Three Remarks On Asset Pricing

Olkhov, Victor

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Victor Olkhov

TVEL, Moscow, Russia

victor.olkhov@gmail.com

ORCID: 0000-0003-0944-5113

Abstract

Asset pricing crucially depends on an averaging time interval Δ of the market trade time-series. The choice of Δ changes the basic pricing equation and determines Taylor series of investor's utility functions over current and future values of consumption. We present current and future values of random consumption as sums of the mean values during the interval Δ and perturbations determined by random variations of the price at current moment *t* and the payoff at day *t+1*. Linear and quadratic Taylor series' approximations of the basic pricing equation describe mean price, mean payoff, their volatilities, skewness and the amount of asset ξ_{max} that delivers max to investor's utility. We believe that the stochasticity of the market trade time-series must define the random properties of the price and introduce the new price probability measure entirely determined by the probability measures of trading value and volume. We define the set of *nth* statistical moments of the price as ratio of the *nth* statistical moment of the value to *nth* statistical moment of the volume of the market trades performed during the averaging interval Δ. The set of price statistical moments determines the price characteristic function and its Fourier transform defines the new price probability measure. Prediction of the price probability measure requires forecasts of all statistical moments of the trades. Definition of the price probability expresses the catch phrase "You can't beat the market".

Keywords: asset pricing, volatility, price probability, market trades

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1. Introduction

Forecasting of asset prices define the main problem and desire of investors, traders and all the participants of financial markets. Investors, traders, academic scholars make their best to outrun and get ahead of others in the treatment, guessing and solution of the price puzzles. Last decades give a great progress in asset price valuation and setting. Starting with Hall and Hitch (1939) many researchers investigate the price theory (Friedman, 1990; Heaton and Lucas, 2000) and the factors those impact markets (Fama, 1965), equilibrium economy (Sharpe, 1964), fluctuations (Mackey, 1989) macroeconomics (Cochrane and Hansen, 1992) and business cycles (Mills, 1946; Campbell, 1998). Muth (1961) initiated studies on the dependence of asset pricing on the expectations and numerous scholars developed his ideas further (Lucas, 1972; Malkiel and Cragg, 1980; Campbell and Shiller, 1988; Greenwood and Shleifer, 2014). Many researchers describe the price dynamics and references (Goldsmith and Lipsey, 1963; Campbell, 2000; Cochrane and Culp, 2003; Borovička and Hansen, 2012; Weyl, 2019) give only a small part of them.

Asset pricing depends on price fluctuations and volatility. The mean price trends and the price volatility are the most important issues that impact investors' expectation. Description of volatility is inseparable from price modeling (Hall and Hitch, 1939; Fama, 1965; Stigler and Kindahl, 1970; Tauchen and Pitts, 1983; Schwert, 1988; Mankiw, Romer and Shapiro, 1991; Brock and LeBaron, 1995; Bernanke and Gertler, 1999; Andersen et.al., 2001; Poon and Granger, 2003; Andersen et.al., 2005). The list of references can be continued as hundreds and hundreds of publications describe different faces of the price-volatility puzzle.

Simple and practical advises on the price modeling and forecasting among the most demanded by investors. Different price models were developed to satisfy and saturate investors' desires. We refer only some pricing models (Ferson et.al., 1999; Fama and French, 2015) and studies on Capital Asset Pricing Model (CAPM) (Sharpe, 1964; Merton, 1973; Cochrane, 2001; Perold, 2004). Cochrane (2001) shows that CAPM includes different versions of asset pricing as ICAPM and consumption-based pricing model (Campbell, 2002) are CAPM variations. Further we consider Cochrane (2001) as clear and consistent presentation of CAPM basis, problems and achievements. His resent study (Cochrane, 2021) complements the rigorous asset price description with deep and justified general considerations of the nature, problems and possible directions for further research.

Despite the fact that asset pricing, risk, uncertainties and financial markets were studied with a great accuracy and solidity there are still "some" problems left. We assume that the core economic difficulties and the fundamental economic relations may still impede further significant development of the price theory. To explain the nature of the existing economic obstacles that may hamper price forecasting we consider three remarks that impact asset pricing. It is convenient consider asset pricing having the single reference that describes almost all extensions and model variations within the uniform frame. We propose that readers are sufficiently familiar with CAPM (Cochrane, 2001) and refer this monograph for any clarifications. In our paper we consider the basic pricing equation (Cochrane, 2001) and show why and how some simple conventional notions may be the origin of the tough problems that prevent successful prediction of asset price.

Equation (4.5) means equation 5 in the Sec. 4 and (A.2) – notes equation 2 in Appendix A. We use roman letters A, B, d to denote scalars and bold $\boldsymbol{B}$, $\boldsymbol{P}$, $\boldsymbol{v}$ – to denote vectors. We assume that readers are familiar with basic notions of probability density functions, statistical moments, characteristic functions and etc.

In Sec.2 we remind main CAPM notions. In Sec.3 we consider remarks on the time scales and introduce an interval Δ that determine averaging of the market trades and price time-series, Sec.4 – remarks on Taylor series generated by the averaging interval Δ. We expand the utility functions by Taylor series and in linear and quadratic approximations by the price and payoff variations consider the idiosyncratic risk, the utility max conditions and the impact of price-volume correlations. In Sec.5 we introduce the new price probability measure and briefly consider its implications on asset pricing. Sec.7 – Conclusion. In App.A. we collect some calculations that define maximum of investor's utility.

2. Brief CAPM Assumptions

The general frame that determines all CAPM versions and extensions states: "All asset pricing comes down to one central idea: the value of an asset is equal to its expected discounted payoff" (Cochrane, 2001; Cochrane and Culp, 2003; Hördahl and Packer, 2007; Cochrane 2021). Let's follow (Cochrane, 2001) and briefly consider CAPM assumptions. The basic consumption-based equation has form:

$$p = E[m\,x] \tag{2.1}$$

In (2.1) p denotes the asset price at moment t, $x=p_{t+1}+d_{t+1}$ – payoff,, p_{t+1} - price and d_{t+1} - dividends at moment $t+1$, m - the stochastic discount factor and E – math expectation at moment $t+1$ made by the forecast under the information available at moment t. Cochrane (2001) considers relations (2.1) in various forms to show that almost all models of asset pricing united by the title CAPM can be described by the similar equations. We shall consider (2.1) and refer (Cochrane, 2001) for all other CAPM extensions. For convenience we briefly reproduce consumption-based derivation of (2.1). Cochrane "models investors by a utility function defined over current c_t and future c_{t+1} values of consumption. c_t and c_{t+1} denotes consumption at date t and $t+1$."

$$U(c_t;\ c_{t+1}) = u(c_t) + \beta E[u(c_{t+1})] \tag{2.2}$$

$$c_t = e_t - p\xi \quad ; \quad c_{t+1} = e_{t+1} + x\xi \tag{2.3}$$

$$x = p_{t+1} + d_{t+1} \tag{2.4}$$

Here (2.3) e_t and e_{t+1} "denotes original consumption level (if the investor bought none of the asset), and ξ denotes the amount of the asset he chooses to buy" (Cochrane, 2001). A payoff x (2.4) is determined by a price p_{t+1} and a dividend d_{t+1} of asset at moment $t+1$. Cochrane calls β as "subjective discount factor that captures impatience of future consumption". $E[...]$ in (2.2) denotes math expectation of the random utility due to the random payoff x (2.4) made at moment $t+1$ by forecast on base of information available at moment t. The first-order maximum condition for (2.2) by amount of asset ξ is fulfilled by putting derivative of (2.2) by ξ equals zero (Cochrane, 2001):

$$\max_{\xi} U(c_t;\ c_{t+1}) \leftrightarrow \frac{\partial}{\partial \xi} U(c_t;\ c_{t+1}) = 0 \tag{2.5}$$

From (2.2-2.5) it is obvious that:

$$p = \beta E\left[\frac{u'(c_{t+1})}{u'(c_t)}\, x\right] = E[mx] \quad ; \quad m = \beta \frac{u'(c_{t+1})}{u'(c_t)} \quad ; \quad \frac{d}{dc} u(c) = u'(c) \tag{2.6}$$

and (2.6) reproduces (2.1) for m (2.6). This completes the brief derivation of the basic equation (2.1; 2.6) and we refer Cochrane (2001) for any further details.

