Determining Capital Structure within Arbitrage-Based Production Framework

Zhao, Guo

Southwest Minzu University,

27 June 2021

Determining Capital Structure within Arbitrage-Based Production Framework

By GUO ZHAO[*]

To explore the interactions between financial decisions and production decisions of representative firm, we propose a dynamic production model under the joint constraints of technology, budget and no arbitrage. Theoretical and numerical analysis shows that dynamic production in no arbitrage equilibrium may undergo a bifurcation into a stable capital-intensive state and an unstable labor-intensive state. Further, it is shown that in no-arbitrage equilibrium a firm's capital structure is endogenously determined by its endowment structure. These findings are consistent with empirical evidences and hence justify the no-arbitrage based production model as a useful framework with methodological advantages. (JEL D24, E23, G32)

As a cornerstone of modern corporate finance, the Modigliani-Miller theorem represents the first formal use of a *no arbitrage proof* in finance (Modigliani and Miller 1958; Miller 1988). Under very general assumptions, the Modigliani-Miller proposition demonstrates that a firm's capital structures do not affect its market value. However, if debt policy were completely irrelevant, actual debt ratios would vary randomly from firm to firm and from industry to industry. This inconsistency with the empirical evidences means that some determinant factors have not been captured. In the literature, this indeterminacy was referred to as the Capital Structure Puzzle (Myers 1984). Despite extensive research in the field, there is no consensus on the determinants of capital structure, and a significant

[*] Zhao: Southwest Minzu University, Chengdu 610041, China (e-mail: zhaoguo@swun.edu.cn). Part of this paper has been presented at The Chinese Economists Society 2015 Conference, held on June 13-14th at Chongqing University. Financial supports from Hunan University are gratefully acknowledged.

gap still exists in understanding how do firms choose their capital structures (Michael J. Barclay and Clifford W. Smith, 2005; Murray Z. Frank and Vidhan K. Goyal, 2015).

An unfortunate consequence of the Modigliani-Miller irrelevancy propositions has been the discarding (and denial) of theories for determining the financial structure (Ross, 1977). If the Modigliani-Miller hypothesis is complete and thought to be correct, then it also means that if firms maximize their market value, the *real* decisions are the only decisions that count, and the financial decisions have no bearing on them (Stiglitz,1974).[1] From this point of view, an answer to the capital structure puzzle requires an analysis of the interactions between the financial and real decisions of representative firm. In fact, researchers in finance have long been interested in the question of how financial decisions influences and is influenced by production decisions (Diamond, 1967; Fama, 1972; Stiglitz, 1972; Hayne E. Leland, 1974; Ross, 1978).

Part of the difficulty in understanding the capital structure puzzle stems from the dichotomy between finance and production. In contrast to the financial view of the criteria of rational decision-making as the maximization of market value (Modigliani and Miller 1958), the neoclassical theory of production is based on the maximization of profits (Goolsbee *et al.*,2019). However, the profit maximization assumption has long been criticized, [2] chiefly on grounds that it lacks realism (Koplin, 1963; Anderson *et al.*, 2005). For example, the profit maximization model only analyzes a single industry, regardless of the existence of risk-free assets in a whole economy with multiple interacting markets. This will inevitably lead to zero-profit equilibrium in the industry alone (Goolsbee *et al.*, 2019). In practice, however, the existence of risk-free assets may help to set a lower bound to the rate of return on investment and hence prevent the return on investment from being driven to zero (James Tobin, 1965).

[1] Stiglitz (1974) divided the decisions of the firm into four groups:

(a) How should the firm finance its investment?
(b) How should the firm distribute its revenue?
(c) How much should the firm invest?
(d) Which projects should the firm undertake (or what techniques of production should the firm employ)?

The first two decisions of the firm are the *financial* decisions of the firm, the latter two the *real* decisions. Stiglitz mentioned that the two financial decisions are closely related, and so are the two real decisions. What is not obvious is the relationship between the real decisions and the financial decisions, which is exactly what this paper shall explore.

[2] Modigliani and Miller (1958) has pointed out that the maximization of profits has serious drawbacks when allowing for the existence of uncertainty: “With the recognition of uncertainty, …, the profit maximization criterion is no longer even well defined. Under uncertainty there corresponds to each decision of the firm not a unique profit outcome, but a plurality of mutually exclusive outcomes which can at best be described by a subjective probability distribution. The profit outcome, in short, has become a random variable and as such its maximization no longer has an operational meaning. Nor can this difficulty generally be disposed of by using the mathematical expectation of profits as the variable to be maximized. For decisions which affect the expected value will also tend to affect the dispersion and other characteristics of the distribution of outcomes.”

