Paces of fiscal consolidations, fiscal sustainability, and welfare: An overlapping generations approach

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Abstract

This study investigates expenditure- and tax-based consolidations under the rule of reductions in debt-to-GDP ratios to the target level as well as the effects of these consolidations on fiscal sustainability and welfare, using an overlapping generations model with exogenous growth settings. We derive (i) the global transition dynamics of the economy, (ii) a threshold (ceiling) of public debt to ensure fiscal sustainability, (iii) sustainable paces of these consolidations, and (iv) optimal pace of consolidations from viewpoints of both social welfare and fairness of each generation's welfare. We find that higher paces or lower targets of debt-to-GDP ratio make fiscal policies more sustainable. The pace required of tax-based consolidation to ensure fiscal sustainability is higher than that required of expenditure-based consolidation. As for welfare, countries may differ in their choice of the type of consolidation. It depends on how large outstanding debts relative to capital are and how large the utility derived by individuals from public goods and services is. By contrast, a common result from the viewpoints of both social welfare and fair distribution of welfare across generations is that very slow pace of fiscal consolidation cannot be supported.

JEL classification: E62; H60; H40

Keywords: Fiscal consolidation, Paces of consolidation, Fiscal sustainability, Welfare

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1 Introduction

The default risk on the Greek government debt in the 2008–2009 global financial crisis raised concern over the sustainability of public debts or deficits among countries with large public debts. Long-term sustainability is one of the largest concerns of both policymakers and academics (e.g., Fatás and Mihov (2010) and D'Erasmo, Mendoza, and Zhang (2016) for recent studies). In fact, the Stability and Growth Pact (SGP) in the EU identifies fiscal sustainability as the main goal of its fiscal framework. The fiscal consolidation rule in the SGP has two directives; it sets the (i) target of debt levels, and (ii) the pace of reduction in debt. The rule states that member states whose current debt-to-GDP ratio is above the 60% threshold must reduce their ratios to 60% at an average rate of one-twentieth per year.

Although the need for fiscal consolidation prevails in OECD countries with high debt, there exists little consensus on the paces of fiscal consolidation (see e.g., Rawdanowicz (2014))[1] Why is determining the pace of fiscal consolidation difficult? Consider the case of countries with very large outstanding debts (e.g., Japan, Greece, Italy, Portugal, and the US). At a very slow pace, fiscal consolidations may fail to sustain fiscal policy due to the large interest payment of public debt as well as the crowding out effect of public debt on capital accumulation (e.g., Diamond (1965) and Chalk (2000) can referred for theoretical literature while Mankiw and Elmendorf (1999) provide a survey of the empirical literature). Furthermore, it postpones the burden of debt payment on future generations, which may not be fair to them. By contrast, a very rapid pace of consolidation may lead to a burden on the current and earlier generations and result in a large loss of social welfare. Then, a common pace of consolidation under the SGP in the EU might be the *hard coordinated consolidation regime*, as classified by Panico and Purificato (2013), for countries with extremely high debts (e.g., Greece, Italy, Portugal).

When considering the pace of consolidation, it is also important to know as to the type of consolidation that is more effective between spending cuts and tax increases. According to the literature survey by Molnar (2012), earlier studies are not conclusive on this issue. Some studies (e.g., Alesina and Ardagna 1998, 2009; von Hagen et al. 2002; Guichard et al. 2007) indicate

[1] Rawdanowicz (2014) states as follows. "While there is generally little controversy about the need for fiscal consolidation, its optimal pace is ..., posing a key dilemma for policymakers in many OECD countries. Some argue for postponing consolidation as a large, frontloaded adjustment that can reduce GDP growth with negative fallout for the fiscal situation . ... The choice of optimal consolidation path depends crucially on the ultimate long-term objective of fiscal policy and market conditions. Estimating optimal consolidation pace is challenging given the nexus of interactions between fiscal policy, financial markets and economic growth. "

that consolidation based on expenditure cuts are found to be more effective while others (e.g., Alesina and Perotti 1995; Tsibouris et al. 2006) find that revenue-based consolidations can be effective.

Accordingly, we tackle the following research questions: (i) How does the pace of fiscal consolidation affect the transition paths of the economy? (ii) How rapid should the pace of fiscal consolidation be to ensure fiscal sustainability? (iii) How does the pace of fiscal consolidation impact each generation's welfare or social welfare? Is there a trade-off between the two? (iv) What would be the different impact on the above three questions under expenditure-based and tax-based fiscal consolidations?

To that end, we study a standard overlapping-generations (OLG) model of a closed economy developed by Diamond (1965). We introduce a debt policy rule under which the government debt relative to the size of the economy is adjusted gradually to a targeted debt/GDP level in the long run.[2] Under expenditure-based (tax-based) consolidations, governments adjust their spending (income tax rates) with fixed income and consumption tax rates (the fixed ratio of government spending to GDP). In OLG models, fiscal sustainability means that the ratio of public debt to GDP (or capital) converges to a stable level in the long run (e.g., Chalk (2000) and de la Croix and Michel (2002)). Thus, we investigate the global transition dynamics and check whether the transition paths converge globally to the steady state. To shed some light on global transitional dynamics, we employ analytically tractable settings with inelastic labor supply and the Cobb-Douglas utility and production functions. For the welfare analyses, we calibrate the model to the data of Japan, the US, Greece, Italy, and Portugal as examples of countries with very high debt-to-GDP ratios.

In this study, the pace of consolidation plays an important role in turning unsustainable transition paths into sustainable ones. Then, welfare effects of the transition from unsustainable to sustainable paths are highlighted in this study while a large body of previous studies focuses only on the steady states between pre and post policy changes or transitions between these two. We also judge the welfare effect of the fiscal consolidation based on both social welfare and fairness of welfare distribution between each generation.

The main findings of this study are summarized as follows.

[2]A recent empirical study by Molnar (2012) finds that fiscal rules are associated with a greater probability of stabilizing debt. Many empirical analyses show that better-designed rules are more likely to reduce fiscal deficits (see the survey by Eyraud, Debrun, Hodge, Lledó, and Pattillo (2018))

(i) A unique stable steady state exists under both, expenditure- and tax-based consolidations with the debt policy rule. Properties of global transition paths are derived analytically and represented in two two-dimensional phase diagrams, each under two types of consolidation plans.

(ii) There is a threshold of public debt for each level of capital in order for the government to sustain fiscal policy, and the threshold of public debt increases in proportion to the size of capital under each type of consolidation plan. A higher pace or lower target of debt-to-GDP ratio makes fiscal policies more sustainable.

(iii) The minimal pace of tax-based consolidation that ensures fiscal sustainability is higher than that of expenditure-based consolidation, indicating that expenditure-based consolidation is more likely to make fiscal policy sustainable.

(iv) Numerical investigations show that Japan, Greece, Italy, and Portugal cannot sustain fiscal policy either without reducing debt or a low pace of reduction in debts. By contrast, the US economy may sustain its fiscal policy even without reducing debt.

(iv) Social welfare increases in all countries (Japan, the US, Greece, Italy and Portugal) by fiscal consolidation. The choice of the type of consolidation (between tax-based or expenditure-based) may differ among countries. It depends on how large the outstanding debts relative to capital are, and how large the utility that individuals derive from public goods and services is. By contrast, a common result from the viewpoints of both social welfare and fair distribution of welfare is that a very slow pace of fiscal consolidation cannot be supported.

Related literature

Fiscal consolidation is shown to be productive in the medium and long term in the literature of exogenous growth models. Some studies (e.g., Coenen, Taylor, Wieland, and Wolters 2008; Forni, Gerali, and Pisani 2010; Bi, Leeper, and Leith 2013; Cogan, Mohr, and Straub 2013; Erceg and Lindé 2013; Philippopoulos, Varthalitis, and Vassilatos 2017) use new Keynesian dynamic stochastic general equilibrium (DSGE) models while others (e.g., Papageorgiou 2012; Hansen and İmrohoroğlu 2016) use real business-cycle (RBC) models. Common features of these studies are that the focal point is the effect of fiscal consolidation on transitional dynamics.

Growth models that examine optimal paces of consolidation include Maebayashi, Hori and Futagami (2017), Morimoto, Hori, Maebayashi, and Futagami (2017), and Futagami and Konishi (2018). These studies consider a debt policy rule in line with the SGP's 60% rule of the debt-to-GDP ratio for welfare analyses of fiscal consolidations, and show that a faster pace of consolidation drives larger welfare gains.[3] Rawdanowicz (2014) also sheds light on the pace of consolidation plan to reduce debt from 90% to 60% of GDP within 20 years and to maximize a discounted sum of real GDP growth (or minimize a discounted sum of squared output gaps).

In the literature, Erceg and Lindé (2013) compare spending-based versus tax-based consolidation in a two-country new Keynesian model. They show that spending-based consolidation has far less costly effects on output than tax-based consolidation in the longer-term. Erceg and Lindé (2013) demonstrate that this finding is consistent with the supply side effects emphasized in Uhlig (2010). Maebayashi et al. (2017) show that spending-based consolidations have larger welfare gains than the tax-based consolidations in an endogenous growth model. Morimoto et al. (2017) assess both sustainability and social welfare in a small open endogenously growing economy and show that expenditure-based consolidation can be preferable for both fiscal sustainability and welfare.[4]

However, previous studies on transitional dynamics, optimal paces of fiscal consolidation, and spending-based versus tax-based consolidation assume an infinitely living agent, and therefore ignore intergenerational welfare losses or gains and the possibility of a Ponzi game by the governments.

Chalk (2000), de la Croix and Michel (2002), and Yakita (2008) investigated the sustainability of public debt (global transitional dynamics of debt) in OLG models and concluded that a Ponzi game by the governments is possible. The sustainability in OLG models is often defined as the convergence of the public debt to a sustainable level in the long term under some fiscal rules. Constant deficit (or deficit to GDP) rules are examined in Chalk (2000) and Yakita (2008), while a constant debt-to-GDP ratio is imposed in de la Croix and Michel (2002).[5] They show that a debt

[3]These studies use endogenous growth models with productive government spending and public debt, which are similar to Greiner and Semmler (2000) and Ghosh and Mourmouras (2004), for example.

[4]Other recent studies on the sustainability of debt in some endogenous growth models include Greiner (2007, 2012) Kamiguchi and Tamai (2012) Miyazawa, Ogawa, and Tamai (2019). These studies consider infinitely lived agent economies.

[5]In Chalk (2000), de la Croix and Michel (2002), and Yakita (2008), factor prices are endogenous, while other studies on fiscal sustainability by Bräuninger (2005) Arai(2011) Teles and Mussolini (2014) Maebayashi and Konishi (2019: forthcoming) use endogenously growing OLG models with constant factor prices. Oguro and Sato (2014)

above the threshold level can explode and cannot sustain fiscal policy. These studies however do not conduct analyses of fiscal consolidations that encompass the timeline effect of reduction in debt-to-GDP ratio.

When we regard pay-as-you-go (PAYG) social security as an implicit debt, privatization of public pension is like a problem of reductions in debt. Recently, Nishiyama and Smetters (2007) (by a simulation methodology in line with Auerbach and Kotlikoff (1987)) and Andersen and Bhattacharya (2020) (by a Diamond-type OLG model) show that decreasing debt can achieve Pareto improvement if the present value of these net resources can be distributed to future generations. However, these analyses sacrifice global transitional dynamics in the sense that they focus only on the steady states with between pre and post policy changes or transitions between these two. Then, these studies ignore how to prevent unsustainable ways of consolidation.

Building on these previous studies, we examine the global transition dynamics of both expenditure- and tax-based consolidations, a sustainable pace of these consolidations, welfare effects, and an optimal pace of consolidations from viewpoints of both social welfare and fairness of welfare across generations in an OLG economy.

examine the relationship between interest rates on government bonds and the fiscal consolidation rule by using an OLG model with endogenous and stochastic growth settings.

2 The Model

The framework is based on an OLG model following Diamond (1965). There are L_t individuals who live for two periods. We assume population grows at an exogenous rate, n, so that $L_t = (1+n)L_{t-1}$. They supply one unit of labor in youth inelastically and retire in old age.

A single final good, Y_t, is produced by using capital, K_t, and labor, L_t, according to a constant-returns-to-scale technology, $Y_t = F(K_t, L_t)$. The intensive form of this function is $y(k_t) = f(k_t)$, where $f'(\cdot) > 0$ and $f''(\cdot) < 0$ with respect to the capital to labor ratio: $k_t \equiv K_t/L_t$. We assume that capital depreciates fully after one period. Profit maximization under perfect competition yields the interest rate, $R_t = f'(k_t) \equiv R(k_t)$, and the wage rate, $w_t = f(k_t) - f'(k_t)k_t \equiv w(k_t)$.

Individuals consume private goods and services when young c_t (old d_{t+1}) and utilize public goods and services provided by the government in both periods S_t^g and S_{t+1}^g. We assume that public goods and services are denoted by $S_t^g = G_t/(L_t + L_{t-1}) = ((1+n)/(2+n))g_t$, where $g_t \equiv G_t/L_t$ and G_t is public spending. The lifetime utility function of an individual born in period t is

$$U_t = \ln c_t + \theta \ln S_t^g + \beta(\ln d_{t+1} + \theta \ln S_{t+1}^g), \tag{1}$$

where β and θ denote the subjective discount factor and the preference weight on public goods and services, respectively. Let s_t be the saving in youth. The lifetime budget constraints of generation t are $(1+\tau_t^c)c_t + s_t = (1-\tau_t^w)w_t$ and $(1+\tau_{t+1}^c)d_{t+1} = (1-\tau_t^R)R_{t+1}s_t$, where τ_t^w, τ_t^R, and τ_t^c, are tax rates on wage income, capital income, and consumption, respectively. The utility maximization yields

$$s_t = \frac{\beta(1-\tau_t^w)}{1+\beta} w(k_t). \tag{2}$$

Next, we move onto the fiscal policy. The governments face their budget constraint, $B_{t+1} = R(k_t)B_t + G_t - T_t$, where B_t, G_t, and $T_t(= \tau_t^w w(k_t)L_t + \tau_t^R R(k_t)(B_t + K_t) + \tau_t^c(c_t L_t + d_t L_{t-1}))$ are government bonds, government expenditure and tax revenue, respectively. Dividing this constraint by L_t, we obtain

$$(1+n)b_{t+1} = R(k_t)b_t + g_t - \tau_t^w w(k_t) - \tau_t^R R(k_t)(b_t + k_t) - \tau_t^c \left(c_t + \frac{d_t}{1+n}\right), \tag{3}$$

where $b_t \equiv B_t/L_t$ and $g_t \equiv G_t/L_t$. Additionally, fiscal policy is subject to the following debt policy rule:

$$b_{t+1} - b_t = -\phi \left(b_t - \bar{b} y(k_t) \right), \tag{4}$$

where, $\phi(> 0)$ and $\bar{b}$ stand for the adjustment coefficient of the rule and the target level of debt-to-GDP ratio, respectively. We consider the case of $\bar{b} > 0$. We rewrite (4) into $\frac{b_{t+1} - b_t}{y_t} = -\phi \left(\frac{b_t}{y_t} - \bar{b} \right)$ to interpret it. If the ratio of debt-to-GDP ratio ($b_t/y_t(= B_t/Y_t)$) is larger than $\bar{b}$, the government has to reduce b_t/y_t by making fiscal surplus a percentage of GDP ($(b_{t+1} - b_t)/y_t$), according to the difference between the current and target levels of debt-to-GDP ratio ($b_t/y_t - \bar{b}$). If the adjustment coefficient (ϕ) takes a large (or small) value, the government adjusts b_t/y_t to the target level ($\bar{b}$) at a fast (or slow) pace.

There are three notable cases as to the values of ϕ. First, when $\phi = 1$, (4) leads to $b_{t+1} - b_t = -(b_t - \bar{b} y(k_t))$. Here, the government will reduce public debt by the difference between the current and target levels of the debt-to-GDP ratio ($b_t/y_t - \bar{b}$) in one period. If we regard one period as 30 years, the fiscal consolidation in the EU may ask for such a tight plan because the plan achieves a fiscal reconstruction in 20 years (within 30 years). Second, when $0 < \phi < 1$, it takes more than one period to achieve a fiscal reconstruction because public debt decreases more gradually. Finally, when applying $\phi = 0$ to (4), we obtain $b_{t+1} = b_t = b_0$, which indicates that the government does not reduce outstanding public debts but keeps its debts at the initial level b_0. Throughout this study, we treat $\phi \leq 1$ as the case where fiscal consolidations are implemented and $\phi = 0$ as the one without fiscal consolidations. We summarize these points in the following Remark 1.

Remark 1. *(i) When $\phi = 1$, a fiscal reconstruction is achieved in one period. (ii) When $0 < \phi < 1$, it takes more than one period to accomplish a fiscal reconstruction. A larger (or lower) ϕ leads to a more rapid (or slower) fiscal consolidation. (iii) when $\phi = 0$, no fiscal consolidations to reduce outstanding debt are implemented, that is, $b_{t+1} = b_t = b_0$.*

The government implements fiscal consolidations with (4) unexpectedly at time 0. The tax rates at $t = 0$ before consolidations are supposed to be given by $(\tau^w_{init}, \tau^R_{init}, \tau^c_{init})$.

3 Equilibrium

Asset market clears as $B_{t+1} + K_{t+1} = (1+n)(b_t + k_t)L_t = s_t L_t$. This together with (2) yields

$$b_{t+1} + k_{t+1} = \frac{\beta(1-\tau_t^w)}{(1+\beta)(1+n)} w(k_t). \tag{5}$$

By substituting (4) into (5), we can derive the difference equation of k_t as

$$k_{t+1} = \Phi(k_t, b_t, \tau_t^w) \equiv \frac{\beta(1-\tau_t^w)w(k_t)}{(1+\beta)(1+n)} - \left[b_t - \phi\left(b_t - \bar{b}y(k_t)\right)\right]. \tag{6}$$

The goods market equilibrium condition is given by $K_{t+1} = Y_t - G_t - c_t L_t - d_t L_{t-1}$ and is rewritten into $c_t + d_t/(1+n) = y(k_t) - g_t - (1+n)k_{t+1}$. This, together with (6), yields the tax revenues from consumption (per capita) as

$$\tau_t^c\left(c_t + \frac{d_t}{1+n}\right) = \tau_t^c[y(k_t) - g_t - (1+n)\Phi(k_t, b_t, \tau_t^w)]. \tag{7}$$

From (3), (4), (6), and (7), we obtain

$$\begin{aligned}(1+\tau_t^c)g_t =&(1+n)\left[b_t - \phi\left(b_t - \bar{b}y(k_t)\right)\right] + \tau_t^w w(k_t) + \tau_t^R R(k_t)k_t + \tau_t^c y(k_t) \\ &- (1-\tau_t^R)R(k_t)b_t - (1+n)\tau_t^c\Phi\left(k_t, b_t, \tau_t^w\right). \end{aligned} \tag{8}$$

The following condition must be satisfied to sustain (keep) fiscal policy (providing public services): $g_t > 0$ for all t:

$$\begin{aligned} b_t <& \frac{\phi\bar{b}(1+n)(1+\tau_t^c)y(k_t) + \tau_t^w w(k_t) + \tau_t^R R(k_t)k_t + \tau_t^c y(k_t) - \tau_t^c \frac{\beta(1-\tau_t^w)}{1+\beta} w(k_t)}{(1-\tau_t^R)R(k_t) - (1+n)(1+\tau_t^c)(1-\phi)} \\ &\equiv \Omega\left(k_t, \tau_t^w, \tau_t^R, \tau_t^c\right), \end{aligned} \tag{9}$$

otherwise ($g_t \leq 0$), $g_t = 0$ binds, meaning that fiscal policy cannot be sustained.[6]

[6] If $g_t = 0$ binds at a certain period, fiscal policy can no longer follow the rule of (4), because large issuance of public bonds is necessary to meet net interest payment of debt: $\tau_t^w w(k_t) + \tau_t^R R(k_t)k_t + \tau_t^c[y(k_t) - (1+n)k_{t+1}] + (1+n)b_{t+1} = (1-\tau_t^R)R(k_t)b_t$. By this government budget constraint and (5), the issuance of public bonds and accumulation of capital are derived as $(1+n)b_{t+1} = (1+\tau_t^c)^{-1}\{(1-\tau_t^R)R(k_t)b_t - \tau_t^w w(k_t) - \tau_t^R R(k_t)k_t - \tau_t^c[y(k_t) - (\beta/(1+\beta))(1-\tau_t^w)w(k_t)]\}$ and $(1+n)k_{t+1} = (1/(1+\tau_t^c))[(\beta/(1+\beta))(1-\tau_t^w)w(k_t) - (1-\tau_t^R)R(k_t)b_t + \tau_t^w w(k_t) + \tau_t^R R(k_t)k_t + \tau_t^c y(k_t)]$, respectively. However, these dynamics under $g_t = 0$ are outside the main scope of our investigation.

In this study, we consider a dynamically-efficient economy (i.e., $R(k_t) > 1+n$) and pay attention to fiscal policies with positive debts: $\Omega(k_t) > 0 \Leftrightarrow k_t < \hat{k}$, where $\hat{k}_t$ satisfies $(1-\tau_t^R)R\left(\hat{k}_t\right) = (1+n)(1+\tau_t^c)(1-\phi)$.

Condition 1. *$R(k_t) > 1+n$ and $k_t < \hat{k}_t$, where $(1-\tau_t^R)R\left(\hat{k}_t\right) = (1+n)(1+\tau_t^c)(1-\phi)$.*

In the next section, we consider fiscal consolidations ($\phi \in (0,1]$: Remark 1-(i) and-(ii)) by adjusting expenditure g_t as

$$\begin{aligned}(1+\tau^c)g_t =& (1+n)\left[b_t - \phi\left(b_t - \bar{b}y(k_t)\right)\right] + \tau^w w(k_t) + \tau^R R(k_t)k_t + \tau^c y(k_t) \\ & - (1-\tau^R)R(k_t)b_t - (1+n)\tau^c \Phi\left(k_t, b_t\right) \end{aligned} \tag{10}$$

with the constant tax rates (i.e., $\tau_t^w = \tau^w$, $\tau_t^R = \tau^R$, and $\tau_t^c = \tau^c$), termed *the expenditure-based consolidation*, hereafter. Furthermore, let us denote $\Phi\left(k_t, b_t, \tau^w\right)$ in (6) and $\Omega\left(k_t, \tau_t^w, \tau_t^R, \tau_t^c\right)$ in (9) simply as $\Phi(k_t, b_t)$ and $\Omega(k_t)$, respectively.

Equations (4) and (6) combined with (9) characterize the dynamic system of the economy under the expenditure-based consolidation.

Before moving onto the following sections, we mention the case of no fiscal consolidation ($\phi = 0$: Remark 1-(iii)) with the tax rates fixed at the level before consolidations ($\tau_t^w = \tau_{init}^w$, $\tau_t^R = \tau_{init}^R$, and $\tau_t^c = \tau_{init}^c$ for all t). Applying $\phi = 0$, $b_{t+1} = b_t = b_0$, and $\tau_t^w = \tau_{init}^w$ $\forall t$ into (6), we obtain the following dynamic system:

$$k_{t+1} = \Phi(k_t, b_0, \tau_0^w) = \frac{\beta(1-\tau_{init}^w)w(k_t)}{(1+\beta)(1+n)} - b_0 \quad \text{and} \quad b_{t+1} = b_t = b_0 \tag{11}$$

for a given $b_0 > 0$. Assuming that $w(k_t)$ is concave in k_t: i.e., $w''(k_t) < 0$, we can derive the following facts.

Remark 2 *Consider the case of no fiscal consolidation ($\phi = 0$, $b_{t+1} = b_t = b_0$, and $\tau_t^w = \tau_{init}^w$ $\forall t$) and define*

$$b^{upper} \equiv [\beta(1-\tau_{init}^w)/(1+\beta)(1+n)]\, w\left(\bar{k}_{no}\right) - \bar{k}_{no}.$$

(i) If the initial public debt b_0 is large enough to satisfy $b_0 > b^{upper}$, where $\bar{k}_{no}$ satisfies $w'\left(\bar{k}_{no}\right) = (1+\beta)(1+n)/\beta(1-\tau_0^w)$, k_t decreases monotonically and eventually takes zero,

meaning that the economy goes bankrupt and public debt cannot be sustainable (Figure 1-(a)).

(ii) If $b_0 < b^{upper}$, public debt can (cannot) be sustainable when $k_0 \geq (<) \underline{k}_{no}$, where $\underline{k}_{no}$ satisfies $\Phi\left(\underline{k}_{no}, b_0, \tau_0^w\right) = \underline{k}_{no}$ and $[\beta(1-\tau_0^w)/(1+\beta)(1+n)]\, w'\left(\underline{k}_{no}\right) > 1$. When $k_0 > \underline{k}_{no}$, k_t converges on a steady-state value, k_{no}^*, which satisfies $\Phi\left(k_{no}^*, b_0, \tau_0^w\right) = k_{no}^*$ and $[\beta(1-\tau_0^w)/(1+\beta)(1+n)]\, w'\left(k_{no}^*\right) < 1$ (Figure 1-(b)).

Remark 2 indicates that fiscal consolidations should be implemented in an economy in which current debt b_0 is larger than b^{upper}. As we will discuss later, the current level of public debt: b_0 in Japan, Greece, Italy, Portugal may be the case with the one in Remark 2.[7] In the following sections, we study the economy under Remark 2 mainly and examine the effects of fiscal consolidation on the transition paths of the economy, fiscal sustainability, and welfare.

[Figure 1]

4 Dynamics under expenditure-based consolidation

In this section, we derive the global transitional dynamics of the economy under the expenditure-based consolidation. For the tractability of analyses, we consider the case of Cobb-Douglas production function: $Y_t = AK_t^\alpha L_t^{1-\alpha}$ $(0 < \alpha < 1)$. Then, equations (4), (6), (9), and Condition 1 for $\phi \in (0,1]$ are written as follows:

$$b_{t+1} - b_t = -\phi\left(b_t - \bar{b}Ak_t^\alpha\right), \tag{12}$$

$$k_{t+1} = \Phi(k_t, b_t) = \eta Ak_t^\alpha - \left[b_t - \phi\left(b_t - \bar{b}Ak_t^\alpha\right)\right], \tag{13}$$

$$b_t < \Omega(k_t) = \frac{Ak_t^\alpha\left[\phi\bar{b}(1+n)(1+\tau^c) + \tilde{\tau} + \tau^c(1-(1+n)\eta)\right]}{(1-\tau^R)\alpha Ak_t^{\alpha-1} - (1+n)(1+\tau^c)(1-\phi)}, \tag{14}$$

$$k_t < \hat{k} \equiv \left\{(1-\tau^R)\alpha A/[(1+n)(1+\tau^c)(1-\phi)]\right\}^{\frac{1}{1-\alpha}}, \tag{15}$$

where $\eta \equiv \frac{\beta(1-\alpha)(1-\tau^w)}{(1+\beta)(1+n)}$ $(1-(1+n)\eta = 1 - \frac{\beta(1-\alpha)(1-\tau^w)}{1+\beta} > 0)$ and $\tilde{\tau} \equiv \tau^w(1-\alpha) + \tau^R\alpha$.

We start with the derivation of the steady-state values of k_t and b_t. Applying $k_{t+1} = k_t$ and

[7] We may not ignore other countries whose outstanding public debts are growing rapidly.

$b_{t+1} = b_t$ to (12) and (13) and solving yields

$$(k^*, b^*) = \left(\left[\left(\eta - \bar{b}\right) A \right]^{\frac{1}{1-\alpha}}, \ \bar{b} \left(\eta - \bar{b}\right)^{\frac{\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} \right), \tag{16}$$

where we assume $\bar{b} < \eta$ to ensure the existence of this steady state. (16) leads to the following proposition:

Proposition 1. *A unique steady state (k^*, b^*) exists if and only if $\bar{b} < \eta$. Both k^* and b^* are independent on the pace of fiscal consolidation ϕ.*

For the tractability of later analyses, we define $y_t/k_t = Ak_t^{\alpha-1} \equiv q(k_t)$ and $b_t/k_t \equiv x_t$ and prepare the expressions of (14), (15), and (16) with $(q(k_t), x_t)$ as follows:

$$x_t < \tilde{\Omega}(q(k_t)) \equiv \frac{q(k_t)\left[\phi\bar{b}(1+n)(1+\tau^c) + \tilde{\tau} + \tau^c(1-(1+n)\eta)\right]}{(1-\tau^R)\alpha q(k_t) - (1+n)(1+\tau^c)(1-\phi)}, \tag{17}$$

$$q(k_t) > q\left(\hat{k}\right) = \frac{(1+n)(1+\tau^c)(1-\phi)}{(1-\tau^R)\alpha}, \tag{18}$$

$$(q(k^*), x^*) = \left(\frac{1}{\eta - \bar{b}}, \frac{\bar{b}}{\eta - \bar{b}}\right). \tag{19}$$

Next, we derive the $k_{t+1} = k_t$ and $b_{t+1} = b_t$ loci on the (k_t, b_t) plane. From (12), $b_{t+1} = b_t$ locus is given by

$$b_t = \bar{b}Ak_t^{\alpha}. \tag{20}$$

It is the concave and strictly increasing function of k_t that takes $k_t = b_t = 0$.

[Figure 2]

Next, $k_{t+1} = k_t$ locus is $k_t = \eta Ak_t^{\alpha} - \left[b_t - \phi\left(b_t - \bar{b}Ak_t^{\alpha}\right)\right]$, which is rewritten as

$$b_t = \frac{\left(\eta - \phi\bar{b}\right)Ak_t^{\alpha} - k_t}{1-\phi} \equiv Z(k_t) \qquad \text{for} \quad \phi \in (0,1), \tag{21}$$

$$k_t = k^* \quad \forall b_t \qquad \text{for} \quad \phi = 1. \tag{22}$$

Here, keep in mind that $\bar{b} < \eta$. Furthermore, $Z(k_t)$ has the following properties. First, $Z(0) = Z(\tilde{k}) = 0$, where $\tilde{k} \equiv [(\eta - \phi\bar{b})A]^{\frac{1}{1-\alpha}}$. Second, $Z(k_t) > 0$ holds for $0 < k_t < \tilde{k}$, and $Z'(k_t) = \frac{(\eta-\phi\bar{b})\alpha q(k_t)-1}{1-\phi} \geq (<) 0$ for $0 \leq k_t \leq \bar{k}$ ($\bar{k} < k \leq \tilde{k}_t$), where $\bar{k} \equiv [(\eta - \phi\bar{b})\alpha A]^{\frac{1}{1-\alpha}}$.

Lemma 1. *Suppose that $\bar{b} < \eta$. The $b_{t+1} = b_t$ and $k_{t+1} = k_t$ loci have the following properties.*

(i) The $b_{t+1} = b_t$ locus is a concave and upward-sloping curve that takes $k_t = b_t = 0$.

(ii) The shape of $k_{t+1} = k_t$ locus depends on the value of ϕ.

(a) When $\phi < 1$, it is an inverted-U shaped curve that takes $k_t = 0$ and $\tilde{k}$ (> 0) when $b_t = 0$.

