preferences* function $\eta^{EP} : E(T) \to 2^{\mathbb{L}(A)} \setminus \{\emptyset\}$ assigns each edge a subset of preferences satisfying the following criteria:

(i) for all distinct $e, e' \in E(T)$ such that $s(e) = s(e')$, we have $\eta^{EP}(e) \cap \eta^{EP}(e') = \emptyset$, and

(ii) for any $v \in V(T) \setminus L(T)$,

 (a) if there exists a path $(v^1, \ldots, v^t)$ from $r(T)$ to v and some $1 \leq r < t$ such that $\eta^{NI}(v^r) = \eta^{NI}(v)$ and $\eta^{NI}(v^s) \neq \eta^{NI}(v)$ for all $s = r+1, \ldots, t-1$, then $\bigcup_{e \in E^{out}(v)} \eta^{EP}(e) = \eta^{EP}(v^r, v^{r+1})$, and

 (b) if there is no such path, then $\bigcup_{e \in E^{out}(v)} \eta^{EP}(e) = \mathbb{L}(A)$.

An ***extensive-form assignment mechanism*** is defined as a tuple $G = \langle T, \eta^{LA}, \eta^{NI}, \eta^{EP} \rangle$, where T is a rooted tree, η^{LA} is a leaves-to-allocations function, η^{NI} is a nodes-to-individuals function, and η^{EP} is an edges-to-preferences function.

Note that for a given extensive-form assignment mechanism G, every preference profile P_N identifies a unique path from the root to some leaf in T in the following manner: for each node v, follow the outgoing edge e from v such that $\eta^{EP}(e)$ contains the preference $P_{\eta^{NI}(v)}$. If a node v lies in such a path, then we say that the preference profile P_N *passes through the node* v. Furthermore, we say two preferences P_i and P'_i of some individual i *diverge at a node* $v \in V(T) \setminus L(T)$ if $\eta^{NI}(v) = i$ and there are two distinct outgoing edges e and e' in $E^{out}(v)$ such that $P_i \in \eta^{EP}(e)$ and $P'_i \in \eta^{EP}(e')$.

For a given extensive-form assignment mechanism G, the ***extensive-form assignment rule*** f^G implemented by G is defined as follows: for all preference profiles P_N, $f^G(P_N) = \eta^{LA}(l)$, where l is the leaf that appears at the end of the unique path characterized by P_N.

In what follows, we define the notion of obvious strategy-proofness.

Definition 2.1. An extensive-form assignment mechanism G is ***Obviously Strategy-Proof (OSP)*** if for all $i \in N$, all nodes v such that $\eta^{NI}(v) = i$, and all $P_N, \tilde{P}_N \in \mathbb{L}^n(A)$ passing through v such that P_i and $\tilde{P}_i$ diverge at v, we have $f^G_i(P_N) R_i f^G_i(\tilde{P}_N)$.

An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is ***OSP-implementable*** if there exists an OSP mechanism G such that $f = f^G$.[9,10]

[9] Definition 2.1 is taken from Troyan (2019). However, his definition has a typo as it does not mention that P_N and $\tilde{P}_N$ must pass through v. We have corrected it here.

[10] An extensive-form assignment mechanism is called an ***OSP mechanism*** if it is OSP.

Remark 2.1. Every OSP-implementable assignment rule is strategy-proof (see Li (2017) for details).

3 Hierarchical exchange rules

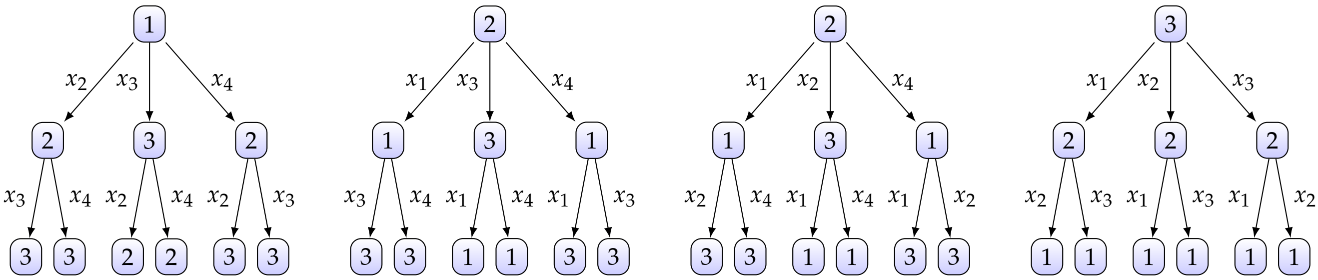
The notion of *hierarchical exchange rules* is introduced in Pápai (2000). We explain how such a rule works by means of an example.[11]

We begin with the notion of a *TTC procedure* with respect to a given endowments of the objects over the individuals. Suppose that each object is owned by exactly one individual (an individual may own more than one objects). A directed graph is constructed in the following manner. The set of nodes is the same as the set of individuals. There is a directed edge from individual i to individual j if and only if individual j owns individual i's most-preferred object. Note that such a graph will have exactly one outgoing edge from every node (though possibly many incoming edges to a node). Further, there may be an edge from a node to itself. It is clear that such a graph will always have a cycle. This cycle is called a *top trading cycle (TTC)*. After forming a TTC, the individuals in the TTC are assigned their most-preferred objects.

Example 3.1. Suppose $N = \{1, 2, 3\}$ and $A = \{x_1, x_2, x_3, x_4\}$. A hierarchical exchange rule is based on a collection of *inheritance trees*, one tree for each object.[12] Figure 3.1 presents a collection of inheritance trees $\Gamma_{x_1}, \ldots, \Gamma_{x_4}$. Consider Γ_{x_1} to have an understanding of their structure. Each maximal path of this tree has $\min\{|N|, |A|\} - 1 = 2$ edges. In any maximal path, each individual appears *at most* once at the nodes. For instance, individuals $1, 2$ and 3 appear at the nodes (in that order) in the left most path of Γ_{x_1}. Each object other than x_1 appears *exactly* once at the outgoing edges from the root (thus there are three edges from the root). For every subsequent node which is not the end node of a maximal path, each object other than x_1, that has *not* already appeared in the path from the root to that node, appears *exactly* once at the outgoing edges from that node. For instance, consider the node marked with 2 in the left most path of Γ_{x_1}. Since this node is not the end node of the left most maximal path and object x_2 has already appeared at the edge from the root to this node, objects x_3 and x_4 appear exactly once at the outgoing edges from this node. Thus, each object other than x_1 appears *at most* once at the edges in any maximal path of Γ_{x_1}. For instance, objects x_2 and x_3 appear at the edges (in that order) in the left most path of Γ_{x_1}. It can be verified that other inheritance trees have the same structure.

[11] See Pápai (2000) for an intuitive explanation of these rules.

[12] We define this notion formally in Subsection 3.1.



(a) Γ_{x_1} (b) Γ_{x_2} (c) Γ_{x_3} (d) Γ_{x_4}

Figure 3.1: Inheritance trees for Example 3.1

Consider the hierarchical exchange rule based on the collection of inheritance trees given in Figure 3.1 and consider the preference profile P_N such that $x_2P_1x_1P_1x_3P_1x_4$, $x_1P_2x_2P_2x_3P_2x_4$, and $x_1P_3x_2P_3x_3P_3x_4$. The outcome is computed through a number of stages. In each stage, endowments of the individuals are determined by means of the inheritance trees, and TTC procedure is performed with respect to the endowments.

Stage 1. In Stage 1, the "owner" of an object x is the individual who is assigned to the root-node of the inheritance tree Γ_x. Thus, object x_1 is owned by individual 1, objects x_2 and x_3 are owned by individual 2, and object x_4 is owned by individual 3. TTC procedure is performed with respect to these endowments to decide the outcome of Stage 1. Individuals who are assigned some object in Stage 1 leave the market with the corresponding objects. It can be verified that for the given preference profile P_N, individual 1 gets object x_2 and individual 2 gets object x_1. So, individuals 1 and 2 leave the market with objects x_2 and x_1, respectively.

Stage 2. As in Stage 1, the endowments of the individuals are decided first and then TTC procedure is performed with respect to the endowments. To decide the owner of a (remaining) object x, look at the root of the inheritance tree Γ_x. If the individual who appears there, say individual i, is remained in the market, then i becomes the owner of x. Otherwise, that is, if i is assigned an object in Stage 1, say y, then follow the edge from the root that is marked with y. If the individual appearing at the node following this edge, say j, is remained in the market, then j becomes the owner of x. Otherwise, that is, if j is assigned an object in Stage 1, say z, then follow the edge that is marked with z from the current node. As before, check whether the individual appearing at the end of this edge is remained in the market or not. Continue in this manner until an individual is found in the particular path who is not already assigned an object and decide that individual as the owner of x.

For the example at hand, the remaining market in Stage 2 consists of objects x_3 and x_4, and individual 3. Consider object x_3. Individual 2 appears at the root of Γ_{x_3}. Since individual 2 is assigned object x_1 in Stage 1, we follow the edge from the root that is marked with x_1 and come to individual 1. Since individual 1 is assigned object x_2, we follow the edge marked with x_2 from this node and come to individual 3. Since individual 3 is remained in the market, she becomes the owner of x_3. For object x_4, individual 3 appears at the root of Γ_{x_4} and she is remained in the market. So, individual 3 becomes the owner of x_4 in Stage 2. To emphasize the process of deciding the owner of an object, we have highlighted the node in red in the corresponding inheritance tree in Figure 3.2.

(a) Γ_{x_3} (b) Γ_{x_4}

Figure 3.2: Stage 2

Once the endowments are decided for Stage 2, TTC procedure is performed with respect to the endowments to decide the outcome of this stage. As in Stage 1, individuals who are assigned some object in Stage 2 leave the market with the corresponding objects. It can be verified that for the current example, individual 3 gets object x_3 in this stage. So, individual 3 leave the market with objects x_3.

Stage 3 is followed on the remaining market in a similar way as Stage 2. For the current example, everybody is assigned some object by the end of Stage 2 and hence the algorithm stops in this stage. Thus, individuals 1, 2, and 3 get objects x_2, x_1, and x_3, respectively, at the outcome of the hierarchical exchange rule.

In what follows, we present a formal description of hierarchical exchange rules.

3.1 Inheritance trees

For a rooted tree T, the *level* of a node $v \in V(T)$ is defined as the number of edges appearing in the (unique) path from $r(T)$ to v.

Definition 3.1. For an object $x \in A$, an ***inheritance tree for*** $x \in A$ is defined as a tuple $\Gamma_x = \langle T_x, \zeta_x^{NI}, \zeta_x^{EO} \rangle$, where

(i) T_x is a rooted tree with

  (a) $\max\limits_{v \in V(T_x)} level(v) = \min\{|N|, |A|\} - 1$, and

  (b) $|E^{out}(v)| = |A| - level(v) - 1$ for all $v \in V(T_x)$ with $level(v) < \min\{|N|, |A|\} - 1$,

(ii) $\zeta_x^{NI} : V(T_x) \to N$ is a nodes-to-individuals function with $\zeta_x^{NI}(v) \neq \zeta_x^{NI}(\tilde{v})$ for all distinct $v, \tilde{v} \in V(T_x)$ that appear in same path, and

(iii) $\zeta_x^{EO} : E(T_x) \to A \setminus \{x\}$ is an edges-to-objects function with $\zeta_x^{EO}(e) \neq \zeta_x^{EO}(\tilde{e})$ for all distinct e, $\tilde{e} \in E(T_x)$ that appear in same path or have same source node (that is, $s(e) = s(\tilde{e})$).

3.2 Endowments

A hierarchical exchange rule works in several stages and in each stage, endowments of individuals are determined by using a (fixed) collection of inheritance trees.

Given a collection of inheritance trees $\Gamma = (\Gamma_x)_{x \in A}$, one for each object $x \in A$, we define a class of endowments $\mathcal{E}^\Gamma$ as follows:

(i) The ***initial endowment*** $\mathcal{E}_i^\Gamma(\emptyset)$ of individual i is given by

$$\mathcal{E}_i^\Gamma(\emptyset) = \{x \in A \mid \zeta_x^{NI}(r(T_x)) = i\}.$$

(ii) For all $N' \subseteq N \setminus \{i\}$ and $A' \subseteq A$ with $|N'| = |A'| \neq 0$, and all $\mu' \in \mathcal{M}(N', A')$, the ***endowment*** $\mathcal{E}_i^\Gamma(\mu')$ of individual i is given by

$$\begin{aligned}\mathcal{E}_i^\Gamma(\mu') = \{ & x \in A \setminus A' \mid \zeta_x^{NI}(r(T_x)) = i, \text{ or} \\ & \text{there exists a path } (v_x^1, \ldots, v_x^{r_x}) \text{ from } r(T_x) \text{ to } v_x^{r_x} \text{ in } \Gamma_x \text{ such that } \zeta_x^{NI}(v_x^{r_x}) = i \\ & \text{and for all } s = 1, \ldots, r_x - 1, \text{ we have } \zeta_x^{NI}(v_x^s) \in N' \text{ and } \mu'(\zeta_x^{NI}(v_x^s)) = \zeta_x^{EO}(v_x^s, v_x^{s+1})\}.\end{aligned}$$

3.3 Iterative procedure to compute the outcome of a hierarchical exchange rule

For a given collection of inheritance trees $\Gamma = (\Gamma_x)_{x \in A}$, the ***hierarchical exchange rule*** f^Γ ***associated with*** Γ is defined by an iterative procedure with at most $\min\{|N|, |A|\}$ number of stages. Consider a preference profile $P_N \in \mathbb{L}^n(A)$.