To study the interaction between financial and production decisions, we propose a dynamic production model under the joint constraints of technology, budget and no arbitrage. In no-arbitrage equilibrium, risk-neutral firms obtain risk-free rate of return on investment,[3] rather than zero economic profit derived from profit maximization. The risk-free return may be zero but is not necessarily so. As a result, the dynamic production in no-arbitrage equilibrium is a natural generalization of profit maximization. When the risk-free return approaches zero, this dynamic production model may degenerate into the classical case of profit maximization. This technical route can be shown in the following scheme in which the horizontal arrows $(\Rightarrow)$ represent implication and the downward arrows $(\downarrow)$ represent generalization

$$\begin{array}{ccccc} profit\ \ Maximization & \Rightarrow & competitive\ \ equilibrium & \Rightarrow & zero\ \ economic\ \ profit \\ \downarrow & & \downarrow & & \downarrow \\ no\ \ arbitrage & \Rightarrow & no-arbitrage\ \ equilibrium & \Rightarrow & risk-free\ \ rate\ \ of\ \ return \end{array}.$$

On the other hand, when the risk-free returns become nonzero, the economic system may undergo a bifurcation into a capital-intensive state and a labor-intensive state. Besides numerical simulation, we also theoretically obtain an estimate region of attraction for these equilibrium states in terms of capital-labor ratio. As a result, the capital-intensive state is asymptotically stable, but the labor-intensive state is unstable. These findings are consistent with the empirical pattern of *multiple equilibria* (Costas Azariadis and Allan Drazen, 1990; Carter and Barrett*, 2006*) and *conditional convergence* (Baumol *, 1986;* De Long*, 1988;* Barro and Xavier *, 1992*).

As an application, we show that in no-arbitrage equilibrium a firm's debt/equity ratio is endogenously determined by its endowment structures, and *vice versa*. As a result, the actual debt/equity ratios would tend to vary from industry to industry, and even between companies within an industry. This result partially solved the long-standing capital structure puzzle (Myers 1984), and hence justify the no-arbitrage production model as a useful framework.

The rest of the paper proceeds as follows. In section **I**, a dynamic production model of a closed economy is constructed under the joint constraints of technology, budget and no arbitrage. Section **II** focuses on the solutions of the model, with emphasis on the tangent bifurcation from the profit maximization

[3] The validity of this statement depends on the assumption that representative firms are risk neutral, so that the degree of uncertainty (measured by Variance) will not affect investment decisions. Otherwise, risk premium of different industry must be considered on the basis of the Capital Asset Pricing Model (see Tobin 1958; Sharpe 1964).

point. The Lyapunov asymptotic stability of these solutions is discussed in Appendix and summarized in section **III**. In section **IV** we give an answer to the long-standing capital structure puzzle. Section **V** concludes this paper with some methodology remarks.

I. Dynamic Production Model

Before we get into the details, let's summarize a few key features of classical theory of production that are shared by our model. Typically, the market structure is characterized by perfect competition with free entry and exit. The representative firm produces a perfectly divisible product using exogenously given technologies that converts inputs into outputs. Further, the representative firm is assumed to be a price taker *in a probabilistic sense* (Sandmo, 1971),[4] so that the expected wage of labor W, the expected rental price of capital i, and the expected price level P will be taken as given.

Contrary to the static characteristic of profit maximization, our dynamic production model comprises two periods. The investment is taking place at the *beginning* of period t, but the resulting output is sold at the *end* of period t, or equivalently, at the *beginning* of period $t+1$.[5] The very bridge that links this time lag is the risk-free interest rate, written as r, which is the bridge between present and future (Fisher, 1930). The risk-free rate of return may be zero but is not necessarily so. Further, in a dynamic world we have to consider the depreciation rate of capital, denoted to be $\delta \in [0,1]$.

In reality, firms like consumers are subject to budget constraints.[6] Formally, we assume that the market value of the total budget at the *beginning* of period t is B_t.

4 This basic assumption is made in order to be consistent with the dynamic essence of our production model, since there exists of a fundamental time lag between expenditure and revenue. In a dynamic world, risk may arise in the gap between investing money and receiving profits because unexpected events may occur which may alter the value of profits. The absence of risk and uncertainty shows itself particularly in the absence of asset preference (see Tobin 1958).

5 We shall assume that in stationary economy the production period coincides with the maturity of risk-free interest rate. Time-to-build technology (see Kydland and Prescott 1982) will not be considered in this paper.

6 Note that in much of standard microeconomic theory, only consumers, not producers, face budget constraints. But the assumption that producers are unconstrained is made merely for convenience, since most of this theory is not concerned with the relationship between finance and production, where such constraints come into play (Kornai *et al.*, 2003). In practice, the existence and importance of a budget constraint becomes patently clear, and the traditional distinction between firms and households is blurred and perhaps vanished (see Becker 1962).

Then the efficient allocation of labor L_t and capital K_t at the beginning of period t must satisfy the budget constraint (cost constraint) imposed by its total wealth

$$iK_t + WL_t = B_t . \tag{1}$$

The no-arbitrage constraint means there is no free lunch (Ross, 2004). Thus, in no-arbitrage equilibrium the rate of return on investment is necessarily equal to the risk-free interest rate, and is the same no matter in terms of what it is measured. To be precise, the total wealth at the end of period t equals $B_t(1+r)$ in terms of money. On the other hand, at the end of period t the firm's total wealth consists of two components: the physical output Q_t and the capital stock $K_t(1-\delta)$. In equilibrium the market value of the physical output and the capital stock must add up to the total wealth at the end of period t. Were this not so an arbitrage process would be set in motion.

Assume that the output is sold at the *end* of period t, then the no-arbitrage constraint gives rise to the following accounting identity

$$PQ_t + iK_t(1-\delta) = B_t(1+r) . \tag{2}$$

In theory, when a particular production function $Q_t = AF_t(K_t, L_t)$ is specified, the solution of the system gives each of the three physical variables (L_t, K_t, Q_t) as multivariable functions of other state variables (Fig. 1(**B**)). No maximum problem need be studied, and no derivatives need be taken.

