Disjointly and jointly productive players and the Shapley value

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Abstract

Central to this study is the concept of disjointly productive players where no cooperation gain occurs when one of two such players joins a coalition containing the other. Our first new axiom states that the payoff to a player does not change when another player, disjointly productive to that player, leaves the game. The second axiom implies that the payoff to a third player does not change if we merge two disjointly productive players into a new player. These two axioms, along with efficiency, characterize the Shapley value and may be advantageous sometimes to improve the runtime for computing the Shapley value. Further axiomatizations are provided, using, for example, a modification of behavior property where the payoff for two players in two new games in which their behavior changes once to total dislike and once to total affection is equal to the payoff in the original game.

Keywords Cooperative game · Shapley value · Disjointly productive players · Mutually dependent players · Merged (disjointly productive) players game · Modification of behavior

JEL Classification: C71 - Cooperative Games

1 Introduction

An area of study within the theory of cooperative games is how payoffs of TU-values change when two or more players merge into a single one (see, e.g., Derks and Tijs (2000)). Ideally, this also leads to axiomatizations like for the Banzhaf value (Banzhaf, 1965) in Lehrer (1988). Lehrer defines an amalgamated game where two players are merged into one who has the same effect in the new game as the two merging players had in the old game. Together with an axiom, known as standardness (Hart and Mas-Colell, 1989), Lehrer axiomatizes the Banzhaf value with a reduction axiom for these games, called 2-efficiency. A TU-value satisfies this axiom if the payoff to the merged player in the new game is equal to the sum of the payoffs to the two merging players in the old game. Since the Shapley value (Shapley, 1953b) also satisfies standardness, the Shapley value is bound to fail for this axiom.

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2-efficiency is also used in the axiomatizations of the Banzhaf value in Nowak (1997), Casajus (2011), and Casajus (2012). As pointed out in Alonso-Meijide et al. (2012), Casajus (2012) uses a somewhat different definition of 2-efficiency and, therefore, comes up with somewhat contradictory solutions for the Banzhaf value compared to Nowak (1997).

Haller (1994) investigates collusion properties of TU-values. Instead of merging two players into one, one player becomes a proxy player with the power of both players and the other player becomes a null player, meaning that this player contributes nothing to any coalition. For general games, the results for the Shapley value can be transferred to the amalgamation of players because this value satisfies for such games the null player out property (Derks and Haller, 1999), i.e., removing a null player does not change the payoffs to the other players.

The study of Haller (1994) was inspired by the joint-bargaining paradox in Harsanyi (1977), also known as the Harsanyi paradox (see Vidal-Puga (2012)). Harsanyi observes that in simple bargaining processes, when two or more players join to form an acting bargaining unit, their bargaining position worsens relative to the remaining players. Moreover, Harsanyi notes that this holds for all solution concepts that satisfy efficiency and the symmetry axiom, hence also for the Shapley value. Chae and Heidhues (2004) explain this paradox with the argument that, by merging, players trade their multiple "rights to talk" for a single one, thereby weakening their power position.

To axiomatize the class of weighted Shapley values (Shapley, 1953a), Nowak and Ratzik (1995) presented the concept of mutually dependent players. These are players who are only jointly productive. Any coalition of mutually dependent players forms a partnership, introduced in Kalai and Samet (1987). For so-called weighted games, which consist of a classical TU-game and a weight vector λ, Ratzik (2012) formulates an amalgamating payoffs axiom. Here, not arbitrary players merge into a new one, as in 2-efficiency, but only those that form a partnership, i.e., only jointly productive players.

Besner (2019) introduces a player splitting axiom to axiomatize the proportional Shapley value (Besner, 2016; Béal et al., 2018). This axiom can also be interpreted as a merging axiom for weakly dependent players who are jointly productive and, in addition, still have a stand-alone worth. Unlike all the studies above, Besner (2019), at least directly, is not concerned with the ratio of the payoff for the merged player versus the sum of the payoffs for the merging players in the original game. His focus is on ensuring that a merger or split does not impact payoffs to unaffected players.

This view is also central to the first main part of our study. For this, as a contrast to mutually dependent players, i.e., jointly productive players, we use the concept of disjointly productive players. Two players are disjointly productive if their marginal contribution to any coalition that does not include the other player is the same as if that coalition had previously been joined by the other one. Therefore, this can be considered as the special case of "interaction of cooperation" in Grabisch and Roubens (1999) without any interaction. Curiously, apart from Grabisch and Roubens, we have not found any other study that uses or refers to disjunctive productive players.

Our first new axiom then states that the payoff to a player does not change when a player who is disjointly productive with that player leaves the game. For our second axiom, we introduce a merged (disjointly productive players) game, corresponding to the merged game[1] in Lehrer (1988), but only for disjointly productive players. The meaning of our axiom in that case is that the payoff does not change for players who are not affected by

[1] Lehrer (1988) introduces also another merged players game that uses as a new worth for certain coalitions the maximum worth of certain subcoalitions from the old game.

the merger.

As the first main result, we show that the Shapley value, along with efficiency, is axiomatized by our two new axioms. In addition to our merging axiom, if a TU-value also satisfies efficiency, like the Shapley value, then the payoff to the merged player is equal to the sum of the payoffs to the merging players from the original game. Thus, the Harsanyi paradox does not apply to the Shapley value for disjointly productive players and they do not lose their "right to talk." The crucial role in the proof of our theorem is played by a new split game that contains one new player for each old player and each coalition containing this player.

When disjointly productive players occur in practice, both new axioms can help to reduce the computational time for the payoff calculation of the Shapley value. This can often be the deciding factor for performing an exact calculation instead of using approximation methods.

In the second main part of this study, we examine games in which the behavior of two players towards each other has changed compared to the original game, but the other players behave as before. In the first game, the two players delete their cooperation in all coalitions containing both players. In the other game, both only want to participate in coalitions that also contain both players. A further new axiom then states that the payoff to the two players in the initial game is fixed by the payoff in the two new games. Along with this axiom, we can also present a new axiomatization of the Shapley value.

Our last axiomatization uses a weak additivity property. The payoff to a player in a game is equal to the sum of the payoff to that player in a game in which two arbitrary players changed their behavior to mutual non-cooperation and the payoff in the complementary game to that game.

The remainder of this paper is organized as follows. Section 2 contains some preliminaries. In Section 3, we introduce the concept of disjointly productive players, related axioms, a first theorem and an additional compact corollary. Besides disjointly productive players, jointly productive players also play a significant role in the axiomatizations of Section 4 and Section 5. Section 6 gives a short discussion and some hints on how to reduce the complexity of computing the Shapley value in some situations. The Appendix (Section 7) shows the logical independence of the axioms used in the axiomatizations.

2 Preliminaries

Let the countably infinite set $\mathfrak{U}$ be the universe of players. We denote by $\mathcal{N}$ the set of all non-empty and finite subsets of $\mathfrak{U}$. A (TU-)game is a pair (N,v) such that $N \in \mathcal{N}$ and v is a **coalition function**, i.e., $v\colon 2^N \to \mathbb{R}$, $v(\emptyset)=0$. We call the subsets $S \subseteq N$ **coalitions** and $v(S)$ is the **worth** of the coalition S, Ω^S denotes the set of all nonempty subsets of S, (S,v) is the **restriction** of (N,v) to the player set $S \in \Omega^N$, and the set of all games (N,v) is denoted by $\mathbb{V}(N)$.

Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$. For all $S \subseteq N$, the **dividends** $\Delta_v(S)$ (Harsanyi, 1959) are defined recursively by

$$\Delta_v(S) := \begin{cases} 0, \text{ if } S=\emptyset, \text{ and} \\ v(S) - \sum_{R \subsetneq S} \Delta_v(R), \text{ if } S \in \Omega^N. \end{cases} \tag{1}$$

A TU-game $(N,u_T) \in \mathbb{V}(N)$, $T \in \Omega^N$, is called a **unanimity game** if for all $S \subseteq N$ we have $u_T(S) := 1$ if $T \subseteq S$ and $u_T(S) := 0$ otherwise. By Shapley (1953b), any coalition function v on N has a unique representation

$$v = \sum_{T \in \Omega^N} \Delta_v(T) u_T. \tag{2}$$

We call a coalition $S \subseteq N$ **inessential** in (N, v) if $\Delta_v(S) = 0$, otherwise S is called **essential** in (N, v). The **marginal contribution** $MC_i^v(S)$ of a player $i \in N$ to a coalition $S \subseteq N \backslash \{i\}$ is given by $MC_i^v(S) := v(S \cup \{i\}) - v(S)$. A player $i \in N$ is called a **dummy player** in (N, v) if $v(S \cup \{i\}) = v(S) + v(\{i\})$, $S \subseteq N \backslash \{i\}$, a dummy player i is called a **null player** in (N, v) if we have $v(\{i\}) = 0$. We call two players $i, j \in N$, $i \neq j$, **symmetric** in (N, v) if for all $S \subseteq N \backslash \{i, j\}$, we have $v(S \cup i) = v(S \cup j)$, they are called **mutually dependent** (Nowak and Radzik, 1995) in (N, v) if

$$v(S \cup \{i\}) = v(S) = v(S \cup \{j\}), \tag{3}$$

which is equivalent (see Casajus (2018)) to

$$\Delta_v(S \cup \{k\}) = 0, \; k \in \{i, j\}. \tag{4}$$

For all $N \in \mathcal{N}$, a **TU-value** φ is an operator that assigns to any $(N, v) \in \mathbb{V}(N)$ a payoff vector $\varphi(N, v) \in \mathbb{R}^N$. The **Shapley value** Sh (Shapley, 1953b) is given by

$$Sh_i(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{\Delta_v(S)}{|S|} \quad \text{for all} \;\; i \in N. \tag{5}$$

We make use of the following standard axioms for TU-values.

Efficiency, E. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, we have $\sum_{i \in N} \varphi_i(N, v) = v(N)$.

Dummy player, D. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and $i \in N$ such that i is a dummy player in (N, v), we have $\varphi_i(N, v) = v(\{i\})$.

Null player, N. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and $i \in N$ such that i is a null player in (N, v), we have $\varphi_i(N, v) = 0$.

Symmetry, S. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and $i, j \in N$ such that i and j are symmetric in (N, v), we have $\varphi_i(N, v) = \varphi_j(N, v)$.

Additivity, A. For all $N \in \mathcal{N}$, $(N, v), (N, w) \in \mathbb{V}(N)$, we have $\varphi(N, v) + \varphi(N, w) = \varphi(N, v + w)$.

3 Disjointly productive players

Nowak and Radzik (1995) introduced the concept of mutually dependent players. These are players who are only jointly productive, i.e., the contribution of each of these players to any coalition that does not contain the other is zero. The following concept represents the opposite: certain players are productive only when a specific other player is not in the group.

Definition 3.1. *For all $N \in \mathcal{N}, (N, v) \in \mathbb{V}(N)$, two players $i, j \in N$, $i \neq j$, are called **disjointly productive** in (N, v) if, for all $S \subseteq N \backslash \{i, j\}$, we have $MC_i^v(S \cup \{j\}) = MC_i^v(S)$ which is equivalent to*

$$v(S \cup \{i, j\}) - v(S \cup \{j\}) = v(S \cup \{i\}) - v(S).$$

Remark 3.2. *Note that* $MC_i^v(S \cup \{j\}) = MC_i^v(S)$ *in Definition 3.1 is equivalent to* $MC_j^v(S \cup \{i\}) = MC_j^v(S)$. *Grabisch and Roubens (1999) use the quantity* $v(S \cup \{i, j\}) - v(S \cup \{i\}) - v(S \cup \{j\}) + v(S)$, *or, respectively, the average of it over all coalitions, for their study of "interaction indices." Thus, our definition is equivalent to this quantity if it is zero for all coalitions.*

If we consider the dividend as "the pure contribution of cooperation in a TU-game" (Billot and Thisse, 2005), it is consequent that any coalition containing only one of two mutually dependent players has a dividend of zero (see (4)). In this sense, the contribution of cooperation made by the formation of coalitions with two disjointly productive players should also be zero.

Lemma 3.3. *Let* $N \in \mathcal{N}, (N, v) \in \mathbb{V}(N)$. *Two players* $i, j \in N$, $i \neq j$, *are disjointly productive in* (N, v) *if and only if for all* $S \subseteq N$, *we have*

$$v(S) = \sum_{R \subseteq S,\, \{i,j\} \nsubseteq R} \Delta_v(R) \text{ or, equivalent by (1), } \Delta_v(S) = 0, \text{ if } \{i, j\} \subseteq S. \tag{6}$$

Proof. Let $N \in \mathcal{N}, (N, v) \in \mathbb{V}(N)$, $i, j \in N$, $i \neq j$, and $T \subseteq N \backslash \{i\}$, $T \ni j$. It is sufficient to show the equivalence

$$v(T \cup \{i\}) = v\big((T \backslash \{j\}) \cup \{i\}\big) - v(T \backslash \{j\}) + v(T) \Leftrightarrow v(T \cup \{i\}) = \sum_{R \subseteq T \cup \{i\},\, \{i,j\} \nsubseteq R} \Delta_v(R). \tag{7}$$

We have

$$\begin{aligned} v\big((T \backslash \{j\}) \cup \{i\}\big) &- v(T \backslash \{j\}) + v(T) \\ &\underset{(1)}{=} \sum_{S \subseteq (T \backslash \{j\}) \cup \{i\}} \Delta_v(S) - \sum_{S \subseteq T \backslash \{j\}} \Delta_v(S) + \sum_{S \subseteq T} \Delta_v(S) \\ &= \sum_{S \subseteq T \cup \{i\},\, \{i,j\} \nsubseteq S} \Delta_v(S), \end{aligned}$$

and (7) and, therefore, Lemma 3.3 is shown. □

Since two disjointly productive players do not mind each other's business, so to speak, they should not mind if the other player leaves the game. This is the statement of our first new axiom.

Disjointly productive players, DP. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and $i, j \in N$ such that i and j are disjointly productive players in (N, v) we have

$$\varphi_i(N, v) = \varphi_i(N \backslash \{j\}, v).$$

The following definition considers games that result from the union of disjointly productive players into a single player.

Definition 3.4. *Let* $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, $i, j \in N$ *be two disjointly productive players in* $(N, v), k \in \mathfrak{U}$, $k \notin N$, *and* $N_{ij}^k := (N \backslash \{i, j\}) \cup \{k\}$. *The TU-game* $(N_{ij}^k, v_{ij}^k) \in \mathbb{V}(N_{ij}^k)$ *is called a* ***merged (disjointly productive) players game*** *to* (N, v) *where* v_{ij}^k *is given by*

$$v_{ij}^k(S) := \begin{cases} v(S), & k \notin S, \\ v\big((S \backslash \{k\}) \cup \{i, j\}\big), & k \in S, \end{cases} \qquad \text{for all } S \subseteq N_{ij}^k. \tag{8}$$

That is, any coalition consisting of the same players in the new and old game has the same worth in both games. Coalitions in which the merged player is a member receive the worth of the corresponding coalition with all the original merging players from the old game. This definition corresponds to the definition of the merged game in Lehrer (1988) with the difference that only disjointly productive players are merged.

The following lemma states that for a player in any game, we have split games where that player is split into two disjointly productive players and the old game is a merged players game to those games. The dividends from coalitions containing some players and the split player in the original game are equal to the sum of the dividends from coalitions containing the same other players and only one each of the two split disjointly productive players in the split game.

