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Bergantiños, Gustavo and Moreno-Ternero, Juan D.

Universidade de Vigo, Universidad Pablo de Olavide

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Gustavo Bergantiños*

Juan D. Moreno-Ternero†

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Abstract

The COVID-19 pandemic forced the partial or total cancellation of most sports competitions worldwide. Sports organizations crucially rely on revenues raised from broadcasting. How should the allocation of these revenues be modified when sports leagues are cancelled? We aim to answer that question in this paper by means of the axiomatic approach.

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*ECOBAS, Universidade de Vigo, ECOSOT, 36310 Vigo, España

†Department of Economics, Universidad Pablo de Olavide

1 Introduction

Over the past years, professional sports leagues have gained momentum with an increasing number of fans paying to watch games on television. According to Statista, in 2019, the NFL's broadcasting rights amounted to 4.4 billion U.S. dollars domestically and 120 million U.S. dollars overseas. The English Premier League raised 2.08 billion U.S. dollars domestically and 1.75 billion U.S. dollars abroad. It can be safely argued that the sale of broadcasting rights is currently the biggest source of revenue for sports clubs.

Most of the sports broadcasting contracts have recently been in jeopardy. The COVID-19 pandemic caused major lockdowns worldwide, confining all the population and maintaining only essential services, for non-negligible periods of time. Among many other things, this forced the partial or total cancellation of an endless list of sports competitions.[1] Most important competitions resumed after lockdowns (in spite of having empty stadiums) to secure broadcasting contracts. But some others did not, and ended up cancelling the season. An important instance was Ligue 1, the French Football League, whose broadcasting rights amounted to 1.37 billion U.S. dollars in 2019, according to Statista.[2] We are concerned in this paper with the renegotiation of those broadcasting contracts after cancelling sports seasons.

The sale of broadcasting rights for sports leagues is often carried out collectively. Afterwards, the revenues collected from the sale have to be shared, which becomes a crucial aspect for the management of sports organizations. We recently introduced a simple formal model (Bergantiños and Moreno-Ternero, 2020a) in which the sharing process is based on the (broadcasting) audiences that games throughout the season generate. We considered two simplifying assumptions: a double round-robin tournament (with a fixed set of clubs), in which all games had a constant pay-per view fee. Thus, the prior of the model was simply a square matrix, whose entries indicate the audience of the game involving the row team and the column team, at the former's stadium (thus, all entries in the diagonal are zero). We extend that model here to account for cancelled seasons.[3] More precisely, the extended model allows for empty (to be distinguished from zero) entries in the matrix and it also considers an external endowment to be allocated (thus avoiding the simplifying assumption of constant pay-per view fee).

[1] See, for instance, https://www.espn.com/olympics/story/_/id/28824781/list-sporting-events-canceled-coronavirus. Last accessed, September, 2021

[2] It was also the case of the football leagues in The Netherlands and Belgium, among others.

[3] Csató (2021) also analyzes cancelled sports seasons, but focussing on obtaining a fair ranking for them.

A natural way to address the extended model would be to ignore the nature of cancelled games, just treating them as games with zero audience, and solve the allocation problem with the actual (lower) endowment after cancellation. But this might be unfair for (weaker) teams whose games with popular teams were cancelled. For instance, in the case of Ligue 1 mentioned above, Nimes was supposed to play PSG in late May 2019, and the game was cancelled, whereas Amiens was able to play PSG in mid February 2019, weeks before confinement measures took place worldwide and Ligue 1 was cancelled. Thus, assigning Nimes zero audience in such a game would hurt it in the allocation process with respect to Amiens, which was lucky to play the audience-boost PSG game. Prompted by this case, we shall also explore an alternative way to address the extended model: assigning to cancelled games the audience of the corresponding game in the first leg of the tournament, provided this was not cancelled.

More precisely, we construct two operators that convert audience matrices for cancelled leagues into audience matrices for (fictional) non-cancelled leagues. Both operators leave audiences of non-cancelled games unchanged. Thus, non-empty entries remain the same. And they treat empty entries differently. The *zero* operator converts them into zero. The *leg* operator converts them into their symmetric entries. If we consider benchmark rules that solve the allocation problem for non-cancelled leagues, then the operators convert them into rules that solve the allocation problem for cancelled leagues.

We then take the axiomatic approach to explore the two routes that the previous two operators convey. To wit, we explore the implications of several basic axioms for allocation rules. The first two axioms we consider are natural extensions of two axioms from the benchmark model. *Null team on non-cancelled games* says that if the audience of each game played by a team is zero, then this team obtains no revenue. *Essential team on non-cancelled games* says that if all games with positive audience are played by a team, then this team obtains all the revenue. The rest of the axioms we consider are new. *Baseline monotonicity* compares the allocation in two problems that have the same audience in all games except for a specific game, say played by i and j at $i's$ stadium. If the audience has increased from the first to the second problem, *baseline monotonicity* says that teams i and j could not receive less whereas the rest of the teams could not receive more, provided that the total revenue is the same in both problems. If the game has only been played at the second problem, then we apply the same idea, but now comparing the audience of that game with the audience given by the operator to the cancelled game in the other problem. *Reallocation proofness* compares two tournaments where the aggregate audience of a given team, as well as the aggregate audience, coincide in both tournaments. The axiom says that this team should receive the same in both tournaments. The last axioms consider leagues divided into conferences. Suppose that only games among teams in the same conference have a positive audience. Then, instead of solving the whole tournament, we can solve each conference tournament separately, assuming that the revenue is divided among the conference tournaments proportionally to their estimated audiences, computed via the operator. We consider two axioms, depending on how we define the conference tournament. In the *single-conference axiom* each team plays a single tournament (the one given by the teams of its conference). In the *multi-conference axiom* each team plays several tournaments. For each conference we consider a tournament where all teams participate, but only the games involving the teams of the conference have been played.

We show that several combinations of the above axioms characterize the extensions of two focal rules from the benchmark model: the *equal-split* rule, which splits the revenue generated from each game equally among the participating teams, and *concede-and-divide*, which concedes each team the revenues generated from its fan base and divides equally the residual. The *extended equal-split* rule via the *zero operator* is characterized by *reallocation proofness* and *single-conference*. The *extended equal-split* rule via the *leg operator* is characterized with *weak reallocation proofness* (we claim reallocation proofness only on tournaments where no game has been cancelled), *single-conference*, and *baseline monotonicity*. The *extended concede-and-divide* via the *zero operator* is characterized by *reallocation proofness*, *essential team on non-cancelled games*, and *multi-conference*. The *extended concede-and-divide* via the *leg operator* is characterized by *weak reallocation proofness*, *essential team on non-cancelled games*, *multi-conference*, and *baseline monotonicity*.

To conclude with the introduction, we mention that the broadcasting problem we consider here is a specific resource allocation problem, akin to well-known problems largely analyzed in the game-theory literature. Instances are land division (e.g., Steinhaus, 1948; Chambers, 2005; Segal-Halevi et al., 2020), claims problems (e.g., O'Neill, 1982; Young, 1987; Thomson, 2019), telecommunications problems (e.g., van den Nouweland et al. 1996), museum pass problems (e.g., Ginsburgh and Zang, 2003; Bergantiños and Moreno-Ternero, 2015), or cost sharing in networks (e.g., Bergantiños and Vidal-Puga, 2007; Bogomolnaia et al., 2010). Some of the insights we obtain here are somewhat reminiscent of some of the results from that literature.

The rest of the paper is organized as follows. In Section 2, we introduce the model (benchmark rules, operators, extended rules and axioms). In Section 3, we provide the characterization results. Section 4 concludes. For a smooth passage, all proofs have been deferred to an appendix.

