Equilibrium of a Search Model with Non-unit Demand

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27 July 2021

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Introduction

Justification

Search models represent an alternative to the assumption of a walrasian auctioneer, as they explicitly model how a decentralized trading economy works: firms are price-makers and consumers search sequentially between firms. One of the limitations of the existing literature is that its focus has been on trading mechanisms, as opposed to traditional consumer-theoretic considerations: in search models, consumer demand is usually inelastic with respect to price. Thus, when firms set their prices, they only take into account their extensive margin (number of customers) and not their intensive margin (quantity consumed by each customer). Therefore, these models are not compatible with general equilibrium models that are the foundations of contemporary macroeconomics, such as the models based on Dixit-Stiglitz utility functions.

In the present article, I build a search model with consumers having a non-unit demand and I prove the existence of an equilibrium. This proof is a stepping stone towards building a "complete" consumption model, in which consumers search for and consume multiple goods. Such a model would be a major building block of a general equilibrium model that would be entirely based on explicit micro-foundations and would not require the assumption of a walrasian auctioneer.

Literature review

The search literature has branched into multiple subfields each having their own core issue of interest and models. The main two such subfields are search in the labor market (for a review, cf. \cite{MOR/PIS/99} and \cite{ROG/SHI/WRI/05}), and search in the market for goods, to which this article belongs. In that latter subfield, once the initial problem set up by \cite{STI/61}, research quickly focused on answering the paradox presented by \cite{DIA/71} : in a model with identical consumers and identical firms, the only equilibrium is the degenerate equilibrium where all firms post the monopoly price. \cite{SAL/STI/77} offered a model where consumers who search can observe the prices posted by all the firms; alternatively, \cite{REI/79} proposed a model where consumers have a non-unit demand and don't search at the equilibrium (they buy from the first firm they encounter); finally, \cite{MAC/80} and \cite{BUR/JUD/83} proposed a model where consumers search through sampling instead of sequentially.

This classic literature, now somewhat old, has given birth to several extensions. One of them is multi-product search; another considers intertemporal models where consumers can search only once per "real" time period (e.g., \cite{ALB/ALI/09}).

None of these models include "realistic" consumers with a price-elastic demand who search sequentially for the lowest price. \cite{BEN/93} offered a partial characterization of the equilibrium for such a model with a single good, and \cite{LEV/YUN/09} did the same for an infinite number of goods (a la Dixit-Stiglitz). But neither prove formally the existence of an equilibrium. The present article expands in that direction by proving the existence of an equilibrium.

Contents

In the next two sections, I describe the behavior of consumers, then of firms. In the last section, I demonstrate the existence of the equilibrium, formulated in game-theoretic terms; this makes the economic mechanisms at play more intuitive and easier to understand (esp. the relationships between the resulting equilibrium and the general competitive equilibrium). However, this proof is not constructive, which limits our understanding of the qualitative properties of the equilibrium.

Consumers

Hypotheses

Let there be a continuum of consumers with measure 1. They have a price-elastic demand: if a consumer decides to buy from a firm with price p, they will buy and consume $x(p)$ units of the good and receive an indirect utility $V(p)$. All consumers share the same functions x and V. Price elasticity of individual demand is constant.

Each consumer is initially assigned at random to a firm, from which they can purchase at no search cost. Should they decide to search, they'll incur a cost s to access another firm. s is expressed directly in terms of individual utility ("utils") and not in monetary terms; searching thus doesn't affect a consumer's available budget for consumption. While potentially more realistic, assuming monetary search costs would result in a consumer's budget decreasing after each draw, which would make the consumer's optimization problem non-stationary.

Consumers are heterogeneous in terms of search costs; search costs follow the uniform distribution *F* over

$$S \equiv [s_{min}, s_{max}]$$

The corresponding density function is *f*. Search is instantaneous, so the discount rate doesn't impact search. Each consumer can search as many times as they wish, and they can return at any time to any of the firms they have already visited (perfect recall).

