Heterogeneous Households' Choices of Departure Time and Residential Location in a Multiple-origin Single-destination Rail System: Market Equilibrium and the First-best Solution

Konagane, Joji and Kono, Tatsuhito

Tohoku University

28 June 2021

Heterogeneous households' choices of departure time and residential location in a multiple-origin single-destination rail system: market equilibrium and the first best solution

Joji Konagane and Tatsuhito Kono[*1, *2]

Tohoku University, Aoba 09, Aoba-ku, Sendai 980-0879, Japan

Abstract. This paper explores heterogenous commuters' train choices with different arrival times and residential locations in a city composed of a single CBD and multiple residential zones. First, we analyze the relation between the cost of train overcrowding and train choice, given three residential location patterns according to income level, which are discussed in Fujita (1989) and Tabuchi (2019). Next, we analyze the necessary conditions for the existence of the residential location patterns on the basis of the relation between train overcrowding and train choice in equilibrium. The obtained necessary conditions depend on the values of time, housing lot sizes, and the overcrowding costs. The overcrowding costs depend on the choices of trains with different arrival times in equilibrium. Finally, in quantitative analysis, we show how much the social welfare improves due to the first-best congestion fares, depending on the residential location patterns. In any residential pattern, households with the lowest income increase their utilities the most among all households with different incomes, whereas households with the highest income lose their utilities.

Key words: Train overcrowding, Schedule delay costs, Heterogeneous households, Muth condition, Congestion fares

[*1] Corresponding author. E-mail: kono@plan.civil.tohoku.ac.jp

[*2] The authors thank discussions with Yuki Takayama about potential functions. This research was supported by grants from the Ministry of Education, Culture, Sports, Science and Technology.

1 Introduction

Heterogeneity in people is an important factor affecting every aspect of our economic activities. We explore the relation between commuters' departure time choices and train overcrowding in a city with many residential zones, considering heterogeneous commuters and their endogeneous locational choices. Heterogeneous people reside in the location of their choice. With the location patterns, they choose the best trains to the CBD for them when they commute. We explore this equilibrium and the welfare effects of the first-best fare policy on heterogeneous residents qualitatively as well as quantitatively.

In many cities across the world, severe train overcrowding during the morning rush is one of the most important problems. Furthermore, due to COVID-19, train passengers will be in more danger, even at lower levels of congestion than previously, due to novel viruses including new variants of COVID-19. Policy makers have to consider new risks of infection which had not previously been considered before the COVID-19 pandemic. Commuters face a trade-off between the arrival time and the congestion level on the train. It is, therefore, important to analyze the relation between the train choice and overcrowding costs, and to seek a solution for the problem. Possible solutions are an increase in the number of trains, construction of new railway lines, etc. But these are not always possible on every line because they take a lot of construction time and cost.

Congestion fares are easier to implement than new railway construction. In fact, these have been implemented in Washington and London subways for the purpose of relieving overcrowding. In Japan, dynamic fares have not been imposed, but are currently under consideration by the relevant ministry. Probably, other countries are also assmued to consider implementing such dynamic fares in the near future. In a city, heterogeneous residents live in different areas. Congestion fares affect them differently because the levels of the fares depend on the commuting lengths as well as the congestion levels of their train choices. This implies that, even for theoretical analyses, we cannot adopt the common assumption of the representative resident, typically supposing that all residents are average residents. Accordingly, to explore the effects of congestion fares, we have to examine the location equilibrium of heterogenous residents and who benefits from congestion fares.

Tian, Huang and Yan (2004) consider a mass transit line with multiple origins and a single CBD, assuming only homogeneous residents. They explore the trade-off between schedule delay costs and train overcrowding costs. They show an intriguing result that the overcrowding costs between any station and the station closest to the CBD are identical among trains used by commuters from the station closest to the CBD. However, they do not take account of the heterogeneity of commuters and endogeneous residential locations. In the real world, people move to the best places. Van den Berg and Verhoef (2014) consider heterogeneous commuters to solve the equilibrium of rail users on many trains. However, they consider only a single origin station and a single destination station. On real railways, there are many stations lying between the CBD and the suburbs. Commuting costs affect the location choices of commuters.

Many papers (e.g., Mills and de Ferranti (1971), Brueckner (2007), and Pines and Kono (2012)) consider residential location choices with static flow road congestion. However, a static transportation model cannot explore the choices of departure times of commuters. Such dynamic situations are explored with a bottleneck model developed by Vickrey (1969). A large body of literature on the economics of road pricing has emerged since Vickrey (1969) (see Small and Verhoef (2007)). However, most papers exploring bottleneck congestion do not consider urban spaces in the city.

Arnott (1998) is the first paper to simultaneously consider location choices and a bottleneck model with two residential zones. Fosgerau and de Palma (2012), and Takayama and Kuwahara (2016) consider a continuous city with a central bottleneck, exploring welfare improvements of heterogeneous residents. Osawa et al. (2018) consider multiple bottlenecks in a corridor road network, and explore the first-best dynamic assignment of heterogeneous commuters with endogeneous location choices. These papers all consider only road congestion. Car bottleneck congestion arises only when drivers traverse bottlenecks. Except Osawa et al. (2018), all the previous papers only assume one bottleneck. Train congestion generates uncomfortable feelings for passengers while they are on a train. So, train congestion externality prevails across the city. The multiple bottlenecks in line in Osawa et al. (2018) generates congestion externality across the city. But they assume only quasi-linear utility function, so there is no income effects with respect to housing lot size. This generates only a unique location pattern in which richer people live closer to the CBD. In addition, the mechanisms of train congestion on commuters are different from that of road bottleneck congestion. So, train congestion influences the location choices in a different way from the case of road bottleneck congestion.

The first novel point of the current paper is that it explores dynamic train congestion in a city with many residential zones, given endogenously determined residential locations. The second novel point is that, in contrast to the previous relevant literature, heterogenous residents choose the best trains and the best locations. Train overcrowding increases as the number of total passengers increases, and differs among all trains with different numbers of passengers. Heterogenous residents choose different trains, generating different commuting costs even if they live in the same zone. Heterogeneity in residents is a key ingredient characterising a city, but from 1964, when Alonso first developed an urban model, until recently, this was not supposed in most urban policy analyses.

The purpose of the current paper is to analyze the relation between train choice and overcrowding costs, and the properties of the endogeneous location in a city composed of multiple residential zones with heterogenous households. To analyze this relation, we assume three residential location patterns, which are explored in Fujita (1989) and Tabuchi (2019). Fujita (1989) shows that, if the income elasticity of lot size is sufficiently large (resp., low) compared to the value of time, households with higher incomes reside farther from (resp., closer to) the CBD. The residential pattern is named Pattern 1 (resp., Pattern 2). Tabuchi (2019) shows that households with both high and low incomes reside closer to the CBD; households with a middle income reside in the suburbs. This residential location is named Pattern 3.

We theoretically analyze the relation between train choice and train overcrowding cost in Patterns 1, 2 and 3. Our results show that, similar to van den Berg and Verhoef (2014), the overcrowding in a train increases as the arrival time of the train approaches the desired time in most cases. In Patterns 2 and 3, there are adverse cases in which the overcrowding in a certain section of a railway decreases on a train as the arrival time of the train approaches the desired time. This is because residents with higher income, who reside closer to the CBD, choose trains arriving around the desired time. This overcrowding blocks commuters with lower incomes from using the trains. This result is novel in this paper and greatly different from both papers, Tian et al. (2004) and van den Berg and Verhoef (2014). These theoretical results are verified in our quantitative simulations.

Furthermore, we investigate the properties of residential location equilibrium considering the choices of the best arrival times. Fujita (1989) does not consider commuters' departure time choices. As a result, he shows that households with different incomes tend to sort their own residential zones completely on the basis of their incomes because the slope of the bid rent function, which is determined by the per-distance commuting cost, differs among heterogenous residents. On the other hand, considering the choices of arrival times, we show that households are not likely to sort themselves completely because heterogenous residents at the same zone can use different trains with different congestion levels. A similar incomplete segregation arises in Osawa et al. (2018) with multiple bottlenecks on roads. These properties of residential location equilibrium are verified quantitatively, too.

Our quantitative analyses simulate welfare changes across heterogeneous households when the first-best policy is implemented in Patterns 1, 2 and 3. The numerical solution is obtained by an equivalent optimization problem representing the equilibrium, which is adopted by Tian et al. (2004), Takayama and Kuwahara (2016), and Osawa et al. (2018). Here, the first-best policy is to charge all commuters the first-best congestion fares, which differ according to the train and the section of the railway. These results show that households with the lowest income increase their utilities the most among all households with different incomes. In contrast, households with the highest income (or the highest schedule

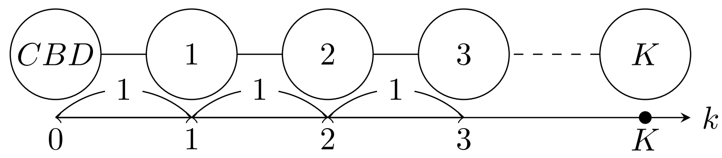


Fig. 1 City model

cost) lose their utilities the most because they choose very overcrowded trains and have to pay high congestion fares after the imposition of congestion fares. In summary, when the first-best policy is implemented, welfare changes are related with train choice for all households with different incomes.

The remainder of this paper is as follows. Section 2 develops an urban model with train commuting with multiple locations. Section 3 explores the relation between arrival times and overcrowding costs depending on Patterns 1, 2 and 3. Section 4 shows the necessary conditions for the existence of Patterns 1, 2 and 3. Quantitative analyses are shown in Section 5. Finally, Section 6 concludes this paper.

2 Model

2.1 City

We consider a closed linear city with a single Central Business District (CBD), in which all job opportunities are gathered. The CBD is located on the left of the city and K residential zones are connected as shown in Fig. 1. The zone number is given from the CBD in numerical order and the set of the residential locations is defined as $\mathcal{K} \equiv \{1, 2, 3, \cdots, K\}$. The land supply in the kth residential zone is shown as A_k $\forall k \in \mathcal{K}$, which are exogenously set.

In the city, as shown in Fig. 2, trains with different arrival times run. All trains depart from the farthest zone (defined as the Kth zone) to the CBD and stop at all zones. Commuters are heterogenous in values of time and schedule costs, but commuters have a common desired arrival time t^*, which is set at 9:00 a.m. in our paper. All trains are indexed by j on the basis of the desired time. The index of the train arriving at the CBD at t^* is defined as 0, and the jth train counted from the train arriving at t^*, defined as train j, arrives at the CBD at t_j. If index j is a negative number, the train arrives at the CBD

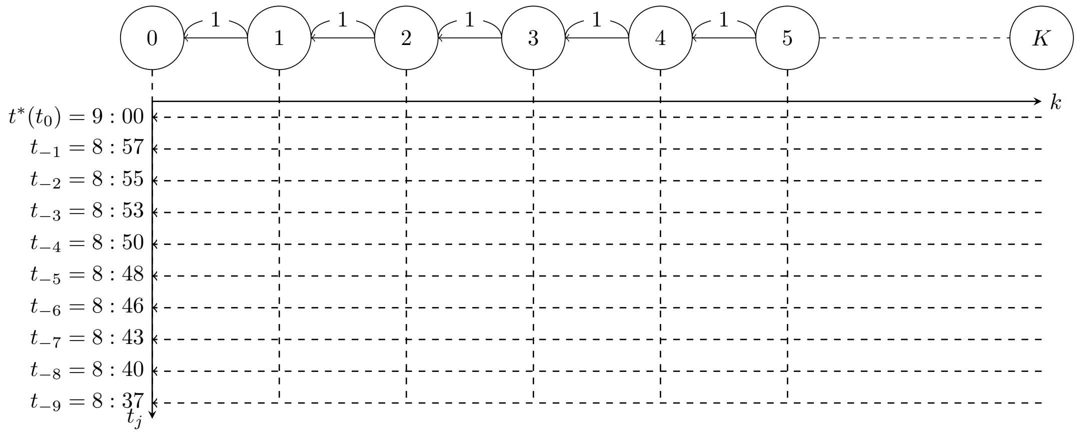


Fig. 2　Railway model

earlier than t^*; if the index is a positive number, the train arrives at the CBD later than t^*. Here, the set of train indices is defined as $\mathcal{J} \equiv \{-a, -a+1, \cdots, -1, 0, 1, \cdots, b-1, b\}$. All indices in set $\mathcal{J}$ are integers. Trains $-a$ and b arrive at the CBD the earliest and latest among all trains, respectively. Commuters incur overcrowding costs which depend on the total number of the train passengers. Per-unit-distance travel time is assumed to be constant as $\phi > 0$ (i.e., train speed is $1/\phi$).

2.2 Heterogeneity of households

There are I groups of households with different incomes y_i, values of time α_i and schedule delay costs per minute for early arrival β_i and for late arrival γ_i. The set of the kinds of households is defined as $\mathcal{I} \equiv \{1, 2, 3, \cdots, I\}$. The number of households belonging to group i, which is referred to as 'type i households' hereafter, is shown as N_i $\forall i \in \mathcal{I}$. We set two assumptions about incomes, values of time and schedule delay costs per minute, as shown in Assumption 1. These assumptions are adopted in previous papers, too (e.g., van den Berg and Verhoef (2014) and Takayama and Kuwahara (2016)).

Assumption 1

(i) $\beta_i \geq \beta_{i-1}$ $\forall i \in \mathcal{I} \backslash \{1\}$

(ii) $\gamma_i / \beta_i \equiv \eta > 1$ $\forall i \in \mathcal{I}$

Assumption 1 (i) requires that as type index i increases, schedule delay cost per minute for early arrival β_i of type i households increases. Assumption 1 (ii) requires that, for all households, the proportion of schedule delay cost per minute for late arrival γ_i to that for early arrival β_i is constant and equal to η, which is a positive real number larger than 1. This assumption means that households with larger type index i, which have larger costs β_i and γ_i, intend to avoid the loss of their opportunity costs by arriving at the CBD closer to the desired time t^*.

