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Abstract

This study considers a three-period overlapping generations model with an endogenous growth setting, in which an agent borrows in the first period and repays the loan in the second period under a perfect credit market. Two educational subsidy schemes are considered: one is provided when an agent borrows and the other is provided when the agent repays their loan. This study compares the growth rates and social welfare under each educational subsidy scheme at a unique balanced growth path equilibrium. The first contribution of this paper is that it provides sufficient conditions under which the growth rate in one scheme is higher than that in the other. A key to determining the size relationship of growth rates is whether the production of goods and services is physical-capital-intensive, which determines the size of the interest rate. The second contribution is that it shows that higher growth and higher social welfare may not be achieved simultaneously. Specifically, this paper presents a case wherein, even if the growth rate when student loans are subsidized is higher than that when the cost of education is subsidized, social welfare defined by the Golden Rule criterion in the former scheme can be lower than that in the latter scheme.

Keywords: Endogenous growth, educational subsidy, balanced growth path equilibrium, growth rate, social welfare, Golden Rule

JEL Classification: O40, I22, H52

1 Introduction

Since the studies of Lucas (1988) and Azariadis and Drazen (1990), human capital externalities have become important for understanding the sources of income differences across countries. However, a laissez-faire equilibrium allocation may not be dynamically efficient under human capital externalities. To correct the inefficiencies caused by these externalities, some form of educational subsidy is needed.

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If individuals need to borrow to pursue education in a perfect credit market, the question is when educational subsidies should be provided: when individuals borrow or when they repay their loans?[1] If the timing of providing educational subsidies differs, an individual's decision-making regarding human capital investment and savings will also differ, and this difference will cause different growth rates for a balanced growth path. Therefore, this study considers a three-period overlapping generations model with physical and human capital, investigating which educational subsidy scheme will lead to a higher growth rate for a balanced growth path. Furthermore, this study investigates whether a higher growth rate leads to higher social welfare.

The model is a three-period overlapping generations model, in which an agent borrows for education in the first period and repays the loan in the second period. A perfect credit market is assumed. Under this setting, two educational subsidy schemes are considered: one is when educational subsidies are provided in the first period, when an agent borrows, and the other is when subsidies are provided in the second period, when an agent repays their loan.[2] The educational subsidy is financed by proportional labor income tax and I assume the government budget is balanced in each period. Hence, if the cost of education is subsidized in the first period of an agent's life, the educational subsidy is an *inter*-generational transfer scheme. If the student loans are subsidized in the second period of an agent's life, the educational subsidy is an *intra*-generational transfer scheme. Under each educational subsidy scheme, a perfect foresight competitive equilibrium for a balanced growth path, called the balanced growth path (BGP) equilibrium, is characterized and the growth rates and social welfare of the BGP equilibrium under both schemes are compared.

The motivation of this study is closely related to that of Eckwert and Zilcha (2014). That is, Eckwert and Zilcha (2014) considers a two-period model where human capital is under-invested and shows that two subsidization schemes similar to those in this paper induce individuals to invest more in terms of human capital and lead to a more socially desirable income distribution. Additionally, Eckwert and Zilcha (2014) compares the two educational subsidy schemes from the perspective of income distribution, whereas this study compares them from the perspectives of growth rate and social welfare in the long run. In contrast to the main results of Eckwert and Zilcha (2014), this study implies that there is no dominance relationship between the two schemes from the perspectives of long-run growth rate and social welfare. Furthermore, this study shows that even if the growth rate when student loans are subsidized is higher than that when the cost of education is subsidized, social welfare in the former scheme can be lower than that in the latter scheme.

In an endogenous growth setting, some studies, such as Blankenau (2005) and Docquier et al. (2007), consider a model for subsidizing the cost of education, while other papers, such as Yakita (2004), Del Rey and Lopez-Garcia (2013), Del Rey and Lopez-Garcia (2016), and Del Rey and Lopez-Garcia (2019), consider a model with subsidizing loans. To the best of my knowledge, this study is the first to compare the effects of different educational subsidy schemes under an endogenous growth setting.

One contribution of this paper is providing sufficient conditions under which one of the two growth rates under the two educational subsidy schemes is larger than the other. A key to determining the size relationship of the two growth rates is whether the output production is physical-capital-intensive. When the production function is sufficiently physical-capital-intensive, the interest rate tends to be high.

[1] As shown by Boldrin and Montes (2005), if a credit market is imperfect, educational subsidies should be provided when individuals borrow. Hence, this study assumes a perfect credit market.

[2] Eckwert and Zilcha (2014) call the first scheme *subsidizing tuition* and the second one *subsidizing student loans*.

If a young agent borrows the cost of education, they need to repay it with interest. A higher interest rate increases the interest payment. If an educational subsidy is provided when a young agent borrows, the interest payment will become small, even if the agent borrows a large amount. This merit is large when the interest rate is high. Therefore, when the production function is sufficiently physical-capital-intensive, the growth rate in an economy subsidizing the cost of education is higher than that in an economy subsidizing student loans. Conversely, when the production function is sufficiently effective-labor-intensive, the opposite is likely to hold because the interest rate tends to be low.

Another contribution of this paper is that it shows that a higher growth rate and higher social welfare may not be achieved simultaneously. Specifically, this paper demonstrates that even if the growth rate when subsidizing student loans is higher than that when subsidizing the cost of education, social welfare defined by the Golden Rule criterion under the latter scheme can be higher than that under the former scheme. This finding implies that policy makers must make their goal when they provide educational subsidy because, depending on their goal, the type of educational subsidy that should be used will differ.

The remainder of this paper is organized as follows. Section 2 characterizes the BGP equilibrium in the model subsidizing the cost of education, and Section 3 describes the BGP equilibrium in the model subsidizing loans. In Section 4, the growth rates and social welfare under the previous BGP equilibria are compared, and Section 5 concludes the paper.

2 Model: Subsidizing the cost of education

Time is discrete and continues forever, namely $t = 1, 2, \ldots$. An agent lives for three periods: young, middle, and old. Therefore, in each period t, three generations coexist. When an agent is young, in period $t-1$, they decide on the educational expenditure, e_{t-1}, to accumulate human capital. Assuming that a young agent has no wealth or income, they must borrow to manage the educational expenditure. Let b_{t-1} denote the real debt of an agent born in period t. Additionally, a young agent receives educational subsidies provided by the government. If a young agent spends e_{t-1} to accumulate human capital, then the agent will receive an educational subsidy, denoted by $\sigma_{t-1}e_{t-1}$ ($\sigma_{t-1} \in [0,1)$). Thus, the first period budget constraint in period $t-1$ is

$$(1-\sigma_{t-1})e_{t-1} = b_{t-1}. \tag{1}$$

A young agent who borrows b_{t-1} in period $t-1$ will repay the debt in the next period with interest rate r_t. Furthermore, I assume a perfect credit market. If a young agent spends e_{t-1} for human capital accumulation in period $t-1$, the agent's human capital in period t will be

$$h_t = \theta e_{t-1}^{\eta} h_{t-1}^{1-\eta}, \tag{2}$$

where $\theta > 0$ and $\eta \in (0,1)$. Following literature such as Azariadis and Drazen (1990), Glomm and Ravikumar (1992), de la Croix and Doepke (2003), and de la Croix and Doepke (2004), I assume that human capital fully depreciates from one period to another. Assume that $e_0 > 0$ and $h_0 > 0$.

A middle-aged agent in period t is endowed with one unit of time and supplies this unit for labor inelastically. A middle-aged agent with human capital h_t receives an effective labor income of $w_t h_t$, consumes c_t, repays debt $(1+r_t)b_{t-1}$, and saves s_t, where w_t is the effective wage in period t and r_t is the interest rate in period t. Additionally, a middle-aged agent will pay a labor income tax whose tax rate is $\tau \in [0,1)$. I assume that tax rate τ is time-invariant and set by the government. Then, the budget constraint for the middle-aged agent is

$$c_t + s_t + (1+r_t)b_{t-1} = (1-\tau)w_t h_t. \tag{3}$$

When an agent becomes old, they retire. An old agent in period $t+1$ has interest income $(1+r_{t+1})s_t$ and consumes d_{t+1}. Therefore, the budget constraint for an old agent is

$$d_{t+1} = (1+r_{t+1})s_t. \tag{4}$$

From Equations (1), (3), and (4), the lifetime budget constraint of an agent born in period $t-1$ is:

$$c_t + \frac{d_{t+1}}{1+r_{t+1}} + (1+r_t)(1-\sigma_{t-1})e_{t-1} = (1-\tau)w_t h_t. \tag{5}$$

An agent's lifetime utility is expressed as

$$U(c_t, d_{t+1}) := \ln(c_t) + \beta \ln(d_{t+1}),$$

where $\beta \in (0,1)$ is a discount factor.