2 The model

Let N describe a finite set of teams. Its cardinality is denoted by n. Without loss of generality, we usually take $N = \{1, 2, \ldots, n\}$. We assume $n \geq 4$. We also assume a round-robin tournament in which each team plays in turn against each other team twice (home and away), which is the typical format of the national football leagues. For each pair of teams $i, j \in N$, we denote by a_{ij} the broadcasting audience (number of viewers) for the game played by i and j at i's stadium. We write $a_{ij} = \varnothing$ if the game was cancelled. For notational simplicity, we also assume that $a_{ii} = \varnothing$ for each $i \in N$. Let $A \in \mathcal{A}_{n \times n}$ denote the resulting matrix of broadcasting audiences generated in the whole tournament involving the teams within N.

Let $\alpha_i(A)$ denote the total audience achieved by team i, i.e.,

$$\alpha_i(A) = \sum_{i \in \{j,k\} \subset N, a_{jk} \neq \varnothing} a_{jk}$$

We take $\alpha_i(A) = 0$ when team i has not played any game. When no confusion arises, we write α_i instead of $\alpha_i(A)$.

For each $A \in \mathcal{A}_{n \times n}$, let $||A||$ denote the total audience of the tournament. Namely,

$$||A|| = \sum_{i,j \in N, a_{ij} \neq \varnothing} a_{ij} = \frac{1}{2} \sum_{i \in N} \alpha_i.$$

We take $||A|| = 0$ when no game has been played.

A (broadcasting) **problem** is a pair (A, E), where $A \in \mathcal{A}_{n \times n}$ is a matrix defined as above and $E \in \mathbb{R}_+$ is an endowment to be allocated among teams in N, based on the audience matrix.[4] The family of all the problems is denoted by $\mathcal{P}$.

[4] In Bergantiños and Moreno-Ternero (2020a) it is assumed that each viewer pays a constant (pay-per-view) fee, which is normalized to 1. Thus, the endowment is not a prior of the model therein as allocating the revenue from broadcasting is the same as allocating the audiences.

Let $\mathcal{P}^c$ denote the subset of $\mathcal{P}$ encompassing the problems corresponding to fully completed seasons. Namely, $(A, E) \in \mathcal{P}^c$ if and only if $a_{ij} \neq \varnothing$, for each pair $i, j \in N$, with $i \neq j$.[5]

2.1 Benchmark rules

A (sharing) **rule** R is a mapping that associates with each problem an allocation indicating the amount each team gets from the endowment. Thus, $R : \mathcal{P} \to \mathbb{R}^N$ is such that, for each $(A, E) \in \mathcal{P}$,

$$\sum_{i \in N} R_i (A, E) = E.$$

We also impose from the outset that, if $||A|| = 0$, then $R_i (A, E) = \frac{E}{n}$, for each $i \in N$. Furthermore, $R_i (A, 0) = 0$, for each $i \in N$.

We now consider two focal rules for problems in $\mathcal{P}^c$, which have been introduced in Bergantiños and Moreno-Ternero (2020a). Viewers of each game can essentially be divided in two categories: those watching the game because they are fans of one of the teams playing and those watching the game because they thought that the specific combination of teams rendered the game interesting. We refer to them as hard-core (team) fans and neutral (football) fans, respectively. We argue that the revenue generated by the first category should be allocated to the corresponding team, whereas the revenue generated by the second category should be divided equally between both teams. The *equal-split* rule and *concede-and-divide* are two extreme rules from the point of view of treating fans. The *equal-split* rule assumes that only neutral fans exist whereas *concede-and-divide* assumes that there are as many hard-core fans as possible (compatible with the audiences).

Bergantiños and Moreno-Ternero (2020a) normalize the revenue generated from each viewer to 1. Thus, they divide $||A||$ among the teams. As we must divide E, the revenue generated by each viewer is $\frac{E}{||A||}$. We now adapt the definitions of equal-split and concede-and-divide to our setting.

The *equal-split* rule splits equally the audience of each game (a_{ij}) among the two teams, thus ignoring the existence of hard-core fans for each team. The total audience assigned to each team is computed as the sum, over all games played by such a team, of the audiences assigned

[5]The domain in Bergantiños and Moreno-Ternero (2020a) is precisely the subset of $\mathcal{P}^c$, for which the endowment coincides with the aggregate audience of the season. Namely, $(A, E) \in \mathcal{P}^c$ such that $E = ||A||$.

to each game. The amount received by each team is obtained multiplying the audience assigned to the team by $\frac{E}{||A||}$. Formally,

Equal-split, ES: for each $(A, E) \in \mathcal{P}^c$, and each $i \in N$,

$$ES_i(A, E) = \frac{E}{||A||}\frac{\alpha_i}{2}.$$

The second rule, the so-called *concede-and-divide*, concedes each team its number of fans and divides equally the rest. For each team i we estimate f_i, the number of fans of team i. Given a game with audience a_{ij}, i receives $f_i + \frac{a_{ij}-f_i-f_j}{2}$ and j receives $f_j + \frac{a_{ij}-f_i-f_j}{2}$. The total audience assigned to each team is computed as the sum over all games played by such a team. The amount received by each team is obtained multiplying the audience assigned to the team by $\frac{E}{||A||}$. Bergantiños and Moreno-Ternero (2020a) prove that this rule could be computed through an easier formula in such a way that we do not need to estimate the number of fans of each team. Formally,

Concede-and-divide, CD: for each $(A, E) \in \mathcal{P}^c$, and each $i \in N$,

$$CD_i(A, E) = \frac{(n-1)\alpha_i - ||A||}{n-2}\frac{E}{||A||}.$$

2.2 Operators

We aim to provide rules that can also address problems with cancelled games. To do so, we extend benchmark rules (such as the ones introduced above) by means of *operators*, associating to each benchmark rule an extended rule resulting from a two-stage procedure in which the matrix of audiences is modified first, to replace its empty entries, and the rule is then used to solve the resulting problem.[6] Formally, an operator is a mapping $o : \mathcal{P} \to \mathcal{P}^c$ assigning to each problem $(A, E) \in \mathcal{P}$ a problem $(A^o, E) \in \mathcal{P}^c$ such that $a^o_{ij} = a_{ij}$ when $a_{ij} \neq \varnothing$.

We concentrate on the following two operators. First, the one associating to a cancelled game a null audience. Second, the one associating to a cancelled game the audience of the game in the first leg of the tournament, or zero if such a game was also cancelled. Formally,

Zero, z: For each pair $i, j \in N$

$$a^z_{ij} = \begin{cases} 0 & \text{if } a_{ij} = \varnothing \\ a_{ij} & \text{if } a_{ij} \neq \varnothing. \end{cases}$$

[6]The concept of operators on the space of allocation rules is explored in detail by Thomson and Yeh (2008) and Thomson (2019). See also Hougaard et al., (2012, 2013a,b) and Moreno-Ternero and Vidal-Puga (2021).

Leg, l: For each pair $i, j \in N$,

$$a_{ij}^{l} = \begin{cases} a_{ji} & \text{if } a_{ij} = \varnothing \text{ and } a_{ji} \neq \varnothing \\ 0 & \text{if } a_{ij} = \varnothing \text{ and } a_{ji} = \varnothing \\ a_{ij} & \text{if } a_{ij} \neq \varnothing. \end{cases}$$

2.3 Extended rules

For each operator o, and each benchmark rule R on $\mathcal{P}^c$, we can define an extended rule R^o on $\mathcal{P}$ in the obvious way. Namely, for each $(A, E) \in \mathcal{P}$,

$$R^o(A, E) = R(A^o, E).$$

We sometimes refer to R^o as the image of the benchmark rule R via the operator o. Some instances are just the images of the two benchmark rules introduced above, via the two operators also defined above: the *zero-extended equal-split rule* (ES^z), the *zero-extended concede-and-divide* (CD^z), the *leg-extended equal-split rule* (ES^l), and the *leg-extended concede-and-divide* (CD^l).