Following previous papers considering heterogeneous households, this paper imposes Assumption 2 that households with higher incomes y_i have higher values of time α_i.

Assumption 2

$$When\ \alpha_i > \alpha_{i-1}, then\ y_i > y_{i-1}\ \forall i \in \mathcal{I} \backslash \{1\}$$

2.3 Commuting costs

The commuting cost of type i households from the kth zone is composed of time cost $\alpha_i \phi k$, fare e_k, total overcrowding cost $\sum_{l=1}^{k} \rho \left(\sum_{m=l}^{K} n_{j,m}^{train} \right)$ and schedule delay cost $s_i(t_j - t^*)$. Time cost is the value of time multiplied by the travel time from the CBD to the kth zone. The total overcrowding cost is the sum of overcrowding costs between the 0th zone (i.e., the CBD) and the kth zone. Overcrowding cost $\rho \left(\sum_{m=l}^{K} n_{j,m}^{train} \right)$ arises between the $(k-1)$th zone and the kth zone, and monotonically increases with the total number of passengers. Because all households residing farther from the kth zone traverse the section on a railway line between the $(k-1)$th zone and the kth zone during their journey, the overcrowding costs arising in this section affect them. This cost can represent fatigue, discomfort, and risks of infection caused by overcrowding. Here, $n_{j,k}^{train}$ shows the total number of passengers boarding train j from the kth zone and consists of the total number of $n_{i,j,k}^{train}$ for all $i \in \mathcal{I}$, which shows the number of type i households boarding train j from the kth zone. Schedule delay cost is determined by the absolute value of schedule delay defined as the difference between the arrival time t_j and the desired time t^*. Mathematically, the commuting cost of type i households boarding train j from the kth zone is shown as

$$C_{i,j,k} = \alpha_i \phi k + e_k + \sum_{l=1}^{k} \rho \left(\sum_{m=l}^{K} n_{j,m}^{train} \right) + s_i(t_j - t^*) \;\; \forall i \in \mathcal{I},\; j \in \mathcal{J},\; k \in \mathcal{K}, \quad \text{and} \tag{1}$$

$$s_i(t_j - t^*) \equiv \begin{cases} \beta_i(t^* - t_j) & \text{if } t_j < t^* \\ \gamma_i(t_j - t^*) & \text{if } t_j > t^*. \end{cases} \tag{2}$$

2.4 Utilities

The utility of a type i household residing in the kth zone and choosing train j is defined as

$$\max_{\mathbf{z},\mathbf{q}} u_i(z_{i,j,k}, q_{i,j,k}) \;\; \forall i \in \mathcal{I} \equiv \{1, 2, \cdots, I\}, \tag{3}$$

where $z_{i,j,k}$ denotes composite goods and $q_{i,j,k}$ is the lot size at the kth zone, which is regarded as a superior good. The budget constraint for the type i household is expressed as

$$y_i + R = z_{i,j,k} + C_{i,j,k} + r_k q_{i,j,k}, \tag{4}$$

where y_i denotes the income and r_k shows the land rent at the kth zone. R is non-working income and identical among all types of households. The bid-rent is defined as

$$\max_{\mathbf{z},\mathbf{q}} \; r_k = \frac{y_i + R - z_{i,j,k} - C_{i,j,k}}{q_{i,j,k}} \quad s.t. \;\; u_i(z_{i,j,k}, q_{i,j,k}) = u_i^* \; \forall k \in \mathcal{K}, \tag{5}$$

where variable u_i^* is the utility level of type i households in equilibrium. By solving the first order condition of the above function, bid rent r_k and lot size $q_{i,j,k}$ are shown as

$$q_{i,j,k} = q_i(C_{i,j,k}, R, u_i^*), \quad r_k = r(C_{i,j,k}, R, u_i^*). \tag{6}$$

2.5 Equilibrium conditions

We analyze the equilibrium, separating it into the short-run and long-run equilibria. In the short-run equilibrium, all households choose the best trains to minimize their commuting costs, given the residential locations. In the long-run equilibrium, all households choose the best residential locations to maximize their utilities on the basis of their equilibrium commuting costs.

In the short-run equilibrium, type i households residing in the kth zone cannot decrease their commuting costs by changing their trains. The short-run equilibrium conditions are given by the following conditions:

$$\begin{cases} C_{i,j,k} = C^*_{i,k} & \text{if } n^{train}_{i,j,k} > 0 \\ C_{i,j,k} \geq C^*_{i,k} & \text{if } n^{train}_{i,j,k} = 0 \end{cases} \quad \forall i \in \mathcal{I},\ j \in \mathcal{J},\ k \in \mathcal{K}, \tag{7}$$

$$n_{i,k} = \sum_{j \in \mathcal{J}} n^{train}_{i,j,k} \quad \forall i \in \mathcal{I},\ k \in \mathcal{K}, \tag{8}$$

where $C^*_{i,k}$ is the equilibrium commuting cost of type i households residing in the kth zone and $n_{i,k}$ denotes the total number of type i households residing in the kth zone.

Condition (7) shows the no-arbitrage condition that no commuter can improve his/her commuting cost by changing his/her train unilaterally. Condition (8) is the flow conservation for commuting demand. These conditions give $C^*_{i,k}$ and $n^{train}_{i,j,k}$ at the short-run equilibrium as functions of $(n_{i,k})_{i\in\mathcal{I},k\in\mathcal{K}}$.[*3]

In the long-run equilibrium, each type of households chooses a kth residential zone so as to maximize their indirect utilities given by eq. (3). Thus, the long-run equilibrium conditions are expressed as the

[*3] Note that the short-run equilibrium conditions depend on $(n_{i,k})_{i\in\mathcal{I},k\in\mathcal{K}}$ but not on N_i.

following complementarity problems:

$$\begin{cases} v_{i,k} = v_i^* & \text{if } n_{i,k} > 0 \\ v_{i,k} \leq v_i^* & \text{if } n_{i,k} = 0 \end{cases} \quad \forall i \in \mathcal{I}, k \in \mathcal{K} \tag{9}$$

$$\begin{cases} \sum_{i \in \mathcal{I}} q_{i,k} n_{i,k} = A_k & \text{if } r_k > 0 \\ \sum_{i \in \mathcal{I}} q_{i,k} n_{i,k} \leq A_k & \text{if } r_k = 0 \end{cases} \quad \forall k \in \mathcal{K} \tag{10}$$

$$N_i = \sum_{k \in \mathcal{K}} n_{i,k} \quad \forall i \in \mathcal{I} \tag{11}$$

where v_i^* is the utility level of type i households and $n_{i,k}$ denotes the total number of type i households residing in the kth zone. r_k is the land rent in the kth zone but not the bid rent shown by eq. (6).

Condition (9) is the equilibrium condition for each type of household's residential location choice. This condition implies that, at the long-run equilibrium, no types of households have an incentive to change his/her residential location unilaterally. The land market clearing condition (10) requires that total land demand $\sum_{i \in \mathcal{I}} q_{i,k} n_{i,k}$ for housing at the kth zone equals supply A_k in equilibrium. Condition (11) expresses the population constraint.

3 Short-run Equilibrium

This section analyzes the short-run equilibrium and clarifies the relation between train overcrowding cost and train choice for all households.

3.1 Short-run equilibrium regarding type i households residing in the kth zone

From condition (7), the commuting cost $C_{i,j,k}$ on train j of type i households residing in the kth zone is equal to the cost $C_{i,(j-1),k}$ on train $(j-1)$ in equilibrium (i.e., $C_{i,j,k} - C_{i,(j-1),k} = 0$), where $\mathcal{J}_{i,k}$ denotes the set of train indices chosen by type i households residing in the kth zone. Thus, the relation between overcrowding cost and schedule delay cost for type i households is shown as

$$\sum_{l=1}^{k}\rho\left(\sum_{m=l}^{K}n_{j,m}^{train}\right)-\sum_{l=1}^{k}\rho\left(\sum_{m=l}^{K}n_{(j-1),m}^{train}\right)=s_i(t_j-t_{(j-1)}) \tag{12}$$

$$=\begin{cases}\beta_i(t_j-t_{(j-1)}) & \text{if } t_j,t_{(j-1)}<t^* \\ -\gamma_i(t_j-t_{(j-1)}) & \text{if } t_j,t_{(j-1)}>t^*\end{cases} \qquad \forall j,(j-1)\in\mathcal{J}_{i,k},\ i\in\mathcal{I},\ k\in\mathcal{K}. \tag{13}$$

All types of households arrive at the CBD earlier or later than the desired time in equilibrium. First, we focus on the households arriving earlier than the desired time in equilibrium. Because train $(j-1)$ arrives at the CBD earlier than train j (i.e., $t_j - t_{(j-1)} > 0$), the right term of eq. (12) is positive from eq. (13). Since the overcrowding cost increases with respect to the number of passengers, the total overcrowding cost for type i households choosing train $(j-1)$ is smaller than the cost for train j (i.e., $\sum_{l=1}^{k}\rho\left(\sum_{m=l}^{K}n_{j,m}^{train}\right) > \sum_{l=1}^{k}\rho\left(\sum_{m=l}^{K}n_{(j-1),m}^{train}\right)$). Consequently, the total overcrowding cost on train j for type i households arriving at the CBD earlier than the desired time increases as index j increases.

Next, we focus on the households arriving later than the desired time in equilibrium. Actually, the same discussion can be applied to this case. As a result, as train index j decreases, the total overcrowding cost on train j for type i households arriving at the CBD later than the desired time increases, instead of a decrease in their schedule delay costs.

No previous paper analyzes the relation between overcrowding cost and train choice considering heterogeneous households and many residential locations simultaneously. Van den Berg and Verhoef (2014) analyze the relation considering heterogeneous households in a city composed of a single CBD and a single residential zone ($K = 1$). Tian et al. (2004) analyze the relation considering homogeneous households in a city composed of a single CBD and many residential zones in a line ($I = 1$).

We consider heterogeneous households in the city composed of a CBD and K residential zones, and analyze the relation between overcrowding cost and choice of train in equilibrium by integrating van den Berg and Verhoef (2014) and Tian et al. (2004), given some location patterns considered by previous papers as follows. Specifically, considering that different types of households residing in different zones choose the same trains, we compare the relations between overcrowding cost and choice of train in equilibrium. Furthermore, in section 4, we analyze the necessary conditions for the existence of these location patterns in the long-run equilibrium.

1. Pattern 1: Households with lower incomes reside closer to the CBD (Fujita, 1989).
2. Pattern 2: Households with higher incomes reside closer to the CBD (Fujita, 1989).
3. Pattern 3: Households with high and low incomes reside close to the CBD; those with middle incomes reside in the suburbs (Tabuchi, 2019).

Patterns 1 and 2 are explored by Fujita (1989). He shows that households with higher incomes reside farther from (resp., closer to) the CBD if the income elasticity of lot size is sufficiently large (resp., low) compared to that of value of time. Pattern 3 is explored by Tabuchi (2019). He shows that, given that the wage elasticity of lot size is less than one, households with both high and low incomes reside closer to the CBD and households with middle income reside in the farthest zone from the CBD.

3.2 Pattern 1: Households with lower incomes reside closer to the CBD (Fujita, 1989)

We analyze the relation between the overcrowding costs between the $(k-1)$th zone and the kth zone and the choice of train for type i households residing in the kth zone. In Pattern 1, type i and $(i-1)$ households reside in the kth and $(k-1)$th zones, respectively. When both types of households choose the same trains j and $(j-1)$, the relation is shown as

$$\rho\left(\sum_{m=k}^{K} n_{j,m}^{train}\right) - \rho\left(\sum_{m=k}^{K} n_{(j-1),m}^{train}\right) = (s_i - s_{i-1})(t_j - t_{(j-1)}) \tag{14}$$

$$= \begin{cases} (\beta_i - \beta_{i-1})(t_j - t_{(j-1)}) \text{ if } t_j, t_{(j-1)} < t^* \\ -(\gamma_i - \gamma_{i-1})(t_j - t_{(j-1)}) \text{ if } t_j, t_{(j-1)} > t^* \end{cases} \quad \forall j, (j-1) \in \mathcal{J}_{i,k}, \mathcal{J}_{(i-1),(k-1)}, i \in \mathcal{I}\backslash\{1\}, k \in \mathcal{K}\backslash\{1\}. \tag{15}$$

Train $(j-1)$ arrives at the CBD earlier than train j (i.e., $t_j - t_{(j-1)} > 0$). From Assumption 1, schedule delay costs per minute for early arrival β_i and for late arrival γ_i of type i households are larger than

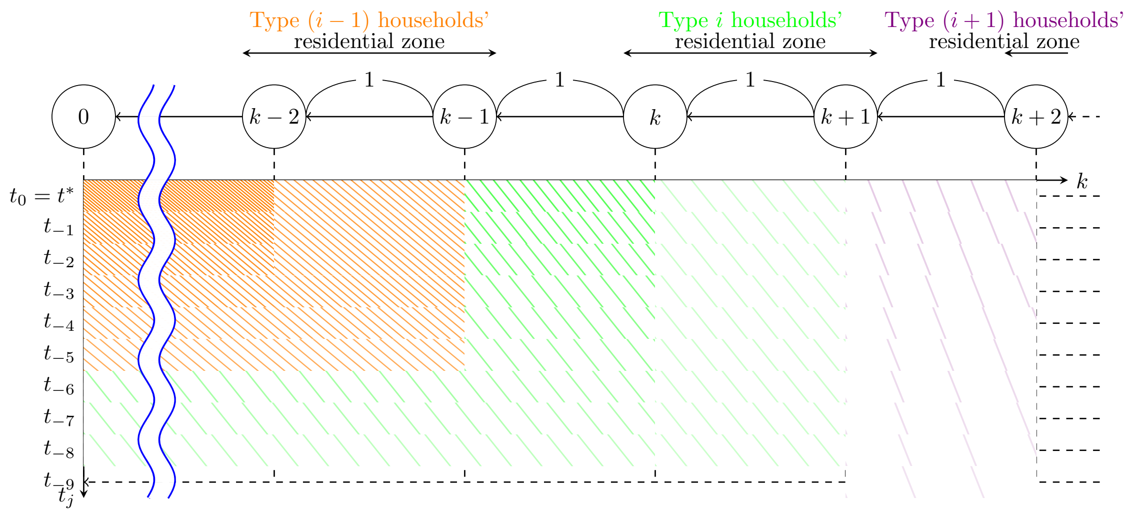


Fig. 3 Pattern 1: The relation between overcrowding costs in each section and arrival times in equilibrium

β_{i-1} and γ_{i-1} of type $(i-1)$ households, respectively. So, in equilibrium, for type i households arriving at the CBD earlier (resp., later) than t^*, the overcrowding cost between the $(k-1)$th zone and the kth zone increases as train index j increases (resp., decreases).[*4]

We demonstrate the relation between overcrowding cost and choice of train for type i households as shown in Fig. 3 with arrival times on the vertical axis and residential zones on the horizontal axis. In Fig. 3, the magnitudes of slopes of diagonal lines show train choices for each type of household. As type index i increases, the diagonal lines become steeper. For example, type $(i-1)$ and i households reside in the $(k-1)$th and kth zones, respectively. Here, the diagonal lines between the $(k-1)$th zone and the kth zone on train 0 are steeper than those between the $(k-2)$th zone and the $(k-1)$th zone. The colors of the diagonal lines also show train choices. All the diagonal lines of the same color have the same gradient. Moreover, the numbers of the diagonal lines show the values of the overcrowding costs. For example, the density of the diagonal lines inside the $(k-2)$th zone on train 0 is the highest among all trains, which means that the overcrowding cost inside the $(k-2)$th zone on train 0 is the largest.

