Two-Person Fair Division of Indivisible Items: Compatible and Incompatible Properties

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Abstract

Suppose two players wish to divide a finite set of indivisible items, over which each distributes a specified number of points. Assuming the utility of a player's bundle is the sum of the points it assigns to the items it contains, we analyze what divisions are fair. We show that if there is an envy-free (EF) allocation of the items, two other desirable properties—Pareto-optimality (PO) and maximinality (MM)—can also be satisfied, rendering these three properties compatible, but other properties—balance (BL), maximum Nash welfare (MNW), maximum total welfare (MTW), and lexicographic optimality (LO)—may fail. If there is no EF division, as is likely, it is always possible to satisfy EFx, a weaker form of EF, but an EFx allocation may not be PO, BL, MNW, MTW, or LO. Moreover, if one player considers an item worthless (i.e., assigns zero points to it), an EFx division may be Pareto dominated by a nonEFx allocation that is MNW. Although these incompatibilities suggest that there is no "perfect" 2-person fair division of indivisible items, EFx and MNW divisions—if they give different allocations when there is no EF-PO-MM division—seem the most compelling alternatives, with EFx, we conjecture, satisfying the Rawlsian objective of helping the worse-off player and MNW, a modification of MTW, suggesting a more Benthamite view.

1. Introduction

In the past 40 years, there have been a number of books and many articles on the fair allocation of divisible items, such as cake, land, or money. The fair allocation of indivisible items, such as physical property in a divorce, has also received attention. Different algorithms for allocating divisible and indivisible items have been analyzed and compared in, among other places, Brams and Taylor (1994, 1999); for a recent review, see Klamler (2021).

Our focus in this paper is entirely on the *properties* of 2-person fair allocation of indivisible items, rather than *algorithms* to allocate the items or incentives of players to manipulate them (see Brams et al., 2015, for an analysis of one indivisible-item algorithm). These are important questions, but we think showing what combinations of properties are compatible or incompatible should precede questions of implementation and manipulation.

Assume two players wish to divide a finite set of indivisible items, over which they can distribute the same number of points. If the utility of each player's bundle is the sum of the points it assigns to the items in it—so there are no synergies among subsets of items—we analyze what is a fair division of these items.

We begin by defining seven properties for the 2-person fair division of indivisible items (an eighth will be defined in section 3), based on the utility of each player's bundle:

1. **Envy-freeness (EF).** *An allocation is EF for a player if its utility for its bundle (assignment) is at least as great as its utility for the other player's bundle. The allocation is EF if it is EF for both players.*
2. **Efficiency or Pareto-optimality (PO).** *If each player's assignment in allocation S is at least as preferable as its assignment in allocation T, and in at least one case strictly preferable, then S is* **Pareto-Superior (PS)** *to T or Pareto-dominates T. An allocation T is Pareto-Optimal (PO) if there exists no allocation S such that S is PS to T.*

3. **Maximinality (MM).** *An allocation is MM if there is no other allocation for which the minimum of the players' utilities is greater.*
4. **Maximum Nash welfare (MNW).** *An allocation is MNW if there is no other allocation for which the product of the players' utilities is greater.*
5. **Maximum total welfare (MTW).** *An allocation is MTW if there is no other allocation for which the sum of the players' utilities is greater.*
6. **Balanced (BL).** *An allocation is BL if each player receives the same number of items.*
7. **Lexicographic Optimality (LO).** *For any allocation, write the players' utilities for their own assignments in ascending order. If each ascending utility for allocation S is at least equal to the corresponding ascending utility for allocation T, and in at least one case strictly greater, then S is* **Lexicographically-Superior (LS)** *to T. An allocation T is* **Lexicographically-Optimal (LO)** *if there exists no allocation S such that S is LS to T.*

Whereas an allocation is PO if there is no other allocation that each player values at least as much and one player values more, an allocation is LO if there is no other allocation for which the minimum and maximum assigned utilities are at least as great, and at least one is greater. PS implies LS, but not vice versa. LO implies PO, but not vice versa.

In section 2 we show that if there is an EF allocation, there is an EF-PO-MM allocation, rendering these three properties compatible. But an EF-PO-MM allocation may not satisfy BL, MNW, MTW, or LO.

In section 3, we show that if there is no EF allocation, it is always possible to satisfy EFx—a weaker form of EF to be defined in section 3—but an EFx allocation also may fail any or all of BL, MNW, MTW, and LO. Moreover, if one player considers an item worthless (i.e., assigns zero points to it), an EFx allocation may be Pareto-dominated by a nonEFx allocation that is MNW. We conjecture, however, that only an EFx allocation is MM.

