Optimal Taxation in the Endogenous Growth Framework with the Private Information

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Abstract:

Differing from taxes of the new dynamic public finance theory without growth, our paper setups an endogenous growth model with the public finance sector which levies heterogeneous non-linear income taxes and linear flat-rate tax on gross outputs to guarantee the optimal investment in the public goods accumulation. Each taxation has individual effect: heterogeneous non-linear income taxes are used to keep standard Euler equation hold; flat-rate tax is used to compensate for the fiscal gap. The paper firstly makes the growth rate endogenous, and show there is a unique steady state growth rate for every aggregate variable by keeping assumptions of the dynamic general equilibrium theory unchangeable. We further prove the growth must exist when externalities are provided by public finance sector. The steady state growth rate can be expressed by coefficients, and the steady state intertemporal relationships of aggregate variables help us simplify simulation equations and calculations on endogenous heterogeneous non-linear income taxes in infinite periods.

Key words: endogenous tax; public finance; growth; uniqueness

I. Introduction

Modern economic researches on the endogenous tax in the dynamic environment have two branches: heterogeneous endogenous non-linear income taxes to remove the influence trigged by idiosyncratic skill shocks and the optimal public finance revenues (in line with Ricardian equivalence, public finance revenues are taxes) in the endogenous growth theory. The heterogeneous endogenous non-linear tax researches inherit the thought of Mirrlees (1971) that tax should be respectively designed according to idiosyncratic labor skills levels. Golosov et al. (2003) shows that agents' private information on their skill shocks lead to reverse Euler equation holding and the wedge existing; Golosov and Tsyvinski (2007) proves taxation of capital income can be employed into private market to introduce wedge as better insurance policy; Findeisen and Sachs (2016) substitutes the skill shocks for educational shocks, and study the optimal design of aggregate education finance and tax schedule. However, at a dynamic social optimum, the Euler equation should intertemporal hold true to make the marginal social benefit equal to marginal social cost, so the social planner, such as government, should design a non-linear income tax schedule to eliminate the wedge and transform the reverse Euler equation into the standard Euler equation. Kocherlakota (2005) illustrates, in an economy with private information on idiosyncratic labor skills, aggregate heterogeneous non-linear capital income tax is zero; this implies taxation is rather a kind of the mean of income redistribution (Werning, 2007) than an impetus to economic growth as in Barro (1990), in which the public finance sector should constantly supply public goods as intermediate products to final goods firms subject to the constraint of the fiscal budget. In Ramsey framework, with sound conceptions, if social planner in an economy having idiosyncratic skill shocks needs to achieve the social welfare maximum, the feasible optimal tax schedule is used not only to eliminate wedges but also to supply optimal size of externalities. In following, we concisely review relevant literatures on heterogeneous endogenous non-linear income taxes and optimal public finance revenues, respectively.

1. Literatures on heterogeneous endogenous non-linear taxes

Concretely, in the framework of new dynamic public finance theory, researches on heterogeneous endogenous non-linear taxes are generally built on the assumptions of existence of idiosyncratic skill shocks, such as Kocherlakota (2005), Albanesi and Sleet (2006), Golosov and Tsyvinski (2006). In their framework, the goal of non-linear taxes is to eliminate wedges triggered by idiosyncratic skill shocks, and make the standard Euler equation hold in a competitive economy. The agent's private information on her idiosyncratic skill shocks cause the appearance of the inverse Euler equation (Golosov et al., 2003), the non-linear capital income tax is committed to make the agent's inverse Euler equation equal to the corresponding standard Euler equation. Similarly, the non-linear labor income tax is designed to eliminate wedge between the agent's consumption-labor marginal rates of substitution and marginal rates of transformation (Golosov and Tsyvinski, 2006). Chamley (2001) finds out that although heterogeneous capital income tax is non-zero under the condition of borrowing constraints, the aggregation is zero when idiosyncratic skill shocks can be fully insured. Besides, Werning (2007) finds out the aggregate heterogeneous labor income tax is non-zero even if idiosyncratic skill shocks are fully insured. Piketty and Saez (2013) develops a model consisting in heterogeneous saving and bequests, and two taxation instruments are simultaneously designed to eliminate both wedges. The new dynamic public finance theory has two disadvantages: first, no matter what the aggregate capital income tax is zero or not, there is no usage of the non-zero aggregate labor income tax, which implies the heterogeneous non-linear taxes cause welfare loss under the pre-assumption of social welfare optimum; second, the simulation calculations increase exponentially to infinity with periods running to infinity. To simplify calculations, simulation works usually deal with 2 periods economy and let $\beta R = 1$, thus boundary conditions are generally regarded as variables to be solved, and rest variables' values are exogenously given rather than endogenously expressed by coefficients.

2. Literatures on optimal public finance revenues.

There are hugely numerous literatures on optimal public finance revenues. In here, we only focus our attentions on optimal public finance revenues in endogenous growth theory. The key difference between heterogeneous endogenous non-linear taxes in new dynamic public finance theory and optimal public finance revenues in endogenous growth theory is the placement of taxation. Heterogeneous endogenous non-linear taxes in new dynamic public finance theory are committed according to expected shock realization and current shock realization, and public expenditures are just treated as the compensation term in the government budget constraint. Whereas, the endogenous growth theory emphasizes the externality of the public finance, central planner uses optimal public finance revenues to constantly invest in the external sector which is crucial for growth and social welfare optimum. The taxation, in an economy with growth, is employed as a funds origin of public expenditure in external sector, such as seminal works of Barro (1990), Lucas (1990), and following works as Tamai (2006), Agénor (2008), Tamai (2008), Greiner (2012), Figuières et al. (2013), Aghion et al. (2013), Novales et al. (2014), Teles and Mussolini (2014), Irmen and Tabakovic (2017).

With reasonable ideas, heterogeneous endogenous non-linear income taxes and optimal public finance revenues can be solved separately in Ramsey framework as Saez and Stantcheva (2018), although the growth rate is exogenously given in their work. Akcigit et. al (2019) attempts to depict heterogeneous endogenous non-linear income taxes and subsidies in an environment with innovations' overspill effect; the heterogeneous taxations make the innovators' marginal profits and marginal costs equal in a partial equilibrium economy. However, the growth rate caused by overspill effect is usually exogenously given as Grossman and Helpman (1991), Aghion and Howitt (1992).

Unluckily, Jones (1995) criticizes that the endogenous growth theory is actually semi-endogenous, so our work setups an environment to make growth rate completely endogenous in the dynamic general equilibrium economy. In contrast to static works such as Ales et. al (2015), Ales and Sleet (2016), Rothschild and Scheuer (2016), Lockwood et. al (2017), we introduce the public finance sector into the environment setting; public finance sector not only eliminates wedges but also dynamically supplies externalities subject to the constraint of the taxation. Unlike Jaimovich and Rebelo (2017), where tax rates are exogenously given, we incorporate heterogeneous endogenous non-linear income taxes with optimal public finance revenues, and separately solve them which are contingent on growth rates. Using the non-expansion fixed point theorem, we prove the existences and uniqueness of the steady state growth rate and prices under the condition of the constant return to scale public goods accumulation.

The rest of the paper is organized as follows. The section II is the economic environment; in this section, we introduce heterogeneous endogenous non-linear income taxes, and show the equilibrium allocation's consistence of the agent and the central planner. The section III is the FOCs by the form of state-contingent. The section IV is the existence of the unique steady state growth rate; we prove the existence and uniqueness of steady state growth rate with incomplete information. The section V is the simulation; firstly, we simulate ratios of optimal public finance revenues to gross output and steady sate growth rates; secondly, under different steady state growth rate levels, we compute optimal quantities and endogenous non-linear income taxes in 2 periods economy; thirdly, in terms of optimal flat-rate tax, we depict trade-off between linear capital income tax and linear labor income tax. The section VI is the conclusion.

