Optimal portfolio under ambiguous ambiguity

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Abstract

A prominent approach to modelling ambiguity about stock return distribution is to assume that investors have multiple priors about the distribution and these priors are distributed according to a certain second-order distribution. Realistically, investors may also have multiple priors about the second-order distribution, thus allowing for *ambiguous ambiguity.* Despite a long history of debates about this idea (Reichenbach [1949], Savage [1954]), there seems to be no formal analysis of investment behavior in the presence of this feature. We develop a tractable portfolio choice framework incorporating ambiguous ambiguity, characterize analytically the optimal portfolio, and examine its properties.

Keywords: ambiguous ambiguity, portfolio choice, smooth ambiguity, third-order probabilities

☆I would like to thank my colleagues at ICEF for numerous conversations on the subject. All errors are solely my responsibility.

1. Introduction

> As a practical matter, if a person cannot express a precise probability, she may not be able to confidently express a second-order distribution either.
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> Camerer and Weber [1992]

In real financial markets, investors do not know precisely the true distribution of asset returns, hereafter referred to as FOD (first-order distribution). A prominent approach to modelling this feature is to assume that investors have *multiple* priors about FOD that are distributed according to a *single* second-order distribution (SOD). For example, this approach is taken in a large finance literature building on the KMM smooth ambiguity framework (Klibanoff, Marinacci, and Mukerji [2005])[1]. There is a long-standing debate, dating back to classical works by Reichenbach [1949] and Savage [1954], about whether it is justified to disregard the possibility that agents may have multiple priors about SOD. The crux of the debate is nicely summarized by Camerer and Weber [1992] in the epigraph above (Atkinson and Peijnenburg [2017] provide more details).

To make progress in this debate, researchers need to start developing finance models with multiple priors about SOD. Taking their predictions to the data will allow us to evaluate the empirical relevance of this mechanism. Hereafter, we refer to multiple priors about SOD as *ambiguous ambiguity* or, briefly, *A-ambiguity*. This note is, to our knowledge, the first to develop a tractable framework for analyzing investment behavior under A-ambiguity. We solve analytically for the optimal portfolio

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[1] Examples include Ju and Miao [2012] Chen, Ju, and Miao [2014] Jahan-Parvar and Liu [2014], Backus, Ferriere, and Zin [2015], Gallant, R Jahan-Parvar, and Liu [2019], and Makarov [2020].

and examine its properties.

We consider a setting in which an investor trades in a risk-free bond and two risky stocks. We view stock 1 as representing a local stock index whose return distribution is known and stock 2 as representing a foreign stock index with an ambiguous (not perfectly known) return distribution. The investor has multiple priors about stock 2's expected return that are distributed according to a normal SOD. The key novelty of this work is that the investor is allowed to have multiple priors about the mean and variance of SOD.

Our findings are as follows. In a special case in which stock 2's return distribution is known (A-ambiguity is absent), we recover the standard two-fund separation result—all investors, regardless of their preferences, hold the same portfolio of risky assets (tangency portfolio). Once A-ambiguity is incorporated, however, this is no longer the case. We present a series of results showing how the composition of risky stocks in the optimal portfolio changes as we vary the investor's attitude towards risk, ambiguity, and A-ambiguity.

Our paper is similar in spirit Taboga [2005], which is one of the first studies to examine portfolio selection under the KMM smooth ambiguity approach (Klibanoff, Marinacci, and Mukerji [2005]).[2] In Taboga [2005], an investor's behavior is smoothly driven by her ambiguity attitude and the amount of ambiguity in the stock market. The key novelty of our model setting is that, in addition to these features, the investor's behavior is affected in a smooth way by her attitude towards A-ambiguity

[2]In addition to the KMM approach, finance models use a number of other ambiguity specifications, such as the maxmin framework developed by Gilboa and Schmeidler [1989] (static case) and Chen and Epstein [2002] (continuous time case). A recent example of a model relying on this framework is Ruan and Zhang [2020].

and the amount of A-ambiguity.