Lemma 3.5. *Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, $k \in N$, $i,j \in \mathfrak{U}$, $i,j \notin N$, $i \neq j$, and $N_k^{ij} := (N\backslash\{k\}) \cup \{i,j\}$. Then, we have some $(N_k^{ij}, v_k^{ij}) \in \mathbb{V}(N_k^{ij})$ such that i,j are disjointly productive in (N_k^{ij}, v_k^{ij})(which we will call* ***split (in disjointly productive) players games****) and (N,v) is a merged players game to each (N_k^{ij}, v_k^{ij}) where, for all $S \subseteq N\backslash\{k\}$, v_k^{ij} is given by*

$$\Delta_{v_k^{ij}}(S) = \Delta_v(S) \text{ and} \tag{9}$$

$$\Delta_{v_k^{ij}}(S \cup \{i\}) + \Delta_{v_k^{ij}}(S \cup \{j\}) = \Delta_v(S \cup \{k\}). \tag{10}$$

Proof. Let $(N,v) \in \mathbb{V}(N)$, $k \in N$, $i,j \in \mathfrak{U}$, $i,j \notin N$, $i \neq j$, $N_k^{ij} := (N\backslash\{k\}) \cup \{i,j\}$, and be $(N_k^{ij}, v_k^{ij}) \in \mathbb{V}(N_k^{ij})$ such that (9) and (10) are satisfied. We define $v_k^{ij}(S \cup \{i,j\})$ such that

$$\Delta_{v_k^{ij}}(S \cup \{i,j\}) := 0 \text{ for all } S \subseteq N_k^{ij}\backslash\{i,j\}. \tag{11}$$

This is always possible, and, by Lemma 3.3, i,j are disjointly productive in (N,v). Therefore, by (8), we have the merged players game $(N,\tilde{v})$ to (N_k^{ij}, v_k^{ij}), given, for all $S \subseteq N\backslash\{k\}$, by

$$\tilde{v}(S) := v_k^{ij}(S), \text{ and} \tag{12}$$

$$\tilde{v}(S \cup \{k\}) := v_k^{ij}(S \cup \{i,j\}). \tag{13}$$

By (1), (9), and (12), we have $v(S) = \tilde{v}(S)$ for all $S \subseteq N\backslash\{k\}$. We will show, by induction on the size $s := |S|$,

$$v(S \cup \{k\}) = \tilde{v}(S \cup \{k\}) \text{ for all } S \subseteq N\backslash\{k\}. \tag{14}$$

Initialization: Let $s = 0$ and therefore $S = \emptyset$. Then, by (1), (10), (11), and (13), (14) is satisfied.

Induction step: Let $s \geq 1$. Assume that (14) is satisfied for all $s', s' < s$, (IH). We have

$\Delta_{\tilde{v}}(S \cup \{k\})$

$$
\begin{aligned}
&\underset{(1)}{=} \quad \tilde{v}(S \cup \{k\}) - \sum_{R \subsetneq (S \cup \{k\})} \Delta_{\tilde{v}}(R) \\
&\underset{(8)}{=} \quad v_k^{ij}(S \cup \{i,j\}) - \sum_{R \subsetneq (S \cup \{k\}), R \ni k} \Delta_{\tilde{v}}(R) - \sum_{R \subsetneq (S \cup \{k\}), k \notin R} \Delta_{\tilde{v}}(R) \\
&\underset{(12)}{=} \quad v_k^{ij}(S \cup \{i,j\}) - \sum_{R \subsetneq S \cup \{k\}, R \ni k} \Delta_{\tilde{v}}(R) - \sum_{R \subseteq S} \Delta_{v_k^{ij}}(R) \\
&\underset{\substack{(IH)\\(10)}}{=} \quad v_k^{ij}(S \cup \{i,j\}) - \sum_{R \subsetneq S \cup \{i\}, R \ni i} \Delta_{v_k^{ij}}(R) - \sum_{R \subsetneq S \cup \{j\}, R \ni j} \Delta_{v_k^{ij}}(R) - \sum_{R \subseteq S} \Delta_{v_k^{ij}}(R) \\
&\underset{(6)}{=} \quad \sum_{R \subseteq S \cup \{i,j\},\, \{i,j\} \not\subseteq R} \Delta_{v_k^{ij}}(R) - \sum_{R \subsetneq S \cup \{i\}, R \ni i} \Delta_{v_k^{ij}}(R) - \sum_{R \subsetneq S \cup \{j\}, R \ni j} \Delta_{v_k^{ij}}(R) - \sum_{R \subseteq S} \Delta_{v_k^{ij}}(R) \\
&= \quad \Delta_{v_k^{ij}}(S \cup \{i\}) + \Delta_{v_k^{ij}}(S \cup \{j\}) \underset{(10)}{=} \Delta_v(S \cup \{k\}),
\end{aligned}
$$

and (14) is satisfied. □

The following new axiom states that if two disjointly productive players merge into one player who has the same impact on the new game as the two players together had previously, the payoff for the other players should not change.

Merged (disjointly productive) players game property, MP. For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, $i,j \in N$ such that i and j are disjointly productive in (N,v), $k \in \mathfrak{U}$, $k, \notin N$, and a merged players game $(N_{ij}^k, v_{ij}^k) \in \mathbb{V}(N_{ij}^k)$ to (N,v), we have

$$\varphi_\ell(N_{ij}^k, v_{ij}^k) = \varphi_\ell(N,v) \text{ for all } \ell \in N_{ij}^k \backslash \{k\}. \tag{15}$$

Our interest is also in the payoffs for the merging players versus the merged player. If the value is efficient, we get an obvious result.

Remark 3.6. *Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, $i,j \in N$ such that i and j are disjointly productive in (N,v), $k \in \mathfrak{U}$, $k, \notin N$, and $(N_{ij}^k, v_{ij}^k) \in \mathbb{V}(N_{ij}^k)$ be a merged players game to (N,v). If φ is a TU-value that satisfies **E** and **MP**, we have, by (8) and (15),*

$$\varphi_k(N_{ij}^k, v_{ij}^k) = \varphi_i(N,v) + \varphi_j(N,v). \tag{16}$$

(16) corresponds to the condition for 2-efficiency in Lehrer (1988), but our merging players must be disjointly productive and the TU-value must be efficient.

Remark 3.7. *Let Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $I \subseteq N$, $|I| \geq 3$, be a coalition of players where each $i \in I$ is mutually disjointly productive with all other players $j \in I \backslash \{i\}$. If we merge two players $i,j \in I$, accordingly to Definition 3.4, into a new player $k \in \mathfrak{U}$, resulting in a merged players game $(N_{ij}^k, v_{ij}^k) \in \mathbb{V}(N_{ij}^k)$, by (6) and (10), we have $\Delta_{v_{ij}^k}(S \cup \{k\}) = 0$ for all $S \cap I \neq \emptyset$. This means, by (6), all $\ell \in (I \backslash \{i,j\}) \cup \{k\}$ are mutually disjointly productive in $\mathbb{V}(N_{ij}^k)$. Therefore, we can apply Definition 3.4 repeatedly to all $i \in I$ and have, finally, for the last merged players game, here denoted by $(\overline{N}, \overline{v})$, a player set $\overline{N} = (N \backslash I) \cup \overline{k}$ and a coalition function $\overline{v}$, given by*

$$\overline{v}(S) := \begin{cases} v(S), & \overline{k} \notin S, \\ v\big((S \backslash \{\overline{k}\}) \cup I\big), & \overline{k} \in S, \end{cases} \qquad \textit{for all } S \subseteq \overline{N}.$$

Accordingly, (10) can be adapted to

$$\Delta_{v_{ij}^{\overline{k}}}(S \cup \{\overline{k}\}) = \sum_{i \in I} \Delta_v(S \cup \{i\}),$$

and (16) can be adapted to

$$\varphi_{\overline{k}}(\overline{N}, \overline{v}) = \sum_{i \in I} \varphi_i(N, v).$$ [2]

The following lemma is similar to Lemma 2 in Besner (2019), where it is shown that a TU-value that is efficient and satisfies a player splitting axiom defined there also satisfies symmetry.

Lemma 3.8. *If a TU-value φ satisfies **E** and **MP**, then φ also satisfies **S**.*

*P*roof. The proof is similar to the proof of Lemma 2 in Besner (2019).