[*4] If, instead of type $(i-1)$ households, type i households reside in the $(k-1)$th zone, the relation for type i households residing in the kth zone is shown as

$$\rho\left(\Sigma_{m=k}^{K} n_{j,m}^{train}\right) = \rho\left(\Sigma_{m=k}^{K} n_{(j-1),m}^{train}\right) \ \forall j, (j-1) \in \mathcal{J}_{i,k}, \mathcal{J}_{i,(k-1)}, i \in \mathcal{I}, k \in \mathcal{K}\backslash\{1\}. \tag{16}$$

This means that their overcrowding costs between the $(k-1)$th zone and the kth zone are identical among all train j for all $\mathcal{J}_{i,k}$ and $\mathcal{J}_{i,(k-1)}$. Tian et al. (2004) also show this relation as shown by eq. (16).

First, we focus on type i households residing in the kth zone. They choose trains from index 0 to index −8 in equilibrium. From eqs. (14) and (12), among these trains, their overcrowding cost increases as train index j approaches 0 (i.e. the desired time). Next, we focus on type i households residing in the $(k + 1)$th zone. They choose trains from index 0 to index −8 in equilibrium. From eq. (16), the overcrowding costs between the kth zone and the $(k + 1)$th zone are identical among these trains. This result is similar to the result of Tian et al. (2004) that the overcrowding costs between any station and the station closest to the CBD are identical among trains used by commuters from the station closest to the CBD.

Accordingly, in Pattern 1, in order to analyze the relation between overcrowding cost and train choice for type i households residing in the kth $(k > 1)$ zone, it matters whether type $(i - 1)$ or i households reside in the $(k-1)$th zone. If type $(i-1)$ households reside in the $(k-1)$th zone, the overcrowding costs between the $(k - 1)$th zone and the kth zone on train j increases as train index j approaches 0 as shown by eq. (14). In contrast, if the same type i households reside in the $(k - 1)$th zone, the overcrowding costs are identical among all trains as shown by eq. (16).

3.3 Pattern 2: Households with higher incomes reside closer to the CBD (Fujita, 1989)

We analyze the relation between the overcrowding costs between the $(k - 1)$th zone and the kth zone and train choices for type i households residing in the kth zone. In Pattern 2, type i and $(i + 1)$ households reside in the kth and $(k - 1)$th zone, respectively. When both types of households choose the same trains j and $(j - 1)$, the relation is shown as

$$\rho\left(\sum_{m=k}^{K} n_{j,m}^{train}\right) - \rho\left(\sum_{m=k}^{K} n_{(j-1),m}^{train}\right) = (s_i - s_{i+1})(t_j - t_{(j-1)}) \tag{17}$$

$$= \begin{cases} -(\beta_{i+1} - \beta_i)(t_j - t_{(j-1)}) \text{ if } t_j, t_{(j-1)} < t^* \\ (\gamma_{i+1} - \gamma_i)(t_j - t_{(j-1)}) \text{ if } t_j, t_{(j-1)} > t^* \end{cases} \quad \forall j, (j-1) \in \mathcal{J}_{i,k}, \mathcal{J}_{(i+1),(k-1)}, i \in \mathcal{I}, k \in \mathcal{K}. \tag{18}$$

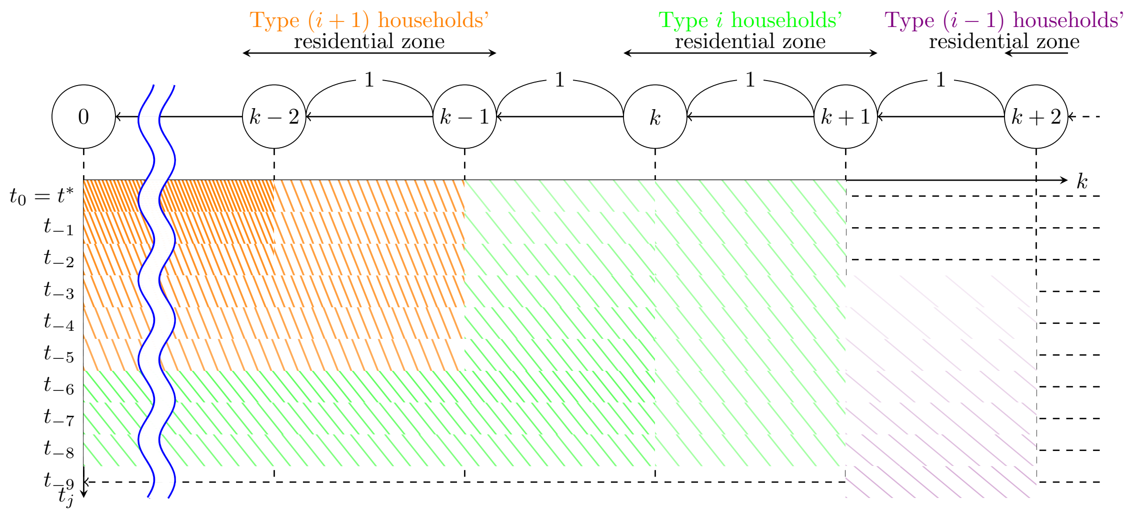


Fig. 4 Pattern 2: The relation between overcrowding costs in each section and arrival times in equilibrium

Train $(j-1)$ arrives at the CBD earlier than train j (i.e., $t_j - t_{(j-1)} > 0$). From Assumption 1, schedule delay costs per minute for early arrival β_i and for late arrival γ_i of type i households are smaller than β_{i+1} and γ_{i+1} of type $(i+1)$ households, respectively. So, in equilibrium, for type i households arriving at the CBD earlier (resp., later) than the desired time t^*, the overcrowding cost between the $(k-1)$th zone and the kth zone decreases as train index j increases (resp., decreases).

The relation between overcrowding cost and train choice is shown in Fig. 4, which has the same characteristics as Fig. 3. We demonstrate the relation for type i households, using Fig. 3.

First, we focus on type i households residing in the kth zone. They choose trains from index 0 to index -8 in equilibrium. From eq. (17), among trains from index 0 to index -5, the overcrowding cost between the $(k-1)$th zone and the kth zone decreases as train index j approaches 0 (i.e., the desired time). In contrast, from eq. (12), the overcrowding cost increases as train index j approaches 0 among trains from index -6 to index -8. Next, we focus on type i households residing in the $(k+1)$th zone. They choose trains from index 0 to index -8 in equilibrium. From eq. (16), the overcrowding costs between the kth zone and the $(k+1)$th zone are identical among all these trains. This result is similar to the result of Tian et al. (2004) as shown in Pattern 1.

Accordingly, in Pattern 2, in order to analyze the relation between overcrowding cost and train choice for type i households residing in the kth $(k > 1)$ zone, it matters whether type $(i+1)$ households residing

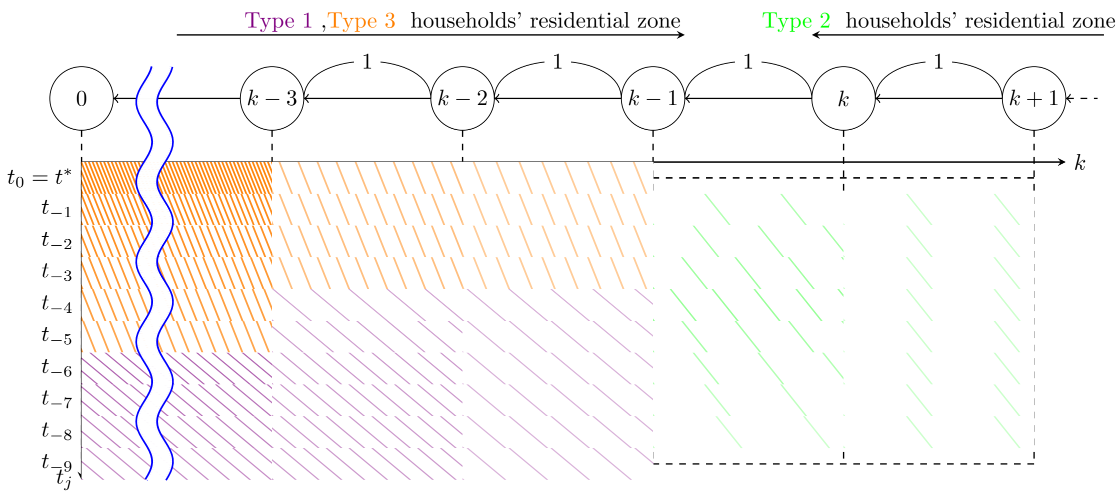


Fig. 5 Pattern 3: The relation between overcrowding costs in each section and arrival times in equilibrium

in the $(k-1)$th zone choose trains or not. If type $(i+1)$ households choose train j, the overcrowding cost between the $(k-1)$th zone and the kth zone on train j decreases as train index j approaches 0 (i.e., the desired time) as shown by eq. (17). In contrast, if the households do not choose train j, the overcrowding cost increases as train index j approaches 0 as shown by eq. (12).

3.4 Pattern 3: Households with high and low incomes reside close to the CBD; those with middle incomes reside in the suburbs (Tabuchi, 2019)

In Pattern 3, we assume set $\mathcal{I} \equiv \{1, 2, 3\}$. We can interpret the equilibrium in Pattern 3 by integrating Patterns 1 and 2. The relation between overcrowding cost and train choice is shown in Fig. 5, which has the same characteristics as Fig. 3.[*5]

We demonstrate the relation between overcrowding cost and train choice for type 2 households residing in the kth zone. It matters whether type 1 or 3 households residing in the $(k-1)$th zone choose the same trains as the type 2 households. If the type 3 households choose the same trains as the type 2 households, the overcrowding cost on train j between the $(k-1)$th zone and the kth zone decreases as train index j approaches 0 (i.e., the desired time) as shown in eq. (17). In contrast, if type 1 households choose the same trains as type 2 households, the overcrowding cost on train j increases as train index

[*5] All types of households departing from the same zone sort themselves on the basis of the values of their per-minute schedule delay costs in equilibrium as shown by van den Berg and Verhoef (2014). As a result, type 3 households residing in the $(k-2)$th and $(k-1)$th zones arrive at the CBD closer to the desired time than type 1 households.

j approaches 0 as shown by eq. (14).

3.5 Summary of short-run equilibrium

From the discussion in this section, we obtain Proposition 1.

Proposition 1. ***(The overcrowding cost per train).***

We suppose without loss of generality that type i households reside in the kth zone. In this case, depending on which type of households reside in the $(k-1)$th zone, the overcrowding cost per train between the kth zone and the $(k-1)$th zone is characterized as follows.

(1) If type $(i+1)$ households reside in the $(k-1)$th zone, the overcrowding cost of train j used by the households decreases between the $(k-1)$th zone and the kth zone as train index j approaches 0 (i.e., the desired time).

(2) If type $(i-1)$ households reside in the $(k-1)$th zone, the overcrowding cost between the $(k-1)$th zone and the kth zone increases as train index j approaches 0.

(3) If type i households reside in both the $(k-1)$th and the kth zones, the overcrowding costs between the $(k-1)$th zone and the kth zone are identical among all trains.

Point (1) is somewhat intriguing in the sense that trains arriving at the CBD closer to the desired time are not necessarily more congested. This point is numerically verified in a later section[*6]. Point (3) is similar to the result of Tian et al. (2004) that the overcrowding costs between any station and the station closest to the CBD are identical among trains used by commuters from the station closest to the CBD. But our result is obtained in the existence of heterogenous commuters.

4 Long-run equilibrium

This section analyzes the necessary conditions for the existence of three residential location patterns given in section 3.1. Fujita (1989) shows that type i households with a bid-rent slope of a larger absolute

[*6] It is hard to identify this property with real data because real data are affected by various factors. But we found the data representing this property in a Tokyu railway line. So, we show the data in our website as the supplement to this paper.

magnitude reside closer to the CBD. So, we compare the Muth conditions of type i and $(i+1)$ households, which reside adjacently, in Patterns 1, 2 and 3.

The bid-rent function considering the short-run equilibrium is shown as

$$\max_{q_{i,k}} \Psi_i(Y_{i,k}, u_i^*) \equiv r_k = \frac{Y_{i,k} - u^{-1}(q_{i,k}, u_i^*)}{q_{i,k}}, \quad \forall i \in \mathcal{I}, k \in \mathcal{K}, \tag{19}$$

where $Y_{i,k} \equiv y_i - C_{i,k}^* + R$ is net income and r_k is the bid-rent. $q_{i,k}$ denotes the lot size for type i households residing in the kth zone. u_i^* denotes the equilibrium utility of type i households. $C_{i,k}^*$ denotes the commuting cost of type i households residing in the kth zone in the short-run equilibrium, and $z_{i,j,k} = u^{-1}(q_{i,k}, u_i^*)$.