(b) When $\phi = 1$, $k_{t+1} = k_t$ locus is a perpendicular line: $k_t = k^* = \left[\left(\eta - \bar{b}\right) A\right]^{\frac{1}{1-\alpha}}$.

From (12) and Lemma 1-(i), Appendix D shows that $b_{t+1} > (\leq) b_t$ below (above) the $b_{t+1} = b_t$ locus at each point of the (k_t, b_t) plane as depicted in Figure 2. Furthermore, from (13) and Lemma 1-(ii), Appendix D shows that when $0 < \phi < 1$, $k_{t+1} > (\leq) k_t$ below (or above) the inverted-U shaped $k_{t+1} = k_t$ locus at each point of the (k_t, b_t) as depicted in Figure 2- (a). When $\phi = 1$, $k_{t+1} > (\leq) k_t$ at the left (or right) side of $k_{t+1} = k_t$ locus represented as a perpendicular line in Figure 2 -(b).

For the later use, we express $b_{t+1} = b_t$ locus ((20)) and $k_{t+1} = k_t$ locus for $\phi \in (0, 1)$ ((21)) with $(q(k_t), x_t)$ as follows:

$$x_t = \bar{b} q(k_t), \tag{23}$$

$$x_t = \tilde{Z}(q(k_t)) \equiv \frac{\left(\eta - \phi \bar{b}\right) q(k_t) - 1}{1 - \phi}. \tag{24}$$

Finally, we examine the region in which $g_t \geq 0$ on the (k_t, b_t) plane. Eq. (9), associate with the value of $\hat{k}_t$ (see (15)), yield the following:

Lemma 2. *$g_t = 0$ locus has the following properties.*

(i) $g_t = 0$ locus is a convex upward-sloping curve that takes $k_t = b_t = 0$ and has asymptote $\lim_{k_t \to \hat{k}} \Omega(k_t) = +\infty$.

(ii) $g_t = 0$ locus intersect with $b_{t+1} = b_t$ locus at a unique point $H(k_H, b_H)$, where $k_H > 0$ and $b_H > 0$ are given by

$$(k_H, b_H) = \left(\left[\frac{\bar{b}(1-\tau^R)\alpha A}{\bar{b}(1+\tau^c)(1+n)+\tilde{\tau}+\tau^c(1-\eta(1+n))} \right]^{\frac{1}{1-\alpha}}, (\bar{b}A)^{\frac{1}{1-\alpha}} \left[\frac{(1-\tau^R)\alpha}{\bar{b}(1+\tau^c)(1+n)+\tilde{\tau}+\tau^c(1-\eta(1+n))} \right]^{\frac{\alpha}{1-\alpha}} \right). \tag{25}$$

(iii) $g_t = 0$ *locus intersect with* $k_{t+1} = k_t$ *locus at a unique point* $P(k_P, b_P)$*, where* $k_P > 0$ *and* $b_P > 0$*. When* $\phi = 1$*,* $k_P = k^*$ *holds.*

Proof: See Appendix A.

From (14) and Lemma 2, fiscal policy above $g_t = 0$ locus in the (k_t, b_t) plane (shaded area in Figure 2) cannot be sustainable because $g_t = 0$ binds there. All transition paths that lead to this area should be avoided and be regarded as unsustainable.

For later use, we notify the properties of $x_t = \tilde{\Omega}(q(k_t))$ ($g_t = 0$ locus expressed by $(q(k_t), x_t)$: see (17)). This together with (23) and (24) yield the following facts. First, at the point $H(k_H, b_H)$, $q(k_H) = \frac{\bar{b}(1+\tau^c)(1+n)+\tilde{\tau}+\tau^c(1-(1+n)\eta)}{\bar{b}(1-\tau^R)\alpha}$ and $x_H(\equiv b_H/k_H) = \frac{\bar{b}(1+\tau^c)(1+n)+\tilde{\tau}+\tau^c(1-(1+n)\eta)}{(1-\tau^R)\alpha}$. Second, at the point $P(k_P, b_P)$, $q(k_P)$ and $x_P \equiv b_P/k_P$ satisfy

$$\frac{\left(\eta - \phi\bar{b}\right) q(k_P) - 1}{1 - \phi} = \frac{q(k_P)\left[\phi\bar{b}(1+n)(1+\tau^c) + \tilde{\tau} + \tau^c(1-(1+n)\eta)\right]}{(1-\tau^R)\alpha q(k_P) - (1+n)(1+\tau^c)(1-\phi)} \tag{26}$$

and

$$\begin{aligned}&(1-\phi)x_P\left[(1-\tau^R)\alpha x_P - \tilde{\tau} - \tau^c - \eta(1+n)\right]\\&= \phi\bar{b}(1+n)(1+\tau^c) + \tilde{\tau} + \tau^c(1-(1+n)\eta) - (1-\tau^R)\alpha x_P,\end{aligned} \tag{27}$$

$$\begin{aligned}&\underline{x}_P < x_P < \bar{x}_P,\\&\underline{x}_P \equiv \left[\tilde{\tau} + \tau^c + \eta(1+n)\right]/(1-\tau^R)\alpha,\\&\bar{x}_P \equiv \left[\phi\bar{b}(1+n)(1+\tau^c) + \tilde{\tau} + \tau^c(1-(1+n)\eta)\right]/(1-\tau^R)\alpha.\end{aligned} \tag{28}$$

Appendix B shows the uniqueness of k_P and derivation of (28).

From (14), (16), and Lemma 2, the following condition must be satisfied to ensure fiscal sustainability in the steady state when $\phi \in (0, 1)$.

Condition 2. $k^* > k_P$ *if and only if*

$$\bar{b} < \bar{b}_1 \equiv \frac{\zeta_1 + \sqrt{\zeta_1^2 + 4(1+n)(1+\tau^c)\zeta_2}}{2(1+n)(1+\tau^c)} \in (0, \eta),$$

where, $\zeta_1 \equiv \tilde{\tau} + \tau^c(1-(1+n)\eta) + (1-\tau^R)\alpha - (1+\tau^c)(1+n)\eta$ *and* $\zeta_2 \equiv \eta[\tilde{\tau} + \tau^c(1-(1+n)\eta)](>$

0).

Appendix C derives Condition 2 by (26) and shows that $\bar{b}_1$ is increasing in $\tilde{\tau}$. Condition 2 indicates that the target level of debt-to-GDP ratio ($\bar{b}$) must be lower than $\bar{b}_1$ (the ceiling level of $\bar{b}$). $\bar{b}_1$ is increasing in the income tax rate because a rise in income tax revenue loosens government's budget.

Figure 2 illustrates a phase diagram of the economy, highlighting that the steady state S is stable. Let us start with the case of $\phi \in (0, 1)$ (case (a)). The saddle arm converging to $P(k_P, b_P)$, labeled "*Threshold* ", represents the threshold of public debt for each level of k_t. An economy, whose initial state is below the threshold curve as represented by Q_1, converges gradually to the steady state S. At the steady state S, the state variables (k_t, b_t) take constant values of (k^*, b^*), and the government can run its fiscal policy with its positive debt $b^* > 0$ permanently.

By contrast, an economy whose initial state is above the threshold curve, will bind $g_t = 0$ and fiscal policy cannot be sustainable. The point Q_2 represents the case where the initial public level is so large that the economy will not converge to any steady states. In such situations, expenditure cut even under the debt policy rule (4) can no longer eliminate outstanding public debts. Particularly, in the early stage of fiscal consolidation, a large public debt crowds out capital accumulation, shrinks the economy, and exacerbates a fiscal condition seriously.

Next, we move onto the case of $\phi = 1$. Applying $\phi = 1$ into (12) leads to $b_{t+1} - b_t = -(b_t - \bar{b}Ak_t^{\alpha})$. Then, a reduction in public debt in each period is the distance between the debt revel b_t and $b_{t+1} = b_t$ locus. Furthermore, a fall in debt is greater as the current outstanding debt is larger, indicating that the fiscal policy is sustainable as long as the initial state is outside of $g_t \leq 0$ (the shaded area).

In summary, we can state the following proposition.

Proposition 2. *Fiscal policy and public debt are unsustainable in either of (i) or (ii).*

(i) The target level of public debt-to-GDP ratio: $\bar{b}$ *is larger than the certain level* b_1.

(ii) Initial public debt is large enough to exceed the threshold level that is represented by the positive function of k_t.

Next, we focus on the properties of the sustainable transition path during the expenditure-based consolidation. They depend on the initial state of the economy and the pace of debt reduction.

We begin with the case of $0 < \phi < 1$. When the initial public debt is large relative to the capital stock (size of the economy), as represented by the point Q_1, a large public debt crowds out capital accumulation. Accordingly, capital decreases in the early stage of fiscal consolidation. However, as b_t steadily declines, capital begins to increase, eventually exceeding its initial level in the long run. Next, when the initial debt lies in the region of HPS, as represented by Q_3, fiscal consolidation reduces b_t and crowds in capital accumulation both in the short and long run. Finally, when capital is large relative to the debt (as in Q_4), public debt becomes low relative to GDP, leading to a small gap between the current and target debt-to-GDP ratio by (12). Therefore, the magnitude of expenditure cut is small enough to make fiscal policy sustainable.

We move onto the case of $\phi = 1$. As in the initial state represented by Q_5, a strong effect of debt reduction would promote capital accumulation both in the short and long run unless its accompanying expenditure cut would induce $g_t = 0$ to bind.

5 Changes in $\bar{b}$ and ϕ under the expenditure-based consolidation

In this section, we investigate how the policy variables $(\bar{b}, \phi)$ that characterize the fiscal consolidation strategy ((12)) affect the steady-state (long-run effects) and fiscal sustainability (short- and medium- run effects).

5.1 Effects on the steady state $S(k^*, b^*)$

Recall that ϕ is neutral to the steady state (by Proposition 1), and then we focus on the effect of $\bar{b}$ on the steady state values: k^* and b^* here. By (16), we obtain the following immediately.

Proposition 3. *(i) A fall in $\bar{b}$ increases the steady-state capital stock per capita k^*. (ii) A fall(or rise) in $\bar{b}$ decreases (or increases) the steady-state public debt per capita b^* for $\bar{b} \leq (>)(1-\alpha)\eta$.*

Intuitive reasons for Proposition 3 are as follows. First, lowering $\bar{b}$ causes a larger gap between the current and the target debt-to-GDP ratio by (12) and then a larger amount of debt reduction accompanies. Thus, more resources are released to private investment and increase k^*. Next, a reduction in $\bar{b}$ has two opposite effects on b^*. Lowering $\bar{b}$ decreases the long-run public debt level directly while it increases $b^*(= \bar{b}y(k^*))$ indirectly through its positive effect on k^* in the long run. Unless $\bar{b}$ is large enough to satisfy $\bar{b} > (1-\alpha)\eta$, the direct effect of decreasing $\bar{b}$ dominates the indirect effect and decreases b^*.

5.2 Effects on fiscal sustainability

Next, we investigate how a rise in the pace of consolidation (ϕ) or a fall in the targeted debt to GDP ratio ($\bar{b}$) affects fiscal sustainability.

We start with a fall in $\bar{b}$. From (14), (20), and (21), a fall in $\bar{b}$ shifts $g_t = 0$ locus $b_{t+1} = b_t$ locus, and $k_{t+1} = k_t$ locus downward, downward, and upward, respectively. Thus, a fall in $\bar{b}$ increases k_P ($dk_p/d\bar{b} < 0$), as represented in Figure 3-(a). Furthermore, Appendix B shows that (27) and the definition of $x_P \equiv b_P/k_P$ yields

$$\begin{aligned} \frac{db_P}{dk_P} &= \frac{\overbrace{\omega_1 - (1-\tau^R)\alpha x_P}^{(+)} + (1-\phi)\left[\tilde{\tau} + \tau^c + \eta(1+n)\right] x_P}{2(1-\phi)(1-\tau^R)\alpha x_P + \omega_2} > 0 \\ \omega_1 &\equiv 2\left[\phi\bar{b}(1+n)(1+\tau^c) + \tilde{\tau} + \tau^c(1-(1+n)\eta)\right] > 0, \\ \omega_2 &\equiv (1-\tau^R)\alpha - (1-\phi)\left[\tilde{\tau} + \tau^c + \eta(1+n)\right] > 0, \end{aligned} \tag{29}$$

where $\omega_1 - (1-\tau^R)\alpha x_P > 0$ and $\omega_2 > 0$ hold from (28). Therefore, we arrive at the following proposition:

Proposition 4. *A fall in $\bar{b}$ shifts $P(k_P, b_P)$ to the upper right direction (Figure 3-(a)), indicating that a reduction in $\bar{b}$ makes fiscal policy more sustainable.*

A lower $\bar{b}$ causes a larger gap between the current and the target debt-to-GDP ratio by (12) and then a larger amount of debt reduction accompanies in the short and medium run.[8] This extends the fiscal space through the following two channels. First, the interest payment of public debt decreases. Second, more resources are released to private investment (crowd in effect), which enhances tax revenues. Therefore, fiscal policy can be more sustainable.

[Figure 3]

[8]Here, note that a fall in $\bar{b}$ makes fiscal policy unsustainable if the initial state (k_t, b_t) is already near the region of $g_t \leq 0$ ((14)). However, since we have focused mainly on a somewhat mature economy without capital shortage, we could ignore such a rare case throughout this study.

We next examine how changes in the pace of fiscal consolidation ϕ impact the sustainability of fiscal policy. Using (14) and (21), we obtain the following:

Lemma 3. *An increase in ϕ shifts the $k_{t+1} = k_t$ locus upward (or downward) for $k_t < (>) k^*$ and the $g_t = 0$ locus downward (or upward) for $k_t > (<) k_H$.*
Proof: See Appendix E.

From Lemma 3, we obtain $dk_P/d\phi > 0$ immediately. This together with $db_P/dk_P > 0$ leads to the following proposition:

Proposition 5. *A rise in ϕ shifts $P(k_P, b_P)$ to the upper right direction (Figure 3-(b)), indicating that a rise in ϕ makes fiscal policy more sustainable.*

As ϕ increases, a decline in public debt (b_t) in the early stage of the transition is large. Then the government can extend fiscal space more rapidly through decreases in interest payment and increases in tax revenues, making fiscal policy more sustainable.

5.3 Numerical analyses

We calibrate the model to the date of Japan, Greece, Italy, Portugal, and the US as examples of countries whose debt-to-GDP ratios are very high among OECD countries. We consider the following scenarios. Expenditure-based consolidation starts at period 0 unexpectedly for given (k_0, b_0). Constant tax rates are assumed to be at the initial levels, $\tau^R = \tau^R_{init}$ $\tau^w = \tau^R_{init}$, $\tau^c = \tau^c_{init}$.

5.3.1 Parameter choices

The targeted debt-to-output ratio is set at 0.6 as the benchmark (the target value of the SGP in the EU). Since B_t is a stock while Y_t is a flow, an appropriate measure of the targeted debt-to-output ratio in the model is $\bar{b} = 0.6/30 (= 0.02)$, taking one period as 30 years.[9] The subjective discount factor is set at $\beta = (0.973)^{30}$ as in Song et al. (2012). We adapt $A^{JPA} = 20$ to the Japanese economy. [10]

We next move onto each country's specific parameter values.

[9]This adjustment between a stock and a flow is in line with Song, Storesletten and Zilibotti (2012) and Andersen and Bhattacharya (2020). They employ OLG models where one period corresponds to 20 or 30 years.
[10]A is simply a scale parameter when the production is Cobb-Douglas and the utility is log-linear (see e.g., the Appendix A.5 of de la Croix and Michel (2002)).

[Table 1]

Japan

We choose $\alpha^{JPA} = 0.38$ following Hansen and İmrohoroğlu (2016).[11] Capital and wage income tax rates are set to $\tau^R_{JPA} = 0.46$ and $\tau^w_{JPA} = 0.31$ based on the estimated values in Gunji and Miyazaki (2011), overall statutory tax rates on dividend income, and average personal income tax and social security contribution rates on gross labor income at the OECD tax database. The (2000–2007) average capital income tax rate by Gunji and Miyazaki (2011) is around 0.53 while the (2000–2007) overall private income tax (PIT) on dividend plus corporate income tax rate (CIT) is around 0.56. Since the (2000–2020) overall PIT plus is around 0.49, the adjusted value of τ^R by Gunji and Miyazaki (2011) from 2000 to 2020 is 0.46. We use the (1995–2007) average wage income tax rate of around 0.31 by Gunji and Miyazaki (2011) since the average personal income tax and social security contribution rates do not change drastically between 2000 and 2019.[12] Consumption tax rate is set to the latest value of $\tau^c_{JPA} = 0.1$ in 2020. The average annual population growth rate between 1990 and 2018 was 0.09% according to the World Development Indicators, and thus we set $n^{JPA} = 0$.[13] The output to capital ratio $(Y/K(= q(k)))$ in Japan from 1990 to 2020 is around 0.32 on average according to the AMECO database.[14] Since K_t is a stock while Y_t is a flow, an appropriate measure of the output to capital ratio in the model is $q(k_0^{JPA}) = 0.32 \times 30 = 9.6$. Solving $q(k_0^{JPA}) = A^{JPA}(k_0^{JPA})^{\alpha^{JPA}-1} = 20(k_0^{JPA})^{0.38-1} = 9.6$ yields $k_0^{JPA} \approx 3.27$. We obtain the output per capita: $y(k_0^{JPA}) = q(k_0^{JPA})k_0^{JPA} \approx 31.36$ and interest rate: $R(k_0^{JPA}) \approx 3.65$ (the annual rate of around 4.4%). Next, let us use the (2014–2018) debt-to-output ratio of 2.37 (OECD (2021)) as the current level. Then, we obtain $b_0^{JPA}/y_0^{JPA} = 2.37/30$ in the model. From $b_0^{JPA} = (b_0^{JPA}/y_0^{JPA})(k_0^{JPA}/q(k_0^{JPA}))$, we obtain $b_0^{JPA} \approx 2.48$.

The US, Greece, Portugal, and Italy

The value of α in the US: $\alpha^{US} = 0.35$, in Greece: $\alpha^{GRE} = 0.4$, in Italy: $\alpha^{ITA} = 0.39$,

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[11]Hansen and İmrohoroğlu (2016) set to 0.3783: the sample (1981–2010) average of the annual ratio of capital income in Japan.

[12]The (2000–2007) total tax wage of a single person (without dependent) at 100% of the average wage is around 29% while the (2000–2019) total tax wage is around 30% according to the OECD tax base (accessed on 09 Feb 2021).

[13]I retrieved from https://data.worldbank.org/indicator/SP.POP.GROW (accessed on 05 Oct 2020).

[14]We have used the data of GDP at constant market prices per unit of net capital stock at the AMECO database (accessed on 13 Feb 2021).

and in Portugal: $\alpha^{PRT} = 0.39$ follow the values in Trabandt and Uhlig (2011). The average annual population growth rate between 2000 and 2019 was 0.97% in the US, 0.21% in Greece, 0.22% in Italy, and 0.09% in Portugal (World Development Indicators), and thus we set $(n^{US}, n^{GRE}, n^{ITA}, n^{PRT}) = (0.01, 0, 0, 0)$.

We employ the values of tax rates (τ^R, τ^w, τ^c) in these four countries: $(0.34, 0.28, 0.05)$ in the US, $(0.16, 0.41, 0.15)$ in Greece, $(0.30, 0.47, 0.15)$ in Italy, and $(0.23, 0.31, 0.23)$ in Portugal based on the estimated values in Trabandt and Uhlig (2011), and the overall statutory tax rates on dividend income, and average personal income tax and social security contribution rates on gross labor income at OECD tax database. The (1995–2007) average capital income tax rate by Trabandt and Uhlig (2011) is around 0.36 in the US, 0.16 in Greece, 0.34 in Italy, and 0.23 in Portugal while the (1995–2007) overall private income tax (PIT) on dividend plus corporate income tax rate (CIT) is around 0.61 in the US, 0.34 in Greece, 0.53 in Italy, and 0.49 in Portugal. Since the (1995–2020) overall PIT plus is around 0.57 in the US, 0.35 in Greece, 0.47 in Italy, 0.48 in Portugal, the adjusted value of τ^R by Trabandt and Uhlig (2011) from 2000 to 2020 is 0.16 in Greece and 0.30 in Italy. Since the average personal income tax and social security contribution rates do not change drastically between 2000 and 2019 (OECD tax base), we adapt the values of τ^w in Trabandt and Uhlig (2011). Consumption tax rate in Greece, Italy and Portugal are based on the actual value in 2020 (OECD tax data base), while the value in the US is based on Trabandt and Uhlig (2011).

The (1990–2020) average output to capital ratios ($q(k)$) in the US, Greece, Italy, and Portugal are 0.41, 0.28, 0.30, and 0.36, respectively (AMECO database), indicating that values of $q(k_0)$ in the model (30 years in one period) are given by $q(k_0^{US}) = 30 \times 0.41 = 12.3$ in the US, $q(k_0^{GRE}) = 30 \times 0.28 = 8.4$ in Greece, $q(k_0^{ITA}) = 30 \times 0.30 = 9.0$ in Italy, and $q(k_0^{PRT}) = 30 \times 0.36 = 10.8$ in Portugal. The (2015–2019) debt-to-output ratio of 1.36 in the US, 1.93 in Greece, 1.53 in Italy, 1.42 in Portugal (OECD (2021)) are adjusted to $b_0^{US}/y_0^{US} = 1.36/30$, $b_0^{GRE}/y_0^{GRE} = 1.93/30$, $b_0^{ITA}/y_0^{ITA} = 1.53/30$, and $b_0^{PRT}/y_0^{PRT} = 1.42/30$ in the model.

Here, we normalize the Japanese economy as the baseline. From data of the actual public debt per capita in 2015 and in 2018 (OECD (2017) and OECD (2019)), the ratios of the public debt per capita in country j to those in Japan are calculated as $(b_0^{US}/b_0^{JPA}) = 0.67$, $(b_0^{GRE}/b_0^{JPA}) = 0.55$, $(b_0^{ITA}/b_0^{JPA}) = 0.66$, and $(b_0^{PRT}/b_0^{JPA}) = 0.50$. Since the public debt per capita in country j in the model is given by $b_0^{JPA} \times (b_0^i/b_0^{JPA})$ $(i = US, GRE, ITA, PRT)$ and $b_0^{JPA} = 2.48$, we obtain

$b_0^{US} = 2.48 \times 0.67 \approx 1.66$, $b_0^{GRE} = 2.48 \times 0.55 \approx 1.36$, $b_0^{ITA} = 2.48 \times 0.66 \approx 1.64$, and $b_0^{PRT} = 2.48 \times 0.50 \approx 1.24$. From the data of GDP per capita between 1990 and 2019 (World Development Indicators), the ratios of the output per capita in country j to those in Japan are calculated as $(y_0^{US}/y_0^{JPA}) \approx 1.13$, $(y_0^{GRE}/y_0^{JPA}) \approx 0.49$, $(b_0^{ITA}/b_0^{JPA}) \approx 0.77$, and $(y_0^{PRT}/y_0^{JPA}) \approx 0.45$. These together with $y(k_0^{JPA}) = 31.36$ yield $y_0^i (i = US, GRE, ITA, PRT)$ in the model as $(y_0^{US}, y_0^{GRE}, y_0^{ITA}, y_0^{PRT}) \approx (35.66, 15.46, 24.30, 14.11)$. From $k_0^i = b_0^j \times \left(\frac{q(k_0^j)}{b_0^i/y_0^i}\right)$, we have $k_0^{US} = 1.66 \times \left(\frac{12.3}{1.36/30}\right) \approx 2.98$, $k_0^{GRE} = 1.36 \times \left(\frac{8.4}{1.93/30}\right) \approx 2.52$, $k_0^{ITA} = 1.64 \times \left(\frac{9.0}{1.53/30}\right) \approx 3.57$, and $k_0^{PRT} = 1.24 \times \left(\frac{10.8}{1.42/30}\right) \approx 2.43$. Substituting the values of y_0^j, k_0^i, and α^j into $y_0^i = A^j (k_0^j)^{\alpha^j}$ yields $(A^{US}, A^{GRE}, A^{ITA}, A^{PRT}) \approx (24.34, 10.68, 14.80, 9.99)$.

These parameter choices yield the plausible values of interest rate of $R(k_0^{JPA}) \approx 3.65$, $R(k_0^{US}) \approx 4.31$, $R(k_0^{GRE}) = 3.36$, $R(k_0^{ITA}) = 3.51$, and $R(k_0^{PRT}) \approx 4.21$ and the ratio of government spending to GDP of $g_0^{JPA}/y(k_0^{JPA}) = 0.3547$, $g_0^{US}/y(k_0^{US}) = 0.2497$, $g_0^{GRE}/y(k_0^{GRE}) = 0.3047$, $g_0^{ITA}/y(k_0^{ITA}) = 0.4131$, and $g_0^{PRT}/y(k_0^{PRT}) = 0.3183$.[15]

5.3.2 Results

Figures 4, 5, 6, 7, and 8 with Tables 2 and 3 indicate the following.

[Figures 4, 5, 6, 7, and 8] [Tables 2 and 3]

Fiscal sustainability

Fiscal policy is unsustainable without decreasing outstanding debt: $\phi = 0$ in Japan ($g_2 = 0$), Greece ($g_1 = 0$), Italy ($g_1 = 0$), and Portugal ($g_1 = 0$) while sustainable in the US. The lowest pace of the example consolidation plans ($\phi = 0.1$) cannot make fiscal policy sustainable in Japan ($g_3 = 0$), Greece ($g_1 = 0$), Italy ($g_1 = 0$), and Portugal ($g_2 = 0$). Only in Greece, fiscal consolidation even under $\phi = 0.3$ does not succeed ($g_1 = 0$ in Greece), indicating that the current/initial fiscal condition in Greece is the worst of the five countries. Outstanding public debts in these countries are so large relative to the size of the economy that the very low paces of consolidation plans cannot ensure their fiscal sustainability.

[15] Annual (long-run) interest rate of between 4% ($(1+0.04)^{30} \approx 3.24$) and 5% ($(1+0.05)^{30} \approx 4.32$). Government consumption + investment + transfer to GDP (the data value in Trabandt and Uhlig (2012) is 0.26 in the US, 0.35 in Greece, 0.40 in Italy, and 0.34 in Portugal, respectively. In Japan, government production costs (% of GDP) between 2007 and 2019 were around 0.21 on average (OECD data accessed on 30 June 2021) and transfer payment from 2000 to 2010 ranged between around 0.15 and 0.17 (see Hansen and İmrohoroğlu (2016)).

In the steady state

The steady-state levels of capital, k^*, government spending, g^*, consumption by the young c^*, and consumption by the old d^* exceed the current/initial ones, k_0, g_0, c_0, and d_0 in Japan and the US whereas fall behind in Greece, Italy and Portugal. In Greece, Italy, and Portugal, resources released to both private and public sectors by a reduction in debt are hampered by resource use to decrease such large outstanding public debt and by a large tax burden on wage.

In sustainable and unsustainable transition paths

We can confirm the properties of transitional dynamics of (k_t, b_t) as we have seen in Section 4. In Japan, a large initial public debt decreases (or increases) capital in the short and medium run when $0 < \phi < 1$ ($\phi = 1$) but begins to increase in the latter stages, eventually exceeding its initial level in the long run. In Greece, Italy, and Portugal, a large tax burden on wage under a low productivity, A, decreases (or increases) capital both when $0 < \phi < 1$ and $\phi = 1$ during fiscal consolidation. A larger decline in capital in the early stage of consolidation is associated with a slower pace of fiscal consolidation in these four countries. By contrast, capital in the US increases during fiscal consolidation owing to high productivity A. It increases rapidly with a faster pace of consolidation.

From (10), larger (or smaller) initial cuts in public expenditures g_0, with a faster (or slower) pace of fiscal consolidation, ϕ (the first term in the RHS of (10)) extend fiscal space more rapidly (or slowly), leading to increases (or further decrease) in public expenditure. From (30) and (31), consumption of both the young and old in time 0 (c_0 and d_0) are not affected by the initial cuts in public expenditures g_0. Since c_t is increasing in k_t ((31)), the dynamics of c_t reflects the dynamics of k_t. Finally, (32) shows that d_t decreases (or increases) in b_t because asset income from bonds decreases (or increases).

$$d_0 = \frac{(1-\tau^R)R(k_0)(k_0 + b_0)}{1+\tau^c} \quad \text{(generation } -1 \text{ in period } 0\text{)}, \tag{30}$$

$$c_t = \frac{(1-\tau^w)w(k_t) - \big(\Phi(k_t, b_t) + (1-\phi)b_t + \phi\bar{b}y(k_t)\big)}{1+\tau^c} \quad \text{(generations } t \geq 0 \text{ in period } t\text{)}, \tag{31}$$

$$d_{t+1} = \frac{(1-\tau^R)R(k_{t+1})(k_{t+1} + b_{t+1})}{1+\tau^c} \quad \text{(generations } t \geq 0 \text{ in period } t+1\text{)}. \tag{32}$$

In unsustainable transition paths, debt increases monotonically, so does the asset income from bonds and consumption by the old (d_t). On the one hand, a large crowding out effect of debt on capital decreases wage income and consumption by the young (c_t). On the other hand, it increases interest rate and the cost of repayment of debt for the government. Then, g_t decreases to zero.[16] From (1) when $g_t = 0$ is binding (no public goods and services), utility of generation $t-1$ takes asymptotically to $-\infty$, we regard the generation that faces this situation as the non-surviving generation (Table 2). These generations can occur in Japan, Greece, Italy and Portugal while cannot in the US.

6 A fiscal consolidation by adjusting the income tax rates

In this section, we examine a fiscal consolidation by adjusting the income tax rates, termed *the tax-based consolidation* hereafter. In tax-based consolidation, the government is assumed to secure its spending by a rate proportional to the rate of GDP: $G_t = \lambda Y_t$ ($\lambda \in (0,1)$), but to adjust the income tax rates (τ_t^w and τ_t^R) endogenously to follow the debt policy rule: (4). As to these endogenous tax rates, we assume that $\tau_t^R = \delta \tau_t^w$ and $\delta > 0$ and simply denote $\tau_t^w = \tau_t$ and $\tau_t^R = \delta\tau_t$, respectively. Finally, we consider the same timeline $\phi \leq 1$, constant consumption tax $\tau_t^c = \tau^c$, and the Cobb-Douglas production function ($Y_t = AK_t^\alpha L_t^{1-\alpha}$), as in the case of expenditure-based consolidation.

Substituting $G_t = \lambda Y_t$ ($g_t = \lambda A k_t^\alpha$), $\tau_t^w = \tau_t$, and $\tau_t^R = \delta\tau_t$ into (8) leads to

$$\begin{aligned}\tau_t =& \frac{[(1+\tau^c)\lambda - \tau^c(1-\tilde{\eta})]k_t + \alpha b_t}{[1+\alpha(\delta-1)+\tilde{\eta}\tau^c]k_t + \delta\alpha b_t} - \frac{(1+\tau^c)(1+n)\left[b_t - \phi\left(b_t - \bar{b}Ak_t^\alpha\right)\right]}{q(k_t)\left\{[1+\alpha(\delta-1)+\tilde{\eta}\tau^c]k_t + \delta\alpha b_t\right\}} \\ \equiv& \, \tau(k_t, b_t),\end{aligned} \tag{33}$$

where $\tilde{\eta} \equiv \frac{\beta(1-\alpha)}{1+\beta} \in (0,1)$ and recall that $q(k)(=Ak^{\alpha-1})$.