Stage 1.

Hierarchical Endowments (Initial Endowments): For all $i \in N$, $E_1(i, P_N) = \mathcal{E}_i^\Gamma(\emptyset)$.

Top Choices: For all $i \in N$, $T_1(i, P_N) = \tau(P_i)$.

Trading Cycles: For all $i \in N$,

$$C_1(i, P_N) = \begin{cases} \{j_1, \ldots, j_g\} & \text{if there exist } j_1, \ldots, j_g \in N \text{ such that} \\ & \text{for all } s = 1, \ldots, g,\ T_1(j_s, P_N) \in E_1(j_{s+1}, P_N), \text{ and} \\ & \text{for some } \hat{s} = 1, \ldots, g,\ j_{\hat{s}} = i; \\ \emptyset & \text{otherwise.} \end{cases}$$

Since each individual can be in at most one trading cycle, $C_1(i, P_N)$ is well-defined for all $i \in N$. Furthermore, since both the number of individuals and the number of objects are finite, there is always at least one trading cycle. Note that $C_1(i, P_N) = \{i\}$ if $T_1(i, P_N) \in E_1(i, P_N)$.

Assigned Individuals: $N_1(P_N) = \{i \mid C_1(i, P_N) \neq \emptyset\}$.

Assignments: For all $i \in N_1(P_N)$, $f_i^\Gamma(P_N) = T_1(i, P_N)$.

Assigned Objects: $A_1(P_N) = \{T_1(i, P_N) \mid i \in N_1(P_N)\}$.

This procedure is repeated iteratively in the remaining reduced market. For each stage t, define $N^t(P_N) = \overset{t}{\underset{u=1}{\cup}} N_u(P_N)$ and $A^t(P_N) = \overset{t}{\underset{u=1}{\cup}} A_u(P_N)$. In what follows, we present Stage $t+1$ of f^Γ.

Stage $t + 1$.

Hierarchical Endowments (Non-initial Endowments): Let $\mu^t \in \mathcal{M}(N^t(P_N), A^t(P_N))$ such that for all $i \in N^t(P_N)$,

$$\mu^t(i) = f_i^\Gamma(P_N).$$

For all $i \in N \setminus N^t(P_N)$, $E_{t+1}(i, P_N) = \mathcal{E}_i^\Gamma(\mu^t)$.

Top Choices: For all $i \in N \setminus N^t(P_N)$, $T_{t+1}(i, P_N) = \tau(P_i, A \setminus A^t(P_N))$.

Trading Cycles: For all $i \in N \setminus N^t(P_N)$,

$$C_{t+1}(i, P_N) = \begin{cases} \{j_1, \ldots, j_g\} & \text{if there exist } j_1, \ldots, j_g \in N \setminus N^t(P_N) \text{ such that} \\ & \text{for all } s = 1, \ldots, g,\ T_{t+1}(j_s, P_N) \in E_{t+1}(j_{s+1}, P_N), \text{ and} \\ & \text{for some } \hat{s} = 1, \ldots, g,\ j_{\hat{s}} = i; \\ \emptyset & \text{otherwise.} \end{cases}$$

Assigned Individuals: $N_{t+1}(P_N) = \{i \mid C_{t+1}(i, P_N) \neq \emptyset\}$.

Assignments: For all $i \in N_{t+1}(P_N)$, $f_i^\Gamma(P_N) = T_{t+1}(i, P_N)$.

Assigned Objects: $A_{t+1}(P_N) = \{T_{t+1}(i, P_N) \mid i \in N_{t+1}(P_N)\}$.

This procedure is repeated iteratively until either all individuals are assigned or all objects are assigned. The hierarchical exchange rule f^Γ associated with Γ is defined as follows. For all $i \in N$,

$$f_i^\Gamma(P_N) = \begin{cases} T_t(i, P_N) & \text{if } i \in N_t(P_N) \text{ for some stage } t; \\ \emptyset & \text{otherwise.} \end{cases}$$

Since for every preference profile P_N and every individual i, there exists at most one stage t such that $i \in N_t(P_N)$, f^Γ is well-defined.

Remark 3.1. Note that a collection of inheritance trees do not uniquely identify a hierarchical exchange rule. More formally, two different collections of inheritance trees Γ and $\overline{\Gamma}$ may give rise to the same hierarchical exchange rule, that is, $f^\Gamma \equiv f^{\overline{\Gamma}}$.

4 A characterization of OSP-implementable assignment rules

In this section, we introduce a property called *dual ownership* of a hierarchical exchange rule and provide a characterization of OSP-implementable, Pareto efficient, and non-bossy assignment rules by means of this property. We also explain the practical usefulness of the dual ownership property.

4.1 Dual ownership

Troyan (2019) introduces the notion of *dual dictatorship* in the context of fixed priority top trading cycles (FPTTC) rules.[13] We introduce a closely related notion for hierarchical exchange rules which we call *dual ownership*. A hierarchical exchange rule satisfies ***dual ownership*** if for any preference profile and any stage of the hierarchical exchange rule at that preference profile, there are at most two individuals who own all the objects that remain in the reduced market in that stage.

4.2 The characterization result

In this subsection, we provide a characterization of OSP-implementable assignment rules under two mild and desirable properties, namely Pareto efficiency and non-bossiness.[14]

Theorem 4.1. *An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is OSP-implementable, Pareto efficient and non-bossy if and only if f is a hierarchical exchange rule satisfying dual ownership.*

The proof of this theorem is relegated to Appendix B.

Since OSP-implementability implies strategy-proofness (see Remark 2.1) and group strategy-proofness is equivalent to strategy-proofness and non-bossiness (see Pápai (2000) for details), we obtain the following corollary from Theorem 4.1.

Corollary 4.1. *A group strategy-proof and Pareto efficient assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is OSP-implementable if and only if f is a hierarchical exchange rule satisfying dual ownership.*

It is worth mentioning that OSP-implementability and non-bossiness together do not imply Pareto efficiency. For instance, any constant assignment rule satisfies the former two properties, but does not satisfy the latter. Furthermore, it follows from Pápai (2000) that non-bossiness and Pareto efficiency together do not imply strategy-proofness. Since OSP-implementability is stronger than strategy-proofness (by Remark 2.1), non-bossiness and Pareto efficiency cannot imply it either. Example 4.1 shows that OSP-implementability and Pareto efficiency together do not imply non-bossiness.

[13] Troyan (2019) uses the term “TTC rule” to refer to an FPTTC rule. In Subsection 5.2, we provide a formal description of FPTTC rules.

[14] Bade and Gonczarowski (2017) characterize OSP-implementable and Pareto efficient assignment rules as the ones that can be implemented via a mechanism they call *sequential barter with lurkers*. Sequential barter with lurkers violates non-bossiness in general, and we do not see any obvious way to relate their result to ours.

Example 4.1. Consider an allocation problem with three individuals $N = \{1, 2, 3\}$ and three objects $A = \{x_1, x_2, x_3\}$. Consider the assignment rule f such that

$$f = \begin{cases} \text{Serial dictatorship with priority } (1 \succ 2 \succ 3) & \text{if } x_2 P_1 x_3 \\ \text{Serial dictatorship with priority } (1 \succ 3 \succ 2) & \text{if } x_3 P_1 x_2 \end{cases}$$

Consider the preference profiles $P_N = (x_1x_2x_3, x_1x_2x_3, x_1x_2x_3)$ and $\tilde{P}_N = (x_1x_3x_2, x_1x_2x_3, x_1x_2x_3)$.[15] Note that only individual 1 changes her preference from P_N to $\tilde{P}_N$. This, together with the facts $f(P_N) = [(1, x_1), (2, x_2), (3, x_3)]$ and $f(\tilde{P}_N) = [(1, x_1), (2, x_3), (3, x_2)]$, implies f violates non-bossiness. However, the OSP mechanism in Figure 4.1 implements f.[16]

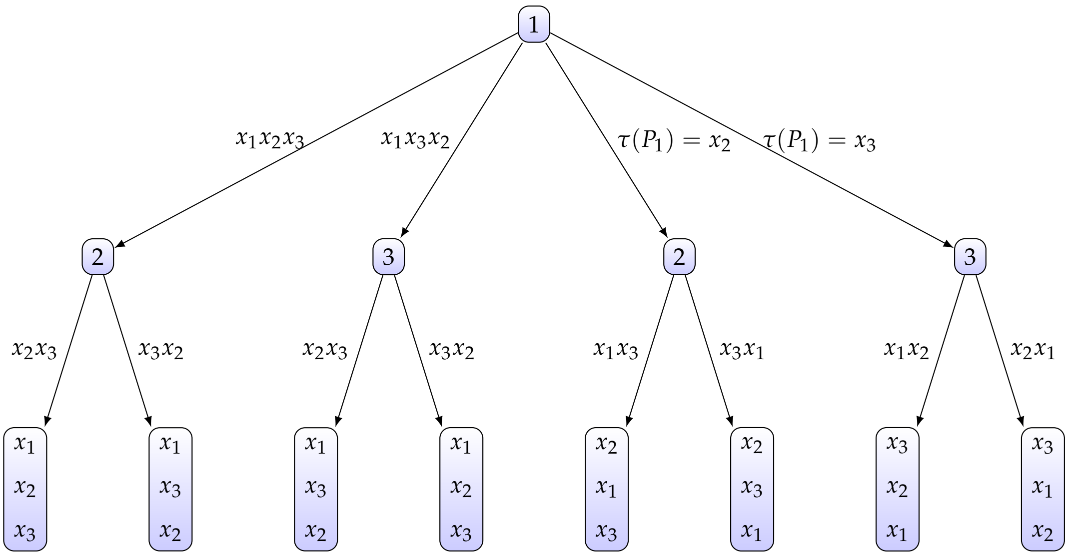


Figure 4.1: Tree Representation for Example 4.1

4.3 Advantage of using hierarchical exchange rules satisfying dual ownership property

In this subsection, we show how a hierarchical exchange rule satisfying the dual ownership property can be explained to the participating individuals and how the explanation helps in convincing individuals that such rules are indeed strategy-proof.[17]

In Stage 1:

(1) We call at most two individuals who will be the owners in this stage.

(2) We tell them their endowed sets.

[15] Here, we denote by $(x_1x_2x_3, x_2x_3x_1, x_3x_2x_1)$ a preference profile where individuals 1, 2 and 3 have preferences $x_1x_2x_3$, $x_2x_3x_1$, and $x_3x_2x_1$, respectively.

[16] We use the following notation in Figure 4.1: by x_1x_2 we denote the set of preferences where x_1 is preferred to x_2 and we denote an allocation $[(1, x_1), (2, x_2), (3, x_3)]$ by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

[17] This explanation does not highlight many of the key features of hierarchical exchange rules satisfying the dual ownership property.

(3) We tell them that each of them can "take" something from her endowed set (and leave the market), or "wait" to see if she gets something better. We additionally mention that if someone chooses to "wait", she can leave the market anytime in the future with an object from her current endowment set.

To see that the owners will act truthfully in (3), first note that the owners are asked to choose between "take" or "wait", in particular, they are not asked to reveal their top choices. Therefore,

(a) if any of the owners has her favorite object in her endowment, then she will "take" that object and leave the market, and

(b) if any of the owners does not have her favorite object in her endowment, then she will "wait" as she can leave the market anytime in the future with an object from her current endowment set.

(4) (i) If any of the owners chooses to "take" in (3). We get a submarket.

(ii) On the other hand, if both of them choose to "wait", we tell each of them to "take" something from other's endowment and leave the market, and again we get a submarket. Clearly, there is no question of manipulation for an individual at this step as she will simply take her favorite object from other's endowment.

In Stage 2:

(1) We call at most two individuals who will be the owners in this stage. If one of the owners in Stage 1 remains in the reduced market in Stage 2, we make her one of the owners in Stage 2.[18]

(2) We tell them their endowed sets. If one of the owners in Stage 2 was also an owner in Stage 1, all the objects in her endowment in Stage 1 must be included in her endowment in Stage 2.

(3) Same as Stage 1. For the same reason as we have discussed in (3) of Stage 1, individuals will act truthfully at this step of Stage 2.

(4) Same as Stage 1.

We continue this procedure until everyone is assigned or all objects are assigned.

The main reason why a hierarchical exchange rule satisfying dual ownership is simpler than an arbitrary hierarchical exchange rule is as follows. The dual ownership property ensures that at most two individuals will get to act in each stage. Therefore, the only way they can trade is to interchange their favorite objects. This makes it easy to see that they cannot strictly benefit by misreporting. For an arbitrary hierarchical exchange rule, there might be a lot more individuals acting in a stage, and consequently it may become harder for an individual to see the consequences of all possible misreports.

[18] Note that both owners in Stage 1 can not remain in the reduced market in Stage 2.

5 Discussion

5.1 OSP-implementability of hierarchical exchange rules and trading cycles rules

In this subsection, we provide a necessary and sufficient condition for a hierarchical exchange rule and a trading cycles rule to be OSP-implementable.

Proposition 5.1. *A hierarchical exchange rule is OSP-implementable if and only if it satisfies dual ownership.*

The proof of this proposition is relegated to Appendix A.[19]

Pycia and Ünver (2017) introduce a general version of hierarchical exchange rules which they call *trading cycles rules*. They show that an assignment rule is group strategy-proof and Pareto efficient if and only if it is a trading cycles rule. Combining this result with Corollary 4.1, we obtain the following corollary.

