Equivalence between fixed fee and ad valorem profit royalty

Colombo, Stefano and Ma, Siyu and Sen, Debapriya and Tauman, Yair

Universita Cattolica del Sacro Cuore, China University of Political Science and Law, Ryerson University, IDC Herzliya Stony Brook University

3 June 2021

Equivalence between fixed fee and ad valorem profit royalty*

STEFANO COLOMBO† SIYU MA‡ DEBAPRIYA SEN§ YAIR TAUMAN¶

July 21, 2021

Abstract

For an outside innovator with a finite number of buyers of the innovation, this paper compares two licensing schemes: (i) fixed fee, in which a licensee pays a fee to the innovator and (ii) ad valorem profit royalty, in which a licensee leaves a fraction of its profit with the innovator. We show these two schemes are equivalent in that for any number of licenses the innovator puts for sale, these two schemes give the same licensing revenue. We obtain this equivalence result in a general model with minimal structure. It is then applied in a Cournot oligopoly for an outside innovator. Finally, in a Cournot duopoly it is shown that when the innovator is one of the incumbent firms rather than an outsider, the equivalence result does not hold.

Corrects some typos of the published version. Specifically: (i) the correct statement in Remark 6 is for linear demand fixed fee is superior for small n while per unit royalty is superior for large n (page 14) and (ii) gives correct statements on the properties of the function $H(p, v)$ (Appendix, page 20).

*We are grateful to the editor Rabah Amir and two anonymous reviewers for their helpful comments and suggestions.

†Università Cattolica del Sacro Cuore.

‡China University of Political Science and Law.

§Ryerson University.

¶IDC Herzliya & Stony Brook University.

1 Introduction

Licensing of innovations is a common practice in industries. It is a key factor in disseminating new technologies. A large literature has extensively studied the impact of various licensing schemes such as upfront fees and per unit royalties on market structure, incentives to innovate and welfare (e.g., Arrow, 1962; Katz and Shapiro, 1986; Kamien and Tauman, 1986). More recently, ad valorem profit royalty licensing, sometimes also called profit-sharing, has received attention due to its widespread use. Under an ad valorem profit royalty policy, a licensee pays a specified proportion of its profit to the innovator. Real world examples of profit-sharing include agreements between Motorola and Universal Display Corporation in 2000, CSIRO and PolyNovo in 2005 and Microsoft and Skinkers in 2006 (see Vishwasrao, 2007; San Martín and Saracho, 2010; Niu, 2017; Hsu et al., 2019).

The theoretical literature of ad valorem profit royalty licensing has mostly looked at the situation in which the innovator is one of the incumbent firms who compete with the potential licensees. In a Cournot duopoly with cost asymmetry between firms, Mukhopadhyay et al. (1999) show that transfer of the least-cost technology through ad valorem profit royalty is beneficial fo both firms. Extending this result to general demand, Niu (2017) shows the optimal ad valorem royalty rate is the one in which the licensee obtains its reservation payoff, which is its profit without a license. San Martín and Saracho (2010) show that ad valorem is superior to per unit royalty for an incumbent innovator in a Cournot duopoly with linear demand; as per unit royalty is superior to fixed fee in this set-up (Wang, 1998), ad valorem royalty is superior to fixed fee as well. Hsu et el. (2019) extend this comparison to general demand. An ad valorem profit royalty contract between competing firms effectively gives passive partial ownership of a firm to its rival; imposing a bound on the rate of ad valorem royalty is considered a desirable policy intervention for consumer welfare (on this, see Proposition 3).

Apart from licensing, the literature has also looked at alternative policies such as selling the patent rights. Tauman and Weng (2012) show that if an external patentee is allowed to sell its patent rights, it is optimal to sell the rights to a single firm in an oligopoly and this policy also gives a higher incentive for innovation to an external innovator compared to an incumbent firm. In a Cournot duopoly with an incumbent patentee in which the patentee has a higher initial cost than its rival, Niu (2019) shows that it may be profitable to use this policy together with "reverse licensing" (that is, the innovator sells its patent rights to its rival and the rival firm licenses back the innovation instead of using it exclusively). Compared to standard licensing, the welfare implications of these policies generally depend on factors such as the extent of initial cost asymmetry.

Going beyond the homogenous good Cournot duopoly, the ranking between per unit and ad valorem royalties is rather ambiguous and is subject to the specification of the model (Colombo and Filippini, 2015). For instance, in a duopoly with differentiated products, Niu (2013) shows that for an incumbent innovator who can combine fixed fees with either ad valorem or per unit royalty, the two kinds of royalty policies are equivalent.

As the real world examples of licensing agreements demonstrate, it can be often the case where the innovator is not in direct competition with the licensees. Such situations are better captured by modeling the innovator as an external entity who is an outsider to the industry. Although the literature has looked at fixed fees, per unit royalties as well as their combinations for both outside and incumbent innovators (e.g., Kamien et al., 1992; Sen, 2005; Sen and Tauman, 2007; 2018), the analysis of ad valorem profit royalty licensing has been mostly limited to the case where the innovator is an incumbent firm. Seeking to fill this gap, this paper presents an analysis of ad valorem profit royalty licensing with an outside innovator in a general setting.

Specifically, we consider an outside patent holder of an innovation who interacts with a finite number of potential buyers. The innovator can license its innovation to some or all of these buyers. We consider two licensing schemes: (i) licensing by means of a fixed fee, in which a licensee pays a fee to the innovator and (ii) ad valorem profit royalty, in which a licensee leaves a fraction of its profit to the innovator. It is shown these two schemes are equivalent for the innovator: for any number of licenses the innovator puts for sale, it obtains the same licensing revenue from these two schemes. This result is obtained in a general model with minimal structure. The intuitive explanation of this equivalence result is that for each of these schemes, given any number of licenses, a licensee can be left with no more than its opportunity cost for a license. Either a suitable fixed fee or a suitable fraction of profit can achieve this outcome.

We look at two alternative methods through which a fixed fee or an ad valorem royalty can be determined: (i) an auction, in which the bids of potential buyers determine the fees or the ad valorem royalties they pay (Section 2.2) and (ii) posted price, in which the innovator sets a fixed fee or an ad valorem royalty at which any buyer can have a license (Section 2.3). The equivalence result holds in both cases.[1] Other than the assumption that a licensee always has a higher profit than a non-licensee, the auction method requires no additional assumption (Theorem 1). The posted price method requires an additional assumption of second order monotonicity and typically yields the innovator a lower licensing revenue (Theorem 2).

If the number of licenses offered is lower than the total number of firms, auction is superior to posted price, but if licenses are offered to all firms, an auction together with a minimum bid coincides with the posted price method. In the latter case, to ensure that all firms become licensees in the *unique* equilibrium, second order monotonicity conditions are required. These conditions are different for fixed fee and ad valorem royalty. For fixed fee, this condition involves the difference of the profits of a licensee and a non-licensee, while for ad valorem royalty it involves the relative difference: the ratio between the difference in profits and the profit of a licensee (see (3) and Lemma 1). In specific applications such as licensing in a Cournot oligopoly with linear demand, when more firms have licenses, the difference falls, but the ratio goes up (see Remark 7). This shows that although the equivalence result may seem intuitively obvious, the way it precisely works does depend on the nature of the relevant strategic interaction.

[1] Katz and Shapiro (1986) also looked at these two alternative methods of selling licenses and had similar assumptions. Our paper is distinct from this early work in two respects: first, we look at ad valorem profit royalties in addition to fixed fees that were considered in their work and second, we establish the equivalence of these two schemes.

The paper is organized as follows. Presenting the model, we obtain the equivalence result in Section 2. In Section 3.1, we apply the result in a Cournot oligopoly to compare fixed fee, per unit royalty and ad valorem royalty for an outside innovator (Proposition 1). Finally in Section 3.2, using a Cournot duopoly it is demonstrated that the equivalence result does not hold for an incumbent innovator (Propositions 2,3). This is because at the Cournot stage (when firms choose quantities), the payoff of an incumbent innovator under ad valorem royalty is the sum of its own operating profit and a fraction of the operating profit of the licensee, while fixed fee, being a lump-sum transfer, results in payoff which is simply its own operating profit.

2 The model

Consider an innovator I who has a patent on an innovation. There are $n \geq 2$ firms who are potential buyers of the innovation. The innovator can license its patented innovation to some or all of these n buyers. The innovator is an outsider and not one of the firms. The objective of the innovator is to maximize its revenue from licensing.

Assumption 1 (anonymity): The operating profit of any firm is completely determined by (i) whether it has a license or not and (ii) the number licensees.

Under Assumption 1, any two licensees obtain the same profit. Similarly any two non-licensees also obtain the same profit. Denote by $\pi^1(k)$ the profit of a licensee and $\pi^0(k)$ the profit of a non-licensee when there are k licensees.

Remark 1 An example in which Assumption 1 holds is a Cournot oligopoly where firms are symmetric in all aspects. Assumption 1 does not hold if firms are asymmetric in some respects such as their initial costs. In that case, the profit of a firm not only depends on whether it has a license or not, but also on the identities of the licensees. Another example in which Assumption 1 does not hold is a Stackelberg oligopoly where some firms are leaders and others followers. There the profit of a firm not only depends on whether it has a license or not, but also on whether it is a leader or a follower.

Assumption 2 $\pi^1(k) > \pi^0(k) \geq 0$ for $k = 1, \ldots, n-1$.

Assumption 3 $\pi^1(k) > \pi^0(k-1) \geq 0$ for $k = 1, \ldots, n$.

Assumption 2 states a licensee always obtains higher profit than a non-licensee and all profits are non-negative. Note that beginning from any number of licensees, if any one licensee switches to being a non-licensee, the number of licensees will drop by one. Assumption 3 says such a switch is never beneficial for a licensee. This assumption was also used by Katz and Shapiro (1986, p.572).

