Vasicek Model Extension. Premature default

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Vasicek Model Extension. Premature default

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Abstract

The IRB approach underlies Basel II and Basel III. The approach is based on the Vasicek distribution. The main advantage of the distribution is simplicity and accounting for default correlation. But the distribution substantially underestimates probability of default due to ignoring of premature defaults. Besides, the IRB approach uses the maturity adjustment, which is a kind of a black box, since there is no clear information about the econometric model and calibration of its parameters. If maturity exceeds one year, the IRB formula leads to negativity and even discontinuity of capital in the neighborhood of zero default probability.

The paper suggests the Vasicek-Black-Cox (VBC) model, which is constructed to fix drawbacks of the IRB approach. The VBC model is constructed on the base of the Vasicek model and the Black-Cox model. The Vasicek model is a special case of the VBC model, designed to evaluate the default distribution taking into account premature defaults.

The VBC model was constructed using the Method of Images, since the firm in the framework of the Black-Cox model is treated as the barrier binary option.

Keywords: IRB; Vasicek; Merton; Black-Cox; barrier options; default distribution

1. Introduction

The IRB approach is based on the Vasicek default distribution. The main advantage of the distribution is its simplicity and taking into account of defaults correlation. But the distribution underestimates probability of default due to ignoring of premature defaults. Meanwhile, if a negative equity is formed before the expiration of the loan term, then the process of bankruptcy of the borrower may begin. This drawback is especially significant for relatively long loan terms. Understanding this, the authors of Basel II and Basel III used the Vasicek formula only for one year maturity, and to account for a longer period, they used the maturity adjustment.

The maturity adjustment is a kind of a black box, since there is no clear information about the econometric model and calibration of its parameters. If maturity exceeds one year, the IRB formula leads to negativity and even discontinuity of capital in the neighborhood of zero probability of default.

Paul Kupiec (2009) underlined: “The results show that the VAIRB (Vasicek-Basel IRB – Author) does not capture the variability in Moody’s default data: there are numerous episodes in which obligors default with much greater frequency than predicted”.

The paper suggests the Vasicek-Black-Cox (VBC) model, which was constructed to fix these drawbacks: to take into account premature defaults and to abandon the application of the maturity adjustment.

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The text of the paper is organized as follows. In Review we consider a set of 3 well-known models, on the basis of which our model is built: the Merton model, the Vasicek model, and the Black-Cox model. Also, we consider the drawbacks of the IRB model. In Section 3 we construct the Vasicek-Black-Cox model using the Method of Images. In Section 4 we answer the question about substantiality of adjustments taking into account premature defaults. The final Section concludes.

2. Review

Consider a set of two models, on the basis of which the VBC model is built: the Vasicek model and the Black-Cox model, and their predecessor – the Merton model. Also, we consider the drawbacks of the IRB model.

2.1. Merton model

On the base of the Black-Scholes model Robert Merton (1974) proposed the first structural credit risk model for assessing the probability of default of a firm and valuation of its debt and equity.

The value of assets V_t of the firm obeys the geometric Brownian motion with the trend r (the risk-free rate) and the volatility σ:

$$dV_t = rV_t dt + \sigma V_t dW_t,$$

where W_t is the Wiener process. Hence

$$V_t = V_0 e^{\nu t + \sigma W_t}$$

where

$$\nu = r - \frac{\sigma^2}{2}$$

Assume $\nu > 0$.

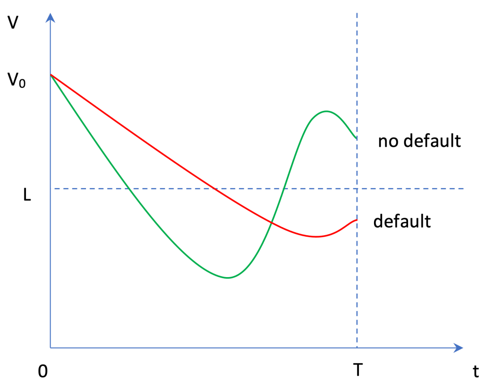

Figure 1. Possible trajectories V_t

The condition of default is $V_T < L$, where T is the maturity of the debt, L is liabilities, V_T is the terminal value of assets. Assume $V_0 > L$. The probability of default

$$PD = \mathbb{P}(V_T < L) = \mathbb{P}\left(W_T < -\frac{x + \nu T}{\sigma}\right) = \Phi(-d_2)$$

where $\Phi(.)$ is the standard normal CDF,

$$d_2 = \frac{ln\left(\frac{V_0}{L}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

is the Distance to Default,

$$x = ln\left(\frac{V_0}{L}\right)$$

2.2. Vasicek distribution

On the base of the Merton model Oldrich Vasicek (1987) constructed the Vasicek distribution, which takes into account a correlation between defaults of individual firms.
Assume

