Factor Supply Elasticities, Returns to Scale, and the Direction of Technological Progress

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Abstract: This paper finds that the steady-state direction of technological progress is determined by the relative size of factor supply elasticities and the returns to scale of the production function, which have so far been ignored. However, the relative price (Hicks, 1932) and relative market size (Acemoglu, 2002) emphasized in the existing literature have only short-term effects. This conclusion is obtained by introducing generalized factor accumulation processes that do not restrict factor supply elasticities, and a generalized production function that does not restrict the returns to scale. It emanates solely from the characterizations of production function, steady-state growth, direction of technological progress and factor supply elasticities. The paper also analyzes a particular micro-founded growth model and uses it to exemplify the conclusions. The findings of this paper provide new explanations to the Uzawa (1961) steady-state theorem puzzle as well as to the Kaldor facts characterization of modern economic growth. It also suggests a way to reconcile falling investment good prices with the Kaldor facts. In addition, it may help explain why technological progress did not increase per capita income before the industrial revolution and what might have led to the modern pattern of economic growth.

Key Words: Economic Growth, Direction of Technological Progress, Returns to Scale, Factor Supply Elasticities, Uzawa's Steady-State Theorem, Industrial Revolution, Adjustment Cost

JEL: O33; O41; E13; E25

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I. Introduction

Technological progress relates not only to its rate but also to its direction. While there is a large and influential body of literature concerning the determinants of the *rate* element (see, e.g., Romer, 1990; Grossman and Helpman, 1991; Aghion and Howitt, 1992), the *direction* is not as well understood.[1] However, for many important questions of economic growth, the direction may be more important than the rate of technological progress. In particular, understanding what determines the direction of technological progress may be key to resolving some salient problems associated with prevalent growth theory.

The first problem relates to the Kaldor (1961) facts which underlie much of the neoclassical growth theory. Specifically, post-industrial-revolution economic growth in developed countries has been characterized by increasing per-capita output and physical capital, whereas the capital/output ratio, the real return to capital, and factor income shares have remained basically constant (See Figure 1 and Jones, 2016, p.5). As post-industrial-revolution technological progress has been very rapid and extensive, this gives rise to the question of what economic factors might have caused only the productivity of labor to improve and not that of capital?[2]

Panel A. Panel B.

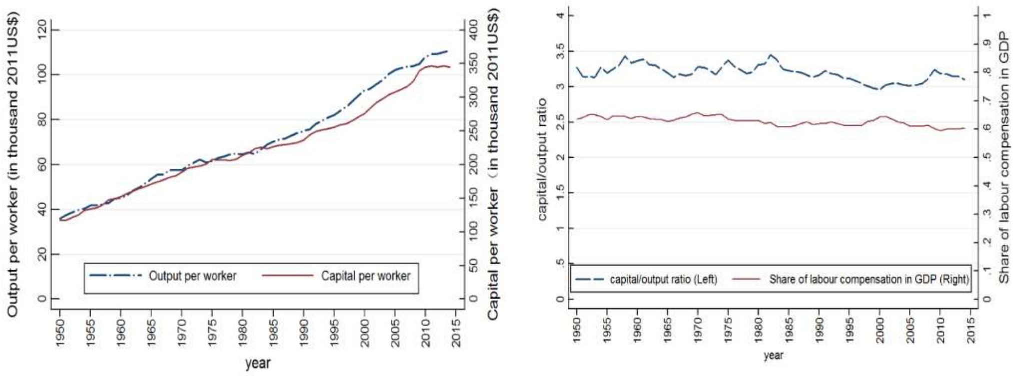

Figure 1: Kaldor (1961) facts in the USA (1950-2014)

Note: Panel A represents the output and capital per worker, and Panel B represents the capital/output and labor share in the USA from 1950-2014. Source: Feenstra, Inklaar and Timmer (2015), PWT 9.0.

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[1] About 20 years ago, Acemoglu (2002) has already made this point. Although Acemoglu and many other authors have done significant work on the direction of technological progress, fundamentally the basic question remains unresolved.

[2] Some authors think that by assuming purely labor-augmenting technological progress the neoclassical growth model is successful in explaining Kaldor's facts (Jones and Romer, 2010; Grossman et al., 2017). However, making the *direction* of technological progress exogenous is as unsatisfactory as modeling per-capita output growth as a consequence of an exogenous *rate* of technological progress.

The second issue concerns the Uzawa (1961) steady-state theorem. It is this theorem that implies that the neoclassical growth models can attain a steady-state equilibrium only if technological progress is purely labor-augmenting.[3] In particular, if investment-specific technological progress takes place and the production function is not Cobb-Douglas, these models do not possess steady states. This knife-edge result has puzzled growth theorists for a long time, yet the implicit assumptions generating it have not been clarified.

Moreover, ongoing investment-specific technological progress seems to be empirically evidenced by the long trend of falling investment goods prices (Gordon, 1990; Greenwood et al., 1997; Jones, 2016; Grossman et al., 2017, see Figure 2). In addition, there is mounting evidence that the (aggregate) substitution elasticity of capital and labor is not unitary, making the Cobb-Douglas specification unrealistic (see, e.g., Karabarbounis and Neiman, 2014; Oberfield and Raval, 2014; Chirinko and Mallick, 2014; Lawrence, 2015; Knoblach, Roessler and Zwerschke, 2020). These observations contradict the implications of the Uzawa theorem. In order to include investment-specific technological progress, some authors (Grossman et al. 2017; Casey and Horii, 2019) introduce capital-augmenting technological progress into the production of final goods, but the Kaldor facts are consistent only with pure labor-augmentation. This raises the question whether investment-specific technological progress can coexist with pure labor-augmentation in steady state?

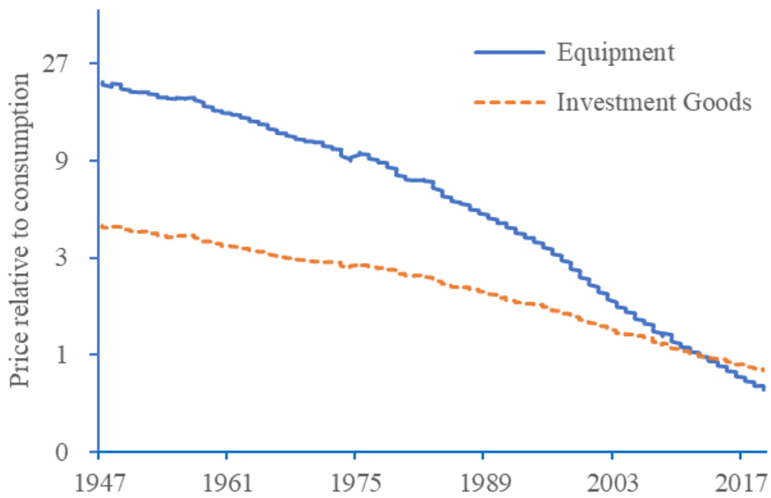

Figure 2 US Relative Price of Equipment and Investment goods, 1947–2019.
Source: Federal Reserve Bank Economic Data (FRED), series PIRIC and PERIC

A closely related question that has been largely ignored concerns the huge

[3] Acemoglu (2003) pointed out that the neoclassical growth model does not require technological progress to be purely labor-augmenting all the time and can include capital-augmentation in the transitional phase. It is only in the steady state where it must be purely labor-augmenting.

difference in the direction of technological progress before and after the industrial revolution, and the reasons for the transition. According to Ashraf and Galor (2011), before the industrial revolution technological progress only improved the productivity of land but not that of labor and for an extended period of time generated population growth and higher density, but not higher per-capita income (the "Malthusian trap"). Therefore, it is not only necessary to explain why modern economic growth is driven by purely labor-augmenting technological progress, but also to explain why technological progress before the industrial revolution did not encompass labor augmentation and what caused the transition.

To answer these questions, we introduce generalized factor accumulation processes that do not restrict the factor supply elasticities, and a generalized production function without constraining its returns to scale. This enables us to investigate the role played by factor supply elasticities and returns to scale, which so far has been ignored, on the direction of technological progress. The key finding is that if production is governed by constant returns to scale, the steady-state direction of technological progress depends on the relative size of the factor supply elasticities, and is biased towards the factor with the lower elasticity. The economic intuition behind this conclusion is that, in the long run, a higher factor price may encourage not only inventions to economize that factor's use, but also its accumulation. If the supply elasticity of the factor is very large, the invention incentive may be reversed. Furthermore, to offset that factor's abundance, balanced growth requires the presence of increased investment in technologies that augment the efficiency of the factor with the smaller supply elasticity. We obtain these core conclusions solely from the production function and the definitions of the direction of technological progress, steady-state growth, and factor supply elasticities.

Following this general characterization, the paper provides a specific micro-founded growth framework by extending Acemoglu's (2002) model. In particular, it introduces investment adjustment costs and investment-specific technological progress into the capital accumulation function to relax the restrictions on the elasticity of the capital supply. In addition, it introduces diminishing returns to scale into the production function. Beyond exemplifying the above results, the specific model also shows that changing relative factor prices (as suggested by Hicks 1932) and the relative market size (as argued by Acemoglu 2002) do affect the direction of technological progress in the short run, but have no impact on that direction in steady state. Furthermore, the model identifies the underlying features that determine the supply elasticities.

Based on these findings, the current paper argues that the answer to the above questions all hinge on the *factor supply elasticities.* Specifically, modern economic growth and the associated Kaldor facts result if the supply elasticity of capital is infinite while that of labor is finite. The Uzawa theorem too is the consequence of this elasticity configuration as implicitly set by the neoclassical growth model. In particular, if the capital supply elasticity is finite, then it is possible to generate *any* combination of capital- and labor-augmenting technologies in steady state. If capital has a finite supply elasticity while labor supply is infinitely elastic, the resulting steady state will be characterized by purely capital-augmenting technological progress and per-capita income will not grow. This is consistent with the aforementioned characterization of the economic growth path before the industrial revolution.[4] Accordingly, the transition to the modern growth path may be a consequence of the changing supply elasticities: that of capital increased while that of labor decreased.

The plan of the paper is as follows. Section **II** discusses the related literature; Section **III** introduces a generalized production function and generalized factor accumulation processes; Section **IV** derives the determinants of technological change; Section **V** develops a specific growth model in which the direction of technological progress is endogenized; Section **VI** focuses on some applications; Section **VII** contains concluding remarks.

II. Related Literature

Although the existing literature on the rate of technological progress is more in-depth than on its direction, the literature on the direction preceded that on the rate. Early in 1932, Hicks (1932) pointed out that changing relative factor prices may affect that direction. Brozen (1953) too pointed out that the direction was endogenously determined by economic forces. Lacking a dynamic growth framework (to be developed by Solow a few years later), these early contributions could not distinguish between short-term and long-term effects.

The neoclassical growth models (Solow, 1956; Swan, 1956; Cass, 1965; Koopmans,1965) provided this perspective and pointed out that technological progress was the key factor of economic growth in the long run. However, not only the rate but also the direction of technological progress is exogenous in these models. Although the models can attain steady-state growth consistent with Kaldor's facts, the direction of technological progress turned out to be a cumbersome issue. That direction was exogenously set and restricted to be purely labor-augmenting. Otherwise, unless the production function is Cobb-Douglas, no steady state exists. However, these models do not provide compelling intuitive reasons as to why technological progress should take this specific form. This is the famous puzzle of Uzawa's theorem (elegantly and intuitively re-proven by Schlicht 2006).

The introduction of an innovation possibilities frontier (von Weizsäcker, 1962;

[4] This interpretation requires identifying "capital" with land before the industrial revolution.

Kennedy, 1964), coupled with cost reduction maximization, has seemingly enabled the induced innovation literature of the 1960s (Samuelson, 1965; Drandakis and Phelps, 1966) to resolve this issue. However, Nordhaus (1973) questioned the validity of this resolution for its lack of micro-mechanisms generating technological progress. The assumption that enterprises maximize the current rate of cost reduction rather than profits was also criticized (Acemoglu, 2001).