Optimal consumer behavior

After presenting the consumer optimization problem, I'll show that it admits a solution in the form of a "reservation value"; I'll then build the corresponding function *k*; finally, I'll build the function *a*, which yields the distribution *G* of reservation values as a function of the distribution *F* of search costs and the distribution *J* of prices posted by the firms. Such a plethora of distributions and functions is undeniably a challenge. I can only offer confused readers the same reassurance as \cite{MAS/84}: "Admittedly, so many layers of distribution may be at first a bit disorienting but it is, actually, all quite simple" (p205).

I model consumers' search behavior recursively, given that their optimization problem is stationary. Let $W_s(p_i)$ be the value function for a consumer with a cost of search *s* and whose best offer on hand is p_i. They can decide to consume directly without searching, which yields utility $V(p_i)$. Should they decide to search, they immediately incur a utility search cost *s* and are assigned to another firm. Let *J* be the current distribution of prices posted by the firms. There are then two possibilities:

- with probability $(1 - J(p_i))$, the newly observed price is higher than p_i and the consumer returns at no cost to the firm with price p_i, so their ending situation is the same as their starting situation; their value function after the assignment is again $W_s(p_i)$.
- The newly observed price *p* is lower than p_i, and the consumer's value function is now $W_s(p)$.

By application of the optimality principle, the consumer's problem can be formulated as

$$W_s(p_i) = \max\{V(p_i), \ - s + (1 - J(p_i))W_s(p_i) + \int_{p_{min}}^{p_i} W_s(p)dJ(p)\}$$

A classical result in search theory is that the solution of this problem is a "reservation value" r, namely the consumer accepts any price lower than *r* and rejects any price higher than *r* (i.e., search if and only if their best price on hand is higher than *r*).[1] More precisely, *r* is defined by

$$W(s) \equiv V(s) = -s + (1 - J(r))W_s(r) + \int_{p_{min}}^{r} W_s(p)dJ(p)$$

In addition, a consumer with search cost *s* will accept any price lower than *r* and reject any price higher than *r*, so we have

$$\int_{p_{min}}^{r} W_s(p)dJ(p) = \int_{p_{min}}^{r} V(p)dJ(p)$$

Building the function *k*

Let's now build the function *k*, which associates to any given search cost *s* the corresponding reservation value *r*, for a given distribution J of posted prices.

First,let's build the reciprocal function and show that it's invertible. By definition of *r*, we have

$$s = \int_{p_{min}}^{r} V(p)dJ(p) - J(r)V(r) \ (eqn.\ 1)$$

Equation 1 has a straightforward economic interpretation: a consumer's reservation value is such that the cost of another search (left-hand side) is equal to the expected utility gain of another search (right-hand side). Need to describe and account for corner solutions.

Let define $k^{-1}(r, J)$the function corresponding to the RHS:

$$s \equiv k^{-1}(r, J)$$

For any given *r* and *J*, the RHS of equation 1 admits one and only one value, so this function is well-defined. Consider then the partial function $k^{-1}|_{J_0}$ such that

$$k^{-1}|_{J_0}$$

This function has the following properties:

- It is continuous as the composition of continuous functions.
- It's injective:

$$\forall s \in S,\ \forall r \in \mathbb{R},\ \forall r' \in \mathbb{R},$$

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[1] For a complete presentation of models of search for the lowest price and their derivation, cf. \cite{MCC/MCC/08}).

$$s = k^{-1}|_{J_0}(r) = k^{-1}|_{J_0}(r') \Rightarrow V(r) = V(r') \Rightarrow r = r'$$

Because V is strictly monotonous w.r.t. its argument.