2.4 Axioms

We now consider several axioms of (extended) rules. First, the axiom that says that if a team has a null audience in all its non-cancelled games, then such a team gets no revenue.[7] Formally,

Null team on non-cancelled games (NTN): For each $(A, E) \in \mathcal{P}$ with $||A|| > 0$ and each $i \in N$, such that for each $j \in N \backslash \{i\}$, $a_{ij} \in \{0, \varnothing\}$ and $a_{ji} \in \{0, \varnothing\}$,

$$R_i(A, E) = 0.$$

The next axiom formalizes a sort of dual principle as it says that if only the games played by one team have positive audience, then such an *essential* team should receive all the endowment. Formally,

Essential team on non-cancelled games (ETN): For each $(A, E) \in \mathcal{P}$, and each $i \in N$ such that $a_{jk} \in \{0, \varnothing\}$ for each pair $\{j, k\} \in N \backslash \{i\}$,

$$R_i(A, E) = E.$$

[7] This axiom and the next one are natural counterparts of those introduced in Bergantiños and Moreno-Ternero (2020a) for the benchmark setting.

We now turn to monotonicity axioms, which are natural in resource allocation.[8] To motivate them, we consider first the following example.

Example 1 *Let* $(A,E),(A',E)\in\mathcal{P}$ *be such that* $N=\{1,2,3\}$, $E=1000$, *and*

$$A=\begin{pmatrix} \varnothing & 230 & \varnothing \\ \varnothing & \varnothing & 210 \\ 220 & \varnothing & \varnothing \end{pmatrix} \text{ and } A'=\begin{pmatrix} \varnothing & 230 & 3 \\ \varnothing & \varnothing & 210 \\ 220 & \varnothing & \varnothing \end{pmatrix}$$

In the latter case, teams 1 and 3 have played one more game than in the former case, with an audience $a'_{13}=3$. *How should the allocation of those teams change from one case to the other? Two reasonable answers are possible:*

1. *As the total audience of team* 1 *increased, then the allocation to team 1 should not decrease.*
2. *As the relative audience (per game) of team* 1 *decreased, then the allocation to team 1 should not increase.*

Prompted by the above discussion, we introduce a monotonicity property that, depending on a vector of *baseline* audiences, suggests several possible scenarios.[9]

Formally, for each pair $(A,E),(A',E)\in\mathcal{P}$, and each $i\in N$, let

$$b_i=\{(b_i((i,j),A,A',E),b_i((j,i),A,A',E)):j\in N\setminus\{i\}\},$$

where

$$b_i((i,j),A,A',E)=b_j((i,j),A,A',E)=a_{ij},$$

for each $a_{ij}\neq\varnothing$. We refer to $B=(b_i)_{i\in N}$ as the vector of *baseline* audiences. We then have the following axiom:

Baseline monotonicity (BM): Let $(A,E),(A',E)\in\mathcal{P}$ for which there exist $i,j\in N$ such that $a'_{ij}\neq\varnothing$ and $a'_{kl}=a_{kl}$ for each $(k,l)\neq(i,j)$. Then, two conditions hold:

[8] *Monotonicity* is a general principle of fair division which states that when the underlying data of a problem change in a specific way, the solution should change accordingly. Early formalizations of this principle, for somewhat related models, can be traced back to Megiddo (1974), Kalai and Smorodinsky (1975), or Thomson and Myerson (1980), among others.

[9] This is reminiscent of the concept of baselines in rationing problems formalized by Hougaard et al., (2012, 2013a,b). See also Ju et al., (2021).

1. For each $k \in \{i, j\}$,

$$R_k(A', E) \geq R_k(A, E) \text{ when } a'_{ij} \geq b_k((i,j), A, A', E),$$
$$R_k(A', E) \leq R_k(A, E) \text{ when } a'_{ij} \leq b_k((i,j), A, A', E).$$

2. For each $k \in N\backslash \{i, j\}$,

$$R_k(A', E) \leq R_k(A, E) \text{ when } a'_{ij} \geq \max\{b_i((i,j), A, A', E), b_j((i,j), A, A', E)\}$$
$$R_k(A', E) \geq R_k(A, E) \text{ when } a'_{ij} \leq \min\{b_i((i,j), A, A', E), b_j((i,j), A, A', E)\}.$$

Bergantiños and Moreno-Ternero (2021b) introduce several monotonicity axioms in the benchmark broadcasting problem. Two of them are closely related with *baseline monotonicity.* *Weak team monotonicity* says that for each A, A' and $i \in N$ such that $a_{ij} \leq a'_{ij}$ for all $j \in N\backslash\{i\}$, $a_{ji} \leq a'_{ji}$ for all $j \in N\backslash\{i\}$, and $a_{jk} = a'_{jk}$ when $i \in N\backslash\{j,k\}$, then $R_i(A') \geq R_i(A)$. Since $b_i((i,j), A, A', E) = a_{ij}$ when $a_{ij} \neq \varnothing$, condition 1 of *baseline monotonicity* is an extension of *weak team monotonicity.* *Reciprocal monotonicity* says that for each A, A' and $i \in N$ such that $a_{ij} = a'_{ij}$ for all $j \in N\backslash\{i\}$, $a_{ji} = a'_{ji}$ for all $j \in N\backslash\{i\}$, and $a_{jk} \leq a'_{jk}$ when $i \in N\backslash\{j,k\}$, then $R_i(A') \leq R_i(A)$. Since $b_i((i,j), A, A', E) = b_j((i,j), A, A', E) = a_{ij}$ when $a_{ij} \neq \varnothing$, condition 2 of *baseline monotonicity* is an extension of *reciprocal monotonicity.*

Obviously, the above crucially relies on the vector of baseline audiences B. We now give some examples of possible baselines, sharing the spirit of the two operators considered above.

1. **Zero**. Let $i \in N$. For each $j \in N\backslash\{i\}$, let B^z be defined as

$$b_i^z((i,j), A, A', E) = \begin{cases} 0 & \text{if } a_{ij} = \varnothing \\ a_{ij} & \text{otherwise} \end{cases} \quad \text{and}$$
$$b_i^z((j,i), A, A', E) = \begin{cases} 0 & \text{if } a_{ji} = \varnothing \\ a_{ji} & \text{otherwise.} \end{cases}$$

   If a rule satisfies $B^z M$, then we say that the rule satisfies *zero baseline monotonicity.* Note that if a rule satisfies $B^z M$ then to play a game with zero audience could not be worse than not playing such a game.

2. **Leg**. Let $i \in N$. For each $j \in N \setminus \{i\}$, let B^l be defined as

$$b_i^l((i,j), A, A', E) = \begin{cases} a_{ji} & \text{if } a_{ij} = \varnothing \text{ and } a_{ji} \neq \varnothing \\ 0 & \text{if } a_{ij} = \varnothing \text{ and } a_{ji} = \varnothing \\ a_{ij} & \text{otherwise} \end{cases} \quad \text{and}$$

$$b_i^l((j,i), A, A', E) = \begin{cases} a_{ij} & \text{if } a_{ji} = \varnothing \text{ and } a_{ij} \neq \varnothing \\ 0 & \text{if } a_{ji} = \varnothing \text{ and } a_{ij} = \varnothing \\ a_{ji} & \text{otherwise.} \end{cases}$$

If a rule satisfies $B^l M$, then we say that the rule satisfies *leg baseline monotonicity.*

We now move to consider axioms preventing manipulations via reallocations.[10]

Suppose first two tournaments such that the aggregate audience of a given team, as well as the aggregate audience of the rest of the games, coincide in both tournaments. Then, such a given team should receive the same in both tournaments. Namely,

Reallocation proofness (RP): Let $(A, E), (A', E) \in \mathcal{P}$ and $i \in N$ be such that $\alpha_i(A) = \alpha_i(A')$ and $||A|| = ||A'||$. Then,

$$R_i(A, E) = R_i(A', E).$$

The following example captures a possibly disturbing feature with the previous axiom.