Our residential zones are discrete, so the distance between residential zones is substantial. However, regarding the distance as marginal by approximation to differentiate eq. (19), the Muth condition is derived as,

$$\frac{\Delta r_k}{\Delta k} = \frac{\Delta \Psi_i(Y_{i,k}, u_i^*)}{\Delta k} = \frac{\Delta \Psi_i(Y_{i,k}, u_i^*)}{\Delta Y_{i,k}} \frac{\Delta Y_{i,k}}{\Delta k} = -\frac{\Delta C_{i,k}^*}{\Delta k} \frac{1}{q_{i,k}} < 0. \tag{20}$$

Arranging eq. (20) yields

$$\Delta r_{i,k} \equiv \frac{\Delta r_k}{\Delta k} = -\frac{\alpha_i \phi + (e_k - e_{k-1}) + \rho \left(\sum_{m=k}^{K} n_{j,k}^{train} \right)}{q_{i,k}} \; \forall j \in \mathcal{J}_{i,k}, \; i \in \mathcal{I}, \; k \in \mathcal{K}. \tag{21}$$

Here, we define eq. (21) as $\Delta r_{i,k}$ because eq. (21) differs according to type and zone.

The Muth condition for the type i households residing in the kth zone is different among the same type of households with different arrival times. Thus, to obtain the Muth condition, we have to explore who is most likely to move from the kth zone to the $(k-1)$th zone, along with their arrival time, and which train the moving households choose after moving.

4.1 Pattern 1: Households with lower incomes reside closer to the CBD (Fujita 1989)

Using Fig. 3, we explore the necessary condition of Pattern 1 that the absolute magnitude of the Muth condition for type i households in the kth zone cannot exceed that for type $(i-1)$ households. The necessary condition is shown as

$$
\begin{aligned}
&|\Delta r_{(i-1),k}| - |\Delta r_{i,k}| > 0 \\
&\frac{\alpha_{i-1}\phi + (e_k - e_{k-1}) + \rho\left(\sum_{m=k}^{K} n_{-5,m}^{train}\right)}{q_{(i-1),k}} - \frac{\alpha_i\phi + (e_k - e_{k-1}) + \rho\left(\sum_{m=k}^{K} n_{0,m}^{train}\right)}{q_{i,k}} > 0 \\
&\phi\left\{\frac{\alpha_{(i-1)}}{q_{(i-1),k}} - \frac{\alpha_i}{q_{i,k}}\right\} + (e_k - e_{k-1})\left\{\frac{1}{q_{(i-1),k}} - \frac{1}{q_{i,k}}\right\} + \left\{\frac{\rho\left(\sum_{m=k}^{K} n_{-5,m}^{train}\right)}{q_{(i-1),k}} - \frac{\rho\left(\sum_{m=k}^{K} n_{0,m}^{train}\right)}{q_{i,k}}\right\} > 0.
\end{aligned}
\tag{22}
$$

Eq. (22) shows that train choices for type $(i-1)$ and i households affects their location choices. In Fig. 3, type i households, which have high schedule delay cost than type $(i-1)$, choose train 0 so as to minimize their commuting costs, and type $(i-1)$ households choose train -5 so as to minimize their commuting costs. The overcrowding cost between the $(k-1)$th zone and the kth zone is the highest on train 0 among all the trains, implying $\rho\left(\sum_{m=k}^{K} n_{-5,m}^{train}\right) < \rho\left(\sum_{m=k}^{K} n_{0,m}^{train}\right)$ in the last term. In Fujita (1989), because his transport model is static, there are no train choices, implying that this last term in our case is smaller than that in the Fujita case. As a result, than Fujita (1989), it is more likely that households with different incomes colocate in the same zone because the absolute magnitude of the Muth condition for type i households is closer to that for type $(i-1)$ households in the kth zone with residents choosing different trains. But note that our model has discrete zones, so $|\Delta r_{(i-1),k}| \simeq |\Delta r_{i,k}|$ even if types i and type $(i-1)$ households both reside in the same zone.

4.2 Pattern 2: Households with higher incomes reside closer to the CBD (Fujita 1989)

Using Fig. 4, we explore the necessary condition that the absolute magnitude of the Muth condition for the type i households in the kth zone cannot exceed that for type $(i+1)$ households. The necessary condition is shown as

$$
\begin{aligned}
&|\Delta r_{(i+1),k}| - |\Delta r_{i,k}| > 0 \\
&\frac{\alpha_{i+1}\phi + (e_k - e_{k-1}) + \rho\left(\sum_{m=k}^{K} n_{0,m}^{train}\right)}{q_{(i+1),k}} - \frac{\alpha_i\phi + (e_k - e_{k-1}) + \rho\left(\sum_{m=k}^{K} n_{-6,m}^{train}\right)}{q_{i,k}} > 0 \\
&\phi\left\{\frac{\alpha_{i+1}}{q_{(i+1),k}} - \frac{\alpha_i}{q_{i,k}}\right\} + (e_k - e_{k-1})\left\{\frac{1}{q_{(i+1),k}} - \frac{1}{q_{i,k}}\right\} + \left\{\frac{\rho\left(\sum_{m=k}^{K} n_{0,m}^{train}\right)}{q_{(i+1),k}} - \frac{\rho\left(\sum_{m=k}^{K} n_{-6,m}^{train}\right)}{q_{i,k}}\right\} > 0. \qquad (23)
\end{aligned}
$$

Similar to eq. (22), eq. (23) shows that train choices for type i and $(i+1)$ households affect their location choices. In Fig. 4, the type i households choose train -6 to minimize their commuting costs, and type $(i+1)$ households choose train 0. Note that, in this rail section, the type i households avoid boarding trains arriving around the desired time because the trains are very crowded close to the CBD with the type $(i+1)$ households. The overcrowding cost between the $(k-1)$th zone and the kth zone is the largest on train -6 among all trains, implying $\rho\left(\sum_{m=k}^{K} n_{0,m}^{train}\right) < \rho\left(\sum_{m=k}^{K} n_{-6,m}^{train}\right)$. Thus, similar to Pattern 1, than the static transport model, it is more likely that households with different incomes colocate in the same zone because the absolute magnitudes of the Muth conditions for type i households is closer to that for type $(i+1)$ households.

4.3 Pattern 3: Households with high and low incomes reside close to the CBD; those with middle income reside in the suburbs (Tabuchi 2019)

Using Fig. 5, we analyze the necessary conditions for the existence of Pattern 3. From eqs. (22) and (23), the conditions are shown as

$$
\begin{aligned}
&|\Delta r_{3,k}| - |\Delta r_{2,k}| > 0 \\
&\frac{\alpha_3\phi + (e_k - e_{k-1}) + \rho\left(\sum_{m=k}^{K} n_{-1,m}^{train}\right)}{q_{3,k}} - \frac{\alpha_2\phi + (e_k - e_{k-1}) + \rho\left(\sum_{m=4}^{K} n_{-4,m}^{train}\right)}{q_{2,k}} > 0 \\
&\phi\left\{\frac{\alpha_3}{q_{3,k}} - \frac{\alpha_2}{q_{2,k}}\right\} + (e_k - e_{k-1})\left\{\frac{1}{q_{3,k}} - \frac{1}{q_{2,k}}\right\} + \left\{\frac{\rho\left(\sum_{m=k}^{K} n_{-1,m}^{train}\right)}{q_{3,k}} - \frac{\rho\left(\sum_{m=k}^{K} n_{-4,m}^{train}\right)}{q_{2,k}}\right\} > 0.
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&|\Delta r_{1,k}| - |\Delta r_{2,k}| > 0 \\
&\frac{\alpha_1\phi + (e_k - e_{k-1}) + \rho\left(\sum_{m=k}^{K} n_{-9,m}^{train}\right)}{q_{1,k}} - \frac{\alpha_2\phi + (e_k - e_{k-1}) + \rho\left(\sum_{m=k}^{K} n_{-4,m}^{train}\right)}{q_{2,k}} > 0 \\
&\phi\left\{\frac{\alpha_1}{q_{1,k}} - \frac{\alpha_2}{q_{2,k}}\right\} + (e_k - e_{k-1})\left\{\frac{1}{q_{1,k}} - \frac{1}{q_{2,k}}\right\} + \left\{\frac{\rho\left(\sum_{m=k}^{K} n_{-9,m}^{train}\right)}{q_{1,k}} - \frac{\rho\left(\sum_{m=k}^{K} n_{-4,m}^{train}\right)}{q_{2,k}}\right\} > 0.
\end{aligned}
\tag{25}
$$

Eqs. (24) and (25) show that the Muth conditions of type 3 and 1 households are larger than that of type 2 households in the kth zone. Similar to the discussions in sections 4.2 and 4.3, it is more likely that different types of households colocate in the same zone than the static transport model.

4.4 Summary of long-run equilibrium

From the analysis in section 4, we obtain Lemma 1.

Lemma 1. ***(The necessary conditions for the existence of each location pattern)***.

(1) The Muth condition which is a necessary condition for the existence of Pattern 1 is shown as

$$
\begin{aligned}
&|\Delta r_{(i-1),k}| - |\Delta r_{i,k}| > 0 \\
&\phi\left\{\frac{\alpha_{(i-1)}}{q_{(i-1),k}} - \frac{\alpha_i}{q_{i,k}}\right\} + (e_k - e_{k-1})\left\{\frac{1}{q_{(i-1),k}} - \frac{1}{q_{i,k}}\right\} + \left\{\frac{\rho\left(\sum_{m=k}^{K} n_{\underline{j}_{(i-1),(k-1)},m}^{train}\right)}{q_{(i-1),k}} - \frac{\rho\left(\sum_{m=k}^{K} n_{0,m}^{train}\right)}{q_{i,k}}\right\} > 0,
\end{aligned}
\tag{26}
$$

where $\underline{j}_{(i-1),(k-1)}$ denotes the train which arrives at the CBD the earliest among all trains used by type $(i-1)$ households residing in the $(k-1)th$ zone.

(2) The Muth condition which is a necessary condition for the existence of Pattern 2 is shown as

$$|\Delta r_{(i+1),k}| - |\Delta r_{i,k}| > 0$$

$$\phi\left\{\frac{\alpha_{i+1}}{q_{(i+1),k}} - \frac{\alpha_i}{q_{i,k}}\right\} + (e_k - e_{k-1})\left\{\frac{1}{q_{(i+1),k}} - \frac{1}{q_{i,k}}\right\} + \left\{\frac{\rho\left(\sum_{m=k}^{K} n_{0,m}^{train}\right)}{q_{(i+1),k}} - \frac{\rho\left(\sum_{m=k}^{K} n_{\underline{j}_{(i+1),(k-1)}-1,m}^{train}\right)}{q_{i,k}}\right\} > 0, \quad (27)$$

where $\underline{j}_{(i+1),(k-1)}$ denotes the train which arrives at the CBD the earliest among all trains used by type $(i+1)$ households residing in the $(k-1)th$ zone.

(3) The Muth conditions which are necessary conditions for the existence of Pattern 3 are shown as

$$|\Delta r_{3,k}| - |\Delta r_{2,k}| > 0$$

$$\phi\left\{\frac{\alpha_3}{q_{3,k}} - \frac{\alpha_2}{q_{2,k}}\right\} + (e_k - e_{k-1})\left\{\frac{1}{q_{3,k}} - \frac{1}{q_{2,k}}\right\} + \left\{\frac{\rho\left(\sum_{m=k}^{K} n_{\bar{j}_{2,k},m}^{train}\right)}{q_{3,k}} - \frac{\rho\left(\sum_{m=k}^{K} n_{\bar{j}_{1,(k-1)},m}^{train}\right)}{q_{2,k}}\right\} > 0. \quad (28)$$

$$|\Delta r_{1,k}| - |\Delta r_{2,k}| > 0$$

$$\phi\left\{\frac{\alpha_1}{q_{1,k}} - \frac{\alpha_2}{q_{2,k}}\right\} + (e_k - e_{k-1})\left\{\frac{1}{q_{1,k}} - \frac{1}{q_{2,k}}\right\} + \left\{\frac{\rho\left(\sum_{m=k}^{K} n_{\underline{j}_{1,(k-1)},m}^{train}\right)}{q_{1,k}} - \frac{\rho\left(\sum_{m=k}^{K} n_{\bar{j}_{1,(k-1)},m}^{train}\right)}{q_{2,k}}\right\} > 0. \quad (29)$$

where $\underline{j}_{i,k}$ denotes the train which arrives at the CBD the earliest among all trains used by type i households residing in the kth zone. $\bar{j}_{i,k}$ denotes the train which arrives at the CBD the closest to the desired time among all trains used by type i households residing in the kth zone.

The above lemma shows that the Muth conditions for the existence of each location pattern are affected by train choice for all households. This result is novel and different from Fujita (1989) considering a static flow transportation system. Because we take account of many trains with different arrival times and different overcrowding costs, the different overcrowding costs have an effect on the choices of residential zones for households with different incomes. From these discussions, we can obtain Proposition 2.

Proposition 2. ***(The relation between the overcrowding costs and residential distribution).***

We suppose, without loss of generality, that type i households reside in the kth zone. Households with different incomes can choose different trains. As a result, than the static transport model, it is more likely that households with different incomes colocate in the same zone because the absolute magnitudes of the Muth conditions for type i households are closer to that for other types of households.

This incomplete segregation pattern has already been pointed out with multiple consecutive bottlenecks on roads by Osawa et al. (2018). Osawa et al. (2018), assuming a quasi-linear utility function, yields only Pattern 2. The current model also produces similar results with overcrowding on multiple trains in various location patterns. In addition, we explain the reason why these incomplete segregations occur based on the Muth conditions.

5 Quantitative analysis

Using numerical simulations, we analyze the relation between the overcrowding costs and train choice for each type of household.