II. Tangent Bifurcation

In general, if the production function is non-linear, then there is no unique equilibrium like the tangent point for cost minimization. On the contrary, in no-arbitrage equilibrium the economic system may have multiple solutions in general.

To see this, substitute the production function $Q_t = AF_t(K_t, L_t)$ in the no-arbitrage constraint equation to obtain the *no arbitrage curve*

$$PAF_t(K_t, L_t) + iK_t(1-\delta) = B_t(1+r) . \tag{3}$$

Because of the principle of diminishing marginal rate of technical substitution between capital and labor (Goolsbee *et al.*, 2019), the production function (Fig. 1(**A**)) and hence the no arbitrage curve (Fig. 1(**B**)) slopes downwards from left to right and is convex to the point of origin. It follows that the no arbitrage curve intersected with the budget line at most twice. Thus multiple equilibria could appear even among economies with *identical* state variables. [7]

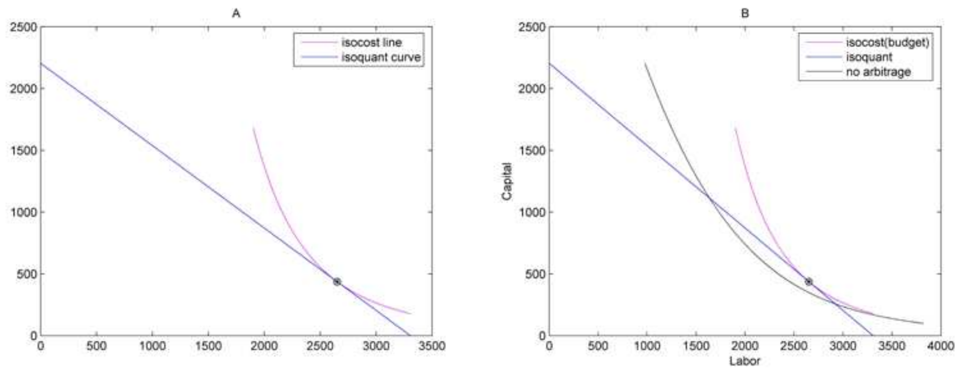

FIGURE 1. THE NO-ARBITRAGE EQUILIBRIUM AS A NATURAL GENERALIZATION OF PROFIT MAXIMIZATION

Notes: (**A**) Shown is to minimize the total cost $TC = 15K + 10L$ subject to the Cobb–Douglas technology $Q = 2K^{0.2}L^{0.81} = 4000$. The cost is minimized at the tangent point where $(L^*, K^*) \approx (2652{,}437)$. The corresponding total cost and marginal cost are $TC \approx 33063$ and $MC \approx 8.1839$, respectively. (**B**) Shown is a plot of the no arbitrage curve with $\delta = 0.2$, $r = 0.05$, $P = MC$ and $B_\infty = TC$. It can be seen clearly that there are two intersections between the no arbitrage curve and the budget line. [8]

It's easy to see that these two solutions have different capital-labor ratio and different level of output, and hence stand for different type of economies. For ease of reference, we denote the solution with low capital-labor ratio as (L_t^1, K_t^1), which corresponds to labor-intensive or capital-poor economy. Similarly, we denote the solution with high capital-labor ratio as (L_t^2, K_t^2), which stands for capital-intensive or capital-rich economy.

The existence of such multiple equilibria is consistent with the pattern of economic growth across and within nations (Costas Azariadis and Allan Drazen,

[7] As a result, heterogeneous firms can coexist in no-arbitrage equilibrium. So, in general an economic system at a particular macroscopic state may occupy a number of microscopic states. This result differs from neoclassical theory of production, but agrees with the thermodynamic equilibrium of idea gas (see Feynman et al. 2013). In a sense, this means that economic system in no-arbitrage equilibrium behaves in much the same way as *thermodynamics* equilibrium (Tao, 2016), rather than *mechanics* equilibrium.

[8] For ease of comparison, we keep these state variables the same in all figures, samples as well as Appendix. By the way, most of these parameters are taken from a cost-minimization example of Goolsbee *et al.* (2019) .

1990; Carter *et al., 2006*). In practice, the economy's actual state is determined by its actual factor endowments.

To proceed, note that there is obviously an asymptotic case where the budget line and the no arbitrage curve are tangent to each other when $\delta \to 1$ and $r \to 0$. In this limiting case our dynamic production model will degenerate into the classical case of profit maximization (Fig. 2). Note that this tangent point always lies *between* the two intersection points of the budget line and the no arbitrage curve for $\delta < 1$ and $r \neq 0$.

Borrowing a terminology from Dynamic System (Robinson,2012), we say that the economic system has undergone a *tangent bifurcation* from the profit maximization point.