Remark 2 This analysis also applies to the special case where one or both of $\pi^1(k)$, $\pi^0(k)$ are constants that do not depend on k. For instance, suppose $\pi^1(k)$ is positive and increasing in k and $\pi^0(k) = 0$ for all k. This can correspond to a situation where the innovation is a new product that has positive network externality. Any user that does not have the new product obtains zero and due to positive network externality, the benefit that a user gets from the product goes up if more people have the product.

2.1 Licensing schemes

We consider two licensing schemes: (i) fixed fee and (ii) ad valorem profit royalty. Under fixed fee, a licensee pays a fee $f \geq 0$ to the innovator. Thus, when there are k licensees, the net profit of each licensee is $\pi^1(k) - f$. Under ad valorem profit royalty, a licensee leaves a fraction v $(0 \leq v \leq 1)$ of its profit to the innovator and retains the remaining fraction $1 - v$. The fraction v is the ad valorem royalty. When there are k licensees, a licensee obtains net profit $(1 - v)\pi^1(k)$.

Under each licensing scheme, the licensing revenue of the innovator is the sum of the payments it receives from all licensees. The objective of the innovator is to maximize this licensing revenue.

For each of the two schemes, there are two ways the fee or the ad valorem royalty can be determined: (i) I announces to sell a certain number of licenses through an auction and bids of firms determine their payments and (ii) I posts a fixed fee f or an ad valorem profit royalty v and firms choose whether to pay that posted price to have a license.[2] We in turn consider each of these two alternatives.

2.2 Licensing payments determined through auction

Fixed fee through auction When I sells licenses using fixed fee through an auction, the game G_F is played. In this game, I chooses the number k of licenses to offer and then announces to sell at most k licenses by fixed fee through an auction. Firms simultaneously place non-negative bids. Any firm that wins a license pays its bid as fixed fee.

If $m \leq k$ firms place bids, each of the bidding firms wins a license. If $m > k$ firms place bids, bids are arranged in ascending order as $f_1 \geq \ldots \geq f_k \geq \ldots \geq f_m$. If $f_k > f_{k+1}$, firms with k highest bids win licenses. If $f_k = f_{k+1}$, then (a) firms with bids strictly higher than f_k win licenses and (b) a random tie breaking process is run among the firms who place bid f_k to determine who get the remaining licenses. When m firms are granted licenses, the payoff of I is its licensing revenue $\sum_{i=1}^{m} f_i$. A firm that wins a license with bid f_i obtains $\pi^1(m) - f_i$ and any firm that does not win a license obtains $\pi^0(m)$.

Ad valorem profit royalty through auction When I sells licenses using ad valorem profit royalty through an auction, the game G_V is played. In this game, I chooses the number k of licenses to offer and then announces to sell at most k licenses by ad valorem royalty through an auction. Firms simultaneously place bids that are non-negative fractions. A firm that wins a license with bid v leaves fraction v of its profit with I and retains the remaining fraction $1 - v$.

As before, if at most k firms place bids, each of the bidding firms win a license. If more than k firms place bids, the same process as in the case of fixed fee is followed to determine which firms win licenses. When m firms are granted licenses, the payoff of I is its licensing revenue $\sum_{i=1}^{m} v_i\pi^1(m)$. A firm that wins a license with bid v_i obtains $(1 - v_i)\pi^1(m)$ and any firm that does not win a license obtains $\pi^0(m)$.

[2] Katz and Shapiro (1986) call the first method "quantity sales strategy" and the second method "price strategy".

Remark 3 Note that when $k = n$, a firm is guaranteed to win a license by placing any bid, so it will be optimal for each firm to place a zero bid. To ensure positive licensing revenue, I needs to specify a minimum bid for $k = n$. In that case, it is not optimal for any firm to place a bid more than the specified minimum bid, so the minimum bid effectively becomes a posted price of a license (see Observation 1). As licensing through a posted price is studied in Section 2.3, to avoid repetation the choice of k for I is restricted to $k = 1, \ldots, n-1$ for both G_F, G_V.

For $k = 1, \ldots, n-1$, denote by $G_F(k)$ the subgame of G_F that follows the announcement of I to sell at most k licenses using fixed fee. Let $G_V(k)$ be the corresponding subgame for G_V. The players of each of these games are the n firms. Let a be strategy of a firm that corresponds to "not place any bid". For $G_F(k)$, any firm can either place no bid or place any non-negative bid, so the strategy set of each firm is $\{a\} \cup \mathbb{R}_+$. Similarly, for $G_V(k)$, any firm can either place no bid or can place any bid that is a non-negative fraction, so the strategy set of each firm is $\{a\} \cup [0, 1]$. The payoffs of firms in $G_F(k)$, $G_V(k)$ are determined by the rules described before. Denote

$$\Delta(k) := \pi^1(k) - \pi^0(k) \text{ and } \phi(k) := 1 - \pi^0(k)/\pi^1(k) \tag{1}$$

By Assumption 2, $\Delta(k) > 0$ and $0 < \phi(k) \leq 1$ for $k = 1, \ldots, n-1$. Also observe that

$$\phi(k)\pi^1(k) = \Delta(k) \text{ and } \pi^1(k) - \Delta(k) = [1 - \phi(k)]\pi^1(k) = \pi^0(k) \tag{2}$$

The next result establishes the payoff equivalence between G_F and G_V.

Theorem 1 (payoff equivalence between G_F and G_V)

(I) *For any $k = 1, \ldots, n-1$, each of the games $G_F(k), G_V(k)$ has a unique equilibrium outcome. The properties of the equilibrium outcome for each game is as follows.*

  (a) *The highest bid is $\Delta(k)$ in $G_F(k)$ and $\phi(k)$ in $G_V(k)$.*

  (b) *For both games, at least $k + 1$ firms place the highest bid and k of them are chosen at random to be licensees.*

  (c) *For both games, the innovator obtains the same licensing revenue $k\Delta(k)$ and any firm (regardless of whether it has a license or not) obtains $\pi^0(k)$.*

(II) *Let $M = max_{k \in \{1,\ldots,n-1\}} k\Delta(k)$. For both G_F, G_V, in any subgame-perfect equilibrium, the number of licensees must be in $argmax_{k \in \{1,\ldots,n-1\}} k\Delta(k)$ and the innovator obtains M.*

Proof See the Appendix for the proof of parts (I)(a)-(b).

For part (I)(c), note by (a)-(b) that in $G_F(k)$, the licensing revenue of the innovator is $k\Delta(k)$ and in $G_V(k)$, it is $k\phi(k)\pi^1(k)$ which equals $k\Delta(k)$ (by (2)). In both games, any firm that does not have a license obtains $\pi^0(k)$. Any firm that has a license obtains $\pi^1(k) - \Delta(k) = \pi^0(k)$ in $G_F(k)$ and $[1 - \phi(k)]\pi^1(k) = \pi^0(k)$ in $G_V(k)$.

Finally, part (II) is immediate from part (I). ∎

2.3 Licensing payments determined through posted price

Let us now look at the alternative procedure in which for each of the two policies, the licensing payments are determined through a posted price by the innovator rather than through an auction. Denote by $\widehat{G}_F$ the game in which I chooses a fixed fee $f \geq 0$ and posts the fee f; any firm can buy a license by paying the fee f to I. If k firms buy licenses, any firm that has a license obtains $\pi^1(k) - f$, any firm that does not have a license obtains $\pi^0(k)$ and I obtains licensing revenue kf.

Denote by $\widehat{G}_V$ the game in which I chooses an ad valorem profit royalty $v \in [0, 1]$ and posts the ad valorem royalty v; any firm can buy a license by agreeing to leave the fraction v of its profit to I. If k firms buy licenses, any firm that has a license obtains $(1-v)\pi^1(k)$, any firm that does not have a license obtains $\pi^0(k)$ and I obtains licensing revenue $kv\pi^1(k)$.

For any $f \geq 0$, let $\widehat{G}_F(f)$ be the subgame of $\widehat{G}_F$ that follows the posted fee f. For any $v \in [0, 1]$, let $\widehat{G}_V(v)$ be the corresponding subgame of $\widehat{G}_V$. The players of each of these subgames are the n firms. In each of these games, any firm has two strategies: (i) buy a license and (ii) not buy. The payoffs of the firms depend on the number of firms buying licenses, as described before. Denote

$$\widehat{\Delta}(k) := \pi^1(k) - \pi^0(k-1) \text{ and } \widehat{\phi}(k) := 1 - \pi^0(k-1)/\pi^1(k) \tag{3}$$

By Assumption 3, $\widehat{\Delta}(k) > 0$ and $0 < \widehat{\phi}(k) \leq 1$ for $k = 1, \ldots, n$. Observe that

$$\widehat{\phi}(k)\pi^1(k) = \widehat{\Delta}(k) \text{ and } \pi^1(k) - \widehat{\Delta}(k) = [1 - \widehat{\phi}(k)]\pi^1(k) = \pi^0(k-1) \tag{4}$$

It will be shown that if both $\widehat{\Delta}(k)$ and $\widehat{\phi}(k)$ are decreasing, then payoff equivalence between $\widehat{G}_F$ and $\widehat{G}_V$ holds. To see this, we begin with the following lemma which looks at the implications of monotonicity of these functions.