Let $N = \{1, 2, ..., n\}$, $n \geq 2$, $(N, v) \in \mathbb{V}^N$, φ be a TU-value that satisfies **E** and **MP**, and, w.l.o.g., player 1 and player 2 be symmetric in (N, v). If we split player 1, in accordance to **MP** and Lemma 3.5, into two new disjointly productive players, player $n+1$ and player $n+2$, and define $N_1^{n+1,n+2} := \{2, 3, ..., n, n+1, n+2\}$, we have

$$\varphi_2(N_1^{n+1,n+2}, v_1^{n+1,n+2}) = \varphi_2(N, v). \tag{17}$$

If we split player 2, in accordance to **MP** and Lemma 3.5, into the same players as before, player $n+1$ and player $n+2$, instead, and define $N_2^{n+1,n+2} := \{1, 3, 4, ..., n, n+1, n+2\}$, we have

$$\varphi_1(N_2^{n+1,n+2}, v_2^{n+1,n+2}) = \varphi_1(N, v), \tag{18}$$

where we choose, for all $S \subseteq N \backslash \{1, 2\}$,

$$\begin{aligned} v_2^{n+1,n+2}(S \cup \{n+1\}) &:= v_1^{n+1,n+2}(S \cup \{n+1\}), \\ v_2^{n+1,n+2}(S \cup \{n+2\}) &:= v_1^{n+1,n+2}(S \cup \{n+2\}), \\ v_2^{n+1,n+2}(S \cup \{1\} \cup \{n+1\}) &:= v_1^{n+1,n+2}(S \cup \{2\} \cup \{n+1\}), \text{ and} \\ v_2^{n+1,n+2}(S \cup \{1\} \cup \{n+1\}) &:= v_1^{n+1,n+2}(S \cup \{2\} \cup \{n+1\}). \end{aligned}$$

This is possible because players 1 and 2 are symmetric in (N, v).

In the same way, now in the game $(N_1^{n+1,n+2}, v_1^{n+1,n+2})$, we split player 2 into two new disjointly productive players, player $n+3$ and player $n+4$, and, analogously, in the game $(N_2^{n+1,n+2}, v_2^{n+1,n+2})$ player 1 into the same players as before, player $n+3$ and player $n+4$. Note that we have $N_{1^2}^{(n+1,n+2)^{n+3,n+4}} = N_{2^1}^{(n+1,n+2)^{n+3,n+4}} = \{3, 4, ..., n, n+1, n+2, n+3, n+4\}$, and, since players 1 and 2 are symmetric in (N, v), we can choose

$$v_{2^1}^{(n+1,n+2)^{n+3,n+4}}(S) = v_{1^2}^{(n+1,n+2)^{n+3,n+4}}(S) \text{ for all } S \subseteq N_{1^2}^{(n+1,n+2)^{n+3,n+4}} = N_{2^1}^{(n+1,n+2)^{n+3,n+4}}.$$

[2] Note that the Banzhaf value would not satisfy a related general super additivity for the amalgamation axiom in Lehrer (1988).

By **E**, we obtain

$$\varphi_{n+3}\Big(N_{1^2}^{(n+1,n+2)^{n+3,n+4}}, v_{1^2}^{(n+1,n+2)^{n+3,n+4}}\Big) + \varphi_{n+4}\Big(N_{1^2}^{(n+1,n+2)^{n+3,n+4}}, v_{1^2}^{(n+1,n+2)^{n+3,n+4}}\Big) \underset{Rem.\,3.6}{=} \varphi_2(N_1^{n+1,n+2}, v_1^{n+1,n+2}) \underset{(17)}{=} \varphi_2(N,v),$$

$$\varphi_{n+3}\Big(N_{2^1}^{(n+1,n+2)^{n+3,n+4}}, v_{2^1}^{(n+1,n+2)^{n+3,n+4}}\Big) + \varphi_{n+4}\Big(N_{2^1}^{(n+1,n+2)^{n+3,n+4}}, v_{2^1}^{(n+1,n+2)^{n+3,n+4}}\Big) \underset{Rem.\,3.6}{=} \varphi_1(N_2^{n+1,n+2}, v_2^{n+1,n+2}) \underset{(18)}{=} \varphi_1(N,v).$$

It follows, $\varphi_1(N,v) = \varphi_2(N,v)$, and **S** is shown. □

We present our first main result.

Theorem 3.9. *Sh is the unique TU-value that satisfies **E**, **DP**, and **MP**.*

Proof. Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$.
I. Existence: It is well-known that Sh satisfies **E**. By (5) and Lemma 3.3, it is obvious that Sh satisfies **DP**.

• **MP**: Let $i, j \in N$ be such that i and j are disjointly productive in (N,v) and $(N_{ij}^k, v_{ij}^k) \in \mathbb{V}(N_{ij}^k)$ be a merged players game to (N,v). We have

$$\begin{aligned}
Sh_\ell(N_{ij}^k, v_{ij}^k) &\underset{(5)}{=} \sum_{\substack{S\subseteq N_{ij}^k,\\ S\ni\ell}} \frac{\Delta_{v_{ij}^k}(S)}{|S|} \\
&\underset{\substack{(9)\\(10)}}{=} \sum_{\substack{S\subseteq(N\setminus\{i,j\}),\\ S\ni\ell}} \frac{\Delta_v(S)}{|S|} + \sum_{\substack{S\subseteq(N\setminus\{j\}),\\ \{i,\ell\}\subseteq S}} \frac{\Delta_v(S)}{|S|} + \sum_{\substack{S\subseteq(N\setminus\{i\}),\\ \{j,\ell\}\subseteq S}} \frac{\Delta_v(S)}{|S|} \\
&= \sum_{\substack{S\subseteq N, \{i,j\}\not\subseteq S,\\ S\ni\ell}} \frac{\Delta_v(S)}{|S|} \underset{(6)}{=} Sh_\ell(N,v) \quad \text{for all } \ell \in N_{ij}^k\setminus\{k\},
\end{aligned}$$

and **MP** is shown.
II. Uniqueness: Let φ be a TU-value that satisfies all axioms of Theorem 3.9 and, therefore, by Lemma 3.8, also **S**. By **MP**, applying Lemma 3.5 and Remark 3.7, we split succesively each player $i \in N$ of the $n := |N|$ players in 2^{n-1} disjointly productive players i_S, $S \subseteq N$, $S \ni i$. In the final split game, we call it $(\overline{N}, \overline{v})$, we have $n \cdot 2^{n-1}$ players. Each of the coalitions containing all players with the same coalition $S \subseteq N$ as a subscript get a worth of the dividend $\Delta_v(S)$ and all other coalitions are defined as inessential in $(\overline{N}, \overline{v})$, i.e., we have, for all $T \in \Omega^{\overline{N}}$,

$$\Delta_{\overline{v}}(T) := \begin{cases} \Delta_v(S), & T = \bigcup_{i\in S}\{i_S\},\ S \in \Omega^N, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

We illustrate our procedure with a small example: Let $(N', w) \in \mathbb{V}(N')$, $N' = \{1,2,3\}$. At first, we split for a new game (N_1', w_1) player 1, using Lemma 3.5 and Remark 3.7, into four players $1_{\{1\}}, 1_{\{1,2\}}, 1_{\{1,3\}}, 1_{\{1,2,3\}}$ with the player set $N_1' := \{2, 3, 1_{\{1\}}, 1_{\{1,2\}}, 1_{\{1,3\}}, 1_{\{1,2,3\}}\}$. By Remark 3.7, we define w_1 by

$$\begin{aligned}
\Delta_{w_1}(\{1_{\{1\}}\}) &:= \Delta_w(1), & \Delta_{w_1}(\{1_{\{1,2\}}, 2\}) &:= \Delta_w(\{1,2\}),\\
\Delta_{w_1}(\{1_{\{1,3\}}, 3\}) &:= \Delta_w(\{1,3\}), & \Delta_{w_1}(\{1_{\{1,2,3\}}, 2, 3\}) &:= \Delta_w(\{1,2,3\}),
\end{aligned}$$

$w_1(S) := w(S)$ for all $S \subseteq \{2,3\}$, and all other coalitions are defined as inessential in (N'_1, w_1).

