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Abstract

The hawk–dove game admits two types of equilibria: an asymmetric pure equilibrium in which players in one population play "hawk" and players in the other population play "dove," and a symmetric mixed equilibrium. The existing literature on dynamic evolutionary models shows that populations will converge to playing one of the asymmetric pure equilibria from any initial state. By contrast, we show that plausible sampling dynamics, in which agents occasionally revise their actions by observing either opponents' behavior or payoffs in a few past interactions, can induce the opposite result: global convergence to a symmetric mixed equilibrium.

Keywords: Chicken game, learning, evolutionary stability, bounded rationality, payoff sampling dynamics, action sampling dynamics. **JEL codes:** C72, C73.

1 Introduction

The hawk–dove game is often applied to study situations of conflict between strategic participants. As a simple motivating example, consider a situation in which a buyer (Player 1) and a seller (Player 2) have to bargain over the price of an asset (e.g., a house). Each player has two possible bargaining strategies (actions): insisting on a more

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Table 1: Payoff Matrix of the Standard Hawk–Dove Game ($g \in (0,1)$)

		Player 2: h_2	Player 2: d_2
Player 1	h_1	$0, 0$	$1+g, 1-g$
	d_1	$1-g, 1+g$	$1, 1$

favorable price (referred to as being a "hawk"), or agreeing to a less favorable price in order to close the deal (being a "dove"). The payoffs of the game are presented in Table 1. Two doves agree on a price that is equally favorable to both sides, and obtain a relatively high payoff, which is normalized to 1. A hawk obtains a favorable price when being matched with a dove, which yields her an additional gain of $g \in (0, 1)$, at the expense of her dovish opponent. Finally, two hawks obtain the lowest payoff of 0, due to a substantial probability of bargaining failure.[1]

The hawk–dove game (also known as the chicken game; see, e.g., Rapoport and Chammah, 1966; Aumann, 1987) has been employed in modeling various strategic situations, such as: provision of public goods (Lipnowski and Maital, 1983), nuclear deterrence between superpowers (Brams and Kilgour, 1987; Dixit et al., 2019), industrial disputes (Bornstein et al., 1997), bargaining problems (Brams and Kilgour, 2001), conflicts between countries over contested territories (Baliga and Sjöström, 2012, 2020), and task allocation among members of a team (Herold and Kuzmics, 2020).

The hawk–dove game admits three Nash equilibria: two asymmetric pure equilibria, and one symmetric mixed equilibrium. In the pure equilibria (in which one of the players plays hawk while the opponent plays dove), all conflicts are avoided at the cost of inequality, as the payoff of the hawkish player is substantially higher than that of the dovish opponent. By contrast, in the symmetric mixed equilibrium both players obtain the same expected payoff, yet this payoff is relatively low due to the positive probability of a conflict arising between two hawks.

A natural question is to ask which equilibrium is more likely to obtain. Standard game theory is not helpful in answering this question, as all these Nash equilibria satisfy all the standard refinements (e.g., perfection). By contrast, the dynamic (evolutionary) approach

[1]Our one-parameter payoff matrix is equivalent to the commonly used two-parameter matrix (Maynard-Smith, 1982), according to which a dove obtains $\frac{V}{2}$ against another dove and 0 against a hawk, and a hawk obtains $\frac{V-C}{2}$ against another hawk and V against a dove. Specifically, our one-parameter matrix is obtained from the two-parameter matrix by the affine transformation of adding the constant $\frac{C-V}{2}$ and dividing all payoffs by $\frac{C}{2}$, followed by substituting $g \equiv \frac{V}{C}$.

can yield sharp predictions (for textbook expositions, see Weibull, 1997; Sandholm, 2010).

Revision dynamics Consider a setup in which pairs of agents from two infinite populations are repeatedly matched at random times (each such match of an agent from population 1 is against a new opponent from population 2).[2] Agents occasionally die (or, alternatively, agents occasionally receive opportunities to revise their actions). New agents observe some information about the aggregate behavior and the payoffs, and use this information to choose the action they will play in all future encounters. We are interested in characterizing the stable rest points of such revision dynamics, which can be used as an equilibrium refinement.

Most existing models assume that the revision dynamics are *monotone* (also known as sign-preserving) with respect to the payoffs: the frequency of the strategy that yields the higher payoff (among the two feasible strategies) increases. A key result in evolutionary game theory is that all monotone (two-population) revision dynamics converge to the asymmetric pure equilibria from almost any initial state (henceforth, *global convergence*; see Maynard-Smith and Parker, 1976, for the classic analysis, Maynard-Smith, 1982, for the textbook presentation, Sugden, 1989, for the economic implications, and Oprea et al., 2011, for the general dynamic result, which, for completeness, is presented in Proposition 1 below.) Thus, the existing literature predicts that an efficient convention will emerge in which trade always occurs and most of the surplus goes to one side of the market. Casual observation suggests that this prediction might not fit well the behavior in situations such as the motivating example, in which the surplus of trade is typically divided relatively equally between the two sides of the market, and in which bargaining frequently fails.

In many applications, precise information about the aggregate behavior in the population may be difficult or costly to obtain. In such situations, new agents have to infer the aggregate behavior in the population from a small sample of other players. In what follows, we present two plausible inference procedures, both of which violate monotonicity. The first procedure is the *action-sampling dynamics* (also known as sampling best-response dynamics; Sandholm, 2001; Osborne and Rubinstein, 2003). In these dynamics, each new agent observes the behavior of k random opponents, and then adopts

[2] Our paper focuses on two-population dynamics. The predictions of one-population dynamics in which agents are matched within a single population and they cannot condition their play on their role in the game are discussed at the end of Section 3.2.

the action that is a best reply to her sample (with an arbitrary tie-breaking rule).

In some applications, new agents may not be able to observe opponents' actions, or they may lack information about the payoff matrix. Plausible revision dynamics in such situations are the *payoff-sampling dynamics* (also known as best experienced payoff dynamics; Osborne and Rubinstein, 1998; Sethi, 2000). In these dynamics, each new agent observes the payoffs obtained by incumbents of her own population in k interactions in which these incumbents played hawk, and in k interactions in which these incumbents played dove. Following these observations, the new agent adopts the action that yielded the higher mean payoff (with an arbitrary tie-breaking rule).

We analyze both sampling dynamics in the hawk–dove game. In our analysis, we allow agents to have heterogeneous sample sizes (i.e., each new agent is endowed with a sample size of k that is randomly chosen from an exogenous distribution). It is simple to show that these dynamics admit at least three rest points (henceforth, equilibria): two asymmetric pure equilibria, and a symmetric mixed equilibrium.[3,4]

Main result and intuition We show that sampling dynamics can yield qualitatively different results compared to monotone dynamics. Specifically, we show that there is a large domain of parameter values for which the payoff-sampling dynamics yield global convergence to the symmetric mixed equilibrium. A similar result holds for the action-sampling dynamics, albeit, in a somewhat narrower domain. Thus, if buyers and sellers have limited information about the aggregate behavior, then an egalitarian, yet inefficient, convention may arise in which bargaining will frequently fail.

We say that an equilibrium is *asymptotically stable* if a population beginning nearby would converge to playing this equilibrium. In what follows, we briefly present the intuition why the sampling dynamics may alter the asymptotic stability of the asymmetric pure equilibria and the symmetric mixed equilibrium.

Consider an asymmetric pure equilibrium in which all buyers play hawk and all sellers play dove. Assume that a small perturbation changes the behavior of $\epsilon << 1$ of the agents in each population. Such a small perturbation does not change the best reply of agents under monotone dynamics. As a result, new buyers (resp., sellers) will play hawk (resp.,

[3]The symmetric mixed equilibrium (i.e., the rest point of the sampling dynamics) has a different mixing probability compared to the symmetric mixed Nash equilibrium.

[4]The dynamics might also admit asymmetric mixed equilibria.

dove) and the population will converge back to the pure equilibrium. By contrast, when a new buyer makes a decision based on a sample of k actions, there is a probability of about $k\epsilon > \epsilon$ that one of these sampled actions is the rare action induced by the perturbation. In some cases (depending on the value of g), a single occurrence of the rare action in the sample is sufficient to change the perceived best reply of the agent. In such cases, the effect of the small perturbation will gradually increase, and the population will move away from the pure equilibrium.

Next consider the symmetric mixed equilibrium in which $p \in (0, 1)$ of the agents in each population plays hawk. Assume that a perturbation slightly increases the shares of hawkish buyers and dovish sellers by ϵ. Under monotone dynamics, the new agents (who were indifferent between the two actions before the perturbation) have a strict best reply: new buyers (resp., sellers) strictly prefer to play hawk (resp., dove). As a result, the small perturbation gradually increases until the population converges to the asymmetric pure equilibrium in which all buyers (resp., sellers) play hawk (resp., dove). By contrast, when new agents base their choices on small samples, their perceived best reply depends on the realized sample. It is still true that the share of new buyers (resp., sellers) with a perceived best reply of playing hawk (resp., dove) increases due to having slightly more dovish sellers (resp., hawkish buyers), but this increase might be smaller than ϵ (e.g., the share of new buyers with a perceived best reply of playing hawk might be $p + 0.9\epsilon$). In this case, the population will gradually converge back to the mixed equilibrium.

Structure and brief summary of results Section 2 presents the related literature. Section 3 formally presents our model. Specifically, it describes a broad class of hawk–dove games, and it defines revision dynamics and various notions of stability. Section 4 characterizes the asymptotic stability of pure equilibria for both sampling dynamics (Theorem 1). The characterization is "complete" in the sense of presenting a simple "iff" condition for asymptotic stability in all generic cases. Section 5 presents interesting necessary and sufficient conditions for the symmetric mixed equilibrium being asymptotically stable under each of the two sampling dynamics (Theorems 2–4). The analytic results of the paper are supplemented by a numeric analysis (Section 6) that shows which parameter values lead to convergence to the symmetric mixed equilibrium, and which lead to convergence to the asymmetric pure equilibria. We conclude in Section 7.

2 Related Literature and Contribution

Related theoretical literature The action-sampling dynamics were pioneered by Sandholm (2001) and Osborne and Rubinstein (2003). Oyama et al. (2015) applied these dynamics to prove global convergence results in supermodular games. Recently, Heller and Mohlin (2018) studied the conditions on the expected sample size that implies global convergence for all payoff functions and all sampling dynamics.

Salant and Cherry (2020) (see also Sawa and Wu, 2021) generalized the action-sampling dynamics by allowing new agents to use various procedures to infer from their samples the aggregate behavior of the opponents. Salant and Cherry pay special attention to *unbiased* inference procedures in which the agent's expected belief about the share of opponents who play hawk coincides with the sample mean. Examples of unbiased procedures are maximum likely estimation, beta estimation with a prior representing complete ignorance, and a truncated normal posterior around the sample mean. In our setup, the payoffs are linear in the share of agents who play hawk, which implies that the agent's perceived best reply depends only on the expectation of her posterior belief. This implies that our results with respect to the action-sampling dynamics hold for any unbiased inference procedure.

The present paper, similar to the papers cited above, studies deterministic dynamics in infinite populations. When there is convergence to a stable equilibrium in such dynamics, the convergence is fast, and the population always remains in this equilibrium's neighborhood (Oyama et al., 2015). By contrast, stochastic evolutionary models (see, e.g., the seminal contribution of Young, 1993, and the recent hawk–dove application in Bilancini et al., 2021), which are also based on revising agents observing a finite sample of opponents' actions, focus on the very long-run behavior of stochastic processes when players' choice rules include the possibility of rare "mistakes."

The payoff-sampling dynamics were pioneered by Osborne and Rubinstein (1998) and Sethi (2000) and later generalized in various respects by Sandholm et al. (2020). It has been used in a variety of applications, including price competition with boundedly rational consumers (Spiegler, 2006), common-pool resources (Cárdenas et al., 2015), contributions to public goods (Mantilla et al., 2018), centipede games (Sandholm et al., 2019), finitely repeated games (Sethi, 2019) and the prisoner's dilemma (Arigapudi et al., 2021). The existing literature assumes that all agents have the same sample size. A methodological contribution of the present paper is in extending the setup of payoff-sampling dynamics to analyze heterogeneous populations in which new agents differ in their sample sizes, and this heterogeneity leads to new results.

Related experimental literature Selten and Chmura (2008) experimentally tested the predictive power of various solution concepts in two-action, two-player games with a unique completely mixed Nash equilibrium. They show that both both the payoff-sampling equilibrium and the action-sampling equilibrium outperform the predictions of both the Nash equilibrium and the quantal-response equilibrium.

Recently, Stephenson (2019) tested the predictive validity of various evolutionary models in coordinated attacker–defender games.[5] Stephenson's experimental design is very favorable for monotone dynamics because each participant is shown the exact (population-dependent) payoff that would be obtained by each action at each point in time. Nevertheless, subjects frequently violate monotonicity: 10%–20% of the subjects switch from higher-performing strategies to lower-performing strategies.