3. Remarks on Time Scales

We start with simple remarks on mathematical expectations and time scales. Any mathematical expectations of the market trade and price time-series delivers the mean values averaged during certain time interval Δ. The averaging procedure can be different but any such procedure aggregates the time-series during certain interval Δ. The choice of the averaging interval Δ may define different mean values. The choice of the averaging interval Δ defines the *internal* time scale of the problem under consideration. The time-horizon T of the asset pricing at "the next day" $t+1 = t+T$ defines the *external* time scale of the problem. Relations between the *internal* Δ and *external* T scales determine evolution of the averaged variables, sustainability and accuracy of the model description. The financial variables – price, volatility, beta – averaged during the interval Δ can behave irregular or randomly on time scales T for $T \gg \Delta$. This effect mentioned, for example, by Cochrane (2021): "Another great puzzle is how little we know about betas. In continuous-time diffusion theory, 10 seconds of millisecond data should be enough to measure betas with nearly infinite precision. In fact, betas are hard to measure and unstable over time". It's clear that if market disturbing factors have a time scale d and $d > \Delta$ then averaging during the interval Δ smooth only the perturbations with scales less than Δ. If market is under impact of the perturbations with the scales d and $\Delta < d < T$, then variables averaged during the interval Δ will be disturbed over the scales $d > \Delta$ and will demonstrate irregular or random properties during the term T. It is clear that dynamics of the price, payoff and discount factor are under impact of the factors with different time scale disturbances. Eventually, the choice of the averaging interval Δ is important for asset pricing modeling, but sadly it is not the main trouble.

As we note the averaging interval Δ defines the *internal* time scale of the problem. In simplest case averaging of the price time-series during the interval Δ that equals 1 hour, 1 day, 1 week establish the least time divisions of "the Clocks" equals 1 hour, 1 day, 1 week.

It is reasonable to use the same time scale divisions "to-day" at moment t and the "next-day" at $t+1$. Time scale divisions can't be measured "to-day" in hours and "next-day" in weeks. Hence time-series of investor's utility function should be aggregated during the interval Δ "to-day" at moment t and the "next-day" at $t+1$ and hence the utility (2.2) "to-day" at moment t and the "next-day" at $t+1$ should have the same time divisions. If (2.2) is averaged at the "next-day" at $t+1$ using the interval Δ then it should be averaged "to-day" at moment t with the same interval Δ. Averaging of any time-series at the "next-day" at $t+1$ undoubtedly implies averaging "to-day" at moment t and vise-versa. Thus the utility (2.2) should be averaged at moment t and take form

$$U(c_t;\ c_{t+1}) = E_t[u(c_t)] + \beta E[u(c_{t+1})] \tag{3.1}$$

We denote $E_t[..]$ in (3.1) as mathematical expectation "to-day" at moment t during interval Δ. It doesn't matter how one considers the price time-series "to-day" – as random or as irregular. Mathematical expectation $E_t[..]$ performs smoothing of the random or irregular time-series via aggregating data during the interval Δ under a particular price probability measure. Mathematical expectations $E_t[..]$ and $E[..]$ within identical averaging intervals Δ establish identical time division of the problem at moment t at moment $t+1$ in (3.1). Hence, relations similar to (2.5; 2.6) derive the basic equation in the form:

$$E_t[p\, u'(c_t)] = \beta E\, [x\, u'(c_{t+1})] \tag{3.2}$$

Cochrane (2001) takes "subjective discount factor" β as non-random and we follow here same assumption. Important note – mathematical expectations in the left and in the right sides of (3.2) are determined by different probability measures but with identical averaging interval Δ. In the left side $E_t[...]$ assesses mean price p at moment t. In the right side $E[...]$ on base of data available at moment t forecasts the average of $[xu'(c_{t+1})]$ at moment $t+1$ within the averaging interval Δ.

4. Remarks on Taylor series

The relation (2.5) presents first-order condition at point ξ_{max} that delivers maximum to investor's utility (2.2) or (3.1). Let us choose the averaging interval Δ and take the price p at moment t during the interval Δ and the payoff x at moment $t+1$ during the interval Δ as:

$$p \;=\; p_0 + \delta p\,; \quad x = x_0 + \delta x \tag{4.1}$$

$$E_t[p] = p_0\,;\; E[x] = x_0\,;\; E_t[\delta p] = E[\delta x] = 0\,;\; \sigma^2(p) = E_t[\delta^2 p]\;\;;\;\; \sigma^2(x) = E[\delta^2 x] \tag{4.2}$$

The relations (4.1; 4.2) give the average price p_0 and its volatility $\sigma^2(p)$ at moment t and the average payoff x_0 its volatility $\sigma^2(x)$ at moment $t+1$. We underline that consider averaging during the interval Δ as averaging of a random or as smoothing of an irregular behavior of any variable. Thus $E_t[p]$ – at moment t smooth random or irregular price p during the interval Δ and $E[x]$ – averages the random payoff x during Δ at moment $t+1$. We call both procedures as mathematical expectations. We remind that $E_t[..]$ is averaging during the interval Δ at moment t and $E[..]$ is a forecast of averaging during Δ at moment $t+1$ using data available at moment t. We assume that the price fluctuations δp at moment t during Δ and the payoff fluctuations δx at moment $t+1$ during Δ are small to compare with their mean values during Δ. We present the derivatives of utility functions in (3.2) by Taylor series in linear approximation by δp and δx during Δ:

$$u'(c_t) = u'(c_{t;0}) - \xi u''(c_{t;0})\delta p \quad;\quad u'(c_{t+1}) = u'(c_{t+1;0}) + \xi u''(c_{t+1;0})\delta x \tag{4.3}$$

$$c_{t;0} = e_t - p_0\xi \quad;\quad c_{t+1;0} = e_{t+1} + x_0\xi$$

Now substitute (4.3) into (3.2) and obtain equation (4.4):

$$u'(c_{t;0})p_0 - \xi u''(c_{t;0})\sigma^2(p) = \beta u'(c_{t+1;0})x_0 + \beta\xi u''(c_{t+1;0})\sigma^2(x) \quad (4.4)$$