5.1 Setting

We conduct numerical simulations with the following settings. The hypothetical line is a railway line in Tokyo. We do not calibrate the parameters using the exact data. Actually, it is hard to collect data such as incomes by location.

We consider the city composed of a single CBD and five residential zones as Fig. 6. The total length is $25km$. For example, the length of the Tokyu Den-en-toshi Line between Nagatsuta and Shibuya is approximately $25km$.

In this city, there are three types of household with low, middle and high incomes. The numbers of types 1, 2 and 3 households are $10,000$, $20,000$ and $10,000$, respectively. Then, the total number is $40,000$, which is equal to that of total passengers riding on the Den-en-toshi Line during the peak-period

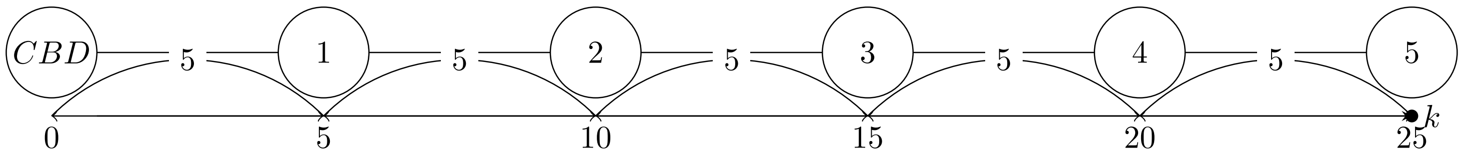


Fig. 6 City model for quantitative analysis

between 7:50 and 8:50 in the morning. The number of days of commuting to the CBD is set at 250 days per household per year. The supply of land is $4km^2$ in each zone and the housing occupancy rate of the land is approximately 30%. Thus, the supply of land for housing is approximately $1km^2$.

Overcrowding cost function on train j between the kth zone and the $(k-1)$th zone is defined as

$$\rho\left(\sum_{m=k}^{K} n_{j,m}^{train}\right) \equiv 0.45\left(\frac{\sum_{m=k}^{K} n_{j,m}^{train}}{Cap}\right)^3 \forall j \in \mathcal{J}, k \in \mathcal{K}, \tag{30}$$

where $\sum_{m=k}^{K} n_{j,m}^{train}$ is the number of total passengers boarding train j at the kth zone. Cap is transport capacity per train and is set as $1,500$ $(passengers/train)$. Based on the public data of a Tokyo railway line (specifically, Tokyu Corporation), the mean-velocity of a train and marginal operation cost are set as 40 (km/h) and \$$0.1(/km/person))$, respectively.[*7]

Next, a household's indirect utility function is set as a Gorman function as[*8]

$$v_i(r_k, y_i) \equiv \xi_i(r_k) + \eta(r_k)(y_i - C_{i,k}^*), \tag{31}$$

where $\xi_i(r_k) \equiv \theta_i r_k^{-\mu}$ and $\eta(r_k) \equiv r_k^{-\mu}$. r_k are the land rent but not the bid rent as shown by eqs. (6) and (19). μ is the distribution parameter and is set as 0.2. θ_i is the parameter for changing the elasticity of lot size with income for households of type i.

In our context, the merits of using the Gorman utility function are twofold. The first merit is that we can set the income elasticity of lot size lower than one. We need an income elasticity of less than one in

---

[*7] The financial statement on Tokyu Corporation shows that the total profits and the total gains per year with respect to railway business are \$1,548,700,000 and \$246,090,000, respectively. Thus, substracting the total gains from the total profits, the total costs per year with respect to railway business are \$1,302,610,000. Also, the total travel distance on Tokyu Den-en-toshi Line per year is 11,281,000,000(person · km). Thus, the marginal operation cost is set as \$1,302,610,000 ÷ 11,281,000,000(person · km) = \$0.1154 ≈ \$0.10 (/(person · km)).

[*8] When θ_i is zero for all types of households, this indirect utility function is equal to that of Cobb · Douglas.

order to obtain the location equilibrium as shown in Tabuchi (2019). Second, if the utility function is a Gorman function, we can set an optimization problem equivalent to a market equilibrium as shown by conditions (9), (10) and (11), using a potential function. Gorman functions can represent quasi-linear, Cobb-Douglas, and C.E.S forms, which are often used. [*9]

Finally, we demonstrate the data of incomes y_i, values of time α_i, schedule delay costs per minute for early arrival β_i and for late arrival γ_i, and parameters θ_i. Incomes y_1 and y_2 of type 1 and 2 households are set based on the mean values of the incomes of people in Tokyo aged 20-29 and 30-39, respectively. Since the mean income in Tokyo is approximately \$620,000, we set income y_3 of type 3 households as \$1,100,000.[*10] Because the mean value of the working hours per year in Japan is approximately 1,643$hours$, value of time α_i of type i households is their income y_i divided by the mean value of the working hours per year for all types. We set β_i of type i households as their income y_i multiplied by 0.6 for all types.[*11] We set γ_i of type i households as β_i multiplied by 4 for all types. Thus, we can set these parameters as Table. 1 [1]. When we use data in Table. 1 [1], households with lower incomes reside closer to the CBD in equilibrium (Pattern 1). In order to analyze the equilibrium in Pattern 2, we use the data set as Table. 1 [2] with setting values of time of type 1 and 3 households lower and larger than that in Table. 1 [1], respectively. Furthermore, when we analyze the equilibrium in Pattern 3, we use Table. 1 [3], which sets θ_i as 45,000, 200,000 and 180,000 instead of 0 for all types of households in Patterns 1 and 2, respectively.

We demonstrate the market equilibria in Patterns 1, 2 and 3. Next, we impose the first-best congestion fares, which is called the first-best regime hereafter. We evaluate equivalent variation (EV) and the total equivalent variation (TEV) defined as the sum of the EV for type i households residing in the kth zone

[*9] See Mas Colell (1995, pp.119-120) for the properties of the Gorman function.

[*10] Income y_3 can be calculated as follows by using the mean value of the income and the number of each type of households.

$$\frac{20000 \times 10000 + 40000 \times 20000 + y_3 \times 10000}{10000 + 20000 + 30000} = 620000 \quad \therefore y_3 \approx 1100000$$

[*11] See Small et al. (1982) for the empirical results about the value of time and schedule delay costs per minute for early arrival.

Table. 1 Parameter set

[1] Pattern 1

Type	1	2	3
y_i (\$)	20,000	40,000	110,000
α_i (\$/h)	18	30	72
β_i (\$/h)	10.8	18	43.2
γ_i (\$/h)	43.2	72	172.8
θ_i	0	0	0

[2] Pattern 2

Type	1	2	3
y_i (\$)	20,000	40,000	110,000
α_i (\$/h)	9	30	120
β_i (\$/h)	5.4	18	72
γ_i (\$/h)	21.6	72	288
θ_i	0	0	0

[3] Pattern 3

Type	1	2	3
y_i (\$)	20,000	40,000	110,000
α_i (\$/h)	18	30	72
β_i (\$/h)	10.8	18	43.2
γ_i (\$/h)	43.2	72	172.8
θ_i	45,000	200,000	180,000

multiplied the number of the households under the market equilibrium.[*12]

In order to obtain the equilibria regarding Patterns 1, 2 and 3, we optimize the following three objective functions by Frank-Wolfe algorithm. When we set households' indirect utility function as a Gorman function, we can have the following proposition as the optimization problems with respect to the short-run and long-run equilibria. See Appendix A for the proof.

Lemma 2. ***(The optimization problems with respect to the short-run and long-run equilibria)***.

(1) We can set an optimization problem equivalent to the short-run equilibrium conditions given by eqs. (7) *and* (8) *as*

$$\min_{\mathbf{n}^{train}} z_s(\mathbf{n}^{train}) \equiv \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} \left\{ \int_0^{\sum_{m=k}^{K} \sum_{i\in\mathcal{I}} n_{i,j,m}^{train}} \rho(x)dx + \sum_{i\in\mathcal{I}} n_{i,j,k}^{train} \{\alpha_i \phi k + e_k + s_i(t_j - t^*)\} \right\} \tag{32}$$

$$s.t.\ \ n_{i,k} = \sum_{j\in\mathcal{J}} n_{i,j,k}^{train}\ ,\ \ n_{i,j,k}^{train} \geq 0 \ \ \forall i \in \mathcal{I},\ j \in \mathcal{J},\ k \in \mathcal{K}, \tag{33}$$

where $\mathbf{n}^{train} \equiv \{n_{i,j,k}^{train}\}$ *denotes a train choice pattern of all types of households at all zones.*

(2) If we set households' indirect utility function as a Gorman function given by eq. (31)*, we can set an optimization problem equivalent to the long-run equilibrium conditions given by eqs.* (9), (10) *and* (11) *as*

$$\max_{\mathbf{n}} z_l(\mathbf{n}) \equiv \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} v_i(Y_{i,k}, r_k(\mathbf{n}))n_{i,k} + \sum_{k\in\mathcal{K}} A_k \int_0^{r_k(\mathbf{n})} \eta(x)dx \tag{34}$$

$$s.t.\ \ N_i = \sum_{k\in\mathcal{K}} n_{i,k}\ ,\ \ n_{i,k} \geq 0\ \ \forall i \in \mathcal{I},\ k \in \mathcal{K}, \tag{35}$$

where $\mathbf{n} \equiv \{n_{i,k}\}$ *denotes a location choice pattern of all types of households and* $Y_{i,k} (= y_i - C^*_{i,k})$ *is net income.*

[*12] Because the EVs are different among the residential zones and all types of households, the value of TEV is different from that of social welfare improvements. However, the difference in the both values is very small.

We can obtain the train choice pattern $\mathbf{n}^{train}$ of all types of households at all zones in the short-run equilibrium if and only if it satisfies the KKT conditions of the optimization problem as shown by Lemma 1 (1). Tian et al. (2004) show optimization problem similar to Lemma 1 (1) with only homogeneous commuters. Because we use a monotonically increasing overcrowding function with respect to the number of total passengers, $\mathbf{n}^{train}$ is uniquely determined.

For the long-run equilibrium, we can obtain the residential location choice pattern $\mathbf{n}$ of all types of households in the long-run equilibrium if and only if it satisfies the KKT conditions of the optimization problem as shown by Lemma 1 (2). The marginal utility $\eta(r_k)$ of income for type i households residing in the kth zone depends on only the rent r_k in the kth zone because we set households' indirect utility function as a Gorman function. Thus, this optimization problem can be set as shown by Lemma 1 (2) because the Hessian of the potential function shown by eq. (34) is symmetry. See Appendix B for the uniqueness of the solution of this problem shown by Lemma 1 (2). Takayama (2018) and Osawa et al. (2018) set similar optimization problems, assuming the Cobb-Douglas function and quasi-linear function, respectively, to solve the equilibrium. We assume a Gorman function, which can represent both functions, CES function and others to set this problem. This is an advantage.[*13]

The short-run optimization problem representing the first-best regime is shown as follows.

$$\min_{\mathbf{n}^{train}} z_{so}^{m}(\mathbf{n}^{train}) \equiv \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}}\left\{\left(\sum_{m=k}^{K}\sum_{i\in\mathcal{I}} n_{i,j,m}\right)\cdot\rho\left(\sum_{m=k}^{K}\sum_{i\in\mathcal{I}} n_{i,j,m}\right)+\sum_{i\in\mathcal{I}} n_{i,j,k}^{train}\{\alpha_i\phi k + e_k + s_i(t_j - t^*)\}\right\} \tag{36}$$

$$s.t.\ n_{i,k} = \sum_{j\in\mathcal{J}} n_{i,j,k}^{train},\ \ n_{i,j,k}^{train} \geq 0 \quad \forall i\in\mathcal{I},\ j\in\mathcal{J},\ k\in\mathcal{K} \tag{37}$$

This short-run optimization problem shows the minimization of the total overcrowding costs in this city. The KKT conditions of this problem implies the imposition of congestion fares. The long-run optimization problem for the first-best regime is the same as the market equilibrium.

[*13] Previous papers regarding the car bottleneck model with heterogenous commuters verify to be able to formulate an optimization problem equivalent to the short-run equilibrium conditions for the car bottleneck model (Arnott et al. (1994); Lindsey (2004); Iryo and Yoshii (2007); Liu et al. (2015)). The optimization problem is used by Takayama and Kuwahara (2016) and Osawa et al. (2018) considering a single bottleneck and multiple bottlenecks, respectively.

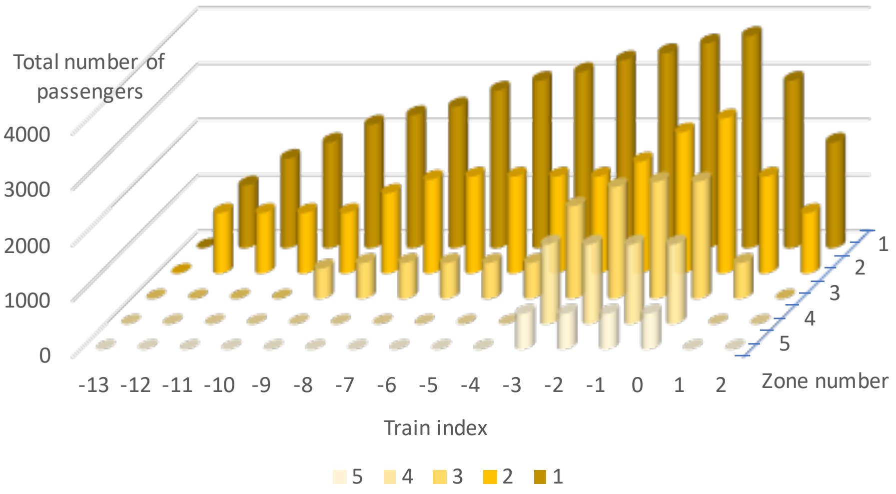
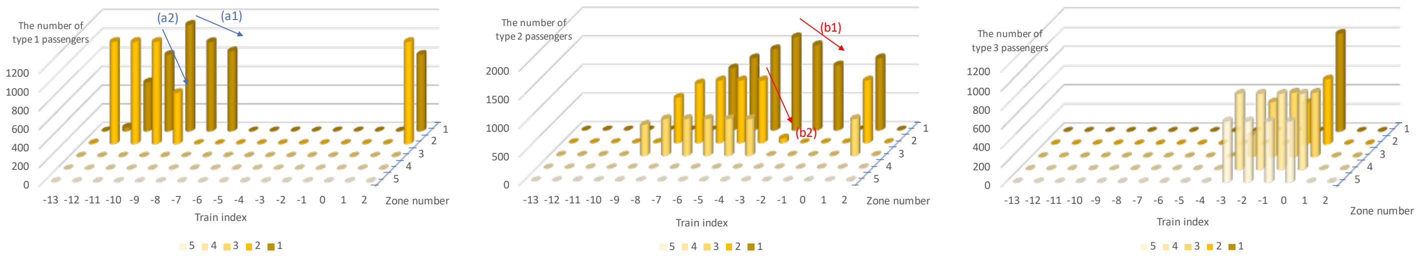


Fig. 7 Total number of passengers in each zone by train in Pattern 1



[1] Type 1 passengers [2] Type 2 passengers [3] Type 3 passengers

Fig. 8 The number of each type of passengers boarding in each zone by train in Pattern 1

5.2 Results

First, we show the relation between the overcrowding costs on trains and the choices of a train in the market equilibria in Patterns 1, 2 and 3. In Pattern 1, total number of passengers in each zone by train is shown in Fig. 7, which has train index on the horizontal axis, total number of passengers on the vertical axis and zone number on the depth axis. Total number of passengers on train j increases at all stations as train index j approaches the desired time.