II. Economic Environment

As in Kocherlakota (2005), Albanesi and Sleet (2006), Golosov and Tsyvinski (2006), agents' private information on their idiosyncratic skill shocks lead to the reverse Euler equation hold, and the heterogeneously endogenous non-linear capital income tax is used to eliminate idiosyncratic skill shocks to keep the standard Euler equation hold. The taxation is the only channel for central planner, government, to finance for supplies of public goods. Public goods are freely used in public goods sector, and nonexclusive between public finance sector and final goods firms. Public goods, which can be treated as intermediate products, are competitively used among identical final goods firms. The government shares the same utility function with agents' and makes the aggregate utility maximization. According to Greiner and Semmler (2000), the government determines the optimal public finance revenues in every period and constantly invests them into public goods accumulation to keep economic growth.

1. Preferences

We assume there are many homogenous preference's agents with the unit measure in the finite or infinite periods economy. The agent maximizes her expected utility:

$$\sum_{t=0}^{T} E_0[\beta^t \left(u(c_t) - v(l_t)\right)], \qquad 0 < \beta < 1 \tag{1}$$

where T stands for the period which may converge to $+\infty$, β is discount rate. c_t

represents the consumption and l_t stands for the work effort of the agent in period t. The expected utility function is bounded, and u', $-u''$, v', v'' are all strictly positive.

2. Idiosyncratic shocks on skills and the effective labor

We rule out multiple idiosyncratic shocks and the assumption of incomplete market, the only idiosyncratic shocks come from labor skills. Inheriting the definition of skills as Golosov et al. (2003), agents' skills differ across agents and over all periods. The structure of skill shocks work as follows. Let Θ be a Borel set in R_+, and let μ_Θ is a probability measure defined on the Borel subsets of Θ^T. At the beginning of period 0, Nature chooses a random skill shock sequence of θ^T for each agent in the Θ^T; θ_t is skill realization for the agent in the period t. The choosing processes of the Nature are independent across agents, then we assume the large number law holds, which implies that the proportion of agent with the skill history θ^T is $\mu_\Theta(\theta^T)$. Every agent will not learn the skill realization θ_t until the beginning of the period t; in other words, every agent only knows her skill history $\theta^t = (\theta_1, \cdots, \theta_t)$ at the beginning of the period t. The agent with skill history θ^T can generate effective labor y_t in every period as follows:

$$y_t(\theta^T) = \theta_t l_t \tag{2}$$

where y_t can be observed as the public information; θ_t and l_t are private information that can only be observed by the agent herself.

3. Labors in the final goods production and the public goods accumulation

Every agent in every period allocates her total work effort (l_t) between public goods accumulation ($l_{1,t}$) and final goods production ($l_{2,t}$), respectively, and $l_{1,t} + l_{2,t} = l_t$. In addition, every agent has same labor endowment allocated between production and accumulation in every period, which implies $l_{1,t} + l_{2,t} = l$. For every period, aggregate effective labor the agent supplies is $Y_t = \int_{\theta^T \in \Theta^T} y_t(\theta^T)\, d\mu_\Theta = l \int_{\theta^T \in \Theta^T} \theta^T\, d\mu_\Theta = l\bar{C}$, where $\bar{C}$ represents the expected value of θ^T in absolute value, so the aggregate effective labor allocated respectively in the public goods accumulation and final goods production are $Y_{1,t} = \int_{\theta^T \in \Theta^T} y_{1,t}(\theta^T) = l_{1,t} \int_{\theta^T \in \Theta^T} \theta^T\, d\mu_\Theta = l_{1,t}\bar{C}$ and $Y_{2,t} = \int_{\theta^T \in \Theta^T} y_{2,t}(\theta^T) = l_{2,t} \int_{\theta^T \in \Theta^T} \theta^T\, d\mu_\Theta = l_{2,t}\bar{C}$.

4. The optimal public finance revenues

The key difference between our work and Golosov et al. (2003), Kocherlakota (2005) is the definition of T_t, which can be understood as the optimal public finance revenues in our work. In their setup, the T_t is treated as an exogenous variable, and the government's task is only to eliminate wedges. Since aggregate heterogeneous non-linear capital income tax is zero, naturally, the T_t only includes the aggregate heterogeneous non-linear labor income tax which fills up the gap between the agent's consumption-labor marginal rates of substitution and marginal rates of transformation. Unlike the setup of Irmen and Tabakovic (2017), in which public finance revenues is just a compensation term in the capital transitional equation, the T_t in our work is endogenously determined. In here, the government's tasks are eliminating the effective labor's wedge as well as funding sufficient revenues to optimally invest in the public goods accumulation, and the optimal public finance revenues in every period composes of two parts: one is the aggregate heterogeneous endogenous non-linear labor income tax; the other is the flat-rate tax which can be treated as the difference between optimal public finance revenues and aggregate heterogeneous endogenous non-linear labor income tax. In other words, $T_t = {T_t}^W + {T_t}^F$, where ${T_t}^W$ is the aggregate heterogeneous endogenous non-linear labor income tax; ${T_t}^F$ is the flat-rate tax.

5. The feasible allocation

An allocation (c, y, k) is a mapping that can be expressed as follows:

$$c: \Theta^T \to R_+^T \tag{3}$$

$$y: \Theta^T \to R_+^T \tag{4}$$

$$k: \Theta^T \to R_+^{T-1} \tag{5}$$

$$(c_t, y_t, k_{t+1}) \text{ is } \theta^t\text{-measurable}$$

where k stands for capital that the agent owns under a given skill realization.

Let $F(K_t, G_t \cdot Y_{2,t})$ be an aggregate production function that is constant return to scale with respect to its two arguments, δ be the capital depreciation rate. An allocation (c, y, k) is feasible if and only if :

$$C_t + K_{t+1} + T_t \leq F(K_t, G_t \cdot Y_{2,t}) + (1 - \delta)K_t, \qquad \forall t \tag{6}$$

$$C_0, K_0, G_0 > 0$$

$$C_t = \int_{\theta^T \in \Theta^T} c_t(\theta^T)\, d\mu_\Theta \tag{7}$$

$$Y_{2,t}=\int_{\theta^T\in\Theta^T} y_{2,t}(\theta^T)\,d\mu_\Theta \tag{8}$$

$$K_t=\int_{\theta^T\in\Theta^T} k_t(\theta^T)\,d\mu_\Theta \tag{9}$$

Here, C_t, $Y_{2,t}$ and K_t stand respectively for per capita consumption, per capita effective labor allocated in the final goods production and per capita capital. G_t is a variable that depends on its initial value, we call it the stock of public goods in period t.

6. Competitive markets

There are many identical final goods firms with the unit measure, these firms use K_t and $G_t \cdot Y_{2,t}$ as factors to produce final goods with constant return to scale production. In every period, firms decide the quantities of K_t and $Y_{2,t}$ they rent to maximize their profits at given prices level in completely competitive markets, so marginal outputs of K_t and $Y_{2,t}$ with respect to F_t respectively equal to interest rate (r_t) and wage rate (w_t).