The key prediction of monotone dynamics for hawk–dove games (in which agents from one population are randomly matched with agents from another population) is experimentally tested in Oprea et al. (2011) and Benndorf et al. (2016). Both experiments apply an interface that is favorable to monotonicity (i.e., each participant is shown the exact population-dependent payoff of each action). Both experiments show that the prediction of monotone dynamics holds in this setup, and that the populations converge to an asymmetric pure equilibrium in which one population (say, the buyers) plays hawk and the other population (say, the sellers) plays dove.

Consider a revised experimental design, where an agent observes only the behavior of her own opponent, rather than the aggregate behavior of the opposing population. An interesting testable prediction of our model is that in this experimental design, the populations are likely to converge to the symmetric mixed equilibrium in the relevant parameter domain (in particular, when g is not too far from 1; see Figure 6.1).[6]

[5] Experiments that directly test the dynamic predictions of evolutionary game theory are quite scarce. Two notable exceptions are the experiments showing the good fit of the dynamic predictions in the rock–paper–scissors game (Cason et al., 2014; Hoffman et al., 2015).

[6] Benndorf et al. (2016, 2021) studied a more general setup in which each participant in each round is randomly matched with an opponent from the other population with probability $\kappa \in (0, 1)$, and is randomly matched with an opponent from her own population with the remaining probability $1 - \kappa$. Our theoretical predictions fit the setup of κ close to one.

Table 2: Payoff Matrix of a Generalized Hawk–Dove Game $g_i, l_i \in (0,1)$

		Player 2: h_2	Player 2: d_2
Player 1	h_1	$0, 0$	$1+g_1, 1-l_2$
	d_1	$1-l_1, 1+g_2$	$1, 1$

3 Model

3.1 The Hawk–Dove Game

Let $G = \{A, u\}$ denote a two-player hawk–dove game, where:

1. $A = A_1 \times A_2$, where $A_i = \{h_i, d_i\}$ is the set of actions of each player $i \in \{1, 2\}$, and
2. $u = (u_1, u_2)$, where $u_i : A \to \mathbb{R}$ is the payoff function of each player $i \in \{1, 2\}$.

Let $i \in \{1, 2\}$ be an index referring to one of the players, and let $j = \{1, 2\} \setminus \{i\}$ be an index referring to the opponent. We interpret action h_i as the hawkish (more aggressive) action and d_i as the dovish action. The payoff matrix $u(\cdot, \cdot)$ of a generalized hawk–dove game is given in Table 2. When both agents are dovish, they obtain a relatively high payoff, which is normalized to 1. When both agents are hawkish, they obtain their lowest feasible payoff, which is normalized to 0. Finally, when player i is hawkish and player j is dovish, the hawkish player i *gains* $g_i \in (0, 1)$ (relative to the payoff 1 obtained by two dovish players), while her dovish opponent *looses*[7] $l_j \in (0, 1)$. Clearly, the set of pure Nash equilibria of G are (h_1, d_2) and (d_1, h_2). The game admits a unique mixed Nash equilibrium $\left(p_1^N, p_2^N\right)$ in which player i plays h_i with probability $p_i^N = \frac{g_j}{l_j + g_j}$, and obtains a relatively low expected payoff of $\frac{l_i + l_i g_i}{l_i + g_i} < 1$.

We say that the hawk–dove game is *symmetric* if $g_1 = g_2$ and $l_1 = l_2$, and in this case we write the parameters as g and l. An important special case of a symmetric hawk–dove game is the *standard* hawk–dove game in which $g = l$ (see Table 1), i.e., the gain of the hawkish player is equal to the loss of her dovish opponent.

[7] Herold and Kuzmics (2020) allow a broader parameter domain in which the assumption of $g_i, l_i \in (0, 1)$ is replaced with the weaker assumption of $g_i > 0$, $l_i < 1$, and $l_i + g_j > 0$. All of our results hold in this extended setup; Theorem 2, requires some modifications to cover the additional case of $g_i > 1$.

3.2 Evolutionary Dynamics

We assume that there are two unit-mass continuums of agents and that agents in population 1 are randomly matched with agents in population 2 to play the hawk–dove game. Aggregate behavior in the populations at time $t \in \mathbb{R}^+$ is described by a *state* $p(t) = (p_1(t), p_2(t)) \in [0,1]^2$, where $p_i(t)$ represents the share of agents playing the hawkish action h_i at time t in population i. We extend the payoff function u to states (which have the same representation as mixed strategy profiles) in the standard linear way. With slight abuse of notation we use d_i (resp., h_i) to denote the degenerate state $p_i = 0$ (resp., $p_i = 1$) in which all agents in population i play d_i (resp., h_i). A state $p = (p_i, p_j)$ is symmetric if $p_i = p_j$, and in this case we write it as (p, p).

Agents occasionally die and are replaced by new agents (equivalently, agents occasionally receive opportunities to revise their actions). Let $\delta_i > 0$ denote the death rate of agents in population i, which we assume to be independent of the currently used actions. In symmetric games we assume $\delta \equiv \delta_1 = \delta_2$, and, in this case, δ does not have any effect on the dynamics except to multiply the speed of convergence by a constant.

The evolutionary process is represented by a continuous function $w : [0,1]^2 \to [0,1]^2$, which describes the frequency of new agents in each population who adopt action h_i as a function of the current state. That is, $w_i(p)$ describes the share of new agents of population i who adopt action h, given state p. Thus, the dynamics are given by $\dot{p}_i = \delta_i \cdot (w_i(p) - p_i)$.

The most widely studied family of dynamics are those that are monotone with respect to the payoffs. A dynamic is monotone in a two-action game (also known as sign-preserving or payoff positive), if the share of agents playing an action increases iff the action yields a higher payoff than the alternative action.[8,9]

Definition 1. The evolutionary dynamic $w : [0,1]^2 \to [0,1]^2$ is *monotone* if for any player i, any interior $p_i \in (0,1)$, and any $p_j \in [0,1]$: $\dot{p}_i > 0 \;\Leftrightarrow\; u_i(h_i, p_j) > u_i(d_i, p_j)$.

[8]In games with more than two actions, there are various definitions that capture different aspects of monotonicity. All these definitions coincide for two-action games. In particular, Definition 1 coincides in two-action games with Weibull's (1997, Section 5.5) textbook definitions of payoff monotonicity, payoff positivity, and weak payoff positivity.

[9]The best-known example of payoff monotone dynamics is the standard replicator dynamic (Taylor, 1979), which is given by $\dot{p}_i = w_i(p_j) - p_i = p_i(u_i(h_i, p_j) - u_i(p_i, p_j))$.

Two-Population vs. One-Population Dynamics for Symmetric Games When the underlying game is symmetric, the dynamics presented above are commonly referred to as "two-population" dynamics. These dynamics fit situations in which a player can condition her play on her role in the game (being Player 1 or Player 2). That is, each player observes a payoff-irrelevant signal, which determines whether she is the row player or the column player. Common examples of such payoff-irrelevant signals are those indicating whether the player (1) is a seller or a buyer, as in the motivating example, and (2) has arrived slightly earlier or slightly later at a contested resource (Maynard-Smith, 1982).

By contrast, in one-population dynamics of symmetric games an agent cannot condition her play on her role in the game. It is well known that all monotone one-population dynamics converge to the unique mixed Nash equilibrium in hawk–dove games (see, e.g., Weibull, 1997, Section 4.3.2). It is relatively straightforward to establish that one-population sampling dynamics lead to qualitatively similar results (convergence is to a somewhat different interior point than in the mixed Nash equilibrium, but comparative statics with respect to the payoff parameters remain similar).

3.3 Dynamic Stability

The following notions of stability are standard (see, e.g., Weibull, 1997, Chapter 5). A state is a (dynamic) equilibrium if it is a rest point of the dynamics.

Definition 2. State $p^* \in [0,1]^2$ is an *equilibrium* if $w_i(p^*) = p_i^*$ for each $i \in \{1,2\}$. $\mathcal{E}(w)$ denotes the set of equilibria of the dynamics w, i.e., $\mathcal{E}(w) = \{p^* | w_i(p^*) = p_i^*\}$.

Under monotone dynamics (as defined below) an interior state $p^* \in (0,1)$ is a (dynamic) equilibrium iff it is a Nash equilibrium. By contrast, under nonmonotone dynamics (such as the sampling dynamics analyzed below) the two notions differ.

A state is Lyapunov stable if a population beginning near it remains close, and it is asymptotically stable if, in addition, it eventually converges to the equilibrium. A state is unstable if it is not Lyapunov stable. It is well known (see, e.g., Weibull, 1997, Section 6.4) that every Lyapunov stable state must be an equilibrium. Formally:

Definition 3. State $p^* \in [0,1]^2$ is *Lyapunov stable* if for every neighborhood U of p^* there is a neighborhood $V \subseteq U$ of p^* such that if the initial state $p(0) \in V$, then $p(t) \in U$ for all $t > 0$. A state is *unstable* if it is not Lyapunov stable.

Definition 4. A Lyapunov stable state $p^* \in [0,1]^2$ is *asymptotically stable* if there is some neighborhood U of p^* such that all trajectories initially in U converge to p^*, i.e., $p(0) \in U$ implies $\lim_{t\to\infty} p(t) = p^*$.

A set of asymptotically stable states P^* is globally stable if the populations converge to one of the states in the set from any initial state, possibly after a small perturbation during their convergence trajectory. Specifically, we require that (1) all trajectories converge to equilibria (i.e., there are no cycles), and (2) for any neighborhood around any equilibrium p' outside P^*, there exists a trajectory beginning in this neighborhood that converges to one of the equilibria in P^*. Thus, if a trajectory has converged to p', then an arbitrarily small perturbation at p' will take it away from p' and into one of the states in P^*.

Definition 5. A set of asymptotically stable states P^* is *globally stable* if (1) $\lim_{t\to\infty} p(t)$ exists for any $p(0) \in [0,1]^2$, and (2) for any equilibrium $\hat{p} \in \mathcal{E}(w) \setminus P^*$ and for any neighborhood U around $\hat{p}$, there exists $p(0) \in U$, such that $\lim_{t\to\infty} p(t) \in P^*$.

A state p^* is *globally stable* if the singleton set that contains it $\{p^*\}$ is globally stable.

3.4 Sampling Dynamics

In what follows, we define two plausible nonmonotone dynamics (action-sampling dynamics and payoff-sampling dynamics), in which agents base their choice on inference from small samples. The action-sampling dynamics fit situations in which agents do not know the exact distribution of actions being played in the opponent's population. A new agent in population i observes the actions of a few randomly sampled opponents from population j. The agent views the empirical distribution of actions in her sample as an unbiased estimate of the distribution of actions in the population, and chooses the optimal action against this empirical distribution for all future encounters.

The payoff-sampling dynamics fit situations in which agents either do not know the payoff matrix or do not have feedback about the actions being played in the opponent's population. A new agent in population i observes the payoffs obtained by (1) a random sample of incumbents of population i who have played hawk, and (2) a random sample of incumbents of population i who have played dove. Following these observations, the new agent adopts the action that yielded the higher mean payoff. Another interpretation of the payoff-sampling dynamics is that each new agent tests each of the available actions a few times and then adopts the action that has given her the highest mean payoff during the testing phase.

Distribution of sample sizes We allow heterogeneity in the sample sizes used by new agents. Let $\theta_i \in \Delta(\mathbb{Z}_+)$ denote the distribution of sample sizes of new agents of population i. A share of $\theta_i(k)$ of the new agents of population i have a sample of size k. Let $\text{supp}(\theta_i)$ denote the support of θ_i, and let $\max(\text{supp}(\theta_i))$ denote the maximal sample size in the support of θ_i, and let $\max(\text{supp}((\theta_i)) = \infty$ if θ_i's support is unbounded. Let $\theta_i(\geq k) = \sum_{m \geq k} \theta_i(m)$ denote the frequency of new agents in either population who have a sample size of at least k.

If there exists some k, for which $\theta_i(k) = 1$, then we use k to denote the degenerate (homogeneous) distribution $\theta_i \equiv k$. In the case of an asymmetric game, we let $\theta = (\theta_1, \theta_2)$, where θ_i is the distribution of sample sizes of new agents of population i. In the case of a symmetric game, we assume that both populations have the same distribution of types and, with a slight abuse of notation, we let θ denote the common distribution of types, i.e., $\theta \equiv \theta_1 = \theta_2$. In symmetric games, a share $\theta(k)$ of the new agents in either population have a sample of size k.

Action-sampling dynamics In the action-sampling dynamics, a new agent with sample size k (henceforth, a k-agent) samples k randomly drawn agents from the opponent population and then adopts the action that has yielded the highest mean payoff against the sample. To simplify the notation below, we assume that in case of a tie, the new agent adopts the action d_i. All of our results are *independent* of the tie-breaking rule.