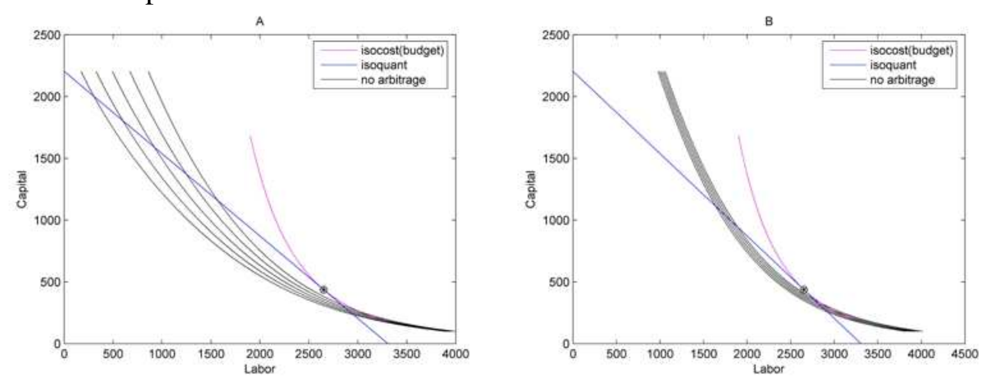

FIGURE 2. TANGENT BIFURCATION FROM THE PROFIT MAXIMIZATION POINT

Notes: Shown is a plot of the no arbitrage curve with $P = MC \approx 8.1839$ and $B_\infty = TC \approx 33063$. (**A**) The no arbitrage curve with $r = 0.05$ and $\delta =$ 0.1, 0.2, 0.3, 0.4, 0.5 respectively, from left to right. (**B**) The no arbitrage curve with $\delta = 0.6$ and $r =$ 0.01, 0.02, 0.03, 0.04, 0.05 respectively, from left to right.

III. Asymptotic Stability

Since the economic system in no-arbitrage equilibrium may have multiple steady-state solutions, none of them can have global asymptotic stability in the sense of Lyapunov (Robinson, 2012). This means that different starting point for the growth path may lead to different final state. [9] Fortunately, the dynamic nature of our model enables us to discuss the asymptotic stability of these solutions.

For simplicity, consider a Cobb–Douglas technology with *constant return to scale*. In such a case the product function has the form $Q_t = K_t^{\alpha} L_t^{1-\alpha}$ for some

[9] See Appendix for a possible definition of growth path in multiple equilibria.

$\alpha \in (0,1)$. To proceed, eliminate B_t from the system of equations (1) and (2) to get

$$PK_t^{\alpha} L_t^{1-\alpha} = i(r+\delta)K_t + W(1+r)L_t. \tag{4}$$

Dividing L_t and rewrite it in terms of per capita variable $k_t = K_t / L_t$, we have

$$Pk_t^{\alpha} = i(r+\delta)k_t + W(1+r). \tag{5}$$

Rearrange equation (5) as follows

$$k_t = \frac{P}{i(r+\delta)} k_t^{\alpha} - \frac{W(1+r)}{i(r+\delta)} := g(k_t). \tag{6}$$

It is easy to see that $k_t = g(k_t)$ is a *fixed point* of the nonlinear function $g(\cdot)$. To analyze the stability of k_t under iteration, take the first-order derivative with respect to k_t

$$g^{'}(k_t) = \frac{\alpha}{i(r+\delta)} Pk_t^{\alpha-1}. \tag{7}$$

Since equation (5) always holds in equilibrium, we can solve and then substitute for $Pk_t^{\alpha-1}$ to get

$$g^{'}(k_t) = \alpha\left(1 + \frac{W(1+r)}{i(r+\delta)} \times \frac{1}{k_t}\right). \tag{8}$$

It is well known (Robinson, 2012) that the fixed point $k_t = g(k_t)$ is stable under iteration if its first-order derivative satisfies

$$g^{'}(k_t) = \alpha\left(1 + \frac{W(1+r)}{i(r+\delta)} \times \frac{1}{k_t}\right) < 1. \tag{9}$$

Solving for k_t we obtain a threshold for the endowment structure to be asymptotically stable

$$k_t > \frac{\alpha W(1+r)}{(1-\alpha)i(r+\delta)} := \underline{k}. \tag{10}$$

Since this threshold $\underline{k}$ does *not* explicitly contain B_t, it is robust in that it is independent of specific growth process (see Appendix for details).

A limiting case of important interest is when $\delta \to 1$ and $r \to 0$. In such a case the threshold approaches to $\alpha W/(1-\alpha)i$, which is exactly the *expansion path* derived from cost minimization under Cobb–Douglas production function (Goolsbee *et al.*, 2019).

Further, if $r+\delta<1$ and $r>0$, then it is easy to see that the threshold value satisfies $\underline{k} > \alpha W/(1-\alpha)i$. It follows that the labor-intensive equilibrium (L_t^1, K_t^1) is unstable at any period t,[10] since the *expansion path* always lies *between* the two no-arbitrage equilibrium solutions (property of tangent bifurcation, see section II).

To illustrate, consider the Cobb–Douglas production function $Q_t = AK_t^{0.2}L_t^{0.81}$, which *approximately* satisfies constant return to scale. So we can obtain a *rough* estimate of the corresponding threshold value

$$\underline{k} \approx \frac{\alpha W(1+r)}{(1-\alpha)i(r+\delta)} \approx 0.7\,. \tag{11}$$