2. Model

We consider a setting with a risk-free bond and two stocks 1 and 2. Stock 1 is purely risky and stock 2 is A-ambiguous. In line with Maccheroni, Marinacci, and Ruffino [2013] (Section 6.3), we interpret stock 1 as a local stock index whose return distribution is known by the local investor. Stock 2 denotes a foreign stock index whose return distribution is not perfectly known by the investor due to her being less familiar with the foreign economy. In Maccheroni et al., the ambiguity about stock 2 is modeled via a *single* SOD (distribution over return distributions). Our novelty is to propose a more general framework by allowing the investor to have *multiple* priors about SOD distributed according to a certain TOD (third-order distribution).

Here are the details. The return on the bond is normalized to zero (cash). The returns on stocks 1 and 2, denoted by R_1 and R_2, are given by

$$R_1 = \mu_1 + \sigma_1 \varepsilon_1, \quad R_2 = \mu_2 + \sigma_2 \varepsilon_2, \tag{1}$$

where ε_1 and ε_2 have standard normal distribution $N(0,1)$ with correlation $corr(\varepsilon_1, \varepsilon_2) = \rho$. As discussed above, stock 1 stands for a local index and so the investor knows its characteristics μ_1 and σ_1. Stock 2 stands for a (less familiar) foreign index and so the investor is ambiguous about its expected return μ_2. Stock 2's volatility σ_2 and the stocks' correlation ρ are known.[3] The investor has multiple priors about μ_2 dis-

[3]The assumption that stock volatility is not ambiguous is common in the literature. It is often justified by Merton [1980]'s result that "reasonably accurate estimate of the variance rate can be obtained using daily data while the estimates for expected return taken directly from the sample

tributed according to a normal SOD:

$$\mu_2 \sim N(\mu_\mu, \sigma_\mu^2). \tag{2}$$

Our contribution is that we allow the investor to have multiple priors about each of the two moments of SOD, μ_μ and σ_μ. In our terminology, her decision-making is affected by A-ambiguity, the property that she is unsure about a unique right way to model ambiguity. The priors are given by two independent normal distributions:

$$\mu_\mu \sim N(\mu_{\mu A}, \sigma_{\mu A}^2), \sigma_\mu^2 \sim N(\mu_{\sigma A}, \sigma_{\sigma A}^2), \tag{3}$$

where "A" in the subscripts means that these variables pertain to A-ambiguity. Notice that our A-ambiguity specification (3) is more general than ambiguity (2)—the latter posits that multiple priors apply only to the first moment μ_2 but not to the second moment σ_2^2. The reason is that Merton [1980]'s findings, seemingly, cannot be used to justify the assumption of a single prior about σ_μ^2 (unlike σ_2^2).

We see from eq. (3) that, according to the investor's belief, σ_μ^2 can be negative with some probability, which of course cannot happen. As in other finance works using normal distribution to model the second moment of a distribution (e.g., Scott [1987], Stein and Stein [1991]), normality considerably simplifies our analysis and allows us to obtain explicit solutions. We verify via an extensive numerical analysis that our predictions remain valid if σ_μ^2 has a truncated normal distribution with a positive support.

will be subject to so much error as to be almost useless."

The investor's initial wealth is normalized to unity. Denoting by θ_i the weight of stock i in the portfolio, the terminal wealth w is

$$w = \theta_1 R_1 + \theta_2 R_2 + (1 - \theta_1 - \theta_2). \tag{4}$$

2.1. Preferences of investor

Our investor treats A-ambiguity, ambiguity, and risk as separate portfolio characteristics, and so may display different attitude towards each of them. In a different context in which individuals make bets on urns, Epstein and Halevy [2019] also document two separate dimensions of ambiguity that can be perceived differently by an individual. To model this, they propose a specification of preferences obtained by combining three "utility" functions. The details are in Section A.3 in Epstein and Halevy. Following their insights, we introduce three "utility" functions, $u_1()$ for risk, $u_2()$ for ambiguity, and $u_3()$ for A-ambiguity, and combine them to obtain the objective function $u(w)$:

$$u(w) = \overbrace{u_3^{-1}(E_{\mu_\sigma,\sigma_\sigma}[u_3}^{\text{A-ambiguity}}\overbrace{(u_2^{-1}(E_{\mu_2}[u_2(u_1^{-1}(}^{\text{KMM}}E_{R_1,R_2}[u_1(w)]))])]). \tag{5}$$