In the next step, we split for a new game (N'_{12}, w_{12}) player 2, using Lemma 3.5 and Remark 3.7, into four players $2_{\{2\}}, 2_{\{1,2\}}, 2_{\{2,3\}}, 2_{\{1,2,3\}}$ with the player set $N'_{12} := (N'_1 \backslash \{2\}) \cup \{2_{\{2\}}, 2_{\{1,2\}}, 2_{\{2,3\}}, 2_{\{1,2,3\}}\}$. By Remark 3.7, we define w_{12} by

$$\begin{array}{rclrcl} \Delta_{w_{12}}(\{2_{\{2\}}\}) & := & \Delta_w(2), & \Delta_{w_{12}}(\{1_{\{1,2\}}, 2_{\{1,2\}}\}) & := & \Delta_w(\{1,2\}), \\ \Delta_{w_{12}}(\{2_{\{2,3\}}, 3\}) & := & \Delta_w(\{2,3\}), & \Delta_{w_{12}}(\{1_{\{1,2,3\}}, 2_{\{1,2,3\}}, 3\}) & := & \Delta_w(\{1,2,3\}), \end{array}$$

$w_{12}(S) := w_1(S)$ for all $S \subseteq N'_1 \backslash \{2\}$, and all other coalitions are defined as inessential in (N'_{12}, w_{12}).

Finally, we split for a new game (N'_{123}, w_{123}) player 3, using Lemma 3.5 and Remark 3.7, into four players $3_{\{3\}}, 3_{\{1,3\}}, 3_{\{2,3\}}, 3_{\{1,2,3\}}$ with the player set $N'_{123} := \{1_{\{1\}}, 1_{\{1,2\}}, 1_{\{1,3\}}, 1_{\{1,2,3\}}, 2_{\{2\}}, 2_{\{1,2\}}, 2_{\{2,3\}}, 2_{\{1,2,3\}}, 3_{\{3\}}, 3_{\{1,3\}}, 3_{\{2,3\}}, 3_{\{1,2,3\}}\}$. By Remark 3.7, we define w_{123} by

$$\begin{array}{rclrcl} \Delta_{w_{123}}(\{1_{\{1\}}\}) & := & \Delta_w(1), & \Delta_{w_{123}}(\{2_{\{2\}}\}) & := & \Delta_w(\{2\}), \\ \Delta_{w_{123}}(\{3_{\{3\}}\}) & := & \Delta_w(3), & \Delta_{w_{123}}(\{1_{\{1,2\}}, 2_{\{1,2\}}\}) & := & \Delta_w(\{1,2\}), \\ \Delta_{w_{123}}(\{1_{\{1,3\}}, 3_{\{1,3\}}\}) & := & \Delta_w(\{1,3\}) & \Delta_{w_{123}}(\{2_{\{2,3\}}, 3_{\{2,3\}}\}) & := & \Delta_w(\{2,3\}), \end{array}$$
$$\Delta_{w_{123}}(\{1_{\{1,2,3\}}, 2_{\{1,2,3\}}, 3_{\{1,2,3\}}\}) := \Delta_w(\{1,2,3\}),$$

and all other coalitions are defined as inessential in (N'_{123}, w_{123}).

Back to our original split game $(\overline{N}, \overline{v})$, by **E**, **MP**, Lemma 3.5, and Remark 3.7, we have that the sum of the payoffs to all players who are split from the same player in the original game equals the payoff to that player in the original game, i.e.,

$$\sum_{i_S,\, S \in \Omega^N,\, S \ni i} \varphi_{i_S}(\overline{N}, \overline{v}) = \varphi_i(N, v) \text{ for all } i \in N. \tag{20}$$

By (19), for each $T \in \Omega^{\overline{N}}, T = \bigcup_{i \in S}\{i_S\}$, $S \in \Omega^N$, all $j \in \overline{N} \backslash T$ are disjointly productive with any $k \in T$. Therefore, repeatedly using **DP**, we have, for each $T \in \Omega^{\overline{N}}, T = \bigcup_{i \in S}\{i_S\}$, $S \in \Omega^N$,

$$\varphi_k(\overline{N}, \overline{v}) = \varphi_k(T, \overline{v}) \text{ for all } k \in T.$$

All $k \in T$ are symmetric in $(T, \overline{v})$. Thus, by **E** and **S**, φ is unique for all $j \in \overline{N}$ in $(\overline{N}, \overline{v})$, and therefore, by (20), for all $i \in N$ in (N, v), and Theorem 3.9 is shown. □

The crucial step in our theorem is to replace an arbitrary game with the split game, which contains one new player for each original player and each coalition containing that player. Then, each of the new players is a member of (at most) one essential coalition. To derive the same unique solution, satisfying also efficiency and the merged players game property, one could use weaker axioms than the disjointly productive players property. In what follows, however, we will use an axiom that implies both the dummy player property and efficiency.

In the context of coalition structures (Aumann and Drèze, 1974; Owen, 1977), Hart and Kurz (1983) presented an axiom, called dummy coalition, which can be seen as a generalization of the dummy player property for games with a coalition structure. We adapt this axiom for TU-games and call a coalition $S \in \Omega^N$ a **dummy coalition** in (N, v) if $v(T \cup R) = v(T) + v(R)$ for all $T \subseteq N \backslash S$ and $R \subseteq S$.

Dummy coalition, DC. For all $(N,v) \in \mathbb{V}(N)$ and $S \in \Omega^N$ such that S is a dummy coalition in (N,v), we have $\sum_{i\in S} \varphi_i(N,v) = v(S)$.

By this axiom, all players of a dummy coalition receive together as a payoff what the coalition alone generates for itself.

Remark 3.10. *Obviously,* ***DC*** *implies* ***D****. Note that the grand coalition is always a dummy coalition. Therefore,* ***DC*** *implies* ***E****.*

Remark 3.11. *It is well-known and easy to show that $i \in N$ is a dummy player in (N,v) if and only if we have $\Delta_v(S) = 0$ for all $S \subseteq N, \{i\} \subsetneq S$.*

A similar result holds for a dummy coalition.

Lemma 3.12. *Let $N \in \mathcal{N}, (N,v) \in \mathbb{V}(N)$. $S \in \Omega^N$ is a* ***dummy coalition*** *in (N,v) if and only if we have $\Delta_v(T) = 0$ for all $T \subseteq N$, $T \nsubseteq S, (T \cap S) \neq \emptyset$.*

Proof. Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $S \in \Omega^N$. We have to show, for all $R \in \Omega^S$ and all $T \in \Omega^{N\setminus S}$,

$$v(R \cup T) = v(R) + v(T) \Leftrightarrow \Delta_v(R \cup T) = 0; \tag{21}$$

We use a first induction I_1 on the size $s := |S|$.

Initialization I_1: Let $s = 1$. Then, $R = S$ and (21) follows by Remark 3.11.

Induction step I_1: Let $s \geq 2$. Assume that (21) is satisfied for all s', $s' < s$, (IH_1). We use a second induction I_2 on the size $t := |T|$.

Initialization I_2: Let $t = 1$ and, therefore, $T = \{i\}$, $i \in N\setminus S$. We have

$$\begin{aligned} 0 = v(R \cup \{i\}) - v(R) - v(\{i\}) &\underset{(1)}{=} \sum_{Q \subseteq (R\cup\{i\})} \Delta_v(Q) - \sum_{Q\subseteq R} \Delta_v(Q) - \Delta_v(\{i\}) \\ &= \sum_{\substack{Q \subseteq (R\cup\{i\}) \\ Q \ni i}} \Delta_v(Q) + \sum_{Q\subseteq R} \Delta_v(Q) - \sum_{Q\subseteq R} \Delta_v(Q) - \Delta_v(\{i\}) \underset{(IH_1)}{=} \Delta_v(R \cup \{i\}), \end{aligned}$$

and (21) is shown.