$$W_t = \sqrt{1-\rho}W_t^{(1)} + \sqrt{\rho}W_t^{(2)}$$

where the idiosyncratic shock $W_t^{(1)}$ and the systematic shock $W_t^{(2)}$ are uncorrelated Wiener processes, and ρ is the correlation coefficient. The probability of default conditional on the shock z equals

$$\boldsymbol{PD}(z) = \mathbb{P}\left(W_T < -\frac{x + \nu T}{\sigma} | W_T^{(2)} = \sqrt{T}z\right) = \mathbb{P}\left(W_T^{(1)} < -\frac{\frac{x + \nu T}{\sigma} + \sqrt{\rho}\sqrt{T}z}{\sqrt{1-\rho}}\right)$$

$$= \Phi\big(-d_2(z)\big) = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right)$$

where the distance to default conditional on the shock z equals

$$d_2(z) = \frac{d_2 + \sqrt{\rho}z}{\sqrt{1-\rho}}$$

Note that on the base of the Merton model we can easily construct the Vasicek model just changing d_2 with $d_2(z)$.

The equality holds

$$\mathbb{E}\big(\boldsymbol{PD}(Z)\big) = PD$$

since

$$\int_{-\infty}^{\infty} \Phi\big(-d_2(z)\big)d\Phi(\mathrm{z}) = \Phi(-d_2) = PD$$

due to the well-known equality

$$\int_{-\infty}^{\infty} \Phi(a + bx)d\Phi(x) = \Phi\left(\frac{a}{\sqrt{1+b^2}}\right)$$

which follows from simple probabilistic considerations:

$$\int_{-\infty}^{\infty} \Phi(a+bx)d\Phi(x) = \iint_{-\infty}^{\infty} d\Phi(x)d\Phi(y)\, \mathbb{I}_{\{y\le a+bx\}} = \mathbb{P}_Z(Z \le a)$$

where $X, Y \sim N(0,1)$, $Z = Y - bX$, $Z \sim N(0,1+b^2)$.

Consider a portfolio of bonds issued by an infinite number of such firms. The expected share of defaults in a portfolio conditional on the shock z is $\boldsymbol{PD}(z)$. Since $Z \sim N(0,1)$, we get the Vasicek CDF:

$$F_V(x) = \mathbb{P}(\boldsymbol{PD}(Z) \le x) = \mathbb{P}\left(\Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right) \le x\right)$$
$$= \mathbb{P}\left(Z \ge \frac{\Phi^{-1}(PD) - \sqrt{1-\rho}\Phi^{-1}(x)}{\sqrt{\rho}}\right) = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right)$$

2.3. IRB model

The Vasicek model underlies the IRB approach (BIS, 2005) of Basel II and Basel III. The IRB model is the mix of 3 components:

(1) ab initio Vasicek CDF;
(2) empiric parameter LGD (Loss Given Default);
(3) maturity adjustment constructed on the base of some uncovered econometric calculations.

The IRB formula is

$$K = \frac{1 + (T - 2{,}5) * b(PD_1)}{1 - 1{,}5 * b(PD_1)} * LGD_1 * \left(\Phi\left(\frac{\sqrt{\rho}\Phi^{-1}(0{,}999) + \Phi^{-1}(PD_1)}{\sqrt{1-\rho}}\right) - PD_1\right)$$

where
K – capital requirement,
$PD_1 - PD$ at maturity $T = 1$,
$LGD_1 - LGD$ at maturity $T = 1$,
the maturity adjustment

$$\frac{1 + (T - 2{,}5) * b(PD_1)}{1 - 1{,}5 * b(PD_1)}$$

$$b(PD) = (0{,}11852 - 0{,}05478 * ln(PD))^2$$

The correlation ρ is considered as a function of PD_1, but not as an independent parameter. The maturity adjustment has discontinuity when $b(PD_1) = \frac{2}{3} \Leftrightarrow PD_1 \approx 2.927 \cdot 10^{-6}$ and it is negative if $0 < PD_1 < 2.927 \cdot 10^{-6}$ for all $T > 1$.
The maturity adjustment is “a black box”: there is no clear information how this adjustment was constructed and calibrated except the remark “The actual form of the Basel maturity adjustments has been derived by applying a specific MtM credit risk model, similar to the KMV Portfolio ManagerTM, in a Basel consistent way. This model has been fed with the same bank target solvency (confidence level) and the same asset correlations as used in the Basel ASRF model” (BIS, 2005, Note 4.6).

