MPRA
Munich Personal RePEc Archive

A Note on the Two-fund Separation Theorem

Jan Wenzelburger

Centre for Economic Research, Keele University

8. February 2008

Online at http://mpra.ub.uni-muenchen.de/11014/
MPRA Paper No. 11014, posted 14. October 2008 04:52 UTC

A Note on the Two-fund Separation Theorem*

Jan Wenzelburger

Center for Economic Research
Keele University
Keele, ST5 5BG, UK
j.wenzelburger@econ.keele.ac.uk

Keele Economic Research Paper No. 1

Abstract

Keywords: Portfolio choice, CAPM, Risk aversion, Equilibrium, Market Participation

JEL Classification: G10, C62

First version: Feb. 2008, this version: Sep. 2008.

This note contains two remarks on the traditional capital asset pricing model (CAPM) with one risk-free asset. Firstly, an elementary proof of the two-fund separation theorem is provided showing that asset-demand may become undefined if the limiting slope of the investor's indifference curves is finite. Secondly, it is shown that an additional limiting condition on the risk aversion is generally necessary to guarantee existence of an equilibrium in the CAPM with one risk-free asset. The role of these two limiting conditions seems to have been overlooked in the established literature. A generalized existence result is formulated which allows for the case in which in equilibrium not all investors participate in the market for risky assets.

*Acknowledgment. I would like to thank Tim Worrall and Gauthier Lanot for stimulating discussions.

1 Introduction

One the most central results of the capital asset pricing model (CAPM), developed by Sharpe (1964), Lintner (1965), and Mossin (1966), is the *two-fund separation theorem*. In an agent-based modelling framework, Böhm & Chiarella (2005) have recently investigated an intertemporal model which is founded on a modern formulation of the two-fund separation theorem (Lemma 2.3) along with an existence and uniqueness result of (intertemporal) CAPM equilibria (Lemma 2.5). For the proofs of these two results the reader is referred to Böhm (2002). Unfortunately, however, the proof of the separation theorem there is incomplete, leaving their existence result in suspension. There are two core issues. Firstly, the asset demand function may become undefined if the limiting slope of the investor's indifference curves is finite. Secondly, the risk involved in the market portfolio may be greater than the aggregate willingness of all investors to take on risk if all limiting slopes are infinite.

The first issue seems to have been overlooked by Tobin (1958), Lintner (1965), Merton (1972) and Fama & Miller (1972) who provide formal proofs of the separation principle which are all based on the implicit assumption that the limiting slope of the indifference curves is infinite. In the vast amount of recent research into agent-based models of financial markets asset demand functions play a central role in characterizing agents investment behavior, e.g., see the forthcoming volume of the handbook of finance edited by Hens & Schenk-Hoppé (2008). Since various aspects of multiple risky assets in an agent-based framework are at the center of ongoing research, it seems to be justified to formulate a modern and complete proof of the two-fund separation theorem. This should be of particular interest for disequilibrium models in which asset prices are not necessarily market-clearing but set by a market maker, e.g., see Chiarella, Dieci & He (2008) and references therein. While the intuitive proof of the separation principle is found in almost any finance textbook, e.g., see Cuthbertson (1996), or Copeland, Weston & Shastri (2005), its mathematically rigorous treatment has mostly been abandoned.

In this note, we will demonstrate that the asset demand function derived from mean-variance preferences is well defined and hence the separation principle is meaningful only, if the market price of risk is less than the limiting slope of the investor's indifference curves. This limiting slope may be finite for a range of examples, implying that an investor is willing to take on an infinite amount of risk for a finite price of risk. Secondly, adopting a notion of risk aversion introduced by Nielsen (1987), it will be shown that existence of an asset-market equilibrium in the standard CAPM with one risk-free asset, generally, requires a certain limiting condition on the risk aversion of at least one of the investors. This limiting condition is essential in the case in which all limiting slopes of the investors are infinite. Despite the fact that in this case the separation principle holds for all prices and all investors, an asset-market equilibrium will fail to exist if the risk of the market portfolio is higher than the aggregate willingness to take risk. On the contrary, if the limiting slope of one of the investors is finite, then at least one investor is willing to take on an infinite amount of risk, implying that an asset-market equilibrium exists for any possible market portfolio.