For tractability, we assume that all functions $u_i()$ are exponential:

$$u_1(x) = -\exp(-\gamma_1 x),\ u_2(x) = -\exp(-\gamma_2 x),\ u_3(x) = -\exp(-\gamma_3 x). \tag{6}$$

Parameters $\gamma_1,\ \gamma_2, \gamma_3 > 0$ capture the investor's sensitivity to risk, ambiguity, and A-ambiguity, respectively.

Definition 1. *The investor's optimal portfolio* (θ_1, θ_2) *is a solution of the maximization problem:*

$$\max_{\theta_1,\theta_2} \quad u(\theta_1 R_1 + \theta_2 R_2 + (1 - \theta_1 - \theta_2)), \tag{7}$$

where $u(\cdot)$ *is as defined in eqs. (5) and (6).*

3. Optimal portfolio

Proposition 1 characterizes analytically the optimal portfolio.

Proposition 1. *The optimal portfolio exists and is unique. The optimal holding of stock 2 is given by*

$$\mu_{\mu A} - \frac{\sigma_1 \left(2\theta_2 \left(\gamma_2 \mu_{\sigma A} + \gamma_3 \sigma_{\mu A}^2 - \gamma_1 \left(\rho^2 - 1\right) \sigma_2^2\right) + \gamma_2^2 \gamma_3 \theta_2^3 \sigma_{\sigma A}^2 + 2\right) + 2\left(\mu_1 - 1\right) \rho \sigma_2}{2\sigma_1} = 0, \tag{8}$$

The optimal holding of stock 1 is

$$\theta_1 = \frac{\mu_1 - 1 - \gamma_1 \theta_2 \rho \sigma_1 \sigma_2}{\gamma_1 \sigma_1^2}, \tag{9}$$

where θ_2 *is the optimal weight of stock 2. In the special case when stock 2's return distribution is known, i.e., when* $\mu_{\sigma A} = \sigma_{\sigma A} = \sigma_{\mu A} = 0$, *the optimal stock holdings are given by*

$$\theta_1 = \frac{\sigma_2(\mu_1 - 1) - \rho\sigma_1\left(\mu_{\mu A} - 1\right)}{\gamma_1\left(1 - \rho^2\right)\sigma_1^2\sigma_2}, \ \theta_2 = \frac{\sigma_1\left(\mu_{\mu A} - 1\right) - \rho\sigma_2(\mu_1 - 1)}{\gamma_1\left(1 - \rho^2\right)\sigma_1\sigma_2^2}. \tag{10}$$