Let $\boldsymbol{b}_k(p_j) \sim Bin(k, p_j)$ denote a random variable with binomial distribution with parameters k (number of trials) and p_j (probability of success in each trial). Then the action-sampling dynamics with a distribution profile of sample sizes θ are given by

$$\begin{aligned} w^A_{\theta_i}(p_j) &= \Pr(h_i \text{ has a higher mean payoff in the sample}) \\ &= \sum_{k \in \text{supp}(\theta)} \theta_i(k) \cdot \Pr\left(\frac{\boldsymbol{b}_k(p_j)}{k} < \frac{g_i}{g_i + l_i}\right). \end{aligned} \tag{3.1}$$

Payoff-sampling dynamics In the payoff-sampling dynamics, a new k-agent from population i observes for each of her feasible actions the payoff obtained by incumbents of population i who played this action in k interactions (with each play of each action being against a newly drawn opponent from the opponent population j), and then chooses the action whose mean payoff was highest during the testing phase. As above, we simplify the notation by assuming that in case of a tie, the new agent adopts the action d_i (and all of our results are independent of which tie-breaking rule is used). We refer to the sample against which action h_i (resp., d_i) is tested as the h_i-sample (resp., d_i-sample). Let $\boldsymbol{b}_k^h(p_j), \boldsymbol{b}_k^d(p_j) \sim Bin(k, p_j)$ denote two iid random variables with a binomial distribution with parameters k and p_j. Let $X_{h_i}(\theta)$ be action h_i's mean payoff against the h_i-sample and $X_{d_i}(\theta)$ be action d_i's mean payoff against the d_i-sample. Then the payoff-sampling dynamics with a distribution profile of sample sizes θ are given by

$$\begin{aligned} w_{\theta_i}^P(p_j) &= \Pr\left(X_{h_i}(\theta) > X_{d_i}(\theta)\right) \\ &= \sum_{k \in \operatorname{supp}(\theta)} \theta_i(k) \cdot \Pr\left((1+g_i)\left(k - \boldsymbol{b}_k^h(p_j)\right) > k - (1-l_i)\,\boldsymbol{b}_k^d(p_j)\right). \end{aligned} \tag{3.2}$$

3.5 Benchmark: Stability under Monotone Dyanamics

Oprea et al. (2011) showed that under monotone dynamics, from any initial state, the populations converge to one of the two asymmetric pure equilibria in which one population plays h_i and the other population always plays d_j (generalizing the seminal analysis of Maynard-Smith and Parker, 1976). For completeness, we state and prove an equivalent result, using the notation of our model.

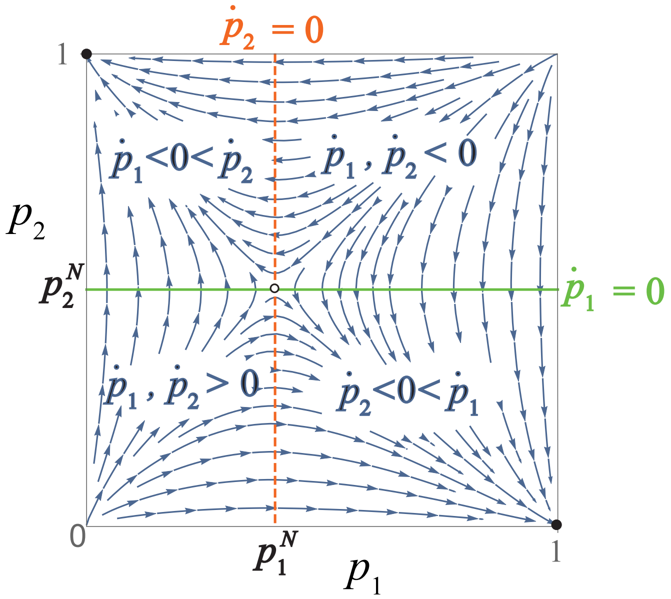
Proposition 1 (Adaptation of Oprea et al., 2011, Proposition 1)**.** *The set* $\{(d_1, h_2), (h_1, d_2)\}$ *is globally stable for any monotone dynamics.*

Proof. Observe that $u_i(h_i, p_j) > u_i(d_i, p_j)$ iff $p_j < p_j^N = \frac{g_i}{l_i+g_i}$. Due to payoff monotonicity this implies that $\dot{p}_i > 0$ iff $p_j < p_j^N$. This implies that one can divide the unit square into four rectangles (as illustrated in Figure 3.1 below):

1. Upper-left rectangle ($p_1 < p_1^N$, $p_2 > p_2^N$) in which the dynamics move upward and to the left (i.e., $\dot{p}_1 < 0 < \dot{p}_2$) until converging to $(0,1)$.

2. Upper-right rectangle ($p_1 > p_1^N$, $p_2 > p_2^N$) in which the dynamics move downward and to the left (i.e., $\dot{p}_1, \dot{p}_2 < 0$) until converging to either the upper-left rectangle, the lower-right rectangle, or the unstable equilibrium $\left(p_1^N, p_2^N\right)$.

Figure 3.1: Global Stability of $\{(0,1),(1,0)\}$ for Monotone Dynamics.



The figure illustrates the four rectangles described in the proof of Proposition 1. A solid (resp., hollow) dot represents an asymptotically stable (resp., unstable) equilibrium.

3. Lower-right rectangle ($p_1 > p_1^N$, $p_2 < p_2^N$) in which the dynamics move downward and to the right (i.e., $\dot{p}_2 < 0 < \dot{p}_1$) until converging to $(1,0)$.

4. Lower-left rectangle ($p_1 > p_1^N$, $p_2 < p_2^N$) in which the dynamics move upward and to the right (i.e., $\dot{p}_1, \dot{p}_2 > 0$) until converging to either the upper-left rectangle, the lower-right rectangle, or the unstable equilibrium $\left(p_1^N, p_2^N\right)$.

This implies that $\{(d_1,h_2),(h_1,d_2)\}$ is globally stable. □

4 Stability of Pure Equilibria

In this section, we characterize the asymptotic stability of the pure equilibria.

4.1 Auxiliary Definitions and Lemma

In order to be able to state a result that is independent of the specific tie-breaking rule, we focus on generic games. The genericity condition implies that there cannot be ties in samples that are relevant to the stability of the pure equilibria (as proven in Lemma 1).

Definition 6. A hawk–dove game G is called *generic* if none of the following expressions is an integer: $\frac{1+g_i}{1-l_i}$, $\frac{1+g_i}{g_i}$, $\frac{1+g_i-l_i}{1-l_i}$, $\frac{1+g_i-l_i}{g_i}$ for each $i \in \{1,2\}$.

Observe that if the values of g_i and l_i are randomly chosen from a continuous (atomless) distribution, then the game is generic with probability one.

Next we present a lemma that characterizes when a single appearance of a rare action in a new agent's sample can change the agent's behavior.

Lemma 1. *Consider a new agent in population i with a sample size of k.*

1. *Action-sampling dynamics: (I) Action h_i induces a higher payoff against a sample with a single d_j iff*[10] $k \leq \left\lfloor \frac{1+g_i-l_i}{1-l_i} \right\rfloor$*; and (II) Action d_i induces a higher payoff against a sample with a single h_j iff $k \leq \left\lfloor \frac{1+g_i-l_i}{g_i} \right\rfloor$. Moreover, in both cases, the mean payoffs of h_i and d_i against the sample cannot be equal if the game is generic.*

2. *Payoff-sampling dynamics: (I) an h_i-sample with a single d_j induces a higher mean payoff than a d_i-sample with no d_j-s iff $k \leq \left\lfloor \frac{1+g_i}{1-l_i} \right\rfloor$; and (II) a d_i-sample with no h_j-s induces a higher mean payoff than an h_i-sample with a single h_j iff $k \leq \left\lfloor \frac{1+g_i}{g_i} \right\rfloor$. Moreover, in both cases, the two mean payoffs cannot be equal if the game is generic.*

Proof.

1. (I) The sum of payoffs of action h_i (resp., d_i) against a sample with a single d_j is $1+g_i$ (resp., $1+(k-1)(1-l_i)$). The mean payoff of h_i is greater than (resp., equal to) the mean payoff of d_i iff $k < \frac{1+g_i-l_i}{1-l_i}$ (resp., $k = \frac{1+g_i-l_i}{1-l_i}$ and $\frac{1+g_i-l_i}{1-l_i}$ is an integer). (II) The sum of payoffs of d_i (resp., h_i) against a sample with a single h_j is $k-1+1-l_i$ (resp., $(k-1)(1+g_i)$). The mean payoff of h_i is greater than (resp., equal to) the mean payoff of d_i iff $k < \frac{1+g_i-l_i}{g_i}$ (resp., $k = \frac{1+g_i-l_i}{g_i}$ and $\frac{1+g_i-l_i}{g_i}$ is an integer).

2. (I) The sum of payoffs of an h_i-sample with a single d_j is equal to $1+g_i$. The sum of payoffs of a d_i-sample with no d_j is equal to $k \cdot l_i$. The former sum is greater than (resp., equal to) the latter iff $k < \frac{1+g_i}{1-l_i}$ (resp., $k = \frac{1+g_i}{1-l_i}$ and $\frac{1+g_i}{1-l_i}$ is an integer). (II) The sum of payoffs of a d_i-sample with no h_j-s is equal to k. The sum of payoffs of an h_i-sample with a single h_j is equal to $(k-1)(1+g_i)$. The former sum is greater than (resp., equal to) the latter iff $k < \frac{1+g_i}{g_i}$ (resp., $k = \frac{1+g_i}{g_i}$ and $\frac{1+g_i}{g_i}$ is an integer). □

Lemma 1 allows us to define the maximal sample sizes in which a single appearance of a rare action can change the behavior of a new agent.

[10] $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Definition 7. Let $m^P_{h_i} = \left\lfloor \frac{1+g_i}{1-l_i} \right\rfloor$, $m^P_{d_i} = \left\lfloor \frac{1+g_i}{g_i} \right\rfloor$, $m^A_{h_i} = \left\lfloor \frac{1+g_i-l_i}{1-l_i} \right\rfloor$, $m^A_{d_i} = \left\lfloor \frac{1+g_i-li}{g_i} \right\rfloor$, where:

1. $m^P_{h_i}$ (resp., $m^A_{h_i}$) is the maximal sample size for which a single appearance of d_j (resp., d_i) in the sample, when all other sampled actions are h_j (resp., h_i), can induce a new agent to adopt h_i under payoff-sampling (resp., action-sampling) dynamics.

2. $m^P_{d_i}$ (resp., $m^A_{d_i}$) is the maximal sample size for which a single appearance of h_j (resp., h_i) in the sample, when all other sampled actions are d_j (resp., d_i), can induce a new agent to adopt d_i under payoff-sampling (resp., action-sampling) dynamics.

We conclude this subsection by presenting a definition of *m-bounded expectation* of a probability distribution with support on the set of positive integers. It is the expected value of the probability distribution by restricting its support to m. Formally, we have:

Definition 8. The *m-bounded expectation* $\mathbb{E}_{\leq m}$ of distribution θ with support on positive integers is defined as $\mathbb{E}_{\leq m}(\theta) = \sum_{k=1}^{m} \theta(k) \cdot k$.

4.2 Characterization Result

Our next result characterizes the asymptotic stability of the pure states. It shows that the asymptotic stability depends only on whether the product of the bounded expectations of the distribution of sample sizes in each population is larger or smaller than one, where the bound of each distribution is the maximal sample size for which a single appearance of a rare action can change the behavior of a new agent. Formally:

Theorem 1. *Assume that the hawk–dove game is generic.*

1. *Action-sampling dynamics:*

   (a) $\mathbb{E}_{\leq m^A_{h_i}}(\theta_i) \cdot \mathbb{E}_{\leq m^A_{d_j}}(\theta_j) < 1 \Rightarrow (d_i, h_j)$ *is asymptotically stable.*

   (b) $\mathbb{E}_{\leq m^A_{h_i}}(\theta_i) \cdot \mathbb{E}_{\leq m^A_{d_j}}(\theta_j) > 1 \Rightarrow (d_i, h_j)$ *is unstable.*

2. *Payoff-sampling dynamics:*

   (a) $\mathbb{E}_{\leq m^P_{h_i}}(\theta_i) \cdot \mathbb{E}_{\leq m^P_{d_j}}(\theta_j) < 1 \Rightarrow (d_i, h_j)$ *is asymptotically stable.*

   (b) $\mathbb{E}_{\leq m^P_{h_i}}(\theta_i) \cdot \mathbb{E}_{\leq m^P_{d_j}}(\theta_j) > 1 \Rightarrow (d_i, h_j)$ *is unstable.*

Sketch of Proof. Consider a slightly perturbed state $(\epsilon, 1-\epsilon)$ near the pure equilibrium $(0,1)$. Observe that almost all agents in population 1 (resp., 2) play d_1 (resp., h_2). We refer to the other action (namely, h_1 in population 1 and d_2 in population 2) as the rare action. The event of two rare actions appearing in a sample of a new agent has a negligible probability of $O(\epsilon^2)$. If a new agent has a sample size of k, then the probability of a rare action appearing in the sample is approximately $k \cdot \epsilon$. This rare appearance changes the perceived best reply of a new agent of population 1 iff k is smaller than the relevant maximal sample size, which is either $m^A_{h_1}$ or $m^P_{h_1}$, depending on the underlying dynamics; henceforth we denote it by m_{h_1}; similarly we let m_{d_2} denote the respective relevant maximal sample size for new agents in population 2.