Example 2 *Let A^1, A^2 and A^3 be such that $N = \{1, 2, 3\}$, $E = 1000$, and*

$$A^1 = \begin{pmatrix} \varnothing & \mathit{100} & 60 \\ \varnothing & \varnothing & \mathit{30} \\ \mathit{60} & 30 & \varnothing \end{pmatrix}, \ A^2 = \begin{pmatrix} \varnothing & \mathit{20} & 140 \\ \varnothing & \varnothing & \mathit{30} \\ \mathit{60} & 30 & \varnothing \end{pmatrix}, \ \text{and } A^3 = \begin{pmatrix} \varnothing & \mathit{100} & 30 \\ \varnothing & \varnothing & \mathit{30} \\ \mathit{90} & 30 & \varnothing \end{pmatrix}.$$

The audience of game $(1, 2)$ is 100 in A^1 and A^3, and 20 in A^2. But a rule satisfying reallocation proofness should ignore these numbers (and take into account only the total audience of team 1). Thus, if the rule is defined through an operator o, such an operator cannot depend on the audience of game $(1, 2)$. But it might be reasonable to do otherwise, therefore distinguishing between A^2 and the other two problems.

[10] Ju et al., (2007) analyze the implications of this kind of axioms in general allocation problems. A seminal contribution for the notion is Moulin (1985) and a more recent instance is Csóka and Herings (2021).

Prompted by the previous example, we weaken *reallocation proofness* by claiming that the allocation to team i should be independent of reallocations of audiences but only when no game has been cancelled (namely, on $\mathcal{P}^c$).

Weak reallocation proofness (WRP): Let $(A, E), (A', E) \in \mathcal{P}^c$ and $i \in N$ be such that $\alpha_i(A) = \alpha_i(A')$ and $||A|| = ||A'||$. Then,

$$R_i(A, E) = R_i(A', E).$$

The last axioms we consider refer to leagues divided into conferences.[11] We assume that only games among teams in the same conference have a positive audience. An alternative interpretation is that each conference can be seen as a different tournament. Then, instead of solving the whole tournament, we can solve each conference tournament separately, under the assumption that the endowment is divided among the conference tournaments proportionally to their estimated audiences, computed via operator o. We consider two axioms, depending on how we define the conference tournament. More precisely, for each $(A, E) \in \mathcal{P}$, and each $S \subset N$, we consider two ways of modeling the tournament induced by A among teams in S:

1. We denote by $A^S \in \mathcal{A}_{|S| \times |S|}$ the matrix obtained from A by considering that the set of teams is S and the audiences are given by A. Namely, $a_{ij}^S = a_{ij}$ for all $i, j \in S$.

2. We denote by $A^{S,\varnothing} \in \mathcal{A}_{n \times n}$ the matrix obtained from A by assuming that only the games between teams within S have been played. Namely,

$$a_{ij}^{S,\varnothing} = \begin{cases} a_{ij} & \text{if } i, j \in S \\ \varnothing & \text{otherwise.} \end{cases}$$

Notice that in A^S the set of teams is S whereas in $A^{S,\varnothing}$ the set of teams is N.

For each $(A, E) \in \mathcal{P}$ and each $S \subset N$, we denote by $||A(S)||$ the aggregate audience of all games played among teams within S. Namely,

$$||A(S)|| = \sum_{i,j \in S, a_{ij} \neq \varnothing} a_{ij}.$$

We say that $\{N_1, ..., N_p\}$ is a partition of N if $N = \bigcup_{k=1}^{p} N_k$, $N_k \cap N_{k'} = \varnothing$ for each pair $k \neq k'$, and $N_k \neq \varnothing$ for each $k = 1, ..., p$.

[11] This is the case, for instance, of the four major sports leagues in North America.

We are now ready to state the two axioms.

O-single-conference (SC^o): Let $(A, E) \in \mathcal{P}$, $\{N_1, ..., N_p\}$ a partition of N such that if $a_{ij} > 0$ with $i \in N_{i'}$ and $j \in N_{j'}$ then $i' = j'$. For each $i \in N_{i'}$,

$$R_i(A, E) = R_i\left(A^{N_{i'}}, \frac{||A^o(N_{i'})||}{\sum\limits_{k=1}^{p} ||A^o(N_k)||} E\right).$$

O-multi-conference (MC^o): Let $(A, E) \in \mathcal{P}$, $\{N_1, ..., N_p\}$ a partition of N such that if $a_{ij} > 0$ with $i \in N_{i'}$ and $j \in N_{j'}$ then $i' = j'$. For each $i \in N$,

$$R_i(A, E) = \sum_{k=1}^{p} R_i\left(A^{N_k,\varnothing}, \frac{||A^o(N_k)||}{\sum\limits_{k=1}^{p} ||A^o(N_k)||} E\right).$$

Note that the condition of *zero-single-conference* can be rewritten only in terms of A as

$$R_i(A, E) = R_i\left(A^{N_{i'}}, E\frac{||A(N_{i'})||}{||A||}\right).$$

Similarly, for the zero operator, *zero-multi-conference* can be rewritten only in terms of the matrix with the original audiences A. Namely, the condition to be satisfied is

$$R_i(A, E) = \sum_{k=1}^{p} R_i\left(A^{N_k,\varnothing}, E\frac{||A(N_k)||}{||A||}\right).$$

3 Characterization results

In this section we present characterization results for the *zero-extended equal-split* rule, *leg-extended equal-split* rule, *zero-extended concede-and-divide*, and *leg-extended concede-and-divide.* The proofs can be found in the Appendix.

Our first result is a characterization of the *extended equal-split* rule, via the *zero operator*, combining two axioms.

Theorem 1 *A rule satisfies reallocation proofness and zero-single-conference if and only if it is the zero-extended equal-split rule ES^z.*

An alternative characterization of the same rule is obtained upon replacing *zero-single-conference* in Theorem 1 by the pair made of *zero-multi-conference* and *null team for non-cancelled games.* In the appendix, after the proof of Theorem 1, we also explain briefly how to obtain this alternative characterization.

Resorting to *leg-baseline monotonicity*, but weakening *reallocation proofness*, we obtain instead a characterization of the *extended equal-split* rule, via the *leg operator.*

Theorem 2 *A rule satisfies weak reallocation proofness, leg-single-conference and leg-baseline monotonicity if and only if it is the leg-extended equal-split rule* ES^l.

As with the previous result, an alternative characterization of the same rule is obtained upon replacing *leg-single-conference* in Theorem 2 by the pair made of *leg-multi-conference* and *null team for non-cancelled games.* In the appendix, after the proof of Theorem 2, we also explain briefly how to obtain this alternative characterization.

We now present the counterpart results for the *extended concede-and-divide*, via the *zero* and *leg operator*, respectively.

Theorem 3 *A rule satisfies reallocation proofness, essential team on non-cancelled games, and zero-multi-conference if and only if it is the zero-extended concede-and-divide* CD^z.

In contrast with Theorem 1, the pair made of *zero-multi-conference* and *essential team for non-cancelled games* cannot be replaced by *zero-single-conference* (which is not satisfied by CD^z).

Theorem 4 *A rule satisfies weak reallocation proofness, essential team on non-cancelled games, leg-multi-conference and leg-baseline monotonicity if and only if it is the leg-extended concede-and-divide* CD^l.

In contrast with Theorem 2, the pair made of *leg-multi-conference* and *essential team for non-cancelled games* cannot be replaced by *leg-single-conference* (which is not satisfied by CD^l).

The next table summarizes the performance of the rules considered in the paper with respect to the introduced axioms (we avoid the technical, and non-difficult, proofs). The axioms that were used for each of the characterization results stated above are highlighted in each case.