3.1. Analysis of optimal portfolio

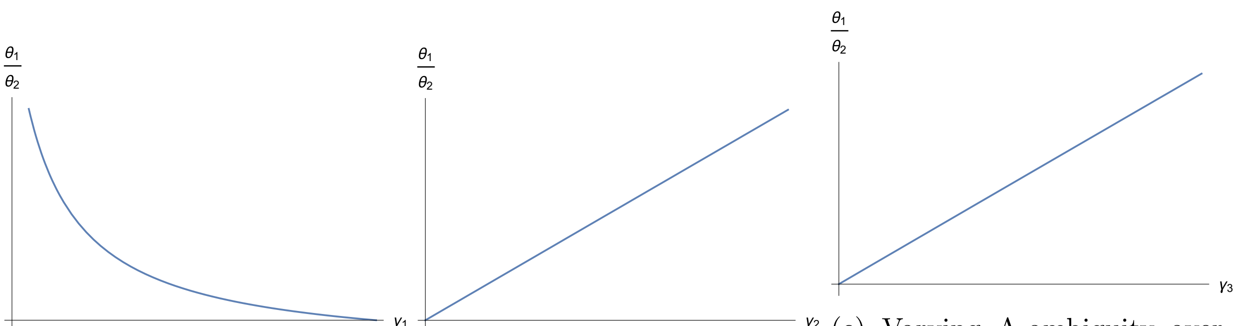
A comprehensive analysis of the optimal portfolio is beyond the scope of this note. We instead focus on just one property of the portfolio—how the ratio of the two stock holdings, θ_1/θ_2, depends on the investor's preference parameters. Our interest in this question is motivated by a classical mean-variance result stating that all investors hold the same portfolio of risky assets, the tangency portfolio, regardless of their risk aversion. If θ_1/θ_2 turns out to be sensitive to any of the investor's preference parameters in our more general setting with A-ambiguity, then the classical result does not hold under A-ambiguity.

First, we note that the mean-variance setting is obtained as a special case of our model in which the distribution of stock 2's return is known. In this case, the above-mentioned result holds: we see from eq. (10) that the ratio θ_1/θ_2 depends only on stock characteristics but not on risk aversion coefficient γ_1 (it cancels when we divide θ_1 by θ_2).

Figure 1 depicts the ratio θ_1/θ_2 for varying levels of risk aversion γ_1 (panel (a)), ambiguity aversion γ_2 (panel (b)), and A-ambiguity aversion γ_3 (panel (c)). The first observation from the Figure is that the ratio θ_1/θ_2 is sensitive to γ_1, γ_2, and γ_3. Therefore, there does not exist a single portfolio of risky stocks (mutual fund) that is held by all investors.

Second, recall that a key difference between the stocks is that stock 1's distribution is known while unknown for stock 2. At the outset, it is not obvious whether increasing risk aversion γ_1 will make the investor to tilt her portfolio towards (purely risky) stock 1 or (A-ambiguous) stock 2. Panel (a) in Figure 1 reveals that the latter is the case , as we see that θ_1/θ_2 decreases in γ_1. In other words, an increase in γ_1 leads to a proportionally higher reduction in the investment in stock 1 relative to stock 2.

Panels (b) and (c) in Figure 1 present natural results the ratio θ_1/θ_2 increases in both ambiguity and A-ambiguity aversion parameters, γ_2 and γ_3. As either of these parameters increases, the investor finds A-ambiguous stock 2 less attractive and so reduces her investment in stock 2 leading to a higher θ_1/θ_2. Interestingly, looking at stock 1 individually, its weight in the optimal portfolio can increase or decrease depending on model parameters. For example, Figure 2 shows that as ambiguity or A-ambiguity aversion goes up, the optimal investment in stock 1, θ_1, may be increasing if the correlation between stocks 1 and 2 is positive but decreasing if the correlation is negative.