Corollary 5.1. *A trading cycles rule is OSP-implementable if and only if it is a hierarchical exchange rule satisfying dual ownership.*

5.2 OSP-implementability of FPTTC rules

In this subsection, we discuss OSP-implementability of FPTTC rules. FPTTC rules are well-known in the literature; we present a brief description for the sake of completeness.

For each object $x \in A$, we define the ***priority*** of x as a "preference" $\succ_x$ over N.[20] We call a collection $\succ_A := (\succ_x)_{x \in A}$ a ***priority structure***. For a given priority structure $\succ_A$, the ***FPTTC rule*** $T^{\succ_A}$ ***associated with*** $\succ_A$ is defined by an iterative procedure as follows. Consider an arbitrary preference profile $P_N \in \mathbb{L}^n(A)$.

Step 1. Each object x is owned by the individual who has the highest priority according to $\succ_x$, that is, the most-preferred individual of $\succ_x$. TTC procedure is performed with respect to these endowments. Individuals who are assigned some object leave the market with their assigned objects.

This procedure is repeated iteratively in the remaining reduced market. We present a general step of $T^{\succ_A}$.

Step t. Consider the reduced market with the remaining individuals and objects. Each remaining object x is owned by the individual who has the highest priority among the remaining individuals according to $\succ_x$, that is, the individual who is remained in the reduced market at this step and is preferred to every other remaining individual according to $\succ_x$. TTC procedure is performed on the reduced market with respect to these endowments, and individuals who are assigned some object at this step leave the market.[21]

[19]Proposition 5.1 follows as a corollary of Theorem 4.1. However, we do not present it as a corollary as we use this proposition in the proof of Theorem 4.1.

[20]That is, $\succ_x \in \mathbb{L}(N)$.

[21]In this TTC procedure, an individual i point to an individual j if j owns i's most-preferred object among the remaining objects.

This procedure is repeated iteratively until either all individuals are assigned or all objects are assigned. The final outcome is obtained by combining all the assignments at all steps. This completes the description of an FPTTC rule.

Since FPTTC rules are special cases of hierarchical exchange rules (see Pápai (2000) for details), the ***dual ownership property of FPTTC rules*** implies the following: for any preference profile and any step of the FPTTC rule at that preference profile, there are at most two individuals who own all the objects that remain in the reduced market at that step. This yields the following corollary from Proposition 5.1.

Corollary 5.2. *An FPTTC rule is OSP-implementable if and only if it satisfies dual ownership.*

Now, we discuss the relation between *dual dictatorship* (Troyan, 2019) and dual ownership of FPTTC rules. It follows from Theorem 1 and Theorem 2 in Troyan (2019) that an FPTTC rule is OSP-implementable if and only if it satisfies dual dictatorship, whereas Corollary 5.2 of our paper says that an FPTTC rule is OSP-implementable if and only if it satisfies dual ownership. In what follows, we clarify the difference between these two (conflicting) results and conclude that while dual dictatorship is a sufficient condition for an FPTTC rule to be OSP-implementable, it is *not* necessary.[22]

Dual dictatorship property of an FPTTC rule requires that in any submarket, at most two individuals will own all the objects in the submarket. In contrast, dual ownership property of an FPTTC rule requires that for every preference profile and every step of that FPTTC rule at that preference profile, at most two individuals will own all the objects that will remain in the reduced market at that step. The difference between these two properties arises from the fact that *not every* submarket arises at some step at some preference profile of an FPTTC rule. In other words, dual dictatorship is stronger than dual ownership. In Appendix C, we clarify this fact by means of an example.

Appendix A Proof of Proposition 5.1

Before we formally start proving Proposition 5.1, to facilitate the proof we introduce the notion of a reduced tree structure and make two observations.

A.1 Reduced tree structure

For an inheritance tree $\Gamma_a = \langle T_a, \zeta_a^{NI}, \zeta_a^{EO}\rangle$ and an edge $(v, v') \in E(T_a)$, we say that an inheritance tree $\tilde{\Gamma}_a = \langle \tilde{T}_a, \tilde{\zeta}_a^{NI}, \tilde{\zeta}_a^{EO}\rangle$ ***is obtained by collapsing the edge*** (v, v') if

(i) $V(\tilde{T}_a) = V(T_a) \setminus \Big(\{v\} \cup \{v'' \mid \text{ there exists a path in } T_a \text{ from } v \text{ to } v'' \text{ which does not contain } v'\}\Big)$,

(ii) $E(\tilde{T}_a) = \Big(E(T_a) \cap \big(V(\tilde{T}_a) \times V(\tilde{T}_a)\big)\Big) \cup \{(\hat{v}, v')\}$, where $\hat{v}$ is the parent node of v in T_a. If $v = r(T_a)$, then $\hat{v}$ does not exist, and consequently, we take $\{(\hat{v}, v')\} = \emptyset$,

[22] In order to prove the "only-if" part of Theorem 1, Troyan (2019) reduces the whole problem to a restricted domain and uses a result from Li (2017). However, for the purpose of Troyan (2019), this reduction step is *not* correct.

(iii) $\tilde{\zeta}_a^{NI}(v) = \zeta_a^{NI}(v)$ for all $v \in V(\tilde{T}_a)$, and

(iv) $\tilde{\zeta}_a^{EO}(e) = \zeta_a^{EO}(e)$ for all $e \in \Big(E(T_a) \cap \big(V(\tilde{T}_a) \times V(\tilde{T}_a)\big)\Big)$ and $\tilde{\zeta}_a^{EO}(\hat{v}, v') = \zeta_a^{EO}(\hat{v}, v)$.

For an inheritance tree $\Gamma_a = \langle T_a, \zeta_a^{NI}, \zeta_a^{EO}\rangle$ and an edge $(v, v') \in E(T_a)$, we say that an inheritance tree $\tilde{\Gamma}_a = \langle \tilde{T}_a, \tilde{\zeta}_a^{NI}, \tilde{\zeta}_a^{EO}\rangle$ ***is obtained by dropping the edge*** (v, v') if

(i) $V(\tilde{T}_a) = V(T_a) \setminus \{v'' \mid$ there exists a path in T_a from v to v'' which contains $v'\}$,

(ii) $E(\tilde{T}_a) = E(T_a) \cap \big(V(\tilde{T}_a) \times V(\tilde{T}_a)\big)$,

(iii) $\tilde{\zeta}_a^{NI}(v) = \zeta_a^{NI}(v)$ for all $v \in V(\tilde{T}_a)$, and

(iv) $\tilde{\zeta}_a^{EO}(e) = \zeta_a^{EO}(e)$ for all $e \in E(\tilde{T}_a)$.

For an inheritance tree $\Gamma_a = \langle T_a, \zeta_a^{NI}, \zeta_a^{EO}\rangle$, we denote an edge $(v, v') \in E(T_a)$ by (i, x) if $\zeta_a^{NI}(v) = i$ and $\zeta_a^{EO}(v, v') = x$ in Γ_a. By the construction of Γ_a, $\zeta_a^{EO}(v, v') = x$ implies $a \neq x$.

For a pair $(i, x) \in N \times A$ and a collection of inheritance trees $\Gamma = (\Gamma_x)_{x \in A}$, we define the ***reduced collection*** $\Gamma \setminus (i, x)$ as follows:

(i) If $a = x$, then drop the inheritance tree Γ_a.

(ii) If $a \neq x$ and $\zeta_a^{NI}(r(T_a)) = i$, then $\Gamma_a \setminus (i, x)$ is obtained by collapsing the edge (i, x) in Γ_a.[23]

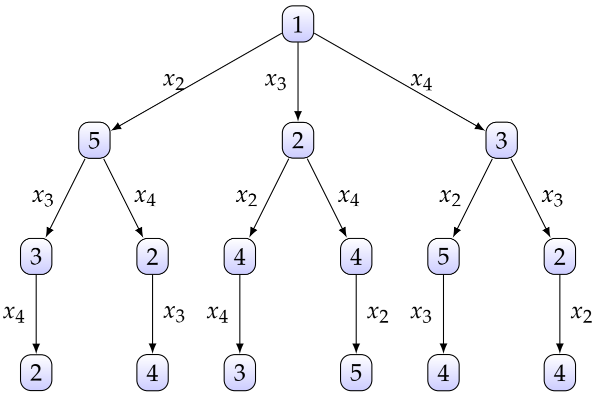
(iii) If $a \neq x$ and $\zeta_a^{NI}(r(T_a)) \neq i$, then $\Gamma_a \setminus (i, x)$ is obtained by collapsing all edges (i, x) and dropping all edges (j, x) with $j \neq i$ in Γ_a.

For $(i, x), (j, y) \in N \times A$ and a collection of inheritance trees $\Gamma = (\Gamma_x)_{x \in A}$, we denote the reduced collection $\Big(\Gamma \setminus (i, x)\Big) \setminus (j, y)$ by $\Gamma \setminus \Big((i, x), (j, y)\Big)$.

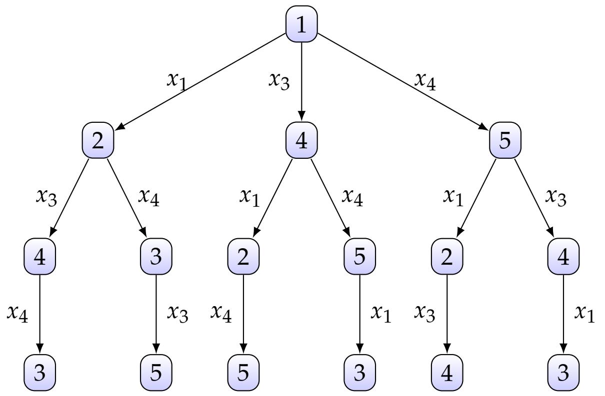
Remark A.1. For $(i, x), (j, y) \in N \times A$ and a collection of inheritance trees $\Gamma = (\Gamma_x)_{x \in A}$, we have $\Gamma \setminus \Big((i, x), (j, y)\Big) = \Gamma \setminus \Big((j, y), (i, x)\Big)$.

Example A.1. Suppose $N = \{1, 2, 3, 4, 5\}$ and $A = \{x_1, x_2, x_3, x_4\}$. Consider the collection of inheritance trees Γ given in Figure A.1.

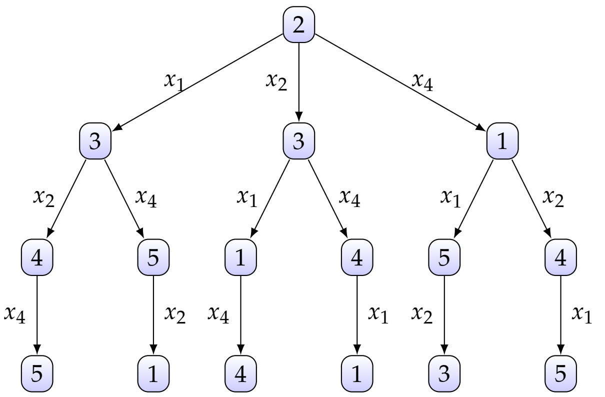
[23] Note that in this case, there is only one such edge (i, x).



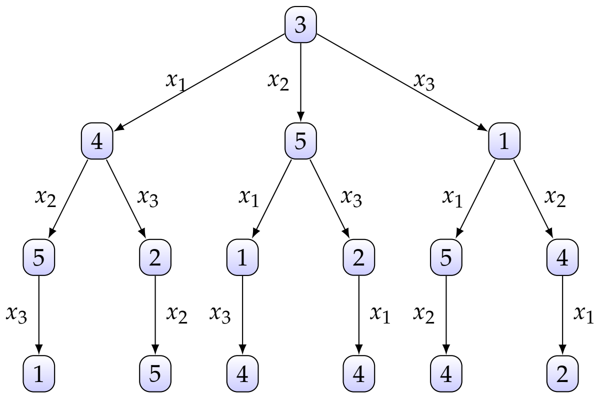
(a) Γ_{x_1}



(b) Γ_{x_2}



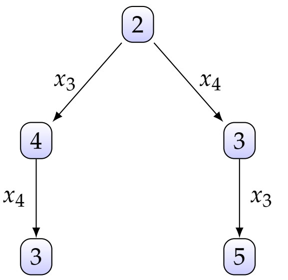
(c) Γ_{x_3}



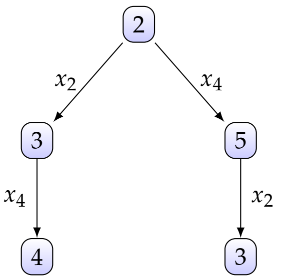
(d) Γ_{x_4}

Figure A.1: Collection of inheritance trees Γ for Example A.1

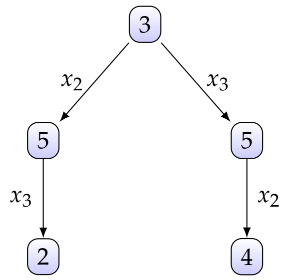
Consider the pair $(1, x_1) \in N \times A$. The reduced collection $\Gamma \setminus (1, x_1)$ is given in Figure A.2.