[16] Transition paths of (k_t, b_t) in the region of $g_t = 0$ on the phase diagram in Figures 4, 5, 6, 7, and 8, result from the dynamic systems that we have shown in footnote 3.

Substituting (33) and (12) into (5), we obtain

$$k_{t+1} = \tilde{\Phi}(k_t, b_t) \equiv \frac{(1+\tau^c)(1-\lambda) + \alpha(\delta-1) + (\delta-1)\alpha(b_t/k_t)}{1+\alpha(\delta-1)+\tilde{\eta}\tau^c+\delta\alpha(b_t/k_t)} \cdot \frac{\tilde{\eta}}{1+n} A k_t^{\alpha} - \frac{1-\tilde{\eta}+\alpha(\delta-1)+\delta\alpha(b_t/k_t)}{1+\alpha(\delta-1)+\tilde{\eta}\tau^c+\delta\alpha(b_t/k_t)} \left[b_t - \phi\left(b_t - \bar{b}Ak_t^{\alpha}\right)\right] \quad \text{for } k_t > 0. \tag{34}$$

Eqs. (12) and (34) characterize the dynamic system of the economy under the tax-based consolidation.

Using (12) and (34), we first investigate the existence of the steady states. Applying $k_{t+1} = k_t = k$ and $b_{t+1} = b_t = \bar{b}Ak^{\alpha}$ into (12) and (34) we obtain

$$\begin{aligned}
&\mu_1 q(k)^2 + \mu_2 q(k) + \mu_3 = 0,\\
&\mu_1 \equiv \alpha\bar{b}\left[(1+n)\delta\bar{b} - \tilde{\eta}(\delta-1)\right],\\
&\mu_2 \equiv (1+n)[\underbrace{1-\tilde{\eta}+\delta\alpha+\alpha(\delta-1)}_{=\frac{1-\alpha}{1+\beta}+2\alpha\delta(>0)}]\bar{b} - \tilde{\eta}[(1+\tau^c)(1-\lambda)+\alpha(\delta-1)],\\
&\mu_3 \equiv (1+n)[1+\alpha(\delta-1)+\tilde{\eta}\tau^c] > 0.
\end{aligned} \tag{35}$$

(35) leads directly to the following proposition:

Proposition 6.

(i) When $0 < \delta \leq 1$, two steady states exist if and only if $\mu_2 < 0$ and $\mu_2^2 - 4\mu_1\mu_3 > 0$.

(ii) When $\delta > 1$ and $\mu_1 > 0$, no steady state exists if $1 + (1+3\beta+\tau^c\beta)\alpha > (1+\tau^c)\beta$.

(iii) When $\delta > 1$ and $\mu_1 \leq 0$, a unique steady-state exists.

Proof: See Appendix F.

The following three points must be noted. First, to ensure the existence of the steady state, we assume that $\mu_2 < 0$ and $\mu_2^2 - 4\mu_1\mu_3 > 0$, when $0 < \delta \leq 1$. Second, when $\delta > 1$ and $\mu_1 > 0$ no steady state exists since $1+(1+3\beta+\tau^c\beta)\alpha > (1+\tau^c)\beta$ holds for reasonable range of parameter sets (α, β, τ^c), indicating that ceiling of $\bar{b}$ is given by ($\mu_1 \leq 0 \Leftrightarrow$) $\bar{b} \leq \bar{b}_2 \equiv \tilde{\eta}(\delta-1)/(\delta(1+n))$.[17]

[17] $1 + (1+3\beta+\tau^c\beta)\alpha > (1+\tau^c)\beta$ is satisfied under $0.2 < \alpha < 1$, $0 < \beta < 1$, and $0 < \tau^c \leq 1$.

Finally, since (35) are independent on ϕ, the pace of tax-based consolidation does not affect the steady state values of k_t and b_t as in the case of expenditure-based consolidation.

Next, we derive $k_{t+1} = k_t$ locus as the function of $b_t = m(k_t)$. From (34), we obtain

$$\begin{aligned} & k_{t+1} - k_t = \tilde{\Phi}(k_t, b_t) - k_t = 0 \\ & \Leftrightarrow h(b_t, k_t) \equiv a_1 b_t^2 + a_2(q(k_t))k_t b_t + a_3(q(k_t))k_t^2 = 0 \ \text{ for } \ k_t > 0, \end{aligned} \tag{36}$$

where $a_1 \equiv (1+n)(1-\phi)\delta\alpha > 0$, $a_2(q(k_t)) \equiv a_{21} + a_{22}q(k_t)$, $a_3(q(k_t)) \equiv a_{31} + a_{32}q(k_t)$, $a_{21} \equiv (1+n)\left\{(1-\phi)\left[\frac{1-\alpha}{1+\beta} + \alpha\delta\right] + \alpha\delta\right\} > 0$, $a_{22} \equiv \left[(1+n)\bar{b}\phi\delta - \tilde{\eta}(\delta-1)\right]\alpha$, $a_{31} = \mu_3 > 0$, and $a_{32} \equiv (1+n)\left[\frac{1-\alpha}{1+\beta} + 2\delta\alpha\right]\bar{b}\phi - \tilde{\eta}\left[(1+\tau^c)(1-\lambda) + \alpha(\delta-1)\right]$. Here, we notify the following condition on the parameters:

Condition 3.

When $\delta > 1$ and $\mu_1 \leq 0$ (Proposition 6-(iii)),

$$\begin{aligned} & a_{32} \leq 0 \quad \textit{if} \quad (1+\tau^c)(1-\lambda)\delta \geq (\delta-1)\left(\frac{1-\alpha}{1+\beta} + \alpha\delta\right) \\ & a_{32} \leq 0 \Rightarrow a_{21}a_{32} - a_{22}a_{31} \leq 0 \quad \textit{if} \quad \alpha\beta \geq \frac{\beta-\alpha}{2} \end{aligned}$$

for $0 < \phi \leq 1$ and $0 \leq \bar{b} \leq \min\left\{\bar{b}^\flat, \frac{(1-\delta)\tilde{\eta}}{\delta}\right\}$, where $\bar{b} = \bar{b}^\flat$ satisfies $a_{21}a_{32} = a_{22}a_{31}$.

Since values of $(\tau^c, \delta, \lambda, \alpha, \beta)$ in countries where $\delta > 1$ (the UK and Denmark in the EU 14 countries[18], Japan, and the US) satisfy Condition 3, we impose it in this study.[19] [20]

We move to examine (36). Appendix G shows that $k_{t+1} = k_t$ locus takes zero when $k_t = 0$ and $\breve{k} \equiv -a_{31}/a_{32} > 0$ for $a_{32} < 0$. To reveal more properties of $k_{t+1} = k_t$ locus, we rewrite (36) into

$$q(k_t) = \Gamma_k(x_t) \equiv -\frac{a_1 x_t^2 + a_{21}x_t + a_{31}}{a_{22}x_t + a_{32}} \quad \text{for} \quad a_{22}x_t + a_{32} \neq 0, \tag{37}$$

[18] see Trabandt and Uhlig (2011)

[19] $(\tau^c, \alpha) = (0.16, 0.36)$ in the UK while $(\tau^c, \alpha) = (0.35, 0.40)$ in Denmark (see Trabandt and Uhlig (2011)). $\beta = 0.973^{30}$ is taken in both countries. δ in the UK and Denmark are $\tau^R/\tau^w = 0.46/0.28 \approx 1.6$ and $0.51/0.47 \approx 1.1$, respectively (See Trabandt and Uhlig (2011)).

[20] Derivation of Condition 3 is available upon request. Even if Condition 3 is relaxed, when $\delta > 1$, $\mu_1 \leq 0$ and add the cases of (i) $a_{32} \leq 0$ and $a_{21}a_{32} - a_{22}a_{31} > 0$ and (ii) $a_{32} > 0$, we can characterize the $b_t = m(k_t)$ on the (k_t, b_t) plane and obtain the same results qualitatively, compared to those under policy changes in $\bar{b}$ and ϕ (as we examine in Section 7). These are available upon request.

where, recall that $x_t \equiv b_t/k_t$. From (36) and (37), we obtain the following properties of $b_t = m(k_t)$ on the (k_t, b_t) plane.

Lemma 4.

(i) *When* $0 < \delta \leq 1$, $b_t = m(k_t)$ *satisfies* $m(0) = m(\breve{k}) = 0$ *and takes the inverted-U shaped curve for* $\phi \in (0, 1]$.

(ii) *When* $\delta > 1$ *and* $\mu_1 \leq 0$, *(a)* $b_t = m(k_t)$ *for* $\phi \in (0, 1)$ *satisfies* $m(0) = m(\breve{k}) = 0$ *and takes the inverted-U shaped curve, while (b)* $b_t = m(k_t)$ *for* $\phi = 1$ *is monotonically decreasing in* k_t *that satisfies* $m(\breve{k}) = 0$ *(*$\breve{k} = q^{-1}(-\mu_3/\mu_2)$*) and has asymptote* $\lim_{q(k_t) \to -\frac{(1+n)\alpha\delta}{\bar{b}^{-1}\mu_1}} m(k_t) = +\infty$.

Proof: See Appendix G.

[Figures 9, and 10]

Figures 9-(a), on the one hand, illustrates the phase diagrams of the economy when $0 < \delta \leq 1$, highlighting that the steady state $S(k_S^*, b_S^*)$ is stable and the steady state $U(k_U^*, b_U^*)$ is saddle-point stable. In those cases, the knife-edge saddle arm converging to $U(k_U^*, b_U^*)$ represents the threshold of the public debt in order for the government to sustain fiscal policy. Figure 9-(b), on the other hand, illustrates the phase diagram of the economy when $\delta > 1$ and $\mu_1 \leq 0$, highlighting that the unique steady state $S(k_S^*, b_S^*)$ is stable and the knife-edge saddle arm converging to $k_t = b_t = 0$ represents the threshold curve.

When $0 < \delta \leq 1$ fiscal policy cannot be sustainable either if $k_{t+1} = 0$ or $\tau_t^w = \tau_t = 1$ binds (i.e., $k_{t+1} \leq 0$ or $\tau_t \geq 1$). $k_{t+1} = 0$ is equivalent to $\tilde{\Phi}(k_t, b_t) = 0$, which we call $k_{t+1} = 0$ locus hereafter, is written by

$$q(k_t) = \Theta(x_t) \equiv -\frac{a_1 x_t^2 + [a_{21} - (1+n)\alpha\delta] x_t}{a_{32} + a_{22} x_t} \quad \text{for} \quad a_{32} + a_{22} x_t \neq 0. \tag{38}$$

Appendix H shows that $k_{t+1} = 0$ locus is always above the $k_{t+1} = k_t$ locus. Next, $k_{t+1} = 0$ locus is above the threshold curve since $k_{t+1} = 0$ realizes eventually only when (k_t, b_t) is above the threshold curve. Furthermore, from (33) and (34), if $\tau_t = 1$, $k_{t+1} = 0$ always binds whereas $\tau_t = 1$ does not always bind if $k_{t+1} = 0$. Then, the condition of $\tau_t = 1$ is above $k_{t+1} = 0$ locus. These positional relationships between the condition of $\tau_t = 1$ and $k_{t+1} = 0$ locus and the threshold curve indicate that if (k_t, b_t) is above the threshold curve, the economy faces $k_{t+1} = 0$ at a certain period t. Then, fiscal policy cannot be sustainable in the next period $t + 1$, where $\tau_{t+1} = 1$ and $\tau^R_{t+1} = \delta\tau_{t+1} = 1$ also bind since (33) with $k_{1+1} = 0$ derives $\tau_{t+1} = 1/\delta > 1$.

When $\delta > 1$, fiscal policy cannot be sustainable either if $k_{t+1} = 0$ or $\tau^R_t = \delta\tau_t = 1$ binds (i.e., $k_{t+1} \le 0$ or $\delta\tau_t \ge 1$). In this case, $\delta\tau_t = 1$ does not bind either if $\delta\left[(1 + \tau^c)\lambda - \tau^c(1 - \tilde{\eta})\right] \le \delta(1+\tau^c)(1+n)\phi\bar{b} + \mu_3$ or if (k_t, b_t) is above the $b_{t+1} = b_t$ locus when $\delta(1+\tau^c)(1+n)\bar{b} + \mu_3 > \delta\left[(1 + \tau^c)\lambda - \tau^c(1 - \tilde{\eta})\right]$ (see Appendix I). Thus, if (k_t, b_t) is above the threshold curve, $k_{t+1} = 0$ binds at a certain period t, and fiscal policy cannot be sustainable in the next period $t + 1$ where $\tau^R_{t+1} = \delta\tau_{t+1} = 1$ also binds. We summarize the results in the following proposition.

Proposition 7.

(i) When $0 < \delta \le 1$ or $\delta > 1$ and $\mu_1 > 0$, the steady state $S(k^*_S, b^*_S)$ is stable while the steady state $U(k^*_U, b^*_U)$ is saddle-point stable, the saddle arm converging to $U(k^*_U, b^*_U)$ represents the threshold of the public debt in order for the government to sustain fiscal policy.

(ii) When $\delta > 1$ and $\mu_1 \le 1$ the unique steady state $S(k^*_S, b^*_S)$ is stable and the arm converging to $k_t = b_t = 0$ represents the threshold of the public debt to sustain fiscal policy.

7 Changes in $\bar{b}$ and ϕ under the tax-based consolidation

In this section, we examine the effects of changes in $\bar{b}$ and ϕ on fiscal sustainability and the steady states.

From (37), we obtain

$$\left.\frac{\partial \Gamma_k(x_t)}{\partial \bar{b}}\right|_{q(k_t)=\Gamma_k(x_t)} = \frac{(1+n)\phi\overbrace{\left(a_1 x_t^2 + a_{21}x_t + a_{31}\right)}^{(+)}}{(a_{22}x_t + a_{32})^2}\left(\delta\alpha x_t + \underbrace{1 - \tilde{\eta} + \alpha(\delta - 1)}_{=\frac{1-\alpha}{1+\beta}+\delta\alpha(>0)}\right) > 0 \quad (39)$$

and

$$\left.\frac{\partial \Gamma_k(x_t)}{\partial \phi}\right|_{q(k_t)=\Gamma_k(x_t)} = \frac{(1+n)\bar{b}\left(\delta\alpha x_t + \frac{1-\alpha}{1+\beta} + \alpha\delta\right)[\Gamma_b(x_t) - \Gamma_k(x_t)]}{a_{22}x_t + a_{32}} > (\leq) 0$$

$$\text{for } \Gamma_b(x_t) > (\leq) \Gamma_k(x_t). \tag{40}$$

Thus, a fall in $\bar{b}$ shifts $q(k_t) = \Gamma_k(x_t)$ ($k_{t+1} = k_t$ locus) downward (upward) whereas $q(k_t) = \Gamma_b(x_t)$ ($b_{t+1} = b_t$ locus) upward (downward) as depicted in Figure 11. Furthermore, a rise in ϕ shifts $q(k_t) = \Gamma_k(x_t)$ ($k_{t+1} = k_t$ locus) downward (upward) for $k_U^* < k_t \leq k_S^*$ ($k_t \leq k_U^*$ and $k_t > k_S^*$) while $q(k_t) = \Gamma_b(x_t)$ ($b_{t+1} = b_t$ locus) and the steady states remain unchanged (as we have seen below Proposition 6) as depicted in Figure 12. These facts together with (35) (see Appendix J in more details) show the following.

Proposition 8.

(i) A fall in $\bar{b}$ or a rise in ϕ shifts the threshold curve leftward, indicating that these policy changes make fiscal policy more sustainable.

(ii) A fall in $\bar{b}$ increases capital stock per capita in the steady state S: k_S^*.

These effects of $\bar{b}$ and ϕ under tax-based consolidation are similar to those under expenditure-based consolidation (Propositions 3, 4, and 5).

[Figures 11 and 12]

A lower $\bar{b}$ causes a larger gap between the current and the target debt-to-GDP ratio by (12) and then a larger amount of debt reduction accompanies it in the early stage of the transition. An increase in ϕ also causes a larger decline in public debt in the early stage of the transition by (12). To achieve a larger amount of debt reduction, income tax rate τ_t is increased in contrast to the case of the expenditure-based consolidation. Large burdens of tax hamper capital accumulation in the early stages of consolidations.

In the long run, reduction in debt extends the fiscal space through the following two channels. First, the interest payment of public debt decreases. Second, more resources are released to private investment (crowd in effect), which enhances tax revenues. However, the latter effects are weakened by increases in distortionary tax rates under the tax-based consolidation. Therefore, readers may imagine that fiscal policy is more likely to be sustainable under expenditure-based consolidation than tax-based consolidation. The next section investigates numerically as to which plan is preferable from the viewpoint of fiscal sustainability or welfare.

8 Numerical studies under tax-based consolidations

In this section, we calibrate the model in the case of tax based-consolidation to the data of the five countries in Subsection 5.3.

8.1 Parameter choices and scenarios

Benchmark parameters and variables follow the ones in Subsection 5.3. λ is set to satisfy $g_{init} = \lambda A k_0^{\alpha}$, where $g_{init} = (1+n)b_0 + \tau_{init}^{w} w(k_0) + \tau_{init}^{R} R(k_0)k_0 + \tau^c y(k_0) - (1-\tau_{init}^{R})R(k_0)b_0 - (1+\tau^c)\Phi(k_0, b_0, \tau_{init}^{w})$ and both τ_{init}^{w} and τ_{init}^{R} take the values in Table 1. Then, we can calibrate the value of λ in each country as $\lambda^{JPA} = 0.3547$, $\lambda^{GRE} = 0.3047$, $\lambda^{ITA} = 0.4134$, $\lambda^{PRT} = 0.3364$, and $\lambda^{US} = 0.2497$. We consider the following scenario. Governments implement tax base consolidations at period 0 unexpectedly before decision-making of the young in period 0. Then, τ_0 and g_0 follows (33) and $g_0 = \lambda A k_0^{\alpha}$, respectively. Consumptions in period 0 are given by $d_0 = \frac{(1-\delta\tau_0)R(k_0)(k_0+b_0)}{1+\tau^c}$ (consumption of the old) and $c_0 = \frac{(1-\tau_0)w(k_0)}{(1+\beta)(1+\tau^c)}$ (consumption of the young). For τ_t ($t \geq 0$) given by (33), we have $g_t = \lambda A k_t^{\alpha}$, $d_t = \frac{(1-\delta\tau_t)R(k_t)(k_t+b_t)}{1+\tau^c}$ $c_t = \frac{(1-\tau_t)w(k_t)}{(1+\beta)(1+\tau^c)}$ with $\{k_t, b_t\}_{t=0}^{\infty}$ following the dynamic equations (12) and (34).

8.2 Results

Figures 13, 14, 15, 16, and 17 with Tables 4 and 5 show the following results.

[Figures 13, 14, 15, 16, and 17] [Tables 4 and 5]

Fiscal sustainability

Fiscal policy is unsustainable without decreasing outstanding debt: $\phi = 0$ in Japan, Greece, Italy, and Portugal while sustainable in the US, which is similar to the results under expenditure-based consolidation. Consolidation plans with a very slow pace, $\phi = 0.1$ in Japan, $\phi = 0.1, 0.3, 0.5$ in Greece and Italy, $\phi = 0.1, 0.3$ in Portugal, cannot sustain fiscal policy. These results are similar to those in the expenditure-based consolidation.

In the steady state $S(k_S^*, b_S^*)$

The steady-state levels of capital, k^*, government spending, g^*, consumption by the young c^*, and consumption by the old d^* exceed the current/initial ones, k_0, g_0, c_0, and d_0 in Japan and the US whereas fall behind in Greece, Italy and Portugal. In Greece, Italy, and Portugal, income tax rates (both $\tau^*(=\tau^{w*})$ and $\delta\tau^*(=\tau^{R*})$) are lower than the initial levels: $\tau_{init}^w(=\tau^w)$ and $\tau_{init}^R(=\tau^R)$ in all five countries, indicating that distortionary effects of income tax on capital accumulation are weaker under tax-based consolidation than under expenditure-based consolidation. (see Tables 3 and 5). Consumptions c^* and d^* (resources in the private sector) are also larger under tax-based consolidation than under expenditure-based while public spending (resources in the public sector) under tax-based consolidation is smaller than that under expenditure-based consolidation. In Greece, Italy and Portugal, resources released to both private and public sectors by fiscal consolidation are hampered by resource use to decrease such a large outstanding public debt and large weight of tax burden on workers.

Transitional dynamics

$\tau_t^R(=\delta\tau_t)$ increase just after the implementation of tax-based consolidation. When the pace of consolidation in Japan, Greece, Italy, and Portugal is high (even when low in the US), the income tax rates turn to decrease in the short run and keep decreasing into the steady state values. By contrast, under a slow pace of consolidation, the income tax rates in these four countries keep increasing in the short and medium run and turn to decrease into the steady state values.

In contrast to the expenditure-based consolidation, tax-based consolidation decreases c_t and d_t just after the implementation of fiscal consolidations with g_t unchanged. Faster (slower) paced consolidations crowd in resources to the private and public sector, c_t, d_t, and g_t strongly (or weakly) in the medium run. However, faster (or slower) paced consolidation decreases d_t strongly (or weakly) in the short run because cuts in debt reduce asset income. c_t and g_t exceed the initial level in Japan and the US while fall behind in the three European countries eventually. d_t exceeds the initial level only in the US.

In unsustainable transition paths, increases in tax rates reduce disposable income, consumption, and capital accumulation, and then erode resource of the government g_t. Unlike the expenditure-based consolidation, consumption by the old (d_t) decreases since tax rate of asset income rises as debt increases monotonically.

Sustainable (or unsustainable) pace of between expenditure- and tax-based consolidation

The rest of this section examines (i) how rapid the pace of consolidation should be to makes the fiscal policy sustainable and (ii) under which consolidation plan is the fiscal policy more likely to be sustainable between expenditure-and tax-based consolidations. We focus on the four countries whose fiscal policy can be unsustainable under the low paces of consolidation plans. Tables 6 and 7 show that the range of ϕ that combines with the sustainable paths is wider under the expenditure-based plans compared to the tax-based ones for all the four countries. Thus, we can conclude that fiscal policy is more likely to become sustainable under expenditure-based consolidation.

[Tables 6 and 7]

This is attributable to the following reasons. As we have seen in Proposition 8, the tax-based plan requires a steep hike in income tax rate τ_t to achieve a larger amount of debt reduction, which deters capital accumulation in the early stages of fiscal consolidation. Additionally, a large distortionary income tax can afford to release less resources that enlarge fiscal capacity under tax-based consolidation.

9 Welfare of each generation and social welfare

Let us begin with the welfare of each generation. The welfare of the initial old (generation -1) is $U^{old}_{init} \equiv \ln d_0 + \theta \ln g_0$ and that of generation $t(\geq 0)$ is given by (1). We set $\theta = 0.8$ as a benchmark in the sense that utility from public goods and services is relatively high. Figures 18 and 19, on the one hand, show the following result in the cases of Japan and the US. Welfare of the initial old (generation -1) and that of initial young (generation 0) decreases as the pace of consolidations increases both under expenditure- and tax-based consolidations. On the other hand, Figures 20, 21, and 22 show the following results in the cases of Greece, Italy, and Portugal. A more rapid tax based-consolidation decreases both welfare of generation -1 and that of generation 0 while a more rapid expenditure-based consolidation decreases welfare of generation -1 but increases welfare of generation 0.

Two common facts are observed among the five countries. First, welfare losses or gains of these early generations are far smaller compared with the differences in welfare of later generations. Second, the welfare of later generations is lower under a slower rapid pace of consolidation.

[Figures 18, 19, 20, 21, and 22]

Next, we evaluate the effect of fiscal consolidation by social welfare. Social welfare is defined as $W \equiv U_{init}^{old} + \sum_{t=0}^{\infty} \lambda^t U_t$, where $\lambda \in (0, 1)$ is a social discount factor. We set $\lambda = 0.7$. Table 8 shows that W is monotonically decreasing in ϕ under expenditure-based consolidation while the relationship between W and ϕ under tax-based consolidation can be hump-shaped in Japan, the US, and Italy among the candidates values of ϕ in Table 8. From the viewpoint of social welfare, tax-based consolidation should be chosen in Japan, Greece, Italy, and Portugal while the expenditure-based consolidation should be chosen in the US. The optimal pace of consolidation is $\phi = 0.9$ in Japan and Italy and $\phi = 1$ in the US, Greece, and Portugal.

[Table 8]

However, social welfare is somewhat problematic if fiscal consolidation causes large welfare inequality between generations, and it obscures this inequality. Then, we need to pay attention to the fairness of welfare distribution between generations for the evaluation of the fiscal consolidation strategy.

To gauge the fairness of welfare distribution, we calculate the Gini coefficient of each generation's welfare.[21] Table 9 shows that tax-based consolidation should be chosen in Greece, Italy, and Portugal while expenditure-based in Japan and the US from the viewpoint of fairness of welfare distribution between generations. The fairest pace of consolidation is $\phi = 1$ in all countries.

Between fairness of welfare and social welfare, the choice of consolidation type (expenditure-based or tax-based consolidation) is different (same) in Japan (in the US and Greece, Italy, and Portugal) while the pace of consolidation is different (same) in Japan and Italy (in the US, Greece, and Portugal).

[Table 9]

Let us consider the case of low utility from public goods and services ($\theta = 0.2$). Results on social welfare in the US and Italy are different from those when $\theta = 0.8$. Table 10 shows that tax-based (expenditure-based) consolidation is better in the US (in Italy), and that optimal pace of consolidation is $\phi = 0.9$ ($\phi = 1$) in the US (in Italy). Table 11 shows that results on fairness of welfare distribution are qualitatively the same as $\theta = 0.8$.

[21] We have taken account of from generation -1 to 19 in a practical calculation.

[Tables 10 and 11]

When the weight on the utility from public goods and services θ is smaller, tax-based consolidation is more likely to be chosen because it releases more resources in the private sector in the steady state. Particularly, in Greece, Italy and Portugal, income tax rates decrease monotonically and resources driven by consolidation (c_t d_t and g_t) are distributed evenly between generations with a rapid pace of tax-based consolidation (see Figures 15, 16, and 17), indicating that tax-base consolidation with $\phi = 1$ results in the fairest outcome. In Japan and the US, expenditure-based consolidation with $\phi = 1$ has the fairest outcome both when $\theta = 0.2$ and 0.8.

In summary, choices of consolidation type between tax-based or expenditure-based may differ among countries and depend on how large outstanding debts relative to capital are and how large the utility derived by individuals from public goods and services is. By contrast, a common result from the viewpoints of both social welfare and fairness of welfare distribution is that a very slow pace of fiscal consolidation cannot be supported.

10 Conclusion

This study investigates the effects of expenditure- and tax-based consolidations on fiscal sustainability and welfare by using an OLG model with endogenous growth settings. Under the debt policy rule of reductions in debts to the targeted debt-to-GDP ratio, we investigate global transition dynamics of the economy and obtain the following results.

First, a unique stable steady state exists both under the expenditure- and tax-based consolidations with the debt policy rule. Properties of global transition paths are derived analytically and represented in two two-dimensional phase diagrams under each of the two types of consolidation plans.

Second, there is a threshold of public debt for each level of capital in order for the government to sustain fiscal policy, and the threshold of public debt is increasing in the size of capital under each of the two types of consolidation plans. A higher pace or lower target of debt-to-GDP ratio makes fiscal policies more sustainable.

Third, the minimal pace of tax-based consolidation that ensures fiscal sustainability is higher than that ensured by expenditure-based consolidation, indicating that expenditure-based consolidation is more likely to make fiscal policy sustainable. Numerical investigations show that Japan, Greece, Italy and Portugal cannot sustain fiscal policy either without reducing debt or low paces of reduction in debts. By contrast, the US economy may sustain its fiscal policy even without reducing debt.

Finally, social welfare increases in all countries (Japan, the US, Greece, Italy and Portugal) by fiscal consolidations. Choices of consolidation type between tax-based or expenditure-based may differ among countries depending upon how large outstanding debts relative to capital are and how large the utility derived by individuals from public goods and services is. By contrast, a common result from the viewpoints of both social welfare and fairness of welfare is that a very slow pace of fiscal consolidation cannot be supported.

Appendix

A Proof of Lemma 2

(i) From (14), we have $\Omega(0) = 0$, and has asymptote $\lim_{k_t \to \hat{k}} \Omega(k_t) = +\infty$, where $\hat{k}$ is defined by (14). Additionally, the first and second derivatives of $g_t = 0$ locus: $b_t = \Omega(k_t)$ are as follows.

$$\begin{aligned}\Omega'(k_t) =& \frac{\left[\phi\bar{b}(1+\tau^c)(1+n) + \tilde{\tau} + \tau^c(1-(1+n)\eta)\right]}{\left[(1-\tau^R)\alpha q(k_t) - (1+n)(1+\tau^c)(1-\phi)\right]^2} \\ &\times \left[(1-\tau^R)q(k_t) - (1+n)(1+\tau^c)(1-\phi)\right] q(k_t) > 0 \quad \text{for} \quad k_t < \hat{k}, \qquad \text{(A.1)} \\ \Omega''(k_t) =& \frac{\left[\phi\bar{b}(1+n)(1+\tau^c) + \tilde{\tau} + \tau^c(1-(1+n)\eta)\right] \alpha(1-\alpha)(1+n)(1-\phi)Ak_t^{\alpha-2}}{\left[(1-\tau^R)\alpha q(k_t) - (1+n)(1-\phi)\right]^3} \\ &\times \left\{2(1-\alpha)q(k_t) + \left[(1-\tau^R)\alpha q(k_t) - (1+n)(1+\tau^c)(1-\phi)\right]\right\} > 0 \quad \text{for} \quad k_t < \hat{k}. \end{aligned} \tag{A.2}$$

These results prove (i).

(ii) In the intersection point between $b_{t+1} = b_t$ and $g_t = 0$ loci, $\bar{b}q(k_t) = \tilde{\Omega}(k_t)$ holds by (17) and (23). Therefore, we have

$$\bar{b}q(k_t) = \frac{q(k_t)\left[\phi\bar{b}(1+n)(1+\tau^c) + \tilde{\tau} + \tau^c(1-(1+n)\eta)\right]}{(1-\tau^R)\alpha q(k_t) - (1+n)(1+\tau^c)(1-\phi)}. \tag{A.3}$$

From (A.3), we obtain

$$k_H = \left[\frac{\bar{b}(1-\tau^R)\alpha A}{\bar{b}(1+\tau^c)(1+n)+\tilde{\tau}+\tau^c(1-\eta(1+n))} \right]^{\frac{1}{1-\alpha}}.$$

Inserting the value of k_H into (20) yields

$$b_H = \left(\bar{b}A\right)^{\frac{1}{1-\alpha}} \left[\frac{(1-\tau^R)\alpha}{\bar{b}(1+\tau^c)(1+n)+\tilde{\tau}+\tau^c(1-\eta(1+n))} \right]^{\frac{\alpha}{1-\alpha}}.$$

(iii) It is evident from Lemma 1-(ii) and Lemma 2-(i).