The role of the limiting condition on the risk aversions seem to have been overlooked by Böhm & Chiarella (2005), Dana (1999), and Hens, Laitenberger & Löffler (2002). The main purpose of these notes therefore is to provide an elementary and self-contained proof of the two-fund separation theorem for mean-standard deviation utility functions of the form $U(\mu, \sigma)$ and arbitrary probability distributions for future returns. From there on the well-established existence and uniqueness result of CAPM equilibria of Dana (1999) is revisited demonstrating the importance of the limiting condition on the risk aversion. Her existence result is generalized in allowing for the case in which some of the investors do not hold risky assets in equilibrium.

This note arose from a survey article (Wenzelburger 2008) I was asked to prepare for the handbook of finance and are a revised and extended version of a lecture that I had to deliver to students in the context of my *Habilitation* at Bielefeld University in December 2002, see Wenzelburger (2002).

2 Prerequisites

In view of the above mentioned agent-based models, the CAPM will formulated in terms of prices rather than in terms of returns.[1] Consider the investment decision of an investor with a one-period planning horizon who needs to transfer wealth from the first to the second period. Her endowment consists of $e > 0$ units of a non-storable consumption good in the first period and none in the second period. Suppose for simplicity that she does only consume in the second period. The investment opportunities consist of $K+1$ assets the proceeds of which she plans to consume in the second period. All prices and payments are denominated in the non-storable consumption good which thus serves as the numéraire. The K risky assets are characterized by stochastic gross returns $\tilde{q} = (\tilde{q}^{(1)}, \dots, \tilde{q}^{(K)})$ per unit which take values in $\mathbb{R}_+^K$. The risk-free bond pays a constant return $R_f = 1 + r_f > 0$ per unit. A portfolio is represented by a vector $(x, y) \in \mathbb{R}^K \times \mathbb{R}$. The vector $x = (x^{(1)}, \dots, x^{(K)})$ represents the portfolio of risky assets with $x^{(k)}$ denoting the number of shares of the k-th risky asset. The scalar y describes the number of risk-free bonds in the portfolio (x, y). The total amount of risky assets is $x_m \in \mathbb{R}_+^K$ and referred to as the *market portfolio* of the economy.

Assume, for simplicity, that there are no short-sale constraints. If $p = (p^{(1)}, \dots, p^{(K)}) \in \mathbb{R}^K$ denotes the price vector of risky assets, the investor's budget constraint is

$$e = y + \langle p, x \rangle = y + \sum_{k=1}^{K} p^{(k)} x^{(k)} \,,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathbb{R}^K$. Substituting for y, the investor's second-period wealth associated with the portfolio $x \in \mathbb{R}^K$ of risky assets becomes

$$w(e, p, \tilde{q}, x) = R_f e + \langle \tilde{q} - R_f p, x \rangle.$$

[1] The setup and notation is based on Böhm, Deutscher & Wenzelburger (2000) and extended to multiple risky assets.

The uncertainty of second-period wealth rests with the random gross return $\tilde{q}$ of risky assets, when the investor treats the asset price p of the first period as a parameter of her decision problem. This uncertainty is described by a probability space $(\mathbb{R}^K, \mathcal{B}, \nu)$, where $\nu \in \mathrm{Prob}(\mathbb{R}^K)$ is a probability distribution for $\tilde{q}$ and $\mathrm{Prob}(\mathbb{R}^K)$ denotes the set of all Borelian probability measures.