Lemma 1 (implications of monotonicity of $\widehat{\Delta}(k), \widehat{\phi}(k)$)

(I) *Suppose $\widehat{\Delta}(k)$ is increasing, that is, $\widehat{\Delta}(1) < \ldots < \widehat{\Delta}(n)$.*

(a) *If $f < \widehat{\Delta}(1)$, then $\widehat{G}_F(f)$ has a unique equilibrium: all n firms buy licenses.*

(b) *If $f > \widehat{\Delta}(n)$, then $\widehat{G}_F(f)$ has a unique equilibrium: no firm buys a license.*

(c) *If $\widehat{\Delta}(1) \leq f \leq \widehat{\Delta}(n)$, then $\widehat{G}_F(f)$ has two equilibria: one in which all n firms buy licenses and one in which no firm does.*

(II) *Suppose $\widehat{\phi}(k)$ is increasing, that is, $\widehat{\phi}(1) < \ldots < \widehat{\phi}(n)$.*

(a) *If $v < \widehat{\phi}(1)$, then $\widehat{G}_V(v)$ has a unique equilibrium: all n firms buy licenses.*

(b) *If $v > \widehat{\phi}(n)$, then $\widehat{G}_V(v)$ has a unique equilibrium: no firm buys a license.*

(c) *If $\widehat{\phi}(1) \leq v \leq \widehat{\phi}(n)$, then $\widehat{G}_V(v)$ has two equilibria: one in which all n firms buy licenses and one in which no firm does.*

(III) *Suppose* $\widehat{\Delta}(k)$ *is decreasing, that is,* $\widehat{\Delta}(1) > \ldots > \widehat{\Delta}(n)$.

(a) *If* $f < \widehat{\Delta}(n)$, *then* $\widehat{G}_F(f)$ *has a unique equilibrium: all* n *firms buy licenses.*

(b) *If* $f > \widehat{\Delta}(1)$, *then* $\widehat{G}_F(f)$ *has a unique equilibrium: no firm buys a license.*

(c) *For* $k = 1, \ldots, n-1$, *if* $\widehat{\Delta}(k+1) < f < \widehat{\Delta}(k)$, *then* $\widehat{G}_F(f)$ *has a unique equilibrium:* k *firms buy a license and the remaining* $n-k$ *firms do not buy.*

(d) *For* $k = 1, \ldots, n$, *if* $f = \widehat{\Delta}(k)$, *then* $\widehat{G}_F(f)$ *has two equilibria: one in which* k *firms buy licenses and the remaining* $n-k$ *firms do not buy; another in which* $k-1$ *firms buy licenses and the remaining* $n-k+1$ *firms do not buy.*

(e) *The maximum revenue* I *can obtain with* k *licensees is* $k\widehat{\Delta}(k)$. *By setting* f *lower than but sufficiently close to* $\widehat{\Delta}(k)$, I *can ensure the unique equilibrium of* $\widehat{G}_F(f)$ *has* k *licensees and obtain a revenue arbitrarily close to* $k\widehat{\Delta}(k)$.

(IV) *Suppose* $\widehat{\phi}(k)$ *is decreasing, that is,* $\widehat{\phi}(1) > \ldots > \widehat{\phi}(n)$.

(a) *If* $v < \widehat{\phi}(n)$, *then* $\widehat{G}_V(v)$ *has a unique equilibrium: all* n *firms buy licenses.*

(b) *If* $v > \widehat{\phi}(1)$, *then* $\widehat{G}_V(v)$ *has a unique equilibrium: no firm buys a license.*

(c) *For* $k = 1, \ldots, n-1$, *if* $\widehat{\phi}(k+1) < v < \widehat{\phi}(k)$, *then* $\widehat{G}_V(v)$ *has a unique equilibrium:* k *firms buy a license and the remaining* $n-k$ *firms do not buy.*

(d) *For* $k = 1, \ldots, n$, *if* $v = \widehat{\phi}(k)$, *then* $\widehat{G}_V(v)$ *has two equilibria: one in which* k *firms buy licenses and the remaining* $n-k$ *firms do not buy; another in which* $k-1$ *firms buy licenses and the remaining* $n-k+1$ *firms do not buy.*

(e) *The maximum revenue* I *can obtain with* k *licensees is* $k\widehat{\phi}(k)\pi^1(k) = k\widehat{\Delta}(k)$. *By setting* v *lower than but sufficiently close to* $\widehat{\phi}(k)$, I *can ensure the unique equilibrium of* $\widehat{G}_V(v)$ *has* k *licensees and obtain a revenue arbitrarily close to* $k\widehat{\Delta}(k)$.

Proof See the Appendix. ■

Recall from Remark 3 that if I wants to sell n licenses using an auction, to ensure positive licensing revenue it has to specify a minimum bid. In that case, any firm willing to have a license will pay exactly the specified minimum bid, so the minimum bid effectively becomes a posted price. The next observation then follows from parts (III)(d)-(e), (IV)(d)-(e) of Lemma 1.

Observation 1 (auction with minimum bid for $k = n$)

(i) *Suppose* $\widehat{\Delta}(k)$ *is decreasing. If* I *offers* n *licenses by fixed fee through an auction with minimum bid* $\underline{f} = \widehat{\Delta}(n)$, *the ensuing game has two equilibria: one in which all* n *firms buy licenses; another in which* $n-1$ *firms buy licenses and one firm does not buy. The maximum revenue* I *can obtain with* n *licensees is* $n\widehat{\Delta}(n)$; *by setting a minimum bid lower than but sufficiently close to* $\widehat{\Delta}(n)$, I *can ensure there are* n *licensees to obtain a revenue arbitrarily close to* $n\widehat{\Delta}(n)$.

(ii) *Suppose* $\widehat{\phi}(k)$ *is decreasing. The same conclusion as* (i) *holds if* I *offers* n *licenses by ad valorem royalty through an auction and minimum bid* $\underline{v} = \widehat{\phi}(n)$.

Using Lemma 1, the next result shows if $\widehat{\Delta}(k)$, $\widehat{\phi}(k)$ are both decreasing, then fixed fee and ad valorem royalty policies are equivalent for the innovator also under licensing by posted price. Moreover, if the operating profit a non-licensee is weakly decreasing in the number of licensees (this happens in a Cournot oligopoly, see Section 3.1), then for both policies, auction is superior to posted price for the innovator.

Theorem 2 (payoff equivalence between $\widehat{G}_F$ and $\widehat{G}_V$) *Suppose* $\widehat{\Delta}(k)$, $\widehat{\phi}(k)$ *are both decreasing.*

(I) *Let* $\widehat{M} = max_{k\in\{1,\ldots,n\}} k\widehat{\Delta}(k)$. *For both* $\widehat{G}_F, \widehat{G}_V$, *in any subgame-perfect equilibrium, the number of licensees must be in* $argmax_{k\in\{1,\ldots,n\}} k\widehat{\Delta}(k)$ *and the innovator obtains* $\widehat{M}$.

(II) *Suppose* $\pi^0(k)$ *is weakly decreasing, that is,* $\pi^0(k) \le \pi^0(k-1)$ *for all* $k = 1, \ldots, n-1$. *Then for both fixed fee and ad valorem royalty policies, licensing by auction* (*where for* $k = n$, *the auction specifies minimum bid* $\underline{f} = \widehat{\Delta}(n)$ *for fixed fee and* $\underline{v} = \widehat{\phi}(n)$ *for ad valorem royalty*) *is superior to licensing by posted price for the innovator.*

Proof (I) By Lemma 1 ((III)(d)-(e), (IV)(d)-(e)), for both $\widehat{G}_F, \widehat{G}_V$, in any subgame-perfect equilibrium the maximum revenue I can obtain is $\widehat{M}$. Take any k such that $k\widehat{\Delta}(k) = \widehat{M}$. For $\widehat{G}_F$, if I sets fixed fee $f = \widehat{\Delta}(k)$, then in any equilibrium of $\widehat{G}_F(f)$ either $k-1$ or k firms buy licenses (Lemma 1(III)(d)). In any subgame-perfect equilibrium of $\widehat{G}_F$, the number of firms buying licenses must be k; otherwise, by Lemma 1(III)(e), I can improve its payoff by deviating to a slightly lower f. This proves the result for $\widehat{G}_F$. The result for $\widehat{G}_V$ follows by similar reasoning by applying parts (IV)(d)-(e) of Lemma 1.

(II) First let $k = 1, \ldots, n-1$. For both fixed fee and ad valorem royalty policies, with k licensees, I can obtain $k\Delta(k)$ under licensing by auction (Theorem 1(I)(c)) and $k\widehat{\Delta}(k)$ under licensing by posted price (Lemma 1(III)(d)-(e), (IV)(d)-(e)). If $\pi^0(k) \ge \pi^0(k-1)$, then by (1) and (3), $k\Delta(k) \ge k\widehat{\Delta}(k)$, so auction gives higher revenue to I when it wants to sell less than n licenses.

If I wants to sell n licensees, then by Observation 1, for both fixed fee and ad valorem royalty policies, using auction with suitable minimum bids I can obtain $n\Delta(n)$ which is the same revenue it can obtain by using posted price.

This shows for both fixed fee and ad valorem royalty policies, by using auction (with suitable minimum bids for $k = n$), I obtains $M_0 = \max\{M, n\widehat{\Delta}(n)\}$ where $M = \max_{k\in\{1,\ldots,n-1\}} k\Delta(k)$. By part (I), I obtains $\widehat{M} = \max_{k\in\{1,\ldots,n\}} k\widehat{\Delta}(k)$ by using posted price. As $M_0 \ge \widehat{M}$, it follows that auction is superior to posted price. ■

Remark 4 When both $\widehat{\Delta}(k)$, $\widehat{\phi}(k)$ are not decreasing, the equivalence result of Theorem 2 may not hold. To see this, consider the example of Remark 2: $\pi^1(k)$ is positive and increasing and $\pi^0(k) = 0$ for all k.

In this case $\widehat{\phi}(k) = 1 - \pi^0(k-1)/\pi^1(k) = 1$ for all k. If I sets $v = 1$, then $\widehat{G}_V(v)$ has $n+1$ equilibrium outcomes: k firms buying and $n-k$ firms not buying is an equilibrium for all $k = 0, 1, \ldots, n$. However, by setting v slightly lower than 1, I can ensure the unique equilibrium of $\widehat{G}_V(v)$ has all users buying and I can obtain a revenue arbitrarily close to the maximum revenue $n\pi^1(n)$.