Axioms / Rules	$\mathbf{ES}^z$	$\mathbf{CD}^z$	$\mathbf{ES}^l$	$\mathbf{CD}^l$
NTN	YES	NO	YES	NO
ETN	NO	YES^{Th3}	NO	YES^{Th4}
B^zM	YES	YES	NO	NO
B^lM	NO	NO	YES^{Th2}	YES^{Th4}
RP	YES^{Th1}	YES^{Th3}	NO	NO
WRP	YES	YES	YES^{Th2}	YES^{Th4}
SC^z	YES^{Th1}	NO	NO	NO
MC^z	YES	YES^{Th3}	NO	NO
SC^l	NO	NO	YES^{Th2}	NO
MC^l	NO	NO	YES	YES^{Th4}

Table 1: Performance of the rules with respect to the axioms and characterization results.

We conclude with characterizations on the restricted domain $\mathcal{P}^c$, i.e., in the case in which no game has been cancelled. Note that, for each $(A, E) \in \mathcal{P}^c$ and each operator o, $A^o = A$. Thus, *reallocation proofness* and *weak reallocation proofness* coincide. Besides, the monotonicity and conference axioms do not depend on the operator and we can remove it from their definitions. Moreover, in the definition of the *multi-conference axiom*, we should change $A^{N_k,\varnothing}$ by $A^{N_k,0}$. Likewise, we avoid adding to the names of the axioms "on non-cancelled games", as it would be redundant in this restricted domain.

Theorem 5 *The following statements hold:*

1. *A rule defined on $\mathcal{P}^c$ satisfies reallocation proofness and single-conference if and only if it is the equal-split rule.*

2. *A rule defined on $\mathcal{P}^c$ satisfies reallocation proofness, multi-conference and null team if and only if it is the equal-split rule.*

3. *A rule defined on $\mathcal{P}^c$ satisfies reallocation proofness, multi-conference and essential team if and only if it is concede-and-divide.*

Theorem 5 provides some characterizations of the *equal-split* rule and *concede-and-divide* in the benchmark broadcasting problem P^c. Bergantiños and Moreno-Ternero (2020a, 2020b, 2021a, 2021b) also provide several characterizations of the *equal-split* rule and *concede-and-divide* in P^c by using, among others, the axioms of *null team* and *essential team*. The axioms of *reallocation proofness*, *single-conference* and *multi-conference* are used in this paper but not in the previous ones.

4 Final remarks

We have explored in this paper the allocation of resources raised from selling broadcasting rights for sports leagues, after the leagues have been cancelled. We have provided normative foundations for the extension of two focal rules (*equal-split* and *concede-and-divide*) via two natural operators: the *zero operator* and the *leg operator*. The former assigns to cancelled games a zero audience. The latter assigns to cancelled games the audience of the corresponding game in the first leg of the tournament. Other operators could also be considered. For instance, to associate each cancelled game the audience of the (non-cancelled) game with highest or lowest audience. Or the average audience of all the (non-cancelled) games in the tournament. Exploring those operators is left for further research.

The two rules we have extended are not only focal but also somewhat extreme in the benchmark setting of complete sports leagues. Compromises among them have been considered (e.g., Bergantiños and Moreno-Ternero, 2021a, 2021, 2021c). It is also left for further research to extend those compromises to the case of cancelled competitions introduced in this paper.

There exist notable differences between the benchmark case and the extended one we have explored here. For instance, *additivity*, a property crucial in the benchmark case to characterize the focal rules, cannot be formalized in this setting. Alternative properties referring to how to allocate extra resources (e.g., Bergantiños and Moreno-Ternero, 2020b), which can also characterize them (dismissing *additivity*), cannot be formalized in this setting either. Conversely, some of the axioms considered here (such as those referring to non-cancelled games) cannot be mimicked for the benchmark setting. On the other hand, the *baseline monotonicity* axioms considered here are natural generalizations of monotonicity axioms considered in the benchmark setting, as mentioned above.

5 Appendix

5.1 Proof of Theorem 1

It is not difficult to show that ES^z satisfies the two axioms in the statement. Conversely, let R be a rule satisfying the two axioms. Let $(A,E)\in\mathcal{P}$ be such that $||A||>0$.[12] Let $i\in N$. Notice that $||A||-\alpha_i(A)$ is the aggregate audience of all games not played by team i. We then define the matrix A^* as follows. Let $i^0,i^1,i^2\in N\backslash\{i\}$. Then,

$$a^*_{jk}=\begin{cases}\frac{\alpha_i(A)}{2} & \text{if } (j,k)\in\{(i,i^0),(i^0,i)\}\\ ||A||-\alpha_i(A) & \text{if } (j,k)=(i^1,i^2)\\ \varnothing & \text{otherwise.}\end{cases}$$

Notice that $||A^*||=||A||>0$ and $\alpha_i(A^*)=\alpha_i(A)$. Therefore, by *reallocation proofness*, $R_i(A,E)=R_i(A^*,E)$.

Let $N_1=\{i,i^0\}$ and $N_2=N\backslash N_1$. By *zero-single-conference*,

$$\begin{aligned}R_i(A^*,E) &= R_i\left((A^*)^{N_1},\frac{||(A^*)^z(N_1)||}{||(A^*)^z(N_1)||+||(A^*)^z(N_2)||}E\right)\\ &= R_i\left((A^*)^{N_1},\frac{\alpha_i(A)}{||A||}E\right).\end{aligned}$$

As

$$R_i\left((A^*)^{N_1},\frac{\alpha_i(A)}{||A||}E\right)+R_{i^0}\left((A^*)^{N_1},\frac{\alpha_i(A)}{||A||}E\right)=\frac{\alpha_i(A)}{||A||}E,$$

it follows that there exists $p_{(i,i_0)}\in\mathbb{R}$ such that

$$\begin{aligned}R_i\left((A^*)^{N_1},\frac{\alpha_i(A)}{||A||}E\right) &= p_{(i,i_0)}\frac{\alpha_i(A)}{||A||}E \text{ and}\\ R_{i^0}\left((A^*)^{N_1},\frac{\alpha_i(A)}{||A||}E\right) &= \left(1-p_{(i,i_0)}\right)\frac{\alpha_i(A)}{||A||}E.\end{aligned}$$

As $R_i(A,E)=R_i\left((A^*)^{N_1},\frac{\alpha_i(A)}{||A||}E\right)=p_{(i,i_0)}\frac{\alpha_i(A)}{||A||}E$, for each $i^0\in N\backslash\{i\}$, we deduce that $p_{(i,i_0)}$ is indeed independent of i_0. Thus, we can just refer to it as p_i.

If $\alpha_i(A)=0$, $R_i(A,E)=0=ES_i(A,E)$.

Therefore, assume now that $\alpha_i(A)>0$. Let $i'\in N\backslash\{i\}$ and construct the problem (A',E) such that $\alpha_{i'}(A')=\alpha_i(A)$ and $||A'||=||A||$.

[12] Otherwise, the proof is trivial as $R_i(A,E)=\frac{E}{n}$, for each $i\in N$.

We then construct the matrix A'^{*}, analogously to how we constructed A^{*} from A, but now assigning the former role of i to i', and the former role of i^0 to i. That is,

$$a'^{*}_{jk} = \begin{cases} \frac{\alpha_i(A')}{2} & \text{if } (j,k) \in \{(i',i),(i,i')\} \\ ||A'|| - \alpha_i(A') & \text{if } (j,k) = (i^1,i^2) \\ \varnothing & \text{otherwise.} \end{cases}$$

Then,

$$p_i \frac{\alpha_i(A)}{||A||} E = R_i\left((A^{*})^{N_1}, \frac{\alpha_i(A)}{||A||} E\right) = R_i\left((A'^{*})^{N_1}, \frac{\alpha_{i'}(A')}{||A'||} E\right) = (1-p_{i'}) \frac{\alpha_{i'}(A')}{||A'||} E,$$

from where it follows that $p_i = 1 - p_{i'}$.