Induction step I_2: Let $t \geq 2$. Assume that (21) is satisfied for all t', $t' < t$, (IH_2). We have

$$\begin{aligned} 0 = \ & v(R \cup T) - v(R) - v(T) \underset{(1)}{=} \sum_{Q \subseteq (R\cup T)} \Delta_v(Q) - \sum_{Q\subseteq R} \Delta_v(Q) - \sum_{Q\subseteq T} \Delta_v(T) \\ = \ & \sum_{\substack{Q \subseteq (R\cup T) \\ R\cap Q \neq \emptyset, T\cap Q\neq\emptyset}} \Delta_v(Q) + \sum_{Q\subseteq T} \Delta_v(Q) + \sum_{Q\subseteq R} \Delta_v(Q) - \sum_{Q\subseteq R} \Delta_v(Q) - \sum_{Q\subseteq T} \Delta_v(T) \\ \underset{\substack{(IH_1) \\ (IH_2)}}{=} & \Delta_v(R \cup T), \end{aligned}$$

and (21), and therefore, Lemma 3.12 is shown. □

Remark 3.13. *Let $N \in \mathcal{N}$ and $(N,v) \in \mathbb{V}(N)$. By Lemma 3.12, each coalition $S \in \Omega^N$ in a unanimity game u_S is a dummy coalition. Since Sh satisfies* ***A*** *and* ***N****, and by (2), it is obvious that Sh also satisfies* ***DC****.*

By Lemma 3.12, all coalitions $T \in \Omega^{\overline{N}}$, $T = \bigcup_{i \in S}\{i_S\}$, $S \in \Omega^N$, in the proof of Theorem 3.9 are dummy coalitions in $(\overline{N}, \overline{v})$. Since, by Remark 3.10, **DC** implies **E**, and, by **DC**, **E**, and (19), $\varphi_k(\overline{N}, \overline{v})$ is unique for all $k \in T$ if φ satisfies **MP** and **DC**. By (20), the proof of Theorem 3.9, and Remark 3.13, the following compact axiomatization is obvious.

Corollary 3.14. *Sh is the unique TU-value that satisfies **DC** and **MP**.*

4 Modification of behavior.

While the previous section focuses on a reduction of the set of players, here, we consider games in which the behavior of two players towards each other changes compared to the initial game. For example, suppose two players would suddenly quarrel and end their previous cooperation in all coalitions that contain both players. In the 2-player case, we would then let the two players keep their stand-alone worth and the worth of the grand coalition is the sum of those two worths. We define the general case.

Definition 4.1. *For all $N \in \mathcal{N}, (N, v) \in \mathbb{V}(N)$, and two players $i, j \in N$, $i \neq j$, we define the (i, j)-**disjointly productive game** (N, v_{ij}^{dp}) to (N, v) by*

$$v_{ij}^{dp}(S) := \begin{cases} v(S\backslash\{i\}) + v(S\backslash\{j\}) - v(S\backslash\{i, j\}), & \text{if } \{i, j\} \subseteq S, \\ v(S), & \text{otherwise,} \end{cases} \quad \text{for all } S \subseteq N. \tag{22}$$

Remark 4.2. *By (22) and Definition 3.1, i, j are disjointly productive in (N, v_{ij}^{dp}).*

In an (i, j)-disjointly productive game, all coalitions containing both players i, j are inessential and all other coalitions have the same dividend as in the initial game.

Lemma 4.3. *Let $N \in \mathcal{N}, (N, v) \in \mathbb{V}(N)$, $i, j \in N$, and (N, v_{ij}^{dp}) be the (i, j)-disjointly productive game to (N, v). Then, we have*

$$\Delta_{v_{ij}^{dp}}(S) := \begin{cases} 0, & \text{if } \{i, j\} \subseteq S, \\ \Delta_v(S), & \text{otherwise,} \end{cases} \quad \text{for all } S \subseteq N.$$

Proof. Let $N \in \mathcal{N}, (N, v) \in \mathbb{V}(N)$, $i, j \in N$, and (N, v_{ij}^{dp}) be the (i, j)-disjointly productive game to (N, v). For all $S \subseteq N$, $\{i, j\} \not\subseteq S$, it is obvious that we have, by $v_{ij}^{dp}(S) = v(S)$ and (1), $\Delta_{v_{ij}^{dp}}(S) = \Delta_v(S)$. If we have $\{i, j\} \subseteq S$, immediately follows, by Remark 4.2 and Lemma 3.3, $\Delta_{v_{ij}^{dp}}(S) = 0$. □

Now, we consider the reverse case, where two players get along so well that they only want to do everything together. In the 2-player case, we would then have the stand-alone worths of the two players equal to zero, and the worth of the grand coalition is just the initial cooperation benefit. Again, we define the general case.

Definition 4.4. *For all $N \in \mathcal{N}, (N, v) \in \mathbb{V}(N)$, and two players $i, j \in N$, $i \neq j$, we define the (i, j)-**mutually dependent game** (N, v_{ij}^{md}) to (N, v) by*

$$v_{ij}^{md}(S) := \begin{cases} v(S) - MC_i^v(S\backslash\{j\}) - MC_j^v(S\backslash\{i\}), \ \text{if } \{i, j\} \subseteq S, \ S \subseteq N, \\ v(S\backslash\{k\}), \ \text{if } S \subseteq (N\backslash\{i, j\}) \cup \{k\}, \ k \in \{i, j\}, \\ v(S), \ \text{if } S \subseteq N\backslash\{i, j\}. \end{cases} \tag{23}$$

In other words, we subtract the marginal contributions of each player to the coalition that does not contain the other from the worth of the coalition that contains both players, and any coalition containing only one of the mutually dependent players has the same worth as the coalition without that player.

Remark 4.5. *By (23) and (3), i, j are mutually dependent in (N, v_{ij}^{md}).*

In an (i, j)-mutually dependent game, all coalitions containing only one of the two players i, j are inessential and all other coalitions have the same dividend as in the initial game.

Lemma 4.6. *Let $N \in \mathcal{N}, (N, v) \in \mathbb{V}(N)$, $i, j \in N$, and (N, v_{ij}^{md}) be the (i, j)-mutually dependent game to (N, v). Then, we have, for all $S \subseteq N$,*

$$\Delta_{v_{ij}^{md}}(S) := \begin{cases} 0 \ \textit{if}\ k \in S,\ S \subsetneq (N \backslash \{i, j\}) \cup \{k\},\ k \in \{i, j\}, \\ \Delta_v(S), \ \textit{otherwise.} \end{cases}$$

Proof. Let $N \in \mathcal{N}, (N, v) \in \mathbb{V}(N)$, $i, j \in N$, and (N, v_{ij}^{md}) be the (i, j)-mutually dependent game to (N, v). For all $S \subseteq N\backslash\{i, j\}$, it is obvious that we have, by $v_{ij}^{md}(S) = v(S)$ and (1),

$$\Delta_{v_{ij}^{dp}}(S) = \Delta_v(S). \tag{24}$$

If $S \subsetneq (N\backslash\{i, j\}) \cup \{k\}$, $k \in \{i, j\}$, immediately follows, by Remark 4.5 and (4),

$$\Delta_{v_{ij}^{md}}(S) = 0. \tag{25}$$

Let now $S \subseteq N$, $\{i, j\} \subseteq S$. We show that

$$\Delta_{v_{ij}^{dp}}(S) = \Delta_v(S) \tag{26}$$

by induction on the size $s := |S|$, $2 \leq s \leq |N|$.

Initialization: Let $s = 2$ and, therefore, $S = \{i, j\}$. We have

$$\Delta_{v_{ij}^{md}}(\{i, j\}) \underset{\substack{(1)\\(23)}}{=} v(\{i, j\}) - v(\{j\}) - v(\{i\}) + 2 \cdot v(\emptyset) - \sum_{R \subsetneq S} \Delta_{v_{ij}^{md}}(R) \underset{(25)}{=} \Delta_v(\{i, j\}).$$

Induction step: Let $s \geq 3$. Assume that (26) is satisfied for all $s', s' < s$, (IH). We have

$$\begin{aligned} \Delta_{v_{ij}^{md}}(S) &\underset{\substack{(1)\\(23)}}{=} v(S) - v(S\backslash\{i\}) - v(S\backslash\{j\}) + 2 \cdot v(S\backslash\{i, j\}) - \sum_{R \subsetneq S} \Delta_{v_{ij}^{md}}(R) \\ &= \sum_{R \subseteq S} \Delta_v(R) - \sum_{R \subseteq S\backslash\{i\}} \Delta_v(R) - \sum_{R \subseteq S\backslash\{j\}} \Delta_v(R) + 2 \cdot \sum_{R \subseteq S\backslash\{i,j\}} \Delta_v(R) - \sum_{R \subsetneq S} \Delta_{v_{ij}^{md}}(R) \\ &= \sum_{R \subseteq S,\, \{i,j\} \subseteq R} \Delta_v(S) + \sum_{R \subseteq S\backslash\{i,j\}} \Delta_v(R) - \sum_{R \subsetneq S} \Delta_{v_{ij}^{md}}(R) \underset{\substack{(IH)\\(24),(25)}}{=} \Delta_v(S), \end{aligned}$$

and Lemma 4.6 is shown. □

The following axiom states that the payoff to any two players i, j in a game is uniquely determined by the payoffs to those two players from the associated (i, j)-disjointly productive and (i, j)-mutually dependent game.