We focus on the number of each type of passengers boarding in each zone by train as shown in Fig. 8. *14 Type 3 passengers in suburbs, who have the highest income among all types of households, arrive at

*14 From eq. (16), the numbers of type 3 passengers in both 4th and 5th zones are identical among trains from index 0 to index -3. Moreover, the numbers of type 2 passengers in both 2 nd and 3rd zones are identical among trains 1 and from index -4 to index -8 and among trains 1 from index -4 to index -6, respectively.

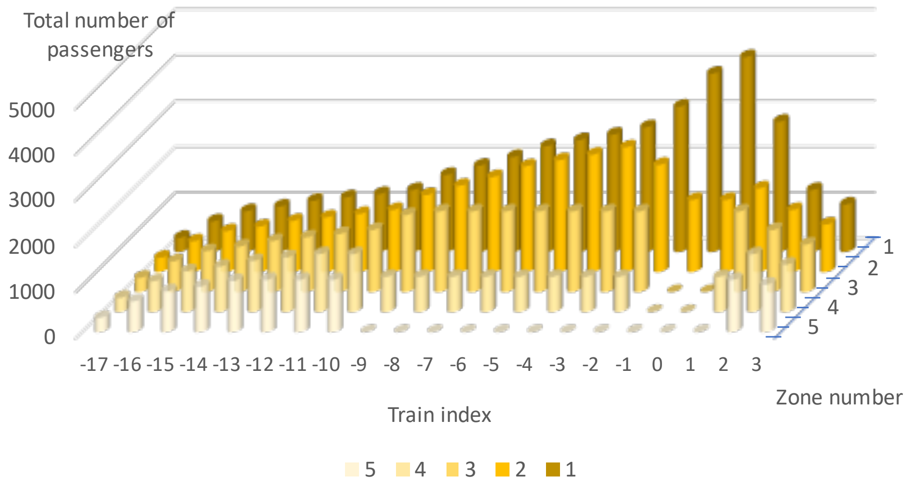


Fig. 9 Total number of passengers in each zone by train in Pattern 2

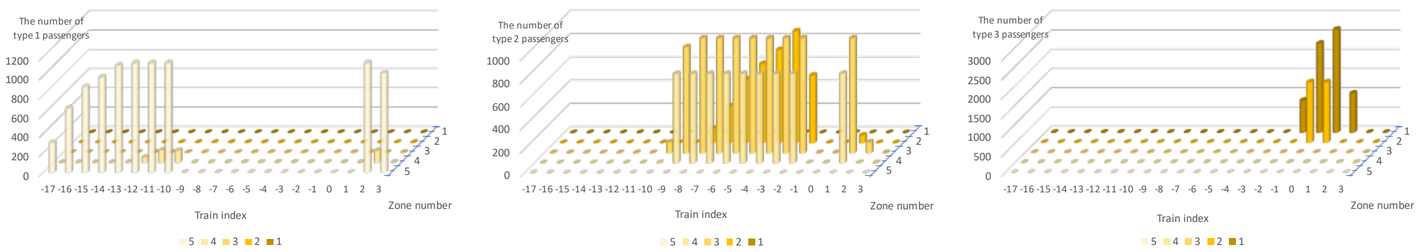
the CBD around the desired time in equilibrium. Because the overcrowding costs on trains from index 0 to index −3 are very severe, type 1 and 2 passengers residing closer to the CBD avoid choosing these trains even if they increase their schedule delay costs.

An interesting point, which is depicted by arrows (b1) and (b2) in Fig. 8[2], is that the number of type 2 passengers in each train at the 1st zone decreases as train index j increases with respect to trains −3, −2 and −1 because type 3 passengers in suburbs choose these trains. Because type 3 passengers arrive at the CBD around the desired time, the overcrowding cost on train j arriving around the desired time is more severe as train index j approaches 0 (i.e., the desired time). As a result, the number of type 2 passengers in the 1st zone decreases as train index j approaches 0. This phenomenon occurs for type 1 passengers in the 1st and 2nd zones as shown by arrows (a1) and (a2) in Fig. 8[1]. We summarize this intriguing result as the following main finding.

Main finging 1. *(The choices of trains in Pattern 1).*

The number of passengers boarding close to the CBD decreases as train index j approaches 0 (i.e., the desired time) when they choose trains which have been chosen by passengers in suburbs, who have higher income than passengers boarding close to the CBD.

Next, we demonstrate the relation between the overcrowding costs on trains and train choices in the



[1] Type 1 passengers [2] Type 2 passengers [3] Type 3 passengers

Fig. 10 The number of each type of passengers boarding in each zone by train in Pattern 2

market equilibrium in Pattern 2. Total number of passengers in each zone by train is shown in Fig. 9, which has train index on the horizontal axis, total number of passengers on the vertical axis and zone number on the depth axis.[*15]

The intriguing point is that no type 1 and 2 commuter from the 3rd to 5th zones chooses trains 0 and −1, which are chosen by type 3 passengers residing in the 1st and 2nd zones. Because the overcrowding costs between the CBD and the 2nd zone are very severe in these trains, passengers residing farther from the 3rd zone avoid choosing these trains even if their schedule delay costs also increase. Likewise, no type 1 commuter from the 5th zone chooses trains from index 1 to index −9, which are chosen by type 2 or 3 passengers residing inside the 4th zone. We can summarize this result as the following main finding.

Main finging 2. *(The choices of trains in Pattern 2).*

When households with higher incomes reside closer to the CBD, few commuters (including zero commuters) from the suburbs choose trains, which are chosen by households residing closer to the CBD.

This result is consistent with theoretical results as shown in Proposition 1. In particular, total number of passengers on train −2 at the 2nd zone is lower than those on trains −3 and −4 as shown in Fig. 10. This means that passengers boarding at the 2nd zone avoid choosing train −2 because the overcrowding between the CBD and the 1st zone on train −2 is larger than those on trains −3 and −4.

[*15] Because type 3 passengers reside closer to the CBD, they have smaller commuting distances to the CBD in Pattern 2 than in Pattern 1. Thus, total numbers of passengers in trains 0 and −1 are larger than those in Pattern 1 as shown in Fig. 7.

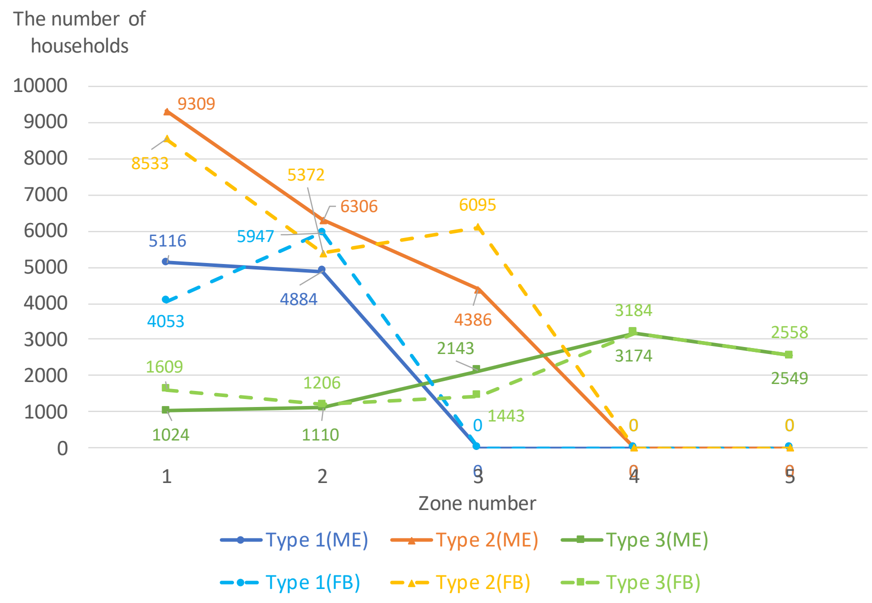


Fig. 11 Location distribution in Pattern 1

We can understand the relation between the overcrowding costs on trains and train choices in the market equilibrium in Pattern 3 by integrating the results in Patterns 1 and 2. Thus, see Appendix 7.2 for the results regarding train choices in Pattern 3.

We demonstrate the properties of the location distributions in equilibria in Patterns 1, 2 and 3 shown in Figs. 11, 12 and 13, respectively, which have the number of each type of households on the vertical axis and zone number on the horizontal axis. The distributions in the market equilibrium (ME) and in the first-best regime (FB) are depicted as the solid line and the dotted line, respectively.

In Pattern 1, type 1 households reside in the 1st and 2nd zones. Type 2 households reside in the 1st, 2nd and 3rd zones. Type 3 households reside in all zones. This means that households with different incomes do not spatially sort their residential zones completely on the basis of their incomes in equilibrium.

In Pattern 2, type 1 households reside in the 4th and 5th zones. Type 2 households reside in the 2nd, 3rd and 4th zones. Type 3 households reside in 1st and 2nd zones. This means that, unlike the result in Pattern 1, households with different incomes sort themselves completely on the basis of their incomes in equilibrium. We can summarize these results as the following main finding.

Main finging 3. *(The properties of residential locations in Patterns 1 and 2).*

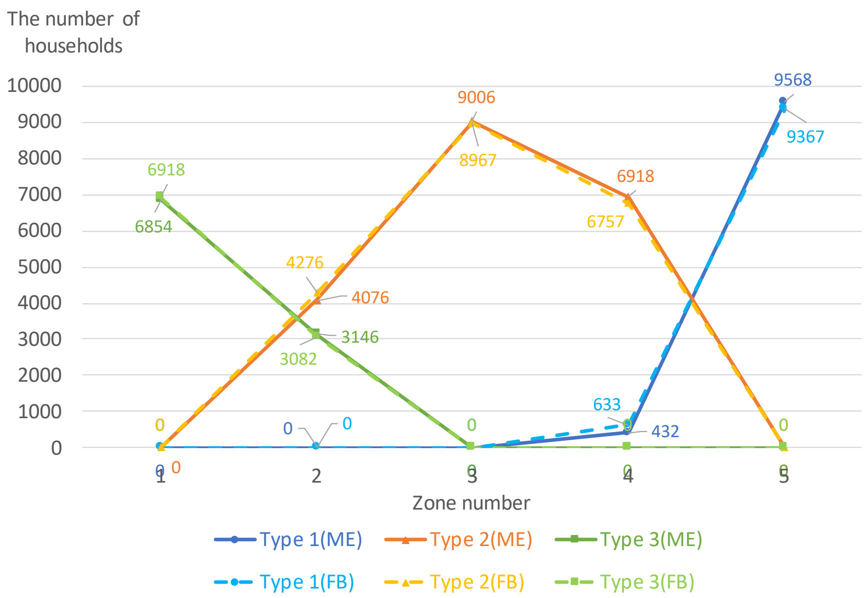


Fig. 12 Location distribution in Pattern 2

(1) When households with lower incomes are likely to reside closer to the CBD (Pattern 1), households do not sort themselves completely on the basis of their incomes because they can use different trains.

(2) When households with higher incomes are likely to reside closer to the CBD (Pattern 2), households with different incomes sort themselves completely on the basis of their incomes.

These results are related with train choices for all types of households as shown by Proposition 2. As we explain Proposition 2 based on the Muth condition, the model with multiple trains is likely to yield incomplete segreation between multiple heterogenous residents. This is numerically verified in Pattern 1. However, as the current Pattern 2 shows, complete segregation also arises depending on the parameters.

Our numerical simulation yields a Pattern 3, in which the type 1 households reside in the 1st and 2nd zones, and the type 3 households reside in 1st and 2nd zones. On the other hand, the type 2 households reside in the 2nd, 3rd, 4th and 5th zones.

Finally, we show equivalent variation (EV) and total EV (TEV) under the first-best regime in Patterns 1, 2 and 3 as Figs.2, 3 and 4, respectively.