7. Public goods accumulation

To keep the non-exclusive property of public goods, let public goods accumulation function be the non-exclusive human capital accumulation function as Brueckner (2006) or the intermediate goods quality function as Akcigit et.al (2019):

$$G_t={T_t}^{v}(G_{t-1}Y_{1,t})^{1-v},\qquad 0<v<1 \tag{10}$$

where v is the elasticity of public finance revenues with respect to the stock of public goods.

Public goods are accumulated only in public finance sector, and the public goods accumulation is constant return to scale. We employ Romer (1990)'s intermediate products' introduction mechanism: in every period, the public finance sector sells G_t to final goods firms at a given price level $P_{G,t}$, and rents $Y_{1,t}$ in the competitive labor market with a given wage rate $\widetilde{w}_t$, and public finance sector's profit is zero. Consequently, the following equation must hold true:

$$P_{G,t}G_t=\widetilde{w}_tY_{1,t} \tag{11}$$

where $\widetilde{w}_t=w_t\cdot G_{t-1}$ is the wage rate that public finance sector pays off in competitive labor market. It's worth noting that this introduction mechanism in competitive economy rules out the extra tax on effective labor income under the condition of disproportionate wage as in Stantcheva (2020). Actually, according to equations (10) and (11), it is easy to deduce that T_t is not contingent on θ^T, which is confirmed in proposition 1.

Proposition 1. In a competitive economy with private information (θ^T), Let $G_t(\theta^T)$ be the stock of public goods in period *t* under the given skill realization state (θ_t), which is expressed as $G_t(\theta^T) = T_t{}^v(G_{t-1}y_{1,t}(\theta^T))^{1-v}$, then $G_t = \int_{\theta^T \in \Theta^T} G_t(\theta^T)\, d\mu_\Theta$.

Proof

According to the above environment setting, since $G_{t-1}(\theta^T)$ is already known at the beginning of period t, $\widetilde{w}_t$ is not contingent on θ^T, so it is easy to check that $P_{G,t}$ is also not contingent on θ^T. Since $G_t(\theta^T)$ and G_t share the same function expression, T_t does not conclude private information, according to equation (10) and (11), we will have the following equations hold true:

$$P_{G,t}G_t = \widetilde{w}_t Y_{1,t} \tag{12}$$

$$P_{G,t}G_t(\theta^T) = \widetilde{w}_t y_{1,t}(\theta^T) \tag{13}$$

According to the definition of $Y_{1,t}$, taking expectations on θ^T of both sides in the equation (13), and comparing with the equation (12), we can easily show that $G_t = \int_{\theta^T \in \Theta^T} G_t(\theta^T)\, d\mu_\Theta$ holds true.

Q.E.D.

8. Aggregate final goods production

There are many identical final goods firms with unit measure, every final goods firm obtains the same quantity of public goods (G_t) because of the non-exclusiveness of public goods, so there exists a representative final goods firm in the competitive economy. Inheriting the setup of Chen (2007) ,the public goods, which can be interpreted as designs in Romer (1990) or knowledge in Lucas and Moll (2014), combining with aggregate effective labor allocated in final goods production $(Y_{2,t})$ as one factor $G_t \cdot Y_{2,t}$ in aggregate production function (F_t), that is to say G_t is an augment to the effective labor.

Proposition 2. In a competitive economy with private information (θ^T), Let $F_t(\theta^T) = F\big(k_t(\theta^T), G_t \cdot y_{2,t}(\theta^T)\big)$ be the representative final goods firm's production function at period t under the given skill realization (θ_t), which has the same functional form as F_t and satisfies constant returns to scale, then $F_t = \int_{\theta^T \in \Theta^T} F_t(\theta^T)\, d\mu_\Theta$.

Proof

Defining $\widehat{w}_t = w_t \cdot G_t$ as the wage rate that final goods firms pay off in competitive labor market. According to the above environment setting, it is easy to check that $\widehat{w}_t$ and r_t are not contingent on θ^T. Since $F_t(\theta^T)$ and F_t share the same function expression and are both constant return to scale, then F_t and $F_t(\theta^T)$can also be rewritten as follows:

$$F_t = r_t K_t + \widehat{w}_t \cdot Y_{2,t} \tag{14}$$

$$F_t(\theta^T) = r_t k_t(\theta^T) + \widehat{w}_t y_{2,t}(\theta^T) \tag{15}$$

According to the definition of $Y_{2,t}$ and K_t, taking expectations on θ^T of both sides in the equation (15), and comparing with the equation (14), we can easily show that F_t=$\int_{\theta^T \in \Theta^T} F_t(\theta^T)\, d\mu_\Theta$ holds true.

Q.E.D.

Since F_t is constant return to scale, without loss of generality, we define F_t as follows:

$$F\left(K_t, G_t \cdot Y_{2,t}\right) = K_t{}^{\alpha} \cdot \left(G_t \cdot Y_{2,t}\right)^{1-\alpha}, \qquad 0 < \alpha < 1 \tag{16}$$

In addition, the section IV shows that there exists a unique balanced growth path that makes r_t and w_t are time-invariant.

9. The agent's equilibrium allocation

At the beginning of the period 0, the government declares a tax schedule $\tau = \{\tau\}_{t=1}^{T}$, where $\tau_t: R^t \times R \times R \to R$ is a mapping from the agent's history of effective labor incomes in final goods production, capital income and aggregate output level in period t to the public finance revenues in period t. Given $(\tau, r, \widehat{w})_{t=0}^{T}$, the agent's problem can be shown as follows:

$$\max_{(c,y,k)} \sum_{t=0}^{T} \beta^t \int_{\theta^T \in \Theta^T} [u\left(c_t(\theta^T)\right) - v\left(\frac{y_{1,t}(\theta^T)}{\theta_t}\right) - v\left(\frac{y_{2,t}(\theta^T)}{\theta_t}\right)]\, d\mu_\Theta \tag{P1}$$

$$s.t. \qquad c_t(\theta^T) + k_{t+1}(\theta^T) + \tau_t\left(\left(\widehat{w}_s \cdot y_{2,s}(\theta^T)\right)_{s=1}^{t}, r_t k_t(\theta^T), F_t\right)$$

$$\leq \ (1-\delta+r_t)k_t(\theta^T) + \widehat{w}_s y_t(\theta^T), \qquad \text{for all } (t, \theta^T)$$

$$(c_t, y_t, k_{t+1}) \text{ is positive and } \theta^T\text{-measurable}$$

Given the choice of government τ, an equilibrium in this economy is an allocation (c, y, k) and prices (r, w) which solves $(P1)$ problem.

10. The government's equilibrium allocation

The government shares the same expected utility function with the agent's, so at the beginning of period 0, he chooses tax schedule τ to maximize his aggregate expected utility on θ^T. In competitive economy, the market clearing condition is automatically satisfied in every period, so the government's problem can be expressed as follows:

$$\max_{(c,y,k,\tau)} \sum_{t=0}^{T} \beta^t \int_{\theta^T \in \Theta^T} [u\big(c_t(\theta^T)\big) - v\left(\frac{y_{1,t}(\theta^T)}{\theta_t}\right) - v\left(\frac{y_{2,t}(\theta^T)}{\theta_t}\right)]\, d\mu_\Theta$$

$$s.t. \text{for all } (t,\theta^T) \qquad C_t + K_{t+1} + T_t = F\big(K_t, G_t \cdot Y_{2,t}\big) + (1-\delta)K_t \qquad (17)$$

$$T_t = \int_{\theta^T \in \Theta^T} \tau_t\left(\left(\widehat{w}_s \cdot y_{2,s}(\theta^T)\right)_{s=1}^{t}, r_t k_t(\theta^T), F_t\right) d\mu_\Theta$$

$$C_0, T_0, G_0, K_1 > 0$$

According to above interpretations, the government's equilibrium allocation (C,Y,K) is actually aggregation of the agent's allocation (c,y,k) on θ^T, which implies that the government treats every agent indifferently. As Atkeson and Lucas (1992), in an economy with idiosyncratic skill shocks, an equilibrium allocation should also satisfy incentive-compatible; furthermore, Kapička (2013) proves that equilibrium allocation is unchangeable by replacing incentive-compatible constraint with envelope condition, consequently, our analysis in next section abandons the incentive-compatible constraint, instead of using FOCs.