Taylor series are simplest entry-level mathematical tool like as ordinary derivatives and we see no sense refer any studies those use Taylor or ordinary derivatives in asset pricing. However, Cochrane (2001) uses Taylor expansions. We underline the important issue: Taylor series and (4.1-4.4) are determined by the choice of the averaging interval Δ. The change of Δ implies change of the mean price p_0, the mean payoff x_0 and their volatilities $\sigma^2(p)$, $\sigma^2(x)$ (4.2). Equation (4.4) is a linear approximation by the price and payoff fluctuations of the first-order max conditions (2.5) and assesses the root ξ_{max} that delivers maximum to the utility $U(c_t;c_{t+1})$ (3.1)

$$\xi_{max} = \frac{u'(c_{t;0})p_0 - \beta u'(c_{t+1;0})x_0}{u''(c_{t;0})\sigma^2(p) + \beta u''(c_{t+1;0})\,\sigma^2(x)} \quad (4.5)$$

We note that (4.5) is not "exact" equation on ξ_{max} as utilities u' and u'' also depend on ξ_{max} as it follows from (4.3). However, (4.5) gives certain assessment of ξ_{max} in a linear approximation by Taylor series δp and δx averaged during Δ. Let's underline that the ξ_{max} (4.5) depends on the price volatility $\sigma^2(p)$ at moment t and on the payoff volatility $\sigma^2(x)$ at moment $t+1$ measured during the interval Δ (4.2).

It is clear that sequential iterations may give more accurate approximations of ξ_{max}. Nevertheless, our approach and (4.5) give a new look on the basic equation (2.6; 3.2). If one follows the standard derivation of (2.6) (Cochrane, 2001) and neglects the averaging at moment t in the left-side of (3.2) then (2.6; 4.5) give

$$\xi_{max} = \frac{u'(c_t)p - \beta u'(c_{t+1;0})x_0}{\beta u''(c_{t+1;0})\sigma^2(x)} \quad (4.6)$$

The relations (4.6) show that even the standard form of the basic equation (2.6) hides dependence of ξ_{max} on the payoff volatility $\sigma^2(x)$ at moment $t+1$. If one has the independent assessment of ξ_{max} then can use it to present (4.6) in a way alike to the basic equation (2.6)

$$p = \frac{u'(c_{t+1;0})}{u'(c_t)}\beta x_0 + \xi_{max}\frac{u''(c_{t+1;0})}{u'(c_t)}\beta\sigma^2(x) \quad (4.7)$$

One can transform (4.7) alike to (2.6):

$$p = m_0 x_0 + \xi_{max} m_1 \sigma^2(x) \quad (4.8)$$

$$m_0 = \frac{u'(c_{t+1;0})}{u'(c_t)}\beta \ ; \ m_1 = \frac{u''(c_{t+1;0})}{u'(c_t)}\beta \quad (4.9)$$

For the given ξ_{max} equation (4.8) in a linear approximation by Taylor series describes dependence of the price p at moment t on the mean discount factors m_0 and m_1 (4.9), the mean payoff x_0 (4.1) and the payoff volatility $\sigma^2(x)$ during the interval Δ. Let us underline that while the mean discount factor $m_0>0$, the mean discount factor $m_1<0$ because the utility $u'(c_t)>0$ and $u''(c_t)<0$ for all t. Hence, for (4.8) valid:

$$p < m_0 x_0 \quad ; \quad \xi_{max} m_1 \sigma^2(x) < 0$$

We underline that (4.6-4.9) have sense for the given value of ξ_{max}. Equation (4.8) in a linear approximation by Taylor series δx during the interval Δ describes the modified CAPM statement: *the value of an asset is equal the mean payoff x_0 discounted by the mean factor m_0 minus payoff volatility $\sigma^2(x)$ discounted by factor $|m_1|$ and multiplied by the amount of asset ξ_{max} that delivers maximum to the investor's utility (2.2).* As the price p in (4.8) should be positive hence ξ_{max} should obey inequality (4.10):

$$0 < \xi_{max} < -\frac{u'(c_{t+1;0})}{u''(c_{t+1;0})} \frac{x_0}{\sigma^2(x)} \tag{4.10}$$

Taking into account (4.3) it is easy to show for (4.10) that for the conventional power utility (Cochrane, 2001) (A.2):

$$u(c) = \frac{1}{1-\alpha} c^{1-\alpha} \quad ; \quad \frac{u'(c)}{u''(c)} = -\frac{c}{\alpha} \quad ; \quad 0 < \alpha \leq 1$$

inequality (4.10) valid always if

$$\frac{\sigma^2(x)}{x_0^2} < \frac{1}{\alpha} \quad ; \quad 0 < \alpha \leq 1$$

For this approximation (4.10) limits the value of ξ_{max}. If one takes (4.5) then obtains equations similar to (4.8; 4.9):

$$m_0 = \frac{u'(c_{t+1;0})}{u'(c_{t;0})}\beta > 0\,; \;\; m_1 = \frac{u''(c_{t+1;0})}{u'(c_{t;0})}\beta < 0\,; \;\; m_2 = \frac{u''(c_{t;0})}{u'(c_{t;0})} < 0 \tag{4.11}$$

$$p_0 = m_0 x_0 + \xi_{max}[m_1 \sigma^2(x) + m_2 \sigma^2(p)] \tag{4.12}$$

We use the same notions m_0, m_1 to denote the discount factors taking into account replacement of $u'(c_t)$ in (4.9) by $u'(c_{t;0})$ in (4.11; 4.12). Modified basic equation (4.12) at moment t describes dependence of the price p_0 averaged during the interval Δ on the price volatility $\sigma^2(p)$ at moment t, the mean payoff x_0 and the payoff volatility $\sigma^2(x)$ at moment $t+1$ averaged during same interval Δ.

Equation (4.15) reproduces well-known practice that high volatility $\sigma^2(p)$ of the price at moment t and the forecast of high payoff volatility $\sigma^2(x)$ at moment $t+1$ may cause decline of the mean price p_0 at moment t. We leave the detailed analysis of (4.5-4.12) for the future.

4.1 The Idiosyncratic Risk

Here we briefly consider the case of the idiosyncratic risk for which the payoff x in (2.6) is not correlated with the discount factor m at moment $t+1$ (Cochrane, 2001):

$$cov(m,x) = 0 \tag{4.13}$$

In this case equation (2.6) takes form:

$$p = E[mx] = E[m]E[x] + cov(m,x) = E[m]x_0 = \frac{x_0}{R_f} \tag{4.14}$$

The risk-free rate R^f in (4.14) is known ahead. Taking into account (4.3) in a linear approximation by δx Taylor series for the derivative of the utility $u'(c_{t+1})$:

$$u'(c_{t+1}) = u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x \tag{4.15}$$

Hence, the discount factor m (2.6) takes form:

$$m = \beta\frac{u'(c_{t+1})}{u'(c_t)} = \frac{\beta}{u'(c_t)}\left[u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x\right]$$

$$E[m] = \bar{m} = \beta\frac{u'(c_{t+1;0})}{u'(c_t)}$$

$$\beta E\left[\frac{u'(c_{t+1})}{u'(c_t)}\right]x_0 = \frac{x_0}{R_f} \quad ; \quad E[u'(c_{t+1})x] = 0$$

and

$$\delta m = m - \bar{m} = \frac{\beta}{u'(c_t)}u''(c_{t+1;0})\xi\delta x$$

Hence, (4.13) implies:

$$cov(m,x) = E[\delta m\delta x] = \beta\frac{u''(c_{t+1;0})}{u'(c_t)}\xi_{max}\sigma^2(x) = 0 \tag{4.16}$$