Observe that $\widehat{\Delta}(k) = \pi^1(k) - \pi^0(k-1) = \pi^1(k)$. As $\pi^1(k)$ is increasing, so is $\widehat{\Delta}(k)$. By Lemma 1(I), I can ensure the unique equilibrium of $\widehat{G}_F(f)$ has all firms buying only by setting a low enough fee $f < \pi^1(1)$. If it sets a higher fee where $\pi^1(1) \leq f \leq \pi^1(n)$, there are two equilibria: one in which I gets zero revenue and another in which I gets positive revenue. The equivalence result does not hold for this example.

3 Application: licensing in a Cournot oligopoly

We apply the results to a specific problem that has been been extensively studied in the literature: licensing of a cost reducing innovation in a Cournot oligopoly. We show that Assumptions 1-3 hold for a Cournot oligopoly. In the case of an outside innovator, the equivalence result holds with some qualifications (Proposition 1). Finally we show when the innovator is one of the incumbent firms in a Cournot duopoly, the equivalence result does not hold (Propositions 2,3).

3.1 Cournot oligopoly with an outside innovator

Consider a Cournot oligopoly with $n \geq 2$ firms where the set of firms is $N = \{1, \ldots, n\}$. For $i \in N$, let q_i be the quantity produced by firm i and $Q = \sum_{i \in N} q_i$. Initially all firms produce with the identical constant marginal cost c, where $0 < c < a$. An outsider innovator I has a patent for a cost reducing innovation of magnitude ε that lowers the cost from c to $c - \varepsilon$, where $0 < \varepsilon < c$. The innovator can license the innovation to some or all firms of the industry.

Assumptions

A1 The price function or the inverse demand function $p(Q)$: $\mathrm{R}_{++} \to \mathrm{R}_{+}$ is non-increasing and $\exists\ \overline{Q} > 0$ such that $p(Q)$ is decreasing and twice continuously differentiable for $Q \in (0, \overline{Q})$.

A2 $\overline{p} \equiv \lim_{Q \uparrow 0} p(Q) > c$ and $\exists\ 0 < Q^c < Q^{c-\varepsilon} < \overline{Q}$ such that $p(Q^c) = c > p(Q^{c-\varepsilon}) = c - \varepsilon > p(\overline{Q})$.

A3 $p(Q)$ is log-concave for $Q \in (0, \overline{Q})$.

Assumptions A1-A3 imply A4.

A4 For $p \in (0, \overline{p})$, the price elasticity $\eta(p) := -pQ'(p)/Q(p)$ is non-decreasing.

We also assume A5, which ensures a certain comparative-statics results.

A5 The revenue function $\gamma(Q) := p(Q)Q$ is strictly concave for $Q \in (0, \overline{Q})$.

The existence and uniqueness of (Cournot-Nash) equilibrium is ensured by Assumptions [A1-A3] (Badia et al., 2014), or alternatively by assumptions [A1-A2, A4-A5]

(Kamien et al., 1992). We assume either [A1-A3, A5] or [A1-A2, A4-A5] holds.[3]

Examples In addition to linear demand, an example of demand functions covered in this analysis include the constant elasticity inverse demand function $p(Q) = s/Q^t$ (where $s > 0$ and $0 < t < 1$) which satisfy [A1-A2, A4-A5]. Another example is $p(Q) = \max\{(a - Q)^t, 0\}$ (where $a, t > 0$ and $c < a^t$), which satisfies [A1-A3, A5].

We compare three licensing policies: (i) fixed fee, (ii) ad valorem profit royalty and (iii) per unit royalty. For any policy, consider the Cournot stage where firms simultaneously choose quantities. For any $i \in N$, let $q_{-i} = (q_j)_{j \neq i}$. The profit function of firm i when it has marginal cost c_i is denoted by ψ_i, that is,

$$\psi_i(q_i, q_{-i}; c_i) = [p(Q) - c_i]q_i \tag{5}$$

If any firm i does not have a license, its marginal cost is c and its payoff is simply its profit, which is

$$[p(Q) - c]q_i = \psi_i(q_i, q_{-i}; c) \tag{6}$$

If any firm i has a license under fixed fee $f \geq 0$, its marginal cost is $c - \varepsilon$. Its payoff is its profit net of fee f, which is

$$[p(Q) - (c - \varepsilon)]q_i - f = \psi_i(q_i, q_{-i}; c - \varepsilon) - f \tag{7}$$

If firm i has a license under ad valorem royalty $v \in [0, 1]$, its marginal cost is $c - \varepsilon$, it leaves fraction v of its profit to I and retains the remaing fraction $1 - v$. So its payoff is

$$(1 - v)[p(Q) - (c - \varepsilon)]q_i = (1 - v)\psi_i(q_i, q_{-i}; c - \varepsilon) \tag{8}$$

Note that for any $f \geq 0$ and $0 \leq v < 1$, the choice of quantities are not affected by the fee or the ad valorem royalty.[4] This shows that when there are k licensees under either fixed fee or ad valorem profit royalty, we have an n-firm Cournot oligopoly in which k firms (licensees) have marginal cost $c - \varepsilon$ and the remaining $n - k$ firms (non-licensees) have marginal cost c. If either [A1-A3, A5] or [A1-A2, A4-A5] holds, this oligopoly has a unique equilibrium; the operating profit of any firm is simply its Cournot profit in this oligopoly which is completely determined by (i) whether the firm has a license or not and (ii) the number of licensees. This shows for both fixed fee and ad valorem profit royalty, Assumption 1 holds.

In this case $\pi^1(k)$ and $\pi^0(k)$ are Cournot profits of a licensee and a non-licensee when there are are k licensees. A licensee obtains $\pi^1(k) - f$ under fixed fee f and $(1 - v)\pi^1(k)$ under ad valorem royalty v. Using assumption A5, along the lines of the comparative statics analsysis of Dixit (1986), it can be shown that

$$\pi^0(k - 1) \geq \pi^0(k) \text{ and } \pi^1(k) > \pi^0(k - 1) \text{ for } k = 1, \ldots, n \tag{9}$$

[3] The assumption A5 is needed to show that the equilibrium profit of a non-licensee is decreasing in the number of licensees.

[4] For $v = 1$, the payoff function in (8) is always zero, but this can be resolved by setting v slightly lower than 1.

This implies Assumptions 2,3 also hold.

Remark 5 Using the first inequality of (9) in (1) and (3), $\Delta(k) \geq \widehat{\Delta}(k)$ and $\phi(k) \geq \widehat{\phi}(k)$ for $k = 1, \ldots, n-1$. This shows when it is optimal to sell less than n licenses under fixed fee or ad valorem profit royalty, auction is superior to posted price for I and by Theorem 1, equivalence between these two policies hold. When I intends to sell n licenses, it can use an auction with minimum bid $\widehat{\Delta}(n)$ for fixed fee and $\widehat{\phi}(n)$ for ad valorem royalty, which will be the same as using posted prices. Such an auction always has an equilibrium in which all n firms buy licenses; however, it may not be the *unique* equilibrium, so the equivalence between these two policies will need some qualification. See Remark 7 for a further clarification of this point in the case of linear demand.

Regarding per unit royalty, note that if any firm i has a license with unit royalty $r \geq 0$, it has to pay r for each unit it produces. So its payoff is

$$[p(Q) - (c - \varepsilon)]q_i - rq_i = [p(Q)q - (c - \varepsilon + r)]q_i = \psi_i(q_i, q_{-i}; c - \varepsilon + r) \tag{10}$$

Under per unit royalty r, the effective marginal cost of a licensee is $c - \varepsilon + r$, so the operating profit of a licensee (and also a non-licensee) not only depends on the number of licenses, but also on the royalty r. Assuming no firm will accept a policy that raises its marginal cost from c, the domain of r is $[0, \varepsilon]$. For any $k \leq n$, when there are k licensees under per unit royalty r, the resulting Cournot oligopoly has a unique equilibrium under our assumptions.

To compare the licensing policies, the notion of *drastic* innovations will be useful. A cost reducing innovation is drastic (Arrow, 1962) if it is significant enough to create a monopoly with the reduced cost. Otherwise it is non drastic. For a drastic innovation, if only one firm has the innovation, it becomes a monopolist and all other firms drop out of the market. Let $\theta \equiv c/\eta(c)$ (recall $\eta(p) = -pQ'(p)/Q(p)$ is the price elasticity). A cost reducing innovation of magnitude ε is drastic if $\varepsilon \geq \theta$ and non drastic if $\varepsilon < \theta$.

Consider a monopolist who produces with unit cost $c - \varepsilon$. The profit of this monopolist at price p is

$$\Omega(p) := [p - (c - \varepsilon)]Q(p) \tag{11}$$

The unique maximum of $\Omega(p)$ is attained at the monopoly price $p = p_M(\varepsilon)$ and $\Omega(p_M(\varepsilon))$ equals $\pi_M(\varepsilon)$ (the monopoly profit under cost $c - \varepsilon$). Also note that $\Omega(p)$ is increasing for $p < p_M(\varepsilon)$ and decreasing for $p > p_M(\varepsilon)$. Observe that $p_M(\varepsilon) \leq c$ for drastic and $p_M(\varepsilon) > c$ for non drastic innovations. If a licensing policy results in Cournot price p, the payoff of I under that policy is bounded above by $\Omega(p)$, so the maximum payoff that I can obtain is $\pi_M(\varepsilon)$.