2.4. Black-Cox model

Black and Cox (1976) developed the Merton model to the case of premature default. Define

$$M_t = \min_{0 \le s \le t} V_s$$

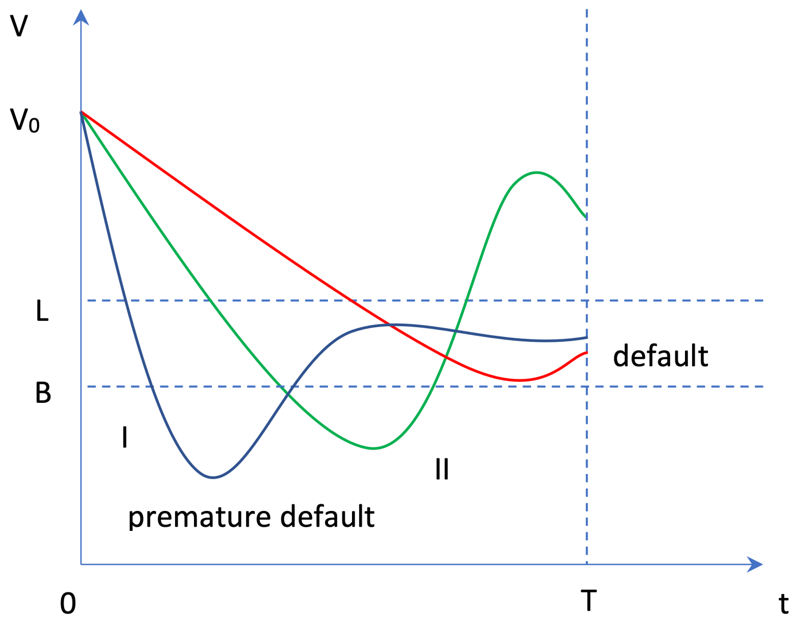

Figure 2. Figure 1. Possible trajectories V_t

Let's expand the concept of default: it happens not only if $V_T < L$, but also if $M_T < B$, where $B < L$ is the barrier. Now default is the event

$$\boldsymbol{D} = V_T < L \vee M_T < B$$

Due to De Morgan's law

$$\mathbb{P}(\boldsymbol{D}) = \mathbb{P}(V_T < L \vee M_T < B) = 1 - \mathbb{P}(V_T > L \wedge M_T > B)$$

Since

$$\mathbb{P}(V_T > L \wedge M_T > B) = \mathbb{P}(V_T > L) - \mathbb{P}(V_T > L \wedge M_T < B)$$

$$\mathbb{P}(\boldsymbol{D}) = 1 - \mathbb{P}(V_T > L) + \mathbb{P}(\mathfrak{B}) = \mathbb{P}(V_T < L) + \mathbb{P}(\boldsymbol{D_B})$$

where the event

$$\boldsymbol{D_B} = V_T > L \wedge M_T < B$$

MoI (Method of Images (Buchen, 2012)) allows to get from the probability of default

$$PD = \Phi(-d_2)$$

probability

$$\mathbb{P}(\boldsymbol{D_B}) = (B/V_0)^\alpha \Phi(\bar{d}_2)$$

where the image distance to default

$$\bar{d}_2 = \frac{ln\left(\frac{B^2}{LV_0}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$\alpha = \frac{r - \frac{\sigma^2}{2}}{\frac{\sigma^2}{2}}$$

The receipt is simple:

(1) to change the domain to its complement (in this context to change $V_T < L$ with $V_T > L$, what is equivalent to the transition from the European binary down-type option to the up-type option, or just to change the sign before the distance to default from minus to plus);
(2) to change V_0 with $\frac{B^2}{V_0}$, what is equivalent to the transition from d_2 to $\bar{d}_2$;
(3) to multiply the result by $\left(\frac{B}{V_0}\right)^{\alpha}$.

Note that application of these rules to

$$\mathbb{P}(\boldsymbol{D_B}) = \left(\frac{B}{V_0}\right)^{\alpha} \Phi(\bar{d}_2)$$

transforms it back into

$$PD = \Phi(-d_2)$$

Hence

$$\mathbb{P}(\boldsymbol{D}) = \Phi(-d_2) + \left(\frac{B}{V_0}\right)^{\alpha} \Phi(\bar{d}_2)$$

The first term is the probability of default if $V_T < L$. It includes those events, when during period (0,T) the value of assets lowered to the barrier B. The second term is the probability of default if $V_T > L$. The proof of this formula see in Appendix.

Let's look at a few simple properties of this formula.

Lemma 1. The inequality

$$\mathbb{P}(\boldsymbol{D}) \leq 1$$

holds.