Let N_t be the population of middle-aged agents in period t. Population growth rate $n > -1$ is given exogenously. Therefore, the population of middle-aged agents in period $t+1$ is

$$N_{t+1} = (1+n)N_t.$$

Assume that $N_0 > 0$.

There exists a representative firm in the economy whose production function is expressed as

$$Y_t = AF(K_t, H_t) := AK_t^{\alpha} H_t^{1-\alpha},$$

where $A > 0$ represents the productivity, $\alpha \in (0,1)$, and K_t and H_t are the aggregate capital stock and aggregate effective labor in the economy, respectively. Given effective wage w_t and real rental rate of capital r_t, the firm's profit in period t is

$$AK_t^{\alpha} H_t^{1-\alpha} - (1+r_t)K_t - w_t H_t.$$

I assume that physical capital is fully depreciated after production. Let $k_t := K_t/N_t$ denote the per capita physical capital in period t.

The government provides educational subsidies for young agents by taxing middle-aged agents. The government balances the budget for every period. Therefore, the government's budget constraint in period t is

$$N_t \tau w_t h_t = N_{t+1} \sigma_t e_t. \tag{6}$$

An equilibrium concept is a standard *perfect foresight competitive equilibrium*. The formal definition is expressed as follows:

Definition 2.1. *Given $N_0 > 0$, $h_0 > 0$, $e_0 > 0$, $s_0 > 0$, $\sigma_0 > 0$, and $\tau \in [0,1)$, an* **equilibrium** *consists of a consumption sequence $(d_1^*, (c_t^*, d_{t+1}^*)_{t=1}^{\infty})$; a sequence of educational expenditure $(e_t^*)_{t=1}^{\infty}$; a sequence of savings $(s_t^*)_{t=1}^{\infty}$; a sequence of human capital $(h_t^*)_{t=1}^{\infty}$; a sequence of production inputs $(K_t^*, H_t^*)_{t=1}^{\infty}$; a sequence of prices $(w_t^*, r_t^*)_{t=1}^{\infty}$; and a sequence of educational subsidy $(\sigma_t^*)_{t=1}^{\infty}$ so that*

1. *For all $t \geq 2$, given $(r_t^*, w_t^*, r_{t+1}^*)$, h_{t-1}^*, and σ_{t-1}^*, $(e_{t-1}^*, c_t^*, d_{t+1}^*)$ is a solution to:*

$$\begin{array}{ll} \max_{e_{t-1}, c_t, d_{t+1}} & \ln(c_t) + \beta \ln(d_{t+1}) \\ \text{s.t.} & c_t + \dfrac{d_{t+1}}{1+r_{t+1}^*} + (1+r_t^*)(1-\sigma_{t-1}^*)e_{t-1} = (1-\tau)w_t^* h_t, \\ & h_t = \theta e_{t-1}^{\eta} (h_{t-1}^*)^{1-\eta}. \end{array}$$

2. *Given (r_1^*, w_1^*, r_2^*), $e_0 > 0$, $h_0 > 0$, and $\sigma_0 > 0$, (c_1^*, d_2^*) is a solution to:*

$$\begin{array}{ll} \max_{c_1, d_2} & \ln(c_1) + \beta \ln(d_2) \\ \text{s.t.} & c_1 + \dfrac{d_2}{1+r_2^*} + (1+r_1^*)(1-\sigma_0)e_0 = (1-\tau)w_1^* h_1, \\ & h_1 = \theta e_0^{\eta} h_0^{1-\eta}. \end{array}$$

3. *For the initial old agent, $d_1^* = (1+r_1^*)s_0$.*

4. *$(w_t^*, r_t^*)_{t=1}^{\infty}$ satisfies:*

$$w_t^* = (1-\alpha)A\left(\frac{k_t^*}{h_t^*}\right)^{\alpha}, \; 1+r_t^* = \alpha A \left(\frac{k_t^*}{h_t^*}\right)^{\alpha-1}.$$

5. *Given τ, $(\sigma_t^*)_{t=1}^{\infty}$ satisfies:*

$$N_t \tau w_t^* h_t^* = N_{t+1} \sigma_t^* e_t^*.$$

6. *The capital and a labor market clear in each period. For all t, $H_t^* = N_t h_t^*$ and $N_t s_t^* = K_{t+1}^* + N_{t+1}(1-\sigma_t^*)e_t^*$.*

A *BGP* equilibrium is an equilibrium where all per capita variables grow at a constant rate.

2.1 Characterizing equilibrium

The first-order conditions for a middle-aged agent's problem in period t induce:

$$\begin{aligned} d_{t+1} &= \beta(1+r_{t+1})c_t, & (7) \\ w_t(1-\tau)\theta\eta e_{t-1}^{\eta-1} h_{t-1}^{1-\eta} &= (1-\sigma_{t-1})(1+r_t). & (8) \end{aligned}$$

Multiplying both sides of Equation (8) by e_{t-1}, I obtain:

$$w_t(1-\tau)\eta h_t = (1+r_t)(1-\sigma_{t-1})e_{t-1}. \tag{9}$$

Substituting Equations (7) and (9) into Equation (5), I have:

$$c_t = \frac{1}{1+\beta}(1-\eta)(1-\tau)w_t h_t.$$

Using Equation (3), a middle-age agent saves:

$$s_t = \frac{\beta}{1+\beta}(1-\eta)(1-\tau)w_t h_t. \tag{10}$$

From Equation (9), a middle-aged agent in period $t+1$ spends:

$$e_t = \frac{w_{t+1}(1-\tau)\eta h_{t+1}}{(1+r_{t+1})(1-\sigma_t)} \tag{11}$$

for human capital accumulation. Note that, without considering the general equilibrium effect, a young agent invests more as the interest rate and tax rate become lower and the subsidy rate becomes larger. The government budget constraint for educational subsidies (Equation (6)) induces:

$$\tau w_t h_t = (1+n)\sigma_t \frac{w_{t+1}(1-\tau)\eta h_{t+1}}{(1+r_{t+1})(1-\sigma_t)}.$$

Rearranging this equation, σ_t in equilibrium satisfies:

$$\sigma_t = \frac{(1+r_{t+1})\tau w_t h_t}{(1+n)w_{t+1}(1-\tau)\eta h_{t+1} + (1+r_{t+1})\tau w_t h_t} = \frac{\tau}{\frac{1+n}{1+r_{t+1}}\frac{w_{t+1}}{w_t}(1-\tau)\eta\frac{h_{t+1}}{h_t} + \tau}. \tag{12}$$

As the educational subsidy is in this case financed *inter*-generationally, the variables in period $t+1$ such as r_{t+1}, w_{t+1}, and h_{t+1} appear in σ_t in Equation (12) because of the general equilibrium effect.