In section 4, we offer some concluding remarks on these compatibilities and incompatibilities and what seem to be the fairest allocations. We compare the views of the 18th century utilitarian theorist Jeremy Bentham (1789/2017) and the 20th century maximin theorist John Rawls (1971/1999), suggesting that Rawls can be interpreted as favoring an EF or EFx allocation that is MM, and Bentham as favoring an MTW allocation but, because of its deficiencies, possibly finding another maximum welfare allocation, MNW, acceptable.

2. Properties of an Envy-Free (EF) Allocation

In the examples that follow, we call the two players A and B, and the items to be allocated a, b, c, …. The points the players assign to the items are non-negative integers 0, 1, 2, …., which sum to the same total for each player.

Thereby the players are on a par in terms of their entitlements—one player cannot "outbid" the other on every item. If a player's value for some item is greater, there must be another item for which its value is less.

We begin with an example that illustrates our notation and then show that an EF allocation may not be PO.

Example 1: *Sum of each player's points is 18.*

	a	**b**	**c**	**d**
A:	9	6	2	1
B:	1	2	6	9

In an allocation S of items to A and B, let S_A be A's bundle (assignment) of items, and let S_B be B's bundle. The utility of player A for S_A will be the sum of A's points for the items in S_A, or $u_A(S_A)$, and similarly for $u_B(S_B)$. We write the utilities of an allocation $S = (S_A, S_B)$ as $u[(S_A, S_B)] = [u_A(S_A), u_B(S_B)]$. Note that quantities in square brackets are utilities.

In Example 1, if $S_A = \{a\}$, or more simply a, and $S_B = \{b, c, d\}$, or more simply bcd, then $u(S_A, S_B) = [9, 17]$ because $u_A(S_A) = u_A(a) = 9$, and $u_B(S_B) = u_B(bcd) = 2 + 6 + 9 = 17$. This allocation is EF, because A is indifferent between item a and items bcd (each bundle gives A 9 points), whereas B prefers its 17 points from items bcd to 1 point from item a. In symbols, $u_A(S_A) = u_A(S_B) = 9$ and $u_B(S_B) > u_B(S_A)$.

Observe that (ac, bd) is also EF, because u(ac, bd) = [11, 11] that gives more than half the 18 points to each player. More specifically, A receives more points (9 + 2 = 11) from its bundle than it would receive from B's bundle (6 + 1 = 7), and B receives more points from its bundle (2 + 9 = 11) than it would receive from A's bundle (1 + 6 = 7).

But (ac, bd) is Pareto-dominated by (ab, cd), because u(ab, cd) = [15, 15]. Like (ac, bd), (ab, cd) is EF, but it is also PO, because there is no other allocation that gives each player utility of at least 15 and one player more. In addition, (ab, cd) is MM, because there is no allocation that gives each player a utility greater than 15.[1] Hence, we call (ac, bd) an EF-PO-MM allocation.

EF allocations may not equally favor the players (in points), as illustrated by our next example.

Example 2: *Sum of each player's points is 18.*

	a	**b**	**c**	**d**
A:	9	4	3	2
B:	7	1	1	9

Observe that u(a, bcd) = [9, 11], so (a, bcd) is not BL but it is EF, because u(bcd, a) = [9, 7]; A's utility for B's assignment is 4 + 3 + 2 = 9, and B's utility for A's assignment is only 7. It is easy to verify that (ab, cd) and (ac, bd) are also EF: u(ab, cd) = [13, 10] and u(ac, bd) = [12, 10].

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[1] There is another EF-PO allocation, (abc, d), that gives the player utilities, u(abd, d) = [17. 9], but it is not MM.

Among the forgoing three EF allocations, (a, bcd) is PO, because there is no other allocation that gives B utility 11 or more, so it is EF-PO. Allocations (ab, cd) and (ac, bd) are MM, because no other allocation gives both players utility greater than 10. However, because (ab, cd) Pareto-dominates (ac, bd)—and no other allocation Pareto-dominates (ab, cd)—only (ab, cd) is EF-PO-MM.