Proof. Since

$$\mathbb{P}(\boldsymbol{D}) = \Phi(-d_2) + \left(\frac{B}{V_0}\right)^{\alpha} \Phi(\bar{d}_2)$$

and

$$\Phi(-x) = 1 - \Phi(x)$$

$$\mathbb{P}(\boldsymbol{D}) = 1 - \Phi(d_2) + \left(\frac{B}{V_0}\right)^{\alpha} \Phi(\bar{d}_2) = 1 - \left(\Phi(d_2) - \Phi(\bar{d}_2)\right) - \left(1 - \left(\frac{B}{V_0}\right)^{\alpha}\right)\Phi(\bar{d}_2)$$

Since

$$B < V_0$$

inequality

$$d_2 > \bar{d}_2$$

holds, hence

$$\Phi(d_2) - \Phi(\bar{d}_2) > 0$$

Consequently, the assertion of the lemma holds. ☐

We also need probability of premature default. The calculation is based on valuation of barrier down-and-in binary option, the result is

$$\mathbb{P}(\boldsymbol{D}^*) = \Phi(-d_2^*) + \left(\frac{B}{V_0}\right)^{\alpha} \Phi(\bar{d}_2^*)$$

$$d_2^* = \frac{ln\left(\frac{V_0}{B}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$\bar{d}_2^* = \frac{ln\left(\frac{B}{V_0}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Hence the probability of "mature" default is $\mathbb{P}(\boldsymbol{D}) - \mathbb{P}(\boldsymbol{D}^*)$.
Easy to see that since $B < L < V_0$

$$\mathbb{P}(\boldsymbol{D}) - \mathbb{P}(\boldsymbol{D}^*) > 0$$

3. Vasicek-Black-Cox model

On the base of the Vasicek model and the Black-Cox model we can easily construct the Vasicek-Black-Cox model

$$\mathbb{P}(\boldsymbol{D}|z) = \Phi\big(-d_2(z)\big) + \left(\frac{B}{V_0}\right)^{\alpha} \Phi\left(\bar{d}_2(z)\right)$$

just changing d_2 with $d_2(z)$, and $\bar{d}_2$ with

$$\bar{d}_2(z) = \frac{\bar{d}_2 + \sqrt{\rho}z}{\sqrt{1-\rho}}$$

The proof of this formula see in Appendix.

The Vasicek model is a special case of the Vasicek-Black-Cox model if the barrier $B = 0$.

Also, the equation

$$\mathbb{P}(\boldsymbol{D}^*|z) = \Phi\big(-d_2^*(z)\big) + \left(\frac{B}{V_0}\right)^{\alpha} \Phi\left(\bar{d}_2^*(z)\right)$$

holds, where

$$d_2^*(z) = \frac{d_2^* + \sqrt{\rho}z}{\sqrt{1-\rho}}$$

$$\bar{d}_2^*(z) = \frac{\bar{d}_2^* + \sqrt{\rho} z}{\sqrt{1-\rho}}$$

The equation

$$\int_{-\infty}^{\infty} \mathbb{P}(\boldsymbol{D}|z) d\mathbb{P}(z) = \mathbb{P}(\boldsymbol{D})$$

holds.

Easy to check that

Lemma 2. The inequality

$$\mathbb{P}(\boldsymbol{D}|z) \leq 1$$

holds.

The function has the following properties:

$$\mathbb{P}(\boldsymbol{D}|-\infty) = 1$$

$$\mathbb{P}(\boldsymbol{D}|+\infty) = \left(\frac{B}{V_0}\right)^{\alpha}$$

It has the only minimum at

$$z^* = \underset{z}{\operatorname{argmin}}\, \mathbb{P}(\boldsymbol{D}|z) = \frac{ln\left(\frac{L}{B}\right) - \rho\left(r - \frac{\sigma^2}{2}\right)T}{\sqrt{\rho}\sigma\sqrt{T}}$$

$$\mathbb{P}(\boldsymbol{D}|z^*) = \Phi\left(\frac{-ln\left(\frac{V_0}{B}\right) - \left(r - (1+\alpha\rho)\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}\sqrt{1-\rho}}\right) + \left(\frac{B}{V_0}\right)^{\alpha} \Phi\left(\frac{-ln\left(\frac{V_0}{B}\right) + \left(r - (1+\alpha\rho)\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}\sqrt{1-\rho}}\right)$$