- It's surjective by property of reservation values: $\forall s \in S$, there exists a reservation value associated with *s*.
- Given that $k^{-1}|_{J_0}$ is continuous, the image of a compact set by $k^{-1}|_{J_0}$ is a compact set (i.e., a closed and bounded set).
- It is strictly monotonous. Let's integrate by parts:

$$\int_{p_{min}}^{r} V(p)dJ(p) = [V(p)J(p)]_{p_{min}}^{r} - \int_{p_{min}}^{r} J(p)dV = J(r)V(r) - \int_{p_{min}}^{r} J(p)dV$$

Replacing in the expression for *s* (eqn. 1) yields:

$$s = \int_{p_{min}}^{r} V(p)dJ(p) - J(r)V(r) = J(r)V(r) - \int_{p_{min}}^{r} J(p)dV - J(r)V(r) = - \int_{p_{min}}^{r} J(p)dV$$

Which is a strictly monotone function.

Finally, per the theorem of local inversion, we can define the function *k*, which associates to any *s* and any *J* the corresponding reservation value:

$$k: S \times [0, 1]^{\mathbb{R}_+} \rightarrow \mathbb{R}_+, \quad (s, J) \rightarrow r$$

Building the function *a*

Let's define the function *a* associating to an exogenous distribution *F* of search costs and an endogenous distribution *J* of prices the endogenous distribution G of reservation values:

$$a: [0, 1]^{S} \times [0, 1]^{\mathbb{R}_+} \rightarrow [0, 1]^{\mathbb{R}_+}, \quad (F, J) \rightarrow G$$

Note that because the function *k* is continuous and strictly increasing, if the exogenous distribution F of search costs is continuous and atomless, then the endogenous distribution *G* of reservation values is also continuous, as long as the distribution *J* of posted prices includes prices that are sufficiently low. Indeed, if the consumers with the lowest search costs have a theoretical reservation value that is lower than the lower bound of the price distribution p_{min}, they'll actually have that lower bound as their practical reservation value. Then we could have an atom at p_{min}, which would cause a discontinuity of the profit function of the firms at this point.

Therefore, I make the assumption that the lowest posted price is low enough to avoid this situation.

Hypothesis 1. *The distribution of search costs is such that the distribution of reservation values doesn't have an atom at* p_{min}.

This issue doesn't occur at the other extremity of the price distribution: consumers who have a search cost so high that they accept any price will spread between firms, and their behavior doesn't create a discontinuity in the profit function of firms.

Firms

Hypotheses

There exists a continuum of firms with measure 1. The production technology is linear. Firms are heterogeneous in terms of production costs; unit production costs follow the uniform distribution *H* over

$$C \equiv [c_{min}, c_{max}]$$

The corresponding density function is *h*.

Optimal firm behavior

In this subsection, I'll set up the firms' optimization problem and show that it has a solution in the form of an optimal price posted by the firm. I'll then build the function *phi* which determines this optimal price as a function of the firm's production cost *c*, the distribution *G* of consumers' reservation values and the distribution *J* of prices posted by the other firms. Finally, I'll build the function *b* which determines the distribution *J'* of prices posted by the firms based on the distribution *H* of production costs, the distribution *G* of reservation values and an initial price distribution *J*.

In order to determine a firm's optimal pricing behavior, let's first calculate the number *N(p)* of consumers buying from a firm with price *p* at the end of their search (the calculations are from \cite{BEN/93}). Let *g* be the density function corresponding to the distribution *G*. A number *g(r)* of consumers have the reservation value *r*. They'll search until they encounter a firm with a posted price less than or equal to *r*, and there are *J(r)* such firms. Therefore each of these firms receive a share $\frac{g(r)}{J(r)}$ of the consumers with reservation value *r*. Calculating the sum over all possible values of *r* above *p* (because consumers with a reservation value below *p* will never stop at a firm posting that price), we get

$$N(p) = \int_{p}^{+\infty} \frac{g(r)}{J(r)} dr$$

And the profit of a firm with production cost *c* posting a price *p* is thus

$$\Pi_c(p) \equiv (p - c)x(p)N(p) = (p - c)x(p) \int_p^{+\infty} \frac{g(r)}{J(r)} dr$$