(a) $\Gamma_{x_2} \setminus (1, x_1)$



(b) $\Gamma_{x_3} \setminus (1, x_1)$



(c) $\Gamma_{x_4} \setminus (1, x_1)$

Figure A.2: Reduced collection $\Gamma \setminus (1, x_1)$

A.2 Two observations

Let $\mathcal{T}(\Gamma) = \{i \mid \zeta_x^{NI}(r(T_x)) = i \text{ for some } x \in A\}$ be the set of individuals who appear at the root-node of some inheritance tree in the collection of inheritance trees Γ. We now make two observations. The first observation is straightforward, and see Step 2.a in the "Necessity Proof" of Pápai (2000) for the second observation.

Observation A.1. *Suppose f^{Γ} satisfies dual ownership. Then, $|\mathcal{T}(\Gamma)| \leq 2$.*

Observation A.2. *Suppose $\zeta_x^{NI}(r(T_x)) = i$ for some $x \in A$ and some $i \in N$. Then, for all $P_N \in \mathbb{L}^n(A)$, $f_i^\Gamma(P_N) R_i x$.*

A.3 The proof

(If part) Suppose f^Γ satisfies dual ownership. We show that f^Γ is OSP-implementable by using induction on the number of individuals, which we refer to as the *size of the market*.

Base Case: Suppose $|N| = 1$.[24] The following extensive-form assignment mechanism, labeled as G^1, implements f^Γ.

Step 1. Ask the (only) individual which object is her top choice and assign her that object.

It is simple to check that the extensive-form assignment mechanism G^1 is OSP. Since the OSP mechanism G^1 implements f^Γ, it follows that f^Γ is OSP-implementable. Now, we proceed to prove the induction step.

Induction Hypothesis: Assume that f^Γ is OSP-implementable for $|N| \leq m$. We show f^Γ is OSP-implementable for $|N| = m + 1$. Since f^Γ satisfies dual ownership, by Observation A.1, we have $|\mathcal{T}(\Gamma)| \leq 2$. We distinguish the following two cases.

Case A: Suppose $|\mathcal{T}(\Gamma)| = 1$.

Let $\mathcal{T}(\Gamma) = \{i\}$. Define the extensive-form assignment mechanism G^{m+1} as follows:

Step 1. Ask individual i which object is her top choice and assign her that object, say x.

Step 2. Consider the reduced market $(N \setminus \{i\}, A \setminus \{x\})$ where individual i is removed from the market together with the object x she is assigned. This reduced market $(N \setminus \{i\}, A \setminus \{x\})$ is of size m.

Claim A.1. *$f^{\Gamma\setminus(i,x)}$ satisfies dual ownership on the reduced market $(N \setminus \{i\}, A \setminus \{x\})$.*[25]

By the induction hypothesis and Claim A.1, it follows that there exists an OSP mechanism G^m that implements $f^{\Gamma\setminus(i,x)}$ on the reduced market $(N \setminus \{i\}, A \setminus \{x\})$. Run the extensive-form assignment mechanism G^m on the reduced market $(N \setminus \{i\}, A \setminus \{x\})$.

By definition, the extensive-form assignment mechanism G^{m+1} implements f^Γ. This extensive-form assignment mechanism is OSP for individual i since she receives her top choice. For every other individual, her first decision node comes after i has been assigned, and hence, her strategic decision is equivalent to that under the OSP mechanism that implements f^Γ restricted to the corresponding reduced market. Thus, the above extensive-form assignment mechanism is OSP for all individuals, and hence, f^Γ is OSP-implementable.

[24] With only one individual, f^Γ trivially satisfies dual ownership.

[25] The proof of Claim A.1 is relegated to Appendix A.3.1.

Case B: Suppose $|\mathcal{T}(\Gamma)| = 2$.

Let $\mathcal{T}(\Gamma) = \{i, j\}$. Let $A_i = \{x \in A \mid \zeta_x^{NI}(r(T_x)) = i\}$ and $A_j = \{y \in A \mid \zeta_y^{NI}(r(T_y)) = j\}$. Define the extensive-form assignment mechanism G^{m+1} as follows:

Step 1. For each $x \in A_i$, ask i if her top choice is x. If i answers "Yes" for some x, assign her this x, and go to Step $1(a)$. Otherwise, jump to Step 2.

Step 1(a). We now have a reduced market $(N \setminus \{i\}, A \setminus \{x\})$ of size m.

Claim A.2. *$f^{\Gamma\setminus(i,x)}$ satisfies dual ownership on the reduced market $(N \setminus \{i\}, A \setminus \{x\})$.*[26]

By the induction hypothesis and Claim A.2, it follows that there exists an OSP mechanism G^m that implements $f^{\Gamma\setminus(i,x)}$ on the reduced market $(N \setminus \{i\}, A \setminus \{x\})$. Run the extensive-form assignment mechanism G^m on the reduced market $(N \setminus \{i\}, A \setminus \{x\})$.

Step 2. For each $y \in A_j$, ask j if her top choice is y. If j answers "Yes" for some y, assign her this y, and go to Step $2(a)$. Otherwise, jump to Step 3.

Step 2(a). We now have a reduced market $(N \setminus \{j\}, A \setminus \{y\})$ of size m. Similar to Claim A.2, we have the following claim.

Claim A.3. *$f^{\Gamma\setminus(j,y)}$ satisfies dual ownership on the reduced market $(N \setminus \{j\}, A \setminus \{y\})$.*

By the induction hypothesis and Claim A.3, it follows that there exists an OSP mechanism G^m that implements $f^{\Gamma\setminus(j,y)}$ on the reduced market $(N \setminus \{j\}, A \setminus \{y\})$. Run the extensive-form assignment mechanism G^m on the reduced market $(N \setminus \{j\}, A \setminus \{y\})$.

Step 3. If the answers to both Step 1 and Step 2 are "No", then i's top choice belongs to A_j, and j's top choice belongs to A_i. Ask i for her top choice x, and j for her top choice y. Assign x to i and y to j, and go to Step $3(a)$.

Step 3(a). We now have a reduced market $(N \setminus \{i, j\}, A \setminus \{x, y\})$ of size $m - 1$.

Claim A.4. *$f^{\Gamma\setminus((i,x),(j,y))}$ satisfies dual ownership on the reduced market $(N \setminus \{i, j\}, A \setminus \{x, y\})$.*[27]

By the induction hypothesis and Claim A.4, it follows that there exists an OSP mechanism G^{m-1} that implements $f^{\Gamma\setminus((i,x),(j,y))}$ on the reduced market $(N \setminus \{i, j\}, A \setminus \{x, y\})$. Run the extensive-form assignment mechanism G^{m-1} on the reduced market $(N \setminus \{i, j\}, A \setminus \{x, y\})$.

[26] The proof of Claim A.2 follows by using similar logic as for the proof of Claim A.1. The only adjustment needed for the proof of Claim A.2 over the proof of Claim A.1 is that instead of $\mathcal{T}(\Gamma) = \{i\}$ (which is an assumption of Case A) meaning that individual i is assigned to the root-node of every inheritance tree, we need to consider $x \in A_i$ (which is an assumption of Step 1 in Case B) meaning that individual i is assigned to the root-node of the inheritance tree for x.

[27] The proof of Claim A.4 is relegated to Appendix A.3.2.

By definition, the extensive-form assignment mechanism G^{m+1} implements f^{Γ}. We show that G^{m+1} is OSP for all individuals by showing it for the case where $|N| = 4$. The proof for other cases is similar.

Consider an allocation problem with four individuals $N = \{i_1, i_2, i_3, i_4\}$ and five objects $A = \{x_1, x_2, x_3, x_4, x_5\}$. Let Γ be a collection of inheritance trees such that $\mathcal{T}(\Gamma) = \{i_1, i_2\}$, $A_{i_1} = \{x_1, x_2\}$, and $A_{i_2} = \{x_3, x_4, x_5\}$. In Figure A.3, we provide the structure of the extensive-form assignment mechanism G^4 which implements the hierarchical exchange rule f^{Γ}.

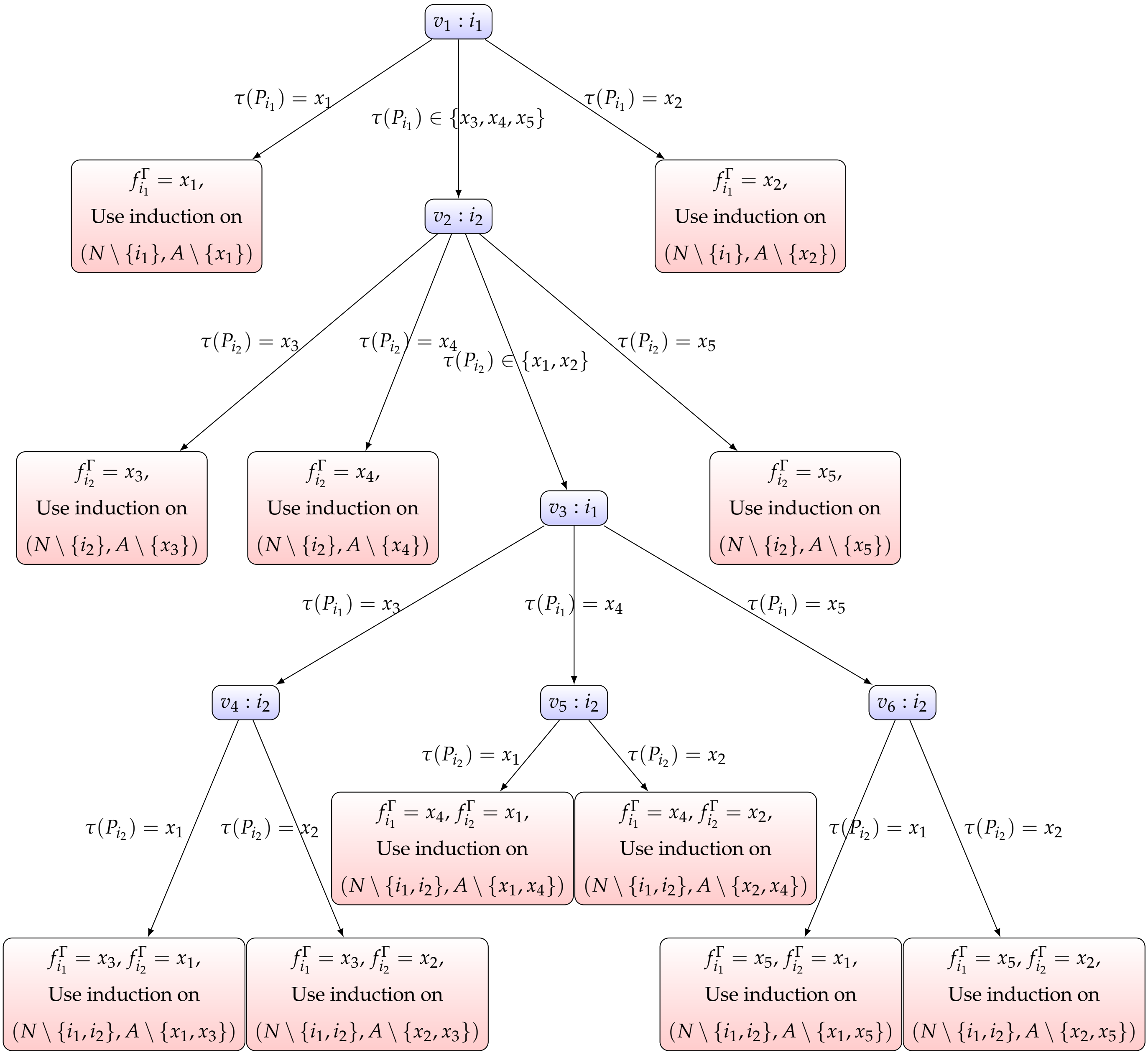


Figure A.3: Structure of G^4

In Figure A.3, node v_1 (which is the root-node of G^4) is assigned to individual i_1 and there are $|A_{i_1}| + 1$ outgoing edges from this node, node v_2 is assigned to individual i_2 and there are $|A_{i_2}| + 1$ outgoing edges from this node, and node v_3 is assigned to individual i_1 and there are $|A_{i_2}|$ outgoing edges from this node. Nodes v_4, v_5, and v_6 are assigned to individual i_2 and there are $|A_{i_1}|$ outgoing edges from each of these nodes.

It follows from the definition of G^4 and Observation A.2 that G^4 satisfies the OSP property at node v_1 (for individual i_1). We distinguish two cases.

(i) Suppose $\tau(P_{i_1}) \in \{x_1, x_2\}$.
Individual i_1 receives her top choice. The first decision node of every other individual comes after i_1 has been assigned, and hence, their strategic decisions are equivalent to that under the OSP mechanism that implements f^Γ restricted to the reduced market.

(ii) Suppose $\tau(P_{i_1}) \in \{x_3, x_4, x_5\}$.
It follows from the definition of G^4 and Observation A.2 that G^4 satisfies the OSP property at node v_2 (for individual i_2).

(a) Suppose $\tau(P_{i_2}) \in \{x_3, x_4, x_5\}$. Individual i_2 receives her top choice. For every other individual, her strategic decision is equivalent to that under the OSP mechanism that implements f^Γ restricted to the reduced market.

(b) Suppose $\tau(P_{i_2}) \in \{x_1, x_2\}$. Both i_1 and i_2 receive their top choices. The first decision node of every other individual comes after i_1 and i_2 have been assigned, and hence, their strategic decisions are equivalent to that under the OSP mechanism that implements f^Γ restricted to the reduced market.

Since Cases (i) and (ii) are exhaustive, it follows that the extensive-form assignment mechanism G^4 is OSP for all individuals, and hence, f^Γ is OSP-implementable for this particular instance.