Although the rate of technological progress has been endogenized, its direction is still exogenous also in the endogenous growth models (Romer, 1990; Aghion and Howitt, 1992). By extending the technological progress from one dimension to two, Acemoglu (2002) provided a framework in which that direction can be endogenized. Within the extended framework, Acemoglu (2002) proposed a market size effect as another key factor affecting the direction of technological progress besides the price effect of Hicks (1932). However, the two production factors are assumed to be inelastically supplied and the production function has constant returns to scale. Therefore, this framework ignores the influence of the relative size of factor supply elasticities and the returns to scale on the direction of technological progress. As a result, it provides just the determinants of the relative technological *level* which, in turn, can be obtained only when technological progress is Hicks neutral in steady state. Acemoglu (2003) incorporated the framework into a neoclassical growth model, which yielded a balanced growth path with purely labor-augmenting technological progress. However, there is no explanation as to why firms would choose only this direction of progress.

Some additional authors (Funk, 2002; Irmen and Tabakovic, 2017) have constructed growth models based on perfect, rather than monopolistic, competition, which endogenize the direction of technological progress. These contributions also ignore the role played by factor supply elasticities and the returns to scale of the production function, thereby failing to identify the factors that determine the direction of technological progress.

Another issue concerns the specification of the production function. As mentioned above, according to Uzawa's theorem the direction of technological progress is not restricted if the production function is Cobb-Douglas. For that reason, some authors (Jones, 2005; León-Ledesma and Satchi, 2018) tried to prove that the production function takes this form, at least in the steady state. However, empirical evidence indicating that the substitution elasticity between capital and labor is not unitary casts doubt on this approach.

Uzawa's theorem does not point out which underlying premise is responsible for the requirement that at the steady state technological progress must be purely labor-augmenting. Nevertheless, some authors have noticed that the factor accumulation processes and production function are the key factors affecting that result, and tried to extend these functions to adjust the result.[5] Sato (1996) showed that a non-linear capital accumulation process, which allows for diminishing returns to investment, is necessary if the steady state is to include also capital-augmenting technological progress. Sato et al. (1999, 2000) proposed specific models of that nature where steady states encompass capital-augmenting technological progress. Irmen (2013) proved that technological progress could include capital-augmentation, provided that the capital accumulation process is affected by adjustment costs. None of the aforementioned papers has pointed out the important influence of the supply elasticities as determined by the factor accumulation processes on the direction of technological progress. Nor have they recognized that the infinite capital supply elasticity implied by the "standard" capital accumulation process is one of the premises underlying the Uzawa theorem.

Suggesting a different approach, Grossman et al. (2017) introduced a schooling variable into the production function in addition to labor and capital. They showed that, in steady state, not only the production technology can admit capital-augmenting progress, but also the capital accumulation process can include investment-specific technological progress. Casey and Horii (2019) built a model which allows for a steady state to encompass capital-augmenting technological progress. They obtained this result by introducing new factors (such as land) into the production function and decreasing returns to scale for capital and labor. However, they have not uncovered the relationship between the returns to scale of the production function and the direction of technological progress.

In a word, the puzzle of Uzawa's theorem is one of the most important motivations for the existing literature to focus on the direction of technological progress. However, two defects of the existing literature, which this paper tries to alleviate, hinder the discovery of the direction's determinants and the resolution of that puzzle.

First, the existing literature usually tries to answer why in steady state technological progress must be purely labor-augmenting. However, posing the question in this fashion is misleading, as it subsumes that Uzawa's proposition is *unconditionally* true despite the missing economic intuition (Acemoglu, 2003; Jones, 2005). As a matter of fact, not all growth models require the steady-state technological progress to be purely labor-augmenting. Specifically, the Malthusian model cannot admit labor-augmentation at all in steady state (Li and Huang, 2016). Therefore, Uzawa's theorem is only a *conditional* proposition. In this vein, and unlike

[5] Already in the 1960s, in an argument akin to the one presented below, Samuelson (1965) pointed out that if factor supplies remain in balance, then technological progress will be Hicks-neutral in steady-state, otherwise it would be tilted towards labor- or capital-augmentation.

the existing literature, this paper asks a more general question, namely, what determines the steady-state direction of technological progress? Accordingly, the puzzle of Uzawa's theorem becomes a subproblem of this question, that is, under what conditions does technological progress have to be purely labor-augmenting?

Second, the existing literature discusses Uzawa's theorem under the same implicit assumptions as those of the neoclassical growth model. As a result, it is impossible to identify which of these implicit assumptions lead to Uzawa's theorem. Here we introduce two generalized functions that embed the neoclassical growth model as a special case. It is through these extensions that the role of the factor supply elasticities and the returns to scale as the determinants of the direction of technological progress are exposed. Moreover, it is precisely because the neoclassical growth model implicitly assumes that the capital supply elasticity is infinite while that of labor is finite that the steady-state technological progress must be purely labor-augmenting.

III. A generalized production function and generalized factor accumulation processes

The returns to scale of the production function and factor supply elasticities constrain economic agents when choosing the direction of technological progress. However, in the existing literature, the returns to scale and the elasticities are implicitly set by a particular formulation of the production function and factor accumulation processes, causing their influence on the direction of technological progress to have inadvertently been ignored. In order to overcome this defect, in this section we construct a generalized production function with constant or diminishing returns to scale that depend on parameter values. We also introduce investment adjustment costs into factor accumulation processes to allow expanding the range of factor supply elasticities from zero to infinity. Later, we will show that the generalization exposes the key influence of the returns to scale and factor supply elasticities on the direction of technological progress.

1. A generalized production function

The returns to scale of the production function may be constant, decreasing or increasing. However, when discussing the direction of technological progress, it is usually assumed that the returns to scale are constant, rather than making them depend on parameter values. For our purpose, we construct a production function that is potentially compatible with different returns to scale for capital, K, and labor, L, as follows:

$$\mathrm{Y}(\mathrm{t}) = \mathrm{F}\left[\mathrm{B}(t)\mathrm{K}(t)^{\phi}, A(t)\mathrm{L}(\mathrm{t})^{\varphi}\right], 0 < \phi \leq 1, 0 < \varphi \leq 1 \tag{1}$$

Here, B and A denote capital-augmenting and labor-augmenting technologies respectively. Accordingly, assuming that the production function has constant returns to scale for K^{ϕ} and L^{φ}, that is, $F[B\lambda(K^{\phi}), A\lambda(L^{\varphi})] = \lambda F[BK^{\phi}, AL^{\varphi}]$, then $F[B(\lambda K)^{\phi}, A(\lambda L)^{\varphi})] \leq \lambda F[BK^{\phi}, AL^{\varphi}]$. If $\phi = \varphi = 1$, the production function takes the usual neoclassical form with constant returns to scale for K and L; when $\phi < 1$ or $\varphi < 1$, it has diminishing returns to scale for K and L.[6] Clearly, whether in reality the returns to scale of K and L are constant or decreasing, is an empirical issue. As the goal of this paper is to theoretically expose the impact of the returns to scale upon the direction of technological progress, its conclusions do not hinge on the empirical outcomes.

Define $\widehat{K} \equiv [BK^{\phi}]$ as representing effective capital, and $\hat{L} \equiv [AL^{\varphi}]$ as representing effective labor. Output per effective labor is expressed by $y \equiv \frac{Y}{AL^{\varphi}}$ and the effective capital to effective labor ratio is expressed by $k \equiv \frac{BK^{\phi}}{AL^{\varphi}}$. Accordingly, the production function in the intensive form takes the form:

$$y(t) = F\left[\frac{B(t)K(t)^{\phi}}{A(t)L(t)^{\varphi}}, 1\right] = f(k(t)) \tag{2}$$

If the market is not completely competitive, then factor prices will be smaller than their marginal products. Therefore, we assume only that the factor prices are proportional to their marginal products. Specifically, by equation (2) the corresponding prices of K and L can be written as:

$$\begin{cases} w(t) = \xi_L \varphi A(t) L(t)^{\varphi-1}\left[f\big(k(t)\big) - kf'\big(k(t)\big)\right] \\ r(t) = \xi_K \phi B(t) K(t)^{\phi-1} f'\big(k(t)\big) \end{cases} \tag{3}$$

where $\xi_L \leq 1$ and $\xi_K \leq 1$. When the market is completely competitive $\xi_L = \xi_K = 1$, otherwise, $\xi_L < 1$ and $\xi_K < 1$.

Equations (3) show that the returns to scale parameters φ and ϕ have important influence on factor prices. When they are smaller than 1, even if k is constant, factor prices decrease with their quantities. Therefore, in this case, only factor-augmenting technological progress can keep factor prices constant or make them increase.

2. Generalized factor accumulation processes

The introduction of generalized factor accumulation processes and the presence

[6] If $\phi > 1$ or $\varphi > 1$, the returns to scale with respect to K and L are increasing, which may lead to negative technological progress in steady state.

of investment adjustment cost are additional important features needed to analyze the direction of technological progress through their impact on the factor supply elasticities. As it turns out, the factor accumulation processes used in the existing literature usually take specific forms which often implicitly limit the size of the ensuing factor supply elasticities, whereas the generalized factor accumulation processes entail no such restrictions.

(1) Factor supply elasticities

The supply elasticities of capital (ε_K) and labor (ε_L) are a key property of the factor supply functions and follow the standard definition:[7]

$$\begin{cases} \varepsilon_K \equiv \dfrac{\dot{\mathrm{K}}(\mathrm{t})/K(t)}{\dot{r}(\mathrm{t})/r(t)} \\ \varepsilon_L \equiv \dfrac{\dot{\mathrm{L}}(\mathrm{t})/L(t)}{\dot{w}(\mathrm{t})/w(t)} \end{cases} \tag{4}$$

Equations (4) clearly imply that the factor accumulation processes $\dot{\mathrm{K}}(\mathrm{t})$ and $\dot{\mathrm{L}}(\mathrm{t})$ are key determinants of the factor supply elasticities. The existing literature, especially the literature on the direction of technological progress, has completely ignored their impact. Acemoglu (2002) assumes that both factors are inelastically supplied. As is well known, the core assumption of the Malthusian model is that the labor supply elasticity is infinite. In the Solow (1956) model, $\dot{\mathrm{K}}(\mathrm{t})/K(t)$ is a positive constant at the steady state while the interest rate is unchanging, rendering $\varepsilon_K = \infty$. Clearly, zero or infinity are both only possible specific elasticities, while an elasticity between zero and infinity may be more realistic. Moreover, a given economy in different periods, or different economies at the same period, may have different factor supply elasticities. As shown below, setting the factor supply elasticities to specific values implicitly restricts technological progress to follow a particular direction in steady state. For this reason, the factor accumulation processes should be allowed to generate factor supply elasticities that range from zero to infinity.

(2) Generalized factor accumulation processes with investment adjustment cost

Although the inclusion of investment adjustment cost has become a basic ingredient of macroeconomic models and economic growth (Barro and Sala-i-Martin, 2004, Ch3; Acemoglu, 2009, Ch8), its impact on the factor supply elasticities has not

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[7] The elasticity of factor supplies is usually defined as $\varepsilon = \frac{\Delta X(t)/X(t)}{\Delta p(t)/p(t)}$. Technically, this is equivalent to equations (4), noting that the numerator and denominator are multiplied by Δt at the same time. Because the factor price and quantity in the growth model are both functions of time, we think that the definition as presented here is more appropriate.

been considered. [8] However, by admitting investment adjustment costs, we can obtain generalized factor accumulation processes that imply factor supply elasticities ranging from 0 to infinity.

The adjustment cost may capture both internal factors and external ones. In the case of material capital, the internal costs may reflect the costs of installing new capital and training workers to operate the new machines, and the external costs may be due to increasing production costs. For labor, the internal adjustment costs may arise from the training of new workers while the external costs may reflect the cost associated with increasing the labor force. Accordingly, it is assumed that both physical capital and labor are the result of investment, and both have adjustment cost of the same form. However, the adjustment cost parameters may be different, implying different supply elasticities. To see this, consider the following standard specification of the factor accumulation process:

$$\dot{X}(t) = I_X(t) - \delta_X X(t) \tag{5}$$

Where X represents K or L, $\mathrm{I_X(t)}$ represents investment in K or L, δ_X is depreciation rate or mortality.