In Table 2, the TEV and the per-capita EV are approximately \$ 16 million and \$ 400 in Pattern 1, respectively. The EVs for type 1 households in the 1st and 2nd zones are about \$ 700. The EVs for type 2

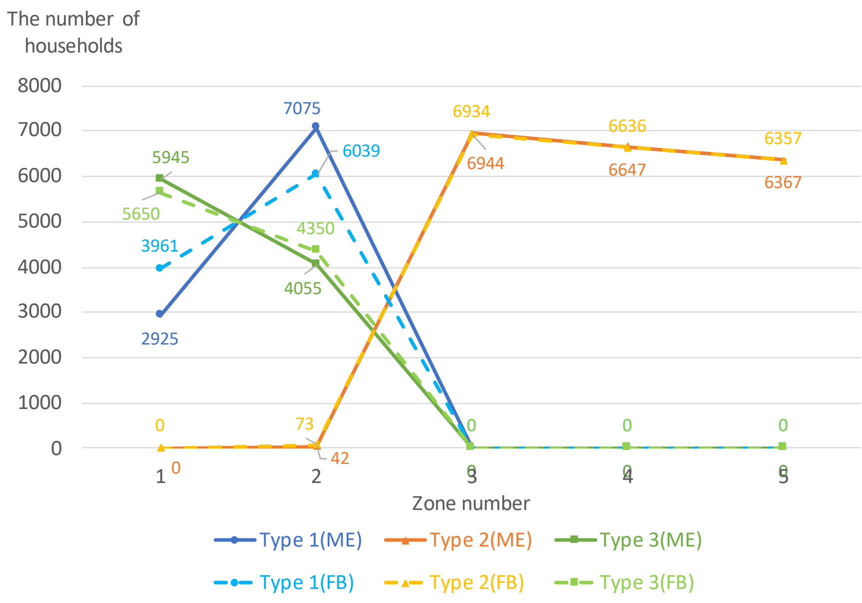


Fig. 13 Location distribution in Pattern 3

Table. 2 Equivalent variation (EV) and total EV (TEV) in Pattern 1 ($ 100/$year$)

Zone	1	2	3	4	5	TEV
EV for Type 1	7.04	6.70	0	0	0	68,736
EV for Type 2	4.48	4.26	4.07	0	0	86,407
EV for Type 3	0.97	0.92	0.88	0.84	0.80	8,579
						163,721

Note: The cells in gray show the EV per person for each type of household in each zone, whereas other cells show the total value in the city.

households in the 1st, 2nd and 3rd zones are about $ 450 – 410. The EVs for type 3 households from the 1st zone to the 5th zone are about $ 95 – 80. This means that the EV for type i households increases as type index i decreases because households with higher incomes have more overcrowding costs. Thus, the benefits for the type 3 households is the lowest among all households. This result is the same in van den Berg and Verhoef (2014). Moreover, looking at a type of households, we realize that the EV for the households increases as the households reside closer to the CBD because households residing closer to the CBD have the lower congestion externalities shown in Fig. 7.

In Table 3, the TEV and the per-capita EV are approximately $ 15 million and $ 390 in Pattern 2, respectively. The EVs for type 1 households in the 4th and 5th zones are about $ 800. The EVs for type 2 households in the 2nd, 3rd and 4th zones are about $ 405 – 370. The EVs for type 3 households in the 1st and 2nd zones are about −$20. The EVs for type 3 households are the lowest among all households

Table. 3 Equivalent variation (EV) and total EV (TEV) in Pattern 2 ($ 100/$year$)

Zone	1	2	3	4	5	TEV
EV for Type 1	0	0	0	8.13	7.91	79,179
EV for Type 2	0	4.08	3.88	3.69	0	77,089
EV for Type 3	- 0.20	- 0.19	0	0	0	-1,982
						154,286

Note: The cells in gray show the EV per person for each type of household in each zone, whereas other cells show the total value in the city.

Table. 4 Equivalent variation (EV) and total EV (TEV) in Pattern 3 ($ 100/$year$)

Zone	1	2	3	4	5	TEV
EV for Type 1	7.09	6.98	0	0	0	70,135
EV for Type 2	0	4.32	4.29	4.25	4.22	85,091
EV for Type 3	- 0.13	- 0.13	0	0	0	-1,316
						153,910

Note: The cells in gray show the EV per person for each type of household in each zone, whereas other cells show the total value in the city.

and negative because they pay much higher congestion fares than other types of households.

In Table 4, the TEV and the per-capita EV are approximately $ 15 million and $ 380 in Pattern 3, respectively. The EVs for type 1 households in the 1st and 2nd zones are about $ 700. The EVs for type 2 households in the 2nd, 3rd, 4th and 5th zones are about $ 430. The EVs for type 3 households in the 1st and 2nd zones are about −$13. Focusing on households residing in the 1st and 2nd zones, we realize that the EVs for type 3 households are negative while those for type 1 households are positive. This is because type 3 and 1 households choose different trains and pay different congestion fares.

6 Conclusion

This paper explores the choices of trains with different arrival times and residential locations in a city composed of a single CBD and many residential zones with heterogeneous households.

We show that the overcrowding cost per train between the kth zone and the $(k-1)$th zone depends on what type of households reside in the $(k-1)$th zone. If households residing in the $(k-1)$th zone have lower income than those residing in the kth zone, the overcrowding cost on train j increases as train index j approaches 0 (i.e., the desired time). In contrast, if households residing in the $(k-1)$th zone have higher income than those residing in the kth zone, the overcrowding cost on train j decreases as train index j approaches 0.

We investigate the properties of residential location equilibrium by comparing the absolute values of the Muth conditions for different types of households residing adjacently. Different types of households residing in the same zone can choose the best trains for them. As a result, the absolute value of the Muth condition for households residing in the kth zone is steeper than for households residing in the $(k-1)$th zone. Thus, it is possible that households with different incomes colocate in the same zone.

Quantitative analysis shows the welfare effects of the first-best policy on heterogenous households in Patterns 1, 2 and 3. The result shows that households with higher incomes lose their utilities more because households with higher incomes have to pay higher congestion fares. If households with the highest income among all households reside in the closest zone to the CBD, the first-best regime decreases their utilities because they have to pay much higher congestion fares than other types of households.

7 Appendix

7.1 A. Proofs of the equivalency of optimization problems to equilibrium conditions

7.1.1 The short-run equilibrium

We verify that the optimization problem given by eqs. (32) and (33) is equivalent to the short-run equilibrium conditions given by conditions (7) and (8) by the Karush-Kuhn-Tucker condition.

$$\min_{\mathbf{n}^{train}} z_s(\mathbf{n}^{train}) \equiv \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} \left\{ \int_0^{\sum_{m=k}^{K} \sum_{i\in\mathcal{I}} n_{i,j,m}^{train}} \rho(x)dx + \sum_{i\in\mathcal{I}} n_{i,j,k}^{train} \{\alpha_i \phi k + e_k + s_i(t_j - t^*)\} \right\} \tag{32}$$

$$s.t.\ \ n_{i,k} = \sum_{j\in\mathcal{J}} n_{i,j,k}^{train}\ ,\ \ n_{i,j,k}^{train} \geq 0\ \ \forall i \in \mathcal{I},\ j \in \mathcal{J},\ k \in \mathcal{K} \tag{33}$$

[Proof]

First, a Lagrangian is defined as follows.

$$L(\mathbf{n}, \mathbf{c}^*) \equiv z_s(\mathbf{n}) + \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} c^*_{i,k} \left\{ n_{i,k} - \sum_{j \in \mathcal{J}} n^{train}_{i,j,k} \right\} \tag{38}$$

We apply the Karush-Kuhn-Tucker condition to the above Lagrangian. When the Lagrangian is differentiated by $n^{train}_{i,j,k}$ $\forall i \in \mathcal{I}, j \in \mathcal{J}, k \in \mathcal{K}$, the Lagrangian is expanded as follows and divided into the term of overcrowding costs and the term of others.

$$\frac{\partial L(\cdot)}{\partial n^{train}_{i,j,k}} = \frac{\partial}{\partial n^{train}_{i,j,k}} \sum_{k \in \mathcal{K}} \left\{ \int_0^{\sum_{m=k}^{K} \sum_{i \in \mathcal{I}} n^{train}_{i,j,m}} \rho(x) dx \right\} + \left\{ \alpha_i \phi k + e_k + s_i (t_j - t^*) - c^*_{i,k} \right\} \tag{39}$$

Here, the term of overcrowding costs is expanded as follows.

$$\begin{aligned}
&\frac{\partial}{\partial n^{train}_{i,j,k}} \sum_{k \in \mathcal{K}} \left\{ \int_0^{\sum_{m=l}^{K} \sum_{i \in \mathcal{I}} n^{train}_{i,j,m}} \rho(x) dx \right\} \\
&= \frac{\partial}{\partial n^{train}_{i,j,k}} \left\{ \int_0^{\sum_{m=1}^{K} \sum_{i \in \mathcal{I}} n^{train}_{i,j,m}} \rho(x) dx + \cdots + \int_0^{\sum_{m=k}^{K} \sum_{i \in \mathcal{I}} n^{train}_{i,j,m}} \rho(x) dx + \cdots + \int_0^{\sum_{m=K}^{K} \sum_{i \in \mathcal{I}} n^{train}_{i,j,m}} \rho(x) dx \right\} \\
&= \rho \left(\sum_{m=1}^{K} \sum_{i \in \mathcal{I}} n^{train}_{i,j,m} \right) + \rho \left(\sum_{m=2}^{K} \sum_{i \in \ \mathcal{I}} n^{train}_{i,j,m} \right) + \cdots + \rho \left(\sum_{m=k}^{K} \sum_{i \in \mathcal{I}} n^{train}_{i,j,m} \right) \quad \left(\because \frac{\partial}{\partial x} \int_0^x g(y) dy = g(x) \right) \\
&= \sum_{l=1}^{k} \rho \left(\sum_{m=l}^{K} \sum_{i \in \mathcal{I}} n^{train}_{i,j,m} \right)
\end{aligned} \tag{40}$$

Therefore, the Karush-Kuhn-Tucker conditions with respect to the above Lagrangian are shown as follows.

$$n^{train}_{i,j,k} \frac{\partial L(\cdot)}{\partial n^{train}_{i,j,k}} = 0, \frac{\partial L(\cdot)}{\partial n^{train}_{i,j,k}} \leq 0, n^{train}_{i,j,k} \geq 0 \quad \forall i \in \mathcal{I},\ j \in \mathcal{J},\ k \in \mathcal{K}$$

$$\frac{\partial L(\cdot)}{\partial c^*_{i,k}} = 0 \quad \forall i \in \mathcal{I},\ k \in \mathcal{K}$$

$$\begin{cases} n^{train}_{i,j,k} \left\{ \alpha_i \phi k + e_k + s_i (t_j - t^*) + \sum_{l=1}^{k} \rho \left(\sum_{m=l}^{K} \sum_{i \in \mathcal{I}} n^{train}_{i,j,m} \right) - c^*_{i,k} \right\} = 0 \\ \alpha_i \phi k + e_k + s_i (t_j - t^*) + \sum_{l=1}^{k} \rho \left(\sum_{m=l}^{K} \sum_{i \in \mathcal{I}} n^{train}_{i,j,m} \right) - c^*_{i,k} \leq 0, n^{train}_{i,j,k} \geq 0 \end{cases} \quad \forall i \in \mathcal{I},\ j \in \mathcal{J},\ k \in \mathcal{K} \tag{41}$$

$$n_{i,k} - \sum_{j \in \mathcal{J}} n^{train}_{i,j,k} = 0 \quad \forall i \in \mathcal{I},\ k \in \mathcal{K} \tag{42}$$

Condition (41) is no arbitrage condition shown as condition (7) and condition (42) is the flow conservation condition shown as condition (8). It is verified that the optimization problem given by eqs. (32) and (33) is the equivalent to equilibrium conditions given by conditions (7) and (8).

[q.e.d]

7.1.2 The long-run equilibrium

We verify that the optimization problem given by eqs. (34) and (35) is equivalent to the long-run equilibrium conditions given by conditions (9), (10) and (11) by the Karush-Kuhn-Tucker condition. The optimization problem given by eqs. (34) and (35) is shown as follows. Here, in equilibrium, the market clearing condition shown as condition (10) can hold. Thus, land rent r_k is a function of a residential location pattern $\mathbf{n}$.

$$\max_{\mathbf{n}} z_l(\mathbf{n}) \equiv \sum_{i\in\mathcal{I}}\sum_{k\in\mathcal{K}} v_i(Y_{i,k}, r_k(\mathbf{n}))n_{i,k} + \sum_{k\in\mathcal{K}} A_k \int_0^{r_k(\mathbf{n})} \eta(x)dx \tag{34}$$

$$s.t.\ \ N_i = \sum_{k\in\mathcal{K}} n_{i,k}\ ,\ \ n_{i,k} \geq 0\ \ \forall i \in \mathcal{I},\ k \in \mathcal{K} \tag{35}$$

[Proof]

First, a Lagrangian is defined as follows.

$$L(\mathbf{n}, \mathbf{V}) \equiv z_l(\mathbf{n}) + \sum_{i\in\mathcal{I}} v_i^* \left\{ -\sum_{k\in\mathcal{K}} n_{i,k} + N_i \right\} \tag{43}$$

We apply the Karush-Kuhn-Tucker condition to the above Lagrangian.

$$n_{i,k}\frac{\partial L(\mathbf{n}, \mathbf{V})}{\partial n_{i,k}} = 0,\ \frac{\partial L(\mathbf{n}, \mathbf{V})}{\partial n_{i,k}} \leq 0,\ n_{i,k} \geq 0,\ \forall k \in \mathcal{K}, i \in \mathcal{I}$$

$$\frac{\partial L(\mathbf{n}, \mathbf{V})}{\partial v_i^*} = 0\ \forall i \in \mathcal{I}$$

$$\begin{cases} n_{i,k}\left\{ v_i(Y_{i,k}, r_k(\mathbf{n})) + \sum_{i\in\mathcal{I}} \dfrac{\partial v_i(Y_{i,k}, r_k(\mathbf{n}))}{\partial n_k} + A_k\eta(r_k(\mathbf{n}))\dfrac{\partial r_k}{\partial n_k} - v_i^* \right\} = 0 \\ v_i(Y_{i,k}, r_k(\mathbf{n})) + \sum_{i\in\mathcal{I}} \dfrac{\partial v_i(Y_{i,k}, r_k(\mathbf{n}))}{\partial n_k} + A_k\eta(r_k(\mathbf{n}))\dfrac{\partial r_k}{\partial n_k} - v_i^* \leq 0,\ n_{i,k} \geq 0 \end{cases} \quad \forall i \in \mathcal{I}, k \in \mathcal{K} \tag{44}$$

$$-\sum_{k\in\mathcal{K}} n_{i,k} + N_i = 0\ \ \ \forall i \in \mathcal{I} \tag{45}$$