B Derivations of (26), (27), (28), and (29)

In the intersection point between $k_{t+1} = k_t$ locus and $g_t = 0$ locus under $\phi < 1$, $\tilde{Z}(q(k_t)) = \tilde{\Omega}(q(k_t))$ holds. Therefore, we have (26).

Next, we move onto the value of x_P. Because $x_t = \tilde{\Omega}(q(k_t))$ and $x_t = \left[(\eta - \phi\bar{b})q(k_t) - 1\right]/(1-\phi)$ are written as

$$q(k_t) = \frac{(1+n)(1+\tau^c)(1-\phi)x_t}{(1-\tau^R)\alpha x_t - \phi\bar{b}(1+n)(1+\tau^c) - \tilde{\tau} - \tau^c(1-(1+n)\eta)} \tag{B.1}$$

and

$$q(k_t) = \frac{(1-\phi)x_t + 1}{\eta - \phi\bar{b}}, \tag{B.2}$$

respectively, we have (27) in $P(q(k_P), x_P)$. Here, we define

$$p_L(x) \equiv (1-\phi)x\left[(1-\tau_R)\alpha x - \tilde{\tau} - \tau^c - \eta(1+n)\right], \tag{B.3}$$

$$p_R(x) \equiv \phi\bar{b}(1+n)(1+\tau^c) + \tilde{\tau} + \tau^c(1-(1+n)\eta) - (1-\tau^R)\alpha x. \tag{B.4}$$

$p_L(x)$ is an downward-sloping line that satisfies $p_L(\bar{x}_P) = 0$, where $\bar{x}_P = \frac{\tilde{\tau}+\tau^c+\eta(1+n)}{(1-\tau^R)\alpha}$ while $p_R(x)$ is a quadratic function of x that satisfies $p_R(x) > (=) 0$ for $x > (=) \underline{x}_P$, where $\underline{x}_P = \frac{\phi\bar{b}(1+n)(1+\tau^c)+\tilde{\tau}+\tau^c(1-(1+n)\eta)}{(1-\tau^R)\alpha}$ and $p'_R(x) > 0$ and $p''_R(x) > 0$ for $x \geq \underline{x}_P$. Therefore, the value of x_P is represented by the intersection between $p_L(x)$ and $p_R(x)$ as represented in Figure 23. Thus, we have (28): $\underline{x}_P < x_P < \bar{x}_P$ and

$$p'_R(x_P) > p'_L(x_P). \tag{B.5}$$

[Figure 23]

Finally, we derive (29). To do this, let us rewrite (27), using the definition of $x_P \equiv b_P/k_P$ into

$$\begin{aligned}&(1-\phi)(1-\tau^R)\alpha b_P^2 + \left\{(1-\tau^R)\alpha - (1-\phi)\left[\tilde{\tau} + \tau^c + \eta(1+n)\right]\right\} k_P b_P \\ &= \left[\phi\bar{b}(1+n)(1+\tau^c) + \tilde{\tau} + \tau^c(1-(1+n)\eta)\right] k_P^2. \end{aligned} \tag{B.6}$$

Taking the total differentials of (B.6) yields (29).

C Derivation of Condition 2

We rearrange (26) into

$$\begin{aligned}&(1-\phi)^{-1}\left[(\eta-\phi\bar{b})q(k_P)-1\right]\underbrace{\left[(1-\tau^R)\alpha q(k_P) - (1+n)(1+\tau^c)(1-\phi)\right]}_{>0 \text{ from Condition 1}} \\ &= \left[\phi\bar{b}(1+\tau^c)(1+n) + \tilde{\tau} + \tau^c(1-(1+n)\eta)\right] q(k_P), \end{aligned} \tag{C.1}$$

Let us define

$$P_L(q(k)) \equiv (1-\phi)^{-1}\left[(\eta-\phi\bar{b})q(k)-1\right]\left[(1-\tau^R)\alpha q(k) - (1+n)(1+\tau^c)(1-\phi)\right] \tag{C.2}$$

$$P_R(q(k)) \equiv \left[\phi\bar{b}(1+\tau^c)(1+n) + \tilde{\tau} + \tau^c(1-(1+n)\eta)\right] q(k). \tag{C.3}$$

$P_L(q(k))$ satisfies $P_L(q(\hat{k})) = P^{LHS}(q(\tilde{k})) = 0$ and is strictly increasing in $q(k)$ for $q(k) \geq q(\tilde{k})$ ($k_t < \tilde{k}$), while $P_R(q(k))$ is upward-sloping line that satisfies $\lim_{k_t\to+\infty} P_R(q(k)) = P_R(0) = 0$. $q(k_P)$ in (C.1) is given by the intersection point between $P_L(q(k))$ and $P_R(q(k))$ as represented in Figure 23.

$k^* > k_P$ if and only if $P_R(q(k^*)) - P_L(q(k^*)) > 0$ from (C.1), (C.2), and (C.3), where $q(k^*) = [\eta-\bar{b}]^{-1}$, $P_R(q(k^*)) = \frac{\phi\bar{b}(1+n)(1+\tau^c)+\tilde{\tau}+\tau^c(1-(1+n)\eta)}{\eta-\bar{b}}$, and $P_L(q(k^*)) = \frac{\bar{b}[(1-\tau^R)\alpha-(1+\tau^c)(1+n)(1-\phi)(\eta-\bar{b})]}{(\eta-\bar{b})^2}$.

Therefore, we have

$$
\begin{aligned}
& P_R(q(k^*)) - P_L(q(k^*)) > 0 \\
& \Leftrightarrow \frac{(\eta-\bar{b})[\tilde{\tau}+\tau^c(1-(1+n)\eta)+\bar{b}(1+\tau^c)(1+n)] - \bar{b}(1-\tau^R)\alpha}{(\eta-\bar{b})^2} > 0 \\
& \Leftrightarrow (\eta-\bar{b})[\tilde{\tau}+\tau^c(1-(1+n)\eta)+\bar{b}(1+\tau^c)(1+n)] > \bar{b}(1-\tau^R)\alpha. \qquad \text{(C.4)}
\end{aligned}
$$

Solving this inequality (C.4) with respect to $\bar{b}$ yields

$$
\begin{aligned}
& \bar{b} < \bar{b}_1 \equiv \frac{\zeta_1 + \sqrt{\zeta_1^2 + 4(1+n)(1+\tau^c)\zeta_2}}{2(1+n)(1+\tau^c)}, \\
& \zeta_1 \equiv \tilde{\tau} + \tau^c(1-(1+n)\eta) + (1-\tau^R)\alpha - (1+\tau^c)(1+n)\eta, \\
& \zeta_2 \equiv \eta[\tilde{\tau}+\tau^c(1-(1+n)\eta)](>0). \qquad \text{(C.5)}
\end{aligned}
$$

Dividing (C.4) by $\eta - \bar{b}$ (>0) rewrite (C.4) into

$$
\tilde{\tau} + \tau^c(1-(1+n)\eta) + \bar{b}(1+\tau^c)(1+n) > \frac{\bar{b}(1-\tau^R)\alpha}{\eta - \bar{b}}. \qquad \text{(C.6)}
$$

The LHS of (C.6) is an upward-sloping line with respect to $\bar{b}$, which takes $\tilde{\tau}$ at $\bar{b} = 0$. By contrast, the RHS of (C.6) is a strictly increasing and convex function of $\bar{b}$ that takes the value zero at $\bar{b} = 0$ and has asymptote $+\infty$ when $\bar{b} \to \eta$. Therefore, $\bar{b}_1 \in (0, \eta)$. Furthermore, because the LHS of (C.6) is increasing in $\tilde{\tau}$, so is $\bar{b}_1$.

D Phase diagram

From (12), $b_{t+1} > (\leq) b_t$ if and only if $b_t < (\geq) \bar{b} A k_t^{\alpha}$. Combining this with Lemma1-(i), we find that $b_{t+1} > (\leq) b_t$ holds below (above) the $b_{t+1} = b_t$ locus at each point of (k_t, b_t).

Next, from (13), when $0 < \phi < 1$, $k_{t+1} > (\leq) k_t$ if and only if $b_t < (\geq) Z(k_t)$ $(= [(\eta - \phi\bar{b})Ak_t^{\alpha} - k_t]/(1-\phi))$. This, associated with Lemma 1-(ii), indicates that $k_{t+1} > (\leq) k_t$ holds below (above) the $k_{t+1} = k_t$ locus: $b_t = [(\eta - \phi\bar{b})Ak_t^{\alpha} - k_t]/(1-\phi)$ at each point of the (k_t, b_t). When $\phi = 1$, $k_{t+1} > (\leq) k_t$ holds if and only if $k_t < (\geq) k^* = \left[\left(\eta - \bar{b}\right) A\right]^{\frac{1}{1-\alpha}}$.

E Proof of Lemma 3

From (21), we obtain

$$\left.\frac{\partial Z(k_t)}{\partial \phi}\right|_{b_t=Z(k_t)} = \frac{\left[\left(\eta-\bar{b}\right)Ak_t^{\alpha-1}-1\right]k_t}{(1-\phi)^2} = \frac{\left[\left(\eta-\bar{b}\right)q(k_t)-1\right]k_t}{(1-\phi)^2} \gtreqless 0 \qquad \text{for} \quad k_t \lesseqgtr k^*. \tag{E.1}$$

From (14), we obtain

$$\begin{aligned}&\left.\frac{\partial \Omega(k_t)}{\partial \phi}\right|_{b_t=\Omega(k_t)}\\ &= \frac{(1+\tau^c)(1+n)Ak_t^{\alpha}\left[\bar{b}(1-\tau^R)\alpha q(k_t)-\bar{b}(1+\tau^c)(1+n)-\tilde{\tau}-\tau^c(1-(1+n)\eta)\right]}{\left[(1-\tau^R)\alpha q(k_t)-(1+\tau^c)(1+n)(1-\phi)\right]^2}.\end{aligned} \tag{E.2}$$

(E.2) together with (25) yields

$$\left.\frac{\partial \Omega(k_t)}{\partial \phi}\right|_{b_t=\Omega(k_t)} \gtreqless 0 \qquad \text{for} \quad k_t \lesseqgtr k_H. \tag{E.3}$$

F Proof of Proposition 6

Both (i) and (iii) are evident. (ii) When $\delta > 1$ and $\mu_1 > 0$, suppose that steady states exist. Then $\mu_2 < 0$ and $\mu_2^2 - 4\mu_1\mu_3 > 0$. From (35), $\mu_2^2 - 4\mu_1\mu_3 > 0$ if and only if

$$\begin{aligned}&\Leftrightarrow (1+n)\bar{b}\left(\frac{1-\alpha}{1+\beta}+2\alpha\delta\right)^2+\tilde{\eta}^2\left[(1+\tau^c)(1-\lambda)+\alpha(\delta-1)\right]^2\\ &\quad > 2\tilde{\eta}\left(\frac{1-\alpha}{1+\beta}+2\alpha\delta\right)\left[(1+\tau^c)(1-\lambda)+\alpha(\delta-1)\right]+4\frac{\mu_1\mu_3}{(1+n)\bar{b}}\\ &\Leftrightarrow \mu_2 > D \equiv \underbrace{\frac{\tilde{\eta}\left[(1+\tau^c)(1-\lambda)+\alpha(\delta-1)\right]}{\frac{1-\alpha}{1+\beta}+2\alpha\delta}}_{(+)}\left\{\frac{1-\alpha}{1+\beta}+2\alpha\delta-\tilde{\eta}\left[(1+\tau^c)(1-\lambda)+\alpha(\delta-1)\right]\right\}\\ &\qquad\quad +4\frac{\mu_1\mu_3}{(1+n)\bar{b}}\end{aligned} \tag{F.1}$$

From $\mu_1 > 0$,

$$\mu_2 > D > \frac{\tilde{\eta}\left[(1+\tau^c)(1-\lambda)+\alpha(\delta-1)\right]}{\frac{1-\alpha}{1+\beta}+2\alpha}\left\{\frac{1-\alpha}{1+\beta}+2\alpha\delta-\tilde{\eta}\left[(1+\tau^c)(1-\lambda)+\alpha(\delta-1)\right]\right\} > 0$$

and it contradicts $\mu_2 < 0$, if $\frac{1-\alpha}{1+\beta} + 2\alpha\delta - \tilde{\eta}\left[(1+\tau^c)(1-\lambda) + \alpha(\delta-1)\right] > 0$.

By $\delta > 1$, $0 < \lambda < 1$, $\tilde{\eta} = \beta(1-\alpha)/(1+\beta)$,

$$\begin{aligned}
&\frac{1-\alpha}{1+\beta} + 2\alpha\delta - \tilde{\eta}\left[(1+\tau^c)(1-\lambda) + \alpha(\delta-1)\right] \\
&= (1+\alpha\beta)\frac{1-\alpha}{1+\beta} + \left(2 - \frac{\beta(1-\alpha)}{1+\beta}\right)\alpha\delta - \frac{\beta(1-\alpha)}{1+\beta}(1+\tau^c)(1-\lambda) \\
&> (1+\alpha\beta)\frac{1-\alpha}{1+\beta} + \left(2 - \frac{\beta(1-\alpha)}{1+\beta}\right)\alpha - \frac{\beta(1-\alpha)}{1+\beta}(1+\tau^c) > 0 \\
&\Leftrightarrow 1 + (1+3\beta+\tau^c\beta)\alpha > (1+\tau^c)\beta
\end{aligned} \tag{F.2}$$

Thus, if $1 + (1+3\beta+\tau^c\beta)\alpha > (1+\tau^c)\beta$, no steady state exists when $\delta > 1$ and $\mu_1 > 0$.

G Phase diagram under the tax-based consolidation

From (34), we have

$$k_{t+1} - k_t \lesseqgtr 0 \Leftrightarrow h(b_t, k_t) \equiv a_1 b_t^2 + a_2(q(k_t))k_t b_t + a_3(q(k_t))k_t^2 \lesseqgtr 0 \ \text{ for } \ k_t > 0, \tag{G.1}$$

where

$$\begin{aligned}
&a_1 \equiv (1+n)(1-\phi)\delta\alpha > 0, \quad a_2(q(k_t)) \equiv a_{21} + a_{22}q(k_t), \quad a_3(q(k_t)) \equiv a_{31} + a_{32}q(k_t) \\
&a_{21} \equiv (1+n)\{(1-\phi)\underbrace{\left[1 - \tilde{\eta} + \alpha(\delta-1)\right]}_{=\frac{1-\alpha}{1+\beta}+\alpha\delta} + \alpha\delta\} > 0, \quad a_{22} \equiv \left[(1+n)\bar{b}\phi\delta - \tilde{\eta}(\delta-1)\right]\alpha, \\
&a_{31} = \mu_3 > 0, \quad a_{32} \equiv (1+n)\underbrace{\left[1 - \tilde{\eta} + \alpha(\delta-1)\right]}_{=\frac{1-\alpha}{1+\beta}+\alpha\delta(>0)}\bar{b}\phi - \tilde{\eta}[(1+\tau^c)(1-\lambda) + \alpha(\delta-1)].
\end{aligned}$$

Here, we define $k_{t+1} = k_t$ locus which is derived from $h(b_t, k_t) = 0$ as $b_t = m(k_t)$.

G.1 The case of $0 < \phi < 1$ ($a_1 > 0$)

$b_t = m(k_t)$ satisfies $b_t = m(0) = 0$ and $b_t = m(\breve{k}) = 0$, where $\breve{k} \equiv A^{\frac{1}{1-\alpha}}\left[-(a_{31}/a_{32})\right]^{\frac{1}{\alpha-1}}$ The former is obvious from $\lim_{k_t \to 0} h(b_t, k_t) = a_1 b_t^2 = 0$. The latter is shown as follows. From $a_3(q(\breve{k})) = a_{31} + a_{32}q(\breve{k}) = 0$, we have $h\left(b_t, \breve{k}_t\right) = \left[a_1 b_t + a_2(q(\breve{k}_t))\right] b_t = 0$. This together with $a_1 b_t + a_2(q(\breve{k}_t)) \neq 0$ lead to $b_t = 0$. Thus, the $k_{t+1} = k_t$ locus takes zero when $k_t = 0$ and $\breve{k}$.

To reveal more properties of $k_{t+1} = k_t$ locus and the dynamics of k_t for $k_t > 0$, we rewrite (G.1) into $k_{t+1} \gtreqless k_t \Leftrightarrow a_1 x_t^2 + a_2(q(k_t))x_t + a_3(q(k_t)) \lesseqgtr 0$ for $k_t > 0$ (recall that $x_t \equiv b_t/k_t$), which leads to

$$k_{t+1} \gtreqless k_t \Leftrightarrow q(k_t) \gtreqless \left(\lesseqgtr\right) \Gamma_k\left(x_t\right) \equiv -\frac{\overbrace{a_1}^{(+)} x_t^2 + \overbrace{a_{21}}^{(+)} x_t + \overbrace{a_{31}}^{(+)}}{a_{22}x_t + a_{32}}$$

$$\text{for} \quad a_{22}x_t + a_{32} < (>)0, \tag{G.2}$$

and the derivative of $\Gamma_k(x_t)$ with respect to x_t is given by

$$\Gamma_k'\left(x_t\right) = \frac{\Lambda\left(x_t\right)}{\left(a_{22}x_t + a_{32}\right)^2},$$

$$\Lambda\left(x_t\right) \equiv -\underbrace{a_1}_{(+)} x_t\left(a_{22}x_t + 2a_{32}\right) - \left(a_{21}a_{32} - a_{22}a_{31}\right). \tag{G.3}$$

$k_{t+1} = k_t$ locus is represented by the relationship between $q(k_t)$ and x_t. In addition to this, by (12) and (23), the motion of debt is

$$b_{t+1} \gtreqless b_t \Leftrightarrow q(k_t) \gtreqless \Gamma_b(x_t) \equiv \bar{b}^{-1}x_t. \tag{G.4}$$

Obviously, $\Gamma_b(x_t)$ is positive linear and takes the value zero when $x_t = 0$. Finally, (G.2) and (G.4) show that the steady states given in Proposition 6 are represented by the intersection points between $q(k_t) = \Gamma_k(x_t)$ and $q(k_t) = \Gamma_b(x_t)$.

Step1: Representation of (G.2) and (G.4) into the $(x_t, q(k_t))$ plane

Examining (G.2), (G.3), and (G.4) yields the following cases (i), (ii), and (iii).

(i) When $0 < \delta \leq 1$, $a_{22} > 0$ holds. Furthermore, $\mu_2 < 0$ (from Proposition 6-(i)) ensures $a_{32} < 0$ and the existence of two steady states, indicating that $q(k_t) = \Gamma_k(x_t)$ and $q(k_t) = \Gamma_b(x_t)$ intersect at the steady states denoted by $S(x_S^*, q(k_S^*))$ and $U(x_U^*, q(k_U^*))$. From these facts,

$$-\underbrace{a_{32}}_{(-)} / \underbrace{a_{22}}_{(+)} > 0, \quad q(\breve{k}) = -\underbrace{a_{31}}_{(+)} / \underbrace{a_{32}}_{(-)} > 0, \quad \underbrace{a_{21}}_{(+)}\underbrace{a_{32}}_{(-)} - \underbrace{a_{22}}_{(+)}\underbrace{a_{31}}_{(+)} \leq 0.$$

Then, $\Gamma_k(x_t) > 0$ for $0 \le x_t \le -a_{32}/a_{22}$ from (G.2). Furthermore, applying $a_{21}a_{32} - a_{22}a_{31} \le 0$ and $-a_{32}/a_{22} > 0$ to $\Lambda(x_t)$ in (G.3), we find that sign $\Gamma_k'(x_t) = \Lambda(x_t) > 0$ for $0 \le x_t \le -a_{32}/a_{22}$.

These facts indicate that $q(k_t) = \Gamma_k(x_t)$ is monotonically increasing in x_t and satisfies $\Gamma_k(0)(= q(\breve{k})) = -a_{31}/a_{32} > 0$ and $\lim_{x_t \to -\frac{a_{32}}{a_{22}}} \Gamma_k(x_t) = +\infty$. Then, $q(k_t) = \Gamma_k(x_t)$ and an upward-sloping line $q(k_t) = \Gamma_b(x_t)$ intersect at x_S^* and x_U^*. both x_S^* and x_U^* lie between 0 and $-a_{32}/a_{22}(> 0)$ as represented in Figure 9-(a).

(ii) When $\delta > 1$ and $\mu_1 \le 0$ hold, we obtain the followings. First, $a_{22} < 0$ is satisfied. Second,

$$\begin{aligned}
&a_{21}a_{32} - a_{22}a_{31} \le 0 \ \text{ (by } \textbf{Condition 3}\text{)}, \\
&-\underbrace{a_{32}}_{(-)}/\underbrace{a_{22}}_{(-)} < 0, \ \ \Gamma_k(0)(= q(\breve{k})) = -\underbrace{a_{31}}_{(+)}/\underbrace{a_{32}}_{(-)} > 0 \ \ (a_{32} < 0 \ \text{ by } \textbf{Condition 3}).
\end{aligned}$$

Third, $\Gamma_k(x_t) > 0$ for $x_t \ge 0 > -a_{32}/a_{22}$. Finally, from Proposition 6-(iii) ensures the uniqueness of steady state, inducing $q(k_t) = \Gamma_k(x_t)$ and $q(k_t) = \Gamma_b(x_t)$ to intersect at $S(x_S^*, q(k_S^*))$. Thus, (G.3) with $-a_{32}/a_{22} < 0$, and $\Lambda(0) = -(a_{21}a_{32} - a_{22}a_{31}) \ge 0$ implies that $\Lambda(x_t) > 0$ for $x_t \ge 0$. Then, $\Gamma_k(x_t)$ is positive and monotonically increasing in x_t for $x_t \ge 0$ and satisfies $\Gamma_k(0)(= q(\breve{k})) = -a_{31}/a_{32}(> 0)$ and $\lim_{x_t \to +\infty} \Gamma_k(x_t) = \lim_{x_t \to +\infty} -2a_1x_t/a_{22} = +\infty$ (Figure 9-(b)).

Step 2: translation of (G.2) **and** (G.4) **into the** (k_t, b_t) **planes**

We translate these relationships between x_t and $q(k_t)$ into the (k_t, b_t) planes.

(i) When $0 < \delta \le 1$, since $q(k_t) \ge 0$ for $0 \le x_t < -a_{32}/a_{22}(> 0)$, $q(k_t) = \Gamma_k(x_t)$ is transformed into $k_{t+1} = k_t$ locus: $b_t = m(k_t)$ as follows. The point $(x_t, q(k_t)) = (0, -a_{31}/a_{32})$ (in the LHS of the Figures 9) which corresponds to the point $(b_t, k_t) = (0, \breve{k})$ (in the RHS of the Figures 9). The trajectory of (b_t, k_t), when $(x_t, q(k_t))$ moves from $(0, -a_{31}/a_{32})$ to the final destination $(x_t, q(k_t)) \to (-a_{32}/a_{22}, +\infty)$ along $q(k_t) = \Gamma_k(x_t)$, represents the $k_{t+1} = k_t$ locus: $b_t = m(k_t)$. As x_t increases from 0 through x_S and x_U to $-a_{32}/a_{22}$ along $q(k_t) = \Gamma_k(x_t)$, $q(k_t)$ increases from $-a_{31}/a_{32}(= q(\breve{k})$ through $q(k_S^*)$ and $q(k_U^*)$ to $+\infty$. At the same time, as k_t decreases from $\breve{k}$ through k_S^* and k_U^* to 0 along the $b_t = m(k_t)$, b_t increases from 0 through b_S^* to the upper level and turns to decrease so as to go through b_U^*, and finally takes 0, as shown in the RHS of Figure 9.

Furthermore, $a_{22}x_t + a_{32} < 0$ is satisfied because of $0 \leq x_t < -a_{32}/a_{22}(> 0)$. Then, (G.2) implies that $k_{t+1} \gtreqless k_t$ if and only if $q(k_t) \gtreqless \Gamma_k(x_t)$ for $0 \leq x_t < -a_{32}/a_{22}$. Thus, $k_{t+1} > (<) k_t$ holds above (bellow) $q(k_t) = \Gamma_k(x_t)$, which satisfies $k_{t+1} > (<) k_t$ bellow (above) $b_t = m(k_t)$ ($k_{t+1} = k_t$ locus) correspondingly.

The translation of (G.2) into $b_t = m(k_t)$ in the rest case (ii) follows that in (i).

G.2 The case of $\phi = 1$ ($a_1 = 0$)

Because of $a_1 = 0$, (G.1) with $a_{21} = (1+n)\alpha\delta > 0$, $a_{22} = \mu_1/\bar{b}$, $a_{31} = \mu_3 > 0$ and $a_{32} = \mu_2$ leads to

$$k_{t+1} \gtreqless k_t \Leftrightarrow q(k_t) \gtreqless \left(\lesseqgtr\right) \Gamma_k\left(x_t\right) = -\frac{(1+n)\alpha\delta x_t + \mu_3}{\bar{b}^{-1}\mu_1 x_t + \mu_2} \quad \text{for} \quad \bar{b}^{-1}\mu_1 x_t + \mu_2 < (>) 0, \quad \text{(G.5)}$$

and

$$\Gamma_k'\left(x_t\right) = -\frac{(1+n)\alpha\delta\mu_2 - b^{-1}\mu_1\mu_3}{(b^{-1}\mu_1 x_t + \mu_2)^2}. \quad \text{(G.6)}$$

(i) When $0 < \delta \leq 1$, $\mu_1 \bar{b}^{-1} > 0$ and $\mu_2 < 0$ derives

$$-\underbrace{\mu_2}_{(-)} / \underbrace{\mu_1 \bar{b}^{-1}}_{(+)} > 0, \quad q(\breve{k}) = -\underbrace{\mu_3}_{(+)} / \underbrace{\mu_2}_{(-)} > 0, \quad (1+n)\alpha\delta \underbrace{\mu_2}_{(-)} - \underbrace{\bar{b}^{-1}\mu_1}_{(+)} \underbrace{\mu_3}_{(+)} \leq 0.$$

Then $\Gamma_k(x_t) > 0$ for $0 \leq x_t \leq -\mu_2/(\mu_1\bar{b}^{-1})$. From Proposition 6-(i), the existence of two steady states indicates that $q(k_t) = \Gamma_k(x_t)$ and $q(k_t) = \Gamma_b(x_t)$ intersect at the steady states denoted by $S(x_S^*, q(k_S^*))$ and $U(x_U^*, q(k_U^*))$. Furthermore, applying $a_{21}a_{32} - a_{22}a_{31} = (1+n)\alpha\delta\mu_2 - \bar{b}^{-1}\mu_1\mu_3 \leq 0$ and $-a_{32}/a_{22} = -\mu_2/(\mu_1\bar{b}^{-1}) > 0$ to (G.6), we find that $q(k_t) = \Gamma_k(x_t)$ is monotonically increasing in x_t for $0 \leq x_t \leq -\mu_2/(\mu_1\bar{b}^{-1})$ and satisfies $\Gamma_k(0) = q(\breve{k}) = -\mu_3/\mu_2 > 0$ and $\lim_{x_t \to \frac{-\mu_2}{\mu_1\bar{b}^{-1}}} \Gamma_k(x_t) = +\infty$ (similar to the case in Figure 9-(a)).

(ii) When $\delta > 1$ and $\mu_1 \leq 0$,

$$a_{21}a_{32} - a_{22}a_{31} = (1+n)\alpha\delta\mu_2 - \bar{b}^{-1}\mu_1\mu_3 \leq 0 \ \text{ (by } \textbf{Condition 3}\text{)},$$
$$-\underbrace{\mu_2}_{(-)} / \underbrace{\mu_1\bar{b}^{-1}}_{(-)} < 0, \ \ \Gamma_k(0)(= q(\breve{k})) = -\underbrace{\mu_3}_{(+)} / \underbrace{\mu_2}_{(-)} > 0 \ \ (a_{32} = \mu_2 < 0 \ \text{ by } \textbf{Condition 3}).$$

Then, $\Gamma_k(x_t) > 0$ for $x_t \geq 0 > -\mu_2/(\mu_1\bar{b}^{-1})$. Applying $(1+n)\alpha\delta\mu_2 - \bar{b}^{-1}\mu_1\mu_3 \leq 0$ and $-\mu_2/(\mu_1\bar{b}^{-1}) < 0$ to (G.6) we find that $q(k_t) = \Gamma_k(x_t)$ is monotonically increasing in x_t for $x_t \geq 0$ and $\Gamma_k(0) = q(\breve{k}) = -\mu_3/\mu_2 > 0$ and $\lim_{x_t\to+\infty}\Gamma_k(x_t) = \lim_{x_t\to+\infty} -\frac{(1+n)\alpha\delta}{\bar{b}^{-1}\mu_1} > 0$. The trajectory of (b_t, k_t), when $(x_t, q(k_t))$ moves from $(0, -\mu_3/\mu_2)$ to the final destination $(x_t, q(k_t)) \to (+\infty, -(1+n)\alpha\delta/(\bar{b}^{-1}\mu_1))$ along $q(k_t) = \Gamma_k(x_t)$, represents the $k_{t+1} = k_t$ locus: $b_t = m(k_t)$. Thus, as k_t decreases from $\breve{k} = q^{-1}(-\mu_3/\mu_2)$ through k_S^* to $q^{-1}(-(1+n)\alpha\delta/(\bar{b}^{-1}\mu_1))$, b_t increases from 0 through b_S^* to $+\infty$ as represented in the RHS of Figure 10.