Thus, the total probability that a new agent of population 1 (resp., 2) adopts a rare action (due to the appearance of a single rare action in her sample) is equal to $\mathbb{E}_{\leq m_{h_1}}(\theta_1)$ (resp., $\mathbb{E}_{\leq m_{d_2}}(\theta_2)$). This implies that the share of new agents of population 1 (resp., 2) who adopt a rare action is $\epsilon \cdot \mathbb{E}_{\leq m_{h_1}}(\theta_1)$ (resp., $\epsilon \cdot \mathbb{E}_{\leq m_{d_2}}(\theta_2)$). Therefore, the product of new agents adopting a rare action in each population is $\epsilon^2 \cdot \mathbb{E}_{\leq m_{h_1}}(\theta_1) \cdot \mathbb{E}_{\leq m_{d_2}}(\theta_2)$. This shows that the share of agents playing rare actions gradually increases (resp., decreases) if $\mathbb{E}_{\leq m_{h_1}}(\theta_1) \cdot \mathbb{E}_{\leq m_{d_2}}(\theta_2) > 1$ (resp., $\mathbb{E}_{\leq m_{h_1}}(\theta_1) \cdot \mathbb{E}_{\leq m_{d_2}}(\theta_2) < 1$), which implies instability (resp., asymptotic stability). See Appendix A.1 for a formal proof. □

Observe that the fact that $l_i > 0$ immediately implies that $m^A_{h_i} < m^P_{h_i}$ and $m^A_{d_i} < m^P_{d_i}$, which, in turn, implies that instability under the action-sampling dynamics holds in a strictly smaller set of distributions than under the payoff-sampling dynamics.

Corollary 1. *If (d_i, h_j) is unstable under the action-sampling dynamics, then it is also unstable under the payoff-sampling dynamics.*

4.3 Implications of Theorem 1 for Symmetric Games

In this section, we study the implications of Theorem 1 for symmetric games. As each game is symmetric, we omit the indices i and j from all the parameters.

4.3.1 Asymptotically Stable Interior Equilibrium

If both pure equilibria are unstable, it immediately implies that the curve $\dot{p}_1 = 0$ is above (resp., below) the curve $\dot{p}_2 = 0$ near the state $(0,1)$ (resp., $(1,0)$), as illustrated in the right panel of Figure 4.1. This implies that there must be an intersection point (equilibrium) of the curves $\dot{p}_1 = 0$ and $\dot{p}_2 = 0$, where the curve $\dot{p}_1 = 0$ is above (resp., below) the curve $\dot{p}_2 = 0$ on the right (resp., left) side of this equilibrium point, which implies that this interior equilibrium is asymptotically stable. Formally:

Corollary 2. *Assume that the hawk–dove game is symmetric and generic. There exists an asymptotically stable interior equilibrium* $p^* \in (0,1)^2$ *if:*

1. *Action-sampling dynamics:* $\mathbb{E}_{\leq m_h^A}(\theta) \cdot \mathbb{E}_{\leq m_d^A}(\theta) > 1$.
2. *Payoff-sampling dynamics:* $\mathbb{E}_{\leq m_{h_i}^P}(\theta_i) \cdot \mathbb{E}_{\leq m_{d_j}^P}(\theta_j) > 1$.

4.3.2 Global Stability with a Maximal Sample Size of Two

It is relatively simple to show that when the maximal sample size is two, then the dynamics admit exactly three equilibria.

Lemma 2. *Assume that the hawk–dove game is generic and symmetric and that* $\max(\text{supp}(\theta)) = 2$. *Then it admits exactly 3 equilibria (two asymmetric pure equilibria and a symmetric mixed equilibrium) under both action-sampling dynamics and payoff-sampling dynamics.*

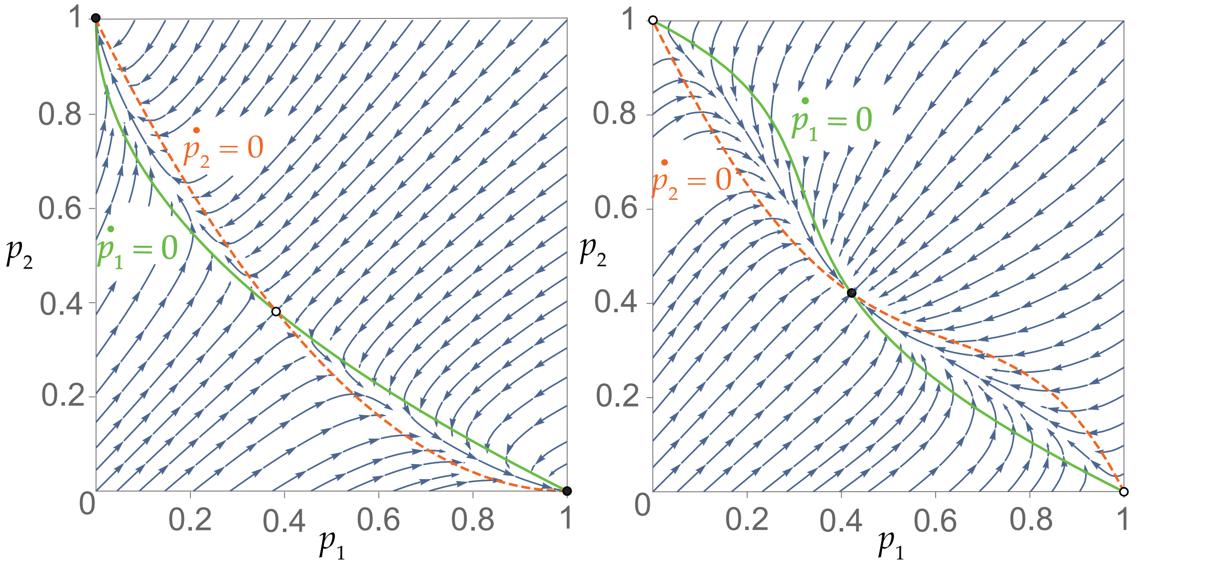
Proof. See Appendix A.2. □

As illustrated in Figure 4.1, dynamics that admit three equilibria have two possible classes of phase potrait. If the asymmetric pure equilibria are asymptotically stable (left panel of Figure 4.1), then they must be globally stable. By contrast, if the asymmetric pure equilibria are unstable (right panel of Figure 4.1), then the symmetric mixed equilibrium must be globally stable.

Thus, we can apply Theorem 1 and characterize the globally stable set when the maximal sample size is two. It turns out that the mixed equilibrium is globally stable under the payoff-sampling dynamics iff the gain of a hawkish player and the loss of a dovish opponent are sufficiently large (namely, $g + 2l > 1$). Under the action-sampling dynamics, the asymmetric pure equilibria are a globally stable set for all parameter values.

Corollary 3. *Assume that the hawk–dove game is symmetric and generic, and that* $\max(\text{supp}(\theta)) = 2$. *Then there exists a globally stable symmetric mixed equilibrium under the payoff-sampling dynamics iff* $g + 2l > 1$. *In all other cases (namely, action-sampling dynamics or* $g + 2l < 1$*), the asymmetric pure equilibria are a globally stable set.*

Figure 4.1: Two Classes of Phase Portraits with Three Equilibria ($\theta \equiv 2$)



Phase portrait with globally stable pure equilibria | Phase portrait with a globally stable symmetric equilibrium

The figure illustrates the two feasible classes of phase portraits for dynamics that admit three equilibria. The left panel shows an example of a phase portrait of dynamics in which the pure equilibria are asymptotically stable (payoff-sampling dynamics with $\theta \equiv 2$ and $g + 2l < 1$). The right panel shows an example of a phase portrait of dynamics in which the pure equilibria are unstable (payoff-sampling dynamics with $\theta \equiv 2$ and $1 < g + 2l < 1 + l$). A solid (resp., hollow) dot represents an asymptotically stable (resp., unstable) equilibrium.

4.3.3 Standard Hawk–Dove Games

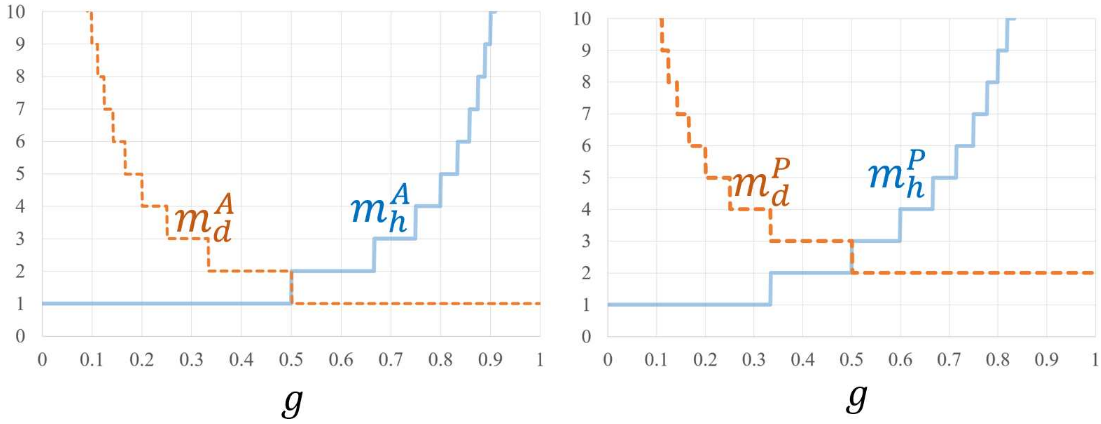
Next we study the implications of Theorem 1 for standard hawk–dove games (in which $g = g_i = l_i$ describes both the gain of a hawkish player and the loss of her dovish opponent). Figure 4.2 illustrates the values of m_h^P, m_d^P, m_h^A, m_d^A as a function of g:

$$m_h^P = \left\lfloor \frac{1+g}{1-g} \right\rfloor, \; m_d^P = \left\lfloor \frac{1+g}{g} \right\rfloor, \; m_h^A = \left\lfloor \frac{1}{1-g} \right\rfloor, \; m_d^A = \left\lfloor \frac{1}{g} \right\rfloor$$

.

Action-sampling dynamics Theorem 1 and Figure 4.2 imply that the pure equilibria are unstable under the action-sampling dynamics iff $\mathbb{E}_{\leq \min\left(\frac{1}{g}, \frac{1}{1-g}\right)} \cdot \mathbb{E}_{\leq \max\left(\frac{1}{g}, \frac{1}{1-g}\right)} = \theta(1) \cdot \mathbb{E}_{\leq \max\left(\frac{1}{g}, \frac{1}{1-g}\right)} > 1$. Thus the asymmetric pure equilibria are unstable if the following two conditions are satisfied: (1) sufficiently many agents have a sample size of 1, and (2) g is not too close to 0.5. Specifically, if $\theta(1)$ is too small, such that $\theta(1) \cdot \mathbb{E}(\theta) < 1$, then the pure equilibria are asymptotically stable for all values of g. By contrast, if $\theta(1)$ is sufficiently large, such that $\theta(1) \cdot \mathbb{E}(\theta) > 1$, then there exists a threshold $x \in (0, 0.5)$,

Figure 4.2: Maximal Sample Sizes in Standard Hawk–Dove Games



such that the pure equilibria are asymptotically stable (resp., unstable) if $|0.5 - g| < x$ (resp., $|0.5 - g| > x$).

Payoff-sampling dynamics Theorem 1 and Figure 4.2 imply that the stability condition for the pure equilibria under the payoff-sampling dynamics is qualitatively similar to the action-sampling dynamics when $g < \frac{1}{3}$ (with a slightly higher bound for the expectation; i.e., the condition for instability is $\theta(1) \cdot \mathbb{E}_{\leq(\frac{1+g}{g})}(\theta) > 1$). By contrast, if $g > \frac{1}{3}$, then a pure equilibrium is unstable iff $(\theta(1) + 2 \cdot \theta(2)) \cdot \mathbb{E}_{\leq \max\left(3, \frac{1+g}{1-g}\right)}(\theta) > 1$.

Thus, the pure equilibria are unstable if there are sufficiently many agents with a sample size of at most 2 (where intermediate frequencies of agents with a sample size of at most 2 further require that g be sufficiently close to one). These conditions hold for many distributions of sample sizes. In particular, the pure equilibria are unstable (1) for any $g > \frac{1}{3}$ if the share of agents with a sample size of 2 is larger than the share of agents with a sample size of at least 3 (i.e., $\theta(2) > \theta(\geq 3)$), (2) for any uniform distribution of types over $\{1, \ldots, k\}$ for any k, if g is sufficiently close to one (i.e., if the surplus from trade in the motivating example is sufficiently close to the cost of bargaining failure).

5 Stability of the Symmetric Equilibrium

In this section, we analyze the stability of the symmetric equilibrium of symmetric games.