That causes zero payoff volatility $\sigma^2(x)=0$. Of course zero payoff volatility does not model market reality but reflect restrictions of the linear approximation (4.15). To overcome this discrepancy let us take into account Taylor series up to the second degree by $\delta^2 x$:

$$u'(c_{t+1}) = u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x + u'''(c_{t+1;0})\xi^2\delta^2 x \tag{4.17}$$

$$m = \beta\frac{u'(c_{t+1})}{u'(c_t)} = \frac{\beta}{u'(c_t)}\left[u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x + u'''(c_{t+1;0})\xi^2\delta^2 x\right] \tag{4.18}$$

For this case the mean discount factor $E[m]$ takes form:

$$E[m] = \bar{m} = \frac{\beta}{u'(c_t)}\left[u'\left(c_{t+1;0}\right) + u'''\left(c_{t+1;0}\right)\xi^2\sigma^2(x)\right] \tag{4.19}$$

and variations of the discount factor δm:

$$\delta m = m - \bar{m} = \frac{\beta}{u'(c_t)}\ \left[u''\left(c_{t+1;0}\right)\xi\delta x + u'''\left(c_{t+1;0}\right)\xi^2\{\delta^2 x - \sigma^2(x)\}\right.$$

In this case

$$cov(m,x) = E[\delta m\delta x] = \ \left[u''\left(c_{t+1;0}\right)\xi\sigma^2(x) + u'''\left(c_{t+1;0}\right)\xi^2\ \gamma^3(x)\ \right] = 0 \tag{4.20}$$

$$\gamma^3(x) = E[\delta^3 x] \quad ; \quad Sk(x) = \frac{\gamma^3(x)}{\sigma^3(x)} \tag{4.21}$$

Sk(x) – denotes normalized payoff skewness at moment *t+1* treated as the measure of asymmetry of the probability distribution during the averaging interval Δ. For approximation (4.18) from (4.20; 4.21) obtain relations on the skewness *Sk(x)* and ξ_{max}:

$$\xi_{max}\ Sk(x)\sigma(x) = -\frac{u''(c_{t+1;0})}{u'''(c_{t+1;0})} \tag{4.22}$$

For the conventional power utility (A.2) and (4.3) relations (4.22) take form

$$\xi_{max} = \frac{e_{t+1}}{(1+\alpha)Sk(x)\sigma(x) - x_0} \tag{4.23}$$

It is assumed that second utility derivative $u''(c_{t+1})<0$ always negative and third derivative $u'''(c_{t+1})>0$ is positive and hence the right side in (4.22) is positive. Hence to get positive ξ_{max} for (4.23) for the power utility (A.2) the payoff skewness *Sk(x)* should obey inequality (4.24) that defines the lower limit of the payoff skewness *Sk(x)*:

$$Sk(x) > \frac{x_0}{(1+\alpha)\sigma(x)} \tag{4.24}$$

In (4.14) R_f denotes known risk-free rate. Hence, (4.19; 4.22; 4.24) define relations:

$$\frac{\beta}{u'(c_t)}\left[u'\left(c_{t+1;0}\right) + u'''\left(c_{t+1;0}\right)\xi_{max}{}^2\sigma^2(x)\right] = \frac{1}{R_f}$$

$$\xi_{max}{}^2\sigma^2(x) = \frac{1}{\beta R_f}\frac{u'(c_t)}{u'''\left(c_{t+1;0}\right)} - \frac{u'\left(c_{t+1;0}\right)}{u'''\left(c_{t+1;0}\right)}$$

$$Sk^2(x) = \frac{R_f}{1 - m_0 R_f}\frac{m_1^2}{m_3} > \frac{x_0^2}{(1+\alpha)^2\sigma^2(x)} \quad ; \quad m_0 < 1/R_f$$

$$\frac{\sigma^2(x)}{x_0^2} > \frac{m_3}{m_1^2}\frac{1 - m_0 R_f}{(1+\alpha)^2 R_f} \tag{4.25}$$

Inequality (4.25) establishes the lower limit on the payoff volatility $\sigma^2(x)$ normalized by the mean payoff x_0^2. The lower limit in the right side of (4.25) is determined by the discount factors (4.26), the risk-free rate R_f and the conventional power utility' factor α (A.2).

$$m_0 = \beta \frac{u'(c_{t+1;0})}{u'(c_t)} \;\; ; \; m_1 = \beta \frac{u''(c_{t+1;0})}{u'(c_t)} \;\; ; \;\; m_3 = \beta \frac{u'''(c_{t+1;0})}{u'(c_t)} \tag{4.26}$$

The coefficients in (4.26) differ a little from (4.1) as (4.26) takes the denominator $u'(c_t)$ instead of $u'(c_{t;0})$ in (4.11) but we use the same letters to avoid extra notations. The similar calculations for (3.1; 3.2) describe both the price volatility $\sigma^2(p)$ and the skewness $Sk(p)$ at moment t and the payoff volatility $\sigma^2(x)$ and the skewness $Sk(x)$ at moment $t+1$. Further approximations by Taylor series of the utility derivative $u'(c_t)$ up to $\delta^3 p$ and $u'(c_{t+1})$ up to $\delta^3 x$ similar to (4.17) give assessments of kurtosis of the price probability at moment t and kurtosis of the payoff probability at moment $t+1$ estimated during interval Δ. We leave these exercises for future.

4.2 The Utility Maximum

The relations (2.5) define the first-order condition that determines the amount of asset ξ_{max} that delivers the max to the utility $U(c_t;\, c_{t+1})$ (2.2; 3.1). To confirm that the utility $U(c_t;\, c_{t+1})$ has max at ξ_{max} the first order condition (2.5) must be supplemented by condition:

$$\frac{\partial^2}{\partial \xi^2} U(c_t;\; c_{t+1}) < 0 \tag{4.27}$$

Usage of (4.27) give interesting consequences. From (2.2–2.4) and (4.27) obtain:

$$p^2 > -\frac{\beta}{u''(c_t)} E[x^2\, u''(c_{t+1})\,] \tag{4.28}$$

Take the linear Taylor series expansion of the second derivative of the utility $u''(c_{t+1})$ by δx

$$u''(c_{t+1}) = u''\big(c_{t+1;0}\big) + u'''\big(c_{t+1;0}\big)\xi\delta x$$

Then (4.28) takes form:

$$p^2 > -\beta \frac{u''(c_{t+1;0})}{u''(c_t)} [{x_0}^2 + \sigma^2(x)] - \beta \frac{u'''(c_{t+1;0})}{u''(c_t)} \xi_{max}\, [2x_0\sigma^2(x) + \gamma^3(x)] \tag{4.29}$$

For the power utility (A.2) simple calculations (see App.A) give relations on (4.27; 4.29). If the payoff volatility $\sigma^2(x)$ normalized by mean payoff x_0^2 is small (4.30; A.5)

$$\frac{\sigma^2(x)}{{x_0}^2} < \frac{1}{1+2\alpha} \quad ; \quad \frac{1}{3} \leq \frac{1}{1+2\alpha} < 1 \tag{4.30}$$

then (4.29) valid always. If payoff volatility $\sigma^2(x)$ is high (A.6)

$$\frac{\sigma^2(x)}{{x_0}^2} > \frac{1}{1+2\alpha}$$

then (4.29) valid only for ξ_{max} (A.6):

$$\xi_{max} < \frac{e_{t+1}[{x_0}^2 + \sigma^2(x)]}{x_0\, [(1+2\alpha)\sigma^2(x) - {x_0}^2]}$$