It is clear that (ab, cd), with utilities *u*(ab, cd) = [13, 10], is LS to (a, bcd) with utilities *u*(a, bcd) = [9, 11], because the players' utilities in ascending order are [10, 13] and [9, 11], and 10 > 9 and 13 > 11. In fact, even (ac, bd) is LS to (a, bcd). No other allocation gives ascending utilities that are at least equal to [10, 13], so (ab, cd) is an EF-LO-MM allocation. But, as we will show later, it is not always the case that an EF-PO-MM allocation is also LO.

It is useful to distinguish PO and MM.

Proposition 1. *PO and MM are independent properties of EF allocations.*

Proof. In Example 2, we showed that (a, bcd) is EF-PO and (ac, bd) is EF-MM, but neither is EF-PO-MM. Hence, an EF allocation may satisfy PO without satisfying MM, and satisfy MM without satisfying PO, so the satisfaction of one property does not imply the satisfaction of the other, making PO and MM independent. Q.E.D.

Of course, Proposition 1 does not preclude an allocation from satisfying all three properties, as does allocation (ab, cd) in Example 2. It also does not imply that any EF allocation satisfies at least one of PO and MM. While all EF allocations in Example 2 are either PO or MM, this is not true for (ac, bd) in Example 1, which is EF but neither PO nor MM.

PO and MM measure, in different ways, how close two players come to benefiting, and benefitting equally, from an EF allocation. Fortunately, there is always an EF-PO-MM allocation in which these properties co-exist.

Proposition 2. *If there is an EF allocation, at least one EF allocation is EF-PO-MM.*

Proof. Suppose there is an EF allocation, so each player values its bundle at least as much as the other player's bundle. Suppose that there is a possible switch of some items in the bundles so that one player values its new bundle at least as much and the other player values its new bundle more. Make this switch, and observe that the minimum of the two players' utilities has not decreased. Repeat. Because the number of items, and therefore the number of possible switches, is finite, at some point no such switch is possible, and the current allocation must be an EF-PO-MM allocation. Q.E.D.

Example 1 illustrates that switching b and c in (ac, bd), where *u*(ac, bd) = [11, 11], yields *u*(ab, cd) = [15, 15], and (ab, cd) is EF-PO-MM. In Example 2, beginning at the allocation (a, bcd), switch c to A, which increases the minimum—*u*(ac, bd) = [12, 10] versus *u*(a, bcd) = [9, 11]—and then again increase the minimum by switching b and c—now achieving *u*(ab, cd) = [13, 10]—resulting in the EF-PO-MM allocation (ab, cd).

It is worth noting that the EF-PO-MM allocations in Examples 1 and 2 satisfy BL—each player receives two items. When two players take turns making choices, using what Brams and Taylor (1999) call "strict alternation," and there is an even number of items, BL will be satisfied. In fact, strict alternation is a common procedure for dividing indivisible items, such as the marital property in a divorce or selecting players for two pickup teams in sports.

A prominent alternative to an EF-PO-MM allocation is an MNW allocation. The two allocations may coincide, but when they differ it is useful to compare their properties. An MNW allocation is always PO (Caragiannis et al., 2019), but it may fail to satisfy both EF and MM, as shown by our next example.

Example 3: *Sum of each player's points is 18.*

	a	**b**	**c**	**d**
A:	9	4	4	1

B: 7 1 2 8

The unique EF-PO-MM allocation is (ab, cd), with utilities u(ab, cd) = [13, 10] and utility product 13 × 10 = 130. By contrast, the MNW allocation, which maximizes the product of the players' utilities, is (abc, d), because u(abc, d) = [17, 8], and 17 × 8 = 136.

But the MNW allocation, while PO, is neither EF (B envies A for getting what it thinks is 10 when it gets only 8) nor MM (the EF-PO-MM allocation (ab, cd) gives a minimum utility of 10, whereas the MNW allocation gives B only 8). However, the MNW allocation does satisfy MTW, giving a maximum sum of 17 + 8 = 25 vs. 13 + 10 = 23 for the EF-PO-MM allocation, showing that in this case that the two welfare properties, MNW and MTW, agree on an allocation (but this is not always the case, as we will see in section 3).

What makes it difficult to say whether the EF-PO-MM or the MNW-PO allocation is "better" is that neither is LS with respect to the other. The ascending utilities of (ab, cd) are [10, 13], and the ascending utilities of (abc, d) are [8, 17], so there is no LS relation between them. Example 3 proves the following proposition.