H The condition of $k_{t+1} \geq 0$ and $k_{t+1} = 0$ locus

(34) indicates that because $1 + \alpha(\delta - 1) + \tilde{\eta}\tau^c + \delta\alpha(b_t/k_t) > 0$ for $k_t \geq 0$ and $b_t \geq 0$, $k_{t+1} = \tilde{\Phi}(k_t, b_t) \geq 0$ is written as

$$[(1+\tau^c)(1-\lambda)+\alpha(\delta-1)+(\delta-1)\alpha x_t]\tilde{\eta}Ak_t^\alpha - [1-\tilde{\eta}+\alpha(\delta-1)+\delta\alpha x_t][b_t - \phi(b_t - \bar{b}Ak_t^\alpha)] \geq 0. \tag{H.1}$$

Dividing (H.1) by $k_t(>0)$, we have

$$q(k_t) \geq (<)\Theta(x_t) \equiv -\frac{\overbrace{a_1}^{(+)} x_t^2 + [(1-\phi)(1-\alpha)(1+\beta)+\alpha\delta]x_t}{a_{32} + a_{22}x_t} \quad \text{for} \quad a_{32} + a_{22}x_t \leq (>)0. \tag{H.2}$$

$k_{t+1} \geq 0$ is satisfied as long as $q(k_t) \geq (<)\Theta(x_t)$ for $a_{32} + a_{22}x_t \leq (>)0$. Furthermore, by (G.2) and (H.2), the difference between $\Gamma_k(x_t)$ and $\Theta(x_t)$ is derived as

$$\Gamma_k(x_t) - \Theta(x_t) = -\frac{\overbrace{a_{31}}^{(+)} + (1+n)\alpha\delta x_t}{a_{32} + a_{22}x_t} \geq (<0) \quad \text{for} \quad a_{32} + a_{22}x_t \leq (>)0. \tag{H.3}$$

(i) When $0 < \delta \leq 1$ ($a_{22} > 0$, and $a_{32} < 0$), $0 \leq x_t < -a_{32}/a_{22}(>0)$ holds along $q(k_t) = \Gamma_k(x_t)$ (in (G.2)). Applying $0 \leq x_t < -a_{32}/a_{22}(>0)$, $a_{22} > 0$, and $a_{32} < 0$ into (H.2) and (H.3), we find that $k_{t+1} \geq 0$ is satisfied as long as $q(k_t) \geq \Theta(x_t)$ and that $\Gamma_k(x_t) \geq \Theta(x_t)$ for $0 \leq x_t < -a_{32}/a_{22}$, respectively.[22]

[22] $\Theta(x_t)$ has the following properties. $\Theta(0) = 0$, $\Theta(x_t) > (\leq)0$ for $0 \leq x_t < -a_{32}/a_{22}$, and $\lim_{x_t\to -\frac{a_{32}}{a_{22}}}\Theta(x_t) = +(-)\infty$ if $a_{21} - (1+n)\alpha\delta = (1+n)(1-\phi)[1-\tilde{\eta}+\alpha(\delta-1)] > 0$ or $a_{21} \leq (1+n)\alpha\delta$ and $-\frac{a_{32}}{a_{22}} < -\frac{a_{21}-(1+n)\alpha\delta}{a_1}$ (if $a_{21} \leq (1+n)\alpha\delta$ and $-\frac{a_{32}}{a_{22}} \geq -\frac{a_{21}-(1+n)\alpha\delta}{a_1}$).

(ii) When $\delta > 1$ and $\mu_1 \leq 0$ ($a_{22} < 0$ and $a_{32} < 0$ (from **Condition 3**)), $\Gamma_k(x_t) > 0$ and $a_{32} + a_{22}x_t \leq 0$ hold for $x_t > 0 > -a_{32}/a_{22}$. Then, (H.2) and (H.3) with $a_{32} + a_{22}x_t \leq 0$ imply that $k_{t+1} \geq 0$ is satisfied as long as $q(k_t) \geq \Theta(x_t)$ and that $\Gamma_k(x_t) \geq \Theta(x_t)$ for $x \geq 0$, respectively.

I Conditions when $\delta\tau_t = 1$ binds

$\delta\tau_t = 1$ binds if and only if $\delta\tau_t \geq 1$. $\delta\tau_t \geq 1$ is rewritten by using (33) and the definition of μ_3 into

$$\left\{\delta\left[(1+\tau^c)\lambda - \tau^c(1-\tilde{\eta})\right] - \delta(1+\tau^c)(1+n)\phi\bar{b} - \mu_3\right\} q(k_t) \geq \delta(1+\tau^c)(1+n)(1-\phi)x_t. \tag{I.1}$$

Thus, $\delta\tau_t \geq 1$ if and only if

$$\delta\left[(1+\tau^c)\lambda - \tau^c(1-\tilde{\eta})\right] > \delta(1+\tau^c)(1+n)\phi\bar{b} + \mu_3 \tag{I.2}$$

and

$$q(k_t) \geq \frac{\delta(1+\tau^c)(1+n)(1-\phi)x_t}{\delta\left[(1+\tau^c)\lambda - \tau^c(1-\tilde{\eta})\right] - \delta(1+\tau^c)(1+n)\phi\bar{b} - \mu_3} \equiv \Upsilon(x_t). \tag{I.3}$$

To ensure $\delta\tau_t < 1$ in the steady state S, the RHS of (I.3) must satisfy

$$\begin{aligned} &\Upsilon(x_t) > \Gamma_b(x_t) = b^{-1}x_t \\ &\Leftrightarrow \delta(1+\tau^c)(1+n)\bar{b} + \mu_3 > \delta\left[(1+\tau^c)\lambda - \tau^c(1-\tilde{\eta})\right]. \end{aligned} \tag{I.4}$$

Thus, if (I.4) is satisfied, we arrive at the following facts. First, $\delta\tau_t = 1$ binds if and only if (I.2) and (I.3). Second, $q(k_t) \geq \Upsilon(x_t)$ is above $q(k_t) = \Gamma_b(x_t)$ in the $(x_t, q(k_t))$ plane. Third, $q(k_t) = \Upsilon(x_t)$ is transformed into the function :$b_t = \upsilon(k_t)$ in the (k_t, b_t) plane and $q(k_t) \geq \Upsilon(x_t) \Leftrightarrow b_t \leq \upsilon(k_t)$. Since $b_t = \upsilon(k_t)$ is increasing in k_t and always below $b_{t+1} = b_t$ locus, $\delta\tau_t \geq 1$ does not bind above $b_{t+1} = b_t$ locus. Finally, from (I.2) and (I.4), when $\phi = 1$, $\delta\tau_t \geq 1$ does not bind either.

J Effects of $\bar{b}$ on $S(k_S^*, b_S^*)$ when $\delta > 1$, $\mu_1 \leq 0$

Taking the total differentials of (35) yields

$$\frac{dq(k_S^*)}{d\bar{b}} = -\frac{q(k_S^*)\left\{\alpha\left[2(1+n)\delta\bar{b} - \tilde{\eta}(\delta-1)\right]q(k_S^*) + (1+n)\left(\frac{1-\alpha}{1+\beta} + 2\delta\alpha\right)\right\}}{\underbrace{2\mu_1 q(k_S^*) + \mu_2}_{(-)}}. \tag{J.1}$$

Let us define the LHS of (35) as $\Xi(q(k)) \equiv \mu_1 q(k)^2 + \mu_2 q(k) + \mu_3$. $q(k_S^*)$ is given by the intersection point between the $q(k_t)$ axis and the inverted U-shaped quadratic function $\Xi(q(k))$ that takes $\Xi(0) = \mu_3 > 0$ and satisfies $\Xi'(q(k_S^*)) = 2\mu_1 q(k_S^*) + \mu_2 < 0$ and $\Xi'(-\mu_2/(2\mu_1)) = 0$. From $\delta > 0$, $\mu_1 \leq 0$, and $\mu_2 = (1+n)\bar{b}\left(\frac{1-\alpha}{1+\beta} + 2\delta\alpha\right) - \tilde{\eta}[(1+\tau^c)(1-\lambda) + \alpha(\delta-1)]$, we have

$$\begin{aligned}
&\alpha\left[2(1+n)\delta\bar{b} - \tilde{\eta}(\delta-1)\right]q(k_t) + (1+n)\left(\frac{1-\alpha}{1+\beta} + 2\delta\alpha\right) \\
&> \frac{2\mu_1}{\bar{b}}q(k_t) + (1+n)\left(\frac{1-\alpha}{1+\beta} + 2\delta\alpha\right) \\
&> \frac{2\mu_1}{\bar{b}}q(k_t) + \frac{\mu_2}{\bar{b}}.
\end{aligned} \tag{J.2}$$

Evaluating (J.2) at $q(k_t) = -\mu_2/(2\mu_1)$, we have $\alpha\left[2(1+n)\delta\bar{b} - \tilde{\eta}(\delta-1)\right](-\mu_2/2\mu_1) + (1+n)\left(\frac{1-\alpha}{1+\beta} + 2\delta\alpha\right) > 0$. Since $q(k_S^*) > -\mu_2/(2\mu_1)$, we have $\alpha\left[2(1+n)\delta\bar{b} - \tilde{\eta}(\delta-1)\right]q(k_S^*) + (1+n)\left(\frac{1-\alpha}{1+\beta} + 2\delta\alpha\right) > 0$. Thus, $dq(k_S^*)/d\bar{b} > 0$.

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Parameter or variable	JPA	US	GRE	ITA	PRT	Source
α	0.38	0.35	0.40	0.39	0.39	Hansen and İmrohoroğlu (2016) and Trabaudt and Uhlig (2011)
$\bar{b}$	0.02	0.02	0.02	0.02	0.02	Set $(0.6/30 = 0.02)$
τ^R	0.46	0.34	0.16	0.30	0.23	Data average with Gunji and Miyazaki (2011) and Trabaudt and Uhlig (2011)
τ^w	0.31	0.28	0.41	0.47	0.31	Gunji and Miyazaki (2011)
τ^c	0.10	0.05	0.15	0.15	0.23	Data and Trabaudt and Uhlig (2011)
n	0	0.01	0	0	0	Data average
Y_0/K_0	0.32	0.41	0.28	0.30	0.36	Data average
B_0/Y_0	2.37	1.36	1.93	1.53	1.42	Data average
A	20	24.34	10.68	14.80	9.99	Calibrated (A of JPA: Set)
k_0	3.27	2.98	2.52	3.57	2.43	Calibrated
b_0	2.48	1.66	1.36	1.64	1.24	Calibrated

Table 1: Benchmark parameters and variables

Expenditure-based consolidation					
		$g_t = 0$ binds	$k_{t+1} = 0$ binds	$d_t = 0$ binds	non-surviving generation
JPA	$\phi = 0$	period 2	period 2 ($k_3 = 0$)	period 3	generation 1
	$\phi = 0.1$	period 3	period 3 ($k_4 = 0$)	period 4	generation 2
GRE	$\phi = 0$	period 1	period 1 ($k_2 = 0$)	period 2	generation 0
	$\phi = 0.1$	period 1	period 1 ($k_2 = 0$)	period 2	generation 0
	$\phi = 0.3$	period 1	period 1 ($k_2 = 0$)	period 2	generation 0
ITA	$\phi = 0$	period 2	period 1 ($k_2 = 0$)	period 2	generation 1
	$\phi = 0.1$	period 2	period 1 ($k_2 = 0$)	period 2	generation 1
PRT	$\phi = 0$	period 1	period 1 ($k_2 = 0$)	period 2	generation 0
	$\phi = 0.1$	period 2	period 2 ($k_3 = 0$)	period 3	generation 1

Table 2: Unsustainable paths under expenditure-based consolidations

Expenditure-based consolidation					
Benchmark case	JPA	US	GRE	ITA	PRT
k^*	3.4069	5.3068	0.9044	1.2861	1.1428
b^*	0.6373	0.8731	0.2052	0.3265	0.2105
g^*	12.9875	13.5552	3.4574	6.9502	3.7982
c^*	8.3572	13.5122	2.1932	3.1874	2.5010
d^*	6.4150	11.2954	3.6775	4.8596	3.0426

Table 3: Values of the steady-state variables under expenditure-based consolidation

Tax-based consolidation					
		$\tau_t = 0$ or $\delta\tau_t = 1$binds	$k_{t+1} = 0$ binds	$d_t = 0$ binds	nonsurviving generation
JPA	$\phi = 0$	$\delta\tau_3 = 1$ binds	period 2 ($k_3 = 0$)	period 3	generation 2
	$\phi = 0.1$	$\delta\tau_3 = 1$ binds	period 2 ($k_3 = 0$)	period 3	generation 2
GRE	$\phi = 0$	$\tau_1 = 1$ binds	period 1 ($k_2 = 0$)	period 2	generation 1
	$\phi = 0.3$	$\tau_2 = 1$ binds	period 1 ($k_2 = 0$)	period 2	generation 1
	$\phi = 0.5$	$\tau_2 = 1$ binds	period 2 ($k_3 = 0$)	period 3	generation 2
ITA	$\phi = 0$	$\tau_2 = 1$ binds	period 1 ($k_2 = 0$)	period 2	generation 1
	$\phi = 0.3$	$\tau_2 = 1$ binds	period 1 ($k_2 = 0$)	period 2	generation 1
	$\phi = 0.5$	$\tau_4 = 1$ binds	period 3 ($k_4 = 0$)	period 4	generation 3
PRT	$\phi = 0$	$\tau_2 = 1$ binds	period 1 ($k_2 = 0$)	period 2	generation 1
	$\phi = 0.3$	$\tau_3 = 1$ binds	period 2 ($k_3 = 0$)	period 3	generation 2

Table 4: Unsustainable paths under tax-based consolidations

Tax-based consolidation					
Benchmark case	JPA	US	GRE	ITA	PRT
k^*	3.9093	6.1108	1.0987	1.3272	1.2691
b^*	0.6715	0.9172	0.2218	0.3315	0.2193
g^*	11.9086	11.4515	3.3791	6.8519	3.6210
c^*	9.4664	15.3669	2.6099	3.2787	2.6806
d^*	8.2922	13.0007	4.0017	5.1168	3.3400
$\tau^*(=\tau^{w*})$	0.2797	0.2206	0.3505	0.4540	0.2689
$\delta\tau^*(=\tau^{R*})$	0.3899	0.2679	0.1368	0.2898	0.1995

Table 5: Values of the steady-state variables under tax-based consolidation

Expenditure-based consolidation					
Benchmark case	JPA	US	GRE	ITA	PRT
sustainable	$\phi \in [0.12, 1]$	$\phi \in [0, 1]$	$\phi \in [0.42, 1]$	$\phi \in [0.30, 1]$	$\phi \in [0.27, 1]$
unsustainable	$\phi \in [0, 0.11]$	-	$\phi \in [0, 0.41]$	$\phi \in [0, 0.29]$	$\phi \in [0, 0.26]$

Table 6: Pace of expenditure-based fiscal consolidation ϕ and sustainability of public debt

Tax-based consolidation					
Benchmark case	JPA	US	GRE	ITA	PRT
sustainable	$\phi \in [0.18, 1]$	$\phi \in [0, 1]$	$\phi \in [0.61, 1]$	$\phi \in [0.53, 1]$	$\phi \in [0.40, 1]$
unsustainable	$\phi \in [0, 0.17]$	-	$\phi \in [0, 0.60]$	$\phi \in [0, 0.52]$	$\phi \in [0, 0.39]$

Table 7: Pace of tax-based fiscal consolidation ϕ and sustainability of public debt

Social welfare W (the benchmark case: $\theta = 0.8$)						
		$\phi = 1$	$\phi = 0.9$	$\phi = 0.7$	$\phi = 0.5$	$\phi = 0.3$
JPA	expenditure base	20.4119	20.3721	20.2391	19.9807	19.3892
	tax base	20.4657	20.5041	20.3409	20.0544	19.3711
US	expenditure base	22.4001	22.3904	22.3583	22.3014	22.1993
	tax base	22.2727	22.3173	22.2759	22.2156	22.1093
GRE	expenditure base	10.0332	9.9101	9.4310	7.9951	-
	tax base	10.1650	9.9845	9.9518	-	-
ITA	expenditure base	15.3716	15.2918	15.0238	14.4323	13.5151
	tax base	15.3373	16.0471	14.6970	-	-
PRT	expenditure base	10.0034	9.9314	9.6792	9.1177	6.8289
	tax base	10.0620	9.9990	9.6644	8.8374	-

Table 8: Social welfare W (the benchmark case: $\theta = 0.8$)

Gini coefficient of welfare Δ (the benchmark case: $\theta = 0.8$)						
		$\phi = 1$	$\phi = 0.9$	$\phi = 0.7$	$\phi = 0.5$	$\phi = 0.3$
JPA	expenditure base	0.0172	0.0177	0.0195	0.0235	0.0339
	tax base	0.0205	0.0205	0.0228	0.0274	0.0404
US	expenditure base	0.0252	0.0254	0.0258	0.0265	0.0279
	tax base	0.0264	0.0261	0.0266	0.0274	0.0289
GRE	expenditure base	0.0331	0.0297	0.0368	0.1014	-
	tax base	0.0180	0.0282	0.0739	-	-
ITA	expenditure base	0.0172	0.0156	0.0149	0.0315	0.1803
	tax base	0.0136	0.0139	0.0294	-	-
PRT	expenditure base	0.0169	0.0146	0.0175	0.0434	0.1812
	tax base	0.0106	0.0135	0.0302	0.0792	-

Table 9: Gini coefficient of welfare Δ (the benchmark case: $\theta = 0.8$)

Social welfare W (when $\theta = 0.2$)						
		$\phi = 1$	$\phi = 0.9$	$\phi = 0.7$	$\phi = 0.5$	$\phi = 0.3$
JPA	expenditure base	14.2897	14.2728	14.2205	14.1246	13.9160
	tax base	14.4187	14.4826	14.3892	14.2187	13.7826
US	expenditure base	16.4211	16.4173	16.4056	16.3861	16.3522
	tax base	16.6052	16.6576	16.6357	16.6029	16.5394
GRE	expenditure base	7.8605	7.8159	7.6536	7.2077	-
	tax base	7.9398	7.8253	6.6634	-	-
ITA	expenditure base	10.5591	10.5294	10.4334	10.2304	9.8415
	tax base	10.4909	9.8623	10.0621	-	-
PRT	expenditure base	7.6096	7.5820	7.4909	7.2993	6.5993
	tax base	7.6427	7.6188	7.4043	6.8523	-

Table 10: Social welfare W (when $\theta = 0.2$)

Gini coefficient of welfare Δ (when $\theta = 0.2$)						
		$\phi = 1$	$\phi = 0.9$	$\phi = 0.7$	$\phi = 0.5$	$\phi = 0.3$
JPA	expenditure base	0.0132	0.0135	0.0149	0.0177	0.0251
	tax base	0.0207	0.0202	0.0222	0.0263	0.0387
US	expenditure base	0.0214	0.0215	0.0217	0.0222	0.0229
	tax base	0.0256	0.0252	0.0257	0.0262	0.0275
GRE	expenditure base	0.0386	0.0368	0.0377	0.0675	-
	tax base	0.0159	0.0258	0.0684	-	-
ITA	expenditure base	0.0208	0.0208	0.0198	0.0292	0.0967
	tax base	0.0144	0.0153	0.0330	-	-
PRT	expenditure base	0.0213	0.0201	0.0209	0.0351	0.0957
	tax base	0.0083	0.0121	0.0283	0.0735	-

Table 11: Gini coefficient of welfare Δ (when $\theta = 0.2$)

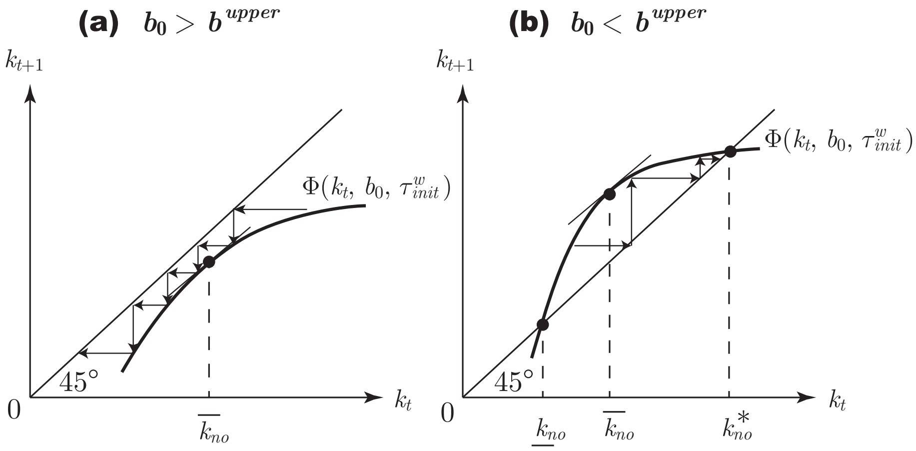


Figure 1: The dynamics of k_t under no fiscal consolidation

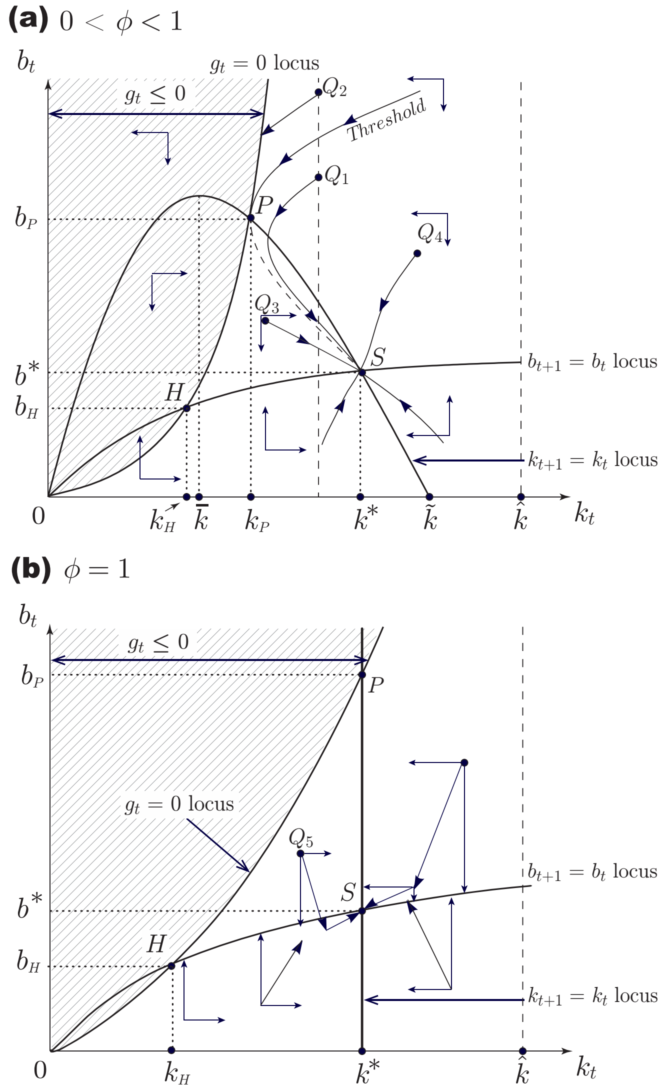


Figure 2: The dynamics of k_t and b_t under expenditure-based consolidation

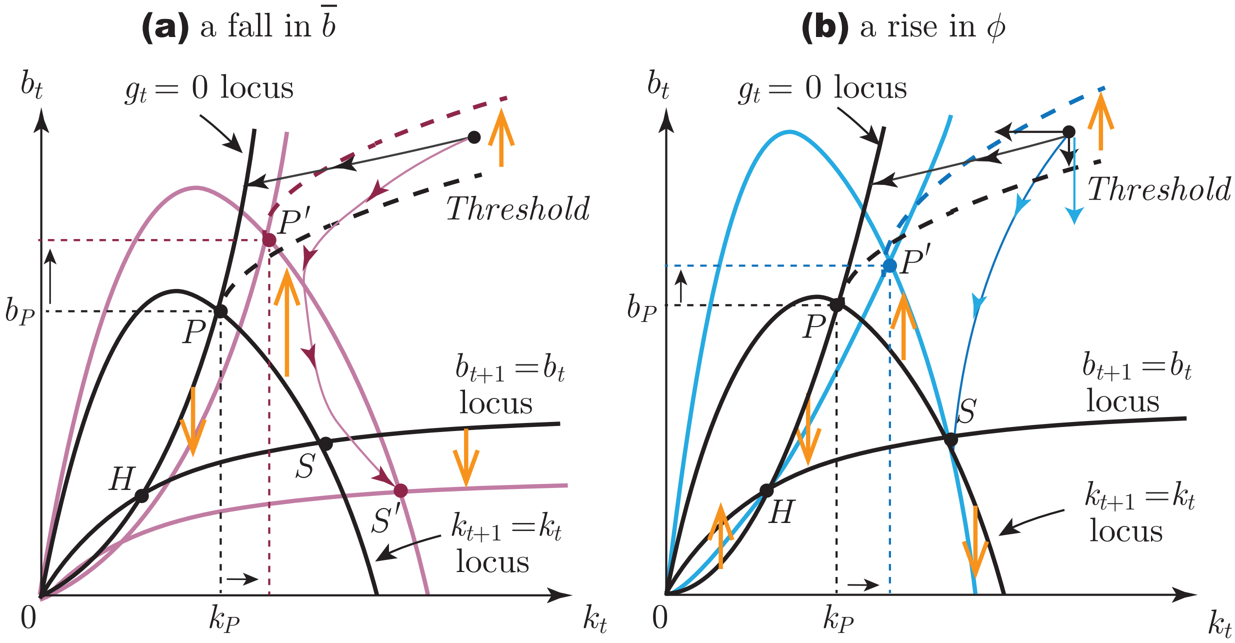


Figure 3: Effects of a fall in $\bar{b}$ and a rise in ϕ in the case of expenditure-based consolidation

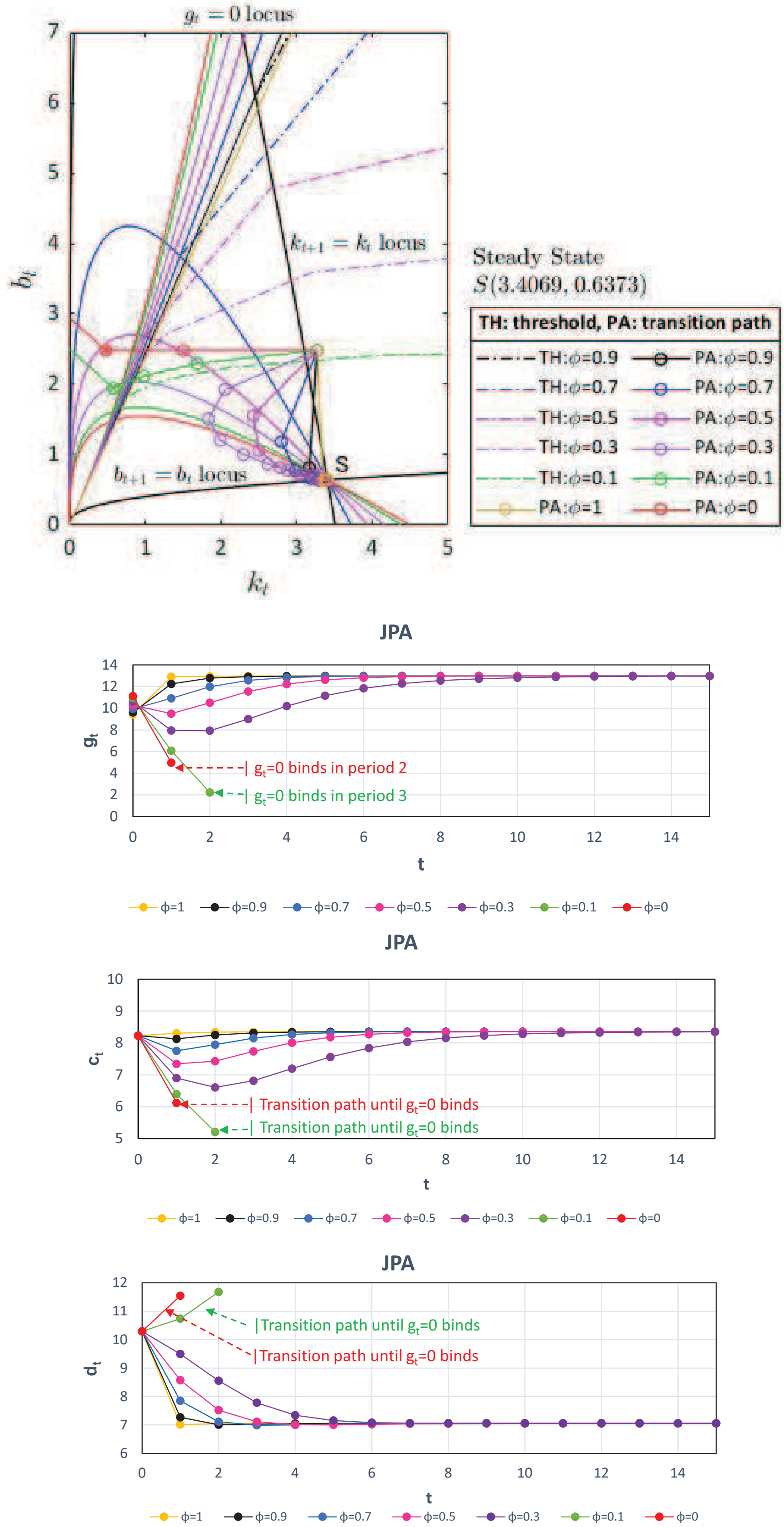


Figure 4: Transitional dynamics for Japan under expenditure-based consolidations

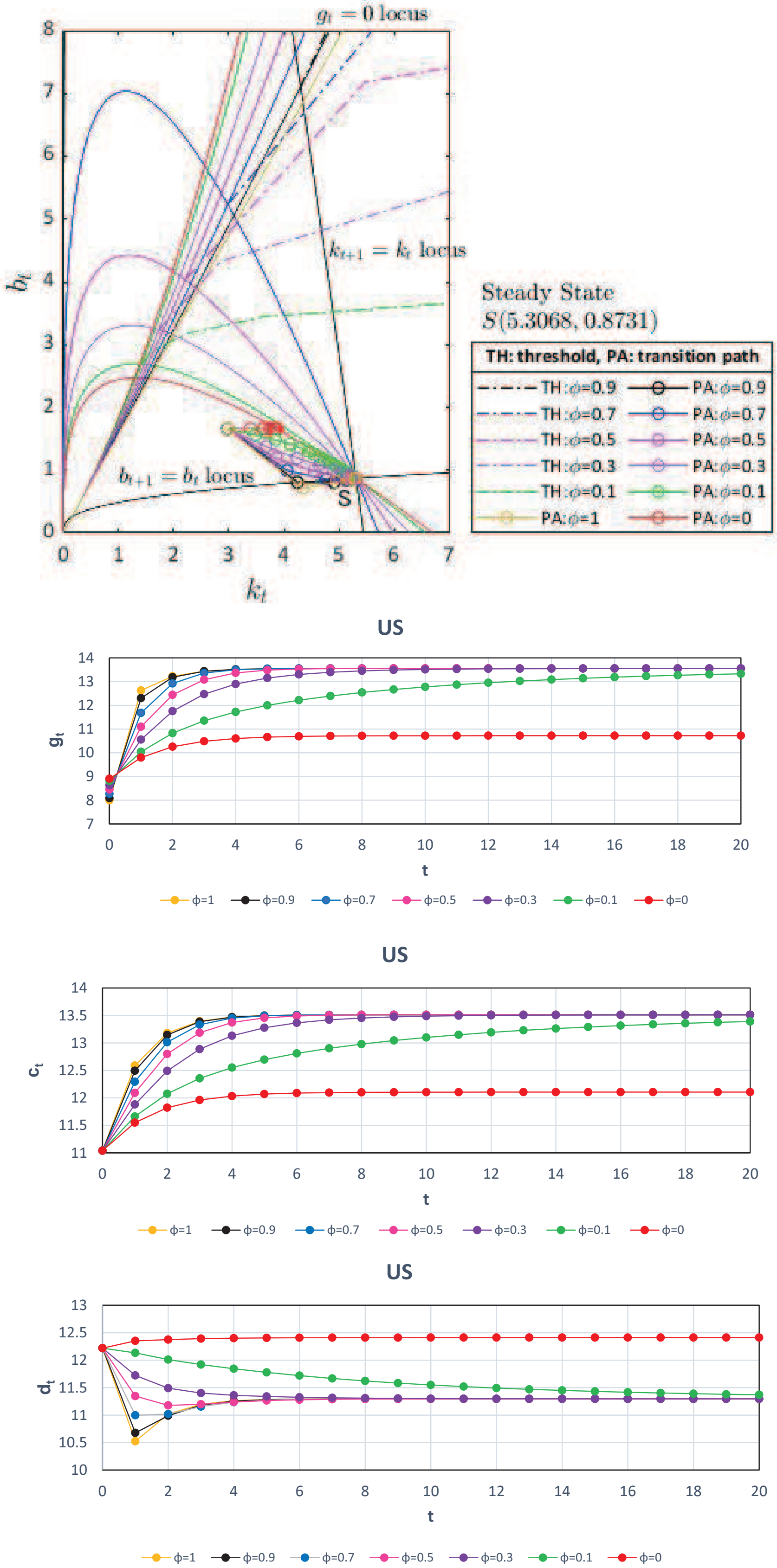


Figure 5: Transitional dynamics for the US under expenditure-based consolidations