However, this upper limit for ξ_{max} can be high enough. The same but more complex considerations can be presented for (3.1).

$$E_t[p^2 u''(c_t)] < -E[\beta x^2\, u''(c_{t+1})\,]$$

4.3 Price-Volume Correlations

Almost all economic and financial notions and relations interfere with others. The basic equation (2.6) is not an exception. Accidently or not but the market trade price-volume correlations impact the basic equation and utility max. The market price-volume correlations are studied for decades (Ying, 1966; Karpoff, 1987; Gallant, Rossi and Tauchen, 1992; DeFusco, Nathanson and Zwick, 2017). Researchers report the evidence as for the positive as well for the negative price-volume correlations for the different time terms, assets and markets. For example, Ying (1966): «A large volume is usually accompanied by a rise in price". Karpoff (1987) collected data from numerous studies since 1963 till 1987 that support positive correlation between the price change and the volume $(\partial p/\partial \xi > 0)$ (Table 1, p.113) and also data that don't support positive correlations (Table 2, p.118). The price-volume correlations reflect the price dynamics with the volume. If the price grows up with the trading volume growth then correlations are positive and $\partial p/\partial \xi > 0$. If the price declines then correlations are negative and $\partial p/\partial \xi < 0$. No price-volume correlations mean $\partial p/\partial \xi = 0$.

CAPM framework (Cochrane, 2001) presupposes no price-volume correlations and takes $\partial p/\partial \xi = 0$. It is assumed that investor at moment t can by any amount of assets ξ at price p. However, reality is much more complex and the market price-volume correlations should be considered at least in a linear approximation. To simplify our consideration let's take into account the price-volume correlation at moment t only. For the utility function (2.2) relations (2.5; 2.6) take form:

$$\frac{\partial p}{\partial \xi}\xi_{max} = \beta E\left[x\frac{u'(c_{t+1})}{u'(c_t)}\right] - p \tag{4.31}$$

If there are no correlations and $\partial p/\partial \xi = 0$ then (4.31) coincides with (2.6). Otherwise we obtain interesting conditions on sign of the price-volume correlations. It is obvious, that the optimal amount of assets ξ_{max} that delivers the max to investor's utility function (2.2; 2.5) should be positive, at least till investor goes long. Hence the sign of the right side determines the sign of price-volume correlations:

$$\text{If} \quad \beta E\left[x\frac{u'(c_{t+1})}{u'(c_t)}\right] - p > 0 \text{ then } \frac{\partial p}{\partial \xi} > 0 \text{ - otherwise } \xi_{max} < 0 \tag{4.32}$$

$$\text{If} \quad \beta E\left[x\frac{u'(c_{t+1})}{u'(c_t)}\right] - p < 0 \text{ then } \frac{\partial p}{\partial \xi} < 0 \text{ - otherwise } \xi_{max} < 0 \tag{4.33}$$

If one takes the utility function (3.1) then equation (4.31) and relations (4.32; 4.33) on the sign of the price-volume correlations take form:

$$E\left[u'(c_t)\frac{\partial p}{\partial \xi}\right]\xi_{max} = \beta E[x\, u'(c_{t+1})] - E[u(c_t)p] \tag{4.34}$$

As $u'(c_t)>0$ then the sign of $\partial p/\partial \xi$ determines the sign of the left side and hence:

$$\text{If} \quad \beta E\left[x\frac{u'(c_{t+1})}{u'(c_t)}\right] - E_t[u(c_t)p] > 0 \text{ then } \frac{\partial p}{\partial \xi} > 0 \text{ - otherwise } \xi_{max} < 0$$

$$\text{If} \quad \beta E\left[x\frac{u'(c_{t+1})}{u'(c_t)}\right] - E_t[u(c_t)p] < 0 \text{ then } \frac{\partial p}{\partial \xi} < 0 \text{ - otherwise } \xi_{max} < 0$$

Thus the effects of the price-volume correlations may change the form of the basic equation (4.31; 4.34) and the sign of the second derivative of the utility (4.28) may violate existence of positive ξ_{max}. For the utility (2.2) relations (4.28) in a linear approximation by price derivative $\partial p/\partial \xi$ take form:

$$2\frac{\partial p}{\partial \xi}\left(u'(c_t) - u''(c_t)\xi_{max}\right) > u''(c_t)p^2 + \beta E\left[x^2\, u''(c_{t+1})\right] \tag{4.35}$$

In (4.35) we neglect all second derivatives and second degrees of the price derivative $\partial p/\partial \xi$. The utility derivative $u'(c_t)>0$ is always positive and the second derivative $u''(c_t)<0$ is always negative. Hence, the right side (4.35) always negative and left side always positive if price-volume correlations $\partial p/\partial \xi >0$. However, (4.35) may be wrong for the case with negative price-volume correlations $\partial p/\partial \xi <0$ if:

$$2\frac{\partial p}{\partial \xi} < \frac{u''(c_t)p^2 + \beta E\left[x^2\, u''(c_{t+1})\right]}{u'(c_t) - u''(c_t)\xi_{max}}$$

High negative $\partial p/\partial \xi <0$ may violate existence of the negative second derivative of utility and thus violate the existence of utility max. We assume that mutual impact of the price-volume correlations and the basic CAPM equation deserves further investigations.

5. Remarks on the Price Probability Measure

As usual the problems that are the most common and "obvious" hide the most difficulties. The price probability measure is exactly the case of such hidden complexity.

All asset pricing models and CAPM in particular assume that it is possible to forecast the probability measures of random price *p* and payoff *x*. Let's consider the price *p* probability measure only as it alone delivers enough complexity. All assets pricing models use certain price probability measures and their forecasts. We consider the choice and forecast of the price probability measure as most interesting, important and complex problem of finance.

The usual and the conventional treatment of the price *p* probability "is based on the probabilistic approach and using A. N. Kolmogorov's axiomatic of probability theory, which is generally accepted now" (Shiryaev, 1999). The conventional definition of the price probability is based on the frequency of events – frequency of trades at a price *p* during the averaging interval Δ. The economic ground for such a choice is simple: it is assumed that each of *N* trades performed during the averaging interval Δ have equal probability ~ $1/N$. Hence if there are $n(p)$ trades at the price *p* then the probability $P(p)$ of the price *p* during the interval Δ equals $n(p)/N$. The frequency of the particular event is absolutely correct, general and conventional approach to the probability definition. The conventional frequency-based approach to the price probability uses different assumptions on form of the price probability measure and checks how almost all known standard probabilities fit the random market price. As standard probabilities we refer (Walck, 2007; Forbes et.al, 2011). Parameters that define standard probabilities permit calibrate each in a manner that increase the plausibility and consistency with the observed market trading time-series. For different assets, options and markets different standard probabilities are tested and applied to fit and predict the random price dynamics as well as possible. Actually, all such hypothesis on the price probability do not model the market origin of trading stochasticity.