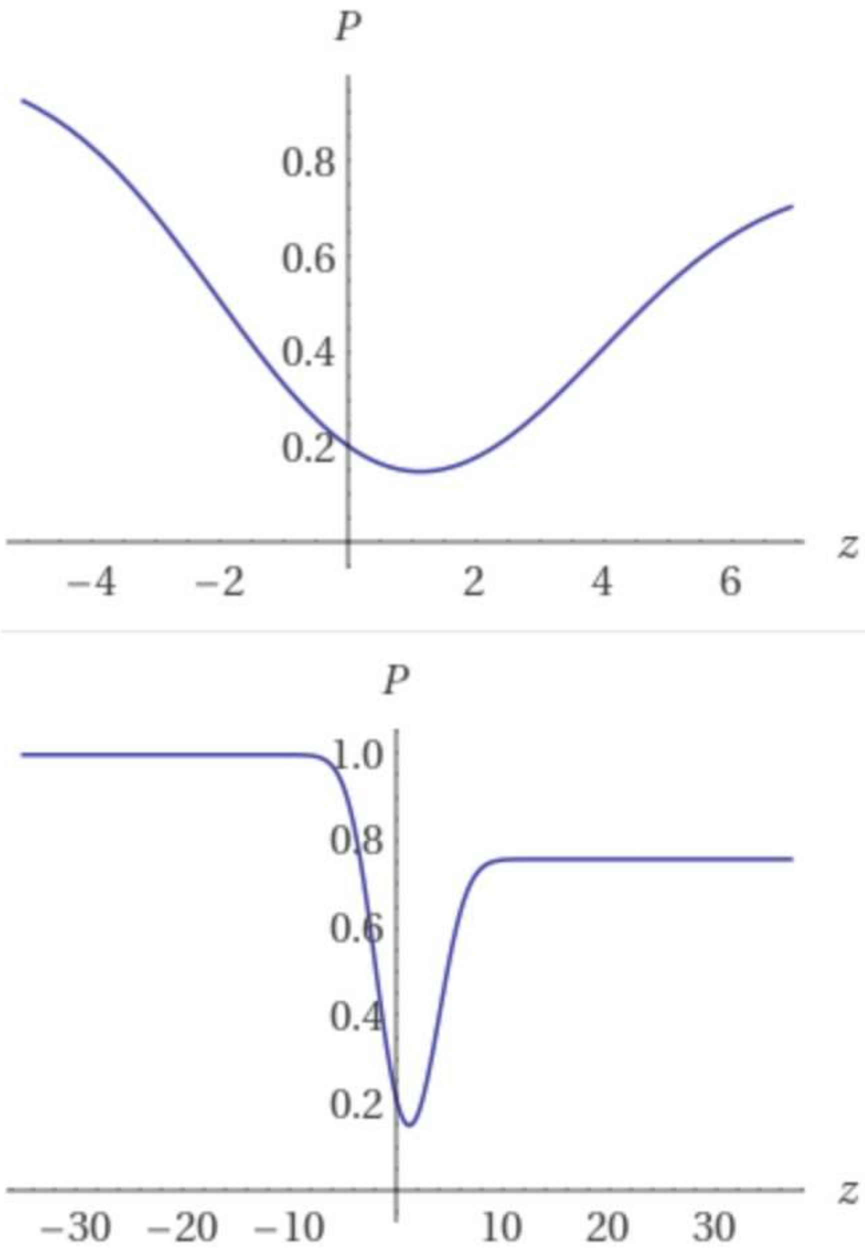

Figure 3. Plots of $\mathbb{P}(\boldsymbol{D}|z)$. Parameters: $\frac{V_0}{B} = 1.2, \frac{V_0}{L} = 1.1, r = 0.05, \sigma = 0.2, \rho = 0.1, T = 1$.

Consider a portfolio of bonds issued by an infinite number of such firms. The expected share of defaults in a portfolio conditional on the shock z is $\mathbb{P}(\boldsymbol{D}|z)$:

$$Loss(z) = \mathbb{P}(\boldsymbol{D}|z)$$

There is no closed form solution $z = z(x)$ of the equation

$$x = \Phi\big(-d_2(z)\big) + \left(\frac{B}{V_0}\right)^{\alpha} \Phi\left(\bar{d}_2(z)\right)$$

but it can be easily found by numerical methods.

Since $Z \sim N(0,1)$, we get the Vasicek-Black-Cox CDF:

If $x < \frac{B}{V_0}$

$$F_{VBC}(x) = \mathbb{P}(Loss(Z) \leq x) = \mathbb{P}\big(z_1(x) \leq Z \leq z_2(x)\big) = \Phi\big(z_2(x)\big) - \Phi\big(z_1(x)\big)$$

where z_1 and z_2 are roots of equation $Loss(z) = x$.

If $x \geq \frac{B}{V_0}$

$$F_{VBC}(x) = \mathbb{P}(Loss(Z) \leq x) = \mathbb{P}\big(Z \geq z(x)\big) = \Phi\big(-z(x)\big)$$

where $z(x)$ is the only root of equation $Loss(z) = x$.