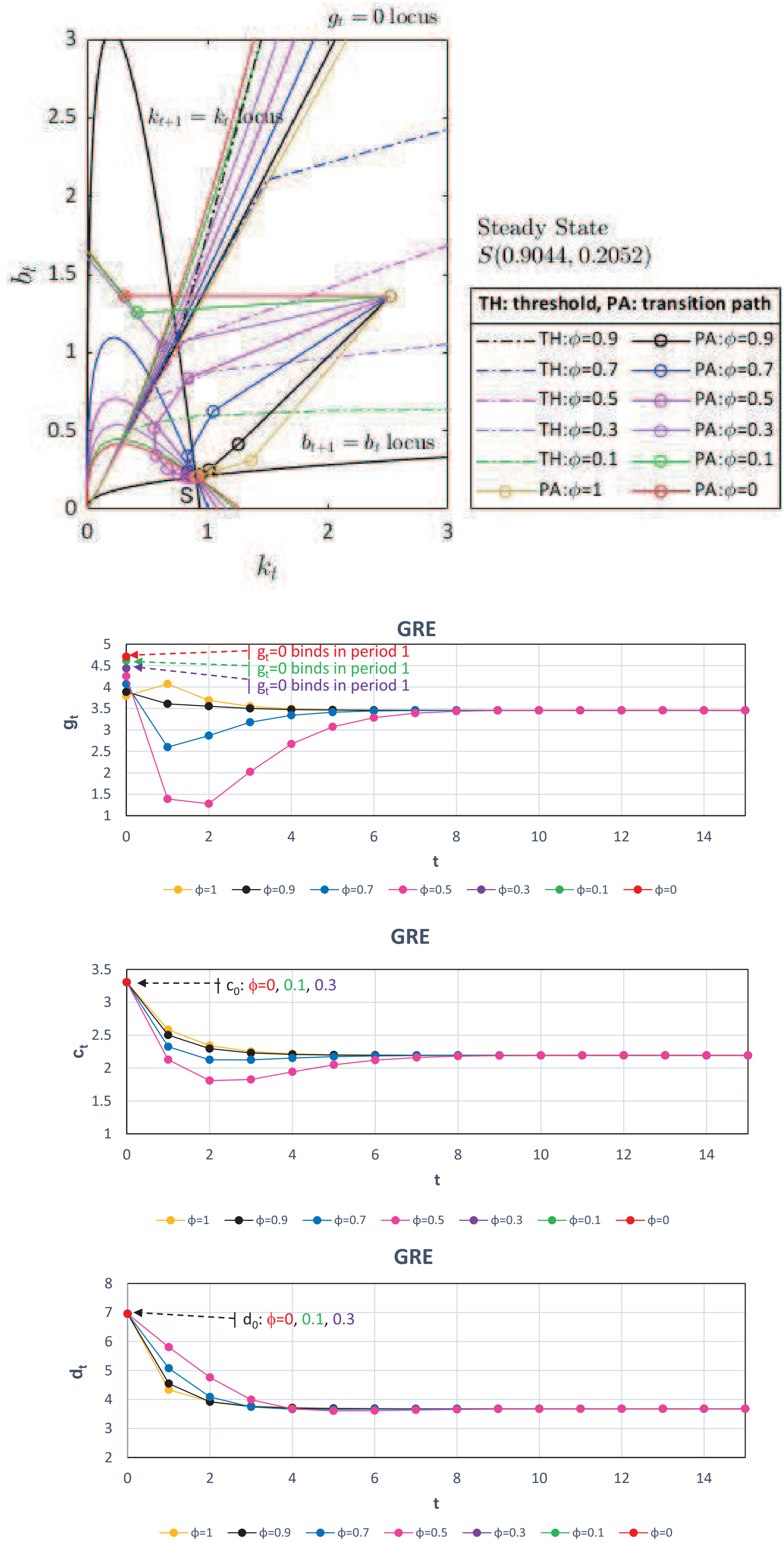


Figure 6: Transitional dynamics for Greece under expenditure-based consolidation

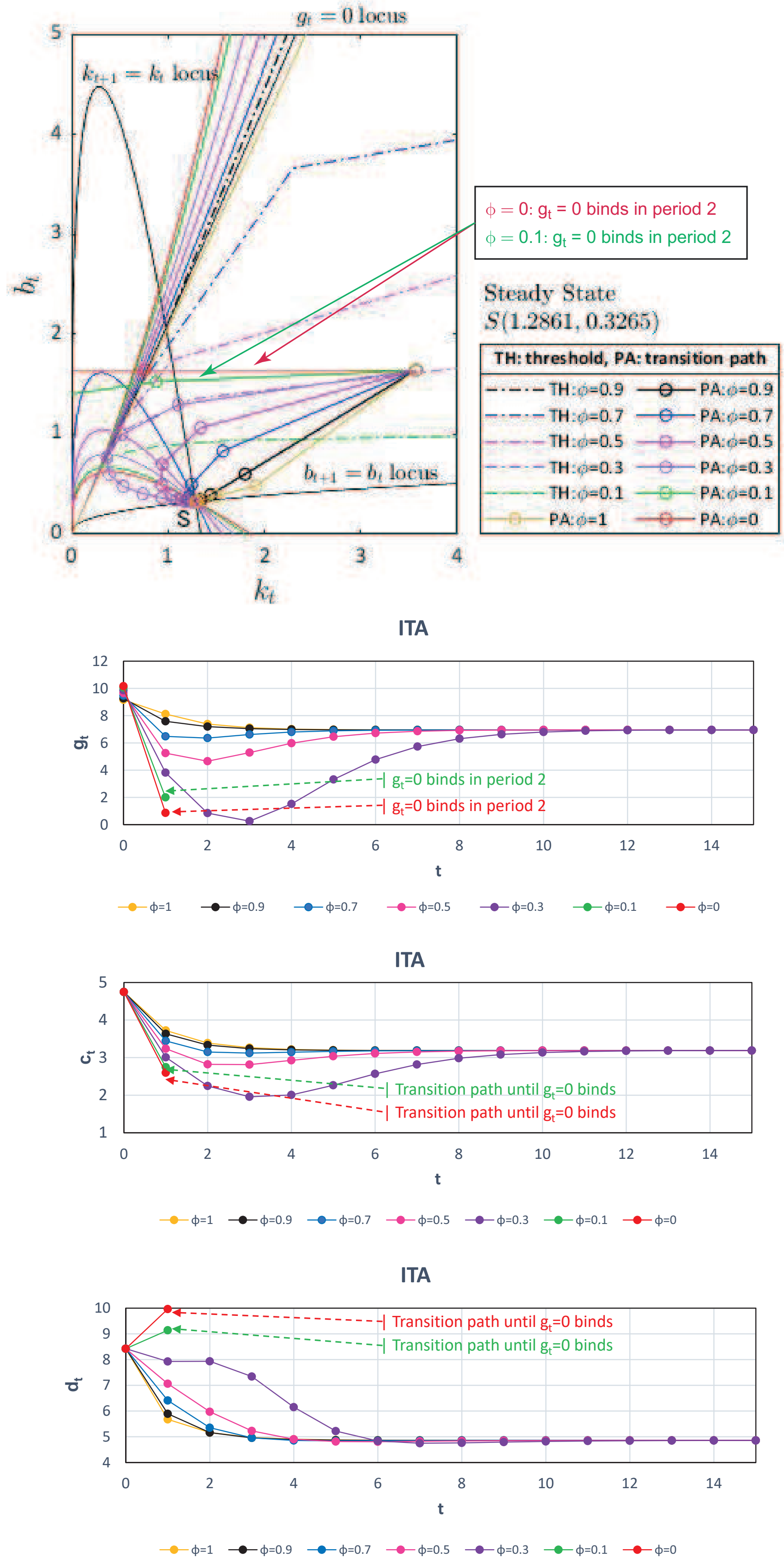


Figure 7: Transitional dynamics for Italy under expenditure-based consolidations

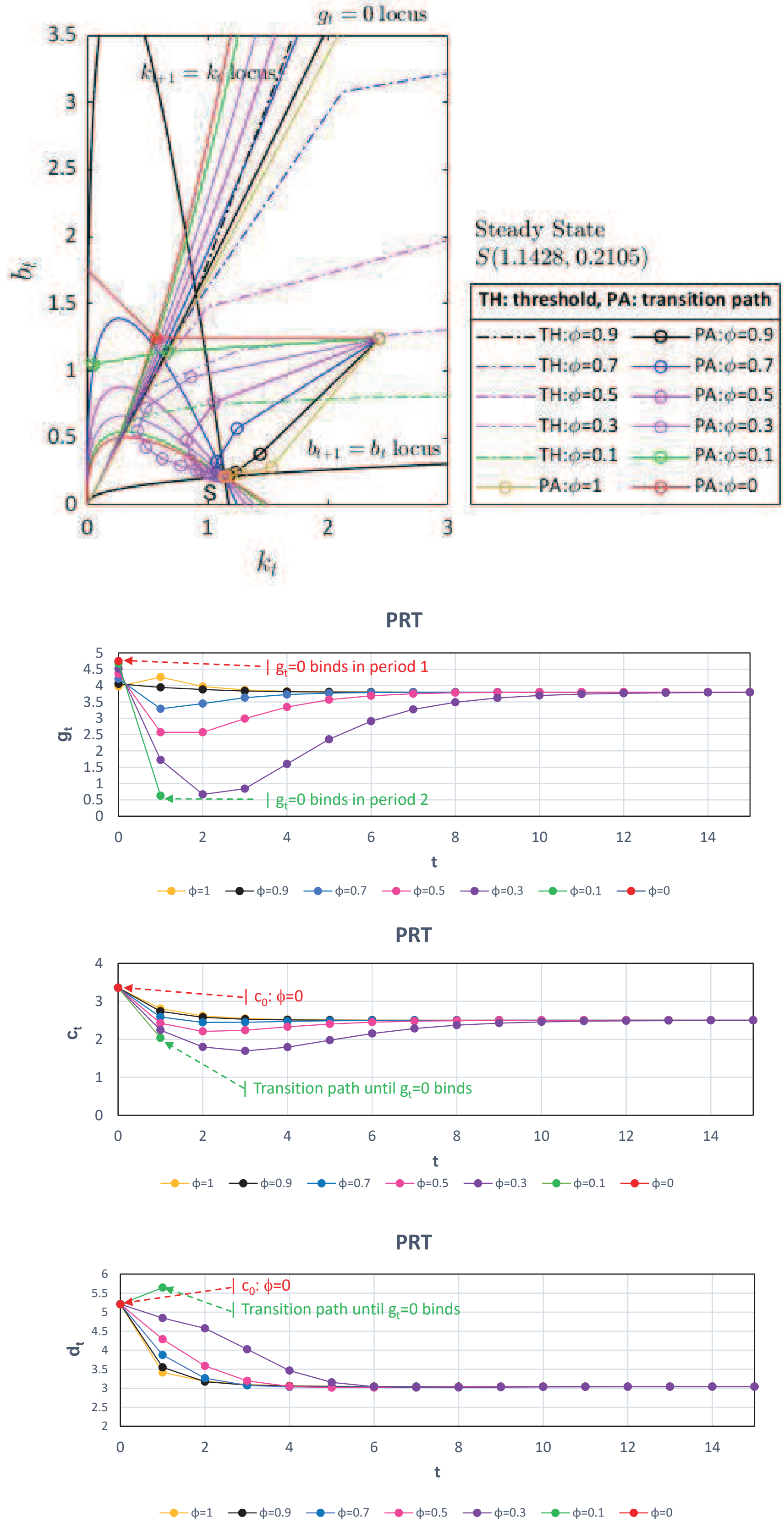
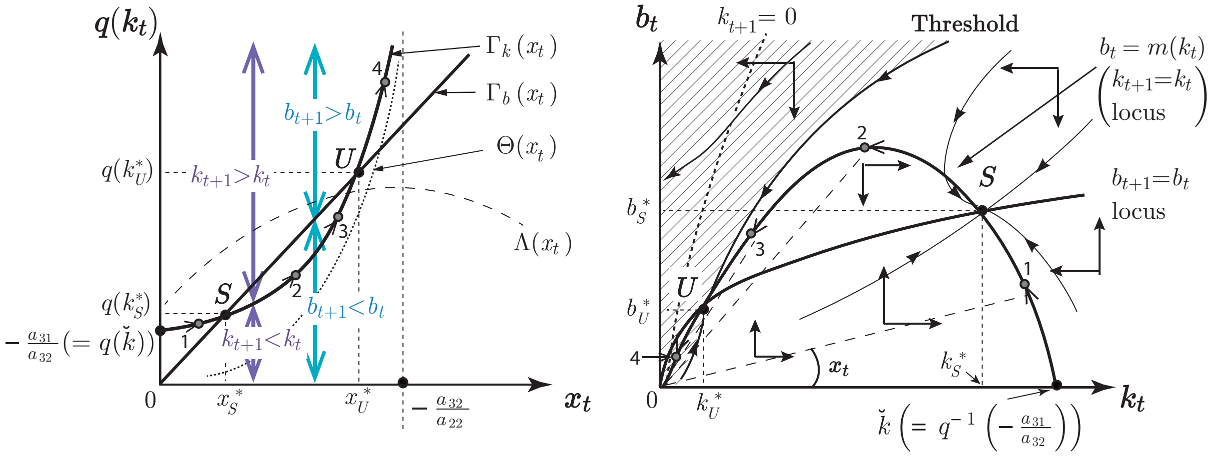


Figure 8: Transitional dynamics for Portugal under expenditure-based consolidations

(a)



(b)

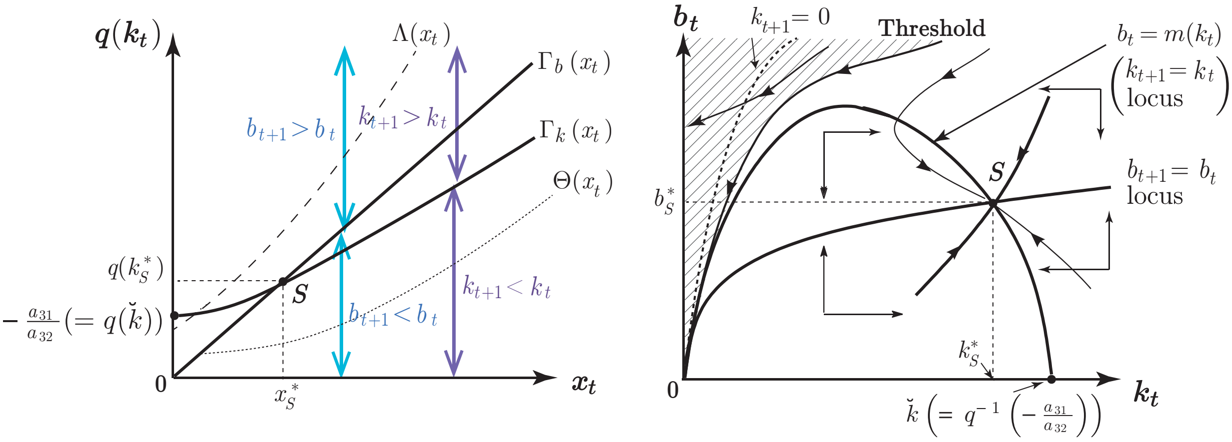


Figure 9: Phase diagram for $0<\phi<1$ in the case of tax-based consolidation: **(a)** $0<\delta\leq 1$ **(b)** $\delta>1$ and $\mu_1\leq 0$

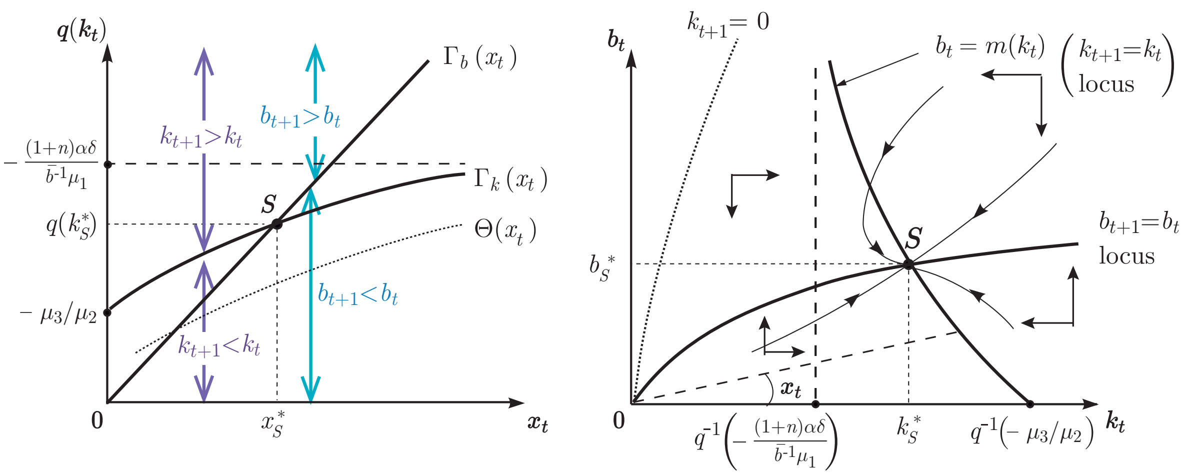


Figure 10: Phase diagram for $\phi=1$: the case of $\delta>1$ and $\mu_1\leq 0$ under tax-based consolidation (Note: the case when $0<\delta\leq 1$ is similar to Figure 9-(a)).

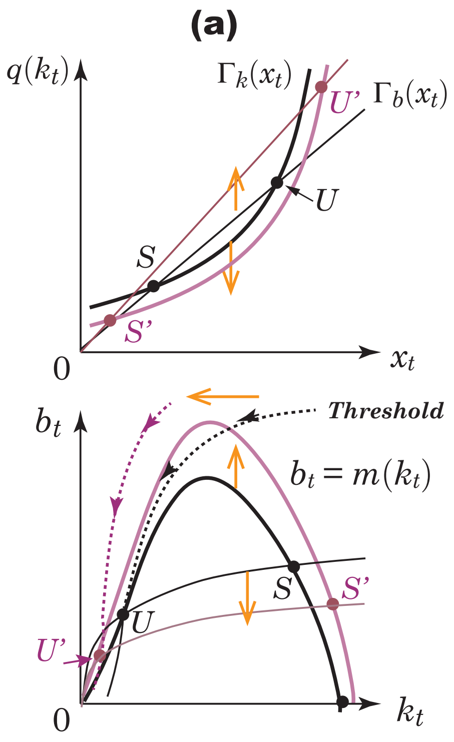


Figure 11: Effects of a reduction in $\bar{b}$ on the sustainability of public debt and the steady state under tax-base consolidation: **(a)** $0 < \delta \leq 1$ **(b)** $\delta > 1$ and $\mu_1 \leq 0$

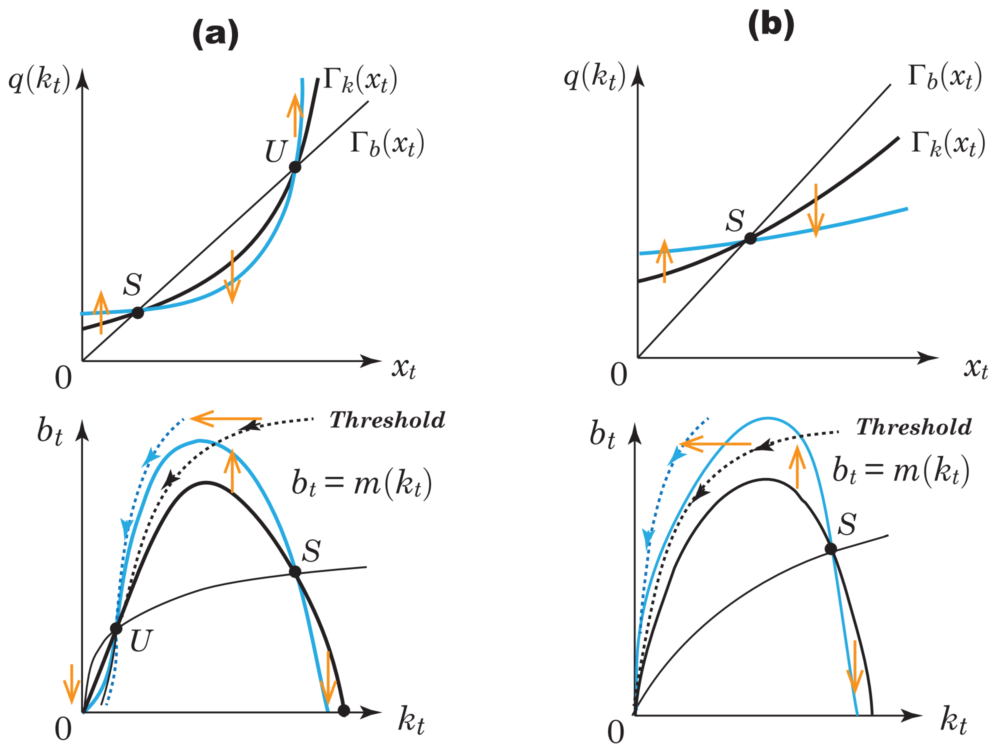


Figure 12: Effects of an increase in ϕ on the sustainability of public debt under tax-based consolidation: **(a)** $0 < \delta \leq 1$ **(b)** $\delta > 1$ and $\mu_1 \leq 0$