The presence of the investment adjustment costs is specified as follows:

$$I(\mathrm{t}) = \mathrm{I_X(t)}\left(1 + \mathrm{h}\big(\mathrm{I_X(t)}\big)\right) \tag{6}$$

That is, investing a total of $I(\mathrm{t})$ units in factor X enhances the amount of the factor only by $\mathrm{I_X}(t)$ units, whereby a proportional addition of $\mathrm{h}(\mathrm{I_X}(t))$ units is spent as adjustment costs. [9] We assume $\mathrm{h}(0) = 0$, $\partial\mathrm{h}/\partial\mathrm{I_X} > 0$, $\partial^2\mathrm{h}/\partial\mathrm{I_X^2} \geq 0$, that is, the adjustment costs are non-decreasing also at the margin. Monotonicity allows us to obtain the inverse function $\mathrm{I_X(t)} = \mathrm{G[I(t)]}$. Substituting it into equation (5) yields the general factor accumulation process that incorporates the investment adjustment costs:

$$\dot{\mathrm{X}}(\mathrm{t}) = \mathrm{G[I(t)]} - \delta_X \mathrm{X(t)} \tag{7}$$

with $\frac{\partial \mathrm{G}}{\partial \mathrm{I(t)}} > 0$ and $\frac{\partial^2 \mathrm{G}}{\partial \mathrm{I(t)}^2} \leq 0$. Equation (7) shows that with adjustment costs, the marginal efficiency of turning investment into a factor is (weakly) decreasing.

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[8] Investment adjustment costs were introduced in order to overcome defects of the neoclassical investment theory (Eisner and Strotz, 1963; Lucas, 1967; Foley and Sidrauski, 1970), specifically the extremely high investment sensitivity to small changes in economic conditions implied by that theory. Hayashi (1982) has introduced investment adjustment costs into a firm's optimal investment program in order to analyze the dynamics of Tobin's Q. The presence of these costs has also been invoked to improve the performance of DSGE models (see, e.g., Cristiano et al., 2005).

[9] Equations (5) and (6) are equations (3.25) and (3.26) in Barro and Sala-i-Martin (2004), Chapter 3, page 152, except that their adjustment cost function is specified as $\mathrm{h}(\mathrm{I_K}/\mathrm{K})$.

Equation (7) implies that the function $\mathrm{G}[\mathrm{I}(\mathrm{t})]$ has a decisive influence on the factor supply elasticity since $\dot{\mathrm{X}}(\mathrm{t})/\mathrm{X}(\mathrm{t}) = \mathrm{G}[\mathrm{I}(\mathrm{t})]/\mathrm{X}(\mathrm{t}) - \delta_X$.

As a special case, consider the following adjustment cost function:

$$\mathrm{h}[\mathrm{I_X}(\mathrm{t})] = \left[\mathrm{I_X}(\mathrm{t})^{(1-\alpha)/\alpha} - 1\right], 0 < \alpha \leq 1 \tag{8}$$

Equation (8) shows that the adjustment costs not only depend on the investment $\mathrm{I_X}(\mathrm{t})$, but also on the parameter α. When $\alpha = 1$, equation (8) implies no adjustment cost, and when α is close to zero, equation (8) implies infinite adjustment cost. Using equation (8) in equations (5) and (6) yields the specific factor accumulation function:

$$\dot{\mathrm{X}}(\mathrm{t}) = I(t)^{\alpha} - \delta_X \mathrm{X}(\mathrm{t}) \tag{9}$$

For $\alpha = 1$, if X represents capital K, equation (9) reduces to the capital accumulation process of the existing neoclassical model, implying a capital supply elasticity of infinity; if X represents labor L, equation (9) reduces to the labor accumulation process of the existing Malthusian model and the labor supply elasticity is infinite.[10] When α approaches 0, the elasticity of the factor supply tends to 0, as in Acemoglu (2002); When $0 < \alpha < 1$, the factor supply has a finite elasticity. Accordingly, the adjustment costs have an important impact on the factor supply elasticities, which, depending on the values of α in equation (9), can take any value between zero to infinity.[11] While the core conclusion does not depend on the specific form of the capital accumulation process, equation (9) will be used in Section V, where a particular model is analyzed.

IV. The Determinants of Technological Progress

For the economy described by the generalized production function (1), this section shows that the *model-free* definitions of steady-state growth, direction of technological progress and factor supply elasticities suffice to draw the core conclusion and its important corollaries concerning the direction of technological progress.

1. Definitions

Definition 1: A *steady-state growth path* obtains when the growth rates of $\mathrm{Y}(\mathrm{t})$, $\mathrm{B}(t)$, $\mathrm{K}(t)$, $A(t)$, $\mathrm{L}(\mathrm{t})$ and factor income shares are constant.

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[10] When $\dot{L} = 0$ we obtain $L^* = 1/\delta_L$. For the special case with $I_L \equiv s_L Y$, $\dot{L} = I_L - \delta_L L = s_L Y - \delta_L L$. This is the labor supply assumption of the Malthusian model, where s_L is endogenously determined by the household's intertemporal optimization.

[11] Irmen (2013) pointed out that equation (9) incorporates adjustment costs in the capital accumulation process. However, he did not provide the underlying adjustment costs function, nor did he discuss the capital supply elasticity implied by this function.

Remark 1: Since $k(\mathrm{t}) \equiv \frac{\mathrm{B}(t)\mathrm{K}(t)^{\phi}}{A(t)\mathrm{L}(\mathrm{t})^{\varphi}}$ is constant at the steady-state growth path, the growth rates of $\mathrm{B}(t)\mathrm{K}(t)^{\phi}$ and $A(t)\mathrm{L}(\mathrm{t})^{\varphi}$ are identical and given by:

$$\phi\frac{\dot{\mathrm{K}}(\mathrm{t})}{\mathrm{K}(\mathrm{t})}+\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})}=\varphi\frac{\dot{\mathrm{L}}(\mathrm{t})}{\mathrm{L}(\mathrm{t})}+\frac{\dot{\mathrm{A}}(\mathrm{t})}{\mathrm{A}(\mathrm{t})} \tag{10}$$

Definition 2: The *direction of technological progress, DTP,* is the ratio between the augmentation rates of capital and labor, i.e.

$$DTP \equiv \frac{\dot{\mathrm{B}}(\mathrm{t})/\mathrm{B}(\mathrm{t})}{\dot{\mathrm{A}}(\mathrm{t})/A(t)} \tag{11}$$

Remark 2: *DTP* can take any value in $[0,\infty]$. When $\dot{B}/B=0$ and $\dot{A}/A>0$ then $DTP=0$, and technological progress is purely labor-augmenting (i.e. Harrod-neutral); when $\dot{B}/B>0$ and $\dot{A}/A=0$ then $DTP\rightarrow+\infty$, and technological progress is purely capital-augmenting (i.e. Solow-neutral); when $\dot{B}/B=\dot{A}/A>0$ then $DTP=1$, and technological progress is Hicks-neutral. Figure 3 shows different directions of technological progress.

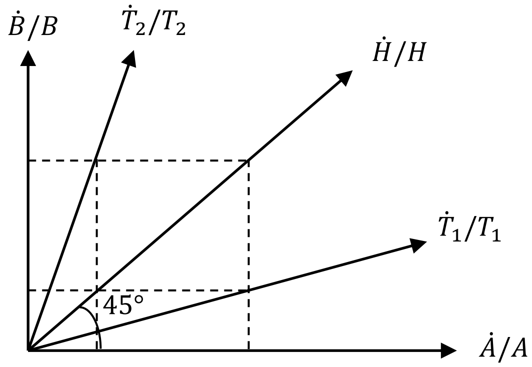

Figure 3: Direction of technological progress

Clearly, the axes represent Harrod-neutral (horizontal) and Solow-neutral (vertical) technological changes. The diagonal $\dot{H}/H$ represents the location of Hicks-neutral technological changes. The ray $\dot{T}_1/T_1$ indicates technological progress which tends to be more labor augmenting, while $\dot{T}_2/T_2$ is more capital augmenting.

Note, that the direction of technological progress is related to the *direction of technology* (*DT*), given by $DT\equiv\mathrm{B}(\mathrm{t})/A(t)$. Obviously, the direction of technological progress determines the direction of technology, but the two terms are fundamentally different. Specifically, when the direction of technological progress is Hicks neutral, the direction of technology remains unchanged. Otherwise, the direction of technology will continuously rise or fall.

2. The Determinants of DTP

In Appendix A we prove the following proposition.

Proposition 1: *If the production function takes the form of equation (1) and factor prices are proportional to their respective marginal products, then the steady-state direction of technological progress is given by:*

$$DTP = \frac{(1+\varepsilon_L)/[1+(1-\varphi)\varepsilon_L]}{(1+\varepsilon_K)/[1+(1-\phi)\varepsilon_K]} \tag{12}$$

Proposition 1 is the core result of this paper. There are three remarkable features of this proposition:

First, according to the proposition, the crucial determinants of the direction of technological progress are the elasticities of the factor supplies (ε_L and ε_K) and the parameters governing the returns to scale of the production function (ϕ and φ), which have so far been missing in the existing literature. The former reflects the factor accumulation processes, and the latter reflect the production function. Given ϕ and φ, technological progress tends towards the factor with the smaller supply elasticity. Similarly, given the supply elasticities, it tends to the factor with the smaller return to scale parameter.

Second, in contrast to Hicks (1932) and Acemoglu (2002), neither the relative price nor the relative market size affect the steady-state direction of technological progress.

Finally, this result is driven only by the generalized production function and the definitions of steady-state growth, direction of technological progress and factor supply elasticities, and does not depend on specific forms of the factor accumulation processes or the production function.

Proposition 1 has an immediate corollary for the constant returns to scale case (CRS, i.e. $\phi = \varphi = 1$).

Corollary 1: *In the CRS case the direction of technological progress is given by:*

$$DTP = \frac{1+\varepsilon_L}{1+\varepsilon_K} \tag{13}$$

Corollary 1 shows that for the CRS case, the steady-state direction of technological progress is determined solely by the relative size of the factor supply elasticities and is biased towards the one with the relatively smaller elasticity.

A further implication, proven in Appendix B, is stated by Corollary 2.

Corollary 2: *Let the production function of a growth model be characterized by equation (1). Then, along the steady-state growth path, capital-augmenting technological progress is ruled out (*$\text{i.e.}\frac{\dot{B}(t)}{B(t)} = 0$*) if and only if capital has an infinite supply elasticity (*$\text{i.e.}\,\varepsilon_K = \infty$*) and the capital returns to scale parameter equals 1 (*$\text{i.e.}\,\phi = 1$*).*

As mentioned above, Uzawa's theorem states that the steady-state technological progress of the neoclassical growth model must be purely labor-augmenting. That result is unexplained and lacks economic intuition, and has puzzled economic growth theorists for decades. Corollary 2 exposes the necessary and sufficient conditions for the steady-state technological progress, if it exists, to admit only labor-augmentation. We will use this corollary in the sequel to further analyze the puzzle of the Uzawa theorem.

Using the symmetry principle, the following holds as well:

Corollary 3: *Let the production function of a growth model be characterized by equation (1). Then, along the steady-state growth path, labor augmenting technological progress is ruled out* ($\text{i.e.}\frac{\dot{A}(t)}{A(t)} = 0$) *if and only if labor has an infinite supply elasticity (*$\text{i.e.}\,\varepsilon_L = \infty$*) and the labor returns to scale parameter is 1 (*$\text{i.e.}\,\varphi = 1$*).*

Li and Huang (2016) proved that there is an analogous "Uzawa theorem" in the Malthusian environment. That is, in the steady state, technological progress can only be purely land-augmenting, and cannot include labor-augmentation. Corollary 3 clarifies that the Uzawa theorem is not an unconditional proposition.

