Convexity in Portfolio Choice under Ambiguity

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Abstract

This paper studies the implication of the Uncertainty Aversion Axiom of Schmeidler (1989) on the problem of portfolio choice under ambiguity, which involves allocating the proportions of an initial wealth to several assets of unknown probability distributions. Our main result shows that if an investor is risk averse and conforms to the uncertainty aversion axiom, then preference under ambiguity in a portfolio space is convex. This means that the convexity in a portfolio choice problem can be guaranteed without restricting preference representation to a particular functional form.

1 Introduction

The Uncertainty Aversion Axiom of Schmeidler (1989) represents the first attempt to formalize the notion that individuals dislike ambiguity... the intuition is that, by mixing two acts, the individual may be able to hedge against variation in utilities, much like, by forming a portfolio consisting of two or more assets, one can hedge against variation in monetary payoffs.

— Machina and Siniscalchi (2014)

The standard definition of convexity in preference requires that if f is preferred to g, then any convex combination of f and g is also preferred to g. The following example illustrates its interpretation in in a portfolio choice problem. Suppose there are two assets f and g and two states of the world. Asset f gives a return of 4 (per unit of money) in state 1 and pays 0 in state 2 while asset g gives a return of 0 in state 1 and gives 4 in state 2. We will use the notation $f = (4, s_1; 0, s_2)$ and $g = (0, s_1; 4, s_2)$. Suppose you have one pound to invest in these two assets and you decide to allocate half of it to f and the other half to g, then you get a safe return $h_1 = (2, s_1; 2, s_2)$. This implies you prefer h_1 to both f and g. Now consider allocating the one pound between h_1 and g equally, then this results in $h_2 = (1, s_1; 3, s_3)$. It is sensible to assume that you would prefer h_2 to g given that it is a mixture of h_1 and g, and you prefer h_1 and g. In general, if you prefer all such mixtures of h_2 and g to g, then we say you conform to *portfolio convexity*, which is formally defined in Definition 3.

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Theoretically, portfolio convexity guarantees that a portfolio choice problem can be represented by maximizing a quasiconcave function. It further means the optimal portfolio is unique or it is an interval, which is a common assumption in the literature. This paper investigates the basic theoretical question that if preference under ambiguity in a portfolio mixture space is convex.

The problem of portfolio choice underlines much of finance and it is commonly adopted by experimentalists to elicit risk preferences. Formally, it involves an investor choosing the proportions $\alpha = (\alpha_1, ..., \alpha_N)$ of an initial wealth w to allocate on $N \geq 2$ assets. Suppose asset n gives a return of z_n per unit of money invested. Then a portfolio α gives rise to a final wealth $x = w(\alpha_1 z_1 + ... + \alpha_N z_N)$. It is commonly assumed that investors only care about the final wealth of a portfolio, therefore preference is defined on x.

Under risk, a risky asset is typically represented by a cumulative probability distribution $F(\cdot)$ over deterministic monetary outcomes, which is often called a *lottery* [1]. Preferences under risk are defined on lotteries and are commonly assumed to yield expected utility (EU) representation in finance: any lottery is evaluated by its probability cumulative distribution $F(\cdot)$ by the von-Neumann-Vorgenstern (vNM) utility function $U(F) = \int u(t)dF(t)$ where $u(\cdot)$ is a Bernoulli function. Conditioning on EU, Risk aversion is defined as follows.

Definition 1 (Risk Aversion)**.** Preferences over lotteries are *risk averse* if and only if $u(\cdot)$ is concave.

Denote $F(z_1, ..., z_N)$ the joint distribution of the return of the N assets. Then the utility of the final wealth of a portfolio of risky assets is

$$U(x) = \int u(\alpha_1 z_1 + ... + \alpha_N z_N) dF(z_1, ..., z_N)$$

Since $u(\cdot)$ is concave, then it follows that $U(\cdot)$ is concave.