Proposition 3. *If an EF-PO-MM allocation differs from an MNW allocation, the MNW allocation may satisfy neither EF nor MM. There may be no LS relation between the PO-MNW allocation and the EF-PO-MM allocation.*

Proposition 3 casts some doubt on the "unreasonable fairness" of MNW (Caragiannis et al., 2019b), because it may fail EF, MM, and LO, even though it is PO (so, of course, is an EF-PO-MM allocation). As we will show in the next section, however, an MNW allocation may Pareto-dominate an EFx allocation—the weaker form of EF that we discuss next—but an EFx allocation may also Pareto-dominate an MNW allocation.

3. Envy-freeness after the Removal of any Item x (EFx)

For indivisible items, there will often be no EF allocation when the players have similar preferences, which precludes an EF-PO-MM allocation. As an alternative, EFx has been conjectured but not proven to exist in the *n*-person fair division of indivisible items, but the conjecture has been proven for 2-person fair division.[2]

8. **Envy-freeness after the removal of any item x (EFx).** *If an allocation is not EF because, say, A envies B, then it is EFx if the removal of any item x from B's assignment eliminates A's envy. (The roles of A and B may be interchanged.)*

In the following example, it is apparent that there is no EF allocation, because if either player receives any two of the three items, the other player will be envious.

Example 4: *Sum of each player's points is 9.*

	a	**b**	**c**
A:	2	3	4
B:	3	2	4

Two allocations, (ab, c) with *u*(ab, c) = [5, 4] and (c, ab) with *u*(c, ab) = [4, 5], are EFx, because the removal of either item a or b eliminates the c-player's envy of the ab-player. On the other hand, other allocations are not EFx, such as (ac, b), for which *u*(ac, b) = [6, 2], because the removal of item a or c from ac leaves B envious, who gets 2 but envies A for getting what B thinks is 3 or 4.

The EFx allocations in Example 4, (ab, c) and (c, ab), do not satisfy MNW. The product of the players' utilities for these allocations is 5 × 4 = 20, whereas two non-EFx allocations, (b, ac) with *u*(b, ac) = [3, 7] and (bc, a), with *u*(bc, a) = [7, 3], both have a greater utility product (3 × 7 = 21). Because that is the highest achievable utility product, these two allocations are MNW. Both the EFx and the MNW allocations are LO.

To summarize, like the unique EF-PO-MM allocation in Example 3, the two EFx allocations in Example 4 are not MNW. In Example 4, the EFx allocations, which give

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[2] In fact, the conjecture has been proven for up to 3 players, but not in the general n-person case whenever players give more than two values to items (Amanatidis et al., 2021; Berger et al., 2021). An allocation is EFx in the *n*-person case if every pair of players satisfies the 2-person EFx property (defined next).

utilities of [5, 4] and [4, 5], are PO and MM. Neither the EFx nor the MNW allocations, which give utilities of [3, 7] and [7, 3], are related by LS—all of them are LO.

EFx allocations exhibit a significant drawback when a player gives zero points to an item, as illustrated by Example 5 (odd number of items) and Example 6 (even number of items).[3]

Example 5 (odd number of items): *Sum of each player's points is 7.*

	a	**b**	**c**
A:	5	1	1
B:	4	3	0

The allocation (a, bc), where *u*(a, bc) = [5, 3], is EFx, because the removal of item a eliminates B's envy of A. But (a, bc) is Pareto-dominated by (ac, b), where *u*(ac, b) = [6, 3]. However, (ac, b) is not EFx—the removal of item c leaves B envious—even though (ac.b) is MNW.

Example 6 (even number of items): *Sum of each player's points is 9.*

	a	**b**	**c**	**d**
A:	6	1	1	1
B:	5	3	1	0

The allocation (a, bcd), where *u*(a, bcd) = [6, 4], is EFx, but it is Pareto-dominated by (ad, bc), where *u*(ad, bc) = [7, 4]. However, (ad, bc) is not EFx, whereas it is MNW.

In both Examples 5 and 6, one item (item a) is a "diamond" that is worth more to A and B than all the other items combined (the "pebbles"). We call such examples, in which

[3] If there is an item that both players consider worthless (i.e., give zero points to), we exclude it from an allocation. But if only one player gives zero points to an item, we do not automatically award it to the other player, because the player who indicated it is worthless may value it positively but want to maximize the number of points it gives to items that it values more. Thereby we do not assume that players' point allocations are necessarily sincere; strategic factors may play a role in their allocations, which we take at face value.

there is a single diamond that is more valued by both players than all the pebbles combined, a *diamond-pebble situation*.[4]

In a diamond-pebble situation, the only EFx allocations are those that assign the diamond to one player and all the pebbles to the other. In particular, if A gets the diamond in an EFx allocation, B must get all of the pebbles, including the one that it considers worthless (item c in Example 5 and item d in Example 6).