Since Case A and Case B are exhaustive, it follows that f^Γ is OSP-implementable for $|N| = m + 1$. This completes the proof of the induction step, and thereby completes the proof of the "if" part of Proposition 5.1.

(Only-if part) Suppose f^Γ does not satisfy dual ownership. We show that f^Γ is not OSP-implementable. Since f^Γ does not satisfy dual ownership, there exist a preference profile P'_N and a stage s^* of f^Γ at P'_N such that there are three individuals i_1, i_2, i_3 and three objects x_1, x_2, x_3 in the reduced market in Stage s^* with the property that for all $h = 1, 2, 3$, individual i_h owns the object x_h in Stage s^*.

Note that if an assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is not OSP-implementable on some restricted domain $\tilde{\mathcal{P}}_N \subseteq \mathbb{L}^n(A)$, then f is not OSP-implementable on the whole domain $\mathbb{L}^n(A)$ (see Li (2017) for details). We distinguish the following two cases.

Case A: Suppose $s^* = 1$.

Consider the restricted domain $\tilde{\mathcal{P}}_N$ defined as follows. Each $l \in N \setminus \{i_1, i_2, i_3\}$ has only one (admissible) preference P'_l, and each individual in $\{i_1, i_2, i_3\}$ has two preferences, defined as follows (the dots indicate that all preferences for the corresponding parts are irrelevant and can be chosen arbitrarily).[28]

Individual i_1	Individual i_2	Individual i_3
$x_2x_3x_1\dots$	$x_3x_1x_2\dots$	$x_1x_2x_3\dots$
$x_3x_2x_1\dots$	$x_1x_3x_2\dots$	$x_2x_1x_3\dots$

Table A.1

In Table A.2, we present some facts regarding the outcome of f^Γ on the restricted domain $\tilde{\mathcal{P}}_N$. These facts are deduced by the construction of $\tilde{\mathcal{P}}_N$ along with the assumptions for Case A.

Preference profile	Individual i_1	Individual i_2	Individual i_3	$f^\Gamma_{i_1}$	$f^\Gamma_{i_2}$	$f^\Gamma_{i_3}$
$\tilde{P}^1_N$	$x_2x_3x_1\dots$	$x_3x_1x_2\dots$	$x_1x_2x_3\dots$	x_2	x_3	x_1
$\tilde{P}^2_N$	$x_2x_3x_1\dots$	$x_1x_3x_2\dots$	$x_1x_2x_3\dots$	x_2	x_1	x_3
$\tilde{P}^3_N$	$x_2x_3x_1\dots$	$x_3x_1x_2\dots$	$x_2x_1x_3\dots$	x_1	x_3	x_2
$\tilde{P}^4_N$	$x_2x_3x_1\dots$	$x_1x_3x_2\dots$	$x_2x_1x_3\dots$	x_2	x_1	x_3
$\tilde{P}^5_N$	$x_3x_2x_1\dots$	$x_3x_1x_2\dots$	$x_1x_2x_3\dots$	x_3	x_2	x_1
$\tilde{P}^6_N$	$x_3x_2x_1\dots$	$x_1x_3x_2\dots$	$x_1x_2x_3\dots$	x_3	x_2	x_1
$\tilde{P}^7_N$	$x_3x_2x_1\dots$	$x_3x_1x_2\dots$	$x_2x_1x_3\dots$	x_1	x_3	x_2
$\tilde{P}^8_N$	$x_3x_2x_1\dots$	$x_1x_3x_2\dots$	$x_2x_1x_3\dots$	x_3	x_1	x_2

Table A.2: Partial outcome of f^Γ on $\tilde{\mathcal{P}}_N$

Assume for contradiction that f^Γ is OSP-implementable on $\tilde{\mathcal{P}}_N$. So, there exists an OSP mechanism $\tilde{G}$ that implements f^Γ on $\tilde{\mathcal{P}}_N$. Note that since $f^\Gamma(\tilde{P}^1_N) \neq f^\Gamma(\tilde{P}^8_N)$, there exists a node in the OSP mechanism $\tilde{G}$ that has at least two edges. Also, note that since each individual $l \in N \setminus \{i_1, i_2, i_3\}$ has exactly one preference in $\tilde{\mathcal{P}}_l$, whenever there are more than one outgoing edges from a node, the node must be assigned to some individual in $\{i_1, i_2, i_3\}$. Consider the first node (from the root) v that has two edges and, without loss of generality, assume $\eta^{NI}(v) = i_1$. Consider the preference profiles $\tilde{P}^3_N$ and $\tilde{P}^5_N$. Note that both of them pass through the node v at which $\tilde{P}^3_{i_1}$ and $\tilde{P}^5_{i_1}$ diverge. Further note that $x_3\tilde{P}^3_{i_1}x_1$, $f^\Gamma_{i_1}(\tilde{P}^3_N) = x_1$, and $f^\Gamma_{i_1}(\tilde{P}^5_N) = x_3$. However, the facts that $x_3\tilde{P}^3_{i_1}x_1$, $f^\Gamma_{i_1}(\tilde{P}^3_N) = x_1$, and $f^\Gamma_{i_1}(\tilde{P}^5_N) = x_3$ together contradict OSP-implementability of f^Γ on $\tilde{\mathcal{P}}_N$.

CASE B: Suppose $s^* > 1$.

Recall that for the preference profile P'_N, $A^{s^*-1}(P'_N)$ is the set of assigned objects up to Stage $s^* - 1$ (including Stage $s^* - 1$) of f^Γ at P'_N. Fix a preference $\hat{P} \in \mathbb{L}(A^{s^*-1}(P'_N))$ over these objects.

Consider the restricted domain $\tilde{\mathcal{P}}_N$ defined as follows. Each $l \in N \setminus \{i_1, i_2, i_3\}$ has only one (admissible) preference P'_l, and each individual in $\{i_1, i_2, i_3\}$ has two preferences, defined as follows.[29]

[28]For instance, $x_1x_2x_3\dots$ indicates (any) preference that ranks x_1 first, x_2 second, and x_3 third.

[29]For instance, $\hat{P}x_1x_2x_3\dots$ denotes a preference where objects in $A^{s^*-1}(P'_N)$ are ranked at the top according to the preference

Individual i_1	Individual i_2	Individual i_3
$\hat{P}x_2x_3x_1\dots$	$\hat{P}x_3x_1x_2\dots$	$\hat{P}x_1x_2x_3\dots$
$\hat{P}x_3x_2x_1\dots$	$\hat{P}x_1x_3x_2\dots$	$\hat{P}x_2x_1x_3\dots$

Table A.3

In Table A.4, we present some facts regarding the outcome of f^Γ on the restricted domain $\tilde{\mathcal{P}}_N$ that can be deduced by the construction of the restricted domain $\tilde{\mathcal{P}}_N$ along with the assumptions for Case B. The verification of these facts is left to the reader.

Preference profile	Individual i_1	Individual i_2	Individual i_3	$f^\Gamma_{i_1}$	$f^\Gamma_{i_2}$	$f^\Gamma_{i_3}$
$\tilde{P}^1_N$	$\hat{P}x_2x_3x_1\dots$	$\hat{P}x_3x_1x_2\dots$	$\hat{P}x_1x_2x_3\dots$	x_2	x_3	x_1
$\tilde{P}^2_N$	$\hat{P}x_2x_3x_1\dots$	$\hat{P}x_1x_3x_2\dots$	$\hat{P}x_1x_2x_3\dots$	x_2	x_1	x_3
$\tilde{P}^3_N$	$\hat{P}x_2x_3x_1\dots$	$\hat{P}x_3x_1x_2\dots$	$\hat{P}x_2x_1x_3\dots$	x_1	x_3	x_2
$\tilde{P}^4_N$	$\hat{P}x_2x_3x_1\dots$	$\hat{P}x_1x_3x_2\dots$	$\hat{P}x_2x_1x_3\dots$	x_2	x_1	x_3
$\tilde{P}^5_N$	$\hat{P}x_3x_2x_1\dots$	$\hat{P}x_3x_1x_2\dots$	$\hat{P}x_1x_2x_3\dots$	x_3	x_2	x_1
$\tilde{P}^6_N$	$\hat{P}x_3x_2x_1\dots$	$\hat{P}x_1x_3x_2\dots$	$\hat{P}x_1x_2x_3\dots$	x_3	x_2	x_1
$\tilde{P}^7_N$	$\hat{P}x_3x_2x_1\dots$	$\hat{P}x_3x_1x_2\dots$	$\hat{P}x_2x_1x_3\dots$	x_1	x_3	x_2
$\tilde{P}^8_N$	$\hat{P}x_3x_2x_1\dots$	$\hat{P}x_1x_3x_2\dots$	$\hat{P}x_2x_1x_3\dots$	x_3	x_1	x_2

Table A.4: Partial outcome of f^Γ on $\tilde{\mathcal{P}}_N$

Using a similar argument as for Case A, it follows from Table A.4 that f^Γ is not OSP-implementable on $\tilde{\mathcal{P}}_N$. This completes the proof of the "only-if" part of Proposition 5.1. ■

A.3.1 Proof of Claim A.1

Assume for contradiction that $f^{\Gamma\setminus(i,x)}$ does not satisfy dual ownership on the submarket $(N\setminus\{i\}, A\setminus\{x\})$. Then, there exist $\tilde{P}_{N\setminus\{i\}} \in \mathbb{L}^{|N\setminus\{i\}|}(A\setminus\{x\})$ and a stage s^* of $f^{\Gamma\setminus(i,x)}$ at $\tilde{P}_{N\setminus\{i\}}$ such that there are three individuals i_1, i_2, i_3 and three objects x_1, x_2, x_3 in the reduced market in Stage s^* of $f^{\Gamma\setminus(i,x)}$ at $\tilde{P}_{N\setminus\{i\}}$ with the property that for all $h = 1, 2, 3$, individual i_h owns the object x_h in Stage s^* of $f^{\Gamma\setminus(i,x)}$ at $\tilde{P}_{N\setminus\{i\}}$.

Consider the preference profile $P_N \in \mathbb{L}^n(A)$ such that $\tau(P_i) = x$ and $P_k = x\tilde{P}_k$ for all $k \in N\setminus\{i\}$.[30] By the assumption of Case A, $\mathcal{T}(\Gamma) = \{i\}$, which implies that individual i is assigned to the root-node of Γ_x. This, together with the construction of P_N and the definition of f^Γ, implies that individuals i_1, i_2, and i_3 own the objects x_1, x_2, and x_3, respectively, in Stage s^*+1 of f^Γ at P_N, a contradiction to the fact that f^Γ satisfies dual ownership. This completes the proof of Claim A.1. □

$\hat{P}$, objects x_1, x_2, and x_3 are ranked consecutively after that (in that order), and the ranking of the rest of the objects is arbitrarily.

[30] $x\tilde{P}_k$ denotes the preference that ranks x first, and follows $\tilde{P}_k$ for the ranking of the rest of the objects.

A.3.2 Proof of Claim A.4

Assume for contradiction that $f^{\Gamma\setminus((i,x),(j,y))}$ does not satisfy dual ownership on the submarket $(N \setminus \{i,j\}, A \setminus \{x,y\})$. Then, there exist $\tilde{P}_{N\setminus\{i,j\}} \in \mathbb{L}^{|N\setminus\{i,j\}|}(A \setminus \{x,y\})$ and a stage s^* of $f^{\Gamma\setminus((i,x),(j,y))}$ at $\tilde{P}_{N\setminus\{i,j\}}$ such that there are three individuals i_1, i_2, i_3 and three objects x_1, x_2, x_3 in the reduced market in Stage s^* of $f^{\Gamma\setminus((i,x),(j,y))}$ at $\tilde{P}_{N\setminus\{i,j\}}$ with the property that for all $h = 1,2,3$, individual i_h owns the object x_h in Stage s^* of $f^{\Gamma\setminus((i,x),(j,y))}$ at $\tilde{P}_{N\setminus\{i,j\}}$.

Consider the preference profile $P_N \in \mathbb{L}^n(A)$ such that $\tau(P_i) = x$, $\tau(P_j) = y$ and $P_k = xy\tilde{P}_k$ for all $k \in N \setminus \{i,j\}$.[31] By the assumption of Step 3 in Case B, $x \in A_j$ and $y \in A_i$, which imply that individuals i and j are assigned to the root-nodes of Γ_y and Γ_x, respectively. This, together with the construction of P_N and the definition of f^Γ, implies that individuals i_1, i_2, and i_3 own the objects x_1, x_2, and x_3, respectively, in Stage $s^* + 1$ of f^Γ at P_N, a contradiction to the fact that f^Γ satisfies dual ownership. This completes the proof of Claim A.4. $\square$

Appendix B Proof of Theorem 4.1

We use Proposition 5.1 (which is presented after Theorem 4.1 in the body of the paper) in the proof of Theorem 4.1. Therefore, we have already presented the proof of Proposition 5.1 in the previous appendix (Appendix A).

We first prove a lemma which says that every OSP-implementable, non-bossy, and Pareto efficient assignment rule is reallocation-proof. Next, we combine this lemma with Proposition 5.1 and two results of Pápai (2000) to complete the proof of Theorem 4.1.

B.1 Lemma B.1 and its proof

Lemma B.1 involves the notion of reallocation-proof assignment rules, which we present first.