Finally, combining corollaries 2 and 3, we obtain:[12]

Corollary 4: *In the CRS case (*$\text{i.e.}\,\phi = 1;\ \varphi = 1$*), along the steady-state growth path, technological progress is Hicks neutral* ($\text{i.e.}\frac{\dot{A}(t)}{A(t)} = \frac{\dot{B}(t)}{B(t)}$) *if and only if both capital and labor have the same finite supply elasticities (*$\text{i.e.}\,\varepsilon_L = \varepsilon_K < \infty$*); in*

[12] To see this, notice that from Corollary 1 we obtain that, if $\phi = \varphi = 1$ and $\varepsilon_L = \varepsilon_K$, then $\frac{\dot{A}(t)}{A(t)} = \frac{\dot{B}(t)}{B(t)}$; from Corollary 2, if $\phi = 1$ and $\varepsilon_K = \infty$, then $\frac{\dot{B}(t)}{B(t)} = 0$; from Corollary 3, if $\varphi = 1$ and $\varepsilon_L = \infty$, then $\frac{\dot{A}(t)}{A(t)} = 0$. Therefore, when $\phi = \varphi = 1$ and $\varepsilon_L = \varepsilon_K = \infty$, it must be that $\frac{\dot{A}(t)}{A(t)} = \frac{\dot{B}(t)}{B(t)} = 0$.

particular, if $\varepsilon_L = \varepsilon_K = \infty$, *there is no technological progress* (i.e. $\frac{\dot{A}(t)}{A(t)} = \frac{\dot{B}(t)}{B(t)} = 0$).

A remark is in order. Acemoglu (2002), when addressing the determinants of the direction of technological progress, assumed that the two factors are inelastically supplied, that is, $\varepsilon_L = \varepsilon_K = 0$. According to corollary 4, the steady-state technological progress *must* then be Hicks neutral. Consequently, Acemoglu (2002) can just discuss the determinants of the steady-state *relative level* of technology (DT) but not those of the *direction* of technological progress (DTP).

The CRS case is summarized in Table 1.

Table 1: the DTP with different relative elasticities of factor supplies under CRS

Capital / Labor	$0 \le \varepsilon_K < \infty$			$\varepsilon_K = \infty$
	$\varepsilon_L < \varepsilon_K$	$\varepsilon_L = \varepsilon_K$	$\varepsilon_L > \varepsilon_K$	
$0 \le \varepsilon_L < \infty$	$\frac{\dot{A}(t)}{A(t)} > \frac{\dot{B}(t)}{B(t)} > 0$	$\frac{\dot{A}(t)}{A(t)} = \frac{\dot{B}(t)}{B(t)} > 0$	$0 < \frac{\dot{A}(t)}{A(t)} < \frac{\dot{B}(t)}{B(t)}$	$\frac{\dot{A}(t)}{A(t)} > \frac{\dot{B}(t)}{B(t)} = 0$
$\varepsilon_L = \infty$	$\frac{\dot{B}(t)}{B(t)} > \frac{\dot{A}(t)}{A(t)} = 0$			$\frac{\dot{A}(t)}{A(t)} = \frac{\dot{B}(t)}{B(t)} = 0$

Table 1 shows that the steady state does not *require* technological progress to be purely labor-augmenting. In fact, there is no restriction and the steady state can be compatible with *any* direction of technological progress, which is determined by the relative size of the factor supply elasticities.

V. A Specific Growth Model

We derived Proposition 1 without specifying the micro-structure of households and enterprises. Next, we provide a well-founded model, verify that it possesses a steady state and determine the corresponding direction of technological progress.

We use for this purpose the Acemoglu (2002) growth model. That model expands the Romer (1990) technology from one dimension to two, making it appropriate for the analysis of potential directions of technological progress. However, as commented above, the production function in the Acemoglu model was assumed to possess constant returns to scale and the two input factors to be inelastically supplied, thereby ignoring the two aforementioned factors that determine the steady-state direction of technological progress. Therefore, the framework needs two fundamental extensions: ***first***, the returns to scale for the two input factors in the production function cannot be assumed to be constant, but rather follow the specification of equation (1); ***second***, the factor accumulation processes should admit supply elasticities ranging from zero to infinity, as in equation (9).

1. The Model

Following Acemoglu (2002), there are two material factors and three production sectors: final goods, intermediate goods, and research and development (R&D). The symbols K, L, S represent the two kinds of material production factors and "scientists" who specialize in research and development of new intermediate products, respectively.

(1) The production function

The final goods sector is competitive, using the following CRS production function:

$$Y = \left[\gamma {Y_L}^{(\varepsilon-1)/\varepsilon} + (1-\gamma){Y_K}^{(\varepsilon-1)/\varepsilon}\right]^{\varepsilon/(\varepsilon-1)}, \quad 0 \le \varepsilon < \infty \tag{14}$$

where Y is output and Y_L and Y_K are the two inputs, with the factor-elasticity of substitution given by ε.

The inputs Y_L and Y_K are also produced competitively by constant elasticity of substitution (CES) production functions using a continuum of intermediate inputs, $X(i)$ and $Z(j)$:

$$Y_L = \left[\int_0^N X(i)^{\varphi\beta} di\right]^{1/\beta} \ and\ Y_K = \left[\int_0^M Z(j)^{\phi\beta} dj\right]^{1/\beta} \tag{15}$$

where the elasticity of substitution is given by $v = 1/(1-\beta)$ and N and M represent the measure of the two types of the intermediate inputs, respectively. The specification of the production functions extends that of Acemoglu's by introducing the parameters φ and ϕ which are assumed to satisfy $0 < \varphi \le 1$ and $0 < \phi \le 1$. When $\varphi = \phi = 1$, equations (14) and (15) degenerate to the form used in Acemoglu (2002). When $\varphi < 1$ or $\phi < 1$, then production of the inputs is characterized by diminishing returns to scale.

The intermediate factors $X(i)$ are produced by labor, whereas $Z(j)$ are produced by capital, where the respective production functions are linear:

$$X(i) = L(i) \text{ and } Z(j) = K(j) \tag{16}$$

Accordingly, Y_L and Y_K represent labor-intensive and capital-intensive inputs respectively.

(2) Factor accumulation processes

Repeating equation (9), the factor accumulation processes are given by:

$$\begin{cases} \dot{K} = q(t)I_K^{\alpha_K} - \delta_K K, & \dot{q}(t)/q(t) = g_q \geq 0, \ 0 \leq \alpha_K \leq 1, \delta_K > 0 \\ \dot{L} = b_L I_L^{\alpha_L} - \delta_L L, & b_L > 0, \ 0 \leq \alpha_L \leq 1, \delta_L > 0 \end{cases} \tag{17}$$

where I_K and I_L are investment into capital and labor accumulation.

Moreover, the capital accumulation process further allows an exogenous investment-specific technological progress, $\dot{q}(t)/q(t) = g_q$ [13]. The latter feature follows Grossman et al. (2017) and enables the model to generate falling investment goods prices.

(3) Other assumptions

Other assumptions are inherited from Acemoglu (2002). The household's goal is to maximize the discounted flow of utility, given by:

$$\mathrm{U} = \int_0^\infty \frac{C(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \tag{18}$$

where $C(t)$ is consumption at time t, $\rho > 0$ is the discount rate, and $\theta > 0$ is a utility curvature coefficient of the household.

The household's periodic budget constraint is given by:

$$C + I_K + I_L \leq Y = wL + rK + \mathrm{w}_S S + \Pi \tag{19}$$

where the LHS stands for expenditures consisting of consumption and investments, I_K and I_L, into capital and labor, and the RHS is income, obtained from renting out labor at the rate w, capital at the rate r, scientists at the rate w_S. Π is total profits, which are positive when the returns to scale of the final good production function are decreasing.

New intermediate inputs are developed by an R&D sector. The innovation possibilities frontier functions are specified as follows:[14]

$$\begin{cases} \dot{N} = d_N N S_N - \delta N \\ \dot{M} = d_M M S_M - \delta M \end{cases} \tag{20}$$

where S_N and S_M represent respectively the number of scientists engaged in innovation of the two kinds of intermediate inputs. The total number of scientists is exogenously set at S, so $\bar{S}_N + \bar{S}_M \leq S$, where $\bar{S}_N = \delta/d_N$ and $\bar{S}_M = \delta/d_M$. Once a

[13] Note that when the embodied technological progress is not taken into account, q(t) is a constant, $\bar{q}$. Therefore, if $\alpha_K = 0$, then at the steady-state value of $\dot{K} = 0$ we obtain $K^* = \bar{q}/\delta_K$.

[14] The extended lab equipment model (Rivera-Natiz and Romer, 1991) can also be used to construct the frontier of innovation possibilities. It does not change the conclusion but adds some intricate knife-edge conditions. The model is available upon request.

new intermediate input is invented, the inventor obtains a permanent patent, as in Romer's (1990) model.

2. The market equilibrium

Given the setting of the model, the final good sector and the intermediate goods sector will be in equilibrium when both the final good firms and the intermediate goods firms maximize their profits and the markets of capital and labor clear. Given the goods market equilibrium, the following two propositions hold:

Proposition 2: *In the market equilibrium, the final output production function takes the form:*

$$Y = \left[\gamma(AL^{\varphi})^{(\varepsilon-1)/\varepsilon} + (1-\gamma)(BK^{\phi})^{(\varepsilon-1)/\varepsilon}\right]^{\varepsilon/(\varepsilon-1)} \tag{21}$$

where $A \equiv N^{(1-\varphi\beta)/\beta}$ *and* $B \equiv M^{(1-\phi\beta)/\beta}$.

Proof: See Appendix C.

The CES production function shown in equation (21) is a specific form of the production function (1), with constant returns to scale to AL^{φ} and BK^{ϕ}. With respect to L and K, it has constant returns to scale if $\varphi = 1$ and $\phi = 1$ and diminishing returns to scale when $\varphi < 1$ or $\phi < 1$.

Proposition 3: *In the intermediate goods market equilibrium, the relative benefits of innovation of capital-intensive and labor-intensive intermediate goods are determined by:*

$$\frac{\pi_Z}{\pi_X} = \frac{(1-\phi\beta)/\phi}{(1-\varphi\beta)/\varphi} \cdot \frac{r}{w} \cdot \frac{K}{L} \cdot \frac{N}{M} \tag{22}$$

where π_Z and π_X represent the monopoly profits of capital-intensive and labor-intensive intermediate goods producers.

Proof: See Appendix D.

Equation (22) shows that, for a given ϕ/φ and a ratio of the technology levels, represented here by *M/N*, relative invention profits are positively related to the relative factor prices (*r/w*) and the relative factor supplies (*K/L*). Accordingly, a change of the relative price encourages innovations directed towards the scarce factor whose price has increased, as suggested by Hicks (1932). Acemoglu (2002) noted that the relative amount of the two factors, (*K/L*), has two countervailing effects on π_Z/π_X. On the one hand, a higher *K/L* causes an increase in π_Z/π_X, which in turn leads to a technological change favoring the abundant factor ("the market size effect"). On the other hand, a higher K/L decreases r/w and π_Z/π_X, which is the price effect of a change in K/L. The total effect of a change in K/L is regulated by the elasticity of substitution ε between the two factors. If $\varepsilon > 1$, the market size effect dominates the price effect, and increasing K/L will encourage favoring improvements of the abundant factor. Otherwise, when $\varepsilon < 1$, improvements of the scarce factor will be favored. Based on the above, Acemoglu (2002) proposed that the relative price and market size are the two key factors affecting the direction of technological progress.[15]

Unlike Acemoglu (2002), equation (22) also exposes the impact the return to scale has on the direction of technological progress. It shows that, other things being equal, relative invention profits π_Z/π_X are negatively related to ϕ/φ. Thus, under the stated condition, a decreased values of ϕ/φ induces a relatively higher profit of R&D in the capital-intensive intermediate input, tilting technological progress in that direction.

However, equation (22) represents only the demand side of technological change. To get the long-run effects, it is necessary to consider also factors affecting the supply of innovations and material factors, in particular that of r/w on K/L and of π_Z/π_X on M/N, within a dynamic general equilibrium framework. As will be shown below, in such a context, the "relative price effect" and the "market size effect" will disappear, while the return to scale parameters will still affect the steady state growth path.

3. The Steady State

When the goods market and the scientists market are in equilibrium and households maximize their utility, the economy arrives at a steady-state growth equilibrium in which each endogenous variable grows at a constant rate. The following proposition shows that the model has a unique steady-state growth equilibrium.