III. FOCs by the form of state-contingent

θ_t can be treated as a constant in every period, let the utility function be $u(c_t(\theta^T)) - v(l_t(\theta^T)) = lnc_t(\theta^T) - \gamma_1 lny_{1,t}(\theta^T) - \gamma_2 lny_{2,t}(\theta^T)$, γ_1 and γ_2 stand respectively for the agent's preferences on labors allocated into public goods accumulation and final goods production. The optimal problem of the government can be expressed as:

$$\underset{(c,y,k,\tau)}{Max} \sum_{t=0}^{\infty} \beta^t \int_{\theta^T \in \Theta^T} lnc_t(\theta^T) - \gamma_1 lny_{1,t}(\theta^T) - \gamma_2 lny_{2,t}(\theta^T)\, d\mu_\Theta \qquad (P2)$$

$$s.t. \qquad K_{t+1} = F_t\big(K_t, G_t \cdot Y_{2,t}\big) + (1-\delta)K_t - C_t - T_t$$

$$G_t = T_t{}^{v}(Y_{1,t}G_{t-1})^{1-v}$$

$$F_t\big(K_t, G_t \cdot Y_{2,t}\big) = K_t{}^{\alpha}(G_t \cdot Y_{2,t})^{1-\alpha}$$

$$T_t = \int_{\theta^T \in \Theta^T} \tau_t\left(\left(\widehat{w}_s \cdot y_{2,s}(\theta^T)\right)_{s=1}^{t}, r_t k_t(\theta^T), F_t\right) d\mu_\Theta$$

$$C_t > 0, T_t > 0, G_t > 0, K_{t+1} > 0, \qquad t = 0,1,2\cdots$$

The Lagrange function of $(P2)$ problem can be represented by state-contingent form as follows:

$$L(c,y,k,\tau)=\sum_{t=0}^{\infty}\beta^t\int_{\theta^T\in\Theta^T}[lnc_t(\theta^T)-\gamma_1 lny_{1,t}(\theta^T)-\gamma_2 lny_{2,t}(\theta^T)$$
$$-\lambda_{1,t}(\theta^T)k_{t+1}(\theta^T)+\lambda_{1,t}(\theta^T)G_t^{1-\alpha}k_t(\theta^T)^{\alpha}y_{2,t}(\theta^T)^{1-\alpha}$$
$$-\lambda_{1,t}(\theta^T)c_t(\theta^T)-\lambda_{1,t}(\theta^T)T_t+\lambda_{1,t}(\theta^T)(1-\delta)k_t(\theta^T)-\lambda_{2,t}(\theta^T)G_t$$
$$+\lambda_{2,t}(\theta^T)G_{t-1}^{1-v}{T_t}^{v}y_{1,t}(\theta^T)^{1-v}]d\mu_{\Theta}$$

First order conditions are:

$$\frac{\partial L}{\partial c_t(\theta^T)}=\frac{1}{c_t(\theta^T)}-\lambda_{1,t}(\theta^T)=0 \tag{18}$$

$$\frac{\partial L}{\partial y_{1,t}(\theta^T)}=-\gamma_1\frac{1}{y_{1,t}(\theta^T)}+\lambda_{2,t}(\theta^T)(1-v)G_{t-1}^{1-v}{T_t}^{v}y_{1,t}(\theta^T)^{-v}=0 \tag{19}$$

$$\frac{\partial L}{\partial y_{2,t}(\theta^T)}=-\gamma_2\frac{1}{y_{2,t}(\theta^T)}+\lambda_{1,t}(\theta^T)(1-\alpha)G_t^{1-\alpha}k_t(\theta^T)^{\alpha}y_{2,t}(\theta^T)^{-\alpha}=0 \tag{20}$$

$$\frac{\partial L}{\partial T_t}=-\lambda_{1,t}(\theta^T)+\lambda_{2,t}(\theta^T)vG_{t-1}^{1-v}{T_t}^{v-1}y_{1,t}(\theta^T)^{1-v}=0 \tag{21}$$

$$\frac{\partial L}{\partial k_t(\theta^T)}=\lambda_{1,t}(\theta^T)(\alpha G_t^{1-\alpha}k_t(\theta^T)^{\alpha-1}y_{2,t}(\theta^T)^{1-\alpha}+1-\delta)-\frac{\lambda_{1,t-1}(\theta^T)}{\beta}=0 \tag{22}$$

Labors of the agent supplies are homogeneous, so the marginal utilities of labors allocated in the final goods production and the public goods accumulation must be equal. Thus we have $\gamma_1/y_{1,t}(\theta^T)=\gamma_2/y_{2,t}(\theta^T)$, and expectations of $y_{1,t}(\theta^T)$ and $y_{2,t}(\theta^T)$ can be arranged as follows:

$$\frac{Y_{1,t}}{Y_{2,t}}=\frac{\gamma_1}{\gamma_2}\quad and\quad Y_t=Y_{1,t}+Y_{2,t}=l\bar{C} \tag{23}$$

According to the equation (23), the optimal aggregate effective labors allocation ratio in every period is constant. Rearranging the equation (18) and equation (20) respectively, we can obtain:

$$\gamma_2 c_t(\theta^T)=(1-\alpha)G_t^{1-\alpha}k_t(\theta^T)^{\alpha}y_{2,t}(\theta^T)^{1-\alpha} \tag{24}$$

Taking expectations on θ^T of both sides in the equation (24), we can obtain:

$$\frac{C_t}{F_t} = \frac{(1-\alpha)}{\gamma_2} \tag{25}$$

The equation (25) implies that C_t and F_t share the same growth rate in the balanced growth path. Substituting $\lambda_{1,t}(\theta^T)$ with $c_t(\theta^T)$ and putting it into equations (19) and (21), we can obtain:

$$\gamma_1 \frac{1}{y_{1,t}(\theta^T)} = \frac{(1-v)G_{t-1}^{1-v}T_t(\theta^T)^v y_{1,t}(\theta^T)^{-v}}{c_t(\theta^T)vG_{t-1}^{1-v}T_t(\theta^T)^{v-1}y_{1,t}(\theta^T)^{1-v}} \tag{26}$$

Taking expectations on θ^T of both sides in the equation (26), we can obtain:

$$\frac{C_t}{T_t} = \frac{(1-v)}{v\gamma_1} \tag{27}$$

The equation (27) means that C_t and T_t share the same growth rate in the balanced growth path. Combining equation (25) with equation (27), we can obtain:

$$\frac{T_t}{F_t} = \frac{\gamma_1 v(1-\alpha)}{(1-v)\gamma_2} \tag{28}$$

Based on equations (25), (27) and (28), it is easy to check that C_t, F_t and T_t share the same growth rate in the balanced growth path. It is worth noting here that T_t is positive and includes a endogeneous nonlinear labor income tax which is contingent on θ^T as shown in Werning (2007).