Proposition 1 *Consider a Cournot oligopoly with $n \geq 3$ firms where an outside innovator I has a cost reducing innovation of magnitude ε. For generic magnitudes of ε, the following results hold.*

(I) *For a drastic innovation ($\varepsilon \geq \theta$), by offering only one license through an auction using either fixed fee or ad valorem profit royalty, I obtains π_ε^M. Under per unit* *royalty policy, I obtains less than π^M_ε, so for I, both fixed fee and ad valorem profit royalty policies are superior to per unit royalty policy.*

(II) *For a non drastic innovation ($\varepsilon < \theta$), if the innovation is relatively significant ($\varepsilon \geq \theta/(n-1)$), fixed fee and ad valorem profit royalty policies are equivalent for I. Moreover for relatively large n, per unit royalty policy is superior to both fixed fee and ad valorem profit royalty policies for I.*

(III) *For both* (I) *and* (II), *fixed fee and ad valorem profit royalty policies result in the same Cournot price, give same payoffs to all firms as well as the innovator and are welfare-equivalent.*

Proof (I) For a drastic innovation, $\pi^1(1) = \pi^M_\varepsilon$, $\pi^0(1) = 0$, so by (1), $\Delta(1) = \pi^M_\varepsilon$ and $\phi(1) = 1$. Using these, the first statement is immediate by taking $k = 1$ in Theorem 1.

To prove the last statement, suppose there are k licensees with per unit royalty r. Let $p^n(k, r)$ be the resulting Cournot price and $Q^n(k, r)$ the industry quantity. Note that the licensing revenue of I can be at most $rQ^n(k, r)$. There are two possibilities.

If $p^n(k, r) > c$, then $Q^n(k, r) < Q(c)$. Since $r \leq \varepsilon$, we have $rQ^n(k, r) < \varepsilon Q(c) = \Omega(c)$ (where $\Omega(p)$ is given in (11)). Since $\Omega(c) \leq \Omega(p_M(\varepsilon)) = \pi^M_\varepsilon$, the licensing revenue is lower than π^M_ε.

If $p^n(k, r) \leq c$, all non-licensees drop out of the market (that is, they produce zero) and there is a k-firm oligopoly with k licensees. Since each of the k licensess has marginal cost $c - \varepsilon + r$, the industry profit is $[p^n(k, r) - (c - \varepsilon + r)]Q^n(k, r) > 0$. The licensing revenue of I is $rQ^n(k, r)$. The sum of industry profit and licensing revenue is $\Omega(p^n(k, r)) \leq \pi^M_\varepsilon$, so again the licensing revenue is lower than π^M_ε.

(II) Suppose $\varepsilon \geq \theta/(n-1)$. In this case, if $n-1$ firms have licenses, the sole non-licensee drops out of the market (so $\pi^0(n-1) = 0$) and an $(n-1)$-firm oligopoly is created with the licensees (see Lemma 1(iii), p.40, Sen and Tauman, 2018). Denoting by $p(n-1)$ the price of this oligopoly, the industry profit is $\Omega(p(n-1))$ (where $\Omega(p)$ is given in (11)). As $\pi^0(n-1) = 0$, by (1) and (3), $\Delta(n-1) = \pi^1(n-1)$, $\widehat{\Delta}(n) = \pi^1(n)$ and $\phi(n-1) = \widehat{\phi}(n) = 1$. By Theorem 1(I), selling $n-1$ licenses through auction by either ad valorem royalty or fixed fee, I obtains $(n-1)\Delta(n-1) = (n-1)\pi^1(n-1)$, which is the industry profit $\Omega(p(n-1))$.

By selling n licenses, I can obtain at most $n\widehat{\Delta}(n) = n\pi^1(n)$ (Observation 2), which is the industry profit when all n firms have a license. Denoting by $p(n)$ the price of this n-firm oligopoly, the industry profit is $\Omega(p(n))$. Since $p(n) < p(n-1) < p_M(\varepsilon)$ (see Observation 1 and Lemma 1(iii), p.40, Sen and Tauman, 2018) and $\Omega(p)$ is increasing for $p < p_M(\varepsilon)$, we have $\Omega(p(n)) < \Omega(p(n-1))$, so it is optimal for I to sell $n-1$ or less licenses. For any $k \leq n-1$, for either fixed fee or ad valorem royalty, I obtains $k\Delta(k)$ using auction (Theorem 1(I)) while it obtains at most $k\widehat{\Delta}(k)$ using posted price (Observation 2). As $\pi^0(k) \leq \pi^0(k-1)$, by (1) and (3), $k\Delta(k) \geq k\widehat{\Delta}(k)$. This shows auction is better than posted price for I and by Theorem 1, equivalence between fixed fee and ad valorem royalty holds.

The last part of (II) is immediate from the result of Sen and Tauman (2018) (see Proposition 3(II), p.42) that with generic magnitudes of the innovation, for any non drastic innovation licensing by per unit royalty is superior to licensing by fixed fee through auction.

Part (III) is immediate from parts (I)-(II). ■

Remark 6 Proposition 1 shows that if $\theta/(n-1) \leq \varepsilon < \theta$, for both fixed fee and ad valorem royalty, it is optimal for I to sell at most $n-1$ licenses and equivalence between these two policies hold by Theorem 1. If $\varepsilon < \theta/(n-1)$, it may be optimal to sell licenses to all n firms. In particular, with linear demand it is shown in Sen and Tauman (2007) (see Table A.5, p.183) that for $n \geq 3$, there is an increasing function $t(n) > n-1$ such that whenever $\varepsilon < \theta/t(n)$, among general licensing policies that are combinations of both fixed fee and per unit royalty, it is optimal for the innovator to sell licenses to all n firms using only a fixed fee (through an auction with minimum bid $\widehat{\Delta}(n)$). This is clearly the optimal fixed fee policy and it is also superior to per unit royalty. As $t(n)$ is increasing, $\varepsilon < \theta/t(n)$ holds for relatively small values of n. Together with the conclusion of Proposition 1(II), it follows that **for linear demand fixed fee is superior for small n while per unit royalty is superior for large n.**

Remark 7 Regarding equivalence between fixed fee and ad valorem royalty in this case ($\varepsilon < \theta/t(n)$), we note that under linear demand, $\widehat{\Delta}(k)$ is decreasing but $\widehat{\phi}(k)$ is increasing (so in particular, $\widehat{\phi}(1) < \widehat{\phi}(n)$). By Observation 1(i), by offering n licenses through fixed fee using a minimum bid marginally lower than $\widehat{\Delta}(n)$, I can ensure that the *unique* equilibrium outcome has all n firms buying licensees. However, by Lemma 1(II)(c), if I offers n licenses by ad valorem profit royalty with minimum bid $\widehat{\phi}(n)$ (or marginally lower than $\widehat{\phi}(n)$), then there is an equibrium where all n firms buy licensees, but there is also *another equilibrium* in which no one buys.

3.2 Cournot duopoly with an incumbent innovator

Consider a Cournot duopoly with two firms $1, 2$ where the demand curve satisfies the assumptions of the last section. Initially both firms produce with the identical constant marginal cost c, where $0 < c < a$. Firm 1 has a patent for a cost reducing innovation of magnitude ε that lowers the cost from c to $c - \varepsilon$, where $0 < \varepsilon < c$. If the innovation is drastic, firm 1 becomes a monopolist with the reduced cost, so it has no incentive to license its innovation to firm 2. So assume the innovation is non drastic, that is, $\varepsilon < \theta$ (recall $\theta = c/\eta(c)$).

Given the assumptions of the last section, the Cournot duopoly has a unique equilibrium under any policy. Since the innovation is non drastic, firm 2 obtains a positive Cournot profit without a license. Denote this profit by $\underline{\pi}$. As before, denote by ψ_i the (operating) profit function of firm i at the Cournot stage when it has marginal cost c_i, that is,

$$\psi_i(q_1, q_2; c_i) = [p(Q) - c_i]q_i. \tag{12}$$

Since firm 1 always uses the innovation, its operating profit is $\psi_1(q_1, q_2; c - \varepsilon)$.

The equivalence between fixed fee and ad valorem royalty does not hold in this case. We have seen in Section 3.1 that for an outside innovator the problems of firms at the Cournot stage do not depend on the fixed fee or the ad valorem royalty, so Assumptions 1-3 hold. This is not the case when the innovator is one of the incumbent firms. At the Cournot stage, the payoff of an incumbent innovator under ad valorem royalty is the sum of its own operating profit and a fraction of the operating profit of the licensee, so the innovator's problem does depend on the rate of ad valorem royalty.

For any fixed fee f, the marginal cost of each firm is $c - \varepsilon$. Denote the Cournot quantities under fixed fee by q_1^F, q_2^F and the Cournot price by p^F. It is optimal for firm 1 to set fee $f = \psi_2(q_1^F, q_2^F; c-\varepsilon) - \underline{\pi}$, making firm 2 just indifferent between accepting and rejecting a license. Using the optimal f and the function $\Omega(p)$ from (11), the payoff of firm 1 under fixed fee policy is

$$\psi_1(q_1^F, q_2^F; c-\varepsilon) + f = \Omega(p^F) - \underline{\pi} \tag{13}$$

Regarding ad valorem policy, note that since firm 2 obtains a positive profit $\underline{\pi}$ without a license, a policy is acceptable to firm 2 only if it gives a payoff of at least $\underline{\pi}$. In particular, firm 2 will not accept an ad valorem royalty policy with $v = 1$. Under any ad valorem royalty $v \in [0, 1)$, firm 2 keeps fraction $1 - v$ of its profit, so its payoff is

$$(1-v)[p(Q) - (c-\varepsilon)]q_2 = (1-v)\psi_2(q_1, q_2; c-\varepsilon) \tag{14}$$

The payoff of firm 1 is the sum of its operating profit and the licensing revenue. As firm 1 receives fraction v of the profit of firm 2 as licensing revenue, its payoff is

$$\psi_1(q_1, q_2; c-\varepsilon) + v\psi_2(q_1, q_2; c-\varepsilon) \tag{15}$$

For any $v \in [0, 1)$, the resulting Cournot duopoly with payoffs (19)-(21) has a unique equilibrium (see Lemma 2 in the Appendix). There always exist acceptable ad valorem policies that are superior to fixed fee for firm 1.