Proposition 4: *An economy characterized by equations (14) - (20) possesses a unique steady-state growth path.*

Proof: See Appendix E.

While Proposition 1 *assumes* that the steady-state equilibrium of the growth model exists, Proposition 4 shows that in the specific model it is in fact the case. Moreover, the steady-state equilibrium is unique.

[15] However, when $\varepsilon > 1$, favoring innovation in the capital-intensive intermediate factor will cause M/N to increase. Equation (22) then shows that as a result π_Z/π_X will decrease, discouraging further inverstment into innovations in the capital-intensive sector. Therefore, the relative price and market size turn out to be irrelevant in the long-run.

The previous section proves that the factor supply elasticities determine the direction of technological progress, but it does not provide the determinants of these elasticities. Here this lacuna can be filled.

Corollary 5: *In the specific growth model, the factor supply elasticities are given by:*

$$\begin{cases} \varepsilon_K = \dfrac{\alpha_K + g_q/g}{(1-\alpha_K) - g_q/g} \\ \varepsilon_L = \dfrac{\alpha_L}{1-\alpha_L} \end{cases} \tag{23}$$

where g denotes the endogenously determined economy's growth rate.

Proof: See Appendix F.

Equation (23) shows that the factor supply elasticities are determined by α_K and α_L reflecting the investment adjustment costs. The labor supply elasticity depends only on α_L, whereas the capital supply elasticity is determined by α_K, the rate of investment-specific technological progress g_q, and all other parameters of the model through their impact on g. However, if there is no investment-specific technological progress, the capital supply elasticity is also determined only by α_K.

By corollary 5, if $0 \le \alpha_L \le 1$, the range of the labor supply elasticity is $0 \le \varepsilon_L \le \infty$. If $0 \le \alpha_K \le 1 - \frac{g_q}{g}$, the range of the capital supply elasticity is $0 \le \frac{g_q}{g-g_q} \le \varepsilon_K \le \infty$. It is worth noting that if $\alpha_K = 1 - \frac{g_q}{g}$, that is, the adjustment cost just offsets the investment-specific technological progress, capital supply will be infinitely elastic with $\varepsilon_K = \infty$. Hence, under a constant returns to scale production function, Uzawa's steady-state theorem still holds and technical progress of the final good production is still purely labor augmenting, despite the presence of investment-specific technological progress. However, if $\alpha_K > 1 - \frac{g_q}{g}$, then the supply elasticity of capital will be negative, that is, $\varepsilon_K < 0$. In other words, the capital supply curve is downwards sloping.

Finally, substituting equations (23) into equation (12), we obtain:

$$DTP = \frac{(1-\phi\alpha_K) - \phi g_q/g}{(1-\varphi\alpha_L)} \tag{24}$$

Equation (24) provides the determinants of the direction of technological progress in the specific growth model. Absent investment-specific technological progress, i.e. $g_q = 0$, it is reduced to $DTP = \frac{1-\phi\alpha_K}{1-\varphi\alpha_L}$. When the returns to scale are constant, i.e. $\phi = \varphi = 1$, then $DTP = \frac{1-\alpha_K}{1-\alpha_L}$. It shows that the direction of technological progress depends on the parameters reflecting the size of the investment adjustment cost, and technological progress tends to the factor with higher adjustment cost. Because equation (23) shows that a greater adjustment cost generates a smaller factor supply elasticity, there must be faster technological progress to maintain the balanced growth of the two effective factors. When there is no investment adjustment cost, i.e $\alpha_K = \alpha_L = 1$, then $DTP = \frac{1-\phi}{1-\varphi}$. This shows that the direction of technological progress depends on the returns to scale parameters ϕ and φ, and tends to the smaller one. This obtains because steady-state growth requires the growth rate of the effective factors (BK^{ϕ} and AL^{φ}) to be balanced. Accordingly, since at the steady state K and L grow at the same rate when $\alpha_K = \alpha_L = 1$ and $g_q = 0$, technological progress must be biased towards the factor with the smaller exponent.

Furthermore, the crucial role equation (24) plays in exposing the factors that determine the direction of technological progress can be exemplified by confronting it with results obtained by Acemoglu (2002, 2003). Specifically, in both papers the returns to scale of the production function are constant (i.e. $\phi = \varphi = 1$), and the size of the factor supply elasticities are explicitly or implicitly set, thereby fixing the direction of the steady-state technological progress. Acemoglu (2002) explicitly assumes that the two material factors are inelastically supplied and does not consider investment-specific technological progress, that is, $\alpha_K = \alpha_L = g_q = 0$, yielding by equation (24) $DTP = 1$ (i.e. Hicks neutral in steady-state). Consequently, while attempting to expose the determinants of the *direction* of technological progress, Acemoglu (2002) in fact provides only the determinants of the relative technological *level* (DT=B/A)[16]. Acemoglu (2003) tries to address the puzzle of Uzawa's theorem using the framework of Acemoglu (2002), but only implicitly replaces $\alpha_K = 0$ with $\alpha_K = 1$. Consequently, equation (24) implies $DTP = 0$. Accordingly, while the steady-state technological progress is purely labor-augmenting endogenously determined in Acemoglu (2003), the fact that setting $\alpha_K = 1$ implies an infinite capital supply elasticity is not exposed. Consequently, the model fails to provide a satisfactory explanation for the Uzawa theorem puzzle. In contrast, the relative price and relative market size do not play a role in equation (24). Accordingly, these factors have no impact on the direction of technological progress in the steady states of the

[16] Moreover, the result is only valid when technological progress is Hicks neutral. Otherwise, the relative technological level (DT) will be a function of time, continuously rising or falling.

Acemoglu (2002, 2003) models.

Notice that by applying $A \equiv N^{(1-\varphi\beta)/\beta}$ and $B \equiv M^{(1-\phi\beta)/\beta}$ to equations (E17) of Appendix E we get:

$$\begin{cases}(\dot{B}/B)^* = (1-\phi\alpha_K)g - \phi g_q \\ (\dot{A}/A)^* = (1-\varphi\alpha_L)g\end{cases} \tag{25}$$

Accordingly, equation (24) can also be obtained by substituting equations (25) into definition (11) of the DTP. It is in this sense that equation (24) confirms that Proposition 1 holds in the specific growth model. Moreover, equation (25) shows that, when $(1-\phi\alpha_K)g - \phi g_q > 0$, then there can be both capital-augmenting and investment-specific technological progress in steady state. Absent investment-specific technological progress, that is when $g_q = 0$, there can be capital-augmenting technological progress as long as $\phi\alpha_K < 1$.[17]

VI. Applications

As shown above, the elasticities of the factor supplies and the returns to scale of the production function are the key determinants of the direction of technological progress. As argued above, these features have been ignored by the existing literature, leaving some important questions of economic growth open. Below we use the findings of the current paper to give our answers to these questions.

1. The puzzle of Uzawa's (1961) steady-state theorem

Acemoglu (2009, pp.59) has already questioned why it is that all forms of technological progress seem equally plausible ex ante, but, in accordance with the Uzawa theorem, only purely labor-augmenting technological progress is compatible with steady-state growth? Schlicht (2006) argues that the theorem looks like an extremely restrictive, and, consequently, extremely decisive requirement, making steady-state growth highly singular and therefore highly improbable. Aghion and Howitt (1998) argue that there is no good reason to think that technological change takes this form, it just leads to tractable steady-state results. Therefore, Jones and Scrimgeour (2008) call the entire issue "a puzzle".

We argue that the problem of Uzawa's theorem is that it does not clearly identify what premise of the neoclassical model implies that technological progress must be purely labor-augmenting, a feature that is therefore mistakenly regarded as a *requirement* for the existence of steady-state growth. Proposition 1 indicates that steady-state growth does not restrict the direction of technological progress *per-se*, but

[17] Irmen (2013) discussed the adjustment costs case, namely, $\phi = 1$ but with $\alpha_K < 1$, while Casey and Horii (2019) analyzed the diminishing returns to scale case, that is, $\alpha_K = 1$, but with $\phi < 1$.

the returns to scale of the production function and the relative size of the factor supply elasticities do. According to Table 1, the necessary and sufficient condition for the steady-state technological progress to be purely labor-augmenting is that the capital supply elasticity be infinite while that of labor finite, i.e. $(\varepsilon_K = \infty, \varepsilon_L < \infty)$. This configuration turns out to emerge from the specific assumptions of the neoclassical growth model. Its capital accumulation process implies the infinite capital supply elasticity, while the exogenous labor growth implies finite elasticity for that factor. Corollary 2 then shows that as long as the elasticity of the capital supply is finite or the production function is characterized by diminishing returns to scale, technological progress in steady state will not be purely labor-augmenting. Therefore, exposing the implicit underlying assumptions which make the neoclassical growth model require steady-state technological progress to be purely labor-augmenting resolves the puzzle of the Uzawa theorem. In particular, there is no restriction on the steady-state direction of technological progress in the neoclassical growth model provided these assumptions are appropriately reset.

Some authors (Sato et al., 1999, 2000; Irmen, 2013) have already modified the capital accumulation process, introduced a production function with diminishing returns to scale for capital and labor (Casey and Horii, 2019), or modified both (Grossman et al., 2017) to obtain steady-state growth with capital-augmenting technological progress. However, none of these authors has pointed out that the infinite capital supply elasticity and the constant returns to scale of the production function are the premises leading to the Uzawa theorem. Acemoglu (2003) and Jones and Scrimgeour (2008) correctly noted that it is the asymmetry between capital and labor in the neoclassical growth model that causes technological progress to be more labor-augmenting in the steady-state. That asymmetry stems from the different accumulation processes; capital is accumulated in terms of units of the output good while labor is not. However, these authors did not further connect this asymmetry to the supply elasticities of capital and labor. They also failed to note that capital supply must be infinitely elastic for the steady-state technological progress to be purely labor-augmenting. Irmen (2018) correctly recognized that the premise of Uzawa's theorem includes also the CRS production function, but he failed to point out, in addition, the condition on the capital supply elasticity.

2. Why is modern economic growth characterized by the Kaldor facts?

Kaldor's (1961) stylized facts characterize principal features of modern economic growth, and have been recently verified by Jones (2016). Why does modern economic growth display these characteristics? This is the basic question that growth theory must answer.

The neoclassical growth model yields steady-state growth which is consistent with the Kaldor facts by *assuming* that technological progress is purely labor-augmenting. Yet, it has not been able to explain why technological progress takes this form. Again, Table 1 lays out the aforementioned conditions on structural parameters that lead to the result. It may be that these conditions are basically in line with the historical circumstances that enabled Western Europe to start the process of modern economic growth. Actually, for quite a long time after the industrial revolution, only few Western European countries entered the phase of capital-based industrialization. The resources needed to produce capital and accumulate it were drawn from the entire world, making the capital supply elasticity quite large. In addition, due to the demographic change, the higher per capita income no longer increased but rather reduced the birth rate. The elasticity of the labor supply became finite.

3. Can investment-specific technological progress and purely labor augmenting technological progress be compatible in steady state?

Although purely labor-augmenting technological progress can explain the Kaldor facts, empirical data also find another important fact in modern economic growth. In particular, the relative price of capital equipment, adjusted for quality, has been falling steadily for decades, as shown in figure 2 above (Grossman et al., 2017; Jones, 2016; Gordon, 1990; Greenwood et al., 1997). This indicates the presence of investment-specific technological progress. Unless the production function is Cobb-Douglas, with such technological progress the steady-state growth path of existing neoclassical growth models will either no longer be purely labor-augmenting or even fail to exist. This poses a new difficult problem for the neoclassical growth model.

At present, there are two solutions for the problem in the literature: one is to argue that the production function is indeed Cobb-Douglas, at least in the steady state (Jones, 2005; León-Ledesma and Satchi, 2018); the other is to introduce capital-augmenting technological progress into the production function (Grossman et al., 2017; Casey and Horii, 2019). The empirical evidence cited in the introduction, showing that the substitution elasticity of capital and labor is not unitary, indicates that the Cobb-Douglas production function specification may be empirically invalid. While the steady-state equilibrium can be obtained by introducing capital-augmenting technological progress, the capital/output ratio will continue to decline in steady-state, which is inconsistent with the Kaldor facts.