5.1 Action-Sampling Dynamics

Our first result shows that any symmetric equilibrium is unstable under the action-sampling dynamics if all agents have the same sample size.

Theorem 2. *Assume that the hawk–dove game is symmetric and generic, and that $\theta \equiv k$. Then the unique symmetric equilibrium is unstable under the action-sampling dynamics.*

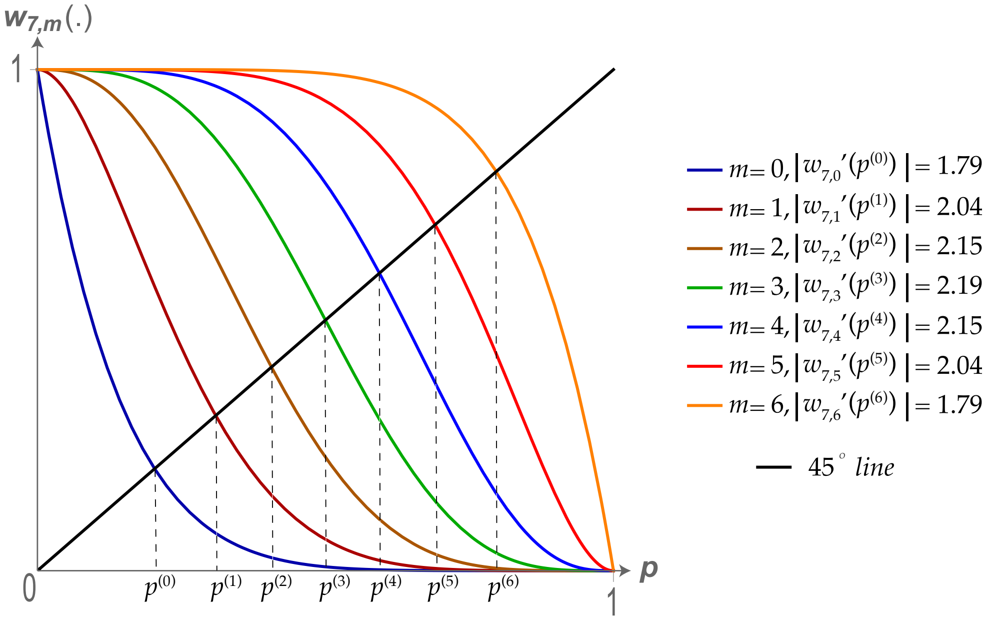
Sketch of Proof. Each pair of parameters g, l induces a threshold $0 \leq m < k$, such that playing hawk is the best reply against a sample of size k iff the sample includes at most m hawkish actions. This implies that a symmetric state $\left(p^{(m)}, p^{(m)}\right)$ is an equilibrium iff $p^{(m)}$ is a fixed point of the function $w_{k,m}(p) \equiv P(X_k(p) \leq m)$, where $X_k(p)$ is a binomial distribution with parameters k (number of trials) and p (probability of success). The fact $w_{k,m}(p)$ is decreasing in p, $w_{k,m}(0) = 1$ and $w_{k,m}(1) = 0$ implies that there exists a unique symmetric equilibrium.

Consider the perturbed state $\left(p^{(m)} - \epsilon, p^{(m)} + \epsilon\right)$ in which population 1 (resp., 2) has slightly more dovish (resp., hawkish) players. The share of new agents of population 1 (resp., 2) who play the hawkish action is approximately equal to $p^{(m)} - |w'_{k,m}(p^{(m)})| \cdot \epsilon$ (resp., $p^{(m)} + |w'_{k,m}(p^{(m)})| \cdot \epsilon$). This implies that the perturbation will gradually increase iff the absolute value of the derivative $|w'_{k,m}(p^{(m)})|$ is greater than 1. It is easy to verify that the function $|w'_{k,m}(p)|$ is unimodal with a peak at $\frac{m}{k-1}$. The formal proof shows that the fixed point $p^{(m)}$ is sufficiently close to the peak, such that $|w'_{k,m}(p^{(m)})| > 1$. See Appendix A.4 for a formal proof. Figure 5.1 illustrates the functions $w_{k,m}(p)$ and their rest points for $k = 7$ and for all values of m. □

Thus, the dynamic behavior under the action–sampling dynamics is similar to monotone dynamics when all agents have the same sample size: the asymmetric pure equilibria are stable, while the symmetric mixed equilibrium is unstable. By contrast, our next result shows that the converse is true if most agents have a small sample size of one, and the remaining few agents have sufficiently large sample sizes. Specifically, we show that for any symmetric hawk–dove game, the symmetric equilibrium is asymptotically stable if the distribution of types θ satisfies the following two conditions: (1) $\theta(1) < 1$ is sufficiently large, and (2) any $1 \neq k \in \text{supp}(\theta)$ is sufficiently large.

Theorem 3. *Fix a symmetric and generic hawk–dove game. Then there exists $\hat{q} \in (0, 1)$ and $\hat{k} \in \mathbb{N}$, such that the unique symmetric equilibrium is asymptotically stable under the*

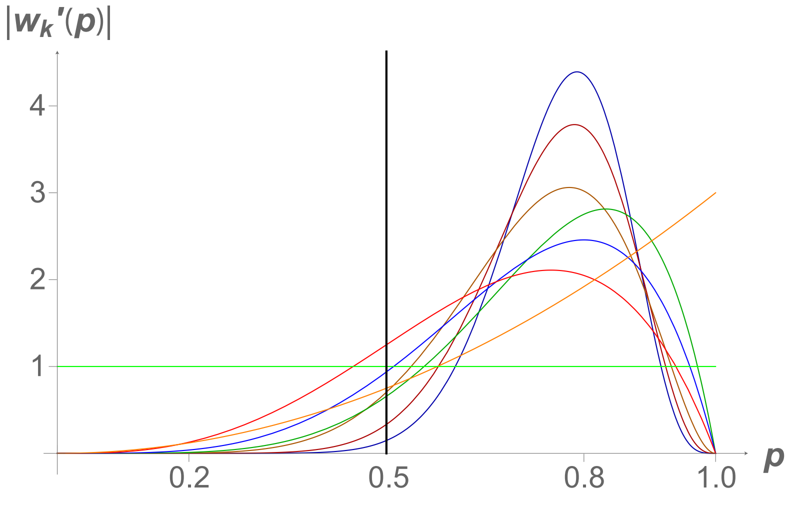
Figure 5.1: The Function $w_{k,m}(p)$ for $k = 7$ and All Values of m



action-sampling dynamics for any distribution of types θ that satisfies $\theta(1) \in (\hat{q}, 1)$, and $\theta(k) = 0$ for each $1 < k < \hat{k}$.

Sketch of Proof. In populations in which all agents have a sample size of one, the unique symmetric equilibrium is $p^{(1)} = 0.5$. By continuity, this implies that the unique symmetric equilibrium $p^{(\theta)}$ is very close to 0.5 if $\theta(1)$ is sufficiently close to one. An analogous argument to the sketch of proof of Theorem 2 shows that a sufficient condition for the symmetric equilibrium to be asymptotically stable is $|w'_\theta(p^{(\theta)})| < 1$.

Figure 5.2: The Function $|w'_k(p)|$ for Various Values of k, where $g = 1 - l = 0.8$



The function $|w'_\theta(p)|$ is a mixture of the functions $|w'_k(p)|$ for the various k-s in the support of θ_q (which are illustrated in Figure 5.2). Observe that $|w'_1(p) \equiv 1|$. The formal proof applies the central limit theorem to show that as k increases, $|w_k'(p)|$ converges to a normal distribution with mean $\frac{g}{g+l}$ and variance $\frac{1}{4k}$. In generic games, $\frac{g}{g+l} \neq \frac{1}{2}$, which implies that $|w_k'(0.5)|$ converges to zero. This, in turn, implies that $|w_{\theta_q}'(0.5)| < 1$ if $\hat{k}$ is sufficiently large. By continuity, $|w_{\theta_q}'(p^{(\theta_q)})| < 1$, which implies that the symmetric equilibrium is asymptotically stable. See Appendix A.5 for a formal proof. □

5.2 Payoff-Sampling Dynamics

For tractability in the analysis of payoff-sampling dynamics, we focus on the cases where the gain of a hawkish player and the loss of her dovish opponent are large, namely, $l, g > \frac{1}{\max(\mathrm{supp}(\theta))}$. Our result shows that in this domain, the symmetric equilibrium is asymptotically stable in the following cases:

1. for any homogeneous distribution of sample sizes $\theta \equiv k < 20$; or

2. for any distribution of sample sizes with a maximal size of at most 5.

The threshold of $k = 20$ is binding. The symmetric equilibrium becomes unstable if the sample size $k \geq 20$. By contrast, the bound of a maximal size of 5 for heterogeneous distributions of sample sizes is a constraint of our proof technique. Numeric analysis suggests that the stability of the mixed equilibrium holds for many distributions of types with larger maximal sample sizes (in particular, it holds for uniform distributions of types over $\{1, \ldots, k\}$ for any $k \leq 20$).

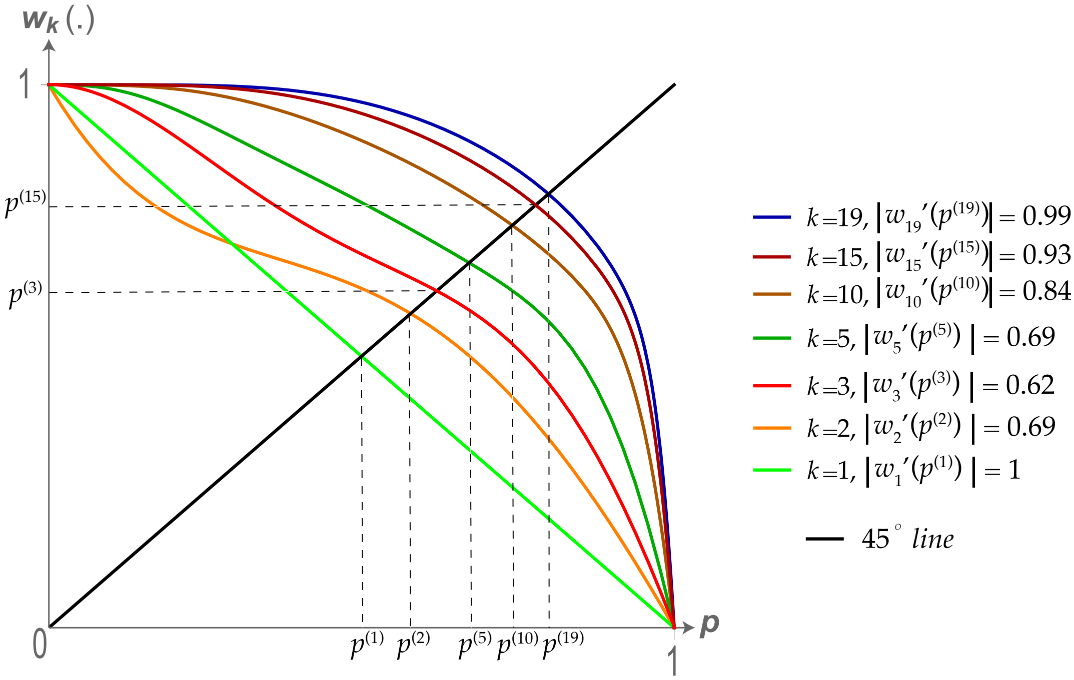
Theorem 4. *Assume that the hawk–dove game is symmetric, $l, g \in \left(\frac{1}{\max(\mathrm{supp}(\theta))}, 1\right)$, and either (1) $\theta \equiv k < 20$, or (2) $\max(\mathrm{supp}(\theta)) \leq 5$. Then, the game admits an asymptotically stable symmetric equilibrium $\left(p^{(\theta)}, p^{(\theta)}\right)$ under the payoff-sampling dynamics.*

Sketch of Proof. When l and g are sufficiently large, the payoff of action h_i is slightly below twice the number of d_j-s in the h_i-sample, and the payoff of action d_i is slightly above the number of d_j-s in the d_i-sample. This implies that action h_i has a higher mean payoff than action d_i iff the number of d_j-s in the h_i-sample is strictly greater than half the number of d_j-s in the d_i-sample.

Thus, we can write $w_k(p)$ as follows:

$$w_k(p) = P\left(\underbrace{k - X_k(p)}_{\#d_j \text{ in } h_i\text{-sample}} > \frac{1}{2}\underbrace{(k - Y_k(p))}_{\#d_j \text{ in } d_i\text{-sample}}\right) = P(2X_k(p) - Y_k(p) < k), \tag{5.1}$$

Figure 5.3: The Function $w_k(p)$ for Various Values of k



where $X_k(p)$ and $Y_k(p)$ are iid binomial random variables with parameters k and p.

In the formal proof (see Appendix A.6), we show that for any $k < 20$, $w_k(p)$ has a unique fixed point $p^{(k)}$ such that $|w_k'(p^{(k)})| < 1$ (see Figure 5.3). This implies, by the same argument as in the sketch of proof of Theorem 2, that the symmetric equilibrium is asymptotically stable. (By contrast, one can verify that $|w_k'(p^k)| > 1$ for $k \geq 20$, which implies that the symmetric equilibrium is unstable for large $k \geq 20$.)