IV. The existence of the unique steady state growth rate

In the section, we make the growth rate completely endogenous in an environment with externalities. Without needs of the extra assumption that growth rate has already existed as Chen (2007), Gonzalez-Eirasa and Niepelt (2020), or the assumption on given prices which guarantees the existence of the growth as Long and Pelloni (2017), the growth rate in our work can be parameterized as endogenous growth works as Lucas (1988) and Romer (1990). Firstly, we prove that every aggregate variable (by the form of expectation) shares the same growth rate in the steady state; secondly, we prove that the growth rate is larger than 1. Defining the growth rate of the aggregate variable X_t in the period $t+1$ as $g_{X,t+1} = X_{t+1}/X_t$, therefore, X_t increases when $g_X > 1$; X_t will converge to one certain non-zero value when $g_X = 1$, and X_t converges to 0 when $g_X < 1$, we rule out this situation. Hence, let $g_X \geq 1$ and assume $g_X \leq \bar{g}$ where $\bar{g}$ is a real number far greater than 1. Then $g_{X,t+1} \in [1, \bar{g}] \equiv A$, where A is a compact convex set.

Assumption 1. Variables C_t, T_t, F_t satisfy equations (25), (27) and (28).

Assumption 2. F_t satisfies the Inada condition: $\lim_{k \to 0} F'_{t,k} \to \infty$, $\lim_{k \to \infty} F'_{t,k} \to c$, c is a positive.

Assumption 3. At the beginning of period 0, the economy is already in the steady state.

According to the assumption 1, let $T_t = a_1 F_t$, where $a_1 = \gamma_1 v(1-\alpha)/(1-v)\gamma_2$. The equation (17) can be rewritten as:

$$[g_{K,t+1} - (1-\delta)]K_t = (1-a_1)F_t - g_{C,t}C_{t-1}$$

Defining $\tilde{g}_{K,t+1} = g_{K,t+1} - (1-\delta)$, the equation (17) can be further transformed as:

$$\tilde{g}_{K,t+1}K_t - (1-a_1)F_t = g_{C,t}\left[\tilde{g}_{K,t}K_{t-1} - (1-a_1)F_{t-1}\right]$$

and

$$\frac{1}{\tilde{g}_{K,t+1}K_t - (1-a_1)F_t} = \frac{1}{g_{C,t}}\left[\frac{1}{\tilde{g}_{K,t}K_{t-1} - (1-a_1)F_{t-1}}\right] \tag{29}$$

Since $g_{X,t} \in [1, \bar{g}] \equiv A$ and assumption 2 holds, it is easy to check that $\tilde{g}_{K,t+1}K_t - (1-a_1)F_t$, $\forall$t, is bounded and belongs to a closed set. Combining with $0 < 1/g_{C,t} \leq 1$, $\forall t$, the Non-expansion fixed point theorem (Stachurski, 2009, pp.52-53) shows the following equation holds:

$$\tilde{g}_{K,t+1}K_t - (1-a_1)F_t = \tilde{g}_{K,t}K_{t-1} - (1-a_1)F_{t-1} \tag{30}$$

The next step shows that all aggregate variables share the same growth rate. Transforming the equation (30) as below:

$$\tilde{g}_{K,t+1} \cdot g_{K,t} - \tilde{g}_{K,t} = (1-a_1)\frac{F_{t-1}}{K_{t-1}}\left(g_{F,t} - 1\right) \tag{31}$$

Situation 1. when $g_{K,t} = 1$, $\forall t$.

Combining $g_{K,t} = 1$ with $\tilde{g}_{K,t+1} = \tilde{g}_{K,t}$, $\forall t$, the left side of equation (30) equals to 0. According to the assumption 2, it is easy to check that $F_{t-1}/K_{t-1} \neq 0$, and we can obtain $g_{F,t} = 1$, $\forall t$. Accoding to assumption 1, the following equations hold true:

$$g_{K,t} = g_{F,t} = g_{C,t} = g_{T,t} = 1 \tag{32}$$

Situation 2. when $g_{K,t} > 1$, $\forall t$

The situation 2 has two parts. In the first part, we will show that the left side of the equation (31) is larger than 0, which is stated in proposition 3. In the second part, we use the monotonic fixed point theorem to show that all aggregate variables share the same growth rate which is larger than 1.

Proposition 3. $\tilde{g}_{K,t+1} \cdot g_{K,t} - \tilde{g}_{K,t} > 0$, when $g_{K,t} > 1$.

Proof

We prove the proposition 3 by contradiction. We firstly assume that the left side of equation (31) is less than or equals to 0, then according to assumption 2, it is easy to check that $g_{F,t} \leq 1$ in the right side of equation (31). Because of $g_{K,t} > 1$ and $g_{F,t} \leq 1$, we can obtain that $\lim_{t\to\infty} \frac{K_t}{F_t} \to \infty$, which implies that $\lim_{t\to\infty} r_t \to 0$. According to the Euler condition: $\lim_{t\to\infty} g_{c,t} = \beta(1-\delta) < 1$, which is out of the domain.

Q.E.D.

According to proposition 3, we have shown that $\tilde{g}_{K,t+1} \cdot g_{K,t} - \tilde{g}_{K,t} > 0$ under situation 2, Let $m_t < 0$ be a exogenous variable with respect to t, which keeps the following equation hold true:

$$\tilde{g}_{K,t+1} \cdot g_{K,t} - \tilde{g}_{K,t} + m_t = 0 \tag{33}$$

It is easy to check that $-m_t/g_{K,t} > 0$ and $1/g_{K,t} < 1$. According to Theorem 18.E in Zeidler (1990, pp.68-69), the equation (33) has a unique fixed point, that is:

$$\begin{aligned} \tilde{g}_{K,t+1} &= \tilde{g}_{K,t} = g - (1-\delta) \\ g_{K,t+1} &= g_{K,t} = g > 1, \quad \forall t \end{aligned} \tag{34}$$

Again, dividing K_t in both sides of the equation (17), we can obtain:

$$\frac{K_{t+1}}{K_t} - (1-\delta) = \frac{F_t}{K_t} - \frac{C_t}{K_t} - \frac{T_t}{K_t}$$

$$g - (1-\delta) = (1 - a_1 - a_2)\frac{F_t}{K_t} \tag{35}$$

where $a_2 = (1-\alpha)/\gamma_2$, we can identify that $g_{F,t} = g_{K,t}$ by equation (35), then according to assumption 1 and equation (34), we can show the following equations hold true:

$$g_{K,t} = g_{F,t} = g_{C,t} = g_{T,t} = g > 1 \tag{36}$$

The situation 2 depicts the steady state with growth. In the steady state, all aggregate variables share the same growth rate which can be expressed by coefficients; and we can show that the interest rate in the steady state is a constant, too. The steady state growth rate and interest rate are shown as follows:

$$g = \beta\left[\alpha \cdot \frac{(1-\delta)(1-\beta)}{\Delta} + (1-\delta)\right] \tag{37}$$

$$r = \alpha \cdot \frac{(1-\delta)(1-\beta)}{\Delta} \tag{38}$$

where $\Delta = [\alpha\beta + \frac{(1-\alpha)}{\gamma_2} + \frac{\gamma_1 v(1-\alpha)}{(1-v)\gamma_2} - 1]$.