We have shown above in section V, that with a constant return to scale production function, (i.e., $\phi = \varphi = 1$), both investment-specific and purely labor-augmenting technological progress can concurrently be present at the steady state, provided the investment adjustment costs increase at a rate that just offsets the investment-specific technological progress, (that is, $\alpha_K = 1 - \frac{g_q}{g}$). The CES specification of the production function implies that the elasticity of factor substitution is not required to be 1; there is a steady state in the model, which is consistent with the long-term stability of factor income shares; at the same time, technological progress in the final good production is purely labor-augmenting which is consistent with the Kaldor facts. Of course, whether this resolution of the neoclassical model's conflict with the empirical findings is more plausible than those suggested by the existing literature (Jones, 2005; León-Ledesma and Satchi, 2018; Grossman et al., 2017; Casey and Horii, 2019) is a matter of further empirical study.

4. Why did technological progress not increase per capita income before the industrial revolution?

According to Ashraf and Galor (2011), before the industrial revolution technological progress only increased land productivity and population density, and had little impact on labor productivity. According to corollary 3, if the production function has constant returns to scale, as long as labor has infinite supply elasticity, there can be no labor-augmenting technological progress in the steady-state. The growth model of Section V also shows that when the $\alpha_L = 1$, the labor supply elasticity is infinite and per capita income remains unchanged in the steady state[18]. In that steady state, technological progress and land growth can only lead to population growth and increased population density, which is consistent with the empirical study of Ashraf and Galor (2011). Therefore, the stagnation of technology and the shortage of land are not the crucial causes of the Malthusian trap. Rather, it is the infinite labor supply elasticity that is fundamental to the result.

5. What led to the industrial revolution?

While this paper does not build a Unified Growth model in the spirit of Galor (2011), it may shed some light on the transformation from the Malthusian trap to modern growth. From the perspective of the direction of technological progress, industrial revolution amounts to a transition from a path that completely excludes labor augmentation to one that includes it, that is, from $\frac{\dot{A}(t)}{A(t)} = 0$ to $\frac{\dot{A}(t)}{A(t)} > 0$. From corollary 1 or table 1, we can see that the fundamental reason for such a transition is that the labor supply elasticity changes from infinite to finite (assuming that the production function has constant returns to scale). While this paper provides no mechanism that may cause such an elasticity change, this is one of the core contents of Galor's (2011) Unified Growth Model.

[18] Even if the production function has diminishing returns to scale, there is labor-augmenting technological progress in the steady state but per capita income is still constant.

VII Concluding Remarks

For the understanding of some important issues of economic growth, the direction of technological progress is at least as important as its rate, or even more important. As it turns out, the elasticities of factor supplies and the characteristics of the returns to scale of the production function are the key variables associated with the economic environment. Indeed, the core conclusion of this paper is that these are the factors that affect the direction of steady-state technological progress. If production is characterized by constant returns to scale, the direction of technological progress depends solely on the relative elasticities of the factor supplies, and tends to the factor with relatively smaller supply elasticity. This conclusion is obtained directly from a generalized production function and the definitions of steady-state growth, direction of technological progress and factor supply elasticities, and has nothing to do with the specific forms of the production function and the factor accumulation processes.

These features also constrain the environment of profit-maximizing innovators when they choose the direction of technological progress. This is demonstrated through a specific micro-founded growth model that extends the Acemoglu (2002) model, and verifies the core conclusion of this paper. The findings presented above can be used to provide new explanations to a series of important problems in economic growth, such as the puzzle of the Uzawa theorem, the Kaldor facts and the Malthusian trap, and issues related to the industrial revolution.

The main innovation of this paper is driven by the introduction of a generalized production function and generalized factor accumulation processes. The returns to scale of the generalized production function can be constant or decreasing while the introduction of investment adjustment costs allows the factor supply elasticities in the generalized factor accumulation processes to range from zero to infinity. These features allow this paper to expose the crucial role the returns to scale and the supply elasticities play in determining the direction of technological progress.

While theoretically the returns to scale of the production function may be constant or diminishing, and the factor supply elasticities may range from zero to infinity, their actual values are subject to an empirical investigation. In view of their important impact on the direction of technological progress, estimating these parameters should be an important task for future empirical research. This is particularly true where the investment adjustment costs are concerned because of their impact on factor supply elasticities and thereby on the direction of technological progress. Moreover, the presence of investment adjustment costs affects the standard perpetual inventory method used to calculate the capital stock, as the contribution of current investment is no longer linear. Consequently, the commonly used assessments of the capital stock are likely to be systematically upwards biased.

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Appendix A: The derivation process of equation (12)

Dividing the denominator of equation (11) by the two sides of equation (10) yields:

$$DTP=\frac{\dot{\mathrm{B}}(\mathrm{t})/\mathrm{B}(\mathrm{t})}{\dot{\mathrm{A}}(\mathrm{t})/A(t)}=\frac{\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})}/\left[\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})}+\phi\frac{\dot{\mathrm{K}}(\mathrm{t})}{\mathrm{K}(\mathrm{t})}\right]}{\frac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)}/\left[\frac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)}+\varphi\frac{\dot{\mathrm{L}}(\mathrm{t})}{L(t)}\right]}=\frac{1+\varphi\frac{\dot{\mathrm{L}}(\mathrm{t})}{L(t)}/\frac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)}}{1+\phi\frac{\dot{\mathrm{K}}(\mathrm{t})}{\mathrm{K}(\mathrm{t})}/\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})}} \quad \text{(A1)}$$

The growth rates of $r(t)$ and $w(t)$ in equations (3) are obtained by taking $k(t)$ as a constant, yielding:

$$\begin{cases}\frac{\dot{w}(\mathrm{t})}{w(t)}=\frac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)}+(\varphi-1)\frac{\dot{\mathrm{L}}(\mathrm{t})}{L(t)}\\ \frac{\dot{r}(\mathrm{t})}{r(t)}=\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})}+(\phi-1)\frac{\dot{\mathrm{K}}(\mathrm{t})}{\mathrm{K}(\mathrm{t})}\end{cases} \quad (A2)$$

Substituting equations (A2) into equations (4) then yields:

$$\begin{cases}\varepsilon_L=\frac{\dot{\mathrm{L}}(\mathrm{t})/L(t)}{\frac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)}+(\varphi-1)\frac{\dot{\mathrm{L}}(\mathrm{t})}{L(t)}}\\ \varepsilon_K=\frac{\dot{\mathrm{K}}(\mathrm{t})/K(t)}{\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})}+(\phi-1)\frac{\dot{\mathrm{K}}(\mathrm{t})}{K(t)}}\end{cases} \quad (A3)$$

From equations (A3) we obtain:

$$\begin{cases}\frac{\dot{\mathrm{L}}(\mathrm{t})}{L(t)}/\frac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)}=\frac{\varepsilon_L}{1+(1-\varphi)\varepsilon_L}\\ \frac{\dot{\mathrm{K}}(\mathrm{t})}{\mathrm{K}(\mathrm{t})}/\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})}=\frac{\varepsilon_K}{1+(1-\phi)\varepsilon_K}\end{cases} \quad (A4)$$

Substituting equations (A4) into equation (A1) and rearranging implies equation (12):

$$DTP=\frac{(1+\varepsilon_L)/[1+(1-\varphi)\varepsilon_L]}{(1+\varepsilon_K)/[1+(1-\phi)\varepsilon_K]}$$

Appendix B: Proof of Corollary 2 (Uzawa's Steady-State Theorem)

First, "If" direction. Let $\varepsilon_K=\infty$ and $\phi=1$. Then $\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})}=0$ in a steady-state equilibrium.

Proof:

According to proposition 1,

$$DTP=\frac{(1+\varepsilon_L)/[1+(1-\varphi)\varepsilon_L]}{(1+\varepsilon_K)/[1+(1-\phi)\varepsilon_K]}$$

If $0<\varphi\leq 1$ and $\varepsilon_L<\infty$, then

$$DTP = \frac{(1+\varepsilon_L)/[1+(1-\varphi)\varepsilon_L]}{\infty} = 0$$

If $\varepsilon_L = \infty$ and $0 < \varphi < 1$, then by l'hospital's rule, $(1+\varepsilon_L)/[1+(1-\varphi)\varepsilon_L] \to \frac{1}{1-\varphi} < \infty$. Hence, in this case,

$$DTP = \frac{1/(1-\varphi)}{\infty} = 0$$

If $\varepsilon_L = \infty$ and $\varphi = 1$, we obtain:

$$DTP = \frac{1+\varepsilon_L}{1+\varepsilon_K} = \frac{\infty}{\infty} \neq 0$$

From (A3) we also know that with $\varepsilon_K = \infty$ and $\phi = 1$,

$$\varepsilon_K = \frac{\dot{\mathrm{K}}(\mathrm{t})/K(t)}{\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})} + (\phi - 1)\frac{\dot{\mathrm{K}}(\mathrm{t})}{K(t)}} = \frac{\dot{\mathrm{K}}(\mathrm{t})/K(t)}{\dot{\mathrm{B}}(\mathrm{t})/\mathrm{B}(\mathrm{t})} = \infty$$

However, $\dot{\mathrm{K}}(\mathrm{t})/K(t)$ must be finite. Hence in this case too $\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})} = 0$.

In sum, when $\varepsilon_K = \infty$ and $\phi = 1$, we obtain that $\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})} = 0$ in a steady-state equilibrium.

Second, "Only If" direction. Let $\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})} = 0$. Then $\varepsilon_K = \infty$ and $\phi = 1$ in a steady-state equilibrium.

Proof:

If $\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})} = 0$, then $DTP = \frac{\dot{\mathrm{B}}(\mathrm{t})/\mathrm{B}(\mathrm{t})}{\dot{\mathrm{A}}(\mathrm{t})/A(t)} = \frac{(1+\varepsilon_L)/[1+(1-\varphi)\varepsilon_L]}{(1+\varepsilon_K)/[1+(1-\phi)\varepsilon_K]} = 0$

Since $\varepsilon_L \geq 0$, and $0 < \varphi \leq 1$, the numerator is strictly positive. Therefore, it must be the case that

$$\frac{1+\varepsilon_K}{[1+(1-\phi)\varepsilon_K]} = \infty$$

Clearly, if $\phi < 1$ and $0 \leq \varepsilon_K < \infty$, then

$$\frac{1+\varepsilon_K}{[1+(1-\phi)\varepsilon_K]} < \infty$$

If $\phi < 1$ and $\varepsilon_K = \infty$, then by l'hospital's rule

$$\lim_{\varepsilon_K = \infty} \frac{1+\varepsilon_K}{[1+(1-\phi)\varepsilon_K]} = \frac{1}{1-\phi} < \infty$$

If $\phi = 1$ and $\varepsilon_K < \infty$, then

$$\frac{1+\varepsilon_K}{[1+(1-\phi)\varepsilon_K]} = 1 + \varepsilon_K < \infty$$

Therefore, $\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})} = 0$ is obtained only if $\phi = 1$ and $\varepsilon_K = \infty$.

In conclusion, $\varepsilon_K = \infty$ and $\phi = 1$ are necessary and sufficient conditions for $\frac{\dot{B}(t)}{B(t)} = 0$ in steady state equilibrium.

Appendix C: Proof of Proposition 2.