From the capital market and labor market clearing conditions,

$$k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{N_t s_t - N_{t+1}(1-\sigma_t)e_t}{N_{t+1}} = \frac{1}{1+n}s_t - (1-\sigma_t)e_t$$

holds in equilibrium. Inserting Equation (10) into this equation yields:

$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\eta)(1-\tau)w_t h_t - (1-\sigma_t)e_t. \tag{13}$$

Because $w_t = (1-\alpha)A\left(\frac{k_t}{h_t}\right)^{\alpha}$, $1+r_t = \alpha A\left(\frac{k_t}{h_t}\right)^{\alpha-1}$, and $(1-\sigma_t)e_t = \frac{w_{t+1}(1-\tau)\eta h_{t+1}}{1+r_{t+1}}$ at equilibrium, the following equation is derived from Equation (13):

$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}\frac{\alpha(1-\eta)(1-\alpha)(1-\tau)A}{\alpha+\eta(1-\tau)(1-\alpha)}k_t^{\alpha}h_t^{1-\alpha}. \tag{14}$$

From Equation (11):

$$\frac{h_{t+1}}{h_t} = \theta^{\frac{1}{1-\eta}}\left[\frac{(1-\tau)\eta}{1-\sigma_t}\frac{1-\alpha}{\alpha}\frac{k_{t+1}}{h_{t+1}}\right]^{\frac{\eta}{1-\eta}}. \tag{15}$$

Using Equation (12), Equation (15) can be rewritten as:

$$\frac{h_{t+1}}{h_t}=\theta^{\frac{1}{1-\eta}}\left[(1-\tau)\eta\frac{1-\alpha}{\alpha}\right]^{\frac{\eta}{1-\eta}}\left[1+\frac{\alpha\tau A}{(1+n)(1-\tau)\eta}\frac{k_t^{\alpha}h_t^{1-\alpha}}{k_{t+1}}\right]^{\frac{\eta}{1-\eta}}\left(\frac{k_{t+1}}{h_{t+1}}\right)^{\frac{\eta}{1-\eta}}. \tag{16}$$

The dynamics of the economy at equilibrium are characterized by Equations (14) and (16). Equation (14) implies that, at equilibrium, $\frac{k_{t+1}}{k_t^{\alpha}h_t^{1-\alpha}}$ is constant, that is:

$$\frac{k_{t+1}}{k_t^{\alpha}h_t^{1-\alpha}}=\widetilde{Z}:=\frac{1}{1+n}\frac{\beta}{1+\beta}\frac{\alpha(1-\eta)(1-\alpha)(1-\tau)A}{\alpha+\eta(1-\tau)(1-\alpha)}. \tag{17}$$

Combining this equation with Equation (16), I obtain:

$$\frac{h_{t+1}}{h_t}=\theta^{\frac{1}{1-\eta}}\left[(1-\tau)\eta\frac{1-\alpha}{\alpha}\right]^{\frac{\eta}{1-\eta}}\left[1+\frac{\alpha\tau A}{(1+n)(1-\tau)\eta}\frac{1}{\widetilde{Z}}\right]^{\frac{\eta}{1-\eta}}\left(\frac{k_{t+1}}{h_{t+1}}\right)^{\frac{\eta}{1-\eta}}. \tag{18}$$

Dividing both sides of Equation (14) by h_{t+1}, I obtain

$$\frac{k_{t+1}}{h_{t+1}}=\widetilde{Z}\left(\frac{k_t}{h_t}\right)^{\alpha}\frac{h_t}{h_{t+1}}.$$

Inserting Equation (18) into this equation, the dynamics of $\overline{k}_t:=\frac{k_t}{h_t}$, which is the physical capital per effective unit of labor in period t, at equilibrium are:

$$\overline{k}_{t+1}=\frac{\widetilde{Z}}{\theta\left(\frac{1-\alpha}{\alpha}\right)^{\eta}\left[(1-\tau)\eta\widetilde{Z}+\frac{\alpha\tau A}{1+n}\right]^{\eta}}\overline{k}_t^{\alpha(1-\eta)}. \tag{19}$$

Therefore, the equilibrium sequence of $\overline{k}_t$ converges to:

$$\widetilde{\overline{k}}^{*}:=\left\{\frac{\widetilde{Z}}{\theta\left(\frac{1-\alpha}{\alpha}\right)^{\eta}\left[(1-\tau)\eta\widetilde{Z}+\frac{\alpha\tau A}{1+n}\right]^{\eta}}\right\}^{\frac{1}{1-\alpha(1-\eta)}}. \tag{20}$$

From Equation (17), the growth factor under the BGP equilibrium is:

$$\begin{aligned}\frac{h_{t+1}}{h_t} &= 1+\widetilde{g}^{*}:=\frac{\widetilde{Z}}{(\widetilde{\overline{k}}^{*})^{1-\alpha}}=\theta^{\frac{1-\alpha}{1-\alpha(1-\eta)}}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{\eta(1-\alpha)}{1-\alpha(1-\eta)}}\left\{\widetilde{Z}^{\alpha}\left[(1-\tau)\eta\widetilde{Z}+\frac{\alpha\tau A}{1+n}\right]^{1-\alpha}\right\}^{\frac{\eta}{1-\alpha(1-\eta)}}\\ &= Q\left[\frac{(1-\alpha)(1-\tau)}{\alpha+\eta(1-\tau)(1-\alpha)}(1-\tau)\eta+\frac{1+\beta}{\beta}\frac{\tau}{1-\eta}\right]^{\frac{(1-\alpha)\eta}{1-\alpha(1-\eta)}}\left[\frac{1-\tau}{\alpha+\eta(1-\tau)(1-\alpha)}\right]^{\frac{\alpha\eta}{1-\alpha(1-\eta)}},\end{aligned} \tag{21}$$

where $Q:=\theta^{\frac{1-\alpha}{1-\alpha(1-\eta)}}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{\eta(1-\alpha)}{1-\alpha(1-\eta)}}\left(\frac{1}{1+n}\frac{\beta}{1+\beta}\alpha A\right)^{\frac{\eta}{1-\alpha(1-\eta)}}(1-\alpha)^{\frac{\alpha\eta}{1-\alpha(1-\eta)}}(1-\eta)^{\frac{\eta}{1-\alpha(1-\eta)}}$. At the BGP equilibrium:

$$\frac{\widetilde{c}^{*}_{t+1}}{\widetilde{c}^{*}_{t}}=\frac{\widetilde{d}^{*}_{t+1}}{\widetilde{d}^{*}_{t}}=\frac{\widetilde{e}^{*}_{t+1}}{\widetilde{e}^{*}_{t}}=1+\widetilde{g}^{*}$$

and

$$\frac{\widetilde{\sigma}^*_{t+1}}{\widetilde{\sigma}^*_t} = 1.$$

The proposition below is a summary of the above results.

Proposition 2.1. *For any $h_0 > 0$, $s_0 > 0$, and $\tau \in [0,1)$, there is a unique BGP equilibrium under which $\frac{\widetilde{c}^*_{t+1}}{\widetilde{c}^*_t} = \frac{\widetilde{d}^*_{t+1}}{\widetilde{d}^*_t} = \frac{\widetilde{e}^*_{t+1}}{\widetilde{e}^*_t} = 1 + \widetilde{g}^*$ and $\frac{\widetilde{\sigma}^*_{t+1}}{\widetilde{\sigma}^*_t} = 1$. Furthermore, it is globally stable.*

For any initial capital and human capital, $k_0 > 0$ and h_0, and $(\bar{k}_t)_{t=0}^{\infty}$ at equilibrium follows the law of motion defined by Equation (19). As $\alpha(1-\eta) \in (0,1)$, a unique steady state, $\widetilde{\bar{k}}^*$, is globally stable.

At the BGP equilibrium, growth rate depends on various parameters. Hence, it is not easy to provide meaningful sufficient conditions for $\widetilde{g}^* > 0$. The next lemma provides a *necessary* condition for $\widetilde{g}^* > 0$.

Lemma 2.1. *If $\widetilde{g}^* > 0$, then $\tau \in [0,1)$ cannot be high.*

Proof. See the Appendix. *Q.E.D.*

Even though high τ leads to high subsidies for education, it also reduces disposable labor income, which discourages a young agent from borrowing and investing in their human capital. This leads to a low economic growth rate. Note that this is a necessary condition. Hence, even if τ is low enough, $\widetilde{g}^* < 0$ is possible. For instance, if θ is low, or if α is high, then from Equation (21), $\widetilde{g}^* < 0$.

Below, I investigate how the changes in τ affect $\widetilde{g}^*$.

Proposition 2.2. *1. At $\tau = 0$, $\left.\frac{\partial \widetilde{g}^*}{\partial \tau}\right|_{\tau=0} > 0$ for all $\alpha \in (0,1)$.*

2. Fix $\tau \in (0,1)$ arbitrarily. Then, for sufficiently large $\alpha \in (0,1)$, $\frac{\partial \widetilde{g}^*}{\partial \tau} < 0$.