We can see the utility function of a portfolio is concave because risk aversion is represented by the concavity of utility function $u(\cdot)$ over monetary outcomes. However, the probability distributions are usually unknown or do not exist (*ambiguity*) in reality. Ellsberg (1969)'s seminal paper argues that people tend to be ambiguous averse, which describes the phenomenon that people prefer betting on known probability to unknown probability. Ambiguity has since been widely studied theoretically, experimentally and its implications on financial market has been developed (See Trautmann and Van De Kuilen (2015) for a survey.).

Similar to risk aversion, decision theorists also strive to derive concave functional representations for ambiguity aversion by introducing a certain set of *axioms*, which are restrictions on binary preference relation. Unlike risk aversion, there is not a unanimous theoretical definition for ambiguity aversion. However most ambiguity models [2] share a basic set of axioms, among which the *Uncertainty Aversion* is a key one.

Definition 2 (Uncertainty Aversion Axiom, Schmeidler 1989)**.** For all acts f and g, preferences $\succsim$ are *uncertainty averse* if $f \succsim g$ implies $\lambda f + (1-\lambda) g \succsim g$ for any $\lambda \in [0,1]$.

where an *act* is a mapping from states to *objective lotteries*, which are probability distributions over deterministic outcomes (henceforth lottery). This type of act is usually

[1] The outcomes for a lottery do not need to be monetary in general. It can be consumption bundles, health status etc. In a financial problem, outcomes are naturally assumed to be monetary.

[2] For example, Schmeidler's (1989)'s Choquet Expected Utility with convex capacity, Gilboa and Schmeidlerl's (1989) max-min Expected Utility, Maccheroni, Marinacci, and Rustichini's (2006) Variational Preference, Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio's (2011) penalization representation, Strzalecki (2011) Multiplier Preferences.

called a horse-roulette act. The addition "$+$" is the probability mixture. A *probability mixture* of two horse-roulette acts f and g is defined as the state-by-state probability mixture of their state-contingent purely objective lotteries.

As stated in the quotation at the beginning of this article, the intuition of uncertainty aversion axiom is that **mixing** acts can give rise to diversification in a similar way of forming a portfolio of several assets. However, *probability mixture* of acts is different from mixture of assets that are characterized by acts in a portfolio, for which we name it as *portfolio mixture*.

To lay out the difference in *probability mixture* and *portfolio mixture* is the meat of this article, which is elaborated in Section 3. Intuitively, probability mixture of two lotteries is about the convex combination of probabilities while portfolio mixture of lotteries is about the convex combination of monetary outcomes. For the same pair of lotteries, portfolio mixture results in a lottery that has a weakly higher risk averse expected utility than the lottery resulted from probability mixture. When defining on acts, the two mixtures are defined as state-by-state mixture of the acts' state-contingent lotteries. Therefore, for the same pair of acts, portfolio mixture would result in an act that is weakly preferred to the act resulted from probability mixture, if preference over primitive lotteries is risk averse.

Our main result, as stated in Proposition 1 in Section 4, shows that the Uncertainty Aversion Axiom, combined with risk aversion, can directly imply *portfolio convexity*, that is convexity of preferences in a portfolio choice problem when assets are characterized by acts . This means portfolio convexity is guaranteed without imposing further axioms on preferences to obtain a particular functional form.

For the rest of the paper, Section 2 sets up the portfolio choice under ambiguity model. Section 3 formally defines portfolio mixture. Section 4 states the main result and provides the proof. Section 5 offers discussions.

2 The Portfolio Choice under Ambiguity Model

Recent decision models under ambiguity are often (see Marchina and Siniscalchi 2014 for a survey) built on a type of Anscombe-Aumann (AA) framework, where $f : S \to \Delta(Z)$ a horse-roulette act that maps states into the linear space $X = \Delta(Z)$. $Z = \mathbb{R}_+$ is the monetary outcome space and Δ is a probability simplex. Therefore, typical element in X is a lottery. The classic Expected utility model is maintained for preferences over primitive lotteries. This objective-subjective approach provides a framework for representing uncertain prospects that involve both objective and subjective uncertainty. In this set-up, ambiguity aversion attitudes featured in the Ellsberg paradox can be incorporated.