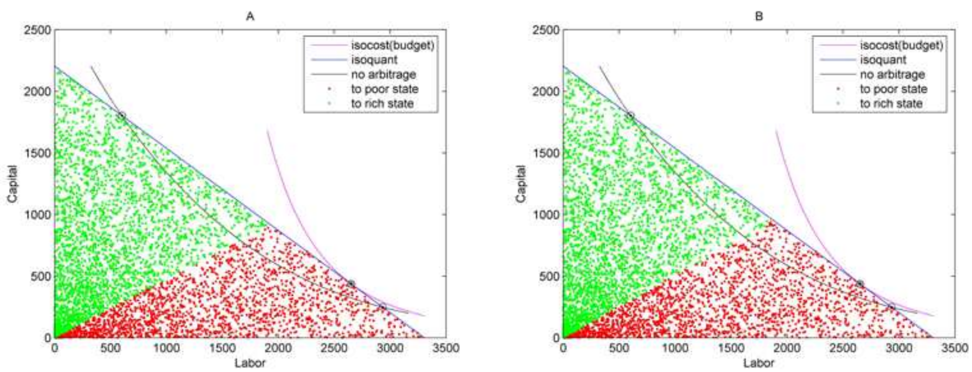

FIGURE 3. MONTE CARLO SIMULATIONS OF THE REGION OF ATTRACTION

Notes: The growth path starting from the red dot will converge to the capital-poor equilibrium (L_∞^1, K_∞^1), and the growth path starting from the green dot will converge to the capital-rich equilibrium (L_∞^2, K_∞^2). The growth rate of Logistic process is fixed at $\lambda = 0.5$. The number of samples is 5000. (**A**) Using Newton's Iteration Method to compute the growth path, the resulting threshold is near 0.5. (**B**) Invoking Matlab's fsolve function to compute the growth path, the resulting threshold is about 0.52.

[10] In a sense the lack of stability for capital-poor state means that labor-intensive economy has a strong incentive to improve its endowment structure, based on its comparative advantage in labor-intensive industries. It is this kind of comparative-advantage-following approach that enables a capital-poor economy to transit into a capital-rich one as capital accumulation reaches certain threshold level (Lin, 2011; Ju *et al.,* 2015).

Example: Consider the following Logistic process

$$B_{t+1} = B_t \exp(\lambda(1 - B_t / TC)), \tag{12}$$

where TC is the capacity of total wealth and $\lambda > 0$ is the usual growth rate. It is well known that for $\lambda < 2$ this Logistic process has one globally stable equilibrium point at $B_\infty = TC$ (May *,1974,1976)*, which gives rise to two final states: $(L_\infty^1, K_\infty^1) \approx (2934,248)$ and $(L_\infty^2, K_\infty^2) \approx (602,1803)$ (Fig. 1(**B**)).

Following this Logistic process, the world may eventually form into two convergence clubs: the poor or developing club (L_∞^1, K_∞^1), and the rich or developed club (L_∞^2, K_∞^2) (Fig. 3). This result of club convergence is consistent with observed pattern of the postwar economic growth (Baumol*, 1986;* De Long*, 1988;* Barro *et al., 1992*).

However, only the capital-rich equilibrium (L_∞^2, K_∞^2) is asymptotically stable. An estimate region of attraction consists of endowments with capital-labor ratios above $\underline{k}$ (Fig. 3). This means that the growth path with initial capital-labor ratio $k_0 > \underline{k}$ always converges to (L_∞^2, K_∞^2).

On the contrary, the capital-poor equilibrium (L_∞^1, K_∞^1) is unstable. More specifically, if the initial capital-labor ratio $k_0 < \underline{k}$, then the corresponding growth path may converge either to (L_∞^1, K_∞^1) or to (L_∞^2, K_∞^2), conditioned on the initial budget and growth rate (Fig. 3). This result of conditional convergence is consistent with observed pattern of the postwar economic growth (Baumol*, 1986;* De Long*, 1988;* Barro *et al., 1992*).

IV. The capital structure puzzle

Arising from the dichotomy between finance and production, the capital structure puzzle calls for a theory to explain "how do firms choose their capital structures." (Myers 1984) In this section, we give an answer to the capital structure puzzle by exploring the interactions between the financial and real decisions of representative firm.

To simplify the analysis, we assume that the total budget at the *beginning* of each period can either be *accumulated* during the past periods or *financed* via perpetual debts with risk-free interest rate. Formally, let E_t and D_t denote the market value of equity and debt respectively, then the budget constraint becomes into

(13) $$iK_t + WL_t = B_t = E_t + D_t .$$

At the *end* of period t, the firm has to pay the risk-free interest payment $D_t r$. So the no arbitrage constraint must be adjusted accordingly[11]

(14) $$PQ_t + iK_t(1-\delta) = (E_t + D_t)(1+r) - D_t r = E_t(1+r) + D_t .$$

Taken together, the equilibrium value of equity and debt can be solved as a function of physical variables

(15) $$\begin{cases} rE_t = PQ_t - iK_t\delta - WL_t \\ rD_t = iK_t(r+\delta) + WL_t(1+r) - PQ_t \end{cases} .$$

Consequently, the equilibrium debt/equity ratio is uniquely determined by the actual endowment of capital and labor when a particular production function $Q_t = AF_t(K_t, L_t)$ is specified,[12] that is,

(16) $$\frac{D_t}{E_t} = \frac{iK_t(r+\delta) + WL_t(1+r) - PQ_t}{PQ_t - iK_t\delta - WL_t} .$$

Since the factor endowments vary by industry, the actual debt/equity ratios will tend to vary significantly from one industry to another, and even between companies within an industry. As the economy develops, the equilibrium capital structure for the economy evolves correspondingly. These results are consistent with the basic patterns of actual capital structure in the real world (Lin *et al.* 2013; Ju *et al.,* 2015).

[11] Now that the no arbitrage constraint is just an accounting identity, it can be easily adjusted to consider more practical factors such as the tax gains of debts (Miller 1977) and dividend policy (Miller and Modigliani 1961).

[12] Another way is to understand Q_t as the final demand of the product. The existence of effective arbitrage roughly equated the supply and demand. So, in no-arbitrage equilibrium the supply roughly equals final demand, and *vice versa*.