4. Discussion

In this section we investigate the question:
Can we neglect adjustments $\left(\frac{B}{V_0}\right)^{\alpha} \Phi\big(\bar{d}_2\big)$ и $\left(\frac{B}{V_0}\right)^{\alpha} \Phi\left(\bar{d}_2(z)\right)$? These adjustments take into account the defaults that occur due to crossing the barrier if $V_T > L$.

Return to the formula

$$\mathbb{P}(\boldsymbol{D}) = \Phi(-d_2) + \left(\frac{B}{V_0}\right)^{\alpha} \Phi\big(\bar{d}_2\big)$$

Since

$$d_2 = \frac{ln\left(\frac{V_0}{L}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$\bar{d}_2 = \frac{ln\left(\frac{B^2}{LV_0}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$\frac{\partial \mathbb{P}(\boldsymbol{D})}{\partial B} = \frac{\partial \mathbb{P}(\boldsymbol{D_B})}{\partial B} = \frac{\left(\frac{B}{V_0}\right)^{\alpha}}{B}\left(\alpha\Phi\big(\bar{d}_2\big) + \frac{2}{\sigma\sqrt{T}}\varphi\big(\bar{d}_2\big)\right) > 0$$

Hence at any L the maximum of $\frac{\mathbb{P}(\boldsymbol{D_B})}{PD}$ is achieved at point $B = L$.
If $B = V_0$ then default is unavoidable:

$$\mathbb{P}(\boldsymbol{D}) = \Phi(-d_2) + \Phi(d_2) = 1$$

since $\bar{d}_2 = d_2$.

Note that in this case $\mathbb{P}(\boldsymbol{D_B}) > PD$ since $V_0 \geq L$. Moreover,

Lemma 3. If $L = B$, then $\mathbb{P}(\boldsymbol{D_B}) > PD$ for any $V_0 > L$.

Proof. Transform

$$\mathbb{P}(\boldsymbol{D}) = \Phi(-d_2) + \left(\frac{B}{V_0}\right)^{\alpha} \Phi\big(\bar{d}_2\big)$$

into

$$\mathbb{P}(\boldsymbol{D}) = \Phi\left(-\frac{x + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) + e^{-\alpha y}\Phi\left(\frac{-2y + x + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

Let

$$x_1 = \frac{x}{\sigma\sqrt{T}}$$
$$y_1 = \frac{y}{\sigma\sqrt{T}}$$
$$\beta = \frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{T}}{\sigma}$$

Hence

$$\alpha y = \frac{r - \frac{\sigma^2}{2}}{\frac{\sigma^2}{2}}\sigma\sqrt{T}y_1 = 2\frac{r - \frac{\sigma^2}{2}}{\sigma}\sqrt{T}y_1 = 2\beta y_1$$

$$\mathbb{P}(\boldsymbol{D}) = \Phi(-x_1 - \beta) + e^{-2\beta y_1}\Phi(-2y_1 + x_1 + \beta)$$

If $L = B$, then $x_1 = y_1$, hence

$$\mathbb{P}(\boldsymbol{D}) = \Phi(-x_1 - \beta) + e^{-2bx_1}\Phi(-x_1 + \beta)$$

Since

$$\Phi(-x_1 + \beta) = \int_{-\infty}^{-x_1+\beta} \varphi(u)\, du = \int_{-\infty}^{-x_1-\beta} \varphi(u + 2\beta)\, du$$

$$\varphi(u) < e^{-2\beta x_1}\varphi(u + 2\beta)$$

$$e^{-\frac{u^2}{2}} < e^{-2\beta x_1}e^{-\frac{(u+2\beta)^2}{2}} = e^{-\frac{u^2}{2}}e^{-2\beta x_1 - 2\beta u - 2\beta^2}$$

$$u < -x_1 - \beta$$

Consequently, the assertion of the lemma holds. ☐

Let's move on to the formula

$$\mathbb{P}(\boldsymbol{D}|z) = \Phi\big(-d_2(z)\big) + \left(\frac{B}{V_0}\right)^{\alpha}\Phi\left(\bar{d}_2(z)\right)$$

The adjustment $\left(\frac{B}{V_0}\right)^{\alpha}\Phi\left(\bar{d}_2(z)\right)$ makes the largest contribution to the sum for large positive shocks z, while the first term $\Phi\big(-d_2(z)\big)$ – for large negative shocks z. Hence the adjustment should not be neglected if $B > 0$.