Here, when the Lagrangian is differentiated by $n_{i,k}$, the Lagrangian is expanded as follows.

$$
\begin{aligned}
&\frac{\partial L(\mathbf{n},\mathbf{V})}{\partial n_{i,k}} \\
&= v_i(Y_{i,k}, r_k(\mathbf{n})) + \sum_{i\in\mathcal{I}} \frac{\partial v_i(Y_{i,k}, r_k(\mathbf{n}))}{\partial r_k(\mathbf{n})}\frac{\partial r_k(\mathbf{n})}{\partial n_{i,k}} n_{i,k} + A_k\eta(r_k(\mathbf{n}))\frac{\partial r_k(\mathbf{n})}{\partial n_{i,k}} - v_i^* \\
&= v_i(Y_{i,k}, r_k(\mathbf{n})) - v_i^* + \sum_{i\in\mathcal{I}} \left\{-q_{i,k}\eta(r_k(\mathbf{n}))\right\}\frac{\partial r_k(\mathbf{n})}{\partial n_{i,k}} n_{i,k} + A_k\eta(r_k(\mathbf{n}))\frac{\partial r_k(\mathbf{n})}{\partial n_{i,k}} \quad \left(\because\ q_{i,k} = -\frac{\partial v_i(Y_{i,k}, r_k(\mathbf{n}))}{\partial r_k(\mathbf{n})}/\eta(r_k(\mathbf{n}))\right) \\
&= v_i(Y_{i,k}, r_k(\mathbf{n})) - v_i^* - A_k\eta(r_k(\mathbf{n}))\frac{\partial r_k(\mathbf{n})}{\partial n_{i,k}} + A_k\eta(r_k(\mathbf{n}))\frac{\partial r_k(\mathbf{n})}{\partial n_{i,k}} \quad \left(\because\ \sum_{i\in\mathcal{I}} q_{i,k}n_{i,k} = A_k\right) \\
&= v_i(Y_{i,k}, r_k(\mathbf{n})) - v_i^* \qquad (46)
\end{aligned}
$$

Thus, the Karush-Kuhn-Tucker conditions with respect to the above Lagrangian are shown as follows.

$$
\begin{cases}
n_{i,k}\left\{v_i(Y_{i,k}, r_k(\mathbf{n})) - v_i^*\right\} = 0 \\
v_i(Y_{i,k}, r_k(\mathbf{n})) - v_i^* \leq 0,\ n_{i,k} \geq 0
\end{cases} \quad \forall i \in \mathcal{I}, k \in \mathcal{K} \qquad (47)
$$

$$
-\sum_{k\in\mathcal{K}} n_{i,k} + N_i = 0 \ \ \forall i \in \mathcal{I} \qquad (48)
$$

Condition (47) is no arbitrage condition shown as condition (9) and condition (48) is the flow conservation condition shown as condition (11). It is verified that the optimization problem given by eqs. (34) and (35) is the equivalent to equilibrium conditions given by conditions (9), (10) and (11).

[q.e.d.]

7.2 B. The uniqueness of these optimization problems

The optimization problem given by eqs. (34) and (35) is equivalent to the following dual problem.

$$
\min_{(r_k)_{k\in\mathcal{K}}} z_{dh}((r_k)_{k\in\mathcal{K}}) \equiv \sum_{i\in\mathcal{I}} N_i \max_{k\in\mathcal{K}} v_i(Y_{i,k}, r_k) + \sum_{k\in\mathcal{K}} A_k \int_0^{r_k} \eta(x)dx \qquad (49)
$$

$$
s.t.\ r_k \geq 0 \quad \forall i \in \mathcal{I}, k \in \mathcal{K} \qquad (50)
$$

Because the objective function of this optimization problem is strictly convex, r_k is uniquely determined. In addition, the uniqueness of r_k implies that the indirect utility function $v_i(Y_{i,k}, r_k)$ is uniquely

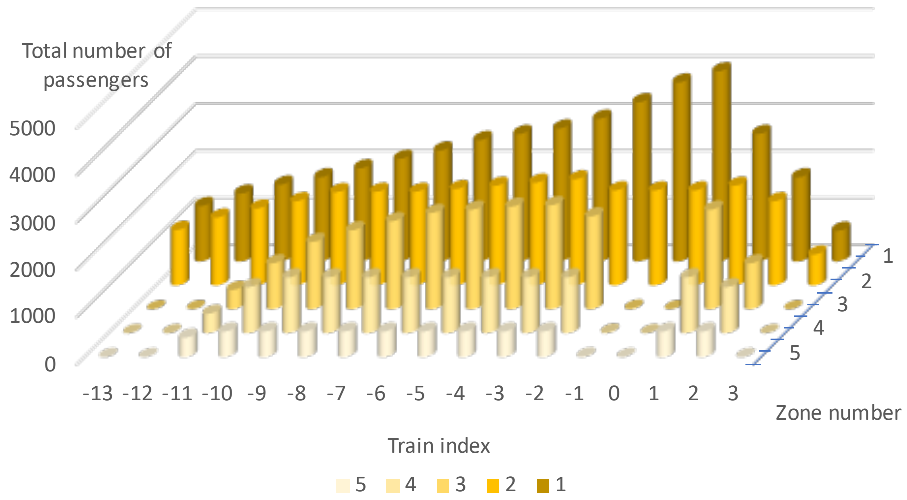
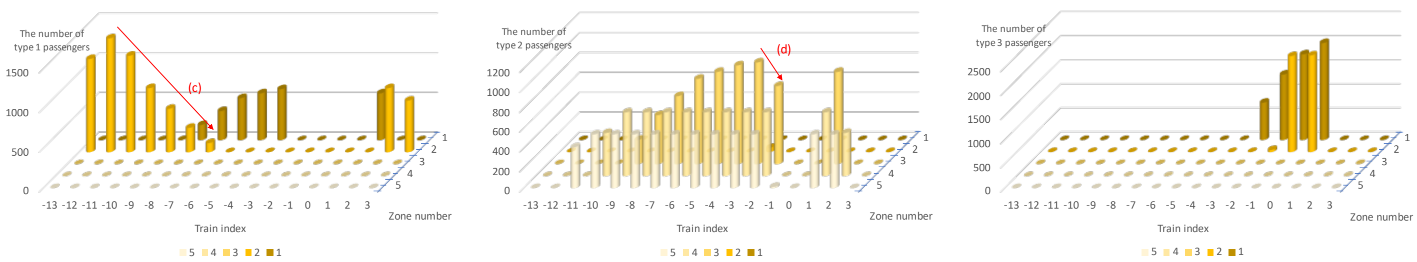


Fig. 14　Total number of passengers in each zone by train in Pattern 3



[1] Type 1 passengers　　[2] Type 2 passengers　　[3] Type 3 passengers

Fig. 15　The number of each type of passengers boarding in each zone by train in Pattern 3

determined. Thus, equilibrium utility of type i households u_i^* is also uniquely determined.

7.3　C. Quantitative results in Pattern 3

In Pattern 3, total number of passengers in each zone by train is shown in Fig. 14, with train index on the horizontal axis, total number of passengers on the vertical axis and zone number on the depth axis. We focus on the number of each type of passengers boarding in each zone by train in Fig. 15. As shown in Fig. 13, type 1 and 3 households colocate in the 1st and 2nd zones. Type 2 households reside in the 2nd, 3rd, 4th and 5th zones.

We can interpret the short-run equilibrium in Pattern 3, considering the analyses in Patterns 1 and 2 in section 5.2.[*16] Looking at arrow (c) in Fig. 15 [1], as main finding 1 shows, arrow (c) shows that

[*16] All types of households departing from the same zone sort themselves on the basis of the values of their per-minute

the number of type 1 households on train j at the 2nd zone decreases as train index j increases with respect to trains from index -12 and index -7. Next, arrow (d) in Fig. 15 [2] shows that the number of type 2 households residing in the 3rd zone on train -2 is lower than that on train -3.

The type 2 households residing in the 3rd, 4th and 5th zones use trains from index -13 to index 2 except index -1 and 0. As main finding 2 shows, they do use trains -1 and 0 because the overcrowding costs between the CBD and the 2nd zone are very severe.

REFERENCES

[1] Alonso, W., 1964. Location and Land Use, Harvard university press, Cambridge, MA:.

[2] Arnott, R., 1998. Congestion tolling and urban spacial structure. J. Reg. Sci. 38 (3), 495-504.

[3] Arnott, R., de Palma, A., and Lindsey, R. 1994. The welfare effects of congestion tolls with heterogeneous commuters. Journal of Transport Economics and Policy, Vol. 28, No. 2, 139-161.

[4] Brueckner, J. K. 2007, Urban Growth Boundaries: An Effective Second-Best Remedy for Unpriced Traffic Congestion?, Journal of Housing Economics 16, 263-73.

[5] Cheung, M. W., and Lahkar, R. 2018. Nonatomic potential games: the continuous strategy case. Games and Economic Behavior, 108, 341-362.

[6] Fosgerau, M., De Palma, A., 2012. Congestion in a city with a central bottleneck. Journal of Urban Economics, 71(3), 269-277.

[7] Fosgerau, M., Kim, J., and Ranjan, A. 2018. Vickrey meets Alonso: Commute scheduling and congestion in a monocentric city," Journal of Urban Economics, Vol. 105, pp. 40-53.

[8] Fujita, M., 1989. Urban Economic Theory: Land use and City Size. Cambridge University Press.

[9] Gubins, S., and Verhoef, E. T. (2014). Dynamic bottleneck congestion and residential land use in the monocentric city. Journal of Urban Economics, 80, 51-61.

schedule delay costs in equilibrium as shown by van den Berg and Verhoef (2014). As a result, type 3 and 1 households residing in the 1st zone use trains from index -3 to index 0, and trains from index -8 to index -4 and index 1, respectively. Type 3 and 1 households residing in the 2nd zone use trains from index -2 to index 0, and trains from index -13 to index -7, index 2 and index 3, respectively.

[10] Iryo, T. and Yoshii, T. 2007. Equivalent optimization problem for finding equilibrium in the bottleneck model with departure time choices. in Heydecker, B. G. ed. Mathematics in Transport: Elsevier, 231-244.

[11] Kono, T., Kawaguchi, H., 2017. Cordon pricing and land use regulation. Scand. J. Econ. 119 (2), 405-434.

[12] Lindsey, R. 2004. Existence, uniqueness, and trip cost function properties of user equilibrium in the bottleneck model with multiple user classes. Transportation Science. Vol. 38, No. 3, 293-314.

[13] Liu, Y., Nie, Y. M., and Hall, J. D. 2015. A semi-analytical approach for solving the bottleneck model with general user heterogeneity. Transportation Research Part B: Methodological, Vol. 71, 56-70.

[14] Mas-Colell, A., Whinston, M.D., Green, J.R., et al., 1995. Microeconomic Theory. Oxford University Press.

[15] Osawa, M., Fu. H., Akamatsu, T., 2018. First-best dynamics assignment of commuters with endogenous heterogeneities in a corridor network. Transportation Research Part B. 117, 811-831.

[16] Pines, D. and Kono, T. (2012), FAR Regulations and Unpriced Transport Congestion, Regional Science and Urban Economics 42, 931-937.

[17] Small, K., 1982. The Scheduling of Consumer Activaties: Work Trips. Am. Econ. Rev. 72 (3), 467-479.

[18] Tabuchi, T., 2019. Do the rich and poor colocate in large cities? J. Urban Econ. 113, 103-186.

[19] Takayama, Y., 2018. Time-varying congestion tolling and urban spacial structure. MPRA Paper.

[20] Takayama, Y., Kuwahara, M., 2016. Bottleneck congestion and residential location of heterogeneous commuters. J. Urban Econ. 100, 65-79.

[21] Tian, Q., Huang, H.J., Yang, H., 2004. Equilibrium properties of the morning peak-period commuting in a many-to-one mass transit system. Transportation Research Part B. 41. 616-631.

[22] XIE, C., Fukuda, D., 2014. Scheduling preference modeling of rail passengers in the Tokyo metropolitan area and evaluation of time-varying fare policy for a congested urban railway line. J. JSCE. Ser. D3, 70 (3), 535-548.

[23] van den Berg, V., Verhoef, E.T., 2011. Winning or losing in the from dynamic bottleneck congestion pricing? The distributional effects of road pricing with heterogeneity in values of time and schedule delay. J. Public Econ. 95(7), 983-992.

[24] van den Berg, V., Verhoef, E.T., 2014. Congestion Pricing in a Rail and Rail Network with Heterogeneous Values of Time and Schedule Delay. Transportmetrica A: Transport Science, 10(5), 377-400.

[25] Vickrey, W.S., 1969. Congestion theory and transport investment. Am. Econ. Rev. 59 (2), 251-260.

[26] Yu, X., van den Berg, V., Verhoef, E.T., 2019. Carpooling with heterogeneous users in the bottleneck model. Transportation Research Part B. 127. 178-200.

—-Data source for numerical simulation—-

[27] Ministry of Land, Infrastructure, Transport and Tourism. Retrieved January 2021 from
`https://www.mlit.go.jp/report/press/tetsudo04_hh_000068.html`

[28] Tokyo Metropolitan Government Bureau of Environment. Retrieved January 2021 from
`http://www.kankyo.metro.tokyo.jp/vehicle/management/price/country/singapore.html`

[29] Statistic Bureau, Ministry of Internal Affairs and Communications. Retrieved January 2021 from
`https://www.stat.go.jp/data/shakai/2016/kako.html`

[30] Ministry of Health, Labour and Welfare. Retrieved January 2021 from
`https://www.mhlw.go.jp/index.html`

[31] Tokyu Corporation. Retrieved January 2021 from
`https://www.tokyu.co.jp/index.html`

[32] Ministry of Land, Infrastructure, Transport and Tourism and Tourism Railway Bureau January 2021 from
`https://www.mlit.go.jp/common/000224631.pdf`