Applying this AA framework to portfolio choice under ambiguity, then the return of ambiguous assets would be characterized by acts. Denote the set of states by S that the outcome of the ambiguous assets will depend on. Slightly abusing notation, let S also denote the finite number of states. The return of an asset is $f : S \to X$. In state s, its return is denoted by f_s, which is a lottery. Hence, ambiguity is expressed in this way: the subjective uncertainty (states) will solve and, depending on how it resolves the return of the ambiguous asset is a lottery. While the information about the probability of the subjective uncertainty is not available, the specification and the parameters of the lotteries are known objectively or they can be estimated using statistics.

For example, consider how investors may formulate the effect of international travel restrictions on an airline company' return: in state 1 (with travel restrictions), the return is uniformly distributed on the region of two times the standard deviation of 0.2 around the mean of 0.4 ; in state 2 (without travel restrictions), the region is then two times the standard deviation of 0.2 around a higher mean of 1.4. While the mean and standard deviations can be calculated based on statistical data, there is not enough information to assign a probability on the event of travel restrictions.

Let $f^n, n = 1, ..., N$ denote asset n. Denote $\alpha = (\alpha_1, .., \alpha_N) \in \mathbb{R}^N$ the portfolio. We can normalize the initial wealth $w \in \mathbb{R}_+$ to 1.Then preference $\succsim$ is defined on the final wealth x, which is

$$x = \alpha_1 f^1 \oplus ... \oplus \alpha_N f^N$$

where

$$x_s = \alpha_1 f_s^1 \oplus ... \oplus \alpha_N f_s^N \tag{1}$$

for all $s \in S$. We name the addition operation "$\oplus$" in (1) *portfolio mixture* and its interpretation is vital. Note f_s^n is a lottery. Let $R^n, n = 1, ..., N$ denote N independent real-valued random variable that distributes as f_s^n. A portfolio mixture of lotteries yields a lottery x_s that the random variable $\alpha_1 R^1 + ... + \alpha_N R^N$ distributes as. A portfolio mixture of acts is defined as the state-by-state portfolio mixture of state-contingent purely objective lotteries of those acts.

Hence, portfolio mixture is similar to the one in Gollier (2013)'s portfolio choice under uncertainty model. Since it involves convex combination of lotteries instead of real numbers, it is different from algebraic addition in a classic portfolio choice model within Arrow-Debreu framework. It is different from the probability mixture because it involves convex combinations in outcomes whereas probability mixture only involves convex combinations in probabilities. For example, let $p = (0, 0.5; 2, 0.5)$ denote a lottery that yields two outcomes of 0 and 2 with the same probability 0.5. Then we have $0.5p \oplus 0.5p = (0, 0.25; 0.5, 1; 0.25, 2)$ and $0.5p + 0.5p = p$. The intuition is that even an investor allocates her wealth among several identical assets, this is still a diversification and can reduce the variations in monetary outcomes.

Hence, the budget set can be written as

$$B(f^1, .., f^N) = \{\mathbf{x} \in X^S : x = \alpha_1 f^1 \oplus ... \oplus \alpha_N f^N, \sum_1^N \alpha_n = 1.\} \tag{2}$$

Definition 3 (Portfolio Convexity)**.** A preference relation $\succsim$ satisfies portfolio convex if $x \succsim x'$ implies $\lambda x \oplus (1 - \lambda) x' \succsim x'$ for any $\lambda \in [0, 1]$.

If preferences satisfies *portfolio convexity*, then the portfolio choice can be represented by the following

$$max\ U(x), \quad x \in B(f^1, .., f^N) \tag{3}$$

where $U(\cdot)$ is a quasiconcave function.