Proposition 2 *Consider a Cournot duopoly with two firms* $1, 2$ *where firm* 1 *has a non drastic cost reducing innovation that it can license to firm* 2 *using an ad valorem royalty* $v \in [0, 1)$. *There exist ad valorem royalty policies that are acceptable to firm* 2 *and superior to the fixed fee policy for firm* 1.

Proof Without a license firm 2 obtains positive profit $\underline{\pi}$, so it will not accept any policy where it obtains lower than $\underline{\pi}$. Under ad valorem royalty v, firm 2 obtains $(1-v)\Omega(p^v)/(2-v)$ (see Lemma 2), where p^v is the resulting Cournot price and $\Omega(p)$ is given in (11). When $v = 0$, this payoff is $\Omega(p^F)/2$ (the Cournot profit of firm 2 under a fixed fee policy). Note that $\Omega(p^F)/2 > \underline{\pi}$ (see Lemma A.2, p.44, Sen and Tauman, 2018). As $\lim_{v\uparrow 1}(1-v)\Omega(p^v)/(2-v) = 0 < \underline{\pi}$, it follows that there is one or more $v \in (0, 1)$ such that $(1-v)\Omega(p^v)/(2-v) = \underline{\pi}$. Any such v is acceptable to firm 2 and under such an ad valorem policy, firm 1 obtains

$$\Omega(p^v)/(2-v) = \Omega(p^v) - (1-v)\Omega(p^v)/(2-v) = \Omega(p^v) - \underline{\pi} \tag{16}$$

Since $\Omega(p)$ is increasing for $p < p_M(\varepsilon)$ and $p^F = p^0 < p^v < p_M(\varepsilon)$ (see Lemma 2), we have $\Omega(p^v) > \Omega(p^F)$. Then the result follows by (13) and (16). ■

In the case of linear demand, we can identify (the unique) optimal ad valorem royalty profit royalty policy and show that it is superior to both fixed fee and per unit royalty for an incumbent innovator in a Cournor duopoly. Different papers have compared per unit royalties with fixed fees and ad valorem with per unit royalties (e.g., Wang, 1998; San Martín and Saracho, 2010; Niu, 2017; Hsu et al. 2019). Here we give a self contained explanation of the superiority of ad valorem royalty using the function $\Omega(p)$ of (11) which gives a clear ranking of the Cournot prices under different policies.

Proposition 3 *Consider a Cournot duopoly with firms* $1, 2$ *with linear demand where firm* 1 *has a patent on a non drastic cost reducing innovation. Denote by* p^V, p^R, p^F *the respective Cournot prices under the* (*unique*) *optimal ad valorem, per unit royalty and fixed fee policies.*

(i) *Firm* 1 *obtains* $\Omega(p^V) - \underline{\pi}$ *under the optimal ad valorem royalty policy,* $\Omega(p^R) - \underline{\pi}$ *under the optimal per unit royalty policy and* $\Omega(p^F) - \underline{\pi}$ *under the optimal fixed fee policy.*

(ii) $p^F < p^R < p^V < p^M(\varepsilon)$*. Consequently* $\Omega(p^V) > \Omega(p^R) > \Omega(p^F)$ *and ad valorem royalty is superior to both per unit royalty and fixed fee for firm* 1.

Proof See the Appendix. ∎

Proposition 3 shows that while the ad valorem royalty is the most preferred for an incumbent innovator, it also results in the highest Cournot price out of the three policies, so it is the worst preferred for consumers. As ad valorem profit royalty can be viewed as partial passive ownership of firm 2 by firm 1 (see, e.g., Niu, 2017; Hsu et al. 2019), imposing a bound on the ownership share (which is represented by ad valorem royalty v in our model) may be a desirable policy intervention.

Appendix

Proof of parts (I)-(II) of Theorem 1 Observe that for each of $G_F(k), G_V(k)$, the following outcome is an equilibrium for any $m \geq k+1$: m firms place bids, the highest bid is $\Delta(k)$ for $G_F(k)$ and $\phi(k)$ for $G_V(k)$ and at least $k+1$ firms place the highest bid. In this case the number of licensees is k. A firm that wins a license obtains $\pi^1(k) - \Delta(k) = \pi^0(k)$ in $G_F(k)$ and $[1-\phi(k)]\pi^1(k) = \pi^0(k)$ in $G_V(k)$, so each firm, regardless of whether it wins a license or not, obtains $\pi^0(k)$. Any unilateral deviation by a firm does not alter the number of licensees and the deviating firm cannot obtain more than $\pi^0(k)$. In what follows, we show that this is the unique equilibrium outcome.

Note that for each of $G_F(k), G_V(k)$, there is no equilibrium in which k or less firms place bids. If there is an equilibrium in which $m \leq k$ firms place bids, each of the bidding firms wins a license with certainty, so each of these bids must be zero (otherwise a bidding firm can improve its payoff by a slightly lower bid). As $m \leq k < n$, there is at least one firm who does not place a bid. Such a firm obtains $\pi^0(m)$. If $m < k$, let this firm unilaterally deviate by placing zero bid to obtain $\pi^1(m+1)$. By Assumption 3, this deviation is gainful, so there is no equilibrium where $m < k$.

If $m = k$, any non-bidding firm obtains $\pi^0(k)$. Let such a firm unilaterally deviate by placing a sufficiently small positive bid. Then it will win a license with certainty (since all other bids are zero), the number of licensees will be still k and the deviating firm will obtain a payoff which is only slightly lower than $\pi^1(k)$. By Assumption 2, $\pi^1(k) > \pi^0(k)$, so the deviation is gainful. This shows there is no equilibrium where $m = k$.

Therefore for each of $G_F(k), G_V(k)$, in any equilibrium the number of firms placing bid must be $m \geq k+1$, so there are k licensees. Arrange the bids in ascending order as $b_1 \geq \ldots \geq b_m$. If $b_i > b_{i+1}$ for some $i = 1, \ldots, k$, then the firm that places bid b_i wins a license with certainty and it can improve its payoff by placing a slightly lower bid. So in any equilibrium, it must be the case that $b_i = b_{i+1}$ for all $i = 1, \ldots, k$, that is, $b_1 = \ldots = b_{k+1}$. This shows that in any equilibrium the highest bid must be placed by at least $k+1$ firms. Let this highest bid be $\overline{f}$ for $G_F(k)$ and $\overline{v}$ for $G_V(k)$. Let $\lambda \in (0,1)$ be the probability of winning a license for a firm that places the highest bid. Then in $G_F(k)$ such a firm obtains

$$\lambda[\pi^1(k) - \overline{f}] + (1-\lambda)\pi^0(k) = \pi^0(k) + \lambda[\Delta(k) - \overline{f}] \tag{17}$$

and in $G_V(k)$ such a firm obtains

$$\lambda(1-\overline{v})\pi^1(k) + (1-\lambda)\pi^0(k) = \pi^0(k) + \lambda[\phi(k) - \overline{v}]\pi^1(k) \tag{18}$$

where $\Delta(k), \phi(k)$ are defined in (1). By not placing any bid a firm obtains $\pi^0(k)$. If $\overline{f} > \Delta(k)$, then by (17), for $G_F(k)$, any firm who places the bid $\overline{f}$ is better off by unilaterally deviating to not placing any bid. Similarly if $\overline{v} > \phi(k)$, then by (18), for $G_V(k)$, any firm who places bid $\overline{v}$ is better off by unilaterally deviating to not placing any bid. So we must have

$$\overline{f} \leq \Delta(k) \text{ and } \overline{v} \leq \phi(k).$$

If $\overline{f} < \Delta(k)$, then any firm that places bid $\overline{f}$ can unilaterally deviate to a slightly higher bid f with $\overline{f} < f < \Delta(k)$. Then it wins a license with certainty to obtain $\pi^1(k) - f = \pi^0(k) + \Delta(k) - f$, which is more than the payoff in (17) for f close enough to $\overline{f}$.

Similarly if $\overline{v} < \phi(k)$, then any firm that places bid $\overline{v}$ can unilaterally deviate to a slightly higher bid v with $\overline{v} < v < \phi(k)$. Then it wins a license with certainty to obtain $(1-v)\pi^1(k) = \pi^0(k) + [\phi(k) - v]\pi^1(k)$, which is more than the payoff in (18) for v close enough to $\overline{v}$.

This shows for each of $G_F(k), G_V(k)$, in any equilibrium at least $k+1$ firms place the highest bid, the highest bid is $\Delta(k)$ for $G_F(k)$ and $\phi(k)$ for $G_V(k)$. Taking $\overline{f} = \Delta(k)$ in (17) and $\overline{v} = \phi(k)$ in (18), any licensee obtains $\pi^0(k)$, which is the same payoff that any non-licensee obtains. This completes the proof. ■

The following result will be useful to prove Lemma 1.

Observation 2

The following hold for the games $\widehat{G}_F(f), \widehat{G}_V(v)$.

(i) *For $\widehat{G}_F(f)$:*

(a) *No firms buying a license is an equilibrium if and only if* $f \geq \widehat{\Delta}(1)$.

(b) *All* n *firms buying licenses is an equilibrium if and only if* $f \leq \widehat{\Delta}(n)$.

(c) *For* $k = 1, \ldots, n-1$, k *firms buying licenses and the remaining* $n-k$ *firms not buying is an equilibrium if and only if* $\widehat{\Delta}(k+1) \leq f \leq \widehat{\Delta}(k)$.

(d) *For* $k = 1, \ldots, n$, *with* k *licensees, the maximum revenue* I *can obtain is* $k\widehat{\Delta}(k)$.

(ii) *For* $\widehat{G}_V(v)$*:*

(a) *No firms buying a license is an equilibrium if and only if* $v \geq \widehat{\phi}(1)$.

(b) *All* n *firms buying licenses is an equilibrium if and only if* $v \leq \widehat{\phi}(n)$.

(c) *For* $k = 1, \ldots, n-1$, k *firms buying licenses and the remaining* $n-k$ *firms not buying is an equilibrium if and only if* $\widehat{\phi}(k+1) \leq v \leq \widehat{\phi}(k)$.