Next, we verify in the formal proof that for any $k \in \{1, 2, 3, 4, 5\}$ it holds that (I) the fixed points are all in the interval $(0.5, 0.68)$, and (II) $|w_k'(p)| < 1$ for any $k \in \{1, ..., 5\}$ and any $p \in (0.5, 0.68)$. Let θ be any distribution with $\max(\text{supp}(\theta)) \leq 5$. The fact that $w_\theta(p)$ is a weighted average of the various $w_k(p)$ implies that (I) the fixed point $p^{(\theta)}$ of $w_\theta(p)$ is in $(0.5, 0.68)$, and (II) $|w_\theta'(p^{(\theta)})| < 1 \Rightarrow (p^{(\theta)}, p^{(\theta)})$ is asymptotically stable. □

6 Numeric Analysis

We present numeric results that complement the analytic results of the previous sections.

Methodology and Parameter Values The analysis focuses on the standard hawk–dove games, in which the gain of a hawkish player is equal to the loss of her dovish opponent, i.e., $g = g_i = l_i$ for each $i \in \{1, 2\}$. We have tested the following $270 = 10 \times 27$

combinations of parameter values for each of the two sampling dynamics:

1. 10 values for the ratio g: 0.05, 0.15, 0.25, 0.35, 0.45, 0.55, 0.65, 0.75, 0.85, 0.95.

2. 27 distributions of sample sizes:

   (a) 9 *degenerate distributions*, in which all agents have sample size k, for each $2 \leq k \leq 10$ (the case of $k = 1$ is analytically analyzed in Appendix A.3).

   (b) 9 *uniform distributions* over $\{1, .., k\}$, for each $2 \leq k \leq 10$.

   (c) 9 1-*biased distributions*, in which a share $q \in \{10\%, 20\%, ..., 90\%\}$ of the new agents have sample size one, while the sample sizes of the remaining agents are distributed uniformly over {1,2,. . . , 10}. That is, the frequency of sample size 1 is $\frac{1-q}{10} + q$ and the frequency of each $k \in \{2, \ldots, 10\}$ is $\frac{1-q}{10}$.

For each set of parameters, we have numerically calculated the phase portrait and the curves for which $\dot{p}_1 = 0$ and $\dot{p}_2 = 0$, and used this to determine the dynamic behavior. The code is provided in the online supplementary material.[11,12]

Results The numeric results are summarized in Figure 6.1. The action-sampling dynamics typically yield global convergence to the pure equilibria (orange shaded region in Figure 6.1). The exceptions are consistent with Theorems 1 and 3. Specifically, the action-sampling dynamics admit (almost) global convergence to the symmetric equilibrium (green shaded region) if (1) most agents have sample size 1, and (2) g is sufficiently far from 0.5.

The payoff-sampling dynamics typically yield global (or almost global) convergence to either the asymmetric pure equilibria or the symmetric mixed equilibrium, where each kind of convergence holds, roughly, in half of the parameter combinations. Global convergence to the symmetric mixed equilibrium occurs for all parameter values for which it occurs under the action-sampling dynamics. In addition, the payoff-sampling dynamics globally converge to the symmetric mixed equilibrium for all distributions of types, provided that the ratio g is sufficiently close to one.

[11]We have arbitrarily chosen a tie-breaking rule that favors d in the event of a tie, in order to be consistent with Eqs. (3.1) and (3.2), but the results remain essentially the same with any tie-breaking rule.

[12]Our numeric analysis is based on deterministic dynamics in a continuum population. We have randomly chosen 10 of these combinations of parameter values, and tested each of them by running it 100 times in the stochastic dynamics induced by a finite population of 1,000 agents, using ABED software (Izquierdo et al., 2019). The results for finite populations are qualitatively the same.

The shapes of the phase portraits in the common cases of global convergence to either the asymmetric pure equilibria or to the symmetric mixed equilibrium have been illustrated in Figure 4.1. The bottom part of Figure 6.1 illustrates the phase portraits in two other cases:

1. Left panel: Most initial states (>90%) converge to the symmetric mixed equilibrium, while the remaining states converge to asymmetric pure equilibria (olive-green shaded region).

2. Right panel: Almost all initial states (>95%) converge to equilibria that are neither asymmetric pure equilibria nor symmetric mixed equilibria (blue shaded region).

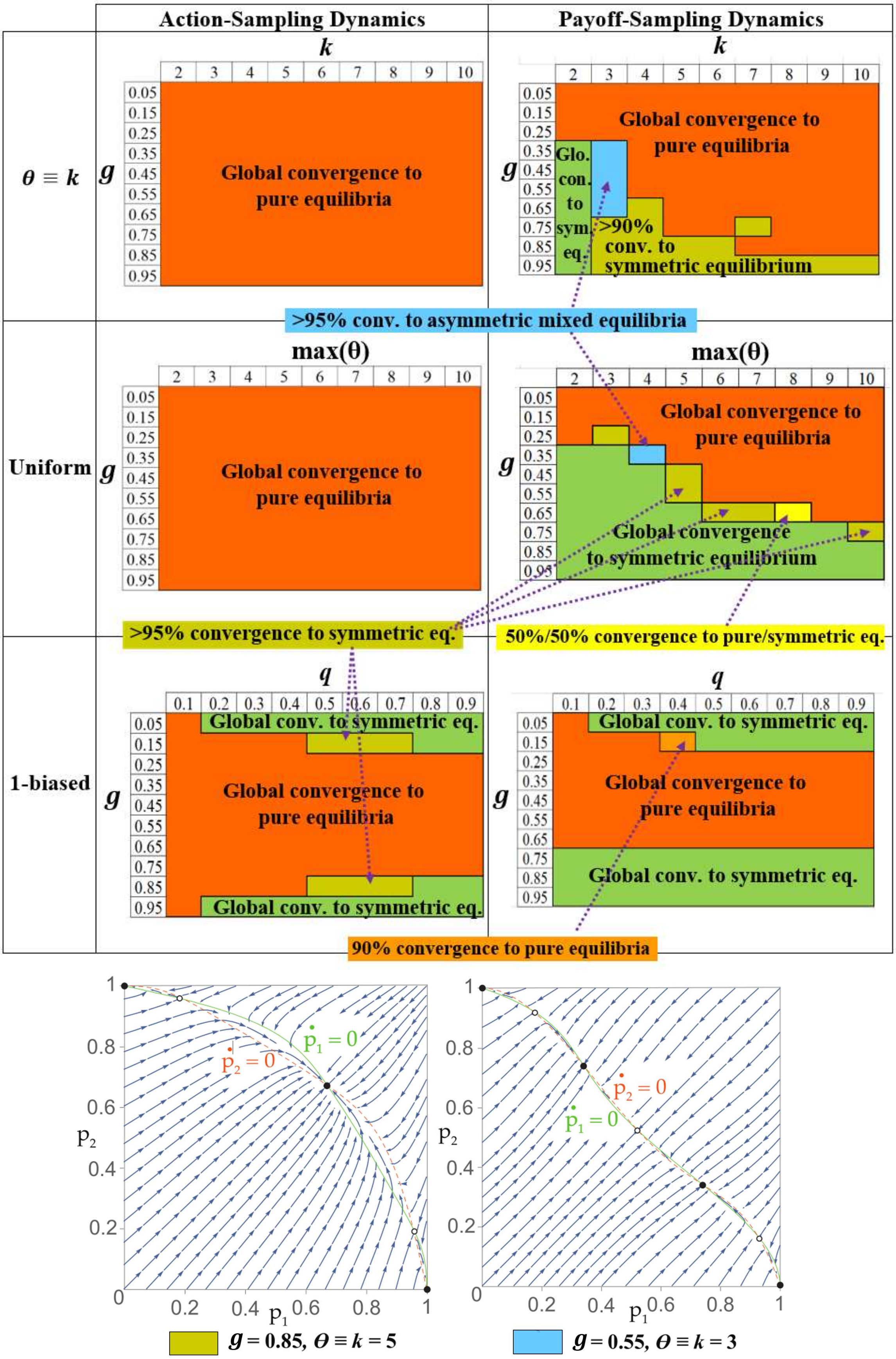
7 Conclusion

A key result in evolutionary game theory is the prediction that only the asymmetric pure equilibria in hawk–dove games are evolutionarily stable. We show that this result crucially depends on the revision dynamics being monotone. We show that two plausible classes of dynamics, in which new agents base their chosen actions on sampling the actions of a few agents in the opponent population (action-sampling dynamics) or on sampling the payoffs of a few agents in their own population (payoff-sampling dynamics) can lead to the opposite prediction: convergence to the symmetric mixed equilibrium.

The convergence to the mixed equilibrium occurs under both classes of dynamics when most new agents have a small sample size of one, and when the gain of a hawkish player against a dovish opponent (g) is not too close to 0.5. Under the payoff-sampling dynamics, this convergence also occurs for many distributions of sample sizes if the gain g is sufficiently large. Our results might help to explain why in bargaining situations, such as the motivating example, players in both populations tend to play hawkish strategies, and bargaining frequently fails.

Our model assumes that all players in each population have the same payoff matrix. Heterogeneity in the payoffs, and private information regarding one's payoff, are important aspects of many real-life bargaining situations. An interesting direction for future research is to apply the analysis of sampling dynamics in more complicated models that incorporate heterogeneous payoffs.

Figure 6.1: Summary of Results of the Numeric Analysis



Examples of two phase portraits in two of the cases studied in the numeric analysis.
A solid (resp., hollow) dot represents an asymptotically stable (resp., unstable) equilibrium.

A Technical Proofs

A.1 Proof of Theorem 1

We are interested in deriving conditions for the asymptotic and Lyapunov stability of the pure action profile (d_i, h_j). In what follows, we compute the Jacobian of the sampling dynamics at the state $(0,1)$. For this, we consider a state $(\epsilon_i, 1-\epsilon_j)$ which is infinitesimally close to the state $(0,1)$ i.e., $0 < \epsilon_i, \epsilon_j << 1$. In words, we consider the state with a "very small" ϵ_i share of hawks in population i and a "very small" ϵ_j share of doves in population j. By "very small," we mean that higher-order terms of ϵ are neglected.

1. *Action-sampling dynamics:*

   Consider a new agent of population i with a sample size of k_i. Action h_i has a higher mean payoff against a sample size of k_i iff (neglecting rare events of having multiple d_j-s in the sample): (1) the sample includes the single action d_j of an opponent, and (2) $k_i \leq m^A_{h_i}$ (due to Lemma 1). The probability of (1) is $k_i \cdot \epsilon_j + o(\epsilon_j)$, where $o(\epsilon_j)$ denotes terms that are sublinear in ϵ_j, and, thus, it will not affect the Jacobian as $\epsilon_j \to 0$. This implies that the probability that a new agent of population i (with a random sample size distributed according to θ_i) has a higher mean payoff for action h_i against her sample is $w^A_{\theta_i}(1-\epsilon_j) = \epsilon_j \cdot \sum_{k_i=1}^{m^A_{h_i}} \theta_i(k_i) \cdot k_i + o(\epsilon_j)$. An analogous argument implies that the probability that a new agent of population j has a higher mean payoff for action d_j against her sample is $w^A_{\theta_j}(\epsilon_i) = \epsilon_i \cdot \sum_{k_j=1}^{m^A_{d_j}} \theta_j(k_j) \cdot k_j + o(\epsilon_i)$. Therefore, the action-sampling dynamics at $(\epsilon_i, 1-\epsilon_j)$ can be written as follows (ignoring the higher-order terms of ϵ_i and ϵ_j):

   $$\dot{\epsilon}_i = \epsilon_j \cdot \sum_{k_i=1}^{m^A_{h_i}} \theta_i(k_i) \cdot k_i - \epsilon_i, \qquad \dot{\epsilon}_j = \epsilon_i \cdot \sum_{k_j=1}^{m^A_{d_j}} \theta_j(k_j) \cdot k_j - \epsilon_j. \tag{A.1}$$

   Define: $a^A_{\theta_i} = \sum_{k_i=1}^{m^A_{h_i}} \theta_i(k_i) \cdot k_i$ and $b^A_{\theta_j} = \sum_{k_j=1}^{m^A_{d_j}} \theta_j(k_j) \cdot k_j$. The Jacobian of the above system of Equations (A.1) is then given by $J = \begin{pmatrix} -1 & a^A_{\theta_i} \\ b^A_{\theta_j} & -1 \end{pmatrix}$. The eigenvalues of J are $-1-\sqrt{a^A_{\theta_i} b^A_{\theta_j}}$ and $-1+\sqrt{a^A_{\theta_i} b^A_{\theta_j}}$. Observe that: (1) if $a^A_{\theta_i} b^A_{\theta_j} < 1$ then both eigenvalues are negative, which implies that the state (d_i, h_j) is asymptotically stable, and (2) if $a^A_{\theta_i} b^A_{\theta_j} > 1$ then one of the eigenvalues is positive, which implies that the state (d_i, h_j) is unstable (see, e.g., Perko, 2013, Theorems 1 and 2 in Section 2.9).