The transformation

$$\mathbb{P}(\boldsymbol{D}|z) = \Phi\left(-\frac{x_1 + \beta + \sqrt{\rho}z}{\sqrt{1-\rho}}\right) + e^{-2\beta y_1}\Phi\left(\frac{-2y_1 + x_1 + \beta + \sqrt{\rho}z}{\sqrt{1-\rho}}\right)$$

$$\mathbb{P}(\boldsymbol{D}^*|z) = \Phi\left(-\frac{y_1 + \beta + \sqrt{\rho}z}{\sqrt{1-\rho}}\right) + e^{-2\beta y_1}\Phi\left(\frac{-y_1 + \beta + \sqrt{\rho}z}{\sqrt{1-\rho}}\right)$$

demonstrates that instead of the Vasicek distribution with two parameters $PD = \Phi(-x_1 - \beta)$ and ρ we get two distributions with 4 parameters x_1, y_1, β and ρ.

If $\rho \to 1$, the Vasicek distribution $Vas(PD;\rho)$ degenerates into the Bernoulli distribution $Ber(PD)$, and the Vasicek-Black-Cox distribution $VBC(x_1, y_1, \beta; \rho)$ degenerates into the Bernoulli distribution $Ber(\mathbb{P}(\boldsymbol{D}))$.

5. Conclusion

The Vasicek-Black-Cox model is suggested as an extension of the Vasicek model and as an alternative to the IRB approach. The Vasicek model is a special case of the Vasicek-Black-Cox model.

The Black-Cox formula

$$\mathbb{P}(\boldsymbol{D}) = \Phi(-d_2) + \left(\frac{B}{V_0}\right)^{\alpha} \Phi(\bar{d}_2)$$

allows us to take a fresh look at the Vasicek model. The first term is the probability of default $PD = \Phi(-d_2)$. It includes premature defaults if $V_T < L$. Premature defaults if $V_T > L$ are described by the second term. However, it is not possible in practice to separate premature defaults of the first type from premature defaults of the second type, because in the case of premature default, we usually do not receive information about V_T.

Therefore, we can conclude that the Vasicek formula is based on the false assumption that there are no premature defaults if $V_T > L$.

Moreover, If $L = B$, then the adjustment $\mathbb{P}(\boldsymbol{D_B}) > PD$ for any $V_0 > L$.

Main conclusion of the paper: revision of the IRB approach is necessary. The Vasicek-Black-Cox model is proposed as an alternative.

6. Appendix

6.1. Proof of Black-Cox formula

$$V_t = V_0 e^{\nu t + \sigma W_t}$$

Let's change the process W_t with the process

$$\widetilde{W}_t = \frac{\nu}{\sigma}t + W_t$$

The event

$$\boldsymbol{D_B} = (V_T > L \wedge M_T < B) = \left(\widetilde{W}_T > \frac{x}{\sigma} \wedge \min_{0 \le t \le T} \widetilde{W}_t < \frac{y}{\sigma}\right)$$

where

$$x = ln\left(\frac{V_0}{L}\right)$$

$$y = ln\left(\frac{V_0}{B}\right)$$

$$\mathbb{P}(\boldsymbol{D_B}) = \mathbb{E}\left(\mathbb{I}_{\boldsymbol{D_B}}\right) = \widetilde{\mathbb{E}}\left(exp\left\{-\frac{1}{2}\left(\frac{\nu}{\sigma}\right)^2 T + \frac{\nu}{\sigma}\widetilde{W}_T\right\}\mathbb{I}_{\boldsymbol{D_B}}\right)$$

Now use the Reflection Principle and replace $\widetilde{W}_T$ with $2\frac{y}{\sigma} - \widetilde{W}_T$:

$$\mathbb{P}(\boldsymbol{D_B}) = \widetilde{\mathbb{E}}\left(exp\left\{-\frac{1}{2}\left(\frac{\nu}{\sigma}\right)^2 \mathrm{T} + \frac{\nu}{\sigma}\left(2\frac{y}{\sigma} - \widetilde{W}_T\right)\right\}\mathbb{I}_{2\frac{y}{\sigma}-\widetilde{W}_T > \frac{x}{\sigma}}\right)$$
$$= exp\left\{\frac{2\nu y}{\sigma^2}\right\}\widetilde{\mathbb{E}}\left(exp\left\{-\frac{1}{2}\left(\frac{\nu}{\sigma}\right)^2 \mathrm{T} - \frac{\nu}{\sigma}\widetilde{W}_T\right\}\mathbb{I}_{\widetilde{W}_T < \frac{2y-x}{\sigma}}\right)$$