As we showed in Examples 5 and 6, EFx allocations (a, bc) and (a, bcd) are Pareto-dominated by non-EFx allocations that are MNW. To be sure, there are *alternative* EFx allocations in which B obtains the diamond—(bc, a), with *u*(bc, a) = [2, 4], in Example 5, and (bcd, a), with *u*(bcd, a) = [3, 5] in Example 6—that are PO. But they are neither MNW nor MM and, therefore, not as appealing as the nonEFx allocations, which give [6, 3] and [7, 4] in Examples 5 and 6 and are also LO.

The underlying reason why the EFx allocations in Examples 5 and 6 are not PO is that one pebble in these allocations is 0-valued by B but positive-valued by A. If A is given this pebble, the resulting allocation, which is not EFx, is PO as well as MNW.

Proposition 4. *In a diamond-pebble situation, an allocation in which one player (say, A) receives the diamond and the other (B) all the pebbles is EFx. Assume there is one pebble that is 0-valued by B but positive-valued by A. If this pebble is reassigned to A, the resulting allocation is not EFx but Pareto-dominates the original EFx allocation.*

Proof. Assume in an allocation that A receives the diamond. Because B envies A, this allocation cannot be EFx if A also receives any pebble, because then the removal of the pebble from A's allocation would not eliminate B's envy. Therefore, the allocation is EFx only if B receives all the pebbles. If there is a pebble that is 0-valued by B but positive-valued by A, then reassigning it to A will increase A's total value but not affect B's total

[4] In a divorce, the diamond might be the house, and all the other indivisible property the pebbles.

value. This new allocation Pareto-dominates the EFx allocation, but it is not EFx because A now receives a pebble. Q.E.D.

Proposition 4 does not preclude the existence of a second EFx allocation that is not Pareto-dominated by a non-EFx allocation.

Example 7: *Sum of each player's points is 9.*

	a	**b**	**c**	**d**
A:	5	2	1	1
B:	6	2	1	0

In Example 7, the allocation (a, bcd), where *u*(a, bcd) = [5, 3], is EFx. It is Pareto-dominated by the nonEFx allocation (ad, bc) where *u*(ad, bc) = [6, 3]. But a second EFx allocation, (bcd, a), gives utilities [4, 6]. It is LO-MM, and therefore PO, proving the next proposition.

Proposition 5. *In a diamond-pebbles situation with a 0-valued item, an EFx allocation may be LS to a non-EFx allocation that Pareto dominates another EFx allocation.*

In Example 7, the EFx-PO-MM allocation is also MNW (utility product of $4 \times 6 = 24$ vs. $6 \times 3 = 18$ for the nonEFx allocation). However, in a diamond-pebbles situation, it is not always the case that an EFx allocation is MNW.

Example 8: *Sum of utilities is 11.*

	a	**b**	**c**
A:	6	3	2
B:	6	4	1

Notice that both players value the diamond the same, but they value the two pebbles differently.

Allocation (ac, b), which gives utilities [8, 4], has the largest utility product of any allocation (32) and so is MNW. It also has the largest utility sum and so is MTW. By contrast, the EFx allocations, yielding utilities of [6, 5] or [5, 6], have a utility product of only 30. These two allocations are EFx-PO-MM, but they are not LO, which proves the next proposition.

Proposition 6. *In a diamond-pebbles situation, an EFx-PO-MM allocation may be different from, and not LS to, an MNW and MTW allocation.*

This proposition echoes Proposition 3 that, together with Proposition 6, shows that neither EF nor EFx (if there is no EF allocation) may be MNW. But, in a diamond-pebbles situation, one EFx allocation (there are no EF allocations) is always PO-MM, but it may not be LS to the MNW allocation if it is different.

We next show that an EFx allocation may be incompatible with BL.

Proposition 7. *In a diamond-pebbles situation, if there is an even number of items, there may be no EFx-PO-MM-BL allocation.*

Proof. Assume there is a diamond and three pebbles, and assign the diamond to A and the pebbles to B. This is a 1-3 unbalanced allocation. Move one of the pebbles to A to create a 2-2 balanced allocation. This new allocation is no longer EFx, because if the pebble is removed from A's allocation, B is still envious. In this example, the properties of EFx and balance (BL) cannot both be satisfied. Q.E.D.