As i and i' were arbitrary members of N, it follows that $p_j = \frac{1}{2}$ for all $j \in N$. Hence,

$$R_i(A,E) = R_i\left((A^1)^{N_1}, \frac{\alpha_i(A)}{||A||} E\right) = p_i \frac{\alpha_i(A)}{||A||} E = \frac{1}{2}\frac{\alpha_i(A)}{||A||} E = ES_i^z(A,E). \qquad \square$$

5.2 Remark from Theorem 1

We now explain how to obtain the alternative characterization at Theorem 1 by replacing *zero-single-conference* by *zero-multi-conference* and *null team on non-cancelled games*.

It is straightforward to prove that ES^z also satisfies *zero-multi-conference* and *null team on non-cancelled games*. As for the converse implication, note that *zero-multi-conference* is used only once in the previous proof. We explain now how to derive the same conclusion therein with *zero-multi-conference* and *null team on non-cancelled games*.

By *zero-multi-conference*,

$$R_i(A^{*},E) = R_i\left((A^{*})^{N_1,\varnothing}, \frac{\alpha_i(A)}{||A||} E\right) + R_i\left((A^{*})^{N_2,\varnothing}, \frac{||A|| - \alpha_i(A)}{||A||} E\right).$$

By *null team on non-cancelled games*, $R_i\left((A^{*})^{N_2,\varnothing}, \frac{||A||-\alpha_i(A)}{||A||} E\right) = 0$. Hence, $R_i(A^{*},E) = R_i\left((A^{*})^{N_1,\varnothing}, \frac{\alpha_i(A)}{||A||} E\right)$.

By *null team on non-cancelled games*, $R_j\left((A^{*})^{N_1,\varnothing}, \frac{\alpha_i(A)}{||A||} E\right) = 0$ for all $j \in N^2$. Using arguments similar to those used above for $\left((A^{*})^{N_1}, \frac{\alpha_i(A)}{||A||} E\right)$ we can prove that $R_i\left((A^{*})^{N_1,\varnothing}, \frac{\alpha_i(A)}{||A||} E\right) = \frac{\alpha_i(A)}{2||A||} E = ES_i^z(A,E)$.

5.3 Proof of Theorem 2

It is straightforward to show that ES^l satisfies *weak reallocation proofness* and *leg-single-conference*. Regarding *leg-baseline monotonicity*, let $(A,E),(A',E) \in \mathcal{P}$ and $i,j \in N$ be as in the definition of this axiom. We consider several cases.

1. $a_{ij} = \varnothing \neq a_{ji}$.

   In this case, $b_i^l\left(\left(i,j\right),A,A',E\right) = b_j^l\left(\left(i,j\right),A,A',E\right) = a_{ji}$. Thus,

$$\begin{aligned}
\alpha_i\left(A'^l\right) &= \alpha_i\left(A^l\right) + a'_{ij} - a_{ji} \\
\alpha_j\left(A'^l\right) &= \alpha_j\left(A^l\right) + a'_{ij} - a_{ji} \\
\alpha_k\left(A'^l\right) &= \alpha_k\left(A^l\right) \text{ for all } k \in N\backslash\left\{i,j\right\} \text{ and} \\
\left|\left|A'^l\right|\right| &= \left|\left|A^l\right|\right| + a'_{ij} - a_{ji}.
\end{aligned}$$

   We consider two sub-cases.

   (a) $a'_{ij} \geq a_{ji}$.

   In this case, $ES_k\left(A'^l,E\right) \geq ES_k\left(A^l,E\right)$, for each $k \in \left\{i,j\right\}$. Besides,

$$a'_{ij} \geq a_{ji} = \max\left\{b_i^l\left(\left(i,j\right),A,A',E\right), b_j^l\left(\left(i,j\right),A,A',E\right)\right\}$$

   and $ES_k\left(A'^l,E\right) \leq ES_k\left(A^l,E\right)$ for each $k \in N\backslash\left\{i,j\right\}$.

   (b) $a'_{ij} \leq a_{ji}$.

   In this case, $ES_k\left(A'^l,E\right) \leq ES_k\left(A^l,E\right)$, for each $k \in \left\{i,j\right\}$. Besides,

$$a'_{ij} \leq a_{ji} = \min\left\{b_i^l\left(\left(i,j\right),A,A',E\right), b_j^l\left(\left(i,j\right),A,A',E\right)\right\}$$

   and $ES_k\left(A'^l,E\right) \geq ES_k\left(A^l,E\right)$ for each $k \in N\backslash\left\{i,j\right\}$.

2. $a_{ij} = \varnothing = a_{ji}$.

   In this case, $b_i^l\left(\left(i,j\right),A,A',E\right) = b_j^l\left(\left(i,j\right),A,A',E\right) = 0$. Thus,

$$\begin{aligned}
\alpha_i\left(A'^l\right) &= \alpha_i\left(A^l\right) + 2a'_{ij} \\
\alpha_j\left(A'^l\right) &= \alpha_j\left(A^l\right) + 2a'_{ij} \\
\alpha_k\left(A'^l\right) &= \alpha_k\left(A^l\right) \text{ for all } k \in N\backslash\left\{i,j\right\} \text{ and} \\
\left|\left|A'^l\right|\right| &= \left|\left|A^l\right|\right| + 2a'_{ij}.
\end{aligned}$$

   Therefore, $ES_k\left(A'^l,E\right) \geq ES_k\left(A^l,E\right)$ for each $k \in \left\{i,j\right\}$. Besides,

$$a'_{ij} \geq 0 = \max\left\{b_i^l\left(\left(i,j\right),A,A',E\right), b_j^l\left(\left(i,j\right),A,A',E\right)\right\}$$

   and $ES_k\left(A'^l,E\right) \leq ES_k\left(A^l,E\right)$ for each $k \in N\backslash\left\{i,j\right\}$.

3. $a_{ij} \neq \varnothing \neq a_{ji}$.

   In this case, $b_i^l((i,j), A, A', E) = b_j^l((i,j), A, A', E) = a_{ij}$. Thus,

$$\begin{aligned} \alpha_i\left(A'^l\right) &= \alpha_i\left(A^l\right) + a'_{ij} - a_{ij} \\ \alpha_j\left(A'^l\right) &= \alpha_j\left(A^l\right) + a'_{ij} - a_{ij} \\ \alpha_k\left(A'^l\right) &= \alpha_k\left(A^l\right) \text{ for all } k \in N\backslash\{i,j\} \text{ and} \\ \|A'^l\| &= \|A^l\| + a'_{ij} - a_{ij}. \end{aligned}$$

   The rest of the proof is similar to that of Case 1.

4. $a_{ij} \neq \varnothing$ and $a_{ji} = \varnothing$.

   In this case, $b_i^l((i,j), A, A', E) = b_j^l((i,j), A, A', E) = a_{ij}$. Thus,

$$\begin{aligned} \alpha_i\left(A'^l\right) &= \alpha_i\left(A^l\right) + 2\left(a'_{ij} - a_{ij}\right) \\ \alpha_j\left(A'^l\right) &= \alpha_j\left(A^l\right) + 2\left(a'_{ij} - a_{ij}\right) \\ \alpha_k\left(A'^l\right) &= \alpha_k\left(A^l\right) \text{ for all } k \in N\backslash\{i,j\} \text{ and} \\ \|A'^l\| &= \|A^l\| + 2\left(a'_{ij} - a_{ij}\right). \end{aligned}$$

   The rest of the proof is similar to that of Case 1.

Conversely, let R be a rule satisfying all the properties in the statement. Let $(A, E) \in \mathcal{P}$ be such that $||A|| > 0$.[13] Let $i \in N$.