According to equation (10), the steady-state relations of the rest aggregate variables are shown as follows:

$$\frac{T}{G} = (\frac{\gamma_1}{\gamma_1+\gamma_2} l\bar{C})^{\frac{v-1}{v}} \cdot g^{\frac{1-v}{v}} \tag{39}$$

$$\frac{F}{G} = \frac{(1-v)\gamma_2}{\gamma_1 v(1-\alpha)} \cdot \left(\frac{\gamma_1}{\gamma_1+\gamma_2} l\bar{C}\right)^{\frac{v-1}{v}} \cdot g^{\frac{1-v}{v}} \tag{40}$$

From equation (39) and (40), it is easy to check that $g_{K,t} = g_{F,t} = g_{C,t} = g_{T,t} = g_{G,t}$ hold true in the steady rate. Furthermore, $r_t K_t = \alpha F_t$ *and* $(1-\alpha)F_t = w_t G_t Y_{2,t}$ imply that r_t and w_t are time-invariant in the steady state.

In addition, we can give proposition 4 to show that the central planner, government, will rule out the situation 1 under the above environment setting, that is to say there exists a unique balanced growth path.

Proposition 4. The central planner, government, will always keep the steady state growth rate larger than 1 under the above environment setting.

Proof

According to equation (11), $y_{1,t}(\theta^T)$ is not equals to zero in every period, combining with $\gamma_1/y_{1,t}(\theta^T) = \gamma_2/y_{2,t}(\theta^T)$ and $y_t(\theta^T) = y_{1,t}(\theta^T) + y_{2,t}(\theta^T) = \theta_t l$ in every period, it is easy to induce that the optimal effective labor allocation ${y_{1,t}}^*(\theta^T)$ and ${y_{2,t}}^*(\theta^T)$ are already known under a given θ^T and not contingent on steady state growth rate.

Situation 1 (competitive economy). Because of non-competitiveness of the public goods, no competitive firm wants to supply this product; then exogenously given level of G_0 will not change in the following period, the aggregate final goods production function can be expressed by variables K_t and $Y_{2,t}$, satisfying constant return to scale, then the steady growth rate automatically converges to 1 (Stokey and Lucas, 1989). The social welfare level can be expressed as:

$$W_1 = \sum_{t=0}^{\infty} \beta^t \int_{\theta^T \in \Theta^T} ln \prod_{i=1}^{t} g_i(\theta^T)\, c_0^*(\theta^T) - \gamma_1 lny_{1,t}{}^*(\theta^T) - \gamma_2 lny_{2,t}{}^*(\theta^T)\, d\mu_\Theta$$

where $c_0^*(\theta^T)$ represents the steady state consumption level according to assumption 3, $g_i(\theta^T) = c_i(\theta^T)/c_{i-1}(\theta^T)$ stands for the agent's consumption growth rate under situation 1, $g_i(\theta^T)$ may be larger than 1 or smaller than 1 with different θ^T, and $\int_{\theta^T \in \Theta^T} g_i(\theta^T)\, d\mu_\Theta = 1$ holds true for $\forall i$. Every $g_i(\theta^T)$ can be expressed as:

$$g_i(\theta^T) = (g(\theta^T))^n \tag{41}$$

where $g(\theta^T) \epsilon (1-\varepsilon, 1+\varepsilon)$ and n is a finite value.

Situation 2 (the economy with public finance sector). If the economy has the public finance sector to constantly supply public goods as intermediate products, the social welfare level under this situation will be:

$$W_2 = \sum_{t=0}^{\infty} \beta^t \int_{\theta^T \in \Theta^T} ln \prod_{i=1}^{t} g_i'(\theta^T)\, c_0^*(\theta^T) - \gamma_1 lny_{1,t}{}^*(\theta^T) - \gamma_2 lny_{2,t}{}^*(\theta^T)\, d\mu_\Theta$$

where $c_0^*(\theta^T)$ represents the steady state consumption level with growth, $g_i'(\theta^T) = c_i(\theta^T)/c_{i-1}(\theta^T)$ stands for the agent's consumption growth rate under situation 2, $g_i'(\theta^T)$ may also be larger than 1 or smaller than 1 with different θ^T, and $\int_{\theta^T \in \Theta^T} g_i'(\theta^T)\, d\mu_\Theta = g > 1$ holds true for $\forall i$. Every $g_i'(\theta^T)$ can also be expressed as:

$$g_i'(\theta^T) = (g'(\theta^T))^{n'} \tag{42}$$

where $g'(\theta^T) \epsilon (1-\varepsilon, 1+\varepsilon)$, n' is a finite value and $n' = n$.

The welfare level difference between the two situation is:

$$\begin{aligned} W_2 - W_1 &= \sum_{t=0}^{\infty} \beta^t \int_{\theta^T \in \Theta^T} \sum_{i=1}^{t} (lng_i'(\theta^T) - \ln g_i(\theta^T))\, d\mu_\Theta \\ &= \sum_{t=0}^{\infty} \beta^t \int_{\theta^T \in \Theta^T} \sum_{i=1}^{t} (ln(g'(\theta^T))^n - \ln(g(\theta^T))^n)\, d\mu_\Theta \\ &= \sum_{t=0}^{\infty} \beta^t \int_{\theta^T \in \Theta^T} \sum_{i=1}^{t} n(ln\widetilde{g}'(\theta^T) - \ln\tilde{g}(\theta^T))\, d\mu_\Theta \end{aligned}$$

where $1+\widetilde{g'}(\theta^T)=g'(\theta^T)$ and $1+\tilde{g}(\theta^T)=g(\theta^T)$.

According to equation (41) and (42), the following equations hold true:

$$\int_{\theta^T\in\Theta^T}(g(\theta^T))^n\,d\mu_\Theta=1$$

$$\int_{\theta^T\in\Theta^T}(g'(\theta^T))^n\;\;d\mu_\Theta=g>1$$

Substituting $g(\theta^T)$ and $g'(\theta^T)$ respectively with $1+\tilde{g}(\theta^T)$ and $1+\widetilde{g'}(\theta^T)$, due to equivalent infinitesimal transformation $(1+x)^n\sim 1+nx$, it is easy to check that $\int_{\theta^T\in\Theta^T}n\widetilde{g'}(\theta^T)\,d\mu_\Theta>\int_{\theta^T\in\Theta^T}n\tilde{g}(\theta^T)\,d\mu_\Theta$ holds true, then W_2-W_1 is automatically larger than 0.

Under the above environment setting, we know the domain of g is $[1,\bar{g}]$, since the goal for the central planner, government, is to maximize the social welfare level, he will naturally rule out the situation 1, in other words, there exists a unique balanced growth path.

Q.E.D.

V. Simulation

In this section, we simulate the optimal public finance revenues and endogenous heterogeneous non-linear income taxes, respectively. As the above sections show, the choice of optimal allocation for government is equivalent to the choice of optimal public finance revenues which make the economy grow and eliminate wedges. The taxation should be divided into two parts: one is the endogenous heterogeneous non-linear income taxes, the other is the linear flat-rate tax to compensate the difference between the optimal public finance revenues and the endogenous heterogeneous non-linear income taxes.

1. Ratios of the optimal public finance revenues to the output and steady state growth rates.

We set steady state growth rate as $g^*=g-1$. Before simulations, we should set values of coefficients. According to the Mankiw et.al (1992), Barro et.al (1995), we set $\alpha=0.33$. Capital accumulation only comes from the final goods sector in our model; according to the structural growth theory, such as Kongsamut et.al (2001), the final goods sector can be regarded as the manufactural sector, and the public goods sector in our model can be thought as the R&D sector which supplies externalities; γ_1/γ_2 is the ratio of human capitals investing into the R&D sector to that of manufactural sector, Guo and Li (2015) shows the ratio is around $1/2$, so we set $\gamma_1 = 0.47$ and $\gamma_2 = 1.08$. The rest coefficients' values are respectively $\beta = 0.98$, $\delta = 0.05$, and ν is a free coefficient, we assign different values to ν and respectively simulate steady state growth rates and ratios of the optimal public finance revenues to the output with different ν. Figure 1 shows simulating trajectories of the steady state growth rates and ratios of the optimal public finance revenues to the output.