Definition B.1 (Pápai 2000)**.** An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is *manipulable through reallocation* if there exist $P_N \in \mathbb{L}^n(A)$, distinct individuals $i, j \in N$, and $\tilde{P}_i \in \mathbb{L}(A)$, $\tilde{P}_j \in \mathbb{L}(A)$ such that

(i) $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) R_i f_i(P_N)$,

(ii) $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$, and

(iii) $f_i(P_N) = f_i(\tilde{P}_i, P_{-i}) \neq f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ and $f_j(P_N) = f_j(\tilde{P}_j, P_{-j}) \neq f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$.

An assignment rule is ***reallocation-proof*** if it is not manipulable through reallocation.

Lemma B.1. *Suppose an assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is OSP-implementable, non-bossy, and Pareto efficient. Then, f is reallocation-proof.*

[31] $xy\tilde{P}_k$ denotes the preference that ranks x first, y second, and follows $\tilde{P}_k$ for the ranking of the rest of the objects.

Proof of Lemma B.1. Since f is OSP-implementable, by Remark 2.1, f is strategy-proof. Assume for contradiction that f is not reallocation-proof. Then, there exist $P_N \in \mathbb{L}^n(A)$, distinct individuals $i, j \in N$, and $\tilde{P}_i \in \mathbb{L}(A)$, $\tilde{P}_j \in \mathbb{L}(A)$ such that

(i) $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) R_i f_i(P_N)$,

(ii) $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$, and

(iii) $f_i(P_N) = f_i(\tilde{P}_i, P_{-i}) \neq f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ and $f_j(P_N) = f_j(\tilde{P}_j, P_{-j}) \neq f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$.

Using non-bossiness, $f_i(P_N) = f_i(\tilde{P}_i, P_{-i})$ implies $f(P_N) = f(\tilde{P}_i, P_{-i})$, and $f_j(P_N) = f_j(\tilde{P}_j, P_{-j})$ implies $f(P_N) = f(\tilde{P}_j, P_{-j})$. Combining the facts that $f(P_N) = f(\tilde{P}_i, P_{-i})$ and $f(P_N) = f(\tilde{P}_j, P_{-j})$, we have

$$f(P_N) = f(\tilde{P}_i, P_{-i}) = f(\tilde{P}_j, P_{-j}). \tag{B.1}$$

Claim B.1. $\left\{f_i(P_N), f_j(P_N), f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}), f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})\right\} \subseteq A$.

Proof of Claim B.1. Assume for contradiction that $f_i(P_N) = \emptyset$. By (B.1), we have $f_i(P_N) = f_i(\tilde{P}_j, P_{-j})$. Because $f_i(P_N) = \emptyset$ and $f_i(P_N) = f_i(\tilde{P}_j, P_{-j})$, we have $f_i(\tilde{P}_j, P_{-j}) = \emptyset$. Since f is strategy-proof, $f_i(\tilde{P}_j, P_{-j}) = \emptyset$ implies $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = \emptyset$. However, as $f_i(P_N) = \emptyset$ and $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = \emptyset$, we have a contradiction to $f_i(P_N) \neq f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$. So, it must be that

$$f_i(P_N) \neq \emptyset. \tag{B.2}$$

Using a similar argument, we have

$$f_j(P_N) \neq \emptyset. \tag{B.3}$$

Since $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$, (B.3) implies $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) \neq \emptyset$. Also, the fact $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) R_i f_i(P_N)$, together with (B.2), implies $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) \neq \emptyset$. This completes the proof of Claim B.1. □

Claim B.2. $f_i(P_N) = f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$.

Proof of Claim B.2. Assume for contradiction that $f_i(P_N) \neq f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$. Let $f_i(P_N) = w$, $f_j(P_N) = x$, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = y$, and $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = z$. By Claim B.1, we have $w, x, y, z \neq \emptyset$. Since $f_i(P_N) = w$ and $f_j(P_N) = x$, we have $w \neq x$. Similarly, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = y$ and $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = z$ together imply $y \neq z$. Since $f_i(P_N) \neq f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$, we have $w \neq y$. Similarly $f_j(P_N) \neq f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ implies $x \neq z$, and $f_i(P_N) \neq f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ implies $w \neq z$. Moreover, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$ implies $x \neq y$. However, the facts $w, x, y, z \neq \emptyset$, $w \neq x$, $y \neq z$, $w \neq y$, $x \neq z$, $w \neq z$, and $x \neq y$ together imply w, x, y, and z are all distinct objects.

Since $f_i(P_N) \neq f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$, $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) R_i f_i(P_N)$ implies $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_i f_i(P_N)$. The facts $f_i(P_N) = w$, $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = z$, and $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_i f_i(P_N)$ together imply $z P_i w$. Since $z P_i w$ and $f_i(P_N) = w$, by strategy-proofness, we have

$$f_i(P'_i, P_{-i}) \neq z \text{ for all } P'_i \in \mathbb{L}(A). \tag{B.4}$$

By (B.1) we have $f_i(P_N) = f_i(\tilde{P}_j, P_{-j})$. This, along with the fact that $f_i(P_N) = w$, yields $f_i(\tilde{P}_j, P_{-j}) = w$. Since f is strategy-proof, the facts $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = y$ and $f_i(\tilde{P}_j, P_{-j}) = w$ together imply $y\tilde{R}_i w$, which, along with the fact that $w \neq y$, yields $y\tilde{P}_i w$. Also, combining the facts that $f_i(P_N) = w$ and $f_i(P_N) = f_i(\tilde{P}_i, P_{-i})$, we have $f_i(\tilde{P}_i, P_{-i}) = w$. Since $y\tilde{P}_i w$ and $f_i(\tilde{P}_i, P_{-i}) = w$, by strategy-proofness, we have

$$f_i(P'_i, P_{-i}) \neq y \text{ for all } P'_i \in \mathbb{L}(A). \tag{B.5}$$

Moreover, since $zP_i w$ and $f_i(\tilde{P}_j, P_{-j}) = w$, by strategy-proofness, we have

$$f_i(P'_i, \tilde{P}_j, P_{-i,j}) \neq z \text{ for all } P'_i \in \mathbb{L}(A). \tag{B.6}$$

Let $\hat{P}_i$ rank z first, y second, and w third. Since f is strategy-proof and non-bossy, the fact $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = y$ and (B.6) imply

$$f(\hat{P}_i, \tilde{P}_j, P_{-i,j}) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.7}$$

Similarly, by strategy-proofness and non-bossiness, the fact that $f_i(P_N) = w$ along with (B.4) and (B.5), yields

$$f(\hat{P}_i, P_{-i}) = f(P_N). \tag{B.8}$$

By (B.8) we have $f_j(\hat{P}_i, P_{-i}) = f_j(P_N)$. This, along with the fact $f_j(P_N) = x$, yields $f_j(\hat{P}_i, P_{-i}) = x$. Also, the facts $f_j(P_N) = x$, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = y$, and $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$ together imply $yP_j x$. Since $yP_j x$ and $f_j(\hat{P}_i, P_{-i}) = x$, by strategy-proofness, we have

$$f_j(\hat{P}_i, P'_j, P_{-i,j}) \neq y \text{ for all } P'_j \in \mathbb{L}(A). \tag{B.9}$$

Let $\hat{P}_j$ rank y first and z second. By (B.7) we have $f_j(\hat{P}_i, \tilde{P}_j, P_{-i,j}) = f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$. This, along with the fact $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = z$, yields $f_j(\hat{P}_i, \tilde{P}_j, P_{-i,j}) = z$. Since f is strategy-proof and non-bossy, the fact $f_j(\hat{P}_i, \tilde{P}_j, P_{-i,j}) = z$ and (B.9) imply $f(\hat{P}_i, \hat{P}_j, P_{-i,j}) = f(\hat{P}_i, \tilde{P}_j, P_{-i,j})$. This, along with (B.7), yields

$$f(\hat{P}_i, \hat{P}_j, P_{-i,j}) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.10}$$

Because $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = y$ and $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = z$, (B.10) implies $f_i(\hat{P}_i, \hat{P}_j, P_{-i,j}) = y$ and $f_j(\hat{P}_i, \hat{P}_j, P_{-i,j}) = z$. However, since $z\hat{P}_i y$ and $y\hat{P}_j z$, the facts $f_i(\hat{P}_i, \hat{P}_j, P_{-i,j}) = y$ and $f_j(\hat{P}_i, \hat{P}_j, P_{-i,j}) = z$ together contradict Pareto efficiency.

So, it must be that $f_i(P_N) = f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$. This completes the proof of Claim B.2. □

Since f is Pareto efficient, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$ implies that there exists $k \in N \setminus \{j\}$ such that $f_k(P_N) = f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$. Also, the facts $f_k(P_N) = f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ and $f_i(P_N) \neq f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ together imply $k \neq i$. Let $f_i(P_N) = a, f_j(P_N) = b$, and $f_k(P_N) = c$. Combining the facts that $f_k(P_N) = f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ and $f_k(P_N) = c$, we have $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$. Also the fact $f_i(P_N) = a$ along with Claim B.2, implies $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$. Let $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = d$.

Claim B.3. *a, b, and c are distinct objects, $d \in A$, and a, c, and d are distinct objects.*

Proof of Claim B.3. Since $f_i(P_N) = a, f_j(P_N) = b$, and $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$, by Claim B.1, we have $a \neq \emptyset$, $b \neq \emptyset$, and $c \neq \emptyset$. Moreover, since $f_i(P_N) = a, f_j(P_N) = b$, and $f_k(P_N) = c$, it follows that a, b, and c are all distinct objects.

Now, we show $d \in A$. Assume for contradiction that $d = \emptyset$. Consider the preference profiles presented in Table B.1. In addition to the structure provided in the table, suppose that $P_j^1 = P_j^3$, $P_j^2 = P_j^4$, and $P_k^1 = P_k^2$. Here, l denotes an individual (might be empty) other than i, j, k. Note that such an individual does not change her preference across the mentioned preference profiles.

Preference profiles	Individual i	Individual j	Individual k	$\ldots$	Individual l
P_N^1	$\tilde{P}_i$	$ca\ldots$	$bc\ldots$	$\ldots$	P_l
P_N^2	$\tilde{P}_i$	$cba\ldots$	$bc\ldots$	$\ldots$	P_l
P_N^3	$\tilde{P}_i$	$ca\ldots$	P_k	$\ldots$	P_l
P_N^4	$\tilde{P}_i$	$cba\ldots$	P_k	$\ldots$	P_l

Table B.1: Preference profiles for Claim B.3

The facts $f_j(P_N) = b$, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$, and $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$ together imply $c P_j b$. Moreover, $f_j(P_N) = b$ and (B.1) yield $f_j(\tilde{P}_i, P_{-i}) = b$. Since $c P_j b$ and $f_j(\tilde{P}_i, P_{-i}) = b$, by strategy-proofness, we have

$$f_j(\tilde{P}_i, P'_j, P_{-i,j}) \neq c \text{ for all } P'_j \in \mathbb{L}(A). \tag{B.11}$$

By strategy-proofness and non-bossiness, the fact $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$ and (B.11) imply

$$f(P_N^3) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.12}$$

The facts $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = d$ and $d = \emptyset$ together imply $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = \emptyset$. Moreover, $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = \emptyset$ and (B.12) imply $f_k(P_N^3) = \emptyset$. Since f is strategy-proof and non-bossy, $f_k(P_N^3) = \emptyset$ yields $f(P_N^1) = f(P_N^3)$. This, together with (B.12), implies

$$f(P_N^1) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.13}$$

Similarly, by strategy-proofness and non-bossiness, the fact $f_j(\tilde{P}_i, P_{-i}) = b$ and (B.11) imply $f(P_N^4) = f(\tilde{P}_i,$

P_{-i}). This, along with (B.1), yields

$$f(P_N^4) = f(P_N). \tag{B.14}$$

Since $f_j(P_N) = b$ and $f_k(P_N) = c$, by (B.14) we have $f_j(P_N^4) = b$ and $f_k(P_N^4) = c$. By strategy-proofness, $f_k(P_N^4) = c$ implies $f_k(P_N^2) \in \{b, c\}$. Suppose $f_k(P_N^2) = c$. Since $f_k(P_N^2) = c$ and $f_k(P_N^4) = c$, by non-bossiness and the fact that $f_j(P_N^4) = b$, we have $f_j(P_N^2) = b$. However, $f_j(P_N^2) = b$ and $f_k(P_N^2) = c$ together contradict Pareto efficiency. So, it must be that

$$f_k(P_N^2) = b. \tag{B.15}$$

Since $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$, by (B.13) we have $f_j(P_N^1) = a$. Also, by (B.15) we have $f_j(P_N^2) \neq b$. By strategy-proofness, the facts $f_j(P_N^1) = a$ and $f_j(P_N^2) \neq b$ imply $f_j(P_N^2) = a$. Since $f_j(P_N^1) = a$ and $f_j(P_N^2) = a$, by non-bossiness and (B.13), we have

$$f(P_N^2) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.16}$$

However, since $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = \emptyset$, by (B.16) we have $f_k(P_N^2) = \emptyset$, a contradiction to (B.15). So, it must be that

$$d \in A. \tag{B.17}$$

Since $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$, $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$, and $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = d$, it follows that a, c, and d are all distinct objects. This completes the proof of Claim B.3. □

Claim B.4. *cP_kd.*

Proof of Claim B.4. Assume for contradiction that dR_kc. By Claim B.3, this means dP_kc. Suppose $b = d$. Because dP_kc, this implies bP_kc. Also, the facts $f_j(P_N) = b$, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$, and $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})P_jf_j(P_N)$ together imply cP_jb. However, since cP_jb and bP_kc, the facts $f_j(P_N) = b$ and $f_k(P_N) = c$ together contradict Pareto efficiency. So, it must be that $b \neq d$. This, along with Claim B.3, yields that a, b, c, and d are all distinct objects.