Let $m(A)$ be the number of cancelled games in tournament A. We prove that $R(A, E) = ES^l(A, E)$ by induction on $m(A)$. Assume first that $m(A) = 0$. Thus, $a_{ij} \neq \varnothing$ for all $i, j \in N$ with $i \neq j$.

Let $i^0, i^1, i^2 \in N\backslash\{i\}$. We define A^* as follows.

$$a^*_{jk} = \begin{cases} \frac{\alpha_i(A)}{2} & \text{if } (j,k) \in \{(i,i^0),(i^0,i)\} \\ ||A|| - \alpha_i(A) & \text{if } (j,k) = (i^1,i^2) \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $||A^*|| = ||A|| > 0$ and $\alpha_i(A^*) = \alpha_i(A)$. Therefore, by *weak reallocation proofness*, $R_i(A^*, E) = R_i(A, E)$.

[13] Otherwise, the proof is trivial as $R_i(A, E) = \frac{E}{n}$, for each $i \in N$.

Let $N_1 = \{i, i^0\}$ and $N_2 = N \backslash N_1$. As $m(A^*) = 0$, we deduce that $A^{*l} = A^*$. Besides, $||A|| = ||A^*(N_1)|| + ||A^*(N_2)||$ and $||A^*(N_1)|| = \alpha_i(A)$. By *leg-single-conference*,

$$R_i(A^*, E) = R_i\left((A^*)^{N_1}, \frac{\alpha_i(A)}{||A||}E\right).$$

Now, using arguments similar to those used in the proof of Theorem 1 we can deduce that

$$R_i\left((A^*)^{N_1}, \frac{\alpha_i(A)}{||A||}E\right) = \frac{\alpha_i(A)}{2\,||A||}E = ES_i(A, E) = ES_i^l(A, E).$$

Therefore, $R(A, E) = ES^l(A, E)$ for $m(A) = 0$. Assume then that $R(A, E) = ES^l(A, E)$ for $m(A) = p$. Let (A, E) be such that $m(A) = p + 1$. Then, there exists a pair $i, j \in N$ such that $a_{ij} = \varnothing$. We consider two cases.

1. $a_{ji} \neq \varnothing$.

   In this case, let A' be such that

   $$a'_{kl} = \begin{cases} a_{ji} & \text{if } (k,l) = (i,j) \\ a_{kl} & \text{otherwise.} \end{cases}$$

   Thus, $m(A') = p$. By the induction hypothesis,

   $$R(A', E) = ES^l(A', E) = ES\left(A'^l, E\right) = ES\left(A^l, E\right) = ES^l(A, E).$$

   As $b_i^l((i,j), A, A', E) = a_{ji}$, and R satisfies *leg-baseline monotonicity*, we deduce that $R_i(A, E) = R_i(A', E)$. Similarly, we can argue that $R_j(A, E) = R_j(A', E)$.

   As $b_i^l((i,j), A, A', E) = b_j^l((i,j), A, A', E) = a_{ji}$ and $a'_{ij} = a_{ji}$, we deduce that

   $\max\left\{b_i^l((i,j), A, A', E), b_j^l((i,j), A, A', E)\right\} \leq a'_{ij} \leq \min\left\{b_i^l((i,j), A, A', E), b_j^l((i,j), A, A', E)\right\}.$

   By *leg-baseline monotonicity*, for each $k \in N \backslash \{i, j\}$,

   $$R_k(A', E) \leq R_k(A, E) \leq R_k(A', E),$$

   Thus, $R(A, E) = R(A', E) = ES^l(A, E)$.

2. $a_{ji} = \varnothing$.

   In this case, let A' be such that

   $$a'_{kl} = \begin{cases} 0 & \text{if } (k,l) = (i,j) \\ a_{kl} & \text{otherwise.} \end{cases}$$

   Replicating the arguments used in Case 1, we can argue that $R(A, E) = R(A', E) = ES^l(A, E)$. $\square$

5.4 Remark from Theorem 2

We now explain how to obtain the alternative characterization at Theorem 2 by replacing *leg-single-conference* by *leg-multi-conference* and *null team on non-cancelled games.*

It is straightforward to prove that ES^l also satisfies *leg-multi-conference* and *null team on non-cancelled games.* As for the converse implication, note that *leg-single-conference* is used only once in the previous proof. We explain now how to derive the same conclusion therein with *leg-multi-conference* and *null team on non-cancelled games.*

By *leg-multi-conference*,

$$R_i\left(A^*,E\right)=R_i\left(\left(A^*\right)^{N_1,\varnothing},\frac{\alpha_i\left(A\right)}{||A||}E\right)+R_i\left(\left(A^*\right)^{N_2,\varnothing},\frac{||A||-\alpha_i\left(A\right)}{||A||}E\right).$$

By *null team on non-cancelled games,* $R_i\left(\left(A^*\right)^{N_2,\varnothing},\frac{||A||-\alpha_i(A)}{||A||}E\right)=0$. Hence, $R_i\left(A^*,E\right)=R_i\left(\left(A^*\right)^{N_1,\varnothing},\frac{\alpha_i(A)}{||A||}E\right)$.

By *null team on non-cancelled games,* $R_j\left(\left(A^*\right)^{N_1,\varnothing},\frac{\alpha_i(A)}{||A||}E\right)=0$ for all $j\in N_2$. Using arguments similar to those used above for $\left(\left(A^*\right)^{N_1},\frac{\alpha_i(A)}{||A||}E\right)$ we can prove that $R_i\left(\left(A^*\right)^{N_1,\varnothing},\frac{\alpha_i(A)}{||A||}E\right)=\frac{\alpha_i(A)}{2||A||}E=ES_i^l\left(A,E\right).$

5.5 Proof of Theorem 3

It is straightforward to show that CD^z satisfies *reallocation proofness* and *essential on non-cancelled games.* Regarding *zero-multi-conference*, let $(A,E)\in\mathcal{P}$ and $\{N_1,...,N_p\}$ be as in its definition. Given $i\in N_{i'}$,

$$\sum_{k=1}^{p}CD_i^z\left(A^{N_k,\varnothing},\frac{||A^z\left(N_k\right)||}{\sum_{k=1}^{p}||A^z\left(N_k\right)||}E\right)=\sum_{k=1}^{p}\frac{\left(n-1\right)\alpha_i\left(\left(A^{N_k,\varnothing}\right)^z\right)-\left|\left|\left(A^{N_k,\varnothing}\right)^z\right|\right|\frac{\frac{||A^z(N_k)||}{\sum_{k=1}^{p}||A^z(N_k)||}E}{||(A^{N_k,\varnothing})^z||}}{n-2}$$

As

$$\begin{aligned}\alpha_i\left(\left(A^{N_k,\varnothing}\right)^z\right)&=\begin{cases}\alpha_i\left(A^z\right)&\text{if }i\in N_k\\0&\text{otherwise,}\end{cases}\\||A^z\left(N_k\right)||&=\left|\left|\left(A^{N_k,\varnothing}\right)^z\right|\right|,\text{ and}\\||A^z||&=\sum_{k=1}^{p}||A^z\left(N_k\right)||\end{aligned}$$

it follows that

$$\begin{aligned}
\sum_{k=1}^{p} CD_i^z\left(A^{N_k,\varnothing},\frac{\|A^z(N_k)\|}{\sum_{k=1}^{p}\|A^z(N_k)\|}E\right) &= \frac{(n-1)\,\alpha_i(A^z)-\|A^z(N_{i'})\|}{n-2}\frac{E}{\|A^z\|}-\sum_{k\neq i'}\frac{\|A^z(N_k)\|}{n-2}\frac{E}{\|A^z\|}\\
&= \frac{(n-1)\,\alpha_i(A^z)-\sum_{k=1}^{m}\|A^z(N_k)\|}{n-2}\frac{E}{\|A^z\|}\\
&= \frac{(n-1)\,\alpha_i(A^z)-\|A^z\|}{n-2}\frac{E}{\|A^z\|}\\
&= CD_i(A^z,E)=CD_i^z(A,E).
\end{aligned}$$

Conversely, let R be a rule satisfying all the axioms in the statement. Let $(A,E)\in\mathcal{P}$ be such that $\|A\|>0$.[14] Let $i\in N$.