Proof. See the Appendix. *Q.E.D.*

An increase in τ decreases the disposable labor income, which in turn lowers the return from education. Then, an increase in τ discourages a young agent from investing in their human capital, which causes a decreases in the growth rate. The role of educational subsidies mitigates this negative effect on growth. Based on the first statement, an introduction of educational subsidies accelerates growth for *all* $\alpha \in (0,1)$. This may not hold anymore when $\tau > 0$. When $\tau > 0$, if α is high enough, an increase in τ decelerates growth. A high value of α implies that the labor income share is small, indicating that labor income for a middle-aged agent is small. Additionally, an increase in τ decreases the disposable labor income. These two effects discourage a young agent from investing in their human capital, which lowers the growth rate in the BGP equilibrium.

3 Model: Subsidizing student loans

Here, I examine another way to provide educational subsidies. In the previous section, in period $t-1$, a *young* agent born in period $t-1$ receives educational subsidy $\sigma_{t-1}e_{t-1}$. Therefore, a young agent borrows $b_{t-1}(=(1-\sigma_{t-1})e_{t-1})$ in period $t-1$ and repays debt $(1+r_t)b_{t-1}$ in period t. Another way to provide educational subsidies is to provide subsidies when a *middle-aged* agent repays their debt.

All but a young agent's budget constraint, a middle-aged agent's budget constraint and the government's budget constraint for educational subsidies are the same as before. The young agent's budget constraint changes from Equation (1) to:

$$e_{t-1} = b_{t-1}. \tag{22}$$

The middle-aged agent's budget constraint changes from Equation (3) to:

$$c_t + s_t + (1-\sigma_t)(1+r_t)b_{t-1} = (1-\tau)w_t h_t. \tag{23}$$

Because an old agent's budget constraint is the same as in Equation (4), an agent's lifetime budget constraint is:

$$c_t + \frac{d_{t+1}}{1+r_{t+1}} + (1-\sigma_t)(1+r_t)e_{t-1} = (1-\tau)w_t h_t.$$

The government's budget constraint for educational subsidies changes from Equation (6) to:

$$N_t \tau w_t h_t = N_t \sigma_t (1+r_t)e_{t-1}.$$

The equilibrium concept is the same as that described in the previous section except for the capital market clearing condition. The capital market clearing condition in the equilibrium definition is:

$$K_t^* + N_t e_{t-1}^* = N_{t-1} s_{t-1}^*.$$

for all t.

3.1 Characterizing equilibrium

The first-order conditions for a young agent's problem induce Equation (7) and:

$$w_t(1-\tau)\theta\eta e_{t-1}^{\eta-1} h_{t-1}^{1-\eta} \quad = \quad (1-\sigma_t)(1+r_t). \tag{24}$$

Multiplying both sides of Equation (24) by e_{t-1}, I obtain:

$$w_t(1-\tau)\eta h_t = (1-\sigma_t)(1+r_t)e_{t-1}. \tag{25}$$

Substituting Equations (7) and (25) into an agent's lifetime budget constraint, I have:

$$c_t = \frac{1}{1+\beta}(1-\eta)(1-\tau)w_t h_t.$$

Using Equation (23), a middle-aged agent saves:

$$s_t = \frac{\beta}{1+\beta}(1-\eta)(1-\tau)w_t h_t. \tag{26}$$

From Equation (25), a middle-aged agent in period t spends:

$$e_{t-1} = \frac{w_t(1-\tau)\eta h_t}{(1+r_t)(1-\sigma_t)} \tag{27}$$

for human capital accumulation. Since, the government's budget constraint must be satisfied at equilibrium,

$$\tau w_t h_t = \sigma_t \frac{w_t(1-\tau)\eta h_t}{1-\sigma_t}$$

holds. At equilibrium, σ_t is set to satisfy this equation. By rearranging the equation, σ_t must satisfy the following:

$$\sigma_t = \frac{\tau}{(1-\tau)\eta+\tau}. \tag{28}$$

In contrast to Equation (12) in the previous section, since the educational subsidy is in this case financed *intra*-generationally, the variables in period $t+1$ do not appear in σ_t in Equation (28) and σ_t is constant over time.

From the capital market and labor market clearing conditions,

$$k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{N_t s_t - N_{t+1} e_t}{N_{t+1}} = \frac{1}{1+n} s_t - e_t$$

holds at equilibrium. Substituting Equation (26) into this equation yields:

$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\eta)(1-\tau)w_t h_t - e_t. \tag{29}$$

Since $w_t = (1-\alpha)A\left(\frac{k_t}{h_t}\right)^{\alpha}$, $1+r_t = \alpha A\left(\frac{k_t}{h_t}\right)^{\alpha-1}$, and $e_t = \frac{w_{t+1}(1-\tau)\eta h_{t+1}}{(1-\sigma_{t+1})(1+r_{t+1})}$ at equilibrium, from Equation (29), I obtain:

$$k_{t+1} = \widehat{Z}\left(\frac{k_t}{h_t}\right)^{\alpha} h_t, \tag{30}$$

where

$$\widehat{Z} := \frac{1}{1+n}\frac{\beta}{1+\beta}\frac{\alpha(1-\eta)(1-\alpha)(1-\tau)A}{\alpha+(1-\alpha)[(1-\tau)\eta+\tau]}. \tag{31}$$

From Equation (27):

$$\frac{h_{t+1}}{h_t} = \theta^{\frac{1}{1-\eta}}\left[\frac{(1-\tau)\eta}{1-\sigma_{t+1}}\frac{1-\alpha}{\alpha}\frac{k_{t+1}}{h_{t+1}}\right]^{\frac{\eta}{1-\eta}}. \tag{32}$$

Using Equation (28), Equation (32) can be rewritten as:

$$\frac{h_{t+1}}{h_t} = \theta^{\frac{1}{1-\eta}}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{\eta}{1-\eta}}[(1-\tau)\eta+\tau]^{\frac{\eta}{1-\eta}}\left(\frac{k_{t+1}}{h_{t+1}}\right)^{\frac{\eta}{1-\eta}}. \tag{33}$$

Combining Equations (30) and (33) yields:

$$\bar{k}_{t+1} = \frac{\widehat{Z}^{1-\eta}}{\theta\left(\frac{1-\alpha}{\alpha}\right)^{\eta}[(1-\tau)\eta+\tau]^{\eta}}\bar{k}_t^{\alpha(1-\eta)}.$$

Therefore, the equilibrium sequence of $\bar{k}_t$ converges to:

$$\widehat{\bar{k}}^* := \frac{\widehat{Z}^{\frac{1-\eta}{1-\alpha(1-\eta)}}}{\theta^{\frac{1}{1-\alpha(1-\eta)}}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{\eta}{1-\alpha(1-\eta)}}[(1-\tau)\eta+\tau]^{\frac{\eta}{1-\alpha(1-\eta)}}}. \tag{34}$$

From Equation (31), the growth factor at BGP equilibrium is:

$$\begin{aligned}
\frac{h_{t+1}}{h_t} = 1+\widehat{g}^* \quad := \quad & \frac{\widehat{Z}}{(\widehat{\bar{k}}^*)^{1-\alpha}} = \theta^{\frac{1-\alpha}{1-\alpha(1-\eta)}}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{\eta(1-\alpha)}{1-\alpha(1-\eta)}}[(1-\tau)\eta+\tau]^{\frac{(1-\alpha)\eta}{1-\alpha(1-\eta)}}\widehat{Z}^{\frac{\eta}{1-\alpha(1-\eta)}} \\
= \quad & Q\left\{\frac{(1-\alpha)(1-\tau)}{\alpha+(1-\alpha)[(1-\tau)\eta+\tau]}[(1-\tau)\eta+\tau]\right\}^{\frac{(1-\alpha)\eta}{1-\alpha(1-\eta)}}\left\{\frac{1-\tau}{\alpha+(1-\alpha)[(1-\tau)\eta+\tau]}\right\}^{\frac{\alpha\eta}{1-\alpha(1-\eta)}} \\
= \quad & Q\{(1-\alpha)[(1-\tau)\eta+\tau]\}^{\frac{(1-\alpha)\eta}{1-\alpha(1-\eta)}}\left\{\frac{1-\tau}{\alpha+(1-\alpha)[(1-\tau)\eta+\tau]}\right\}^{\frac{\eta}{1-\alpha(1-\eta)}}.
\end{aligned} \tag{35}$$

At the BGP equilibrium:

$$\frac{\widehat{c}^*_{t+1}}{\widehat{c}^*_t} = \frac{\widehat{d}^*_{t+1}}{\widehat{d}^*_t} = \frac{\widehat{e}^*_{t+1}}{\widehat{e}^*_t} = 1+\widehat{g}^*$$

and

$$\frac{\widehat{\sigma}^*_{t+1}}{\widehat{\sigma}^*_t} = 1.$$

Note that the ratio of educational subsidies, σ_t, is constant at the BGP equilibrium to satisfy the government's budget constraint. However, since educational expenditure e_t increases over time, the amount of educational subsidy, $\sigma_t e_t$, also increases over time.