Letting the final good serve as numeraire, the representative competitive final good producer faces the input prices p_L and p_K and selects the respective Y_K and Y_L so as to maximize

$$\pi_Y = Y - p_L Y_L - p_K Y_K \qquad (C1)$$

subject to the production function (14), yielding the demand functions:

$$\begin{cases} p_K = (1-\gamma)[\gamma + (1-\gamma)(Y_K/Y_L)^{(\varepsilon-1)/\varepsilon}]^{1/(\varepsilon-1)}(Y_K/Y_L)^{-1/\varepsilon} \\ p_L = \gamma[\gamma + (1-\gamma)(Y_K/Y_L)^{(\varepsilon-1)/\varepsilon}]^{1/(\varepsilon-1)} \quad . \end{cases} \qquad (C2)$$

The reperesentative producers of Y_K and Y_L maximize their profits by choosing $Z(j)$ and $X(i)$, given the intermediate input prices $p_Z(j)$ and $p_X(i)$:

$$\begin{cases} \pi_K = p_K Y_K - \int_0^M p_Z(j)Z(j)dj \\ \pi_L = p_L Y_L - \int_0^N p_X(i)X(i)di \end{cases} \qquad (C3)$$

subject to their respective production functions (15). This generates the demand functions

$$\begin{cases} Z(j) = (Y_K)^{\frac{1-\beta}{1-\phi\beta}}(\phi p_K/p_Z(j))^{1/(1-\phi\beta)} \\ X(i) = (Y_L)^{\frac{1-\beta}{1-\varphi\beta}}(\varphi p_L/p_X(i))^{1/(1-\varphi\beta)} \end{cases} \qquad (C4)$$

The intermediate input producers, who hold the exclusive right to produce their particular type of input, face the prices of the primary inputs and choose, respectively, $(p_Z(j), K(j))$ and $(p_X(i), L(i))$ to maximize

$$\begin{cases} \pi_Z(j) = p_Z(j)Z(j) - rK(j) \\ \pi_X(i) = p_X(i)X(i) - wL(i) \end{cases} \qquad (C5)$$

subject to their technologies (16) and the demand functions (C4).

From the maximization (C5) we obtain:

$$\begin{cases} p_Z(j) = p_Z = r/\phi\beta \\ p_X(i) = p_X = w/\varphi\beta \end{cases} \qquad (C6)$$

which imply that all intermediate inputs have the same mark-up over marginal cost. Substituting equations (C6) into (C4), we find that all capital-intensive and all labor-intensive intermediate goods are produced in equal (respective) quantities.

$$\begin{cases} Z(j) = (Y_K)^{\frac{1-\beta}{1-\phi\beta}}(\beta\phi^2 p_K/r)^{1/(1-\phi\beta)} \\ X(i) = (Y_L)^{\frac{1-\beta}{1-\varphi\beta}}(\beta\varphi^2 p_L/w)^{1/(1-\varphi\beta)} \end{cases} \tag{C7}$$

By the production functions of the intermediate inputs (16), all monopolists have the same respective demand for labor and capital.

The material factor market clearing conditions imply:

$$\begin{cases} Z(j) = K/M \\ X(i) = L/N \end{cases} \tag{C8}$$

Substituting equations (C8) into (15), we obtain the equilibrium quantities of the labor-intensive and capital-intensive inputs as equations (C9):

$$\begin{cases} Y_L = \left[\int_0^N X(i)^{\varphi\beta} di\right]^{1/\beta} = N^{(1-\varphi\beta)/\beta} L^{\varphi} \\ Y_K = \left[\int_0^M Z(j)^{\phi\beta} dj\right]^{1/\beta} = M^{(1-\phi\beta)/\beta} K^{\phi} \end{cases} \tag{C9}$$

Substituting equations (C9) into equation (14), we obtain equations (21) as follows:

$$Y = \left[\gamma (N^{(1-\varphi\beta)/\beta} L^{\varphi})^{(\varepsilon-1)/\varepsilon} + (1-\gamma)(M^{(1-\phi\beta)/\beta} K^{\phi})^{(\varepsilon-1)/\varepsilon}\right]^{\varepsilon/(\varepsilon-1)}$$

Appendix D: Proof of Proposition 3.

Letting $k \equiv \frac{BK^{\phi}}{AL^{\varphi}} = \frac{M^{(1-\phi\beta)/\beta} K^{\phi}}{N^{(1-\varphi\beta)/\beta} L^{\varphi}}$, the factor-intensive production function becomes:

$$f(k) \equiv Y/AL^{\varphi} = \left[\gamma + (1-\gamma)k^{(\varepsilon-1)/\varepsilon}\right]^{\varepsilon/(\varepsilon-1)} \tag{D1}$$

Using equation (D1), we transform the market prices of the capital-intensive and labor-intensive inputs (C2) into the following forms:

$$\begin{cases} p_K = \dfrac{\partial Y}{\partial Y_K} = f'(k) \\ p_L = \dfrac{\partial Y}{\partial Y_L} = f(k) - kf'(k) \end{cases} \tag{D2}$$

Substituting (C8) and (C9) into (C7), we obtain

$$\begin{cases} K/M = \left(M^{(1-\phi\beta)/\beta} K^{\phi}\right)^{\frac{1-\beta}{1-\phi\beta}}(\beta\phi^2 p_K/r)^{1/(1-\phi\beta)} \\ L/N = \left(N^{(1-\varphi\beta)/\beta} L^{\varphi}\right)^{\frac{1-\beta}{1-\varphi\beta}}(\beta\varphi^2 p_L/w)^{1/(1-\varphi\beta)} \end{cases} \tag{D3}$$

Substituting (D2) into (D3) and rearranging, we obtain the market prices of capital and labor:

$$\begin{cases} r = \beta\phi^2 K^{\phi-1} M^{(1-\phi\beta)/\beta} f'(k) \\ w = \beta\varphi^2 L^{\varphi-1} N^{(1-\varphi\beta)/\beta}[f(k) - kf'(k)] \end{cases} \quad (D4)$$

Substituting equation (16) and (C6) into (C5), we obtain:

$$\begin{cases} \pi_Z(j) = (r/\phi\beta - r)Z(j) \\ \pi_X(i) = (w/\varphi\beta - w)X(i) \end{cases} \quad (D5)$$

Substituting (C8) into (D5) yield:

$$\begin{cases} \pi_Z = (r/\phi\beta - r)K/M \\ \pi_X = (w/\beta\varphi - w)L/N \end{cases} \quad (D6)$$

The monopoly profit of each producer of an intermediate product is obtained by substitutitng (D4) into (D6):

$$\begin{cases} \pi_Z = (1 - \phi\beta)\phi M^{(1-\phi\beta-\beta)/\beta} K^{\phi} f'(k) \\ \pi_X = (1 - \varphi\beta)\varphi N^{(1-\varphi\beta-\beta)/\beta} L^{\varphi}[f(k) - kf'(k)] \end{cases} \quad (D7)$$

From equations (D7) and (D4) we finally obtain equation (22):

$$\frac{\pi_Z}{\pi_X} = \frac{(1-\phi\beta)/\phi}{(1-\varphi\beta)/\varphi} \cdot \frac{r}{w} \cdot \frac{K}{L} \cdot \frac{N}{M}$$

Appendix E：Proof of Proposition 4.

First, consider the market for scientists which determines the supply of innovations. Free-entry into the R&D sector implies that the marginal innovation value of scientists should be equal across technologies. Using the innovation possibilities frontier function (20), this implies

$$d_N N \pi_X = d_M M \pi_Z \quad (E1)$$

From the innovation profit equation (D7) we obtain

$$\frac{d_N}{d_M} \frac{(1-\varphi\beta)\varphi}{(1-\phi\beta)\phi} = \frac{kf'(k)}{f(k) - kf'(k)} \quad (\text{E2})$$

Applying equation (D1) in (E2) yields

$$k^* = \left[\frac{\gamma d_N}{(1-\gamma)d_M} \frac{(1-\varphi\beta)\varphi}{(1-\phi\beta)\phi}\right]^{\frac{\varepsilon}{\varepsilon-1}} \quad (\text{E3})$$

Equation (E3) shows that market clearing implies that k^* is a constant, determined solely by the parameters $\gamma, d_M, d_N, \beta, \varphi, \phi$ and ε.

Second, we solve the Euler equations.

Let the Hamiltonian associated with the household optimization problem be:

$$H = U(C)e^{-\rho t} + \lambda_K\left(q(t)I_K^{\alpha_K} - \delta_K K\right) + \lambda_L\left(b_L I_L^{\alpha_L} - \delta_L L\right)$$
$$+\mu[wL + rK + w_S S - C - (I_K + I_L)] \qquad (E4)$$

The first-order conditions are:

$$\begin{cases} C^{-\theta}e^{-\rho t} = \lambda_K \alpha_K q(t) I_K^{\alpha_K - 1} \\ C^{-\theta}e^{-\rho t} = \lambda_L \alpha_L b_L I_L^{\alpha_L - 1} \\ C^{-\theta}e^{-\rho t} = \mu \end{cases} \qquad (E5)$$

Taking log-derivatives of both sides of (E5) over time, we obtain

$$\begin{cases} -\theta\dfrac{\dot{C}}{C} - \rho = \dfrac{\dot{\lambda}_K}{\lambda_K} + (\alpha_K - 1)\dfrac{\dot{I}_K}{I_K} + g_q \\ -\theta\dfrac{\dot{C}}{C} - \rho = \dfrac{\dot{\lambda}_L}{\lambda_L} + (\alpha_L - 1)\dfrac{\dot{I}_L}{I_L} \\ -\theta\dfrac{\dot{C}}{C} - \rho = \dfrac{\dot{\mu}}{\mu} \end{cases} \qquad (E6)$$

The motion equations of λ_K and λ_L are:

$$\begin{cases} \dot{\lambda}_K = -\partial H/\partial K = \lambda_K \delta_K - \mu r \\ \dot{\lambda}_L = -\partial H/\partial L = \lambda_L \delta_L - \mu w \end{cases} \qquad (E7)$$

Based on (E5) and (E7), we obtain

$$\begin{cases} \dot{\lambda}_K/\lambda_K = \delta_K - r\alpha_K q(t) I_K^{\alpha_K - 1} \\ \dot{\lambda}_L/\lambda_L = \delta_L - w\alpha_L b_L I_L^{\alpha_L - 1} \end{cases} \qquad (E8)$$

Using (E8) in (E6), we obtain the Euler equations (E9).

$$\begin{cases} \dfrac{\dot{C}}{C} = \dfrac{1}{\theta}\left\{r\alpha_K q(t) I_K^{\alpha_K - 1} - (\alpha_K - 1)\dfrac{\dot{I}_K}{I_K} - g_q - \rho - \delta_K\right\} \\ \dfrac{\dot{C}}{C} = \dfrac{1}{\theta}\left\{w\alpha_L b_L I_L^{\alpha_L - 1} - (\alpha_L - 1)\dfrac{\dot{I}_L}{I_L} - \rho - \delta_L\right\} \end{cases} \qquad (E9)$$

Third, we solve for the steady state equilibrium.

We first conjecture that there exists a steady-state growth path (hereafter SSGP) then verify it indeed exists by explicitly solving for it.

From the budget constraint (19) and the definition of an SSGP, we obtain

$$\frac{\dot{Y}}{Y} = \frac{\dot{I}}{I} = \frac{\dot{I}_L}{I_L} = \frac{\dot{I}_K}{I_K} = \frac{\dot{C}}{C} \qquad (E10)$$

Then, according to the primary factor accumulation functions (17), along an SSGP the following must hold:

$$\begin{cases} \dfrac{\dot{K}}{K} = \alpha_K \dfrac{\dot{I}_K}{I_K} + \dfrac{\dot{q}}{q} = \alpha_K \dfrac{\dot{C}}{C} + \dfrac{\dot{q}}{q} \\ \dfrac{\dot{L}}{L} = \alpha_L \dfrac{\dot{I}_L}{I_L} = \alpha_L \dfrac{\dot{C}}{C} \end{cases} \tag{E11}$$

From equation (D1) we obtain:

$$Y = N^{\frac{1-\varphi\beta}{\beta}} L^{\varphi} f(k) = M^{\frac{1-\phi\beta}{\beta}} K^{\phi} f(k)/k \tag{E12}$$

Since k is constant along the SSGP, from (E12) we get:

$$\frac{\dot{Y}}{Y} = \frac{1-\varphi\beta}{\beta}\frac{\dot{N}}{N} + \varphi\frac{\dot{L}}{L} = \frac{1-\phi\beta}{\beta}\frac{\dot{M}}{M} + \phi\frac{\dot{K}}{K} \tag{$E13$}$$

Equations (E10), (E11), (E13), together with the innovation possibilities frontier (20), yield:

$$\begin{cases} (1-\varphi\alpha_L)\dfrac{\dot{C}}{\mathrm{C}} = \dfrac{1-\varphi\beta}{\beta}\{\mathrm{d_N S_N} - \delta\} \\ (1-\phi\alpha_K)\dfrac{\dot{C}}{\mathrm{C}} = \phi\dfrac{\dot{q}}{q} + \dfrac{1-\phi\beta}{\beta}\{\mathrm{d_M(S - S_N)} - \delta\} \end{cases} \tag{E14}$$

From (E14) and $S_M + S_N = S$, we obtain the allocation of scientists between the two kinds of intermediate R&D processes:

$$\begin{cases} S_N{}^* = \dfrac{\chi_2(d_M S - \delta) + \chi_1\delta + \beta(1-\varphi\alpha_L)\phi g_q}{\chi_1 d_N + \chi_2 d_M} \\ S_M{}^* = \dfrac{\chi_1(d_N S - \delta) + \chi_2\delta - \beta(1-\varphi\alpha_L)\phi g_q}{\chi_1 d_N + \chi_2 d_M} \end{cases} \tag{E15}$$

where $\chi_1 \equiv (1-\varphi\beta)(1-\phi\alpha_K)$ and $\chi_2 \equiv (1-\phi\beta)(1-\varphi\alpha_L)$.