Each firm maximizes its profit taking as given the distribution *G* of reservation values and the distribution *J* of prices posted by the other firms. The profit function is continuous over a closed, bounded interval of $\mathbb{R}$ so it admits at least one maximum. A sufficient condition for this maximum to be unique is that Π_c be strictly concave. Let's define $\pi_c(p) \equiv (p - c)x(p)$ the profit per consumer. Therefore $\Pi_c(p) = \pi_c(p)N(p)$ and we have (omitting the index *c* for readability)

$$\Pi' = \pi' N + \pi N' \; and \; \Pi'' = \pi'' N + 2\pi' N' + \pi N''$$

Profit per consumer is strictly concave by property of the individual demand function, so $\pi'' < 0$. Let π^m be the monopoly price maximizing π; no firm has an incentive to post a price strictly higher than its monopoly price (because lowering its posted price to the monopoly price would increase its profit per consumer while increasing the number of consumers it serves). For $p < p^m$, π is strictly increasing so $\pi' > 0$. *N* is strictly decreasing by property of the reservation values, so $N' < 0$. Ultimately, the overall profit function Π_c is strictly concave if *N* is concave, or at least sufficiently little convex compared to π. I'll assume that's the case.

Hypothesis 2. *The profit function Π_c is strictly concave.*

In this case, the firm's optimization problem has one and only one solution for each cost *c*.

Building the function *phi*

Let *phi* be the counterpart of the function *k* for consumers. It associates to a production cost *c*, a distribution of reservation values *G* and a distribution of prices posted by the other firms *J*, the optimal price p* posted by the firm:

$$phi: C \times [0, 1]^{\mathbb{R}_+} \times [0, 1]^{\mathbb{R}_+} \rightarrow \mathbb{R}_+, \quad (c, G, J) \rightarrow p^*$$

The function *k* only included the individual exogenous variable (i.e., the search cost *s* of a consumer) and the endogenous variable from the other side of the market (i.e., the price distribution). On the contrary, the function *phi* includes the endogenous variables from both sides of the market (the distribution of the reservation values and the distribution of prices posted by other firms) in addition to the individual exogenous variable (here, the production cost). Indeed the number of consumers buying from a firm depends on the relative position of that firm in the price distribution.

Building the function *b*

Let *J* be the distribution of posted prices. We have for any given price p_0

$$J(p_0) = \mathbb{P}\{p \leq p_0\} = \mathbb{P}\{phi(c, G, J) \leq p_0\} = \mathbb{P}\left\{c \leq (phi|_{G,J})^{-1}(p_0)\right\} = H\left((phi|_{G,J})^{-1}(p_0)\right)$$

Let b be the function defined by

$$b: [0, 1]^C \times [0, 1]^{\mathbb{R}_+} \times [0, 1]^{\mathbb{R}_+} \rightarrow [0, 1]^{\mathbb{R}_+}, \quad (H, G, J) \rightarrow J^*$$

This function associates to a distribution of production costs *H*, a distribution of reservation values *G* and an initial distribution of prices *J* the distribution of optimal prices *J**.

If the distribution of production costs is continuous and atomless, the endogenous distribution of optimal posted prices is also continuous and atomless.

Game-theoretic equilibrium

From a game-theoretic perspective, the model presented in this article belongs to the category of non-atomic games, i.e. games with a continuum of players. The founding article for this literature is \cite{SCH/73}. The proof of the existence of an equilibrium is essentially an extension of Nash's theorem, and relies on the fact that the graph of the best response function for each player is closed and that for any profile (i.e. distribution) of strategies played, the set of the corresponding best response profiles is non-empty and convex. It is then possible to use the Glicksberg-Fan theorem, which proves the existence of a fixed point for the best response mapping. This result is extended by \cite{MAS\84} to continuous sets of possible actions. I'll walk through the logic of that article to show that it applies to the model presented here.