The following proposition summarizes the above results.

Proposition 3.1. *For any $h_0 > 0$, $s_0 > 0$, $\tau \in [0,1)$, and $\lambda \in [0,1]$, there exists a unique BGP equilibrium, in which $\frac{\widehat{c}^*_{t+1}}{\widehat{c}^*_t} = \frac{\widehat{d}^*_{t+1}}{\widehat{d}^*_t} = \frac{\widehat{e}^*_{t+1}}{\widehat{e}^*_t} = 1+\widehat{g}^*$ and $\frac{\widehat{\sigma}^*_{t+1}}{\widehat{\sigma}^*_t} = 1$. Furthermore, it is globally stable.*

Analogous to Proposition 2.1, a unique BGP equilibrium is globally stable. Moreover, I provide a necessary condition for which $\widehat{g}^* > 0$ below.

Lemma 3.1. *If $\widehat{g}^* > 0$, then $\tau \in [0,1)$ cannot take high value.*

Proof. See the Appendix. *Q.E.D.*

The next proposition explains how a change in τ affects $\widehat{g}^*$.

Proposition 3.2. *There exists a unique $\widehat{\eta} \in (0,1)$ such that:*

1. *for $\eta < \widehat{\eta}$, there is a unique $\widehat{\tau} \in (0,1)$ such that:*

$$\frac{\partial \widehat{g}^*}{\partial \tau} \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0 \text{ if } \tau \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} \widehat{\tau}.$$

2. *for $\eta = \widehat{\eta}$, $\left.\frac{\partial \widehat{g}^*}{\partial \tau}\right|_{\tau=0} = 0$ and $\frac{\partial \widehat{g}^*}{\partial \tau} > 0$ for all $\tau > 0$.*

3. *for $\eta > \widehat{\eta}$, $\frac{\partial \widehat{g}^*}{\partial \tau} < 0$ for all $\tau \in [0,1)$.*

Proof. See the Appendix. *Q.E.D.*

Different from Proposition 2.2, the effects of τ on growth rate is fully characterized. Depending the value of η, the functional form of $\widehat{g}^*(\tau)$ changes. Notice that η expresses the output elasticity in the production function of human capital represented by Equation (2). That is, a small value of η implies that a 1% change of e_t does not considerably change h_{t+1}. In such a case, $\widehat{g}^*(\tau)$ is inverted U-shaped in τ. Intuitively, when η is small, despite that e_t is invested, h_{t+1} does not increase significantly. Then, a young agent's incentive to invest in their human capital is low. In such a case, providing educational subsidies is more effective. Therefore, $\widehat{g}^*$ is inverted U-shaped in τ when η is small.

4 Comparison of the two educational subsidy schemes

4.1 Growth rate comparison

This section analyzes which educational subsidy leads to a higher growth rate in the BGP equilibrium.

Proposition 4.1.

1. *If $\tau = 0$, then $\widetilde{g}^* = \widehat{g}^*$.*

2. *Assume $\tau > 0$. Then, $\widetilde{g}^* > \widehat{g}^*$ for all $\alpha \geq \alpha^\star$, where $\alpha^\star := \frac{1 - \frac{1+\beta}{\beta}\frac{\eta}{1-\eta}}{1 + \frac{1+\beta}{\beta}\frac{1}{1-\tau}\frac{1-\eta(1-\tau)}{1-\eta}}$.*

3. *Assume $\tau > 0$. Assume further that η is sufficiently small. If τ is sufficiently small, then $\widehat{g}^* > \widetilde{g}^*$ for all $\alpha < \frac{1}{1+\frac{1+\beta}{\beta}}$.*

Proof. See the Appendix. *Q.E.D.*

When $\tau = 0$, there is no educational subsidy. Given that both models are exactly the same, there is no difference between the two growth rates.

When $\tau > 0$, the result depends on the parameter values. When the cost of education is subsidized, from Equation (12),

$$\widetilde{\sigma}_t^* = \frac{\tau}{\frac{1+n}{1+\widetilde{r}^*}(1+\widetilde{g}^*)(1-\tau)\eta + \tau}$$

at the BGP equilibrium. When the student loans are subsidized, from Equation (28),

$$\widehat{\sigma}_t^* = \frac{\tau}{(1-\tau)\eta + \tau}.$$

at the BGP equilibrium. If σ_t is large, then educational investment becomes large, which accelerates economic growth. When α is large, production is physical-capital-intensive. Therefore, the interest rate tends to be high. Given that higher $1+\widetilde{r}^*$ makes $\widetilde{\sigma}_t^*$ higher, a young agent invests more in their human capital, leading to $\widetilde{g}^* > \widehat{g}^*$. By contrast, for a small $\alpha \in (0,1)$ and with additional parameters, the opposite is likely to hold because the interest rate tends to be low.[3]

4.2 Social welfare comparison

Among two systems, which one is "better" for agents? Does higher growth rate induce higher welfare? To answer these questions, I compare welfare in the BGP equilibrium under each system.

First, following Docquier et al. (2007), social welfare is defined as follows. Given a social discount factor, $\gamma \in (0,1)$, let

$$\widetilde{W}(\tau) := \sum_{t=1}^{\infty} \gamma^{t-1} \left[\ln(\widetilde{c}_t^*) + \beta \ln(\widetilde{d}_{t+1}^*) \right],$$

and

$$\widehat{W}(\tau) := \sum_{t=1}^{\infty} \gamma^{t-1} \left[\ln(\widehat{c}_t^*) + \beta \ln(\widehat{d}_{t+1}^*) \right],$$

be social welfare when the cost of education is subsidized and that when student loans are subsidized, respectively. Note that, at the BGP equilibrium, an agent's lifetime utility is

$$U(c_t^*, d_{t+1}^*) = (1+\beta)\ln(c_t^*) + \beta \ln(1+r_{t+1}^*) + \beta \ln(\beta).$$

Additionally, at the BGP equilibrium, a middle-aged agent's consumption is

$$c_t^* = \frac{1}{1+\beta}(1-\eta)(1-\tau)A(1-\alpha)(\overline{k}_t^*)^{\alpha}(1+g^*)^t h_0.$$

When the cost of education is subsidized, an agent's lifetime utility in period t in the BGP equilibrium is

$$\begin{aligned} U(\widetilde{c}_t^*, \widetilde{d}_{t+1}^*) &= (1+\beta)\ln(\widetilde{c}_t^*) + \beta \ln(1+\widetilde{r}_{t+1}^*) + \beta \ln(\beta) \\ &= (1+\beta)\ln\left(\frac{(\widetilde{\overline{k}}^*)^{\alpha}}{1+\beta}(1-\eta)(1-\tau)A(1-\alpha)(1+\widetilde{g}^*)^t h_0 \right) + \beta \ln(A\alpha(\widetilde{\overline{k}}^*)^{\alpha-1}) + \beta \ln(\beta) \\ &= [(1+\beta)\alpha + \beta(\alpha-1)]\ln(\widetilde{\overline{k}}^*) + (1+\beta)\ln((1-\eta)(1-\tau)A(1-\alpha)) + t\ln(1+\widetilde{g}^*) + C, \end{aligned}$$

[3] Note that the condition on α in the third statement is a limit point of $\alpha^\star$ as η and τ go to 0.

where $C := (1+\beta)\ln\left(\frac{h_0}{1+\beta}\right) + \beta\ln(A\alpha\beta)$ is constant. Thus,

$$\begin{aligned}(1-\gamma)\widetilde{W}(\tau) &= [(1+\beta)\alpha+\beta(\alpha-1)]\ln(\widetilde{\bar{k}}^{*}) + (1+\beta)\ln((1-\eta)(1-\tau)A(1-\alpha)) \\ &+ (1-\gamma)\ln(1+\widetilde{g}^{*})D + C, \end{aligned} \tag{36}$$

where $D = \sum_{t=1}^{\infty} t\gamma^{t} < +\infty$ is some positive finite number.