2. *Payoff-sampling dynamics:*

   The agent's h_i-sample has a higher mean payoff than her d_i-sample iff (neglecting rare events of having multiple d_j-s in her samples): (1) the h_i-sample has a single d_j, (2) her d_i-sample does not have any d_j-s, and (3) $k_i \leq m^P_{h_i}$ (by Lemma 1). The probability of both independent events (1) and (2) occurring is $k_i \cdot \epsilon_j + o(\epsilon_j)$. This implies that the probability that a new agent of population i (with a random sample size) has an h_i-sample with a higher mean payoff than her d_i-sample is $w^P_{\theta_i}(1-\epsilon_j) = \epsilon_j \cdot \sum_{k_i=1}^{m^P_{h_i}} \theta_i(k_i) \cdot k_i + o(\epsilon_j)$. Similarly, the probability that a new agent of population j has a d_i-sample with a higher mean payoff than her h_i-sample is $w^P_{\theta_j}(\epsilon_i) = \epsilon_i \cdot \sum_{k_j=1}^{m^P_{d_j}} \theta_j(k_j) \cdot k_j + o(\epsilon_i)$. Let $a^P_{\theta_i} = \sum_{k_i=1}^{m^P_{h_i}} \theta_i(k_i) \cdot k_i$ and $b^P_{\theta_j} = \sum_{k_j=1}^{m^P_{d_j}} \theta_j(k_j) \cdot k_j$. An analogous argument to the previous paragraph shows that (1) if $a^P_{\theta_i} b^P_{\theta_j} < 1$ then the state (d_i, h_j) is asymptotically stable, and (2) if $a^P_{\theta_i} b^P_{\theta_j} > 1$ then (d_i, h_j) is unstable.

A.2 Proof of Lemma 2

Part 1: Proof of the special case in which $\theta(2) = 1$.

Action-sampling dynamics: Consider a new agent of population i with sample size two when the state is (p_1, p_2). Action h_i has a higher mean payoff iff: (1) her sample includes two d_j-s (which happens with probability $(1-p_j)^2$), or (2) her sample includes a single d_j (probability $2p_j(1-p_j)$) and $g + l > 1$. Thus, the action-sampling dynamics are given by

$$\textbf{Case 1}\,(g+l<1): \qquad \dot{p}_i = \delta\left((1-p_j)^2 - p_i\right),$$
$$\textbf{Case 2}\,(g+l>1): \qquad \dot{p}_i = \delta\left((1-p_j)^2 + 2p_j(1-p_j) - p_i\right) = \delta\left(\left(1-p_j^2\right) - p_i\right).$$

It is easy to verify that in both cases the curves intersect three points, which are the equilibria. Two of the equilibria are $(1,0)$, $(0,1)$. The third equilibrium is an interior symmetric state: $(0.38, 0.38)$ in Case 1, and $(0.62, 0.62)$ in Case 2.

Payoff-sampling dynamics: Consider a new agent of population i with sample size 2 in state (p_1, p_2). Her h_i-sample has a higher mean payoff iff:

1. her h_i-sample includes two dovish actions (probability $(1-p_j)^2$), or
2. her h_i-sample includes one dovish action (probability $2p_j(1-p_j)$), and, in addition:
   (a) the d_i-sample includes two hawkish actions (probability p_j^2) and $g+2l>1$, or
   (b) the d_i-sample includes a single h_j (probability $2p_j(1-p_j)$) and $g+l>1$.

This implies that the payoff-sampling dynamics are given by

$$\begin{aligned}
&\textbf{Case 1}\,(g+2l<1): && \dot{p}_i = \delta\left((1-p_j)^2 - p_i\right),\\
&\textbf{Case 2}\,(g\in(1-2l,1-l)): && \dot{p}_i = \delta\left((1-p_j)^2 + 2p_j^3(1-p_j) - p_i\right),\\
&\textbf{Case 3}\,(g+l>1): && \dot{p}_i = \delta\left((1-p_j)^2 + 2p_j^2(1-p_j)(2-p_j) - p_i\right).
\end{aligned}$$

It is easy to verify that in all these cases there exist three equilibria. Two of the equilibria are $(1,0)$, $(0,1)$. The third equilibrium is an interior symmetric state: $(0.38, 0.38)$ in Case 1, $(0.42, 0.42)$ in Case 2, and $(0.58, 0.58)$ in Case 3.

Part 2: Extension of the proof of $\max(\text{supp}(\theta)) = 2$. The sampling dynamics (both payoff-sampling and action-sampling) for a new agent of population i with sample size one at state (p_1, p_2) is $\dot{p}_i = \delta(f_1(p_j) - p_i)$, where $f_1(p_j) = 1 - p_j$ in all cases. The sampling dynamics for $\theta(2) = 1$ is $\dot{p}_i = \delta(f_2(p_j) - p_i)$, where the function f_2 depends on whether we are under action-sampling or payoff-sampling dynamics and whether $g+2l<1$ or $1<g+2l<1+l$ or $g+l>1$ (as calculated above). In all cases, it is straightforward to verify that $f_2(0)=1$, $f_2(1)=0$ and that the function $f_2(\cdot)$ is strictly decreasing on $[0,1]$. The sampling dynamics for $\max(\text{supp}(\theta)) = 2$ can be written as follows:

$$\dot{p}_i = \delta(\theta(1)f_1(p_j) + \theta(2)f_2(p_j) - p_i). \tag{A.2}$$

For $p\in[0,1]$, consider the function $f(p) = \theta(1)f_1(p) + \theta(2)f_2(p)$. Since $\theta(1), \theta(2) \geq 0$ and $\theta(1)+\theta(2) = 1$, it follows that $f(0) = 1$, $f(1) = 0$ and that the function $f(\cdot)$ is strictly decreasing. $f_1(p) = 1-p$ implies that $f_1''(p) = 0$ and thus $f''(p) = \theta(2)f_2''(p)$. For $\theta(2)\neq 0$, it follows that $f''(p)$ and $f_2''(p)$ have the same sign. These properties show that the phase portraits of both sampling dynamics for $\max(\text{supp}(\theta)) = 2$ are qualitatively the same as the corresponding phase portraits for the case of $\theta(2) = 1$. In particular, there are three rest points (namely, $(1, 0), (0, 1)$, and another interior symmetric state) of the sampling dynamics when $\max(\text{supp}(\theta)) = 2$.

A.3 Stability Analysis of Sample Size 1

In this appendix we show that if all players have sample size 1, then any state in which the sum of the shares of hawks in each population is equal to one (i.e., states of the form $(p, 1-p)$) is Lyapunov stable, and that the population converges to this set of states from any initial state. As demonstrated in Theorems 1, 3, and 4, this result is not robust to the presence of an arbitrarily small share of agents with higher sample sizes.

Proposition 2. *Let $\theta \equiv 1$. Then the set of Lyapunov stable states is $L_1 \equiv \{(p_i, 1-p_i) \,|p_i \in [0,1]\}$ for both action-sampling dynamics and payoff-sampling dynamics, and $\lim_{t\to\infty} p(t) \in L_1$ for each $p(t) \in [0,1]^2$.*

Proof. Consider an arbitrary state (p_i, p_j). Consider a new agent of population i. Suppose that this agent has a sample size of 1; then under both action-sampling and payoff-sampling dynamics, the probability that h_i yields a higher payoff is $1-p_j$. This is because with a sample size of 1, h_i yields a higher payoff iff the sampled action from the opponent population is d_j, which occurs with probability $1 - p_j$. Both the action-sampling and payoff-sampling dynamics in this case can be written as $\dot{p}_i = 1 - p_j - p_i$ for $i = \{1, 2\}$ and $j \neq i$. It is now straightforward to verify that the set of Lyapunov stable states is L_1. □

A.4 Proof of Theorem 2

Recall that the action-sampling dynamics in state (p_1, p_2) are given by

$$\dot{p}_1 = \delta(w_k(p_2) - p_1) \qquad \text{and} \qquad \dot{p}_2 = \delta(w_k(p_1) - p_2), \tag{A.3}$$

where, for brevity we omit the superscript A, i.e., we write $w_k \equiv w_k^A$.

Observe that a symmetric state (r, r) is an equilibrium of the dynamics iff r is a fixed point of the function $w_k(p)$. Let $X_k(p)$ denote a binomial distribution with parameters k and p. Let $m = \left\lfloor \frac{kg}{1+g-l} \right\rfloor$. Note that the possible values of m are $\{0, 1, \ldots, k-1\}$. To make the dependence of the function $w_k(p)$ on m explicit, we write as follows:

$$w_k(p) \equiv w_{k,m}(p) = P(X_k(p) \leq m) = F(m; k, p), \tag{A.4}$$

where $F(\cdot\,; k, p)$ is the cumulative distribution function of a binomial distribution with parameters k and p. For all m, it follows that $w_{k,m}(0) = 1, w_{k,m}(1) = 0$, and $w_{k,m}(p)$ is decreasing in p, which implies that $w_{k,m}(p)$ has a unique interior fixed point $p^{(m)}$.

In order to assess the asymptotic stability, we compute the Jacobian J of Eq. (A.3) at the symmetric rest point $(p^{(m)}, p^{(m)})$ (ignoring the constant δ, which plays no role in the dynamics, except multiplying the speed of convergence by a constant):

$$J = \begin{pmatrix} -1 & w'_{k,m}(p^{(m)}) \\ w'_{k,m}(p^{(m)}) & -1. \end{pmatrix}.$$

The eigenvalues of J are $-1 + w'_{k,m}(p^{(m)})$ and $-1 - w'_{k,m}(p^{(m)})$. A sufficient condition for instability at $(p^{(m)}, p^{(m)})$ is that $|w'_{k,m}(p^{(m)})| > 1$.

From Eq. (A.4), we now compute as follows:

$$\begin{aligned} w_{k,k-m-1}(1-p) + w_{k,m}(p) &= P(X_k(1-p) \leq k-m-1) + P(X_k(p) \leq m) \\ &= P(X_k(p) \geq k-(k-m-1)) + P(X_k(p) \leq m) \\ &= P(X_k(p) \geq m+1) + P(X_k(p) \leq m) = 1. \end{aligned}$$

The fact that $w_{k,k-m-1}(1-p) = 1 - w_{k,m}(p)$ implies that $p^{(k-m-1)} = 1 - p^{(m)}$ and $w'_{k,k-m-1}(p^{(k-m-1)}) = w'_{k,m}(p^{(m)})$. Without loss of generality, we can therefore focus on analyzing the cases of m for which $m \leq \left\lfloor \frac{k-1}{2} \right\rfloor$. To ease notation, we fix $k \geq 2$ and define a new function $f_m : [0,1] \to [0,1]$ as $f_m(p) \equiv w_{k,m}(p)$ and let $r_m = p^{(m)}$ be the fixed point of the function $f_m(p)$, i.e., $f_m(r_m) = r_m$. Since $f_m(0) = 0$, $f_m(1) = 1$, and $f_m(\cdot)$ is a strictly decreasing function, it follows that the fixed point $r_m \in (0,1)$ and that it is unique. To complete the proof we need to show that $|f'_m(r_m)| > 1$ for nonnegative integer values of m such that $m \leq \frac{k-1}{2}$. In what follows, we show this.

It is well known (see, e.g., Green, 1983) that

$$f_m(p) \equiv w_{k,m}(p) = \sum_{i=0}^{m} \binom{k}{i} p^i (1-p)^{k-i} = \binom{k}{m} (k-m) \int_0^{1-p} t^{k-m-1} (1-t)^m dt.$$

We now compute as follows:

$$f'_m(p) = -\binom{k}{m}(k-m)p^m(1-p)^{k-m-1} = -k\binom{k-1}{m}p^m(1-p)^{k-m-1} \tag{A.5}$$
$$f''_m(p) = \binom{k}{m}(k-m)p^{m-1}(1-p)^{k-m-2}((k-m-1)p - m(1-p)).$$

Fix an m such that $1 \le m \le \left\lfloor \frac{k-1}{2} \right\rfloor$. From the above computations, it follows that the function $f_m(p)$ is concave for values of $p \le p^*$ and convex for values of $p \ge p^*$ where $p^* = \frac{m}{k-1}$. This is because $f''_m(p^*) = 0$. Either the concave part or the convex part of the function $f_m(p)$ intersects the 45^o line. Suppose that the concave part of the function $f_m(p)$ intersects the 45^o line from the origin, i.e., $r_m \le \frac{m}{k-1}$. In this case, $f_m(p)$ intersects the -45^o line joining the points (1,0) and (0,1) at q^*, where $q^* < r_m$. This is because $m \le \left\lfloor \frac{k-1}{2} \right\rfloor$ implies that $r_m \le 0.5$ as $f_m(0.5) \le 0.5$ for such values of m. Since the function $f_m(p)$ intersects the -45^o line from above, we can conclude that $f'_m(q^*) < -1$. The function $f_m(p)$ is concave between q^* and r_m; therefore, we have $f'_m(r_m) < f'_m(q^*) < -1$, i.e., $|f'_m(r_m)| > 1$. Therefore we are done in cases where $r_m \le \frac{m}{k-1}$.