Modification of behavior, MoB. For all $(N,v) \in \mathbb{V}(N)$, $i,j \in N$, the (i,j)-disjointly productive game (N, v_{ij}^{dp}), and the (i,j)-mutually dependent game (N, v_{ij}^{md}) to (N,v), we have $\varphi_k(N,v) = \varphi_k(N, v_{ij}^{dp}) + \varphi_k(N, v_{ij}^{md})$, $k \in \{i,j\}$.

For this axiom, consider for two players, e.g., games played multiple times in which the other players do not change their behavior. If these two players now stop cooperating in all coalitions in which they both participate, they must additionally play another game in which they cooperate only in the coalitions in which they both participate, to the same extent as in the initial game, to receive the same total payoff as before.

Note that modification of behavior is not a weakening of additivity since, in general, we have $v \neq v_{ij}^{dp} + v_{ij}^{md}$. It follows a new axiomatization of the Shapley value.

Theorem 4.7. *Sh is the unique TU-value that satisfies* ***E****,* ***DP****,* ***S****, and* ***MoB****.*

Proof. I. Existence: It is well-known that Sh satisfies **E** and **S**. By Theorem 3.9, Sh satisfies **DP**, and, by (5), Lemma 4.3, and Lemma 4.6, it is obvious that Sh satisfies **MoB**.
II. Uniqueness: Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and φ be a TU-value that satisfies all axioms of Theorem 4.7. If $|N| = 1$, φ is unique by **E**.

Let now, w.l.o.g., $N = \{1, 2, ..., n\}$, $n \geq 2$. In the following, we replace the game (N,v) by the $(1,2)$-mutually dependent and the $(1,2)$-disjointly productive game, these new games, in the next step, by the $(1,3)$-mutually dependent and the $(1,3)$-disjointly productive games of the new games and so on. In the last and $(n-1)$th step, we have 2^{n-2} $(1,n)$-mutually dependent games and 2^{n-2} $(1,n)$-disjointly productive games from the $(n-2)$th step, in total 2^{n-1} games.

For the $(k-1)$th step, $2 \leq k \leq n$, let the (R,Q)-game, $R \subseteq N \backslash \{1, k+1, ..., n\}$, $Q \subseteq (N \backslash \{1, k+1, ..., n\}) \backslash R$, be the game which is obtained by replacing through a $(1,i)$-mutually dependent game in the $(i-1)$th step for all $i \in R$ and through a $(1,j)$-jointly productive game in the $(j-1)$th step for all $j \in Q$.

In the first step, we replace (N,v) by the related games $(N, v_{1,2}^{md})$ and $(N, v_{1,2}^{dp})$. By Remark 4.5, in the $(1,2)$-mutually dependent game, players 1 and 2 are mutually dependent, and, by Remark 4.2, in the $(1,2)$-disjointly productive game, players 1 and 2 are disjointly productive. For each further step, we obtain an analogous result which can be easily shown by induction on the size k, $2 \leq k \leq n$: in the $(k-1)$th step, in each (R,Q)-game, all players $i \in R \cup \{1\}$ are mutually dependent and all $j \in Q$ are disjointly productive with player 1.

In the $(n-1)$th step, we have $R \cup Q \cup \{i\} = N$. If $R = \emptyset$, all players $i \in N \backslash \{1\}$ are disjointly productive with player 1 and for this game, by **DP** and **E**, φ is unique for player 1. Otherwise, all players $i \in T$, $T := R \cup \{1\}$, are mutually dependent and all players $j \in Q = N \backslash T$, are disjointly productive with the player 1. By **DP**, we can remove all players $j \in Q$ and, by **S** and **E**, φ is unique for player 1 in these games. Using **MoB** repeatedly, it holds that the sum of the payoffs to player 1 in all final games is equal to 1's payoff in the game (N,v) and the proof is complete. □

Remark 4.8. *Similar to Corollary 3.14, for a new corollary, in Theorem 4.7, we could replace* ***E*** *and* ***DP*** *by* ***DC****.*

Nowak and Radzik (1995) introduced for a weight system ω an axiom, called ω-mutual dependence. If we remove the weights or use only equal weights respectively, we finally end up with the following property. The obvious term 'mutual dependence' is not applicable since this expression is already occupied by another axiom in Nowak and Radzik (1995).

Joint productivity, JP. For all $(N, v) \in \mathbb{V}(N)$ and two mutually dependent players $i, j \in N$ in (N, v), we have $\varphi_i(N, v) = \varphi_j(N, v)$.

This property means that players who are only jointly productive should also receive the same payoff. Since mutually dependent players are always symmetric but not vice versa, this axiom can be seen as a weakening of symmetry. As the proof of Theorem 4.7 shows, we can replace symmetry by the weaker joint productivity property and obtain an axiomatization of the Shapley value where, besides efficiency, only properties based on jointly and disjointly productive players are crucial.

Corollary 4.9. *Sh is the unique TU-value that satisfies **E**, **DP**, **JP**, and **MoB**.*

5 Additivity for a disjointly productive game and its complement

This section is closely related to the previous one. Shapley (1953b) introduced the Shapley value with an axiomatization using efficiency, linearity and a carrier axiom. Nowadays, the following version of this axiomatization is common.

Theorem 5.1. *(Shapley, 1953b) Sh is the unique TU-value that satisfies **E**, **N**, **S**, and **A**.*

Often, in this axiomatization, the null player property is also replaced by the stronger dummy player property. We next introduce a weakening of additivity.

Disjointly productive game additivity, DPA. For all $(N, v) \in \mathbb{V}(N)$, $i, j \in N$, and the (i, j)-disjointly productive game (N, v_{ij}^{dp}) to (N, v), we have

$$\varphi(N, v) = \varphi(N, v_{ij}^{dp}) + \varphi(N, v - v_{ij}^{dp}).$$

.

Remark 5.2. *Let $N \in \mathcal{N}, (N, v) \in \mathbb{V}(N)$, $i, j \in N$, and (N, v_{ij}^{dp}) be the (i, j)-disjointly productive game to (N, v). By Lemma 4.3 and (2), we have for the **complement game** $v_{ij}^{cdp} := v - v_{ij}^{dp}$*

$$\Delta_{v_{ij}^{cdp}}(S) := \begin{cases} \Delta_v(S) \text{ if } \{i, j\} \subseteq S,\ S \subseteq N, \\ 0, \text{ otherwise.} \end{cases}$$

The following is an alternative to Theorem 5.1 in which the additivity is weakened and the null player property is strengthened.

Theorem 5.3. *Sh is the unique TU-value that satisfies **E**, **D**, **S**, and **DPA**.*

Proof. I. Existence: Since, obviously, **DPA** is implied by **A**, existence follows immediately by Theorem 5.1.

II. Uniqueness: The proof is similar to the proof of Theorem 4.7. Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and φ be a TU-value that satisfies all axioms of Theorem 5.3. If $|N| = 1$, φ is unique by **E**.

Let now, w.l.o.g., $N = \{1, 2, ..., n\}$, $n \geq 2$. In the following, we replace the game (N,v) by the complement game to the $(1, 2)$-disjointly productive game and the $(1, 2)$-disjointly productive game, these new games, in the next step, by the the complement games to the $(1,3)$-disjointly productive games and the $(1,3)$-disjointly productive games of the new games and so on. In the last and $(n-1)$th step, we have 2^{n-2} complement games to the $(1,n)$-disjointly productive games and 2^{n-2} $(1,n)$-disjointly productive games from the $(n-2)$th step, in total 2^{n-1} games.

For the $(k-1)$th step, $2 \leq k \leq n$, let the (R,Q)-game, $R \subseteq N\backslash\{1, k+1, ..., n\}$, $Q \subseteq (N\backslash\{1, k+1, ..., n\})\backslash R$, be the game which is obtained by replacing through a complement game to the $(1,i)$-disjointly productive game in the $(i-1)$th step for all $i \in R$ and through a $(1,j)$-disjointly productive game in the $(j-1)$th step for all $j \in Q$.