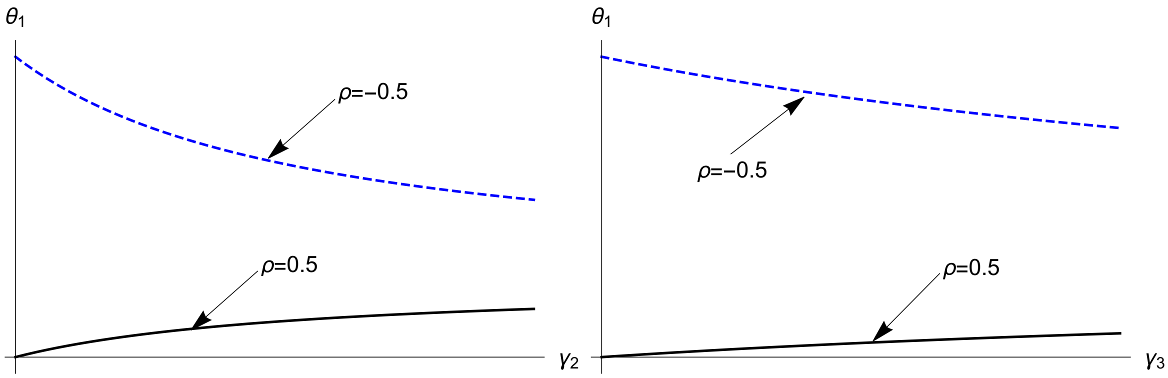


(a) Varying risk aversion

(b) Varying ambiguity aversion

(c) Varying A-ambiguity aversion

Figure 1: Effect of preference parameters on the ratio of the two stock holdings. The parameter values are $\gamma_1 = 1$, $\gamma_2 = 2$, $\gamma_3 = 3$, $\sigma_1 = 0.3$, $\sigma_2 = 0.3$, $\mu_1 = 1.1$, $\rho = 0.5$, $\mu_{\mu A} = 1.1$, $\sigma_{\mu A} = 0.3$, $\mu_{\sigma A} = 0.3$, $\sigma_{\sigma A} = 0.3$. In each panel, the variable on the x-axis varies from 0.5 to 5.



(a) Varying ambiguity aversion

(b) Varying A-ambiguity aversion

Figure 2: Effect of ambiguity and A-ambiguity aversion on investment in (purely risky) stock 1. In each panel, the solid line corresponds to a positive correlation $\rho = 0.5$ and the dashed line to a negative correlation $\rho = -0.5$. Othe parameter values are as in Figure 1.

4. Conclusion

This paper develops a tractable investment model for analysing portfolio choice implications of ambiguous ambiguity. We characterize analytically the optimal portfolio and examine its properties. We show that a classical mean-variance result that all investors hold the same portfolio of risky assets (tangency portfolio) does no longer hold once the setting is generalized to incorporate ambiguous ambiguity. We examine how the composition of the risky stocks in the optimal portfolio depends on the investor's preference parameters.

Appendix A. Proofs

Proof of Proposition 1. First, we calculate all the expectations in the objective function eq. (5). We start from the inner-most expectation and then move outwards one step at a time. The derivations are presented below. Using the stocks' distributions given in (1), we have

$$E_{R_1,R_2}[u_1(w)] = -e^{\frac{1}{2}\gamma_1\left(\theta_1(2\gamma_1\theta_2\rho\sigma_1\sigma_2-2\mu_1+2)+\theta_2\left(\gamma_1\theta_2\sigma_2^2-2\mu_2+2\right)+\gamma_1\theta_1^2\sigma_1^2-2\right)},$$

$$u_1^{-1}(E_{R_1,R_2}[u_1(w)]) = \theta_1(\mu_1-\gamma_1\theta_2\rho\sigma_1\sigma_2-1)+\theta_2(\mu_2-1)-\frac{1}{2}\gamma_1\theta_1^2\sigma_1^2-\frac{1}{2}\gamma_1\theta_2^2\sigma_2^2+1 \tag{A.1}$$

Given that μ_2 has distribution (2), we have after some algebra

$$\begin{aligned} u_2^{-1}\left(E_{\mu_2}[u_2(u_1^{-1}(E_{R_1,R_2}[u_1(w)]))]\right) &= \theta_1(-\gamma_1\theta_2\rho\sigma_1\sigma_2+\mu_1-1)- \\ -\frac{1}{2}\theta_2\left(\gamma_2\theta_2\sigma_\mu^2+\gamma_1\theta_2\sigma_2^2-2\mu_\mu+2\right)-\frac{1}{2}\gamma_1\theta_1^2\sigma_1^2+1. \end{aligned} \tag{A.2}$$