When student loans are subsidized, an agent's lifetime utility in period t in the BGP equilibrium is

$$U(\widehat{c}_t^{*}, \widehat{d}_{t+1}^{*}) = [(1+\beta)\alpha+\beta(\alpha-1)]\ln(\widehat{\bar{k}}^{*}) + (1+\beta)\ln((1-\eta)(1-\tau)A(1-\alpha)) + t\ln(1+\widehat{g}^{*}) + C.$$

Thus,

$$\begin{aligned}(1-\gamma)\widehat{W}(\tau) &= [(1+\beta)\alpha+\beta(\alpha-1)]\ln(\widehat{\bar{k}}^{*}) + (1+\beta)\ln((1-\eta)(1-\tau)A(1-\alpha)) \\ &+ (1-\gamma)\ln(1+\widehat{g}^{*})D + C. \end{aligned} \tag{37}$$

From Equations (36) and (37),

$$(1-\gamma)[\widetilde{W}(\tau) - \widehat{W}(\tau)] = [(1-\beta)\alpha+\beta(\alpha-1)]\ln\left(\frac{\widetilde{\bar{k}}^{*}}{\widehat{\bar{k}}^{*}}\right) + (1-\gamma)D\ln\left(\frac{1+\widetilde{g}^{*}}{1+\widehat{g}^{*}}\right). \tag{38}$$

Does higher growth rate induce higher welfare? The answer to this question is "not necessarily." For instance, assume that α is large enough to satisfy $\alpha \geq \alpha^{\star}$. Then, from Proposition 4.1, $1+\widetilde{g}^{*} > 1+\widehat{g}^{*}$. This implies that the second term in Equation (38) is positive. From Equations (20) and (34),

$$\begin{aligned}\frac{\widetilde{\bar{k}}^{*}}{\widehat{\bar{k}}^{*}} &= \left(\frac{(1-\tau)\{\alpha+(1-\alpha)[(1-\tau)\eta+\tau]\}}{\alpha+\eta(1-\tau)(1-\alpha)}\right)^{\frac{1-\eta}{1-\alpha(1-\eta)}} \\ &\times \left(\frac{(1-\tau)\eta\frac{(1-\eta)(1-\alpha)(1-\tau)}{\alpha+\eta(1-\tau)(1-\alpha)} + \tau\frac{(1-\eta)(1-\alpha)(1-\tau)}{\alpha+\eta(1-\tau)(1-\alpha)}}{(1-\tau)\eta\frac{(1-\eta)(1-\alpha)(1-\tau)}{\alpha+\eta(1-\tau)(1-\alpha)} + \tau\frac{1+\beta}{\beta}}\right)^{\frac{\eta}{1-\alpha(1-\eta)}}. \end{aligned} \tag{39}$$

If α is larger, both the first and second terms in Equation (39) is smaller than 1, which implies that the first term in Equation (38) is negative. Depending on other parameters, such as β and γ, Equation (38) can be negative.

Given that it is not easy to determine analytically which social welfare is larger, I use the Golden Rule criterion suggested by Del Rey and Lopez-Garcia (2013) as social welfare hereinafter. Let

$$\mathbb{U}\left(\frac{c_t}{h_t}, \frac{d_{t+1}}{h_t}\right) := \ln\left(\frac{c_t}{h_t}\right) + \beta\ln\left(\frac{d_{t+1}}{h_t}\right).$$

Under the Golden Rule criterion, each agent's welfare is measured by using consumption *per effective unit of labor*. Note that, on the BGP, given that all generations' welfare under this criterion are equal, all generations are treated equally in this sense. Then, when the cost of education is subsidized, social welfare is

$$\mathbb{U}\left(\frac{\widetilde{c}_t^{*}}{\widetilde{h}_t^{*}}, \frac{\widetilde{d}_{t+1}^{*}}{\widetilde{h}_t^{*}}\right) = [(1+\beta)\alpha+\beta(\alpha-1)]\ln(\widetilde{\bar{k}}^{*}) + (1+\beta)\ln((1-\eta)(1-\tau)A(1-\alpha)) + E, \tag{40}$$

where $E := (1+\beta)\ln\left(\frac{1}{1+\beta}\right) + \beta\ln(A\alpha\beta)$ is constant. When the student loans are subsidized, welfare is

$$\mathbb{U}\left(\frac{\widehat{c}_t^*}{\widehat{h}_t^*}, \frac{\widehat{d}_{t+1}^*}{\widehat{h}_t^*}\right) = [(1+\beta)\alpha + \beta(\alpha-1)]\ln(\widehat{\overline{k}}^*) + (1+\beta)\ln((1-\eta)(1-\tau)A(1-\alpha)) + E. \tag{41}$$

Proposition 4.2. *Assume that $\tau > 0$. Further, assume further that η is sufficiently small. If τ is sufficiently small, then $\mathbb{U}\left(\frac{\widetilde{c}_t^*}{\widetilde{h}_t^*}, \frac{\widetilde{d}_{t+1}^*}{\widetilde{h}_t^*}\right) > \mathbb{U}\left(\frac{\widehat{c}_t^*}{\widehat{h}_t^*}, \frac{\widehat{d}_{t+1}^*}{\widehat{h}_t^*}\right)$ for all $\alpha < \frac{1}{1+\frac{1+\beta}{\beta}}$.*

Proof. See the Appendix. *Q.E.D.*

Note that the conditions guarantee $1+\widehat{g}^* > 1+\widetilde{g}^*$ from Proposition 4.1. From Equations (21) and (35), when $1+\widehat{g}^* > 1+\widetilde{g}^*$, $\widetilde{\overline{k}}^* > \widehat{\overline{k}}^*$ holds because $\widetilde{Z} > \widehat{Z}$. The growth rate at the BGP equilibrium when student loans are subsidized is higher than that when the cost of education is subsidized, whereas social welfare defined by the Golden Rule criterion when the cost of education is subsidized is higher than that when the student loans are subsidized.

5 Concluding remarks

This study considered a three-period OLG model with educational subsidies, in which a young agent borrows for their education in a perfect credit market and a middle-aged agent repays these loans. Two educational subsidy schemes were considered: one is to provide subsidies when a young agent borrows and the other is to provide subsidies when a middle-aged agent repays their loan. I characterized a unique BGP equilibrium under each educational subsidy scheme and compared their growth rates and welfare. Regarding the comparison of growth rates, I found that the size relationship of growth rates depends on whether the production is sufficiently physical-capital-intensive. If the production is highly physical-capital-intensive, the interest rate tends to be high in the credit market. This discourages a young agent from borrowing because the interest payments will become a burden for the agent. In this case, the existence of an educational subsidy is helpful for a young agent, allowing them to borrow and invest in human capital. Therefore, in this case, an educational subsidy provided when a young agent borrows leads to a higher growth rate than that the one provided when a middle-aged agent repays their loan. If the production is not highly physical-capital-intensive, the opposite is likely to hold. Regarding the comparison of welfare, I consider two concepts of welfare criteria. Specifically, I demonstrated that even if educational subsidies for the student loans lead to a higher growth rate in the BGP equilibrium, social welfare defined by the Golden Rule criterion can be lower that when the cost of education is subsidized.

When educational subsidies are provided, several educational subsidy schemes can be considered. This study indicates that a higher growth rate and higher welfare may not be achieved simultaneously. Therefore, when a policy maker provides an educational subsidy, making the policy maker's aim clear is quite important.