3 Probability Mixture "+" and Portfolio Mixture "$\oplus$"

In this section, we formally introduce how portfolio mixture and probability mixture are defined on lotteries and acts. We use the following notional convention. Let f and g denote two acts. The outcome of an act in a state $s \in S$ is denoted by f_s, which is a lottery. $(f + g)_s$ should be read as the act $f + g$'s outcome in state s. $f_s(z)$ should be read as the probability that f_s gives to the monetary payoff z. $(f_s + g_s)(z)$ should be read as the probability that the mixed lottery $f_s + g_s$ gives to the monetary payoff z.

3.1 Probability Mixture $+$

The definition of probability mixture on a pair of lotteries f_s and g_s is standard as in literature:

$$(\lambda f_s + (1-\lambda)g_s)(z) = \lambda f_s(z) + (1-\lambda)g_s(z) \tag{4}$$

where $\lambda \in [0,1]$.

A probability mixture of two acts f and g is defined as the state-by-state probability mixture of their state-contingent purely objective lotteries:

$$(\lambda f + (1-\lambda)g)_s = \lambda f_s + (1-\lambda)g_s \qquad for\ all\ s \in S.$$

3.2 Portfolio Mixture $\oplus$

The definition of portfolio mixture on a pair of lotteries f_s and g_s is:

$$(f_s \oplus g_s)(z) = \int f_s(y)g_s(z-y)dy. \tag{5}$$

Let R_1 and R_2 denote two independent real-valued random variables that distribute as f_s and g_s, respectively. Define a new random variable $R_3 := R_1 + R_2$. Equation (5) describes the probability density function of R_3.

It follows that for any $\alpha, \beta \in \mathbb{R}$

$$(\alpha f_s \oplus \beta g_s)(z) = \int f_s(y)g_s(\frac{z-\alpha y}{\beta})dy,$$

which describes the probability density function for the random variable $\alpha R_1 + \beta R_2$.

A portfolio mixture of acts is defined as the state-by-state portfolio mixture of state-contingent purely objective lotteries of those acts. That is

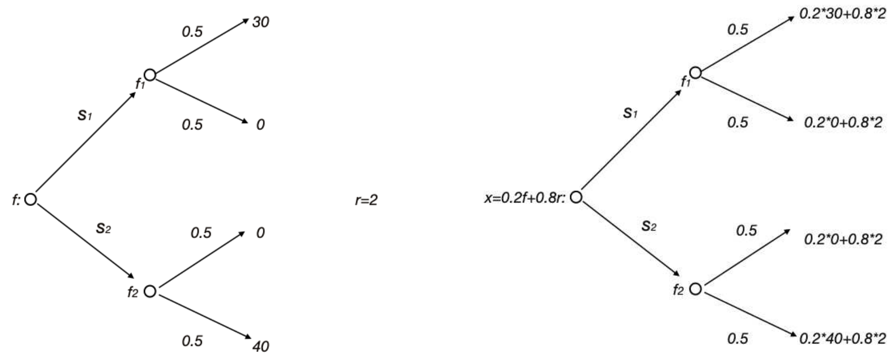
$$(\alpha f \oplus \beta g)_s = \alpha f_s \oplus \beta g_s$$

for all $s \in S$.

3.3 Examples

An Example of Portfolio Mixture

Consider the special case when the act g is a constant real value $r \in \mathbb{R}$. Let $x = 0.2f \oplus 0.8r$. The operation is rather simple: the probability distribution of x_s is the same as f_s while the original outcome z of f_s becomes $0.2z + 0.8r$.



(a) Ambiguous Asset f and Safe Asset r

Suppose there are two states s_1 and s_2. In state 1, f_1 is a lottery that returns 30 and 0 with the same probability of 0.5. In state 2, f_2 returns 40 and 0 with the same probability of 0.5. The safe asset pays a constant r of 2 in either of the two states.