We complete this section with a conjecture that we have not been able to prove (or disprove).

Conjecture. *If there are no 0-valued items and no EF allocation, at least one EFx-PO allocation, and no nonEFx allocations, are MM.*

Put another way, we conjecture that when there are no EF allocations, only an EFx allocation can be MM. This would be a significant finding, because it would rule out, say, an MNW allocation that is not EFx from being MM. As a case in point, the maximin of the EFx allocation in Example 8 is 5, whereas that of the MNW allocation, which is not EFx, is 4.

To recapitulate, if there is no EF-PO-MM allocation, it is always possible to find an EFx allocation for two players. But if at least one player considers an item worthless, a non-EFx allocation may Pareto-dominate an EFx allocation and be MNW. However, there are diamond-pebbles situations in which an EFx allocation is MNW but not MM. Finally, EFx may be incompatible with BL and MNW, but we conjecture that there is always at least one EFx-PO allocation that is MM, maximizing the value of the worse-off player.

We have not so far discussed in detail compatible and incompatible properties of MTW, which we associate with Bentham's (1789/2017) principle that "it is the greatest happiness of the greatest number that is the measure of right and wrong." Putting aside the moral implications of "right and wrong," we associate Bentham's principle with MTW and illustrate it with the following example:

Example 9: *Sum of utilities is 18.*

	a	**b**	**c**	**d**
A:	9	4	4	1
B:	4	3	3	8

The allocation (ab, cd) is EF-PO-MM-BL, with utilities [13, 11]. The utility sum is 24. But an unbalanced allocation, (abc, d), with utilities [17, 8], is MTW, with a utility sum of 25.[5] Not only is (abc, d) unbalanced, but it is not EF (B receives only 8 out of 18), MM (its maximin is 8 vs. 11 for the EF-PO-MM-BL allocation), or MNW (its utility product is 126 vs. 136 for the EF-PO-MM-BL allocation). Yet another unbalanced allocation, (a, bcd), with utilities [9, 14], is EF, but it has a smaller utility sum (23 vs. 24 and 25), utility product (126 vs. 136 and 143), and is not MM.

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[5] MTW is found by assigning each item to the player who puts more points on it, so A is assigned abc and B is assigned d.

In short, an MTW allocation may fail several of our properties. If there is an EF-PO-MM allocation that is also MNW, as there is in Example 9, it is clearly preferable. If there is no EF allocation, there is a trade-off between an EFx-PO-MM allocation, on the one hand, and an MNW-PO allocation on the other, if it is different. We regard the former allocations to be Rawlsian because, as we conjecture, they are always MM, whereas the latter are more Benthamite without the manifest failings of MTW.

4. Conclusions

The 2-person fair division of indivisible items is a ubiquitous problem, from allocating marital property in a divorce to putting together competitive teams in a sports league. Insofar as two players can allocate points to the items being divided, and these points are additive and determine the values of bundles, it is not obvious which allocation is the fairest.

First, there may be different EF allocations that favor different players. This leaves open which allocation is the fairest, especially if one allocation is PO and the other is MM, which we showed was possible. Happily, there is always at least one EF allocation that is both PO and MM, rendering all three properties compatible.

Second, if there is no EF-PO-MM allocation, which we consider the ideal, there is always an EFx allocation, but it may differ from an MNW allocation and not be BL. In addition, it may be Pareto-dominated by a nonEFx allocation if there is an item that one player thinks is worthless (0-valued). It is possible that there exists another EFx allocation that is MNW, and LS to the nonEFx allocation. We conjecture that at least one EFx-PO allocation will be MM, which is our preferred choice when there is no EF-PO-MM allocation.

Although incompatibilities among some of the desirable properties we postulated suggest—in the absence of an EF-PO-MM allocation—that there is no "perfect" 2-person fair division of indivisible items, EFx and MNW divisions are the most compelling alternatives for the 2-person fair division of indivisible items. By contrast, we argued that an MTW allocation is not attractive, especially when it differs from an EF, EFx, or MNW allocation.

An EF or EFx allocation, when it satisfies MM, reflects the Rawlsian view of helping the worse-off player, whereas an MNW allocation—when it differs from an EF or EFx allocation—reflects, to some degree, the Benthamite view of MTW without some of its flagrant deficiencies. To be sure, there is no conflict when EF or EFx allocations coincide with MTW allocations, but when they do not, the choice is not so clear as to which allocation is the fairest.

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