In the first step, we replace (N,v) by the related games $(N, v - v_{1,2}^{dp})$ and $(N, v_{1,2}^{dp})$. By Remark 5.2 and (4), in the complement game to the $(1,2)$-disjointly productive game, players 1 and 2 are mutually dependent and all coalitions not containg both players i and j are inessential, and, by Remark 4.2, in the $(1,2)$-disjointly productive game, players 1 and 2 are disjointly productive. For each further step, we obtain an analogous result which can be easily shown by induction on the size k, $2 \leq k \leq n$: in the $(k-1)$th step, in each (R,Q)-game, all players $i \in R \cup \{1\}$ are mutually dependent, all coalitions not containing all players $i \in R \cup \{1\}$ are inessential and all $j \in Q$ are disjointly productive with player 1.

In the $(n-1)$th step, we have $R \cup Q \cup \{i\} = N$. If $R = \emptyset$, all players $i \in N\backslash\{1\}$ are disjointly productive with player 1 and for this game, by **D**, φ is unique for player 1. Otherwise, all players $i \in T$, $T := R \cup \{1\}$, are mutually dependent and all players $j \in Q = N\backslash T$, are null players. All players $i \in T$ are symmetric in these games and, by **D**, **E**, and **S**, φ is unique for player 1 in these games too. Using **DPA** repeatedly, it holds that the sum of the payoffs to player 1 in all final games is equal to 1's payoff in the game (N,v) and the proof is complete. □

Remark 5.4. *As the proof shows, in Theorem 5.3,* ***S*** *could also be replaced by* ***JP****. That we cannot replace* ***D*** *with the weaker* ***N*** *shows the following TU-value ϕ, which satisfies* ***E****,* ***N****,* ***S****, and* ***DPA*** *but not* ***D*** *and is given by*

$$\phi_i(N,v) := \frac{\sum_{j \in N, v(\{j\}) \neq 0} v(\{j\})}{|\{j \in N : v(\{j\}) \neq 0\}|} + \sum_{\substack{S \subseteq N, S \ni i, \\ |S| \geq 2}} \frac{\Delta_v(S)}{|S|}, \text{ for all } i \in N. \tag{27}$$

6 Discussion and a note on the complexity of computing the Shapley value

The disjointly productive players property can also be understood as 'loyalty' to a group of players. If a player is disjointly productive with all players outside a group of players, the player is loyal to that group in the sense that the player does not engage in productive activities outside the group. Therefore, a loyal player's payoff is determined by the payoff on a subgame on his/her group.

Grabisch and Roubens (1999) refer to two players as 'acting independently' when their joint cooperation gain is zero. We deliberately use the term 'disjointly productive' because our concern in forming coalitions with other players is not so much to be independent of each other but to express that we are not able to make cooperative gains with that player. For example, the term 'mutually independent players' is used in Hou et al. (2018) for probability games in a completely different meaning. Although we find the term 'jointly productive' more appropriate than 'mutually dependent' for the purposes of our research, we stick with 'mutually dependent' because the term is now well established.

We would like to point out that we consider productivity to be a very important perspective within the cooperative game theory, which, in our opinion, apart from Grabisch and Roubens (1999), has perhaps been too little perceived as such in all its facets so far. For example, null players are 'non-productive players,' dummy players are players who are 'only for themselves productive players,' a dummy coalition is a group of players who are 'only within the group productive players,' and symmetric players are 'equally productive players.'

The merged players game property, when interpreted as a 'split into disjointly productive players property,' means that if a player splits into several other disjointly productive players, the payoff to the players non-involved should not change. In this age of ever-increasing online activity, it is often impossible for participants to know whether different user accounts always trace back to different users. If people in various groups cooperate to different degrees and the resulting cooperation gains are to be distributed, it should not matter under a fair solution concept whether a person participates with only one account or with multiple accounts as long as he or she makes the same total contribution overall. However, these multiple accounts of a single individual may just be considered disjointly productive since this participant does not generate coalitional gains only with him/herself. Therefore, satisfying our merged players game property seems desirable for a fair solution concept in this regard.

Just as an aside, it should be mentioned that of our new properties, the Banzhaf value satisfies the disjointly productive players, the modification of behavior, and the joint productivity property, and the disjointly productive game additivity. The merged players game and the dummy coalition property are not satisfied.

Due to the steadily increasing use of artificial intelligence and machine learning, the cooperative game theory is gaining importance. This is especially true for the Shapley value (see, e.g., Štrumbelj and Kononenko (2014), Takeishi (2019), and Rodríguez-Pérez and Bajorath (2020)). Often, different (input) features are used as players and the payoffs are calculated via the different interactions or effects of the features among each other using the Shapley value. For reasons of complexity, approximation methods for the Shapley value must already be used for relatively small sets of players. Nevertheless, if the players have certain structures or properties among them, it may be possible to use exact algorithms.

It is well-known that null players and dummy players can be easily removed from a (general) game to calculate the Shapley value. If we know that certain features or players do not influence each other, i.e., they are disjointly productive, two of our new axioms come into play. Due to the disjointly productive players property, the payoff to the other disjointly productive players does not change when we remove a disjointly productive player. Especially, dummy coalitions can be removed from the game, reducing the complexity of the calculation for both the remaining players and the players within the dummy coalition. By the merged players property, we can simply merge all mutually disjointly productive players into one player and then compute the payoffs to the others with less complexity.

With a simple trick, mutually dependent players can also be valuable for calculating the Shapley value. If we have a group of such only jointly productive players, we combine them into a single player. For the payoff calculation, we then have to apply a weighted Shapley value, where the merged player, e.g., gets as weight the number of merged players and all others keep a weight of one (see Kalai and Samet (1987), Corollary 2). If we reduce the number of players needed for a calculation, the accuracy of the results obtained by approximation methods also improves.

7 Appendix

We show the logical independence of the axioms in our main results.

Remark 7.1. *The axioms in Theorem 3.9 are logically independent:*

- ***E***: *The TU-value* $\varphi := 2Sh$ *satisfies* ***DP*** *and* ***MP*** *but not* ***E***.
- ***DP***: *The TU-value* ϕ, *defined for all* $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}^N$, *by*

$$\phi_i(N,v) := \begin{cases} \dfrac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N), & \sum_{j\in N} v(\{j\}) \neq 0, \\ Sh_i(N,v), & \text{otherwise}, \end{cases} \quad \text{for all } i \in N.$$

  satisfies ***E*** *and* ***MP*** *but not* ***DP***.

- ***MP***: *The TU-values* ϕ^c, *defined in Besner (2020) for all* $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}^N$, *and all* $c > 0$, *by*

$$\phi_i^c(N,v) := \sum_{S\subseteq N,\, S\ni i} \frac{|v(\{i\})| + c}{\sum_{j\in S}(|v(\{j\})| + c)} \Delta_v(S) \quad \text{for all } i \in N. \tag{28}$$

  satisfy ***E*** *and* ***DP*** *but not* ***MP***.

Remark 7.2. *The axioms in Theorem 4.7 and Theorem 5.3 are logically independent:*

- ***E***: *The TU-value* $\varphi := 2Sh$ *satisfies* ***DP/D***, ***S***, *and* ***MoB/DPA*** *but not* ***E***.
- ***DP/D***: *The TU-value* ϕ, *defined by (27), satisfies* ***E***, ***S***, *and* ***MoB/DPA*** *but not* ***DP/D***.
- ***S***: *Let* $W := \{f : \mathfrak{U} \to \mathbb{R}_{++}\}$, $w_i := w(i)$ *for all* $w \in W$, $i \in \mathfrak{U}$. *The (positively) weighted Shapley values* Sh^w, *given by*

$$Sh_i^w(N,v) := \sum_{S\subseteq N, S\in i} \frac{w_i}{\sum_{j\in S} w_j} \Delta_v(S) \text{ for all } i \in N,$$

  satisfy ***E***, ***DP/D***, *and* ***MoB/DPA*** *but not* ***S*** *in general.*

- ***MoB/DPA***: *The TU-values* ϕ^c, *given by (28), satisfy* ***E***, ***DP/D***, *and* ***S*** *but not* ***MoB/DPA***.

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