(b) Portfolio Mixture $0.2f \oplus 0.8r$

Now suppose an investor allocates 20% to f and the remain 80% to r. Then this portfolio mix results in a new act. Since f takes four outcomes and the $r = 2$ is a constant, their weighted sum only takes four outcomes.

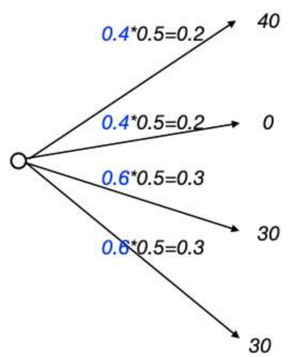
Figure 1: Portfolio Mixture of One Ambiguous Asset and One Safe Asset

An Example of how the two mixtures differ

The following example illustrates how the two mixtures differ. Figure 2 compares how they yield different lotteries in a typical state s.

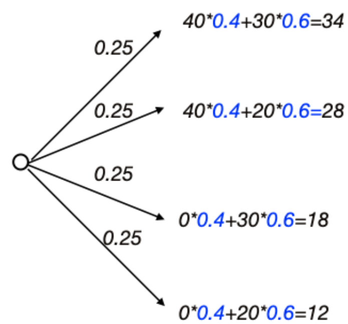
(a) Two Lotteries f_s and g_s

Suppose for some state s, f gives rise to a lottery f_s and g gives rise to a lottery g_s.



(b) Probability Mixture $0.4f_s + 0.6g_s$

The coefficients 0.4 and 0.6 are used for calculating the probabilities.



(c) Portfolio Mixture $0.4f_s \oplus 0.6g_s$

The coefficients 0.4 and 0.6 are used for calculating the weighted average of the outcomes. .

Figure 2: Probability Mixture and Portfolio Mixture of two acts in a typical state s

4 Proof for Proposition 1

Proposition 1. *If a preference relation $\succsim$ is uncertainty averse and risk averse, then $\succsim$ is portfolio convex.*

The following basic axiom of monotonicity will be used for the proof.

Definition 4 (Monotonicity)**.** For any acts f and g, if $f_s \succsim g_s$ for all $s \in S$, then $f \succsim g$

Proof. The key of the proof is Lemma 1. It proves that the portfolio mixture of two lotteries always has at least as great an expected utility as the corresponding probability mixture of these lotteries.

Then, for any pair of **acts**f and g , in each state the portfolio mixture of their state-contingent lotteries always has at least as great an expected utility as the corresponding probability mixture of these lotteries. By the axiom of monotonicity, this implies that the portfolio mixture of two acts is weakly preferred to their corresponding probability mixture: $\lambda f \oplus (1-\lambda) \succsim \lambda f + (1-\lambda)$.

Consider any arbitrary acts f and g such that $f \succsim g$. By the axiom of uncertainty aversion, for any $\lambda \in [0,1]$ we have $\lambda f + (1-\lambda) f \succsim g$. Then by transitivity, $\lambda f \oplus (1-\lambda) \succsim g$. $\square$

Lemma 1. Let f_s and g_s denote two lotteries. If preferences over lotteries are risk averse, then the expected utility of the portfolio mixture $\lambda f_s \oplus (1-\lambda) g_s$ is weakly higher than the expected utility of the probability mixture of $\lambda f_s + (1-\lambda) g_s$ for any $\lambda \in [0,1]$.

Proof. Recall that $f_s(z)$ denotes the probability that f_s gives to the monetary payoff z. Denote $EU(\cdot)$ the expected utility of a lottery. Let L_1 denote $\lambda f_s + (1-\lambda) g_s$ and let L_2 denote $\lambda f_s \oplus (1-\lambda) g_s$.