Consider the preference profiles presented in Table B.2. In addition to the structure provided in the table, suppose $P_j^1 = P_j^3$, $P_j^2 = P_j^4$, and $P_k^1 = P_k^2$.

Preference profiles	Individual i	Individual j	Individual k	...	Individual l
P_N^1	$\tilde{P}_i$	$ca\ldots$	$dbc\ldots$	...	P_l
P_N^2	$\tilde{P}_i$	$cba\ldots$	$dbc\ldots$	...	P_l
P_N^3	$\tilde{P}_i$	$ca\ldots$	P_k	...	P_l
P_N^4	$\tilde{P}_i$	$cba\ldots$	P_k	...	P_l

Table B.2: Preference profiles for Claim B.4

The fact $f_j(P_N) = b$ and (B.1) yield $f_j(\tilde{P}_i, P_{-i}) = b$. Moreover, the facts $f_j(P_N) = b$, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$, and $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$ together imply $cP_j b$. Since $cP_j b$ and $f_j(\tilde{P}_i, P_{-i}) = b$, by strategy-proofness, we have

$$f_j(\tilde{P}_i, P'_j, P_{-i,j}) \neq c \text{ for all } P'_j \in \mathbb{L}(A). \tag{B.18}$$

By strategy-proofness and non-bossiness, the fact $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$ and (B.18) imply

$$f(P^3_N) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.19}$$

The fact $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = d$ and (B.19) imply $f_k(P^3_N) = d$. Since f is strategy-proof and non-bossy, $f_k(P^3_N) = d$ yields $f(P^1_N) = f(P^3_N)$. This, together with (B.19), implies

$$f(P^1_N) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.20}$$

Similarly, by strategy-proofness and non-bossiness, the fact $f_j(\tilde{P}_i, P_{-i}) = b$ and (B.18) imply $f(P^4_N) = f(\tilde{P}_i, P_{-i})$. This, along with (B.1), yields

$$f(P^4_N) = f(P_N). \tag{B.21}$$

Since $f_j(P_N) = b$ and $f_k(P_N) = c$, by (B.21) we have $f_j(P^4_N) = b$ and $f_k(P^4_N) = c$. By strategy-proofness, $dP_k c$ and $f_k(P^4_N) = c$ together imply $f_k(P^2_N) \in \{b, c\}$. Suppose $f_k(P^2_N) = c$. Since $f_k(P^2_N) = c$ and $f_k(P^4_N) = c$, by non-bossiness and the fact that $f_j(P^4_N) = b$, we have $f_j(P^2_N) = b$. However, $f_j(P^2_N) = b$ and $f_k(P^2_N) = c$ together contradict Pareto efficiency. So, it must be that

$$f_k(P^2_N) = b. \tag{B.22}$$

Since $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$, by (B.20) we have $f_j(P^1_N) = a$. Also, by (B.22) we have $f_j(P^2_N) \neq b$. By strategy-proofness, the facts $f_j(P^1_N) = a$ and $f_j(P^2_N) \neq b$ together imply $f_j(P^2_N) = a$. Since $f_j(P^1_N) = a$ and $f_j(P^2_N) = a$, by non-bossiness and (B.20), we have

$$f(P^2_N) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.23}$$

However, since $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = d$, by (B.23) we have $f_k(P^2_N) = d$, a contradiction to (B.22). This completes the proof of Claim B.4. □

Fix a preference $\hat{P} \in \mathbb{L}(A \setminus \{a, b, c\})$ over the objects in $A \setminus \{a, b, c\}$. Consider the preference profiles presented in Table B.3. Assume that $P^5_k = P^{10}_k = P^{11}_k$.

Preference profiles	Individual i	Individual j	Individual k	$\dots$	Individual l
P_N^1	$abc\hat{P}$	$cab\hat{P}$	$acb\hat{P}$	$\dots$	P_l
P_N^2	$abc\hat{P}$	$cba\hat{P}$	$acb\hat{P}$	$\dots$	P_l
P_N^3	$acb\hat{P}$	$cab\hat{P}$	$acb\hat{P}$	$\dots$	P_l
P_N^4	$acb\hat{P}$	$cab\hat{P}$	$cab\hat{P}$	$\dots$	P_l
P_N^5	$acb\hat{P}$	$cab\hat{P}$	$cd\dots$	$\dots$	P_l
P_N^6	$bca\hat{P}$	$cba\hat{P}$	$acb\hat{P}$	$\dots$	P_l
P_N^7	$bca\hat{P}$	$cba\hat{P}$	$cab\hat{P}$	$\dots$	P_l
P_N^8	$cab\hat{P}$	$cab\hat{P}$	$cab\hat{P}$	$\dots$	P_l
P_N^9	$cab\hat{P}$	$cba\hat{P}$	$cab\hat{P}$	$\dots$	P_l
P_N^{10}	$cab\hat{P}$	$cab\hat{P}$	$cd\dots$	$\dots$	P_l
P_N^{11}	$cab\hat{P}$	$cba\hat{P}$	$cd\dots$	$\dots$	P_l
P_N^{12}	$cba\hat{P}$	$cab\hat{P}$	$acb\hat{P}$	$\dots$	P_l
P_N^{13}	$cba\hat{P}$	$cba\hat{P}$	$acb\hat{P}$	$\dots$	P_l
P_N^{14}	$cba\hat{P}$	$cab\hat{P}$	$cab\hat{P}$	$\dots$	P_l
P_N^{15}	$cba\hat{P}$	$cba\hat{P}$	$cab\hat{P}$	$\dots$	P_l

Table B.3: Preference profiles for Lemma B.1

The facts $f_j(P_N) = b$, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$, and $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$ together imply cP_jb. Since cP_jb and $f_j(P_N) = b$, by strategy-proofness, we have

$$f_j(P'_j, P_{-j}) \neq c \text{ for all } P'_j \in \mathbb{L}(A). \tag{B.24}$$

Combining the fact $f_j(P_N) = b$ with (B.1), we have $f_j(\tilde{P}_i, P_{-i}) = f_j(\tilde{P}_j, P_{-j}) = b$. Since f is strategy-proof, the facts $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$ and $f_j(\tilde{P}_i, P_{-i}) = b$ together imply $a\tilde{R}_jb$, which along with Claim B.3, yields $a\tilde{P}_jb$. Since $a\tilde{P}_jb$ and $f_j(\tilde{P}_j, P_{-j}) = b$, by strategy-proofness, we have

$$f_j(P'_j, P_{-j}) \neq a \text{ for all } P'_j \in \mathbb{L}(A). \tag{B.25}$$

However, since $f_j(\tilde{P}_j, P_{-j}) = b$, by strategy-proofness and non-bossiness along with (B.24) and (B.25), we have $f(P^5_j, P_{-j}) = f(\tilde{P}_j, P_{-j})$. By (B.1), this, in particular, means

$$f_i(P^5_j, P_{-j}) = a,\, f_j(P^5_j, P_{-j}) = b, \text{ and } f_k(P^5_j, P_{-j}) = c. \tag{B.26}$$

By moving the preferences of the individuals $l \in \{i, k\}$ from P_l to P^5_l one by one, and by applying strategy-proofness and non-bossiness on (B.26) each time, we conclude

$$f_i(P^5_N) = a, f_j(P^5_N) = b, \text{ and } f_k(P^5_N) = c. \tag{B.27}$$

Using strategy-proofness and non-bossiness, we obtain from (B.27) that

$$f_i(P^4_N) = a, f_j(P^4_N) = b, \text{ and } f_k(P^4_N) = c. \tag{B.28}$$

By strategy-proofness, the facts cP_jb and $f_j(\tilde{P}_i, P_{-i}) = b$ together imply

$$f_j(\tilde{P}_i, P'_j, P_{-i,j}) \neq c \text{ for all } P'_j \in \mathbb{L}(A). \tag{B.29}$$

Since f is strategy-proof, the fact $f_j(\tilde{P}_i, P_{-i}) = b$ and (B.29) imply $f_j(\tilde{P}_i, P^{11}_j, P_{-i,j}) = b$. Moreover, since $f_j(\tilde{P}_i, P_{-i}) = b$ and $f_j(\tilde{P}_i, P^{11}_j, P_{-i,j}) = b$, by non-bossiness, we have $f(\tilde{P}_i, P^{11}_j, P_{-i,j}) = f(\tilde{P}_i, P_{-i})$. This, together with (B.1), yields

$$f(\tilde{P}_i, P^{11}_j, P_{-i,j}) = f(P_N). \tag{B.30}$$

By (B.1) we have $f_i(P_N) = f_i(\tilde{P}_j, P_{-j})$. This, along with the fact that $f_i(P_N) = a$, yields $f_i(\tilde{P}_j, P_{-j}) = a$. Since f is strategy-proof, the facts $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$ and $f_i(\tilde{P}_j, P_{-j}) = a$ together imply $c\tilde{R}_ia$, which along with Claim B.3, yields $c\tilde{P}_ia$. Also, the fact $f_i(P_N) = a$, together with (B.30), implies $f_i(\tilde{P}_i, P^{11}_j, P_{-i,j}) = a$. Since $c\tilde{P}_ia$ and $f_i(\tilde{P}_i, P^{11}_j, P_{-i,j}) = a$, by strategy-proofness, we have $f_i(P^{11}_i, P^{11}_j, P_{-i,j}) = a$. Moreover, since $f_i(\tilde{P}_i, P^{11}_j, P_{-i,j}) = a$ and $f_i(P^{11}_i, P^{11}_j, P_{-i,j}) = a$, by non-bossiness, we have $f(P^{11}_i, P^{11}_j, P_{-i,j}) = f_i(\tilde{P}_i, P^{11}_j, P_{-i,j})$. This, together with (B.30), implies

$$f_i(P^{11}_i, P^{11}_j, P_{-i,j}) = a, f_j(P^{11}_i, P^{11}_j, P_{-i,j}) = b, \text{ and } f_k(P^{11}_i, P^{11}_j, P_{-i,j}) = c. \tag{B.31}$$

Using strategy-proofness and non-bossiness, we obtain from (B.31) that

$$f_i(P^{11}_N) = a, f_j(P^{11}_N) = b, \text{ and } f_k(P^{11}_N) = c. \tag{B.32}$$

Again, using strategy-proofness and non-bossiness, we obtain from (B.32) that

$$f_i(P^9_N) = a, f_j(P^9_N) = b, \text{ and } f_k(P^9_N) = c. \tag{B.33}$$

Since f is strategy-proof, the fact $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$ and (B.29) imply $f_j(\tilde{P}_i, P^{10}_j, P_{-i,j}) = a$. Since $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$ and $f_j(\tilde{P}_i, P^{10}_j, P_{-i,j}) = a$, by non-bossiness, we have $f(\tilde{P}_i, P^{10}_j, P_{-i,j}) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$. This, in particular, means

$$f_i(\tilde{P}_i, P_j^{10}, P_{-i,j}) = c, f_j(\tilde{P}_i, P_j^{10}, P_{-i,j}) = a, \text{ and } f_k(\tilde{P}_i, P_j^{10}, P_{-i,j}) = d. \tag{B.34}$$

From Claim B.4, we have cP_kd. Since f is strategy-proof and cP_kd, (B.34) implies $f_k(\tilde{P}_i, P_j^{10}, P_k^{10}, P_{-i,j,k}) = d$. Moreover, since $f_k(\tilde{P}_i, P_j^{10}, P_{-i,j}) = d$ and $f_k(\tilde{P}_i, P_j^{10}, P_k^{10}, P_{-i,j,k}) = d$, by non-bossiness, (B.34) implies

$$f_i(\tilde{P}_i, P_j^{10}, P_k^{10}, P_{-i,j,k}) = c, f_j(\tilde{P}_i, P_j^{10}, P_k^{10}, P_{-i,j,k}) = a, \text{ and } f_k(\tilde{P}_i, P_j^{10}, P_k^{10}, P_{-i,j,k}) = d. \tag{B.35}$$

Using strategy-proofness and non-bossiness, we obtain from (B.35) that

$$f_i(P_N^{10}) = c, f_j(P_N^{10}) = a, \text{ and } f_k(P_N^{10}) = d. \tag{B.36}$$

By strategy-proofness, (B.33) implies $f_j(P_N^8) \in \{a, b\}$. Suppose $f_j(P_N^8) = b$. Since $f_j(P_N^8) = b$ and $f_j(P_N^9) = b$, by non-bossiness, (B.33) implies $f_k(P_N^8) = c$. However, since $f_k(P_N^8) = c$, (B.36) contradicts strategy-proofness. So, it must be that $f_j(P_N^8) = a$. By strategy-proofness, (B.28) implies $f_i(P_N^8) \in \{a, c\}$. This, along with the fact that $f_j(P_N^8) = a$, yields

$$f_i(P_N^8) = c \text{ and } f_j(P_N^8) = a. \tag{B.37}$$

Using strategy-proofness and non-bossiness, we obtain from (B.37) that

$$f_i(P_N^{14}) = c \text{ and } f_j(P_N^{14}) = a. \tag{B.38}$$

By strategy-proofness, (B.38) implies $f_j(P_N^{15}) \in \{a, b\}$. Suppose $f_j(P_N^{15}) = a$. Since $f_j(P_N^{14}) = a$ and $f_j(P_N^{15}) = a$, by non-bossiness and (B.38), we have $f_i(P_N^{15}) = c$. However, since $f_i(P_N^{15}) = c$, (B.33) contradicts strategy-proofness. So, it must be that $f_j(P_N^{15}) = b$. By strategy-proofness, (B.33) implies $f_i(P_N^{15}) \in \{a, b\}$. This, along with the fact that $f_j(P_N^{15}) = b$, yields $f_i(P_N^{15}) = a$. By non-bossiness, this and (B.33) imply

$$f_i(P_N^{15}) = a, f_j(P_N^{15}) = b, \text{ and } f_k(P_N^{15}) = c. \tag{B.39}$$