Our technical route for understanding the capital structure puzzle can be illustrated by the following extended balance sheet (based on that of Miller 1988).

Assets	Constraints	Liabilities
Productive Capital	Budget No arbitrage Technology	Debts Equity

V. Concluding Remarks

I have reported a dynamic production model under the assumption of no arbitrage. It is shown that dynamic production in no-arbitrage equilibrium may exhibit complex behaviors, including a tangent bifurcation from the profit maximization point. As an application, we give an answer to the long-standing capital structure puzzle by showing that in no-arbitrage equilibrium a firm's capital structure is endogenously determined by its endowment structure, and *vice versa*. As a result, the actual debt/equity ratios would tend to vary by industry due to differences in the endowment structures. These findings are consistent with the empirical evidences.

As we have seen, no-arbitrage equilibrium is a natural generalization of profit maximization. Compared with neoclassical theory of production, this no-arbitrage based dynamic production model confers some methodological advantages:

1. It emphasizes the general equilibrium of the economic system as a whole.
2. Its dynamic essence enables us to investigate the dynamic evolution of the economic system.
3. It is based on no arbitrage assumption and hence bridges production and finance. Due to the central role of no arbitrage assumption in Finance (Ross, 2004), we believe that the no-arbitrage based production model is of a very general nature.

In conclusion, these methodological advantages justify the no-arbitrage based production model as a useful framework, and further effort should be made in order to integrate it in standard framework.

REFERENCES

Anderson, William L. and Ross, Ronald L. 2005. “The Methodology of Profit Maximization: An Austrian Alternative”, *The Quarterly Journal of Austrian Economics* **8**:31–44.

Becker, Gary S. (1962). “Irrational Behavior and Economic Theory”, *Journal of Political Economy* 70(1):1-13.

Michael J. Barclay and Clifford W. Smith. 2005. “The Capital Structure Puzzle: The Evidence Revisited”, *Journal of Applied Corporate Finance* 17(1): 8-17

Barro, Robert J. and Sala-i-Martin Xavier. 1992. “Convergence”, *Journal of Political Economy* **100**(2):223-251.

Baumol, William J. 1986. “Productivity Growth, Convergence, and Welfare: What the Long-Run Data Show”, *American Economic Review* **76**(5):1072-85.

Carter, Michael R. and Barrett, Christopher B. 2006. “The economics of poverty traps and persistent poverty: An asset-based approach”, *The Journal of Development Studies* **42**(2):178-199.

Costas Azariadis and Allan Drazen. 1990. “Threshold Externalities in Economic Development”, *The Quarterly Journal of Economics* **105**(2):501-526.

De Long, J. B. 1988. “Productivity growth, convergence, and welfare: comment”, *American Economic Review* **78**(5):1138-1154.

Diamond, P. 1967. “The Role of a Stock Market in a General Equilibrium Model with Technological Uncertainty.” *The American Economic Review* **57**(4):759-776.

Fama, E. 1972. “Perfect Competition and Optimal Production Decisions under Uncertainty.” *The Bell Journal of Econiomics and Mantagement Science* **3**(2):509-530.

Fisher, Irving. 1930. *The Theory of Interest,* Macmillan: New York.

Feynman R, Leighton R, and Sands M. 2013. *The Feynman Lectures on Physics, Volume I* (online edition), The Feynman Lectures Website. http://feynmanlectures.caltech.edu/I_toc.html

Murray Z. Frank & Vidhan K. Goyal. 2015. “The Profits–Leverage Puzzle Revisited,” *Review of Finance* **19**(4), pages 1415-1453.

Goolsbee Austan, Steven Levitt, Chad Syverson. 2019. *Microeconomics 3rd edition*,Worth Publishers.

Ju J., Lin J.Y., Wang Y. 2015. “Endowment Structures, Industrial Dynamics, and Economic Growth”, *Journal of Monetary Economics*. **76():** 244-263.

Koplin, H. T. 1963. “The profit maximization assumption”, *Oxford Economic Papers (New Series)*. **15**(2):130-9.

Kornai, J. E. Maskin and Roland, G. 2003. “Understanding the Soft Budget Constraint”, *Journal of Economic Literature* **41**(4):1095-1136.

Kydland, F. and E. Prescott. 1982. “Time to build and aggregate fluctuations”, *Econometrica* **50**(6):1345-1371.

Hayne E. Leland. 1974. “Production Theory and the Stock Market”, *The Bell Journal of Economics and Management Science* **5**(1):125-144.

Lin, Justin Yifu. 2011. *New structural economics: a framework for rethinking development*,The World Bank:Washington, D.C.

Lin, J.Y., X. Sun, Y. Jiang. 2013. “Endowment, industrial structure, and appropriate financial structure: a new structural economics perspective.” *Journal of Economic Policy Reform* **16**(2):109-122.

May, Robert M. 1974. “Biological Populations with Nonoverlapping Generations: Stable Points, Stable Cycles, and Chaos”, *Science* **186**(4164):645-647.

May, Robert M. 1976. “Simple mathematical models with very complicated dynamics”, *Nature* **261**(5560):459-467.

Miller, Merton H. 1977. “Debt and Taxes”, *The Journal of Finance* **32**(2):261-75.

Miller, Merton H. 1988. “The Modigliani-Miller Propositions After Thirty Years.” *Journal of Economic Perspectives* **2**(4):99-120.