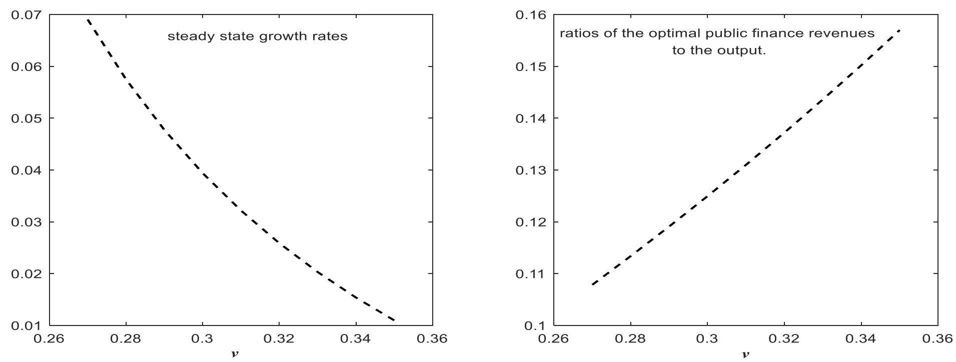

Figure 1. Trajectories of growth rates and ratios.

The figure 1 shows that the steady state growth rates are 1.02, 1.03, 1.04 and 1.05 when the corresponding ratios of the optimal public finance revenues to the output are 0.14, 0.13, 0.125 and 0.118, respectively. That is to say, growth rates share negative correlation with ratios of the optimal public finance revenues to the output as shown in Jaimovich and Rebelo (2017).

2. Endogenous heterogeneous non-linear income taxes.

Inheriting the 2 periods framework of simulations in Kocherlakota (2010) which is based on the assumption of non-growth $(\beta R = 1)$. Our work incorporates growth rate that is derived by the standard Euler equation of intertemporal consumptions, so endogenous heterogeneous non-linear income taxes are also needed to eliminate wedges to keep the standard Euler equation holding. The difference between our work and Kocherlakota (2010) is that the intertemporal relationships of aggregate variables in our work can be expressed by the steady state growth rate. The uniqueness of steady state growth rate implies $t + 1$ period simulation calculations run in the same path as these of t period, which will greatly reduce calculations in the infinite periods economy. In contrast, calculations will increase exponentially when the period goes to infinity in Kocherlakota (2005), Golosov and

Tsyvinski (2006). The simulation details are shown as:

$$U\big(c_i(\theta^T), l_i(\theta^T)\big) = lnc_i(\theta^T) - \gamma_1 ln y_{1,i}(\theta^T) - \gamma_2 ln y_{2,i}(\theta^T), \qquad i = \{0,1\}$$

$$\Theta = \{L = 0.8, H = 1.2\},\ Pr(\theta_0 = 0.8) = \mu,\ Pr(\theta_1 = \theta | \theta_0 = \theta) = \pi$$

$$c_0 = c(\theta_0), y_{2,0} = y_2(\theta_0), c_1 = c(\theta_0, \theta_1), y_{2,1} = y_2(\theta_0, \theta_1)$$

where $\mu = 0.5, \pi = 0.8$. We simulate optimal consumptions and effective labors in situations of $g = 1.01, 1.02, 1.05$. In simulation, we use 13 equations to calculate endogenously heterogeneous non-linear income taxes. 13 equations are 2 intertemporal conditions, 3 incentive compatibility conditions, 2 reverse Euler conditions and 6 first order conditions, respectively. The computed optimal quantities are shown in Table 1.

Table 1. Optimal Quantities in Simulation

	$g = 1.01$	$g = 1.02$	$g = 1.05$
c_H, y_H	141.39, 5.51	261.89, 7.39	390.16, 7.08
c_L, y_L	60.86, 3.24	121.28, 4.54	142.19, 3.75
c_{HH}, y_{HH}	154.15, 5.16	291.37, 7.16	414.74,6.70
c_{HL}, y_{HL}	97.38, 3.93	170.69, 5.18	389.44,6.56
c_{LL}, y_{LL}	55.69, 3.61	114.78, 4.91	115.25,3.58
c_{LH}, y_{LH}	84.53, 4.73	159.62, 6.13	285.53,6.42

Comparing (c_H, y_H) with (c_L, y_L) in the first column of Table 1, computed solutions have two interesting findings. First, high-skill labors supply more labor efforts than that of low skill labors; this finding is consistent with that of Wolcott (2021). Second, high-skill labors consume more than that of low-skill labors; this finding satisfies incentive-compatible constraint. The conclusion still holds in other columns. In addition, from the first 2 rows of Table 1, computed solutions show that both c_H and c_L increase when g increases, then per capita consumption naturally increase with the increasing g. This finding confirms social welfare level is contingent on g, which is stated in proposition 4. The gap between c_{HH} and c_{HL} in situation of $g = 1.01$ is larger than between c_{LH} and c_{LL}, which implies agents who are low-skilled in 0 period have better smoothing consumptions than agents who are high-skilled in 0 period; this finding also holds in situation of $g = 1.02$, which is same as Kocherlakota (2010). Nevertheless, this finding reverses in situation of $g = 1.05$, agents who are high-skilled in 0 period can better smooth their consumptions; this phenome deserves further researches.

Defining $w = 1$, the endogenous heterogeneous non-linear capital income tax rate and labor income tax rate can be solved as ${\tau_1}^k = 1 - \frac{u'(c_0)}{\beta u'(c_1)(1-\delta+r)}$,${\tau_0}^y = 1 - \frac{(\gamma_1+\gamma_2)v'(y_{2,0})}{wG_0\theta_0 u'(c_0)}$ and ${\tau_1}^y = 1 - \frac{(\gamma_1+\gamma_2)v'(y_{2,1})}{wG_1\theta_0\theta_1 u'(c_1)}$, respectively. Endogenous capital income tax rates and endogenous labor income tax rates are respectively shown in Table 2 and Table 3 as below.

Table 2. Endogenous capital income tax rates

	$g = 1.01$	$g = 1.02$	$g = 1.05$
${\tau_{HH}}^k$	0.080	0.090	0.012
${\tau_{HL}}^k$	-0.318	-0.361	-0.049
${\tau_{LL}}^k$	-0.094	-0.073	-0.228
${\tau_{LH}}^k$	0.375	0.290	0.912

According to Table 2, the total heterogeneous non-linear capital income tax in 1 period to eliminate the wedge is zero for any agent in 0 period, which is consistent with results of Kocherlakota (2005), Golosov and Tsyvinski (2006). Variance of endogenous capital income tax rates on agents who are high-skilled in 0 period is lower than that of agents who are low-skilled in 0 period in situations of $g = 1.01$ and $g = 1.05$, the finding reverses in situation of $g = 1.02$. In addition, variance of endogenous capital income tax rates on agents who are high-skilled in 0 period will increase as g increases initially and decrease afterwards; in contrast, variance of endogenous capital income tax rates on agents who are low-skilled in 0 period will decrease as g increases initially and increase afterwards.