(d) *For* $k = 1, \ldots, n$, *with* k *licensees, the maximum revenue* I *can obtain is* $k\widehat{\phi}(k)\pi^1(k) = k\widehat{\Delta}(k)$.

Proof (i) Consider $\widehat{G}_F(f)$.

(a) When no firm buys a license, all firms obtain $\pi^0(0)$. Any firm that unilaterally deviates to buy a license obtains $\pi^1(1) - f$. So no firm buying a license is an equilibrium if and only if $\pi^0(0) \geq \pi^1(1) - f$, that is, $f \geq \widehat{\Delta}(1)$.

(b) When all n firms buy licenses, all firms obtain $\pi^1(n) - f$. Any firm that unilaterally deviates to not buy a license obtains $\pi^0(n-1)$. So all firms buying licenses is an equilibrium if and only if $\pi^1(n) - f \geq \pi^0(n-1)$, that is, $f \leq \widehat{\Delta}(n)$.

(c) Let $k = 1, \ldots, n-1$. When k firms buy licenses and $n-k$ firms do not, any firm that buys a license obtains $\pi^1(k) - f$ and any firm that does not buy obtains $\pi^0(k)$. A firm buying a license has no uniltarel incentive to deviate to not buying if and only if $\pi^1(k) - f \geq \pi^0(k-1)$, that is, $f \leq \widehat{\Delta}(k)$. A firm not buying a license has no uniltarel incentive to deviate to buying if and only if $\pi^0(k) \geq \pi^1(k+1) - f$, that is, $f \geq \widehat{\Delta}(k+1)$. So k firms buying licenses and $n-k$ firms not buying is an equilibrium if and only if $\widehat{\Delta}(k+1) \leq f \leq \widehat{\Delta}(k)$.

(d) Follows from parts (a) and (c).

(ii) Now consider $\widehat{G}_V(v)$.

(a) When no firm buys a license, all firms obtain $\pi^0(0)$. Any firm that unilaterally deviates to buy a license obtains $(1-v)\pi^1(1)$. So no firm buying a license is an equilibrium if and only if $\pi^0(0) \geq (1-v)\pi^1(1)$, that is, $v \geq \widehat{\phi}(1)$.

(b) When all n firms buy licenses, all firms obtain $(1-v)\pi^1(n)$. Any firm that unilaterally deviates to not buy a license obtains $\pi^0(n-1)$. So all firms buying licenses is an equilibrium if and only if $(1-v)\pi^1(n) \geq \pi^0(n-1)$, that is, $v \leq \widehat{\phi}(n)$.

(c) Let $k = 1, \ldots, n-1$. When k firms buy licenses and $n-k$ firms do not, any firm that buys a license obtains $(1-v)\pi^1(k)$ and any firm that does not buy obtains $\pi^0(k)$. A firm buying a license has no uniltarel incentive to deviate to not buying if and only if $(1-v)\pi^1(k) \geq \pi^0(k-1)$, that is, $v \leq \widehat{\phi}(k)$. A firm not buying a license has no uniltarel incentive to deviate to buying if and only if $\pi^0(k) \geq (1-v)\pi^1(k+1)$, that is, $v \geq \widehat{\phi}(k+1)$. So k firms buying licenses and $n-k$ firms not buying is an equilibrium if and only if $\widehat{\phi}(k+1) \leq v \leq \widehat{\phi}(k)$.

(d) Follows from parts (a) and (c) by using (4). ■

Proof of Lemma 1 (I) Since $\widehat{\Delta}(k)$ is increasing, by Observation 2(i)(c), there is no equilibrium of $\widehat{G}_F(f)$ in which k firms buy licenses and $n-k$ firms do not, where $k = 1, \ldots, n-1$. Then the result is immediate by Observation 2(i)(a)-(b).

(II) Since $\widehat{\phi}(k)$ is increasing, by Observation 2(ii)(c), there is no equilibrium of $\widehat{G}_V(v)$ in which k firms buy licenses and $n-k$ firms do not, where $k = 1, \ldots, n-1$. Then the result is immediate by Observation 2(ii)(a)-(b).

(III) Since $\widehat{\Delta}(k)$ is increasing, parts (a)-(d) follow by Observation 2(i). Part (e) follows from parts (c) and (d).

(IV) Since $\widehat{\Delta}(k)$ is increasing, parts (a)-(d) follow by Observation 2(ii). Part (e) follows from parts (c) and (d). ■

Lemma 2 *Consider a Cournot duopoly with two firms $1, 2$ where firm 1 has a non drastic innovation that it licenses to firm 2 using an ad valorem royalty $v \in [0,1)$. The duopoly has a unique equilibrium. Denote the equilibrium quantities by $q_1(v), q_2(v)$, let $Q(v) = q_1(v) + q_2(v)$ and p^v be the Cournot price. The equilibrium has the following properties.*

(i) $q_1(v) = (1-v)q_2(v)$, $Q(v) = (2-v)q_2(v)$.

(ii) *The Cournot price p^v is increasing in v, with $p^0 = p^F$ (the Cournot price at any fixed fee policy) and $\lim_{v \uparrow 1} p^v = p_M(\varepsilon)$ (the monopoly price).*

(iii) *Firm 1 obtains payoff $\Omega(p^v)/(2-v)$, which is increasing in v and firm 2 obtains payoff $(1-v)\Omega(p^v)/(2-v)$, where $\Omega(p)$ is given in (11).*

Proof (i) First note that since the innovation is non drastic, firm 2 obtains a positive profit $\underline{\pi}$ without a license, so it will not accept an ad valorem royalty policy with $v = 1$. Under any ad valorem royalty $v \in [0,1)$, firm 2 keeps fraction $1-v$ of its profit, so using the function ψ_i from (12), its payoff is

$$(1-v)[p(Q) - (c-\varepsilon)]q_2 = (1-v)\psi_2(q_1, q_2; c-\varepsilon) \tag{19}$$

For any $v \in [0,1)$, firm 2 effectively solves the problem of a firm that has profit function

$$u_2(q_1, q_2) = \psi_2(q_1, q_2; c-\varepsilon) = [p(Q) - (c-\varepsilon)]q_2 \tag{20}$$

The payoff of firm 1 is the sum of its operating profit and the licensing revenue. As firm 1 receives fraction v of the profit of firm 2 as licensing revenue, by (21), its payoff is

$$u_1(q_1, q_2) = [p(Q) - (c-\varepsilon)]q_1 + v[p(Q) - (c-\varepsilon)]q_2 \tag{21}$$

Note that

$$\frac{\partial u_1}{\partial q_1} = p'(Q)(q_1 + vq_2) + p(Q) - (c-\varepsilon) \text{ and } \frac{\partial u_2}{\partial q_2} = p'(Q)q_2 + p(Q) - (c-\varepsilon) \tag{22}$$

So $(\partial u_1/\partial q_1)[q_1 = 0] = (\partial u_2/\partial q_2)[q_1 = 0] - (1-v)q_2 p'(q_2)$ and $(\partial u_2/\partial q_2)[q_2 = 0] = (\partial u_1/\partial q_1)[q_2 = 0] - q_1 p'(q_1)$. Thus, for $i \neq j$, if $(\partial u_j/\partial q_j)[q_i = 0] = 0$, then $(\partial u_i/\partial q_i)[q_i = 0] > 0$ for any $q_j > 0$. If there is an equilibrium with $q_i = 0$ and $q_j > 0$, we must have $(\partial u_j/\partial q_j)[q_i = 0] = 0$ and $(\partial u_i/\partial q_i)[q_i = 0] \leq 0$, so there cannot be any such equilibrium. Clearly $q_1 = q_2 = 0$ is also not an equilibrium.

So any equilibrium must have $(q_1 > 0, q_2 > 0)$ and the (unique) equilibrium is determined by the first order conditions $\partial u_i/\partial q_i = 0$ for $i = 1, 2$. By (22), the Cournot quantities $q_i(v)$ satisfy the first order conditions:

$$p'(Q(v))(q_1(v)+vq_2(v))+p(Q(v))-(c-\varepsilon) = 0, p'(Q(v))q_2(v)+p(Q(v))-(c-\varepsilon) = 0 \quad (23)$$

which shows $q_1(v) = (1-v)q_2(v)$, so $Q(v) = (2-v)q_2(v)$.

(ii) Recall the price elasticity $\eta(p) = -pQ'(p)/Q(p)$ is non decreasing by Assumption A4. Let

$$H(p, v) := p[1 - 1/(2-v)\eta(p)] \quad (24)$$

(When $v = 0$, this function coincides with $H^2(p)$ defined in (5), p.40, Sen and Tauman, 2018). Also note that **if $H(p, v)$ is positive, then $H(p, v) < H(\tilde{p}, v)$ for $p < \tilde{p}$ and $H(p, v) > H(\tilde{p}, v)$ for $p > \tilde{p}$.**

Adding the equations of (23) and noting that $Q(v) = (2-v)q_2(v)$, it follows that Cournot price p^v satisfies

$$H(p^v, v) = c - \varepsilon \quad (25)$$

Since $H(p, v)$ is decreasing in v for any positive p, if $v' > v$, by (25): $H(p^{v'}, v) > H(p^{v'}, v') = c - \varepsilon = H(p^v, v)$. Thus, $H(p^{v'}, v) > H(p^v, v) = c - \varepsilon > 0$. Using the property of $H(p, v)$ above, it follows that $p^{v'} > p^v$. This shows that p^v is increasing in v. Note that when $v = 0$, the equations in (23) coincide with the first order conditions under a fixed fee policy, so we have $H(p^F, 0) = c - \varepsilon$ (where p^F is the Cournot price under fixed fee). This shows that $p^0 = p^F$. Next observe that the monopoly quantity $Q_M(\varepsilon)$ satisfies $p'(Q_M(\varepsilon))Q_M(\varepsilon)+p_M(\varepsilon)-(c-\varepsilon) = 0$. When $v = 1$, we note that $q_1 = 0, q_2 = Q_M(\varepsilon)$ is the unique solution of (23) (also note from (24) that $H(p_M(\varepsilon), 1) = c - \varepsilon$). This shows that $\lim_{v\uparrow 1} p^v = p_M(\varepsilon)$.