A Appendix

A.1 Proof of Lemma 2.1

Proof. Suppose, by way of contradiction, that τ is sufficiently high. Then, from Equation (21), as $\tau \to 1$, $1+\widetilde{g}^* \to 0$ for any given parameters. This implies that $\widetilde{g}^* < 0$ for sufficiently high τ for any given parameters. *Q.E.D.*

A.2 Proof of Proposition 2.2

Proof. Taking the derivative of $\widetilde{g}^*$ with respect to τ, I obtain

$$\begin{aligned}
\frac{\partial \widetilde{g}^*}{\partial \tau} &= Q\left[\frac{(1-\alpha)(1-\tau)}{\alpha+\eta(1-\tau)(1-\alpha)}(1-\tau)\eta + \frac{1+\beta}{\beta}\frac{\tau}{1-\eta}\right]^{\frac{(1-\alpha)\eta}{1-\alpha(1-\eta)}-1}\left[\frac{1-\tau}{\alpha+\eta(1-\tau)(1-\alpha)}\right]^{\frac{\alpha\eta}{1-\alpha(1-\eta)}-1} \\
&\times \frac{\eta}{1-\alpha(1-\eta)}\frac{1}{[\alpha+\eta(1-\tau)(1-\alpha)]^3}\widetilde{\Lambda}(\tau), \qquad (42)
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{\Lambda}(\tau) &:= \eta^2(1-\alpha)^3\left(1-\frac{1+\beta}{\beta}\frac{1}{1-\eta}\right)\tau^3 \\
&- \eta(1-\alpha)\left[\begin{array}{c}(1-\alpha)\alpha\left(1-\frac{1+\beta}{\beta}\frac{2}{1-\eta}\right)+3\eta(1-\alpha)^2\left(1-\frac{1+\beta}{\beta}\frac{1}{1-\eta}\right) \\ +\alpha\left(1-\frac{1+\beta}{\beta}\frac{\alpha}{1-\eta}\right)\end{array}\right]\tau^2 \\
&+ \left[\begin{array}{c}2\eta(1-\alpha)^2\alpha\left(1-\frac{1+\beta}{\beta}\frac{2}{1-\eta}\right)+3\eta^2(1-\alpha)^3\left(1-\frac{1+\beta}{\beta}\frac{1}{1-\eta}\right) \\ +\alpha\left\{2\eta(1-\alpha)-\frac{1+\beta}{\beta}\frac{\alpha^2}{1-\eta}\right\}-\frac{1+\beta}{\beta}\frac{\alpha^2}{1-\eta}(1-\alpha)(1+\eta)\end{array}\right]\tau \\
&- (1-\alpha)\left[-(1-\eta)\left(\eta+\frac{1+\beta}{\beta}\right)\alpha^2+2\eta\left(1-\eta-\frac{1+\beta}{\beta}\right)\alpha+\eta^2\left(1-\frac{1+\beta}{\beta}\frac{1}{1-\eta}\right)\right].
\end{aligned}$$

(43)

At $\tau = 0$, the sign of $\frac{\partial \widetilde{g}^*}{\partial \tau}$ is equivalent to that of $\widetilde{\Lambda}(0)$.

Let

$$\widetilde{G}(\alpha) := -(1-\eta)\left(\eta+\frac{1+\beta}{\beta}\right)\alpha^2+2\eta\left(1-\eta-\frac{1+\beta}{\beta}\right)\alpha+\eta^2\left(1-\frac{1+\beta}{\beta}\frac{1}{1-\eta}\right).$$

Note that $\widetilde{G}(0) = \eta^2\left(1-\frac{1+\beta}{\beta}\frac{1}{1-\eta}\right) < 0$ because $\frac{1}{1-\eta} > 1$ and $\frac{1+\beta}{\beta} > 1$. Given that

$$\begin{aligned}
& \eta^2\left(1-\eta-\frac{1+\beta}{\beta}\right)^2+(1-\eta)\left(\eta+\frac{1+\beta}{\beta}\right)\eta^2\left(1-\frac{1+\beta}{\beta}\frac{1}{1-\eta}\right) \\
&= \eta^2\left[\left(1-\eta-\frac{1+\beta}{\beta}\right)^2+\left(\eta+\frac{1+\beta}{\beta}\right)\left(1-\eta-\frac{1+\beta}{\beta}\right)\right] = \eta^2\left(1-\eta-\frac{1+\beta}{\beta}\right) < 0,
\end{aligned}$$

$\widetilde{G}(\alpha) = 0$ has no real number solutions. Therefore, $\widetilde{G}(\alpha) < 0$ for all $\alpha \in (0,1)$. This implies that at $\tau = 0$, $\widetilde{\Lambda}(0) = -(1-\alpha)\widetilde{G}(\alpha) > 0$ for all $\alpha \in (0,1)$, which completes the proof of the first statement.

Fix arbitrary $\tau \in (0,1)$. As $\alpha \to 1$, $\widetilde{\Lambda}(\tau) \to -\frac{1+\beta}{\beta}\frac{1}{1-\eta}\tau < 0$. Hence, for sufficiently large $\alpha < 1$, $\frac{\partial \widehat{g}^*}{\partial \tau} < 0$. *Q.E.D.*

A.3 Proof of Lemma 3.1

Proof. Suppose, by way of contradiction, that τ is high. As $\tau \to 1$, from Equation (35), $1+\widehat{g}^* \to 0$. This implies that if τ is quite high, then $\widehat{g}^* < 0$. *Q.E.D.*

A.4 Proof of Proposition 3.2

Proof. Taking the derivative of $\widehat{g}^*$ with respect to τ, I obtain

$$\begin{aligned}\frac{\partial \widehat{g}^*}{\partial \tau} &= Q\{(1-\alpha)[(1-\tau)\eta+\tau]\}^{\frac{(1-\alpha)\eta}{1-\alpha(1-\eta)}-1}\left\{\frac{1-\tau}{\alpha+(1-\alpha)[(1-\tau)\eta+\tau]}\right\}^{\frac{\eta}{1-\alpha(1-\eta)}-1}\frac{\eta}{1-\alpha(1-\eta)}\\ &\times \frac{1-\alpha}{\alpha+(1-\alpha)[(1-\tau)\eta+\tau]}\left\{(1-\alpha)(1-\eta)(1-\tau)-\frac{(1-\tau)\eta+\tau}{\alpha+(1-\alpha)[(1-\tau)\eta+\tau]}\right\}.\end{aligned}$$

Hence, the sign of $\frac{\partial \widehat{g}^*}{\partial \tau}$ is equivalent to that of $(1-\alpha)(1-\eta)(1-\tau)-\frac{(1-\tau)\eta+\tau}{\alpha+(1-\alpha)[(1-\tau)\eta+\tau]}$. Let

$$\widehat{L}(\tau) := (1-\alpha)(1-\eta)(1-\tau)$$

and

$$\widehat{R}(\tau) := \frac{(1-\tau)\eta+\tau}{\alpha+(1-\alpha)[(1-\tau)\eta+\tau]}$$

Note that $\widehat{L}(\tau)$ is strictly decreasing in τ, $\widehat{L}(0) = (1-\alpha)(1-\eta) > 0$, and $\widehat{L}(1) = 0$. Moreover, $\widehat{R}(\tau)$ is strictly increasing in τ, $\widehat{R}(0) = \frac{\eta}{\alpha+(1-\alpha)\eta} > 0$, and $\widehat{R}(1) = 1$. If $\widehat{L}(0) > \widehat{R}(0)$, then there is a unique $\widehat{\tau} \in (0,1)$ such that $\widehat{L}(\tau) > \widehat{R}(\tau)$ for all $\tau < \widehat{\tau}$ and $\widehat{L}(\tau) < \widehat{R}(\tau)$ for all $\tau > \widehat{\tau}$. If $\widehat{L}(0) < \widehat{R}(0)$, then $\widehat{L}(\tau) < \widehat{R}(0)$ for all $\tau \in [0,1)$. Now, I derive conditions under which $\widehat{L}(0) > \widehat{R}(0)$ holds.

Note that $\widehat{L}(0;\eta) = (1-\alpha)(1-\eta)$ is strictly decreasing in η, $\widehat{L}(0;\eta=0) = 1-\alpha > 0$ and $\widehat{L}(0;\eta=1) = 0$. Notice further that $\widehat{R}(0;\eta)$ is strictly increasing in η, $\widehat{R}(0;\eta=0) = 0$ and $\widehat{R}(0;\eta=1) = 1$. This implies that there is a unique $\widehat{\eta} \in (0,1)$ such that $\widehat{L}(0) > \widehat{R}(0)$ for all $\eta < \widehat{\eta}$ and $\widehat{L}(0) < \widehat{R}(0)$ for all $\eta > \widehat{\eta}$.