However, financial markets do not accept anything standard and one may ask a simple question: does the conventional frequency-based price probability definition fit the random market pricing? The asset price is a result of the market trades and it seems more reasonable that the random trades should govern the market price stochasticity. We propose the new price probability measure that is different from the conventional frequency based probability and is entirely determined by the probabilities of the market trades values and volumes.

Let us note that almost 30 years ago the volume weighted average price (VWAP) was introduced and is widely used now (Berkowitz et.al 1988; Buryak and Guo, 2014; Busseti and Boyd, 2015; Duffie and Dworczak, 2018; CME Group, 2020). The definition of the VWAP during the interval Δ is follows. Let us take that during the interval $\Delta(t)$ (5.3) at moments $t_i,\ i=1,\ldots N(t)$ were performed $N(t)$ market trades.. Then the VWAP $p(1;t)$ (5.1) at moment *t* during the averaging interval Δ equals

$$p(1;t) = \frac{1}{U(1;t)} \sum_{i=1}^{N(t)} p(t_i)U(t_i) = \frac{C(1;t)}{U(1;t)} \quad ; \quad C(t_i) = p(t_i)U(t_i) \tag{5.1}$$

$$C(1;t) = \sum_{i=1}^{N(t)} C(t_i) = \sum_{i=1}^{N(t)} p(t_i)\,U(t_i) \quad ; \quad U(1;t) = \sum_{i=1}^{N(t)} U(t_i) \tag{5.2}$$

$$\Delta(t) = \left[t - \frac{\Delta}{2}\,, t + \frac{\Delta}{2}\right] \quad ; \quad t_i \in \Delta(t)\,,\ i = 1, \dots N(t) \tag{5.3}$$

The relations (5.1) at moment t_i define the price $p(t_i)$ of the trade with the value $C(t_i)$ and the volume $U(t_i)$. The sum $C(1;t)$ of values $C(t_i)$ (5.2) and sum $U(1;t)$ of volumes $U(t_i)$ (5.2) of $N(t)$ trades during the interval $\Delta(t)$ (5.3) define the VWAP $p(1;t)$ (5.1).

It is obvious, that VWAP (5.1) equally determined (5.4) by the mean value $C_m(1;t)$ (5.5) and the mean volume $U_m(1;t)$ (5.6) of N trades performed during the interval Δ:

$$C_m(1;t) = p(1;t)U_m(1;t) \tag{5.4}$$

$$C_m(1;t) = \sum_{i=1}^{N(t)} C(t_i)P\big(C(t_i)\big) = \frac{1}{N}\sum_{i=1}^{N(t)} C(t_i) \quad ; \quad P\big(C(t_i)\big) = 1/N \tag{5.5}$$

$$U_m(1;t) = \sum_{i=1}^{N(t)} U(t_i)\,P\big(U(t_i)\big) = \frac{1}{N}\sum_{i=1}^{N(t)} U(t_i) \quad ; \quad P\big(U(t_i)\big) = 1/N \tag{5.6}$$

The mean value $C_m(1;t)$ (5.5) and the mean volume $U_m(1;t)$ (5.6) of N trades performed during the interval Δ are determined by the conventional frequency-based probabilities (5.6). Each trade has the same probability $1/N$ and probabilities of the value $P(C(t_i))$ and the volume $P(U(t_i))$ of each trade at moment t_i equal $1/N$. The price $p(1;n)$ (5.4) is a coefficient between the mean value $C_m(1;t)$ (5.5) and the mean volume $U_m(1;t)$ (5.6).

It is hard to believe that properties of the random of price $p(t_i)$ may be independent from the trading stochasticity. We propose that trading time-series that record the values $C(t_i)$ and the volumes $U(t_i)$ should determine the price probability measure. It seems reasonable that any asset pricing theory and CAPM in particular should follow laws of the probability distributions of trading value $C(t_i)$ and volume $U(t_i)$. That can make the asset pricing theory more market justified and more market related.

Let us note the trivial relation for each trade as result of (5.1):

$$C^n(t_i) = p^n(t_i)U^n(t_i) \quad ; \quad n = 1, 2, \dots \tag{5.7}$$

We underline that VWAP $p(1;t)$ (5.4) implies that the random trade value $U(t_i)$ and the random price $p(t_i)$ (5.1; 5.7) are not correlated (5.8). We define nth statistical moment $p(n;t)$ of the price similar to (5.4; 5.7) as coefficient between nth statistical moment $C_m(n;t)$ of the value and nth statistical moment $U_m(n;t)$ of the volume

$$C_m(n;t) = p(n;t)U_m(n;t) \tag{5.8}$$

We define *n-th* statistical moments of the value $C_m(n;t)$ and the volume $U_m(n;t)$ by the conventional frequency-based probabilities (5.5; 5.6):

$$C_m(n;t) = \frac{1}{N}\sum_{i=1}^{N(t)} C^n(t_i) \quad ; \quad U_m(n;t) = \frac{1}{N}\sum_{i=1}^{N(t)} U^n(t_i) \qquad (5.9)$$

Averaging of (5.7) and definition (5.8) imply no correlations between *n-th* power of the volume $U^n(t_i)$ and *n-th* power of the price $p^n(t_i)$:

$$C_m(n;t) = E[C^n(t_i)] = E[p^n(t_i)U^n(t_i)] = E[p^n(t_i)]E[U^n(t_i)] = p(n;t)U_m(n;t) \qquad (5.10)$$

We underline that (5.10) does not cause statistical independence between the random volume $U(t_i)$ and the random price $p(t_i)$. Not at all, from (5.7-5.9) it is obvious that *n-th* power of the random volume $U^n(t_i)$ correlates with *m-th* power of the random price $p^m(t_i)$ if $n \neq m$.

Well known that the set of statistical moments of a random variable determines its characteristic function and Fourier transform of the characteristic function determines the probability measure of the random variable (Shephard, 1991; Shiryaev, 1999; Klyatskin, 2005; 2015). The price statistical moments $p(n;t)$ (5.8) determine the price characteristic function $F(x;t)$

$$F(x;t) = 1 + \sum_{i=1}^{\infty} \frac{i^n}{n!} p(n;t)\, x^n \qquad (5.11)$$

and Fourier transform of the price characteristic function $F(x;t)$ (5.11) determines price probability measure $\eta(p;t)$ as:

$$\eta(p;t) = \int dx\, F(x;t) \exp(-ixp) \quad ; \quad F(x;t) = \int dp\, \eta(p;t) \exp(ixp) \qquad (5.12)$$

$$\frac{d^n}{(i)^n dx^n} F_p(x;t)|_{x=0} = \int dp\, \eta(p;t) p^n = p(n;t) \qquad (5.13)$$

For brevity in (5.12) we omit normalizing factors proportional to (2π). Similar to (5.11-5.13) the sets of statistical moments of the value $C_m(n;t)$ and the volume $U_m(n;t)$ define characteristic functions $G(x;t)$ (5.14) of the value and $Q(x;t)$ of the volume (5.14):

$$G(x;t) = 1 + \sum_{i=1}^{\infty} \frac{i^n}{n!} C_m(n;t)\, x^n \quad ; \quad Q(x;t) = 1 + \sum_{i=1}^{\infty} \frac{i^n}{n!} U_m(n;t)\, x^n \qquad (5.14)$$