Based on equation (4), the expected utility of the probability mixture is

$$
\begin{aligned}
EU(L_1) &= \int u(z)\Big(\lambda f_s(z) + (1-\lambda) g_s(z)\Big) dz \\
&= \lambda \int u(z) f_s(z) dz + (1-\lambda) \int u(z) g_s(z) dz \\
&= \lambda EU(f_s) + (1-\lambda) EU(g_s)
\end{aligned}
$$

Based on equation (5), the expected utility of the portfolio mixture

$$
\begin{aligned}
EU(L_2) &= \int u(z) \int f_s(y) g_s\Big(\frac{z-\lambda y}{1-\lambda}\Big) dy dz \\
&= \int \int u\Big(\lambda y + (1-\lambda) z\Big) f_s(y) g_s(z) dy dz.
\end{aligned}
$$

Since $u(\cdot)$ is concave, for any $\lambda \in [0,1]$ and any y, z, there is

$$
u(\lambda y + (1-\lambda) z) \geq \lambda u(y) + (1-\lambda) u(z).
$$

Hence,

$$
\begin{aligned}
EU(L_2) &\geq \int \int \Big(\lambda u(y) + (1-\lambda) u(z)\Big) f_s(y) g_s(z) dy dz \\
&= \int \int \lambda u(y) f_s(y) g_s(z) dy dz + \int \int (1-\lambda) u(z) f_s(y) g_s(z) dy dz \\
&= \lambda \int \Big(\int u(y) f_s(y) dy\Big) g_s(z) dz + (1-\lambda) \int \Big(\int u(z) g_s(z) dz\Big) f_s(y) dy \\
&= \lambda \int EU(f_s) g_s(z) dz + (1-\lambda) \int EU(g_s) f_s(y) dy \\
&= \lambda EU(f_s) \int g_s(z) dz + (1-\lambda) EU(g_s) \int f_s(y) dy \\
&= \lambda EU(f_s) + (1-\lambda) EU(g_s) = EU(L_1)
\end{aligned}
\tag{6}
$$

□

5 Discussions

Proposition 1 states that the combination of Uncertainty Aversion Axiom and risk aversion implies portfolio convexity. Note risk aversion implicitly assumes EU on preferences under risk, which can be be replaced or relaxed. For example, Appendix B shows the popular Mean-Variance theory can also give rise to portfolio convexity.

One might also think about the result in the opposite direction: does portfolio convexity, in turn, implies Uncertainty Aversion Axiom and/or risk aversion. Consider risk aversion first. Let p denote a lottery. Construct a constant act f such that $f_s = p$ for all $s \in S$. Then $\lambda f \oplus (1-\lambda) f$ is a constant act where $(\lambda f \oplus (1-\lambda) f)_s = \lambda f_s \oplus (1-\lambda) f_s = \lambda p \oplus (1-\lambda) p$ for all $s \in S$. By portfolio convexity we have $\lambda f \oplus (1-\lambda) f \succsim f$.Then it must be $\lambda p \oplus (1-\lambda) p \succsim p$ by monotonicity. Suppose $u()$ is non-concave somewhere, then by similar arguments in equation (6), we have

$$
\begin{aligned}
EU(\lambda p \oplus (1-\lambda) p) &< \int \int \Big(\lambda u(y) + (1-\lambda) u(z) \Big) p(y) p(z) dy dz \\
&= \lambda EU(p) + (1-\lambda) EU(p) = EU(p),
\end{aligned}
$$

which is a contradiction to $\lambda p \oplus (1-\lambda) p \succsim p$. This means utility function must be concave everywhere, hence risk aversion is implied by portfolio convexity. However, whether the Uncertainty Aversion Axiom is implied by portfolio convexity is not clear. We leave this for future research.