Modigliani, Franco. 1988. “MM--Past, Present, Future.” *The Journal of Economic Perspectives* **2**(4):149-158.

Modigliani, Franco. and Miller, Merton H. 1958. “The Cost of Capital, Corporation Finance and the Theory of Investment.” *American Economic Review* **48**(3):261-97.

Miller, Merton H. And Modigliani, Franco. 1961. “Dividend Policy, Growth, and the Valuation of Shares”, *Journal Of Business* **34**(4):411-33.

Myers, S. 1984. “The Capital Structure Puzzle”, *Journal of Finance* **39**(3), pp.575-592.

Robinson, R. Clark. 2012. *An Introduction to Dynamical Systems: Continuous and Discrete Second Edition*, American Mathematical Society.

Ross, Stephen A. 1977. "The Determination of Financial Structure: The Incentive-Signalling Approach", *Bell Journal of Economics* **8**(1):23-40.

Ross, Stephen A, 1978. "Some Notes on Financial Incentive-Signalling Models, Activity Choice and Risk Preferences", *Journal of Finance* **33**(3):777-792.

Ross, Stephen A. 2004. *Neoclassical Finance*, Princeton University Press.

Sandmo, Agnar. 1971. On the Theory of the Competitive Firm Under Price Uncertainty, *The American Economic Review* 61(1): 65-73.

Sharpe, William F. 1964. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk", *Journal of Finance* **19**(3):425–42.

Stiglitz, J. 1972. "On the Optimality of the Stock Market Allocation of Investment". *Quarterly Journal of Economics* **86**(1):25-60.

Stiglitz, J.E. 1974. "On the Irrelevance of Corporate Financial Policy", *American Economic Review* **64**(6):851-866.

Tao Y. 2016. "Spontaneous economic order", *Journal of Evolutionary Economics* **26**(3):467–500.

Tobin, James. 1958. "Liquidity Preference as Behaviour Towards Risk", *Review of Economic Studies* **25** (2):65-86.

Tobin, James. 1965. "Money and Economic Growth", *Econometrica* **33** (4):671-684.

Appendix. Lyapunov Asymptotic Stability

We shall consider the Lyapunov asymptotic stability of the system of accounting identities

(A1)
$$\begin{cases} iK_t + WL_t = B_t \\ PK_t^{\alpha} L_t^{\beta} + i(1-\delta)K_t = B_t(1+r) \end{cases},$$

under the exponential Logistic process

(A2)
$$B_{t+1} = B_t \exp(\lambda(1 - B_t / TC)),$$

where $\lambda > 0$ is the usual growth rate and TC is the capacity of total wealth.

It is well known that this Logistic process has a globally stable equilibrium point at $B_{\infty} = TC$ for all $\lambda < 2$ *(*May*, 1974, 1976)*. Correspondingly, for $\lambda < 2$ the economic system has two steady-state solutions $(L_{\infty}^1, K_{\infty}^1)$ and $(L_{\infty}^2, K_{\infty}^2)$ (Fig. 1(**B**)). In theory, none of them can have global asymptotic stability in the sense of Lyapunov (Robinson, 2012).

Fortunately, the dynamic nature of our model enables us to discuss the asymptotic stability of these solutions under Cobb–Douglas technology with *constant return to scale*. In such a case the product function has the form $Q_t = K_t^{\alpha} L_t^{1-\alpha}$ for some $\alpha \in (0,1)$.

To proceed, eliminate B_t from the system to get

(A3)
$$PK_t^{\alpha} L_t^{1-\alpha} = i(r+\delta)K_t + W(1+r)L_t.$$

Dividing L_t and rewrite it in terms of per capita variable $k_t = K_t / L_t$, we have

(A4)
$$Pk_t^{\alpha} = i(r+\delta)k_t + W(1+r).$$

Rearrange equation (A4) as follows

(A5)
$$k_t = \frac{P}{i(r+\delta)} k_t^{\alpha} - \frac{W(1+r)}{i(r+\delta)} := g(k_t).$$

It is easy to see that $k_t = g(k_t)$ is a fixed point of the nonlinear function $g(\cdot)$. To analyze the stability of k_t under iteration, take first-order derivative with respect to k_t

(A6) $$g^{'}(k_t) = \frac{\alpha}{i(r+\delta)} P k_t^{\alpha-1}.$$

Since equation (A4) always holds in equilibrium, we can solve and then substitute for $Pk_t^{\alpha-1}$ to get

(A7) $$g^{'}(k_t) = \alpha\left(1 + \frac{W(1+r)}{i(r+\delta)} \times \frac{1}{k_t}\right).$$

It is well known (Robinson, 2012) that the fixed point $k_t = g(k_t)$ is stable under iteration if its first-order derivative satisfies

(A8) $$g^{'}(k_t) = \alpha\left(1 + \frac{W(1+r)}{i(r+\delta)} \times \frac{1}{k_t}\right) < 1.$$

Solving for k_t we obtain a threshold for k_t to be asymptotically stable

(A9) $$k_t > \frac{\alpha W(1+r)}{(1-\alpha)i(r+\delta)} := \underline{k}.$$

A limiting case of important interest is when $\delta \to 1$ and $r \to 0$. In such a case this threshold approaches to $\alpha W/(1-\alpha)i$, which is exactly the *expansion path* derived from cost minimization under Cobb–Douglas production function (Goolsbee *et al.*, 2019).Further, if $r + \delta < 1$, then it's easy to see that the threshold value satisfies $\underline{k} > \alpha W/(1-\alpha)i$.