Combining (E10), (E14) and (E15), we get the growth rates:

$$\left(\frac{\dot{Y}}{Y}\right)^* = \left(\frac{\dot{I}_L}{I_L}\right)^* = \left(\frac{\dot{I}_K}{I_K}\right)^* = \left(\frac{\dot{C}}{C}\right)^* = g \tag{$E16$}$$

where

$$g = \frac{1-\varphi\beta}{\beta}\frac{(1-\phi\beta)[d_M \mathrm{d_N} S - (\mathrm{d_N} + d_M)\delta] + \phi\beta \mathrm{d_N} g_q}{\chi_1 d_N + \chi_2 d_M}$$

Substituting (E16) into (E11) and (E13) we obtain:

$$\begin{cases} (\dot{\mathrm{K}}/\mathrm{K})^* = \alpha_K g + g_q \\ (\dot{\mathrm{L}}/\mathrm{L})^* = \alpha_L g \\ \left(\dot{\mathrm{M}}/\mathrm{M}\right)^* = \dfrac{\beta}{1-\phi\beta}[(1-\phi\alpha_K)\mathrm{g} - \phi g_q] \\ (\dot{\mathrm{N}}/\mathrm{N})^* = \dfrac{\beta}{1-\varphi\beta}(1-\varphi\alpha_L)\mathrm{g} \end{cases} \tag{E17}$$

Equations (E16) and (E17) confirm that the model has an SSGP[19].

While (E15) shows that there exists also an allocation of scientists which supports the SSGP, it remains to be verified that there exists an appropriate allocation of income into the competing uses.

Using equations (D4), the Euler equations (E9) can be written as:

$$\begin{cases} \dfrac{\dot{C}}{C} = \left[\alpha_K \dfrac{q(t)I_K^{\alpha_K}}{K}\dfrac{Y}{I_K}\dfrac{M^{(1-\phi\beta)/\beta}K^{\phi}}{Y}\beta\phi^2 f'(k) - (\alpha_K - 1)\dfrac{\dot{I}_K}{I_K} - \rho - \delta_K\right]/\theta \\ \dfrac{\dot{C}}{C} = \left[\alpha_L \dfrac{b_L I_L^{\alpha_L}}{L}\dfrac{Y}{I_L}\dfrac{N^{(1-\varphi\beta)/\beta}L^{\varphi}}{Y}\beta\varphi^2 [f(k) - kf'(k)] - (\alpha_L - 1)\dfrac{\dot{I}_L}{I_L} - \rho - \delta_L\right]/\theta \end{cases} \tag{E18}$$

Let $s_K \equiv I_K/Y$, $s_L \equiv I_L/Y$. Substituting equation (15), s_K, s_L, the definitions of k and (D1) into (E18) and rearranging, we get:

$$\begin{cases} \dfrac{\dot{C}}{C} = \left[\alpha_K \dfrac{1}{s_K}\dfrac{kf'(k)}{f(k)}\beta\phi^2\left(\dfrac{\dot{K}}{K} + \delta_K\right) - (\alpha_K - 1)\dfrac{\dot{I}_K}{I_K} - \rho - \delta_K\right]/\theta \\ \dfrac{\dot{C}}{C} = \left[\alpha_L \dfrac{1}{s_L}\dfrac{[f(k) - kf'(k)]}{f(k)}\beta\varphi^2\left(\dfrac{\dot{L}}{L} + \delta_L\right) - (\alpha_L - 1)\dfrac{\dot{I}_L}{I_L} - \rho - \delta_L\right]/\theta \end{cases} \tag{E19}$$

Substituting (E10) and (E11) into (E19), we obtain

$$\begin{cases} \dfrac{\dot{C}}{C} = \left[\alpha_K\beta\phi^2 \dfrac{1}{s_K}\dfrac{kf'(k)}{f(k)}\left(\alpha_K\dfrac{\dot{C}}{C} + g_q + \delta_K\right) - (\alpha_K - 1)\dfrac{\dot{C}}{C} - \rho - \delta_K\right]/\theta \\ \dfrac{\dot{C}}{C} = \left[\alpha_L\beta\varphi^2 \dfrac{1}{s_L}\dfrac{f(k) - kf'(k)}{f(k)}\left(\alpha_L\dfrac{\dot{C}}{C} + \delta_L\right) - (\alpha_L - 1)\dfrac{\dot{C}}{C} - \rho - \delta_L\right]/\theta \end{cases} \tag{E20}$$

Rearranging (E20) yields:

$$\begin{cases} \dfrac{\dot{C}}{\mathrm{C}} = \dfrac{\rho + \delta_{\mathrm{K}} - \left(g_q + \delta_K\right)\{\alpha_K\beta\phi^2 kf'(k)/[s_K f(k)]\}}{\beta\alpha_K{}^2\phi^2 kf'(k)/[s_K f(k)] + 1 - \alpha_K - \theta} \\ \dfrac{\dot{C}}{\mathrm{C}} = \dfrac{\rho + \delta_{\mathrm{L}}\{1 - \beta\varphi^2\alpha_L[f(k) - kf'(k)]/[s_L f(k)]\}}{\beta\alpha_L{}^2\varphi^2[f(k) - kf'(k)]/[s_L f(k)] + 1 - \alpha_L - \theta} \end{cases} \tag{E21}$$

Using k^* in (E3), we obtain:

$$\begin{cases} \dfrac{kf'(k)}{f(k)} = \dfrac{(\varphi - \varphi^2\beta)d_N}{(\varphi - \varphi^2\beta)d_N + (\phi - \phi^2\beta)d_M} \\ \dfrac{f(k) - kf'(k)}{f(k)} = \dfrac{(\phi - \phi^2\beta)d_M}{(\varphi - \varphi^2\beta)d_N + (\phi - \phi^2\beta)d_M} \end{cases} \tag{E22}$$

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[19] The transversality conditions are $\lim_{t\to\infty}[\lambda_K(t)K(t)] = 0$ and $\lim_{t\to\infty}[\lambda_L(t)L(t)] = 0$. At this stage, we cannot provide a rigorous mathematical discussion on stability. However, at $\alpha_K = 1$, $\alpha_L = 0$, $g_g = 0$ and $\varphi = \phi = 1$, our environment degenerates to that of Acemoglu (2003). Therefore, by continuity, at least around that point, by Acemoglu's argument the steady state is stable as long as $\varepsilon \leq 1$.

Substituting (E16) and (E222) into (E21), we then get:

$$\begin{cases} s_K{}^* = \dfrac{(\alpha_K \mathrm{g} + g_q + \delta_{\mathrm{K}})\alpha_K \beta \phi^2}{\rho + \delta_{\mathrm{K}} - (1 - \alpha_K - \theta)\mathrm{g}} \cdot \dfrac{\chi_3 d_N}{\chi_3 d_N + \chi_4 d_M} \\ s_L{}^* = \dfrac{(\alpha_L g + \delta_{\mathrm{L}})\alpha_L \beta \varphi^2}{\rho + \delta_{\mathrm{L}} + (\theta + \alpha_L - 1)g} \cdot \dfrac{\chi_4 d_M}{\chi_3 d_N + \chi_4 d_M} \end{cases} \quad \text{(E23)}$$

where $\chi_3 \equiv (\varphi - \varphi^2\beta)$ and $\chi_4 \equiv (\phi - \phi^2\beta)$.

Let $s_C \equiv C/Y$ so that:

$$s_C + s_K + s_L = 1 \quad (E24)$$

Substituting (E23) into (E24), we obtain that along an SSGP, $\mathrm{s_C}$ is given by:

$$s_C{}^* = 1 - s_K{}^* - s_L{}^* \quad \text{(E25)}$$

Equations (E15), (E23) and (E25) provide the allocation of scientists and income needed to obtain the SSGP.

Fourth, the factor shares.

The income shares of capital, labor and scientists are obtained respectively from (D4), (D7) and (E3) as follows:

$$\sigma_K = \frac{rK}{Y} = \frac{\beta\phi^2 d_N(1 - \varphi\beta)\varphi}{d_M(1 - \phi\beta)\phi + d_N(1 - \varphi\beta)\varphi} \quad (E29)$$

$$\sigma_L = \frac{wL}{Y} = \frac{\beta\varphi^2 d_M(1 - \phi\beta)\phi}{d_M(1 - \phi\beta)\phi + d_N(1 - \varphi\beta)\varphi} \quad (E30)$$

σ_K and σ_L represent income share of capital and labor respectively.

$$\sigma_S = \frac{M\pi_Z + N\pi_X}{Y} = \frac{(1 - \phi\beta)(1 - \varphi\beta)(\varphi\phi d_N + \varphi\phi d_M)}{(1 - \phi\beta)\phi d_M + (1 - \varphi\beta)\varphi d_N} \quad (E31)$$

σ_S represents income share of scientists which include monopoly profit of two kinds of intermediate products.

Equations E(29), E(30) and E(31) establish that the factor income shares are constant in SSGP.

In addition, notice that total factor income share is given by:

$$\sigma_K + \sigma_L + \sigma_S = \frac{\varphi\phi[d_N(1 - \varphi\beta) + d_M(1 - \phi\beta)]}{d_M(1 - \phi\beta)\phi + d_N(1 - \varphi\beta)\varphi} \quad (E32)$$

When $\phi < 1$ or $\varphi < 1$, the total factor income share is less than 1 and there will be net profit for the final product firm. This is because of diminishing returns to scale. Otherwise, the total factor income share will equal to 1.

Finally, notice that the solution process implies that there exists only one allocation of scientists and income that is consistent with an SSGP.

Appendix F: Proof of Corollary 5

From equation (D4), we obtain that in steady state:

$$\begin{cases} \dfrac{\dot{w}}{w} = (\varphi - 1)\dfrac{\dot{L}}{L} + \dfrac{1 - \varphi\beta}{\beta}\dfrac{\dot{N}}{N} \\ \dfrac{\dot{r}}{r} = (\phi - 1)\dfrac{\dot{\mathrm{K}}}{\mathrm{K}} + \dfrac{1 - \phi\beta}{\beta}\dfrac{\dot{M}}{M} \end{cases} \tag{F1}$$

Using equations (E17) in (F1) results in:

$$\begin{cases} \dfrac{\dot{w}}{w} = (1 - \alpha_L)g \\ \dfrac{\dot{r}}{r} = (1 - \alpha_K)g - g_q \end{cases} \tag{F2}$$

Using (F2) and equations (E17) in the factor elasticities yields equation (23), that is:

$$\begin{cases} \varepsilon_K \equiv \dfrac{\dot{\mathrm{K}}/K}{\dot{r}/r} = \dfrac{\alpha_K + g_q/g}{(1 - \alpha_K) - g_q/g} \\ \varepsilon_L \equiv \dfrac{\dot{\mathrm{L}}/L}{\dot{w}/w} = \dfrac{\alpha_L}{1 - \alpha_L} \end{cases}$$