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Appendix A An Counter Example of Convexity without risk aversion

Suppose there are two states of world with two set of priors $p_1 = [0.2, 0.8]$ and $p_2 = [0.6, 0.4]$. Denote $\mathbf{f}$ and $\mathbf{g}$ two assets that map states to monetary return where $f = [0, 4]$ and $g = [4, 0]$. Let $[\alpha, 1-\alpha]$ denote the portfolio choice, which are the proportions of wealth invested in f and g. Let x denote the final wealth of a portfolio, then $\mathbf{x} = \alpha\mathbf{f} + (1-\alpha)\mathbf{g}$. Let the utility function of monetary outcomes be convex $u(x) = x^2$. The Maxmin Expected Utility model postulates that an agent evaluates the portfolio x according two

$$MEU(\mathbf{x}) = min\Big(p_1 u(\mathbf{x}), p_2 u(\mathbf{x})\Big)$$

Consider the following $\mathbf{x}_1 = [2, 2], \mathbf{x}_2 = [4, 0], \mathbf{x}_3 = [3, 1]$ It can be easily verified that $MEU(\mathbf{x}_1) > MEU(\mathbf{x}_2)$ and $MEU(\mathbf{x}_2) > MEU(\mathbf{x}_3)$. This means $\mathbf{x}_1 \succ \mathbf{x}_2$ and $\mathbf{x}_2 \succ \mathbf{x}_3$ while $\mathbf{x}_3 = 0.5\mathbf{x}_1 + 0.5 * \mathbf{x}_2$. A contradiction of convexity.

Appendix B How Uncertainty Averse and Variance Averse implies Portfolio Convexity

Formally, Variance Aversion is defined as follows.

Definition 5. $\succsim$ on lotteries are *variance averse* if for two lotteries with the same mean, the lottery with a smaller variance is preferred.

Proposition 2. *If $\succsim$ is uncertainty averse and variance averse, then $\succsim$ is convex.*

Proof. Lemma 2 proves that the portfolio mixture of two lotteries is preferred to the probability mixture when variance aversion is assumed. Then following similar arguments in the proof of Proposition 1, we have that for $f \succsim g$, there is $\lambda f \oplus (1-\lambda)g \succsim \lambda f + (1-\lambda)g \succsim g$.

Lemma 2. If preferences over objective lotteries are variance averse, then portfolio mix of two lotteries is preferred to the probability mix of two lotteries.

Proof. Let f_s and g_s denote two lotteries. Let P and Q denote two independent random variables that distributes as f_s and g_s, respectively.

Define a new random variable $R_1 := BP + (1-B)Q$, where B is a binary, independent random variable for which the probability that $B = 1$ is α and the probability that $B = 0$ is $1-\alpha$. Define another random variable $R_2 := \alpha P + (1-\alpha)Q$. Thus, R_1 distributes as the probability mixture of f_+g_s. R_2 distributes as the portfolio mixture of $f_\oplus g_s$.

Using Law of Total Variance , we have $Var(R_1) = E_B(Var_B(R_1|B)) + Var_B(E_B(R_1|B))$. Since $Var_B(E_B(R_1)) \geq 0$, we have $Var(R_1) \geq E(Var(R_1|B))$. Recall that $R_1 = P$ if $B = 1$and $R_1 = Q$ if $B = 0$, so $Var(R_1) \geq E_B(Var_B(R_1)) = \alpha Var(P) + (1-\alpha)Var(Q)$. Since $\alpha \in [0, 1]$, we have $\alpha \geq \alpha^2$ and $(1-\alpha) \geq (1-\alpha)^2$.

$$Var(R_1) \geq \alpha^2 Var(P) + (1-\alpha)^2 Var(Q)).$$

Since P and Q are independent, we have

$$E(R_2) = \alpha E(P) + (1-\alpha)E(Q)$$

and

$$Var(R_2) = \alpha^2 Var(P) + (1-\alpha)^2 Var(Q).$$

In summary, $E(R_1) = E(R_2)$ and $Var(R_1) \geq Var(R_2)$. By variance averse we have

$$\alpha f_s + (1-\alpha)g_s \succsim \alpha f_s \oplus (1-\alpha)g_s.$$

□

□