Therefore, if $\eta < \widehat{\eta}$, there is a unique $\widehat{\tau} \in (0,1)$ such that $\frac{\partial \widehat{g}^*}{\partial \tau} > 0$ for all $\tau < \widehat{\tau}$ and $\frac{\partial \widehat{g}^*}{\partial \tau} < 0$ for all $\tau > \widehat{\tau}$. If $\eta > \widehat{\eta}$, $\frac{\partial \widehat{g}^*}{\partial \tau} < 0$ for all τ. *Q.E.D.*

A.5 Proof of Proposition 4.1

Proof. From Equations (21) and (35), $\widetilde{g}^* \geq \widehat{g}^*$ if and only if

$$\left\{\frac{(1-\alpha)(1-\tau)\eta+\frac{1+\beta}{\beta}\frac{\tau}{1-\tau}\frac{\alpha+\eta(1-\tau)(1-\alpha)}{1-\eta}}{(1-\alpha)[(1-\tau)\eta+\tau]}\right\}^{\frac{(1-\alpha)\eta}{1-\alpha(1-\eta)}} \geq \left[\frac{\alpha+\eta(1-\tau)(1-\alpha)}{\alpha+(1-\alpha)[(1-\tau)\eta+\tau]}\right]^{\frac{\eta}{1-\alpha(1-\eta)}}$$

or

$$\left\{\frac{(1-\alpha)(1-\tau)\eta+\frac{1+\beta}{\beta}\frac{\tau}{1-\tau}\frac{\alpha+\eta(1-\tau)(1-\alpha)}{1-\eta}}{(1-\alpha)[(1-\tau)\eta+\tau]}\right\}^{1-\alpha}\geq\frac{\alpha+\eta(1-\tau)(1-\alpha)}{\alpha+(1-\alpha)[(1-\tau)\eta+\tau]}. \tag{44}$$

Note that, if $\tau=0$, both sides of Equation (44) are the same, which completes the proof of statement 1.

Assume that $\tau>0$. Note that the right-hand side of Equation (44) is less than 1. Meanwhile, if the left-hand side of Equation (44) is larger than or equal to 1, $\widetilde{g}^*>\widehat{g}^*$ holds. The term inside the bracket of the left-hand side of Equation (44) is greater than or equal to 1 if

$$(1-\alpha)(1-\tau)\eta+\frac{1+\beta}{\beta}\frac{\tau}{1-\tau}\frac{\alpha+\eta(1-\tau)(1-\alpha)}{1-\eta}\geq(1-\alpha)[(1-\tau)\eta+\tau].$$

This is equivalent to

$$\alpha\geq\alpha^\star=\frac{1-\frac{1+\beta}{\beta}\frac{\eta}{1-\eta}}{1+\frac{1+\beta}{\beta}\frac{1}{1-\tau}\frac{1-\eta(1-\tau)}{1-\eta}},$$

which proves the second statement.

As η goes to 0, the left-hand side of Equation (44) goes to

$$\left[\frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta}\frac{1}{1-\tau}\right]^{1-\alpha} \tag{45}$$

and the right-hand side of it goes to

$$\frac{\alpha}{\alpha+(1-\alpha)\tau}. \tag{46}$$

As τ becomes 0, Equation (45) also leads to

$$\left[\frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta}\right]^{1-\alpha} \tag{47}$$

and Equation (46) goes to 1. As $\alpha<\frac{1}{1+\frac{1+\beta}{\beta}}$,

$$\frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta}<1$$

holds, which implies Equation (47) is less than 1.

Given that Equation (45) is strictly increasing in τ and Equation (46) is strictly decreasing in τ, for sufficiently small τ, $\frac{\alpha}{\alpha+(1-\alpha)\tau}>\left[\frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta}\frac{1}{1-\tau}\right]^{1-\alpha}$ for $\alpha<\frac{1}{1+\frac{1+\beta}{\beta}}$. Therefore, for sufficiently small η, $\widehat{g}^*>\widetilde{g}^*$ for all $\alpha<\frac{1}{1+\frac{1+\beta}{\beta}}$. *Q.E.D.*

A.6 Proof of Proposition 4.2

Proof. From Equations (40) and (41), $\mathbb{U}\left(\frac{\widetilde{c}_t^*}{\widetilde{h}_t^*}, \frac{\widetilde{d}_{t+1}^*}{\widetilde{h}_t^*}\right) > \mathbb{U}\left(\frac{\widehat{c}_t^*}{\widehat{h}_t^*}, \frac{\widehat{d}_{t+1}^*}{\widehat{h}_t^*}\right)$ if and only if

$$[(1+\beta)\alpha + \beta(\alpha-1)] \ln\left(\frac{\widetilde{\overline{k}}^*}{\widehat{\overline{k}}^*}\right) > 0. \tag{48}$$

From Equations (17) and (31), $\widetilde{Z} > \widehat{Z}$ if and only if $\tau > 0$. Given that $1+\widetilde{g}^* = \frac{\widetilde{Z}}{(\widetilde{\overline{k}}^*)^{1-\alpha}}$ and $1+\widehat{g}^* = \frac{\widehat{Z}}{(\widehat{\overline{k}}^*)^{1-\alpha}}$, $1+\widetilde{g}^* < 1+\widehat{g}^*$ implies $\widetilde{\overline{k}}^* > \widehat{\overline{k}}^*$. Hence, $\ln\left(\frac{\widetilde{\overline{k}}^*}{\widehat{\overline{k}}^*}\right) < 0$. When $\alpha < \frac{1}{1+\frac{1+\beta}{\beta}}$, $(1+\beta)\alpha + \beta(\alpha-1) < 0$. Therefore, Equation (48) holds. *Q.E.D.*

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

References

Azariadis, Costas, and Allan Drazen. 1990. "Threshold externalities in economic development." *Quarterly Journal of Economics* 105(2): 501–526. https://doi.org/10.2307/2937797

Blankenau, William. 2005. "Public schooling, college subsidies and growth." *Journal of Economic Dynamics and Control* 29(3): 487–507. https://doi.org/10.1016/j.jedc.2004.04.001

Boldrin, Michele, and Ana Montes. 2005. "The Intergenerational State Education and Pensions." *Review of Economic Studies* 72(3): 651–664. https://www.jstor.org/stable/3700668

Diamond, Perter. 1965. "National debt in a neoclassical growth model." *American Economic Review* 55(4): 1126–1150. https://www.jstor.org/stable/1809231

de la Croix, David, and Matthias Doepke. 2003. "Inequality and growth: Why differential fertility matters." *American Economic Review* 93(4):1091–1113. https://www.aeaweb.org/articles?id=10.1257/000282803769206214

de la Croix, David, and Matthias Doepke. 2004. "Public versus private education when differential fertility matters." *Journal of Development Economics* 73(2):607–629. https://doi.org/10.1016/j.jdeveco.2003.05.005

Del Rey, Elena, and Miguel-Angel Lopez-Garcia. 2013. "Optimal education and pensions in an endogenous growth model." *Journal of Economic Theory* 148(4): 1737–1750. https://doi.org/10.1016/j.jet.2013.01.003

Del Rey, Elena, and Miguel-Angel Lopez-Garcia. 2016. “Endogenous growth and welfare effects of education subsidies and intergenerational transfers.” *Economic Modelling* 52(B): 531–539. https://doi.org/10.1016/j.econmod.2015.09.033

Del Rey, Elena, and Miguel-Angel Lopez-Garcia. 2019. “Optimal public policy *à la Ramsey* in an endogenous growth model.” *Journal of Economics* 128(2): 99–118. https://doi.org/10.1007/s00712-018-0643-z

Docquier, Frédéric, Oliver Paddison, and Pierre Pestieau. 2007. “Optimal accumulation in an endogenous growth setting with human capital.” *Journal of Economic Theory* 134(1): 361–378. https://doi.org/10.1016/j.jet.2006.03.008

Eckwert, Bernhard, and Itzha Zilcha. 2014. “Higher education: Subsidizing tuition versus subsidizing student loans.” *Journal of Public Economic Theory* 16(6): 835–853. https://doi.org/10.1111/jpet.12082

Glomm, Gerhard, and B. Ravikumar. 1992. “Public versus private investment in human capital: Endogenous growth and income inequality.” *Journal of Political Economy* 100(4):818–834. https://www.jstor.org/stable/2138689

Lucas, Robert E. 1988. “On the mechanics of economic development.” *Journal of Monetary Economics* 22(1): 3-42. https://doi.org/10.1016/0304-3932(88)90168-7

Yakita, Akira. 2004. “School loans, subsidies, and economic growth.” *FinanzArchiv*. 60(2): 262–276. https://www.jstor.org/stable/40913040