Fourier transforms of characteristic functions $G(x;t)$ and $Q(x;t)$ (5.14) define probability measures $\nu(C;t)$ of the value and $\mu(U;t)$ of the volume of random trades during Δ:

$$\nu(C;t) = \int dx\, G(x;t) \exp(-ixC) \quad ; \quad \mu(U;t) = \int dx\, Q(x;t) \exp(-ixU) \qquad (5.15)$$

n-th statistical moments of the value $C_m(n;t)$ and the volume $U_m(n;t)$ (5.9) take form:

$$C_m(n;t) = \frac{d^n}{(i)^n dx^n} G(x;t)|_{x=0} = \int dC\, C^n \nu(C;t) \qquad (5.16)$$

$$U_m(n;t) = \frac{d^n}{(i)^n dx^n} Q(x;t)|_{x=0} = \int dU\, U^n \mu(U;t) \qquad (5.17)$$

Due to (5.8-5.10; 5.15-5.17) the probability measures *v(C;t)* of the value and *μ(U;t)* of the volume, the characteristic functions *G(x;t)* of the value and *Q(x;t)* (5.14) of the volume and sets of statistical moments $C_m(n;t)$ and $U_m(n;t)$ (5.9) determine the price probability measure *η(p;t)* (5.12), the price characteristic function *F(x;t)* (5.11) and the price statistical moments *p(n;t)* (5.8).

Description and forecasting of the price probability measure *η(p;t)* (5.12) is a really tough problem. Indeed, prediction of the price probability measure *η(p;t)* requires forecasts of all statistical moments *p(n;t)* (5.8) and thus forecasts of all trading statistical moments *C(n;t)* and*U(n;t)* (5.9). That equals prediction of the market trading probability measures *v(C;t)* and *μ(U;t)* (5.15). In other words – prediction of the price probability measure *η(p;t)* requires prediction of market evolution. One may consider the definition of the new price probability measure *η(p;t)* through probability measures *v(C;t)* and *μ(U;t)* of the market trading (5.8-5.17) as formal mathematical expression of the famous phrase: "You can't beat the market".

Description of the price volatility requires development of a new second-order economic theory. Indeed, the price volatility $\sigma^2(p)$ equals the difference between the price 2-d statistical moment *p(2;t)* and square of the mean price $p^2(1;t)$ (5.8):

$$\sigma^2(p) = E_t\left[\delta^2 p\right] = p(2;t) - p^2(1;t) = \frac{C_m(2;t)}{U_m(2;t)} - \frac{C_m^2(1;t)}{U_m^2(1;t)} \tag{5.18}$$

Thus description of price volatility $\sigma^2(p)$ requires modeling 1-st and 2-d statistical moments of the value and the volume, or equally – modeling the aggregated value *C(n;t)* and the volume *U(n;t)* (5.19) of the trades performed during interval *Δ:*

$$C(n;t) = \sum_{i=1}^{N(t)} C^n(t_i) = \sum_{i=1}^{N(t)} p^n(t_i) U^n(t_i) \;\; ; \;\; U(n;t) = \sum_{i=1}^{N(t)} U^n(t_i) \tag{5.19}$$

It is obvious that the price statistical moments *p(n;t)* (5.8) and the price volatility $\sigma^2(p)$ (5.18) can be presented equally via the aggregated *C(n;t)* and *U(n;t)* (5.19):

$$C(n;t) = p(n;t)U(n;t) \tag{5.20}$$

$$\sigma^2(p) = \frac{C(2;t)}{U(2;t)} - \frac{C^2(1;t)}{U^2(1;t)} \tag{5.21}$$

To forecast the price volatility $\sigma^2(p)$ (5.18; 5.21) one should predict the mean price *p(1;t)* and the mean square price *p(2;t)* and that implies description of *C(1;t), C(2;t), U(1;t) U(2;t)* (5.19-5.21). Forecasting of the mean price *p(1;t)* averaged during the interval *Δ* requires prediction of *C(1;t)* and *U(1;t)* (5.19) – sums of values $C(t_i)$ and volumes $U(t_i)$ of trades performed during *Δ*. That is described by current economic theory that models macroeconomic variables determined as sums of the first degree variables. Indeed, almost all macro variables are composed as sums of agents' variables of the first degree. Macro investment, credits, consumption and etc., are composed as sums (without duplication) of the first-degree investment, credits, consumption of all agents in the economy. Basically to some extend these variables are described by current macroeconomic theory.

However, price volatility (5.21) requires description of the second degree variables *C(2;t), U(2;t)* determined as sums of squares of the values and the volumes of trades performed by agents during the interval *Δ*. The similar are second-degree macroeconomic variables as aggregated squares of agents' investment, credits, consumption and etc., those can be determined as sums (without duplication) of squares of investment, credits, consumption and etc. of all agents in the economy. These second-degree macroeconomic variables can describe volatilities of macro investment, credits, consumption and etc. Description of the second-degree macro variables as well as description of the aggregated squares of the values *C(2;t)* and the volumes *U(2;t)* requires development of the second-order economic theory just because no second-degree variables are considered in the current macroeconomic models at all. Moreover, description of the price skewness *Sk(p)* requires prediction of the *3-d* statistical moments *p(3;t)* of price:

$$C(3;t) = p(3;t)U(3;t)$$

Thus predictions of the price skewness *Sk(p)* requires description of *C(3;t)* and *U(3;t).* Hence one should develop the third-order economic theory that models sums of the 3-d power of the values *C(3;t)* and the volumes *U(3;t).* Forecasts of price kurtosis require development of the forth-order economic theory and so on.

Thus development of the asset pricing as result of random market trading should go through long and difficult line of successive approximations. We start study the second order economic theory in (Olkhov, 2020b) and refer there for detail.

However, above considerations don't determine or choose correct or incorrect price probability measure. Economics is a social science and investors are free in their trade decisions based on personal expectations, beliefs, financial and social "myths & legends". Investors are free to choose any definition of the price probability measure they prefer.

6. Conclusion

Our treatment of asset pricing outlines three critical remarks that may be taken into account by investors and researchers.

1. Any averaging of the market trading time-series is performed during certain time interval Δ. The choice of Δ and description of the dependence of the averaged price, payoff, volatilities and etc., on different intervals Δ are important for any investment strategies.

2. The first-order max condition (2.5) should be complemented by the second condition (4.27) and both define asset amount ξ_{max} that delivers max to investor's utility (2.2; 3.6). The choice of the averaging interval Δ determines the linear approximations of Taylor series of the utility functions (4.3; 4.4) by price and payoff variations δp and δx near the mean values of the price p_0 and the payoff x_0 and gives equations (4.5; 4.6) on the asset amount ξ_{max}, mean price p_0, payoff x_0 and their volatilities $\sigma^2(p)$ and $\sigma^2(x)$. In the case of idiosyncratic risk these equations determine relations (4.14) on price p at moment t, risk-free rate R_f and payoff x_0 at moment $t+1$. For given averaging interval Δ Taylor series of utility derivative $u'(c_{t+1})$ (4.17) with accuracy to squares of payoff $\delta^2 x$ give relations (4.19-4.22) on ξ_{max}, payoff volatility $\sigma^2(x)$ and payoff skewness $Sk(x)$ (4.12). In case of the conventional power utility (A.2) obtain (4.23) and (4.24) that define lower limit on the payoff skewness $Sk(x)$. Further expansion of the utility into Taylor series up to $\delta^3 x$ will describe impact of payoff kurtosis. The price-volume correlations studied for decades may change the basic equation (4.31) and even violate existence of utility max (4.32; 4.33).