Let A^*, i^0, i^1, i^2, N_1 and N_2 be defined as in the proof of Theorem 1. By *reallocation proofness* $R_i(A,E)=R_i(A^*,E)$. By *zero-multi-conference*,

$$R_i(A^*,E)=R_i\left((A^*)^{N_1,\varnothing},\frac{\|A^*(N_1)\|}{\|A\|}E\right)+R_i\left((A^*)^{N_2,\varnothing},\frac{\|A^*(N_2)\|}{\|A\|}E\right).$$

We first analyze the problem induced by conference N_1. We consider two cases.

Case 1.1. $\alpha_i(A)=0$. Then $\|A^*(N_1)\|=0$ and hence

$$R_i\left((A^*)^{N_1,\varnothing},\frac{\|A^*(N_1)\|}{\|A\|}E\right)=0.$$

Case 1.2. $\alpha_i(A)>0$. Then $\|A^*(N_1)\|=\alpha_i(A)>0$. By *essential team on non-cancelled games*,

$$R_i\left((A^*)^{N_1,\varnothing},\frac{\|A^*(N_1)\|}{\|A\|}E\right)=\frac{\|A^*(N_1)\|}{\|A\|}E=\frac{\alpha_i(A)}{\|A\|}E.$$

We now analyze the problem induced by conference N_2. We also consider two cases.

Case 2.1. $\alpha_i(A)=\|A\|$. Then $\|A^*(N_2)\|=0$ and hence

$$R_i\left((A^*)^{N_2,\varnothing},\frac{\|A^*(N_2)\|}{\|A\|}E\right)=0.$$

Case 2.2. $\alpha_i(A)<\|A\|$. Then $\|A^*(N_2)\|=\|A\|-\alpha_i(A)>0$.

For each $j,k\in N$ and $x\in\mathbb{R}_+$, we define $A^{jk,x}$ where

$$a_{lm}^{jk,x}=\begin{cases} x & \text{if } (l,m)=(j,k)\\ \varnothing & \text{otherwise.}\end{cases}$$

[14] Otherwise, the proof is trivial as $R_i(A,E)=\frac{E}{n}$, for each $i\in N$.

By *essential team on non-cancelled games,*

$$R_j\left(A^{jk,x},E\right)=R_k\left(A^{jk,x},E\right)=E.$$

Thus,

$$\sum_{l\in N\backslash\{j,k\}}R_l\left(A^{jk,x},E\right)=-E. \tag{1}$$

Let $l\in N$. Consider $j,k\in N\backslash\{l\}$ and $j',k'\in N\backslash\{l\}$. By *reallocation proofness,*

$$R_l\left(A^{jk,x},E\right)=R_l\left(A^{j'k',x},E\right).$$

As $R_l\left(A^{jk,x},E\right)$ does not depend on j and k,we can define $f(l,x,E)=R_l\left(A^{jk,x},E\right)$ for each $l\in N$ and $x,E\in\mathbb{R}$.

Let $j,k\in N$ and $m\in N\backslash\{j,k\}$. By (1)

$$\begin{aligned}-E &= \sum_{l\in N\backslash\{j,k,m\}}R_l\left(A^{jk,x},E\right)+R_m\left(A^{jk,x},E\right)\text{ and}\\ -E &= \sum_{l\in N\backslash\{j,k,m\}}R_l\left(A^{jm,x},E\right)+R_k\left(A^{jm,x},E\right).\end{aligned}$$

Then, $R_m\left(A^{jk,x},E\right)=R_k\left(A^{jm,x},E\right)$. Hence $f(m,x,E)=f(k,x,E)$. We then denote it as $f(x,E)$.

By (1),

$$f(x,E)=\frac{-E}{n-2}.$$

Notice that

$$\left((A^*)^{N_2,\varnothing},\frac{||A^*(N_2)||}{||A||}E\right)=\left(A^{i^1i^2,||A||-\alpha_i(A)},\frac{||A||-\alpha_i(A)}{||A||}E\right).$$

Thus, for each $k\in N\backslash\{i^1,i^2\}$

$$R_k\left((A^*)^{N_2,\varnothing},\frac{||A^*(N_2)||}{||A||}E\right)=-\frac{||A||-\alpha_i(A)}{(n-2)\,||A||}E.$$

By *essential team on non-cancelled games,*

$$R_{i^1}\left((A^*)^{N_2,\varnothing},\frac{||A^1(N_2)||}{||A||}E\right)=R_{i^2}\left((A^*)^{N_2,\varnothing},\frac{||A^1(N_2)||}{||A||}E\right)=\frac{||A||-\alpha_i(A)}{||A||}E$$

We now conclude as follows.

$\alpha_i(A)=0$. By cases 1.1 and 2.2,

$$R_i(A,E)=-\frac{||A||}{(n-2)\,||A||}E=CD_i^z(A,E).$$

$0 < \alpha_i(A) < ||A||$. By cases 1.2 and 2.2,

$$\begin{aligned} R_i(A,E) &= \frac{\alpha_i(A)}{||A||}E - \frac{||A|| - \alpha_i(A)}{(n-2)||A||}E \\ &= \frac{(n-1)\alpha_i(A) - ||A||}{(n-2)||A||}E = CD_i^z(A,E). \end{aligned}$$

$\alpha_i(A) = ||A||$. By cases 1.2 and 2.1,

$$R_i(A,E) = \frac{\alpha_i(A)}{||A||}E = E = CD_i^z(A,E). \qquad \square$$

5.6 Proof of Theorem 4

We can prove that CD^l satisfies *leg-baseline monotonicity* similarly to the case of ES^l in the proof of Theorem 2. It is straightforward to show that CD^l satisfies the rest of the axioms in the statement.

Conversely, let R be a rule satisfying those axioms. Let $(A,E) \in \mathcal{P}$ be such that $||A|| > 0$.[15]

Let $m(A)$ be the number of cancelled games in tournament A. We prove that $R(A,E) = CD^l(A,E)$ by induction on $m(A)$. Assume first that $m(A) = 0$.

Let $i \in N$. Let A^*, i^0, i^1, i^2, N_1 and N_2 be defined as in the proof of Theorem 2. By *weak reallocation proofness*, replicating the argument therein, we obtain that $R_i(A,E) = R_i(A^*,E)$.

Now, as $m(A^*) = 0$, $A^* = (A^*)^z = (A^*)^l$. This implies that *leg-multi-conference* is equivalent to *zero-multi-conference* for A^*. Besides, *reallocation proofness* is equivalent to *weak reallocation proofness* for A^*.

Thus, similarly to the proof of Theorem 3, we can show $R_i(A^*,E) = CD_i^z(A,E)$. As $m(A) = 0$, $A^z = A^l$. Hence, $R_i(A^*,E) = CD_i^l(A,E)$.

Assume then that $R(A,E) = CD^l(A,E)$ for $m(A) = p$. Let (A,E) be such that $m(A) = p+1$. Then, using similar arguments to those used in the proof of Theorem 2, we can deduce that $R(A,E) = CD^l(A,E)$. $\square$

5.7 Proof of Theorem 5

It is similar to the proofs of Theorem 1 and Theorem 3 and, thus, we omit it.

[15] Otherwise, the proof is trivial as $R_i(A,E) = \frac{E}{n}$, for each $i \in N$.

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