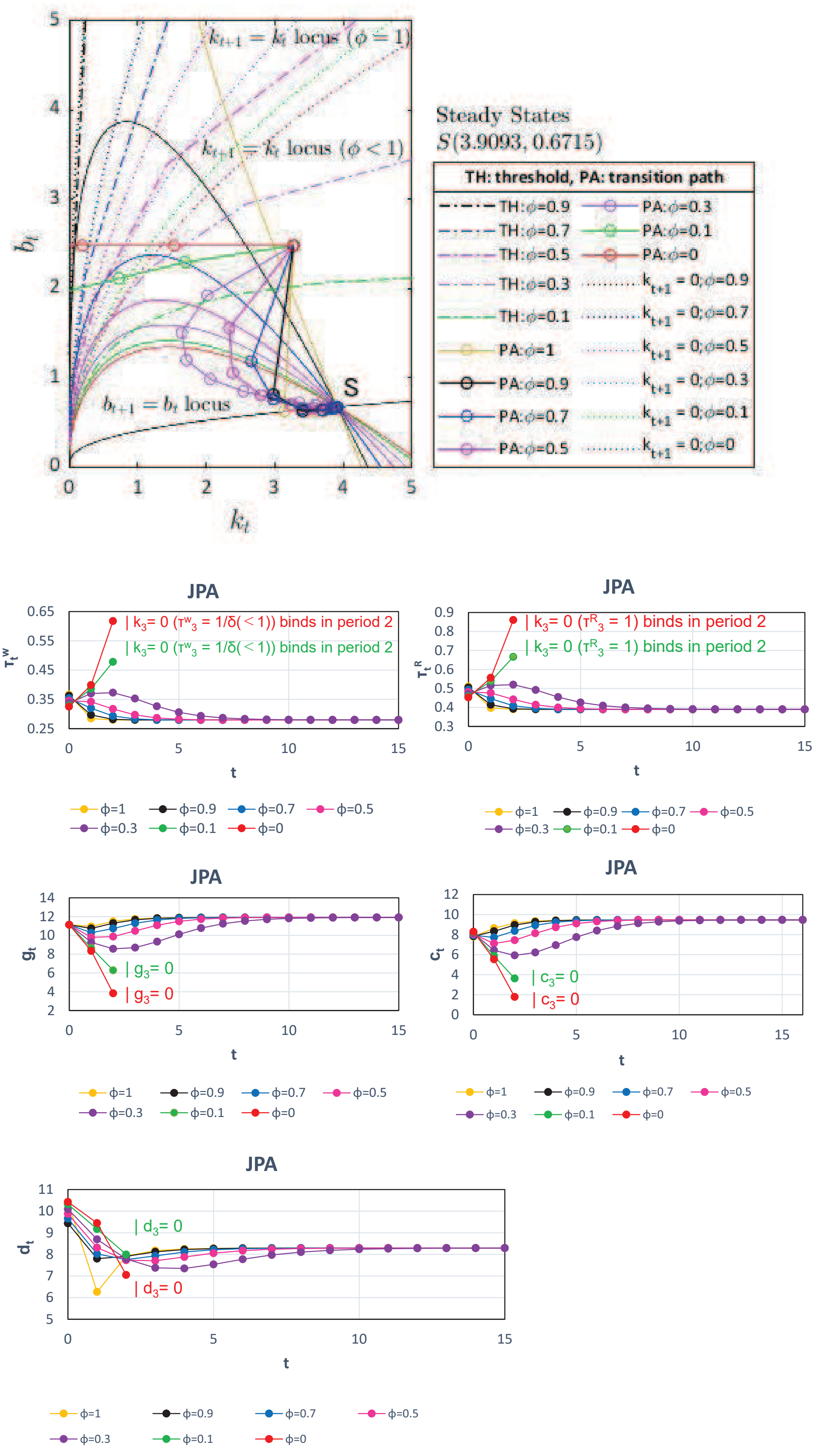


Figure 13: Transitional dynamics for Japan under tax-based consolidations

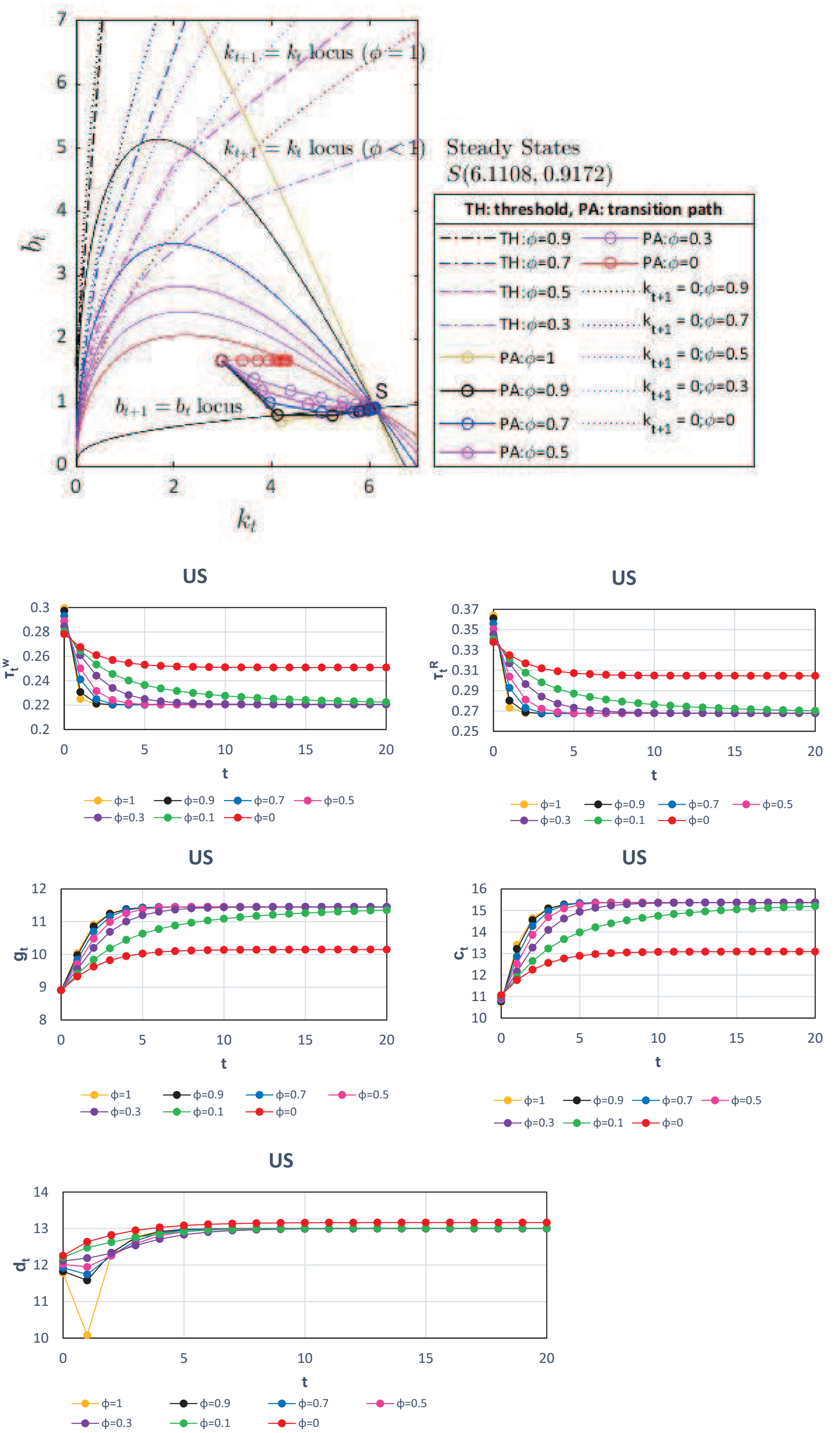


Figure 14: Transitional dynamics for the US under tax-based consolidations

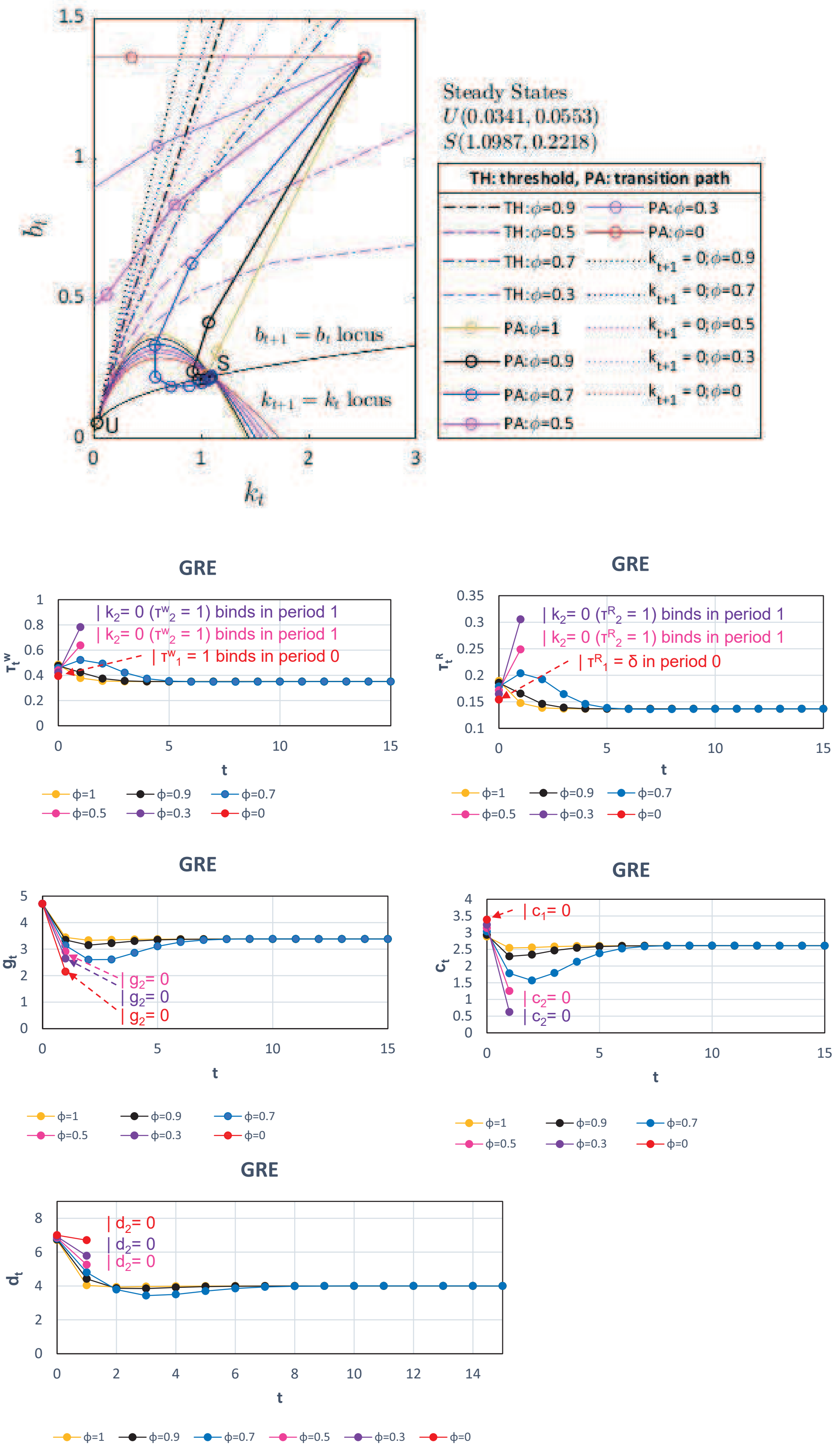


Figure 15: Transitional dynamics for Greece under tax-based consolidations

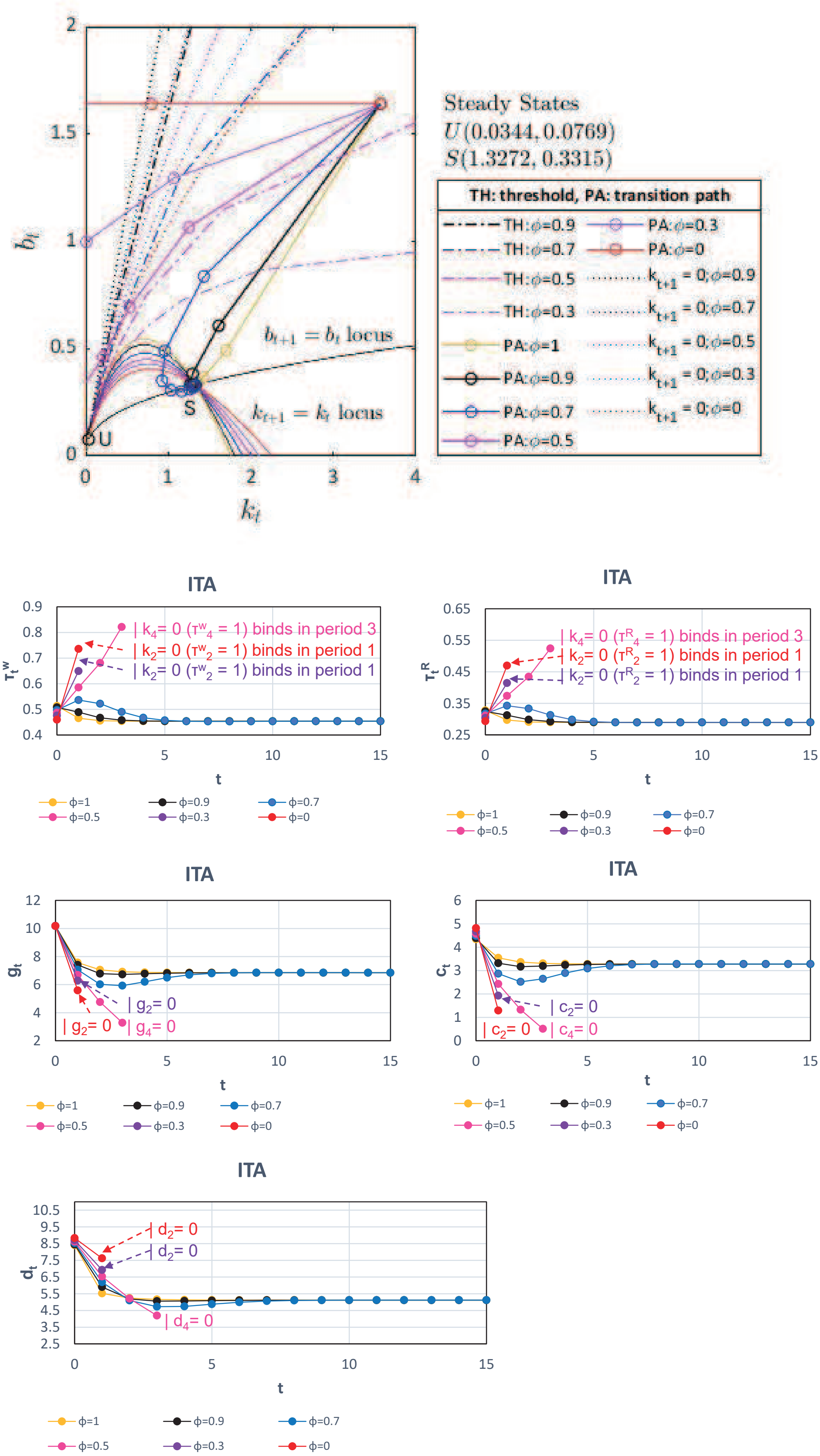


Figure 16: Transitional dynamics for Italy under tax-based consolidations

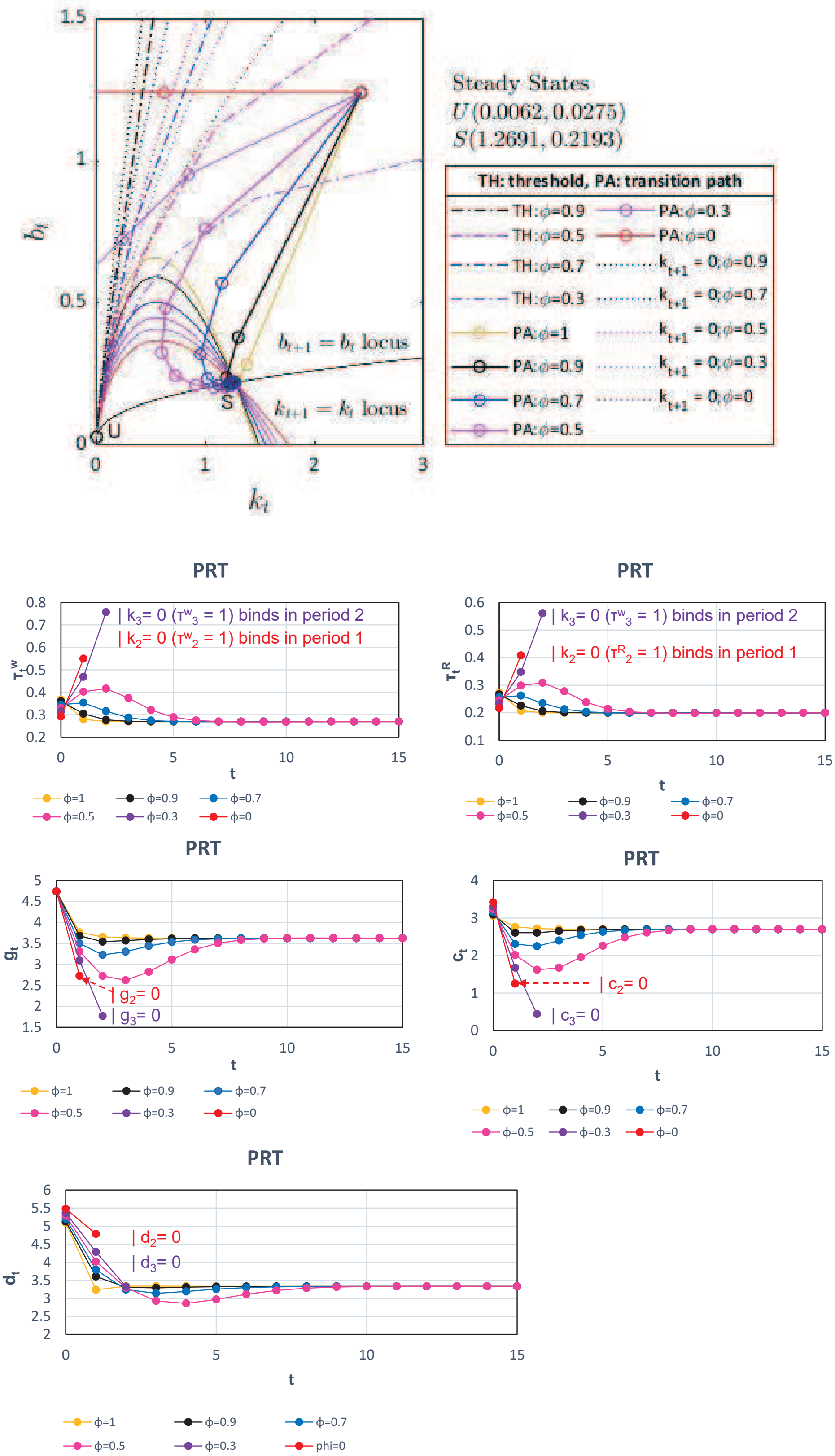


Figure 17: Transitional dynamics for Portugal under tax-based consolidations

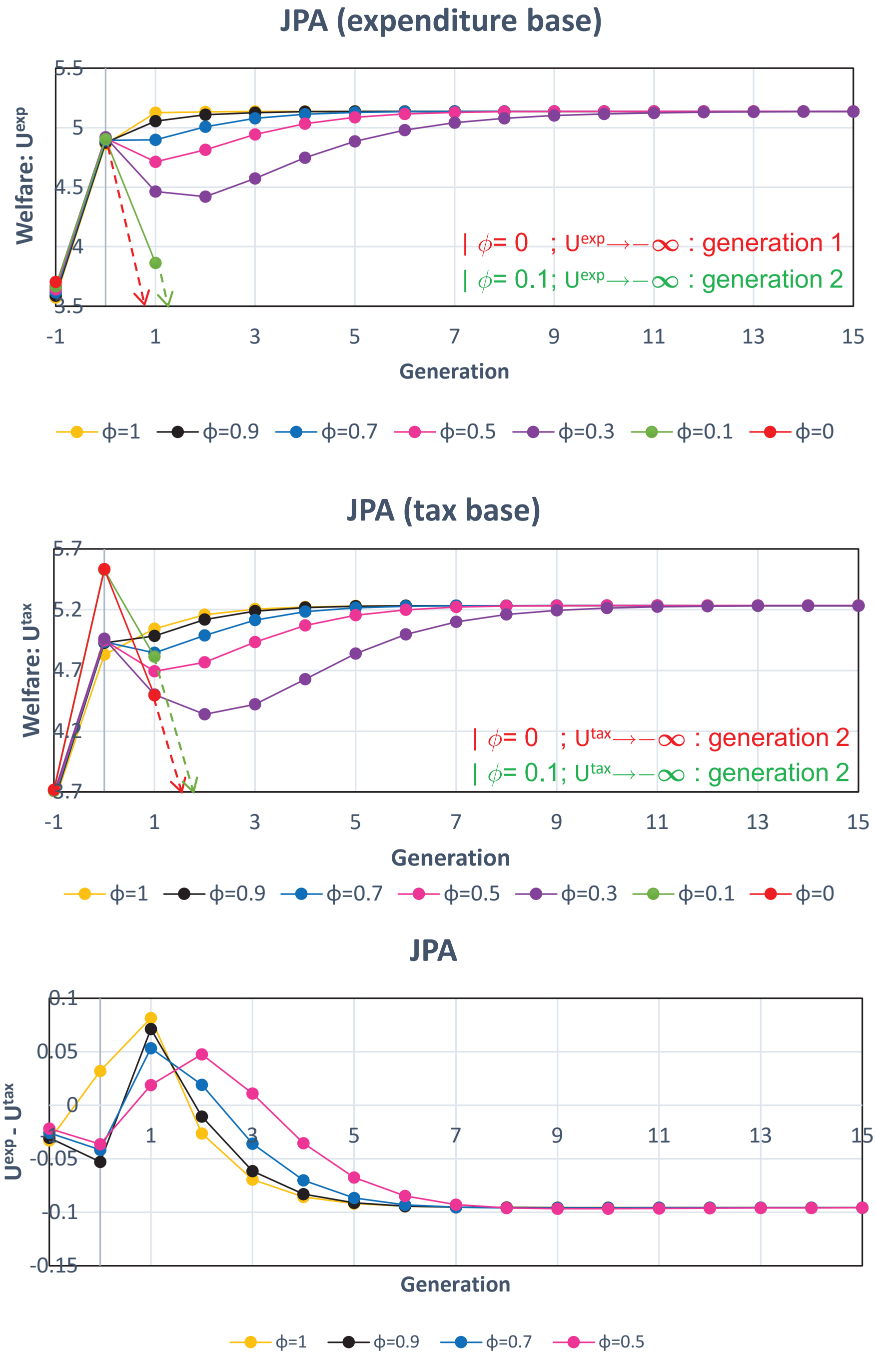


Figure 18: Welfare of each generation: the case of Japan

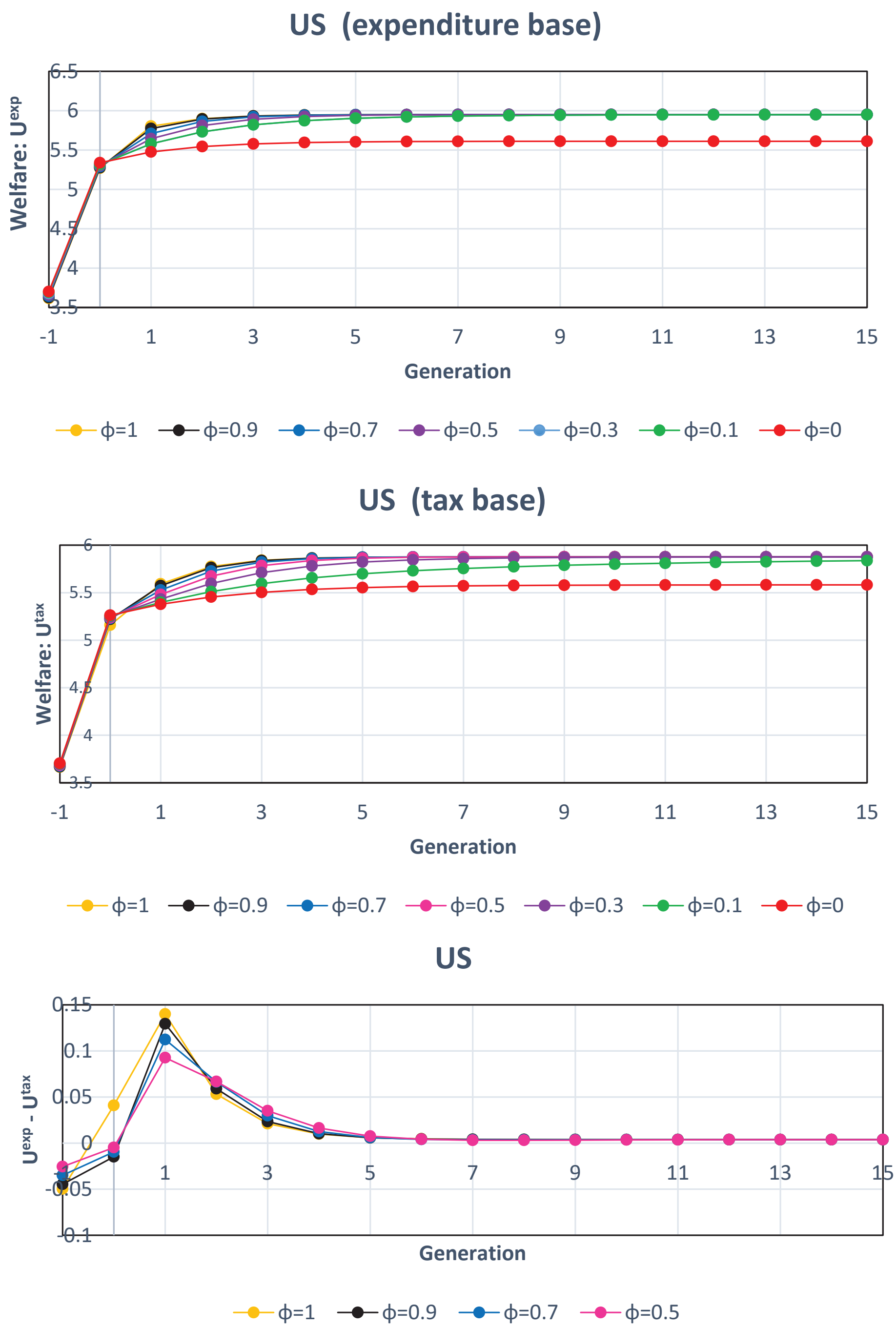


Figure 19: Welfare of each generation: the case of the US

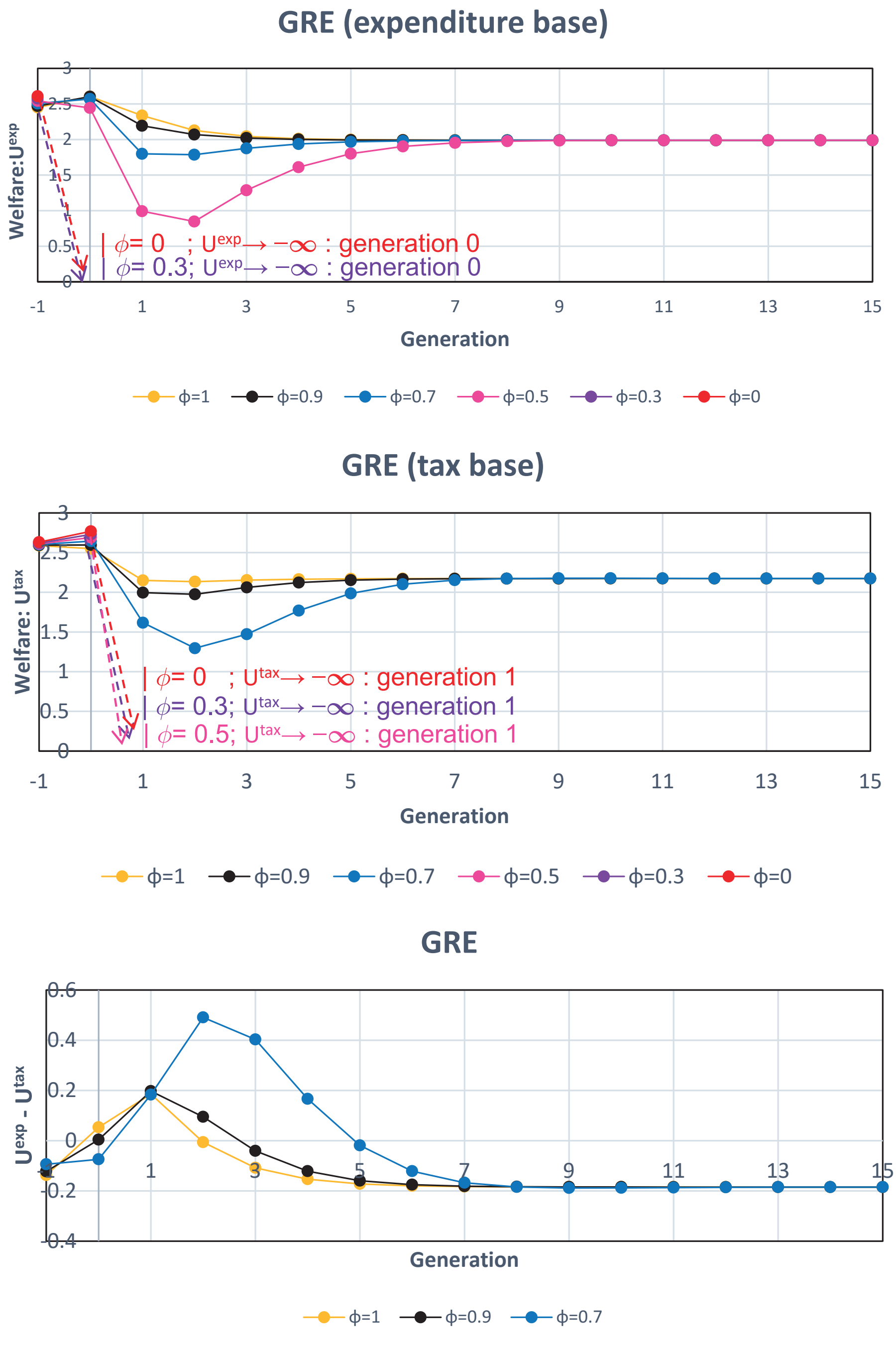


Figure 20: Welfare of each generation: the case of Greece

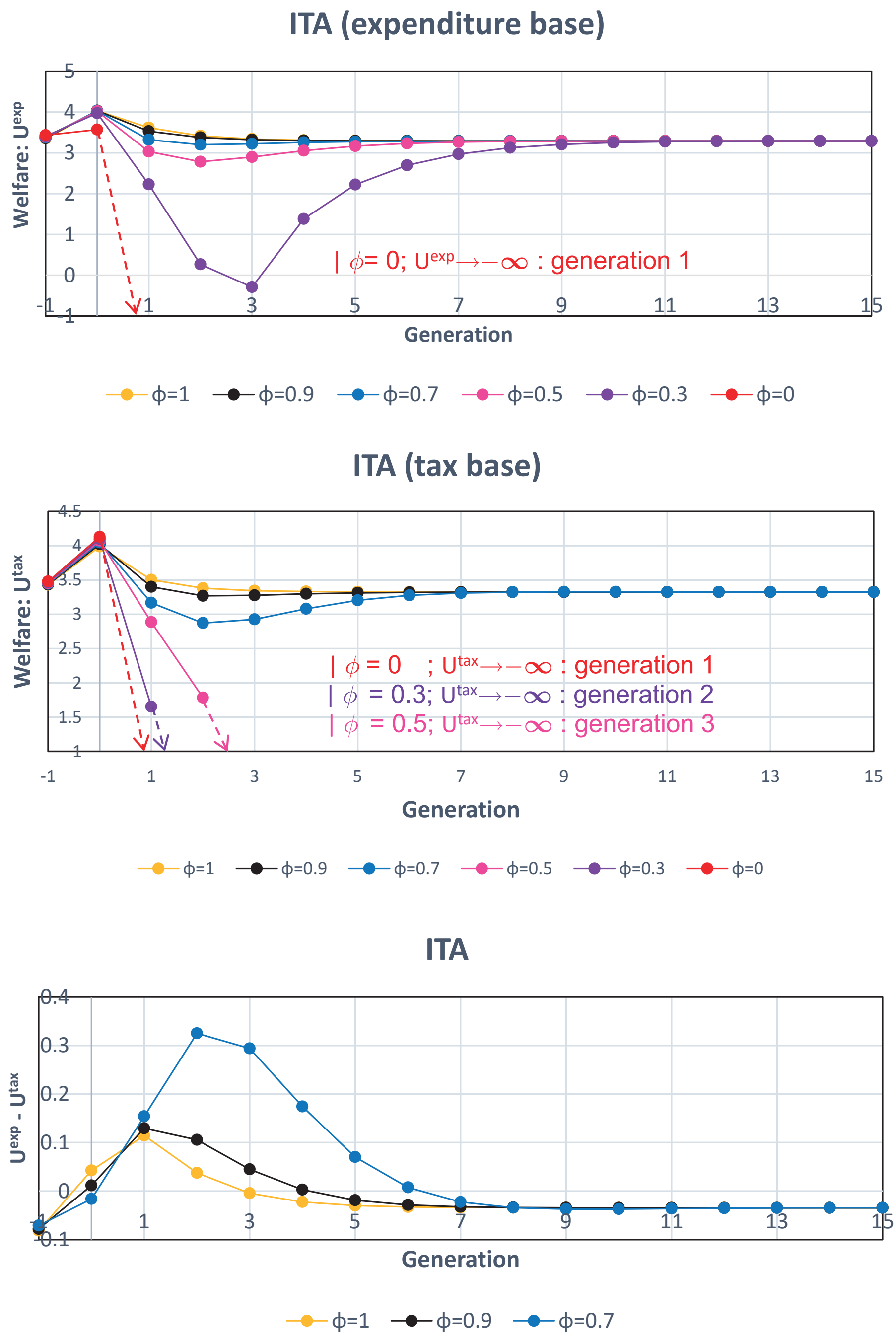


Figure 21: Welfare of each generation: the case of Italy

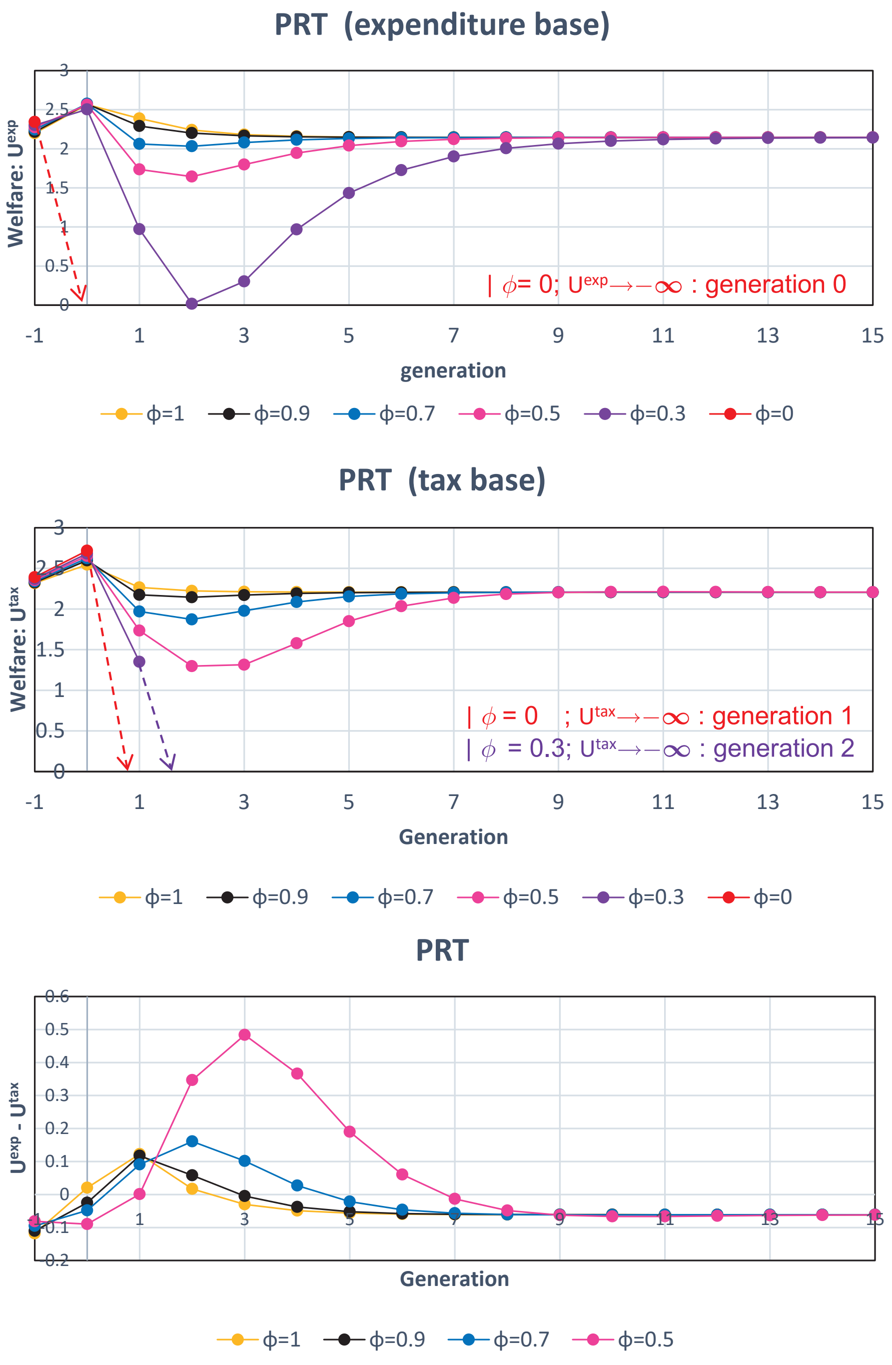


Figure 22: Welfare of each generation: the case of Portugal

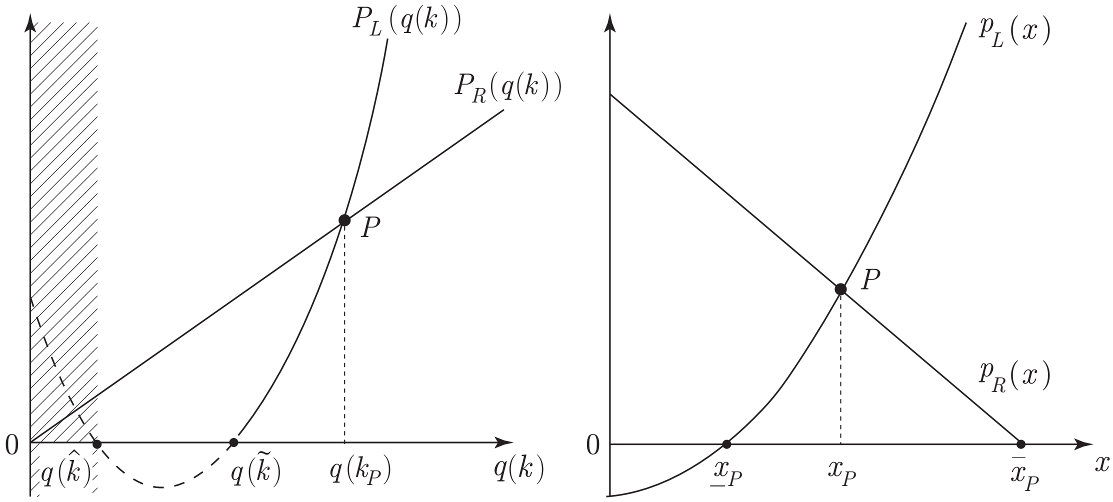


Figure 23: $q(k_P)$ and x_P

Technical Appendix

1 Derivation of Condition 3

$$\mu_1 \leq 0 \Leftrightarrow (1+n)\bar{b} \leq \frac{\tilde{\eta}(\delta-1)}{\delta} \tag{1}$$

Applying this into a_{32} (for $0 < \phi \leq 1$), we have

$$\begin{aligned} a_{32} &= (1+n)\left(\frac{1-\alpha}{1+\beta} + 2\alpha\delta\right)\bar{b}\phi - \tilde{\eta}\left[(1+\tau^c)(1-\lambda) + \alpha(\delta-1)\right] \\ &\leq \frac{\tilde{\eta}(\delta-1)}{\delta}\left(\frac{1-\alpha}{1+\beta} + 2\alpha\delta\right) - \tilde{\eta}\left[(1+\tau^c)(1-\lambda) + \alpha(\delta-1)\right] \\ &= \frac{\tilde{\eta}}{\delta}\left[(\delta-1)\left(\frac{1-\alpha}{1+\beta} + \alpha\delta\right) - (1+\tau^c)(1-\lambda)\delta\right] \leq 0. \end{aligned} \tag{2}$$

Thus $a_{32} \leq 0$ if

$$(1+\tau^c)(1-\lambda)\delta \geq (\delta-1)\left(\frac{1-\alpha}{1+\beta} + \alpha\delta\right). \tag{3}$$

$$\begin{aligned} a_{21}a_{32} =& (1+n)\left[(1-\phi)\left(\frac{1-\alpha}{1+\beta} + \alpha\delta\right) + \alpha\delta\right] \\ & \times \left\{\left(\frac{1-\alpha}{1+\beta} + 2\alpha\delta\right)(1+n)\bar{b}\phi - \tilde{\eta}\left[(1+\tau^c)(1-\lambda) + \alpha(\delta-1)\right]\right\} \end{aligned} \tag{4}$$

$$a_{22}a_{31} = (1+n)\alpha\left[(1+n)\delta\bar{b}\phi - \tilde{\eta}(\delta-1)\right]\left[1 + \alpha(\delta-1) + \tilde{\eta}\tau^c\right] \tag{5}$$

Both $a_{21}a_{32}$ and $a_{22}a_{31}$ are increasing in $\bar{b}$. When $\bar{b} = 0$, $a_{21}a_{32} < a_{22}a_{31}$ for $0 < \phi \leq 1$ if and only if

$$\begin{aligned} &\left[(1-\phi)\left(\frac{1-\alpha}{1+\beta} + \alpha\delta\right) + \alpha\delta\right]\left[(1+\tau^c)(1-\lambda) + \alpha(\delta-1)\right] \\ &> \alpha\delta\left[(1+\tau^c)(1-\lambda) + \alpha(\delta-1)\right] > \alpha(\delta-1)[1 + \alpha(\delta-1) + \tilde{\eta}\tau^c] \end{aligned} \tag{6}$$

Thus, for $0 \leq \bar{b} \leq \min\left\{\bar{b}^\flat, \frac{(1-\delta)\tilde{\eta}}{\delta}\right\}$, where $\bar{b} = \bar{b}^\flat$ satisfies $a_{21}a_{32} = a_{22}a_{31}$.