3. All asset pricing models and the CAPM in particular study and forecast price using certain price probability measure determined during some averaging interval Δ. The choice of the probability measure and predictions of the price probability are the critical issues of any pricing models. We replace the conventional frequency-based price probability and introduce the new price probability measure that is entirely determined by the probabilities measures of trading value and volume. We define *n-th* statistical moment of the price $p(n;t)$ as the coefficient (5.8) between *n-th* statistical moments of the value $C_m(n;t)$ and the volume $U_m(n;t)$ (5.9) of market trades. Aggregated value $C(n;t)$ and volume $U(n;t)$ (5.19) of trades performed during the interval Δ give equal definition of $p(n;t)$ (5.20). The mean price $p(1;t)$ (5.4-5.6) for $n=1$ coincides with the VWAP (5.1; 5.2).

It could be said that replacement of the conventional frequency-based price probability by the new price probability measure $\eta(p;t)$ (5.9; 5.11-5.13) may help forecast price volatility $\sigma^2(p)$ (5.18; 5.21). However, introduction of the new price probability measure $\eta(p;t)$ (5.11-5.13) uncovers the hidden complexity of the price probability forecasting. Due to (5.8; 5.20) each price *n-th* statistical moment $p(n;t)$ is determined by corresponding *n-th* statistical moments of the value $C_m(n;t)$ and the volume $U_m(n;t)$ of market trades (5.9) or by sums of *n-th* power of the value $C(n;t)$ and the volume $U(n;t)$ (5.19) of market trades during Δ. Thus, the forecast of $p(n;t)$ requires the forecasts of $C(n;t)$ and $U(n;t)$ (5.19). We underline that the forecasts of the mean price $p(1;t)$, the value $C(1;t)$ and the volume $U(1;t)$ (5.19) don't allow forecast sum of squares of the value $C(2;t)$ and the volume $U(2;t)$ (5.19) and hence, it is impossible predict 2-d statistical moment of price $p(2;t)$ (5.20) and price volatility $\sigma^2(p)$ (5.18; 5.21). To forecast the price volatility $\sigma^2(p)$ (5.18; 5.21) one needs predictions of the squares of the value $C(2;t)$ and the volume $U(2;t)$ (5.19) of trades aggregated during Δ and that requires development of the second-order economic theory. Description of $C(n;t)$ and $U(n;t)$ (5.19) for each $n=1,2,..$ requires development of additional economic theory of *n-th* order. In other words – prediction of $p(n;t)$ requires prediction of the *n-th* statistical moments of the market trading value $C_m(n;t)$ and volume $U_m(n;t)$. The definition of the new price probability measure $\eta(p;t)$ through probability measures $\nu(C;t)$ and $\mu(U;t)$ of the market trading (5.8-5.17) gives formal mathematical expression of the catch phrase: "You can't beat the market".

Nevertheless, definitions of the new price probability measure (5.11-5.13) open the way for development of different approximations of the price probability. The choice and justification of each approximation are subjects for the further studies.

However, investors are free to choose any price probability they prefer. Investors may choose the conventional frequency-based price measure as ground for their investment decisions and use any available price forecasts without any complex considerations of the market trading via $C(n;t),U(n;t)$. That may be very beneficial for investors and may be not. There's no such thing as a free lunch.

We believe that the asset pricing theory will stay attractive and complex subject for researchers, unsearchable and elusive for investors and will remain so for many years or forever.

Appendix A

Max of Utility

$$p^2 > -\beta \frac{u''(c_{t+1;0})}{u''(c_t)} [{x_0}^2 + \sigma^2(x)] - \beta \frac{u'''(c_{t+1;0})}{u''(c_t)} \xi_{max} [2x_0\sigma^2(x) + \gamma^3(x)] \qquad (A.1)$$

If the right side is negative then it is valid always. If the right side is positive – then there exist a lower limit on the price *p*. For simplicity neglect term $\gamma^3(x)$ to compare with $2x_0\sigma^2(x)$ and take the conventional power utility *u(c)* (Cochrane, 2001) as:

$$u(c) = \frac{1}{1-\alpha} c^{1-\alpha} \qquad (A.2)$$

Let us consider the case with negative right side for (A.1). Simple but long calculations give:

$$-\beta \frac{u''(c_{t+1;0})}{u''(c_t)} [{x_0}^2 + \sigma^2(x)] < \beta \frac{u'''(c_{t+1;0})}{u''(c_t)} \xi_{max}\, 2x_0\sigma^2(x)$$

$$\xi_{max}\, 2x_0\sigma^2(x) < -\frac{u'''(c_{t+1;0})}{u''(c_{t+1;0})} [{x_0}^2 + \sigma^2(x)] \qquad (A.3)$$

Let us take into account (A.2) and for (A.3) obtain:

$$\frac{u''(c)}{u'''(c)} = \frac{-\alpha c^{-\alpha-1}}{\alpha(1+\alpha)c^{-\alpha-2}} = -\frac{c}{1+\alpha} \quad ; \quad \xi_{max}\, 2x_0\sigma^2(x) < \frac{e_{t+1} + x_0\xi_{max}}{1+\alpha} [{x_0}^2 + \sigma^2(x)]$$

$$\xi_{max} x_0 [(1+2\alpha)\sigma^2(x) - {x_0}^2] < e_{t+1}[{x_0}^2 + \sigma^2(x)] \qquad (A.4)$$

Inequality (A.4) determines that the right side (A.1) is negative in two cases.

1. The left side in (A.4) is negative and

$$\frac{\sigma^2(x)}{{x_0}^2} < \frac{1}{1+2\alpha} \quad ; \quad \frac{1}{3} \le \frac{1}{1+2\alpha} < 1 \qquad (A.5)$$

Inequality (A.5) describes small payoff volatility $\sigma^2(x)$. In this case the right side of (A.1) is negative for all ξ_{max} and all price *p* and hence (4.27) that defines max of utility (2.5) is valid.

2. The left side in (A.4) is positive and

$$\frac{\sigma^2(x)}{{x_0}^2} > \frac{1}{1+2\alpha} \quad ; \quad \xi_{max} < \frac{e_{t+1}[{x_0}^2 + \sigma^2(x)]}{x_0 [(1+2\alpha)\sigma^2(x) - {x_0}^2]} \qquad (A.6)$$

This case describes high payoff volatility and defines the upper limit on the value of asset amount ξ_{max} that delivers max to utility (2.5). Take the positive right side in (A.1). Then (A.4) is replaced by the opposite inequality

$$\xi_{max} x_0 [(1+2\alpha)\sigma^2(x) - {x_0}^2] > e_{t+1}[{x_0}^2 + \sigma^2(x)] \qquad (A.7)$$

It is valid for (A.6) only. (A.7) determines a lower limit on ξ_{max} that delivers max to utility (2.5):

$$\xi_{max} > \frac{e_{t+1}[{x_0}^2 + \sigma^2(x)]}{x_0 [(1+2\alpha)\sigma^2(x) - {x_0}^2]}$$

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