To get rid of the Radon-Nikodym derivative

$$exp\left\{-\frac{1}{2}\left(\frac{\nu}{\sigma}\right)^2 \mathrm{T} - \frac{\nu}{\sigma}\widetilde{W}_T\right\}$$

go to a new process

$$\widetilde{W}_t^* = \frac{\nu}{\sigma}\mathrm{t} + \widetilde{W}_t$$

$$\mathbb{P}(\boldsymbol{D_B}) = exp\left\{\frac{2\nu y}{\sigma^2}\right\}\widetilde{\mathbb{E}}^*\left(\mathbb{I}_{\widetilde{W}_T^* < \frac{2y-x+\nu T}{\sigma}}\right) = exp\left\{\frac{2\nu y}{\sigma^2}\right\}\Phi\left(\frac{2y - x + \nu T}{\sigma\sqrt{T}}\right)$$

$$\mathbb{P}(\boldsymbol{D}) = \Phi\left(\frac{x - \nu T}{\sigma\sqrt{T}}\right) + exp\left\{\frac{2\nu y}{\sigma^2}\right\}\Phi\left(\frac{2y - x + \nu T}{\sigma\sqrt{T}}\right)$$

Hence the probability of default

$$\mathbb{P}(\boldsymbol{D}) = \Phi(-d_2) + \left(\frac{B}{V_0}\right)^{\alpha}\Phi(\bar{d}_2)$$

where

$$d_2 = \frac{ln\left(\frac{V_0}{L}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$\bar{d}_2 = \frac{ln\left(\frac{B^2}{LV_0}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$\alpha = \frac{r - \frac{\sigma^2}{2}}{\frac{\sigma^2}{2}}$$

6.2. Proof of Vasicek-Black-Cox formula

Assume in the formula

$$\mathbb{P}(\boldsymbol{D_B}) = exp\left\{\frac{2\nu y}{\sigma^2}\right\}\widetilde{\mathbb{E}}^*\left(\mathbb{I}_{\widetilde{W}_T^* < \frac{2y-x+\nu T}{\sigma}}\right)$$

that

$$\widetilde{W}_t^* = \sqrt{1-\rho}\,\widetilde{W}_t^{*(1)} + \sqrt{\rho}\,\widetilde{W}_t^{*(2)}$$

$$\mathbb{P}(\boldsymbol{D_B}|\bar{z}) = exp\left\{\frac{2\nu y}{\sigma^2}\right\}\mathbb{P}\left(\widetilde{W}_t^* < \frac{2y - x + \nu T}{\sigma} | \widetilde{W}_t^{*(2)} = \sqrt{T}\bar{z}\right)$$
$$= exp\left\{\frac{2\nu y}{\sigma^2}\right\}\mathbb{P}\left(\widetilde{W}_t^{*(1)} < \frac{\frac{2y - x + \nu T}{\sigma} - \sqrt{\rho}\sqrt{T}\bar{z}}{\sqrt{1-\rho}}\right) = exp\left\{\frac{2\nu y}{\sigma^2}\right\}\Phi\left(\frac{\bar{d}_2 - \sqrt{\rho}\bar{z}}{\sqrt{1-\rho}}\right)$$

Since $\widetilde{W}_T$ was replaced with $2\frac{y}{\sigma} - \widetilde{W}_T$, $\bar{z} = -z$. Hence

$$\mathbb{P}(\boldsymbol{D_B}|z) = exp\left\{\frac{2\nu y}{\sigma^2}\right\}\Phi\left(\bar{d}_2(z)\right)$$

6.3. List of Notions

L	Liabilities
V_t	Value of assets
T	Maturity of loan
r	Risk free rate
σ	Assets value volatility
ρ	Assets value correlation
B	Barrier
PD	Probability of default
$\Phi(.)$	Standard normal CDF
$\varphi(.)$	Standard normal PDF
LGD	Loss given default
$d_2 = \frac{ln\left(\frac{V_0}{L}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$	Distance to default
$\bar{d}_2 = \frac{ln\left(\frac{B^2}{LV_0}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$	The image distance to default
$d_2(z) = \frac{d_2 + \sqrt{\rho}z}{\sqrt{1-\rho}}$	Distance to default conditional on the shock z
$\bar{d}_2(z) = \frac{\bar{d}_2 + \sqrt{\rho}z}{\sqrt{1-\rho}}$	The image distance to default conditional on the shock z
$x = ln\left(\frac{V_0}{L}\right)$	
$y = ln\left(\frac{V_0}{B}\right)$	
z	Shock
$\alpha = \frac{r - \frac{\sigma^2}{2}}{\frac{\sigma^2}{2}}$	
$x_1 = \frac{x}{\sigma\sqrt{T}}$	
$y_1 = \frac{y}{\sigma\sqrt{T}}$	
$\beta = \frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{T}}{\sigma}$	
$\mathbb{P}(\boldsymbol{D}) = \Phi(-d_2) + \left(\frac{B}{V_0}\right)^{\alpha}\Phi(\bar{d}_2)$	Probability of default

Declarations of Interest

The author reports no conflicts of interest. The author alone is responsible for the content and writing of the paper.

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