Using strategy-proofness and non-bossiness, we obtain from (B.39) that

$$f_i(P_N^7) = a, f_j(P_N^7) = b, \text{ and } f_k(P_N^7) = c. \tag{B.40}$$

By (B.38) we have $f_k(P_N^{14}) \notin \{a, c\}$. By strategy-proofness, the fact $f_k(P_N^{14}) \notin \{a, c\}$ implies $f_k(P_N^{12}) =$

$f_k(P_N^{14})$. This, by non-bossiness and (B.38), implies

$$f_i(P_N^{12}) = c \text{ and } f_j(P_N^{12}) = a. \tag{B.41}$$

By strategy-proofness, (B.41) implies $f_i(P_N^3) \in \{a,c\}$. Suppose $f_i(P_N^3) = c$. Since $f_i(P_N^{12}) = c$ and $f_i(P_N^3) = c$, by non-bossiness and (B.41), we have $f_j(P_N^3) = a$. However, $f_i(P_N^3) = c$ and $f_j(P_N^3) = a$ together contradict Pareto efficiency. So, it must be that $f_i(P_N^3) = a$. By strategy-proofness, (B.27) implies $f_k(P_N^3) \in \{a,c\}$. This, along with the fact that $f_i(P_N^3) = a$, yields

$$f_i(P_N^3) = a \text{ and } f_k(P_N^3) = c. \tag{B.42}$$

Using strategy-proofness and non-bossiness, we obtain from (B.42) that

$$f_i(P_N^1) = a \text{ and } f_k(P_N^1) = c. \tag{B.43}$$

By (B.43) we have $f_j(P_N^1) \notin \{a,c\}$. By strategy-proofness, $f_j(P_N^1) \notin \{a,c\}$ implies $f_j(P_N^2) = f_j(P_N^1)$. This, by non-bossiness and (B.43), implies

$$f_i(P_N^2) = a \text{ and } f_k(P_N^2) = c. \tag{B.44}$$

By (B.39) we have $f_i(P_N^{15}) = a$ and $f_k(P_N^{15}) = c$. By strategy-proofness, $f_k(P_N^{15}) = c$ implies $f_k(P_N^{13}) \in \{a, c\}$. Suppose $f_k(P_N^{13}) = c$. Since $f_k(P_N^{15}) = c$ and $f_k(P_N^{13}) = c$, by non-bossiness and the fact that $f_i(P_N^{15}) = a$, we have $f_i(P_N^{13}) = a$. However, $f_i(P_N^{13}) = a$ and $f_k(P_N^{13}) = c$ together contradict Pareto efficiency. So, it must be that $f_k(P_N^{13}) = a$. By strategy-proofness, (B.41) implies $f_j(P_N^{13}) \in \{a,b\}$. This, along with the fact that $f_k(P_N^{13}) = a$, yields $f_j(P_N^{13}) = b$. By strategy-proofness, (B.44) implies $f_i(P_N^{13}) \in \{a,b,c\}$. This, together with the facts that $f_j(P_N^{13}) = b$ and $f_k(P_N^{13}) = a$, implies

$$f_i(P_N^{13}) = c, f_j(P_N^{13}) = b, \text{ and } f_k(P_N^{13}) = a. \tag{B.45}$$

By strategy-proofness, (B.40) implies $f_k(P_N^6) \in \{a,c\}$. Suppose $f_k(P_N^6) = c$. Since $f_k(P_N^7) = c$ and $f_k(P_N^6) = c$, by non-bossiness and (B.40), we have $f_i(P_N^6) = a$. However, $f_i(P_N^6) = a$ and $f_k(P_N^6) = c$ together contradict Pareto efficiency. So, it must be that $f_k(P_N^6) = a$. Also, by (B.45) we have $f_i(P_N^{13}) = c$ and $f_j(P_N^{13}) = b$. By strategy-proofness, $f_i(P_N^{13}) = c$ implies $f_i(P_N^6) \in \{b,c\}$. Suppose $f_i(P_N^6) = c$. Since $f_i(P_N^{13}) = c$ and $f_i(P_N^6) = c$, by non-bossiness and the fact that $f_j(P_N^{13}) = b$, we have $f_j(P_N^6) = b$. However, $f_i(P_N^6) = c$ and $f_j(P_N^6) = b$ together contradict Pareto efficiency. So, it must be that $f_i(P_N^6) = b$. Combining the facts that $f_i(P_N^6) = b$ and $f_k(P_N^6) = a$, we have

$$f_i(P_N^6) = b \text{ and } f_k(P_N^6) = a. \tag{B.46}$$

Now we complete the proof of Lemma B.1. Consider the restricted domain $\tilde{\mathcal{P}}_N \subseteq \mathbb{L}^n(A)$ with only three preference profiles as follows.

Preference profiles	Individual i	Individual j	Individual k	...	Individual l
P_N^6	$bca\hat{P}$	$cba\hat{P}$	$acb\hat{P}$	...	P_l
P_N^7	$bca\hat{P}$	$cba\hat{P}$	$cab\hat{P}$	...	P_l
P_N^{14}	$cba\hat{P}$	$cab\hat{P}$	$cab\hat{P}$	...	P_l

Table B.4: Preference profiles of $\tilde{\mathcal{P}}_N$

By (B.38), (B.40), and (B.46), we have

Preference profiles	$f_i(P_N)$	$f_j(P_N)$	$f_k(P_N)$
P_N^6	b		a
P_N^7	a	b	c
P_N^{14}	c	a	

Table B.5: Partial outcome of f on $\tilde{\mathcal{P}}_N$

Since f is OSP-implementable on $\mathbb{L}^n(A)$, it must be OSP-implementable on the restricted domain $\tilde{\mathcal{P}}_N$. Let $\tilde{G}$ be an OSP mechanism that implements f on $\tilde{\mathcal{P}}_N$.

Note that since $f(P_N^6) \neq f(P_N^7)$, there exists a node in the OSP mechanism $\tilde{G}$ that has at least two edges. Also, note that since each individual $l \in N \setminus \{i, j, k\}$ has exactly one preference in $\tilde{\mathcal{P}}_l$, whenever there are at least two outgoing edges from a node, that node must be assigned to some individual in $\{i, j, k\}$. Consider the first node (from the root) v that has two edges.

Suppose $\eta^{NI}(v) = i$. Consider the preference profiles P_N^7 and P_N^{14}. Note that both of them pass through the node v at which P_i^7 and P_i^{14} diverge. Further note that cP_i^7a, $f_i(P_N^7) = a$, and $f_i(P_N^{14}) = c$. However, the facts that cP_i^7a, $f_i(P_N^7) = a$, and $f_i(P_N^{14}) = c$ together contradict OSP-implementability of f on $\tilde{\mathcal{P}}_N$. So, it must be that $\eta^{NI}(v) \neq i$.

Suppose $\eta^{NI}(v) = k$. Consider the preference profiles P_N^6 and P_N^{14}. Note that both of them pass through the node v at which P_k^6 and P_k^{14} diverge. Further note that $f_k(P_N^6) = a$, $f_k(P_N^{14}) \notin \{a, c\}$, and $aP_k^{14}x$ for all $x \in A \setminus \{a, c\}$. Since $aP_k^{14}x$ for all $x \in A \setminus \{a, c\}$, the facts that $f_k(P_N^6) = a$ and $f_k(P_N^{14}) \notin \{a, c\}$ together contradict OSP-implementability of f on $\tilde{\mathcal{P}}_N$. So, it must be that $\eta^{NI}(v) \neq k$.

Since $\eta^{NI}(v) \neq i$ and $\eta^{NI}(v) \neq k$, it must be that $\eta^{NI}(v) = j$. We distinguish the following two cases.

Case 1: $f_j(P^6_N) = c$.

Consider the preference profiles P^6_N and P^{14}_N. Note that both of them pass through the node v at which P^6_j and P^{14}_j diverge. Further note that $cP^{14}_j a$, $f_j(P^6_N) = c$, and $f_j(P^{14}_N) = a$. However, the facts that $cP^{14}_j a$, $f_j(P^6_N) = c$, and $f_j(P^{14}_N) = a$ together contradict OSP-implementability of f on $\tilde{\mathcal{P}}_N$.

Case 2: $f_j(P^6_N) \neq c$.

Consider the preference profiles P^6_N and P^{14}_N. Note that both of them pass through the node v at which P^6_j and P^{14}_j diverge. Further note that $f_j(P^6_N) \notin \{a, b, c\}$, $f_j(P^{14}_N) = a$, and $aP^6_j x$ for all $x \in A \setminus \{a, b, c\}$. Since $aP^6_j x$ for all $x \in A \setminus \{a, b, c\}$, the facts that $f_j(P^6_N) \notin \{a, b, c\}$ and $f_j(P^{14}_N) = a$ together contradict OSP-implementability of f on $\tilde{\mathcal{P}}_N$. This completes the proof of Lemma B.1. ■

B.2 Completion of the proof of Theorem 4.1

We present two results from Pápai (2000), which we use to complete the proof of Theorem 4.1.

Theorem B.1 (Main theorem in Pápai (2000))**.** *An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is group strategy-proof, Pareto efficient, and reallocation-proof if and only if f is a hierarchical exchange rule.*

Lemma B.2 (Lemma 1 in Pápai (2000))**.** *An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is group strategy-proof if and only if it is strategy-proof and non-bossy.*

Proof of Theorem 4.1. ***(If part)*** Let f be a hierarchical exchange rule satisfying dual ownership. By Proposition 5.1, f is OSP-implementable. Moreover, since f is a hierarchical exchange rule, by Theorem B.1, f is group strategy-proof and Pareto efficient. The fact that f is group strategy-proof along with Lemma B.2, implies f is non-bossy. This completes the proof of the "if" part of Theorem 4.1.

(Only-if part) Let f be an OSP-implementable, non-bossy, and Pareto efficient assignment rule. By Lemma B.1, f is reallocation-proof. Since f is OSP-implementable, by Remark 2.1, f is strategy-proof. This, together with Lemma B.2 and the fact that f is non-bossy, implies f is group strategy-proof. Since f is group strategy-proof, Pareto efficient, and reallocation-proof, by Theorem B.1, f is a hierarchical exchange rule. Moreover, since f is an OSP-implementable hierarchical exchange rule, by Proposition 5.1, f is a hierarchical exchange rule satisfying dual ownership. This completes the proof of the "only-if" part of Theorem 4.1. ■

Appendix C Example to clarify the difference between dual dictatorship (Troyan, 2019) and dual ownership of FPTTC rules

Troyan (2019) deals with the case where $|N| = |A|$. Therefore, we explain the difference between dual dictatorship and dual ownership of FPTTC rules for this case only.

Example C.1. Consider an allocation problem with four individuals $N = \{i, j, k, l\}$ and four objects $A = \{w, x, y, z\}$. Let $\succ_A$ be as follows:

$\succ_w$	$\succ_x$	$\succ_y$	$\succ_z$
i	i	l	l
j	j	j	k
k	k	k	j
l	l	i	i

Table C.1: Priority structure for Example C.1

Consider the FPTTC rule $T^{\succ_A}$ associated with the priority structure given in Table C.1. First, we argue that it satisfies dual ownership. Since either individual i or individual l appears at the top position in each priority, it follows that for any preference profile, individuals i and l will own all the objects at Step 1 of $T^{\succ_A}$. Moreover, since there are only four individuals in the original market, for any preference profile, at any step from Step 3 onward of $T^{\succ_A}$, there will remain at most two individuals in the corresponding submarket and hence dual ownership will be vacuously satisfied. In what follows, we show that dual ownership will also be satisfied at Step 2 for any preference profile. We distinguish three cases based on the possible assignments at Step 1.

(i) Suppose only individual i is assigned some object at Step 1. No matter whether individual i is assigned object w or object x, individuals j and l will own all the objects at Step 2.

(ii) Suppose only individual l is assigned some object at Step 1.

  (a) If l is assigned object y, then individuals i and k will own all the objects at Step 2.

  (b) If l is assigned object z, then individuals i and j will own all the objects at Step 2.

(iii) Suppose both i and l are assigned some objects at Step 1. Since there are only four individuals in the original market, only two individuals will remain in the reduced market at Step 2.

Since Cases (i), (ii), and (iii) are exhaustive, it follows that $T^{\succ_A}$ satisfies dual ownership. We now proceed to show that it does not satisfy dual dictatorship. Consider the submarket consisting of individuals i, j, and k and objects x, y, and z. Here, individuals i, j, and k will own objects x, y, and z, respectively, and hence $T^{\succ_A}$ under consideration violates dual dictatorship.

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