Finally, using (3) we obtain the investor's objective function

$$\begin{aligned} u(w) = &-\frac{1}{8}\theta_2\left(4\theta_2\left(\gamma_2\mu_{\sigma A}+\gamma_3\sigma_{\mu A}^2+\gamma_1\sigma_2^2\right)+\gamma_2^2\gamma_3\theta_2^3\sigma_{\sigma A}^2-8\mu_{\mu A}+8\right) \\ &+\theta_1(-\gamma_1\theta_2\rho\sigma_1\sigma_2+\mu_1-1)-\frac{1}{2}\gamma_1\theta_1^2\sigma_1^2+1. \end{aligned} \tag{A.3}$$

Treating $u(w)$ in (A.3) as a function of θ_1 and θ_2 and denoting H its Hessian, we obtain after straightforward computations

$$H = \begin{pmatrix} -\gamma_1\sigma_1^2 & -\gamma_1\rho\sigma_1\sigma_2 \\ -\gamma_1\rho\sigma_1\sigma_2 & \frac{1}{4}\left(-3\gamma_2^2\gamma_3\theta_2^2\sigma_{\sigma\mathrm{A}}^2-4\left(\gamma_2\mu_{\sigma\mathrm{A}}+\gamma_3\sigma_{\mu\mathrm{A}}^2+\gamma_1\sigma_2^2\right)\right)-\frac{3}{4}\gamma_2^2\gamma_3\theta_2^2\sigma_{\sigma\mathrm{A}}^2 \end{pmatrix}. \tag{A.4}$$

H is negative definite by Sylvester's criterion, given that the following two conditions are satisfied. First, the top-left element is negative, $-\gamma_1\sigma_1^2 < 0$. Second, the determinant of H is positive:

$$det(H) = \frac{1}{2}\gamma_1\sigma_1^2\left(3\gamma_2^2\gamma_3\theta_2^2\sigma_{\sigma\text{A}}^2 + 2\gamma_2\mu_{\sigma\text{A}} + 2\gamma_3\sigma_{\mu\text{A}}^2 + 2\gamma_1\left(1-\rho^2\right)\sigma_2^2\right) > 0. \tag{A.5}$$

Therefore, $u(w)$ is strictly concave positive and so its critical point is the global maximum. To find the critical point, we differentiate (A.3) with respect to θ_1 and θ_2 to obtain the two first-order conditions:

$$-\gamma_1\sigma_1\left(\theta_2\rho\sigma_2 + \theta_1\sigma_1\right) + \mu_1 - 1 = 0, \tag{A.6}$$

$$-\gamma_2\theta_2\mu_{\sigma\text{A}} - \gamma_3\theta_2\sigma_{\mu\text{A}}^2 - \gamma_1\sigma_2\left(\theta_1\rho\sigma_1 + \theta_2\sigma_2\right) - \frac{1}{2}\gamma_2^2\gamma_3\theta_2^3\sigma_{\sigma\text{A}}^2 + \mu_{\mu\text{A}} - 1 = 0. \tag{A.7}$$

Solving eq. (A.6) for θ_1 yields eq. (9). Substituting eq. (9) into eq. (A.7) yields eq. (8). Note that eq. (8) is a cubic equation, and we can verify that it has a unique solution by showing that its discriminant D is negative. Computing D and simplifying the obtained expression, we obtain that it is indeed negative:

$$D = -2\gamma_2^2\gamma_3\sigma_{\sigma\text{A}}^2\left(\gamma_2\mu_{\sigma\text{A}} + \gamma_3\sigma_{\mu\text{A}}^2 + \gamma_1\left(1-\rho^2\right)\sigma_2^2\right)^3 - \frac{27\gamma_2^4\gamma_3^2\sigma_{\sigma\text{A}}^4\left(\sigma_1\mu_{\mu\text{A}} + (\mu_1-1)\rho\sigma_2 + \sigma_1\right)^2}{4\sigma_1^2} < 0.$$

As for a special case in which stock 2's return distribution is known, we substitute the corresponding conditions $\mu_{\sigma A} = \sigma_{\sigma A} = \sigma_{\mu A} = 0$ into the optimal portfolio expressions eqs. (8) and (9). This yields eq. (10). □

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