If the convex part of the function intersects the 45^o line from the origin, then $r_m > m/(k-1)$. By definition, we have

$$r_m = (1-r_m)^k + \binom{k}{1}r_m(1-r_m)^{k-1} + \cdots + \binom{k}{m}r_m^m(1-r_m)^{k-m} \tag{A.6}$$

For $j = 0, 1, 2, \ldots, m$, let a_j denote the j^{th} term of the sum on the RHS of Eq. (A.6). For $j = 1, 2, \ldots, m$, we compute as follows:

$$\frac{a_j}{a_{j-1}} = \frac{\binom{k}{j}r_m^j(1-r_m)^{k-j}}{\binom{k}{j-1}r_m^{j-1}(1-r_m)^{k-j+1}} = \left(\frac{k-j+1}{j}\right)\left(\frac{r_m}{1-r_m}\right)$$
$$\frac{a_j}{a_{j-1}} \ge 1 \iff (k-j+1)r_m \ge j(1-r_m) \iff r_m \ge \frac{j}{k+1}.$$

Since $\frac{m}{k-1} > \frac{j}{k+1}$ for $j = 1, 2, \ldots, m$, we have $r_m \ge \frac{j}{k+1}$ and thus $a_j \ge a_{j-1}$. This implies that $a_j \le a_m$ for $j = 1, 2 \ldots, m-1$. From Eq.(A.6), we can thus conclude the following:

$$r_m \le (m+1)\binom{k}{m}r_m^m(1-r_m)^{k-m}. \tag{A.7}$$

From Eqs. (A.5) and (A.7), we have

$$|f'_m(r_m)| = (k-m)\binom{k}{m} r_m^m (1-r_m)^{k-m-1} \geq \left(\frac{k-m}{m+1}\right)\frac{r_m}{1-r_m}.$$

From the above set of equations, a sufficient condition for $|f'_m(r_m)| > 1$ can be written as follows:

$$\left(\frac{k-m}{m+1}\right)\frac{r_m}{1-r_m} > 1 \iff r_m > \frac{m+1}{k+1}. \tag{A.8}$$

We will now establish that $|f'_m\left(\frac{m+1}{k+1}\right)| > 1$. From Eq. (A.5), we have

$$|f'_m(p)| = k\binom{k-1}{m} p^m (1-p)^{k-m-1} = k \cdot \Pr(X_{k-1}(p) = m).$$

where $X_{k-1}(p)$ is a binomial distribution with parameters $k-1$ and p. It is a known fact that the binomial distribution's mode with parameters $k-1$ and p is attained at $\lfloor kp \rfloor$. For $p = \frac{m+1}{k+1}$, we have

$$m \leq k \cdot \left(\frac{m+1}{k+1}\right) < m+1.$$

The above inequalities imply that $\left\lfloor k \cdot \left(\frac{m+1}{k+1}\right)\right\rfloor = m$. The binomial distribution $X_{k-1}(\cdot)$ has k possible values and thus the probability of the occurrence of the mode has to be greater than $\frac{1}{k}$, i.e.,

$$\Pr\left(X_{k-1}\left(\frac{m+1}{k+1}\right) = m\right) > \frac{1}{k} \implies \left|f'_m\left(\frac{m+1}{k+1}\right)\right| > 1.$$

We need to consider the following two possible cases:

Case 1: $\frac{m}{k-1} < r_m \leq \frac{m+1}{k+1}$. For $1 \leq m \leq \frac{k-1}{2}$, we know that $|f'_m(\cdot)|$ attains its maximum value at $\frac{m}{k-1}$ and that it is strictly decreasing for $p > \frac{m}{k-1}$. For $m = 0$, $|f'_m(\cdot)|$ is strictly decreasing for $p > 0$. Thus, we have:

$$|f'_m(r_m)| > \left|f'_m\left(\frac{m+1}{k+1}\right)\right| > 1.$$

Case 2: $r_m > \frac{m+1}{k+1}$. Here, we are done by the sufficient condition of Eq. (A.8).

A.5 Proof of Theorem 3

Let $a = \frac{g}{g+l}$. Since the hawk–dove game is generic, we have $a \neq \frac{1}{2}$. From Eq. (A.5), we have

$$|w'_{k,m}(p)| \equiv |f'_m(p)| = k\binom{k-1}{m}p^m(1-p)^{k-m-1} \Rightarrow |w'_{k,m}(0.5)| = k \cdot \Pr\left(X_{k-1} = m\right),$$

where $m = \lfloor ka \rfloor$ and $X_{k-1} \sim Bin\left(k-1, 0.5\right)$. The central limit theorem implies that the distribution of the binomial random variable X_{k-1} converges to a normal distribution:

$$\lim_{k\to\infty} \frac{X_{k-1}}{k-1} = Z_{k-1} \sim \text{Normal}\left(0.5, \frac{0.25}{k-1}\right).$$

In what follows we omit the subscript m; i.e., we let $w_k \equiv w_{k,m}$. This is without loss of generality as we fix the hawk–dove game and hence the value of m. We now compute

$$\begin{aligned}
&\lim_{k\to\infty} |w'_k(0.5)| = \lim_{k\to\infty} \left(k \cdot \Pr\left(X_{k-1} = \lfloor ka \rfloor\right)\right) = \lim_{k\to\infty} \left(k \cdot \Pr\left(ka \leq X_{k-1} < ka+1\right)\right) \\
&= \lim_{k\to\infty} \left(k \cdot \Pr\left(\frac{ka}{k-1} \leq \frac{X_{k-1}}{k-1} < \frac{ka+1}{k-1}\right)\right) = \lim_{k\to\infty} \left(k \cdot \Pr\left(\frac{ka}{k-1} \leq Z_{k-1} < \frac{ka+1}{k-1}\right)\right) \\
&= \lim_{k\to\infty} \left(\frac{k}{k-1} f_{Z_{k-1}}\left(\frac{ka}{k-1}\right)\right) = f_{Z_{k-1}}(a) = 0.
\end{aligned}$$

where $f_{Z_{k-1}}(\cdot)$ is the probability density function of the normal distribution Z_{k-1}. The last equality in the above series of equations is implied by $a \neq \frac{1}{2}$, and by the fact that $\lim_{k\to\infty} E\left(Z_{k-1}\right) = 0.5$ and $\lim_{k\to\infty}\text{Var}\left(Z_{k-1}\right) = 0$. This implies that there exists $\hat{k}$, such that $|w'_k(0.5)| < 0.5$ for any $k \geq \hat{k}$. By continuity, there exists $\epsilon > 0$, such that $|w'_k(p)| < 1$ for any $p \in [0.5-\epsilon, 0.5+\epsilon]$ and any $k \geq \hat{k}$. This implies that $|w_\theta{}'(p)| < 1$ for any $p \in [0.5-\epsilon, 0.5+\epsilon]$ and any distribution θ satisfying $\theta(k) = 0$ for each $1 < k < \hat{k}$ and $\theta\left(1\right) < 1$. The fact that the fixed point of $w_1(p)$ is 0.5 implies that there exists $\hat{q} < 1$ such that the fixed point of $w_\theta(p)$ is in the interval $[0.5-\epsilon, 0.5+\epsilon]$ for any distribution θ satisfying $\theta\left(1\right) > \hat{q}$. This, in turn, implies that $|w'_k(p)| < 1$ at the fixed point, which implies that the symmetric equilibrium is asymptotically stable.

A.6 Proof of Theorem 4

Recall that the payoff-sampling dynamics in state (p_1, p_2) are given by

$$\dot{p}_1 = \delta(w_\theta(p_2) - p_1) \qquad \text{and} \qquad \dot{p}_2 = \delta(w_\theta(p_1) - p_2), \tag{A.9}$$

where, for brevity we omit the superscript P, i.e., we write $w_\theta \equiv w_\theta^P$.

Observe that a symmetric state $(p^{(\theta)}, p^{(\theta)})$ is an equilibrium of the dynamics iff $p^{(\theta)}$ is a rest point of w_θ, i.e., if $w_\theta(p^{(\theta)}) = p^{(\theta)}$. Such a rest point $p^{(\theta)}$ exists because $w_\theta(1) = 0$, $w_\theta(0) = 1$, and $w_\theta(\cdot)$ is continuous on $[0, 1]$.

In order to assess the asymptotic stability, we compute the Jacobian J of Eq. (A.9) at the symmetric rest point $(p^{(\theta)}, p^{(\theta)})$ (ignoring the constant δ, which plays no role in the dynamics, except multiplying the speed of convergence by a constant):

$$J = \begin{pmatrix} -1 & w'_\theta(p^{(\theta)}) \\ w'_\theta(p^{(\theta)}) & -1. \end{pmatrix}.$$

The eigenvalues of J are $-1 + w'_\theta(p^{(\theta)})$ and $-1 - w'_\theta(p^{(\theta)})$. A sufficient condition for the asymptotic stability at $(p^{(\theta)}, p^{(\theta)})$ is therefore that $|w'_\theta(p^{(\theta)})| < 1$.

We now establish some properties of the payoff-sampling dynamics and the symmetric rest points for symmetric distributions of types $\theta \equiv k$. If $l, 1-g \in \left(0, \frac{1}{\max(\text{supp}(\theta))}\right)$, action h_i has a higher mean payoff iff the number of d_j-s in the h_i-sample is strictly greater than half the number of d_j-s in the d_i-sample. To express $w_k(p)$ concisely in this case, we define $X_k(p)$ and $Y_k(p)$ to be independent and identically distributed binomial random variables with parameters k and p. We can then write $w_k(p)$ as follows:

$$w_k(p) = P\left(k - X_k(p) > \frac{1}{2}(k - Y_k(p))\right) = P(2X_k(p) - Y_k(p) < k). \tag{A.10}$$

Observe that $w_k(p)$ is a polynomial in p of degree at most $2 \cdot k$.

We have verified the following facts about these polynomials for $k < 20$ (for an illustration see Figure 5.3; the Mathematica code is given in the online supplementary material, and the explicit values of the rest points and the derivatives are presented in Table 3):

- For $k \in \{1, 2, \ldots, 18, 19\}$, $w_k(p)$ is decreasing in p.

- For $k \in \{1, 2, \ldots, 18, 19\}$, $w_k(p)$ has a unique fixed point $p^{(k)}$.
  Moreover, $0.5 < p^{(k)} < 0.68$ for any $k \in \{1, 2, 3, 4, 5\}$.

- $|w_1'(p)| \equiv 1$, and $|w_k'(p)| < 1$ for any $k \in \{2, 3, 4, 5\}$ and $0.5 < p < 0.68$.

Recall that $w_\theta(p)$ is a convex combination of the $w_k(p)$ for the k-s in its support (i.e., $w_\theta(p) = \sum_k \theta(k) \cdot w_k(p)$). From the above facts, it follows that:

1. For $\theta \equiv k < 20$, the function $w_k(p)$ has a unique fixed point $p^{(k)}$ such that $|w_k'(p^{(k)})| < 1$, which implies that $(p^{(k)}, p^{(k)})$ is asymptotically stable.

2. For $\max(\text{supp}(\theta)) \leq 5$, the function $w_\theta(p)$ has a unique fixed point $p^{(\theta)}$ such that $p^{(\theta)} \in (0.5, 0.68)$ and $|w_\theta'(p^{(\theta)})| < 1$ if $\theta(1) \neq 1$, which implies that $(p^{(\theta)}, p^{(\theta)})$ is asymptotically stable.

Table 3: Fixed Points of the Function $w_k(p)$ in the Proof of Theorem 4

k	1	2	3	4	5	6	7	8	9	10
$p^{(k)}$	0.500	0.579	0.620	0.649	0.672	0.690	0.706	0.720	0.731	0.741
$\lvert w_k'(p^{(k)})\rvert$	1	0.690	0.618	0.645	0.690	0.730	0.763	0.793	0.818	0.840

k	11	12	13	14	15	16	17	18	19	20
$p^{(k)}$	0.750	0.758	0.765	0.773	0.778	0.784	0.789	0.794	0.799	0.803
$\lvert w_k'(p^{(k)})\rvert$	0.861	0.88	0.899	0.916	0.932	0.948	0.963	0.978	0.991	1.001

Table 4: Values of $|w_k'(p^{(j)})|$ for $k, j \in \{1, 2, 3, 4, 5\}$.

$k \backslash p^{(j)}$	$p^{(1)}$	$p^{(2)}$	$p^{(3)}$	$p^{(4)}$	$p^{(5)}$
1	1	1	1	1	1
2	0.5	0.690	0.812	0.905	0.981
3	0.562	0.560	0.618	0.687	0.759
4	0.625	0.616	0.623	0.645	0.679
5	0.605	0.642	0.659	0.673	0.690

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