From (6),

$$a_{21}a_{32} - a_{22}a_{31} < 0 \qquad \text{if} \qquad (1+\tau^c)(1-\lambda)\delta > (\delta - 1)(1 - \alpha + \tilde{\eta}\tau^c) \tag{7}$$

From (3) and (7), $a_{32} \leq 0 \Rightarrow a_{21}a_{32} - a_{22}a_{31} < 0$ if

$$\frac{1-\alpha}{1+\beta} + \alpha\delta > 1 - \alpha + \tilde{\eta}\tau^c = \frac{(1-\alpha)[1+\beta(1+\tau^c)]}{1+\beta} \Leftrightarrow \alpha\delta > \frac{\beta(1-\alpha)(1+\tau^c)}{1+\beta}. \tag{8}$$

where we have used $\tilde{\eta} \equiv \beta(1-\alpha)/(1+\beta)$. From $\delta > 1$, we find

$$\alpha\delta > \alpha > \frac{\beta(1-\alpha)(1+\tau^c)}{1+\beta} > \frac{\beta(1-\alpha)}{1+\beta} \tag{9}$$

which leads to $(1+\beta)\alpha > (1-\alpha)\beta \Leftrightarrow \alpha\beta > \frac{\beta-\alpha}{2}$.

2 Additional cases when $\delta > 1$ and $\mu_1 \leq 1$ if we relax Condition 3.

As we note in footnote 17, even if we relax Condition 3 when $\delta > 1$, $\mu_1 \leq 0$ and add the cases of (i) $a_{32} \leq 0$ and $a_{21}a_{32} - a_{22}a_{31} > 0$ and (ii) $a_{32} > 0$ for $0 < \phi \leq 1$, I can characterize the $b_t = m(k_t)$ on the (k_t, b_t) plane and obtain the qualitatively same results to those under policy changes in $\bar{b}$ and ϕ.

We begin with the derivation of $b_t = m(k_t)$ on the (k_t, b_t) plane. Recall that when $\delta > 1$ and $\mu_1 \leq 1$, $a_{22} \leq 0$ is satisfied and Proposition 6-(iii) ensures the uniqueness of steady state, inducing $q(k_t) = \Gamma_k(x_t)$ and $q(k_t) = \Gamma_b(x_t)$ to intersect at $S(x_S^*, q(k_S^*))$.

Step 1 Representation of (G.2) and (G.4) into the $(x_t, q(k_t))$ plane when $\phi \in (0,1)$)

(i) If $a_{32} \leq 0$ and $a_{21}a_{32} - a_{22}a_{31} > 0$ hold,

$$-\underbrace{a_{32}}_{(-)} / \underbrace{a_{22}}_{(-)} < 0, \;\; \Gamma_k(0)(= q(\breve{k})) = -\underbrace{a_{31}}_{(+)} / \underbrace{a_{32}}_{(-)} > 0.$$

Then, $\Gamma_k(x_t) > 0$ for $x_t \geq 0 > -a_{32}/a_{22}$ from (G.2). Furthermore (G.3) with $-a_{32}/a_{22} < 0$, and $\Lambda(0) = -(a_{21}a_{32} - a_{22}a_{31}) < 0$ implies that (i) a positive value of x_t, defined as $x^{\dagger\dagger}(> 0)$, satisfies $\Lambda(x^{\dagger\dagger}) = 0$ and (ii) $\Gamma_k'(x_t) < (\geq) 0$ for $0 < x_t < x^{\dagger\dagger}$ ($x^{\dagger\dagger} \leq x_t$). These properties of $\Gamma_k'(x_t)$ and $\Gamma_k(0) = -a_{31}/a_{32} > 0$ show that $\Gamma_k(x_t)$ is positive and is decreasing (increasing) in x_t for $0 < x_t < x^{\dagger\dagger}$ ($x^{\dagger\dagger} \leq x_t$).

(ii) If $a_{32} > 0$ holds,

$$-\underbrace{a_{32}}_{(+)}/\underbrace{a_{22}}_{(-)} > 0, \quad q(\breve{k}) = -\underbrace{a_{31}}_{(+)}/\underbrace{a_{32}}_{(+)} \leq 0, \quad \underbrace{a_{21}}_{(+)}\underbrace{a_{32}}_{(+)} - \underbrace{a_{22}}_{(-)}\underbrace{a_{31}}_{(+)} > 0.$$

Then, $\Gamma_k(x_t) > (<) 0$ for $x_t \geq (<) - a_{32}/a_{22}$, as well as $\lim_{x_t \uparrow -\frac{a_{32}}{a_{22}}} \Gamma_k(x_t) = -\infty$ and $\lim_{x_t \downarrow -\frac{a_{32}}{a_{22}}} \Gamma_k(x_t) = +\infty$. Furthermore, (G.3) with $-a_{32}/a_{22} > 0$, and $\Lambda(0) = -(a_{21}a_{32} - a_{22}a_{31}) < 0$ implies that (i) a positive value of x_t, defined as $x^{\dagger\dagger\dagger}(> 0)$, satisfies $\Lambda(x^{\dagger\dagger\dagger}) = 0$ and (ii) $\Gamma_k'(x_t) < (\geq) 0$ for $0 < x_t < x^{\dagger\dagger\dagger}$ ($x^{\dagger\dagger\dagger} \leq x_t$). In sum, $\Gamma_k(x_t)$ is positive /negative/ asymptote for $x_t > / < / = -a_{32}/a_{22}$, and is decreasing (increasing) in x_t for $0 < x_t < x^{\dagger\dagger\dagger}$ ($x^{\dagger\dagger\dagger} \leq x_t$). A unique steady-state value of x_t in S (intersection between $\Gamma_k(x_t)$ and $\Gamma_b(x_t)$) is larger than $-a_{32}/a_{22}(> 0)$.

Step 2 translation of (G.2) and (G.4) into the (k_t, b_t) **planes when** $\phi \in (0, 1)$

(i) When $a_{32} \leq 0$ and $a_{21}a_{32} - a_{22}a_{31} > 0$ hold, since $q(k_t) \geq 0$ for $-a_{32}/a_{22} < 0 \leq x_t$, $q(k_t) = \Gamma_k(x_t)$ is transformed into $k_{t+1} = k_t$ locus: $b_t = m(k_t)$ as follows. The point $(x_t, q(k_t)) = (0, -a_{31}/a_{32})$ (in the LHS of the Figures 2-(i) below) which corresponds to the point $(b_t, k_t) = (0, \breve{k})$ (in the RHS of the Figures 2-(i) below). The trajectory of (b_t, k_t), when $(x_t, q(k_t))$ moves from $(0, -a_{31}/a_{32})$ to the final destination $(x_t, q(k_t)) \to (-a_{32}/a_{22}, +\infty)$ along $q(k_t) = \Gamma_k(x_t)$, represents the $k_{t+1} = k_t$ locus: $b_t = m(k_t)$. As x_t increases from 0 through x_S to $-a_{32}/a_{22}$ along $q(k_t) = \Gamma_k(x_t)$, $q(k_t)$ increases from $-a_{31}/a_{32}(= q(\breve{k})$ through $q(k_S^*)$ to $+\infty$. At the same time, k_t increases from $\breve{k}$ to $k^{\dagger\dagger}$ and turn to decrease from $k^{\dagger\dagger}$ through k_S^* to 0 along the $b_t = m(k_t)$, while b_t increases from 0 through $b^{\dagger\dagger}$ and b_S^* to the upper level and turns to decrease and finally takes 0, as shown in the RHS of Figure 2-(i) below.

Furthermore, $a_{22}x_t + a_{32} < 0$ is satisfied because of $-a_{32}/a_{22} < 0 \leq x_t$. Then, (G.2) implies that $k_{t+1} \gtreqless k_t$ if and only if $q(k_t) \gtreqless \Gamma_k(x_t)$ for $0 \leq x_t$. Thus, $k_{t+1} > (<) k_t$ holds above (bellow) $q(k_t) = \Gamma_k(x_t)$, which satisfies $k_{t+1} > (<) k_t$ bellow (above) $b_t = m(k_t)$ ($k_{t+1} = k_t$ locus) correspondingly.

(ii) When $a_{32} > 0$ hold, since $q(k_t) \geq 0$ for $0 < -a_{32}/a_{22} \leq x_t$, $q(k_t) = \Gamma_k(x_t)$ is transformed into $k_{t+1} = k_t$ locus: $b_t = m(k_t)$ as follows. The trajectory of (b_t, k_t), when $(x_t, q(k_t))$ moves from $(-a_{32}/a_{22}, +\infty)$ to $(x_t, q(k_t)) \to (+\infty, +\infty)$ along $q(k_t) = \Gamma_k(x_t)$, represents the $k_{t+1} = k_t$ locus: $b_t = m(k_t)$. As x_t increases from $-a_{32}/a_{22}$ through x_S and $x^{\dagger\dagger\dagger}$ along $q(k_t) = \Gamma_k(x_t)$, $q(k_t)$ decreases from $+\infty$ through $q(k_S^*)$ and $q(k^{\dagger\dagger\dagger})$to $+\infty$. At the same time, k_t increases from 0 through k_S^* to $k^{\dagger\dagger\dagger}$ and turn to decrease from $k^{\dagger\dagger\dagger}$ through to 0 along the $b_t = m(k_t)$, while b_t increases from 0 through b_S^* and $b^{\dagger\dagger}$ to the upper level and turns to decrease and finally takes 0, as shown in the RHS of Figure 2-(ii) below.

Furthermore, $a_{22}x_t + a_{32} < 0$ for $0 < -a_{32}/a_{22} \leq x_t$. Then, (G.2) implies that $k_{t+1} \gtreqless k_t$ if and only if $q(k_t) \gtreqless \Gamma_k(x_t)$ for $0 < -a_{32}/a_{22} \leq x_t$. Thus, $k_{t+1} > (<) k_t$ holds above (bellow) $q(k_t) = \Gamma_k(x_t)$, which satisfies $k_{t+1} > (<) k_t$ bellow (above) $b_t = m(k_t)$ ($k_{t+1} = k_t$ locus) correspondingly.

Step 3 $b_t = m(k_t)$ **when** $\phi = 1$

(i) When $a_{32} = \mu_2 \leq 0$ and $a_{21}a_{32} - a_{22}a_{31} = (1+n)\alpha\delta\mu_2 - \bar{b}^{-1}\mu_1\mu_3 > 0$,

$$a_{21}a_{32} - a_{22}a_{31} = (1+n)\alpha\delta\mu_2 - \bar{b}^{-1}\mu_1\mu_3 > 0,$$
$$-\underbrace{\mu_2}_{(-)} / \underbrace{\mu_1\bar{b}^{-1}}_{(-)} < 0, \;\; \Gamma_k(0)(= q(\breve{k})) = -\underbrace{\mu_3}_{(+)} / \underbrace{\mu_2}_{(-)} > 0 \;\; (a_{32} = \mu_2 < 0).$$

Then, $\Gamma_k(x_t) > 0$ for $x_t \geq 0 > -\mu_2/(\mu_1\bar{b}^{-1})$. Applying $(1+n)\alpha\delta\mu_2 - \bar{b}^{-1}\mu_1\mu_3 > 0$ and $-\mu_2/(\mu_1\bar{b}^{-1}) < 0$ to (G.6), we find that $q(k_t) = \Gamma_k(x_t)$ is monotonically deccreasing in x_t for $x_t \geq 0$ and satisfies $\Gamma_k(0) = q(\breve{k}) = -\mu_3/\mu_2 > 0$ and $\lim_{x_t \to +\infty} \Gamma_k(x_t) = \lim_{x_t \to +\infty} -\frac{(1+n)\alpha\delta}{\bar{b}^{-1}\mu_1} > 0$. The trajectory of (b_t, k_t), when $(x_t, q(k_t))$ moves from $(0, -\mu_3/\mu_2)$ to the final destination $(x_t, q(k_t)) \to (+\infty, -(1+n)\alpha\delta/(\bar{b}^{-1}\mu_1))$ along $q(k_t) = \Gamma_k(x_t)$, represents the $k_{t+1} = k_t$ locus: $b_t = m(k_t)$. Thus, as k_t increases from $\breve{k} = q^{-1}(-\mu_3/\mu_2)$ through k_S^* to $q^{-1}(-(1+n)\alpha\delta/(\bar{b}^{-1}\mu_1))$, b_t increases from 0 through b_S^* to $+\infty$ as represented in the RHS of Figure .

(ii) When $a_{32} = \mu_2 > 0$ holds,

$$a_{21}a_{32} - a_{22}a_{31} = (1+n)\alpha\delta \underbrace{\mu_2}_{(+)} - \bar{b}^{-1} \underbrace{\mu_1}_{(-)} \underbrace{\mu_3}_{(+)} > 0,$$
$$-\underbrace{\mu_2}_{(+)} / \underbrace{\mu_1\bar{b}^{-1}}_{(-)} \geq 0, \;\; \Gamma_k(0)(= q(\breve{k})) = -\underbrace{\mu_3}_{(+)} / \underbrace{\mu_2}_{(+)} > 0 \;\; (a_{32} = \mu_2 > 0).$$

Then, $\Gamma_k(x_t) > 0$ for $x_t \geq -\mu_2/(\mu_1\bar{b}^{-1}) > 0$. Applying $(1+n)\alpha\delta\mu_2 - \bar{b}^{-1}\mu_1\mu_3 > 0$ and $-\mu_2/(\mu_1\bar{b}^{-1}) \geq 0$ to (G.6), we find that $q(k_t) = \Gamma_k(x_t)$ is monotonically deccreasing in x_t

for $x_t \geq -\mu_2/(\mu_1\bar{b}^{-1}) > 0$ and satisfies $\lim_{x_t \downarrow -\frac{\mu_2}{\mu_1\bar{b}^{-1}}} \Gamma_k(x_t) = +\infty$ and $\lim_{x_t \to +\infty} \Gamma_k(x_t) = \lim_{x_t \to +\infty} -\frac{(1+n)\alpha\delta}{\bar{b}^{-1}\mu_1} > 0$.

Step 4 Condition of $k_{t+1} \leq 0$ and $k_{t+1} = 0$ locus

Recall again that $a_{22} \leq 0$ when $\delta > 1$ and $\mu_1 \leq 0$.

(i) When $a_{32} \leq 0$ and $a_{21}a_{32} - a_{22}a_{31} > 0$, $\Gamma_k(x_t) > 0$ and $a_{32} + a_{22}x_t \leq 0$ hold for $-a_{32}/a_{22} < 0 \leq x_t$. Then, (H.2) and (H.3) with $a_{32} + a_{22}x_t \leq 0$ imply that $k_{t+1} \geq 0$ is satisfied as long as $q(k_t) \geq \Theta(x_t)$ and that $\Gamma_k(x_t) \geq \Theta(x_t)$ for $x \geq 0$, respectively.

(ii) When $a_{32} > 0$, $\Gamma_k(x_t) > 0$ and $a_{32} + a_{22}x_t \leq 0$ hold for $0 < -a_{32}/a_{22} \leq x_t$. Then, (H.2) and (H.3) with $a_{32} + a_{22}x_t \leq 0$ imply that $k_{t+1} \geq 0$ is satisfied as long as $q(k_t) \geq \Theta(x_t)$ and that $\Gamma_k(x_t) \geq \Theta(x_t)$ for $x \geq 0$, respectively.

Effects of $\bar{b}$ and ϕ

From (38) and (39) (with Appendix J) Proposition 8 holds immediately. See Figures 3 and 4 below.

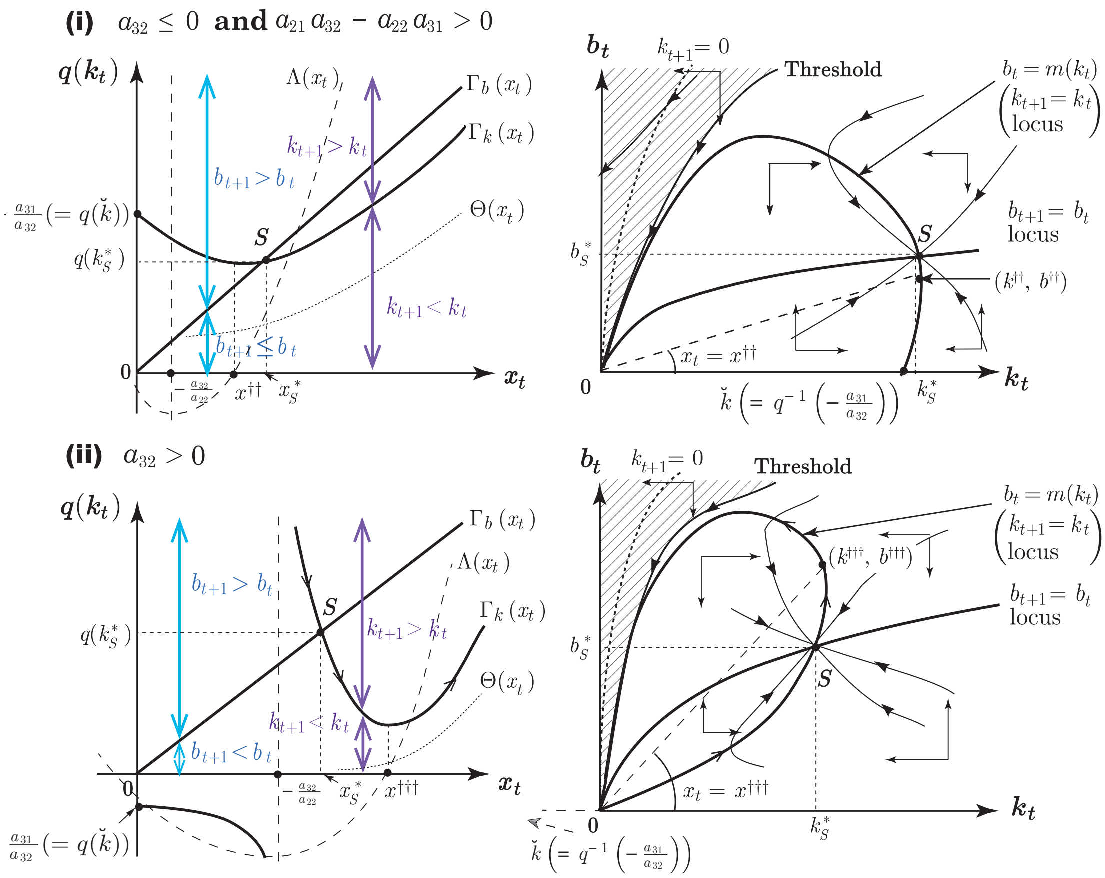


Figure 1: Phase diagram in the case of tax-based consolidation for $\phi \in (0, 1)$: $\delta > 1$ and $\mu_1 \leq 0$ with (i) $a_{32} \leq 0$ and $a_{21}a_{32} - a_{22}a_{31} > 0$ or (ii) $a_{32} > 0$

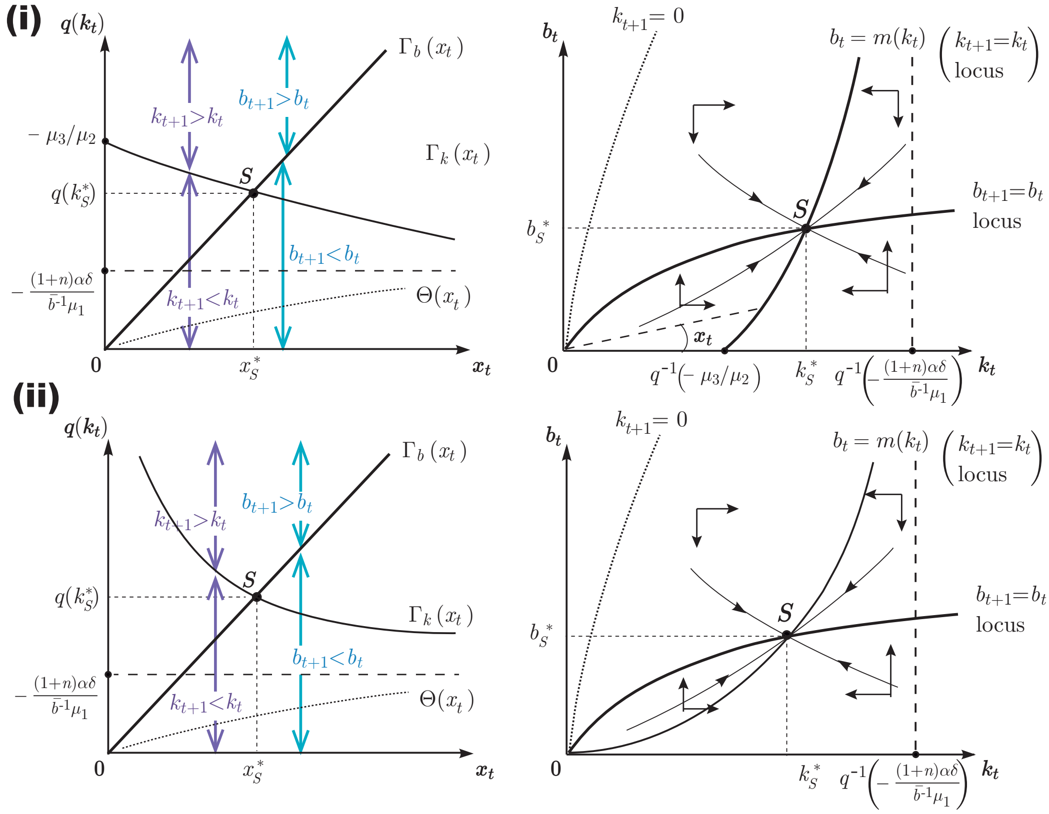


Figure 2: Phase diagram in the case of tax-based consolidation for $\phi = 1$: $\delta > 1$ and $\mu_1 \leq 0$ with (i) $a_{32} \leq 0$ and $a_{21}a_{32} - a_{22}a_{31} > 0$ or (ii) $a_{32} > 0$

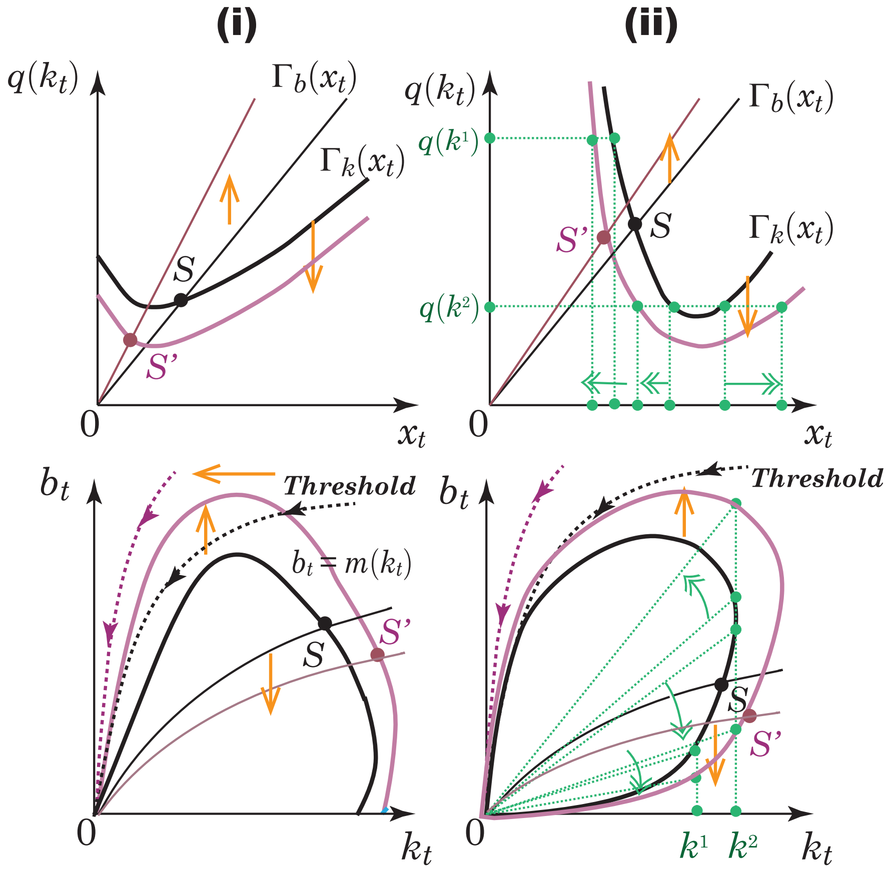


Figure 3: Effects of a reduction in $\bar{b}$ on the sustainability of public debt and the steady state under the tax-base consolidation: $\delta > 1$ and $\mu_1 \leq 0$ with (i) $a_{32} \leq 0$ and $a_{21}a_{32} - a_{22}a_{31} > 0$ or (ii) $a_{32} > 0$

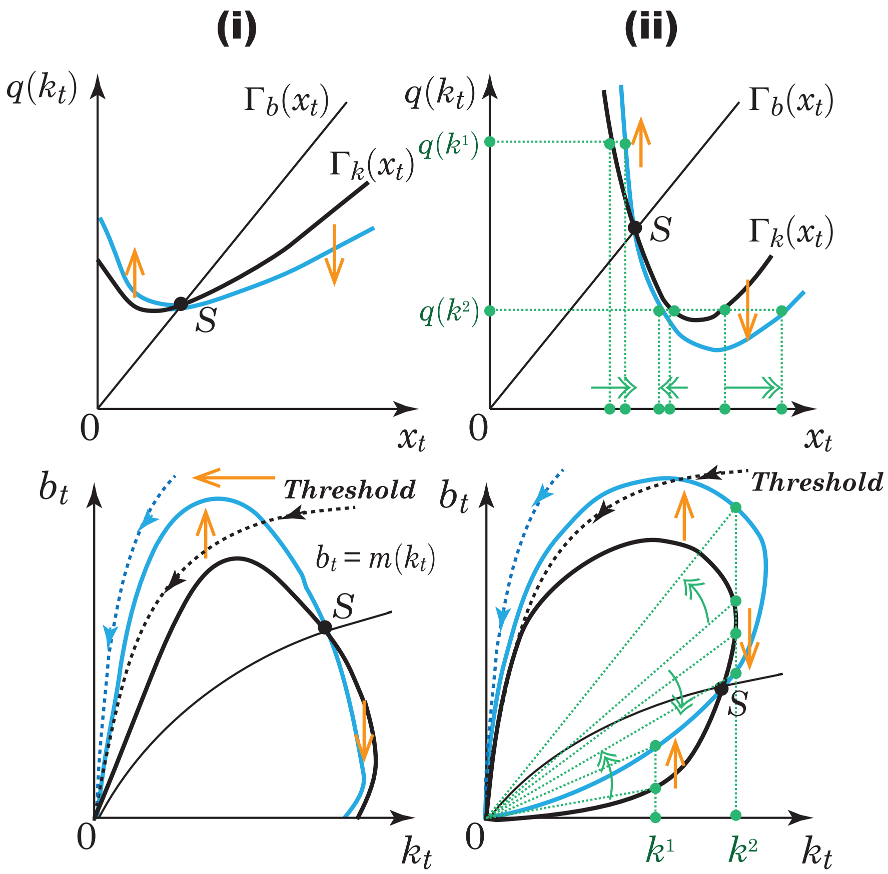


Figure 4: Effects of an increase in ϕ on the sustainability of public debt under the tax-base consolidation: $\delta > 1$ and $\mu_1 \leq 0$ with (i) $a_{32} \leq 0$ and $a_{21}a_{32} - a_{22}a_{31} > 0$ or (ii) $a_{32} > 0$