Let $A \equiv [c_{min}, p^m(c_{max})]$be the set of possible actions. No firm has an incentive to post a price lower than c_{min}(this would lead to a negative profit) or higher than $p^m(c_{max})$, the monopoly price for the highest possible production cost (posting exactly $p^m(c_{max})$ would ensure a strictly higher profit per consumer and an equal or higher number of consumers). If all posted prices are between these two bounds, we can restrict the set of acceptable reservation values to *A* without loss of generality. *A* is an interval of $\mathbb{R}$ and thus a compact metric space. Let *M* be the set of probability distributions on *A*. A player (in the present case a firm or a consumer) is characterized by a utility function $u: A \times M \rightarrow \mathbb{R}$ where *u(a,m)* is the utility for the player of playing action *a* when the distribution of actions played by the other players is *m*. To stay close to the original article, I do not distinguish here between consumers and firms, but that would be a straightforward extension.

Let $U_A$be the space of utility functions $u: A \times M \rightarrow \mathbb{R}$ with the sup norm. It's the space of the players' characteristics. It is a separable and complete metric space. A game is characterized by a distribution μon U_A.We then have the following result:

Theorem 1. *Any game* μ *on* U_A admits a Cournot-Nash equilibrium distribution *t* on U_A *such that*

1. $tU_A = \mu$, the distribution of players characteristics obeys the game;
2. $t\left(\left\{(u, a): u(a, t_A) \geq u(A, t_A)\right\}\right) = 1$,the actions selected are the best responses for all players.

I don't repeat here the proof from \cite{MAS\93} which applies directly. It's an application of the Schauder fixed-point theorem on the space T of probability distributions on $U_A \times A$(i.e., T is the space of strategy profiles of the game).

Discussion and conclusion

It should be possible to extend the proof above to a model with multiple goods, which would allow comparing the conclusions of this category of models with general-equilibrium models based on the walrasian auctioneer.

Bibliography

- Albrecht, J., F. Postel-Vinay, and S. Vroman (2009) : "A Search-Theoretic Model of Equilibrium Cycles with Sales," 2009 Meeting Papers 691, Society for Economic Dynamics.
- Benabou, R. (1993) : "Search Market Equilibrium, Bilateral Heterogeneity, and Repeat Purchases," Journal of Economic Theory, 60(1), 140–158.
- Burdett, K., and K. L. Judd (1983) : "Equilibrium Price Dispersion," Econometrica, 51(4), 955–69.
- Diamond, P. A. (1971) : "A model of price adjustment," Journal of Economic Theory, 3(2), 156–168.
- Levin, A., and T. Yun (2009) : "Reconsidering the Microeconomic Foundations of Price-Setting Behavior," Unpublished Working Paper.
- MacMinn, R. D. (1980) : "Search and Market Equilibrium," Journal of Political Economy, 88(2), 308–27.
- Mas-Colell, A. (1984) : "On a theorem of Schmeidler," Journal of Mathematical Economics, 13(3), 201–206.
- McCall, B., and J. McCall (2008) : The economics of search. Routledge.
- Mortensen, D. T., and C. A. Pissarides (1999) : "New developments in models of search in the labor market," in Handbook of Labor Economics, ed. by O. Ashenfelter, and D. Card, vol. 3, chap. 39, pp. 2567–2627. Elsevier.
- Reinganum, J. F. (1979) : "A Simple Model of Equilibrium Price Dispersion," Journal of Political Economy, 87(4), 851–58.
- Rogerson, R., R. Shimer, and R. Wright (2005) : "Search-Theoretic Models of the Labor Market : A Survey," Journal of Economic Literature, 43(4), 959–988.
- Salop, S., and J. E. Stiglitz (1977) : "Bargains and Ripoffs : A Model of Monopolistically Competitive Price Dispersion," Review of Economic Studies, 44(3), 493–510.
- Schmeidler, D. (1973) : "Equilibrium points of nonatomic games," Journal of Statistical Physics, 7, 295–300.
- Stigler, G. J. (1961) : "The Economics of Information," Journal of Political Economy, 69, 213. 11