(iii) Using $q_1(v) = (1-v)q_2(v)$ and $Q(v) = (2-v)q_2(v)$ in (21), under ad valorem royalty v, the equilibrium payoff of firm 1 is

$$[p(Q(v)) - (c-\varepsilon)](1-v)q_2(v) + v[p(Q(v)) - (c-\varepsilon)]q_2(v) = \psi_2(q_1(v), q_2(v); c-\varepsilon)$$

$$= [p^v - (c-\varepsilon)]Q(v)/(2-v) = \Omega(p^v)/(2-v) \quad (26)$$

where $\Omega(p)$ is given by (11). By (20), the equilibrium payoff of firm 2 is

$$(1-v)\psi_2(q_1(v), q_2(v); c-\varepsilon) = (1-v)\Omega(p^v)/(2-v) \quad (27)$$

Note from (11) that $\Omega(p)$ is increasing for $p < p_M(\varepsilon)$. Since p^v is increasing in v and $p^v < p_M(\varepsilon)$ for all $v \in [0, 1)$ (by (ii)), $\Omega(p^v)$ is increasing in v. As $1/(2-v)$ is positive and increasing for $v \in [0, 1)$, it follows that $\Omega(p^v)/(2-v)$ is increasing in v. ■

Proof of Proposition 3 Note that ψ_i (given in (12)) is the operating profit function of firm i at the Cournot stage when it has marginal cost c_i.

Per unit royalty: The optimal per unit royalty policy for firm 1 is $r = \varepsilon$. Then the effective marginal cost of firm 2 is $c - \varepsilon + r = c$. Denote the Cournot quantities under this policy by q_1^R, q_2^R and let p^R be the Cournot price. Using (12), the Cournot profit of firm 2 is $\psi_2(q_1^R, q_2^R; c) = \underline{\pi}$ and the Cournot profit of firm 1 is $\psi_1(q_1^R, q_2^R; c - \varepsilon)$. The payoff of firm 1 is the sum of its Cournot profit and royalty revenue, which is

$$\psi_1(q_1^R, q_2^R; c-\varepsilon) + \varepsilon q_2^R = \psi_1(q_1^R, q_2^R; c-\varepsilon) + \varepsilon q_2^R + \psi_2(q_1^R, q_2^R; c) - \psi_2(q_1^R, q_2^R; c)$$

$$= [p^R - (c-\varepsilon)]q_1^R + \varepsilon q_2^R + (p^R - c)q_2^R - \underline{\pi} = \Omega(p^R) - \underline{\pi}.$$

Fixed fee: By (13), the payoff of firm 1 under the optimal fixed fee is $\Omega(p^F) - \underline{\pi}$. Note that $p^F < p^R$ and both p^R, p^F are less than the monopoly price $p_M(\varepsilon)$. As $\Omega(p)$ is increasing for $p < p^M(\varepsilon)$, we have $\Omega(p^R) > \Omega(p^F)$, showing the superiority of per unit royalty over fixed fee. In what follows we show that ad valorem royalty is superior to per unit royalty.

Ad valorem royalty: As firm 2 obtains positive payoff $\underline{\pi}$ without a license, it will not accept an ad valorem royalty policy with $v = 1$. Under any ad valorem royalty $v \in [0, 1)$, the payoffs of firms 1, 2 are given in (19) and (21). Let $q_1(v), q_2(v)$ be the (unique) equilibrium quantities and $Q(v) = q_1(v) + q_2(v)$ under linear demand $p(Q) = \max\{a - Q, 0\}$ (where $a > c$ and $\varepsilon < a - c$ ensuring the innovation is non drastic). Denoting $\theta(v) \equiv (a - c + \varepsilon)/(3 - v)$, we note that

$$q_2(v) = \theta(v), q_1(v) = (1-v)\theta(v) \text{ and } Q(v) = (2-v)\theta(v) \tag{28}$$

Also note that $\psi_2(q_1(v), q_2(v); c-\varepsilon) = [\theta(v)]^2$ and $\psi_1(q_1(v), q_2(v); c-\varepsilon) = (1-v)[\theta(v)]^2$. By (19) and (21), under ad valorem royalty the equilibrium payoffs of firms 1, 2 are

$$\pi_1(v) = [\theta(v)]^2 \text{ and } \pi_2(v) = (1-v)[\theta(v)]^2 \tag{29}$$

Firm 2 will not accept any policy where it obtains less than $\underline{\pi} = (a - c - \varepsilon)^2/9$. As $\pi_2(v)$ is decreasing, $\pi_2(0) = (a - c + \varepsilon)^2/9 > \underline{\pi}$ and $\pi_2(1) = 0 < \underline{\pi}$, $\exists\, v^* \in (0, 1)$ such that $\pi_2(v) \gtreqqless \underline{\pi} \Leftrightarrow v \lesseqqgtr v^*$. Noting that $\pi_1(v)$ is increasing, it follows the unique optimal ad valorem royalty for firm 1 is to set $v = v^*$, in which case firm 2 gets $\pi_2(v^*) = \underline{\pi}$. By (19), noting that $Q(v) = (2 - v)q_2(v)$ and denoting $p^V = p(Q(v^*))$, we have

$$\pi_2(v^*) = (1-v^*)\psi_2(q_1(v^*), q_2(v^*); c-\varepsilon) = (1-v^*)\Omega(p^V)/(2-v^*) = \underline{\pi}$$

Using this in (26), the payoff of firm 1 is

$$\Omega(p^V)/(2-v^*) = \Omega(p^V) - (1-v^*)\Omega(p^V)/(2-v^*) = \Omega(p^V) - \underline{\pi}$$

Comparing ad valorem and per unit royalty policies: Note that the industry quantity under the optimal royalty policy is $Q^R = [2(a-c)+\varepsilon]/3$. By (28), $Q(v)$ is decreasing and $Q(0) > Q^R > Q(1)$, so $\exists\, v^R \in (0, 1)$ such that $Q(v) \gtreqqless Q^R \Leftrightarrow v \lesseqqgtr v^R$. Solving $Q(v) = Q^R$, we have $v^R = 3\varepsilon/(a - c + 2\varepsilon)$. By (29), the payoff of firm 2 at ad valorem royalty v^R is $\pi_2(v^R) = (a - c - \varepsilon)(a - c + 2\varepsilon)/9 > (a - c - \varepsilon)^2/9 = \underline{\pi}$. Since $\pi_2(v^*) = \underline{\pi}$

and $\pi_2(v)$ is decreasing, we have $v^* > v_R$, so that $Q(v^*) < Q^R$, implying that $p^V > p^R$. Since p^V, p^R are both less than $p_M(\varepsilon)$, we have $\Omega(p^V) > \Omega(p^R)$, showing that the payoff of firm 1 under ad valorem royalty is higher than its payoff under per unit royalty. ■

References

1. Arrow, K.J. 1962. Economic welfare and the allocation of resources for invention. In: Nelson, R.R. (Ed.), The Rate and Direction of Inventive Activity. Princeton University Press.

2. Badia, B.D., Tauman, Y., Tumendemberel, B., 2014. A note on Cournot equilibrium with positive price. Economic Bulletin 34: 1229-1234.

3. Colombo, S., Filippini, L. 2015. Patent licensing with Bertrand competitors, The Manchester School 83: 1-16.

4. Dixit, A., 1986. Comparative statics for oligopoly. International Economic Review 27, 107–122.

5. Hsu, J., Liu L., Wang X.H., Zeng, C. 2019. Ad valorem versus per-unit royalty licensing in a Cournot duopoly model. The Manchester School 87: 890-901.

6. Kamien, M., Tauman, Y., 1986, Fees versus royalties and the private value of a patent. Quarterly Journal of Economics 101: 471-491.

7. Kamien, M., Oren, S., Tauman, Y., 1992. Optimal licensing of cost-reducing innovation. Journal of Mathematical Economics 21: 483-508.

8. Katz, M., Shapiro, C. 1986. How to license intangible property. Quarterly Journal of Economics 101: 567-589.

9. Mukhopadhyay, S., Kabiraj, T., Mukherjee, A. 1999. Technology transfer in duopoly: the role of cost asymmetry, International Review of Economics and Finance 8: 363-374.

10. Niu, S., 2013. The equivalence of profit-sharing licensing and per-unit royalty licensing, Economic Modelling 32: 10-14.

11. Niu, S., 2017. Profit-sharing licensing, Journal of Economics 121: 267-278.

12. Niu, S. 2019. Licensing versus assignment: Innovation transfer in an asymmetric duopoly, Journal of Public Economic Theory 21:1286-1308

13. San Martín, M., Saracho, A.I. 2010. Royalty licensing, Economics Letters 107: 284-287.

14. Sen, D. 2005. Fee versus royalty reconsidered. Games and Economic Behavior 53: 141-147.

15. Sen, D., Tauman, Y., 2007. General licensing schemes for a cost-reducing innovation. Games and Economic Behavior 59: 163-186.

16. Sen, D., Tauman, Y., 2018. Patent licensing in a Cournot oligopoly: general results. Mathematical Social Sciences 96: 37-48.

17. Tauman, Y., Weng, M-H., 2012. Selling patent rights and the incentive to innovate. Economics Letters 114: 241-244.

18. Wang, X.H., 1998, Fee versus royalty in a Cournot duopoly model, Economics Letters 50: 55-62.

19. Vishwasrao, S., 2007. Royalties vs. fees: How do firms pay for foreign technology?, International Journal of Industrial Organization 25: 741-759.