Table 3. Endogenous labor income tax rates

	$g = 1.01$	$g = 1.02$	$g = 1.05$

${\tau_H}^y$	-0.267	-0.254	-0.298
${\tau_L}^y$	-0.390	-0.419	-0.341
${\tau_{HH}}^y$	-0.217	-0.175	-0.157
${\tau_{HL}}^y$	-0.513	-0.429	-0.663
${\tau_{LL}}^y$	-0.413	-0.519	-0.353
${\tau_{LH}}^y$	-0.091	-0.128	-0.246

According to Table 3, the total heterogeneous non-linear labor income tax in every period to eliminate the wedge is non-zero. Computed solutions in Table 3 provide concrete examples that all endogenous labor income rates should be negative as proved in Stantcheva (2017). In addition, comparing ${\tau_H}^y$ with ${\tau_L}^y$ in the first column of Table 3, computed solutions depict that high-skill agents face lower labor income tax rates than low-skill agents in absolute value, which implies endogenous labor income tax rates provide better degree of consumption insurance for low-skill agents, this finding holds in other columns too.

3. Trade-off between per capita capital tax rate and per capita effective labor tax rate

The optimal public finance revenues (T_t) composes of two parts: one is the taxes for wedge elimination (T_t^w), since the aggregate heterogeneous non-linear capital income tax is zero, the taxes for wedge elimination only includes aggregate heterogeneous non-linear effective labor income tax; the other is the linear flat-rate tax T_t^F. The property of the constant return to scale production and competitive markets guarantee the factor marginal output linear additivity holding, consequently, the T_t^F is the affine of the tax rate on per capita capital income (τ_K) and tax rate on per capita effective labor income (τ_Y); that is to say, T_t^F is the linear combination of these tax rates. The linear relation of these tax rates can be obtained as follows: we respectively calculate T_t^w and T_t in situations of $g = 1.01, 1.02, 1.05$, where T_t^w can be solved according to Table 1-3, and T_t can be solved by equation (28) as well as relevant coefficients; then T_t^F ($T_t^F = T_t - T_t^w$) can be expressed as linear combination of above tax rates. The linear relations of τ_K and τ_Y with respect to different steady state growth rate are respectively shown in Figure 2.

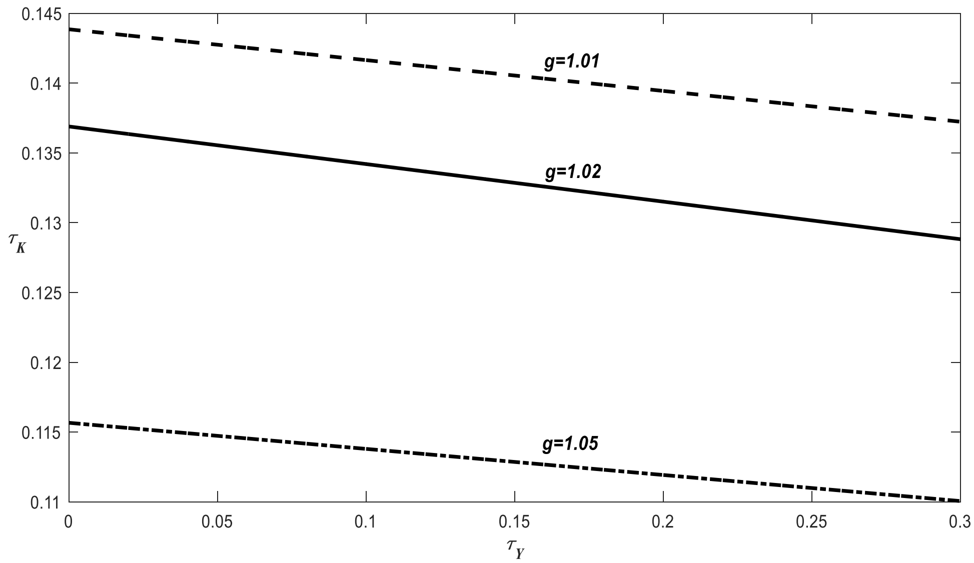

Figure 2. Trade-off between τ_K and τ_Y

Every line in Figure 2 illustrates a significant negative correlation between τ_K and τ_Y with respect to different steady state growth rate level. Let $\tau_Y = 0$, it easy to obtain $\tau_K(g) = rK/T^F(g)$, trajectories in Figure 2 show $rK/T^F(g)$ is monotonously decreasing with g; in other words, the ratio of capital income to flat-rate tax shares negative correlation with g, this phenomenon deserves notice. What's more, τ_K shares negative correlation with g provided arbitrary value of τ_Y as shown in Figure 2.

VI. Conclusion

The motive of our paper is to unify heterogeneous endogenous non-linear income taxes and optimal public finance revenues into Ramsey framework. Under our environment settings, the introduced public finance sector constantly supplies externalities for final goods firms, and we find out the non-exclusiveness of public goods is necessary for existence of growth in the dynamic environment. Taxations in our work consist of two parts, and each part has its own effect: heterogeneous endogenous non-linear income taxes are used for eliminating wedges; flat-rate tax is used to balance the fiscal budget constraint. We prove the existence of unique steady state growth rate by state-contingent FOCs and fixed point principle, and show the steady state growth rate should always larger than 1 through growth rate contingent social welfare level. Comparing with classical endogenous growth works such as Lucas (1988) and Romer (1990), our work makes growth rate completely endogenous without pre-assumption that the steady state growth has already existed.

Environment settings in the new dynamic public finance theory usually adopt 0-1 period economy, thus solutions of 1 period in simulations actually are boundary conditions of 0-1 period dynamic optimization, which are inconsistent with principles of dynamic optimization. Under our environment, 0-1 period economy can be naturally extended to infinite periods economy; the advantage of this setting is that the boundary conditions need no to be specifically defined, as long as utility function is bounded. Therefore, though our simulation analyzes 2 periods economy, our conclusions can be extended to arbitrary periods economy. In addition, the uniqueness of steady state growth rate helps us simplify simulation equations and greatly reduce calculations.

Solutions show that although the aggregate heterogeneous non-linear capital income tax is still zero, growth rates exert huge influences on heterogeneous endogenous non-linear income taxes, and per capita consumptions are positive with growth rates. Concrete conclusions can be obtained as follows: first, high-skill labors supply more labor efforts and consume more, and low-skilled agents in 0 period have better smoothing consumptions in situations of $g = 1.01, 1.02$, while this phenomenon reverses in the situation of $g = 1.05$; second, variance of capital income tax rates on high-skill agents in 0 period is lower than variance of capital income tax rates on low-skill agents in 0 period in situations of $g = 1.01$, 1.05, while this relation reverses in the situation of $g = 1.02$; variance of capital income tax rates on agents with high-skill (low-skill) in 0 period shows reverse U-shape (U-shape) relationship with respect to g; third, all endogenous labor income tax rates are negative, and high-skill agents pay off lower labor income tax rates in absolute value; fourth, in term of linear combinations of every T_t^F, per capita capital tax shares negative correlation with per capita effective labor tax. In addition, we find an interesting result that per capita capital tax rate shares negative correlation with steady state growth rate provided fixed per capita effective labor tax rate.

Our paper setups a primary growth model containing endogenous heterogeneous non-linear income taxes and linear flat-rate taxes, and designs a mechanism (intermediate goods introduction mechanism) making the gross taxes as public finance revenues (T_t) are not contingent on θ^T. Further researches can be launched as following: exploring other mechanisms that guarantee the FOCs are state-contingent; in addition, since taxation in this framework can be regarded as pricing on heterogeneity, our work can be extended into other fields such as asset pricing and monetary economics.

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