As an example, consider the following instance corresponding to $B_\infty = TC \approx 33063$

(A10) $$\begin{cases} 15K_t + 10L_t = 33063 \\ 16.3678K_t^{0.2}L_t^{0.81} + 12K_t = 1.05 \times 33063 \end{cases}.$$

It's routine to check that this system has two solutions $(L_\infty^1, K_\infty^1) \approx (2934, 248)$ and $(L_\infty^2, K_\infty^2) \approx (602, 1803)$ (Fig. 1(**B**)). Here the Cobb–Douglas production function

$Q_t = K_t^{0.2} L_t^{0.81}$ *approximately* satisfies constant return to scale. So we can obtain a *rough* estimate of the threshold value

$$\text{(A11)} \qquad \underline{k} = \frac{\alpha W(1+r)}{(1-\alpha)i(r+\delta)} \approx 0.7 .$$

As a result, the capital-rich equilibrium (L_∞^2, K_∞^2) is asymptotically stable, with an estimate region of attraction consists of endowments whose capital-labor ratios are above $\underline{k}$. This means that the growth path with initial capital-labor ratio $k_0 > \underline{k}$ always converges to the capital-rich equilibrium (L_∞^2, K_∞^2). On the contrary, the capital-poor equilibrium (L_∞^1, K_∞^1) is unstable. More specifically, if the initial capital-labor ratio $k_0 < \underline{k}$, then the corresponding growth path may converge either to (L_∞^1, K_∞^1) or to (L_∞^2, K_∞^2), conditioned on the initial budget and growth rate. These findings are consistent with the empirical pattern of *conditional convergence* (Baumol*, 1986;* De Long*, 1988;* Barro and Xavier*, 1992*).

To investigate the robustness of this threshold, we define the system's growth path starting from (L_0, K_0) in the following way: Using (L_0, K_0) as starting point, we apply numerical method to find the solution (L_1, K_1) for the system at B_0. Recursively, by adopting (L_1, K_1) as new initial value, we continue to apply numerical method to calculate a solution (L_2, K_2) for the system at $B_1 = B_0 \exp(\lambda(1 - B_0 / TC))$. Repeat this iteration process until a satisfactory growth path $\{(L_t, K_t) \mid t = 0,1,2,\cdots\}$ is obtained.

Following this iteration process, the world may eventually form into two convergence clubs: the rich or developed club (L_∞^2, K_∞^2), and the poor or developing club (L_∞^1, K_∞^1) (Fig. 3 and Fig. A1). This result of club convergence is consistent with observed pattern of the postwar economic growth (Baumol*, 1986;* De Long*, 1988;* Barro *et al., 1992*).

To continue, we fix the growth rate at $\lambda \in (0,2)$. Then we choose the initial budget $B_0 < TC$ at random and randomly select the initial endowment (L_0, K_0) along the budget line corresponding to B_0. As is shown in figure 3 and Figure A1, all growth paths with initial capital-labor ratio $k_0 > \underline{k}$ will converge to the capital-rich solution (L_∞^2, K_∞^2). On the contrary, the growth path with initial endowment structure $k_0 < \underline{k}$ may converge either to (L_∞^1, K_∞^1) or to (L_∞^2, K_∞^2). The resulting threshold is not far from the theoretic estimate as far as the nonlinearity of the production function is concerned.

It is worth emphasis that this threshold is robust in that it approximately keeps the same for $0 < \lambda < 2$ (compare subfigures in figure 3 and figure A1).

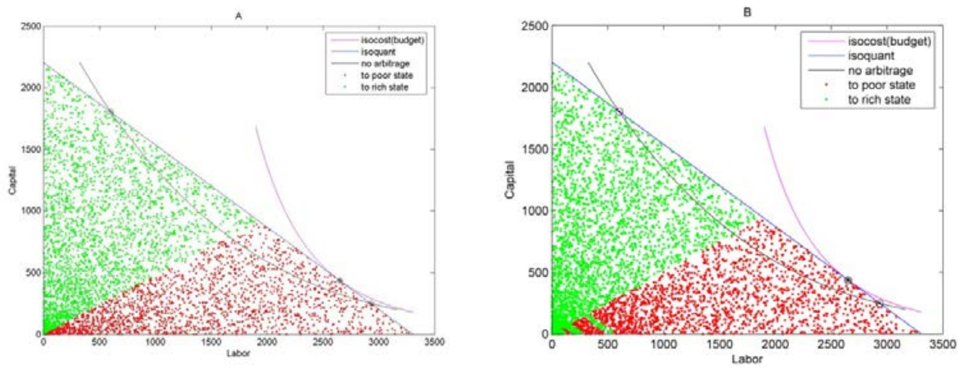

FIGURE A1. MONTE CARLO SIMULATIONS OF THE REGION OF ATTRACTION

Notes: The growth path starting from the red dot will converge to the capital-poor equilibrium (L_∞^1, K_∞^1), and the growth path starting from the green dot will converge to the capital-rich equilibrium (L_∞^2, K_∞^2). The growth rate of Logistic process is fixed at $\lambda = 1.0$. The number of samples is 5000. (**A**) Using Newton's Iteration Method to compute the growth path, the resulting threshold is near 0.5. (**B**) Invoking Matlab's fsolve function to compute the growth path, the resulting threshold is about 0.52.