Rather than using the full probability distribution ν, the CAPM assumes that an investor bases her evaluation of the uncertain return solely on the mean and the variance of second-period wealth. Denote *expected gross returns* by

$$\overline{q} = \mathbb{E}[\tilde{q}] := \int_{\mathbb{R}^K} q\, \nu(\mathrm{d}q) \in \mathbb{R}^K$$

and the *(variance)-covariance matrix* of future returns by

$$V = \mathbb{V}[\tilde{q}] := \int_{\mathbb{R}^K} [q - \overline{q}][q - \overline{q}]^\top \nu(\mathrm{d}q) \in \mathcal{M}_K,$$

where $\mathcal{M}_K$ denotes the set of all symmetric and positive definite $K \times K$ matrices. The kl-th entry of the V is the covariance $V_{kl} = \mathbb{C}\mathrm{ov}[\tilde{q}^{(k)}, \tilde{q}^{(l)}]$ between the gross returns of the k-th and the l-th risky asset. The positive definiteness of V ensures that there are no redundant assets in the market. Based on ν, the expected wealth and its standard deviation associated with a portfolio of risky assets $x \in \mathbb{R}^K$ becomes

$$\mu_w(e,p,x) := \mathbb{E}\big[w(e,p,\cdot,x)\big] = R_f e + \langle \overline{q} - R_f p, x\rangle$$

and

$$\sigma_w(x) := \mathbb{V}\mathrm{ar}\big[w(e,p,\cdot,x)\big]^{\frac{1}{2}} = \langle x, Vx\rangle^{\frac{1}{2}},$$

respectively.[2] The investment behavior of an investor is now based on the following assumption.

Assumption 1
An investor is characterized by an utility function U which is a function of the mean and the standard deviation of future wealth and a probability distribution ν for future gross returns. These satisfy the following:

(i) The utility function $U : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, strictly increasing in μ, strictly decreasing in σ, and strictly concave.[3]

(ii) The probability distribution $\nu \in \mathrm{Prob}(\mathbb{R}^K)$ is parameterized by a pair $(\overline{q}, V)$, where $\overline{q} \in \mathbb{R}^K$ and $V \in \mathcal{M}_K$.

The assumptions on $U(\mu,\sigma)$ are standard except that utility on the boundary of $\mathbb{R}\times\mathbb{R}_+$ may be strictly increasing. This allows to include cases in which the investor requires

[2] To relate the notation to the one adopted in the incomplete markets literature (e.g., LeRoy & Werner 2001), note that each function $q \mapsto w(e,p,q,x)$ is an element of the Hilbert space $L^2(\nu)$.

[3] To include interesting examples, it is sometimes convenient to bound the domain of U from below.

a positive minimum price of risk in order to participate in the asset market thereby investing into risky assets.

With the notation introduced above, the decision problem of an investor takes the form

$$\max_{x\in\mathbb{R}^K} U\big(\mu_w(e,p,x),\sigma_w(x)\big). \tag{1}$$

Assumption 1 implies that the objective function in (1) is strictly concave in x. To ensure boundedness of the asset demand derived from (1), we need the concept of a *limiting slope* of an indifference curve as adopted in Nielsen (1987). Recall that the slope of any indifference curve in the $\mu-\sigma$ plane is given by the marginal rate of substitution between risk and return

$$S(\mu,\sigma) := -\frac{\frac{\partial U}{\partial \sigma}(\mu,\sigma)}{\frac{\partial U}{\partial \mu}(\mu,\sigma)}. \tag{2}$$

S is a measure of the investor's risk aversion. Consider the indifference curve through the point $(R_f e, 0)$ and denote by

$$\rho_U := \sup\left\{S(\mu,\sigma):\ (\mu,\sigma)\in\mathbb{R}\times\mathbb{R}_+ \text{ s.t. } U(\mu,\sigma)=U(R_f e,0)\right\}$$

the limiting slope of this indifference curve. From convex analysis it is well known that ρ_U is either positive and finite or plus infinity. Since U is concave, all indifference curves have the same limiting slope ρ_U, see Rockafellar (1970).

3 Proof of the Two-fund Separation Theorem

In order to formulate and prove the two-fund separation theorem, we first construct the capital market line, assuming that borrowing and lending at the safe rate R_f is allowed. Let $\pi := \overline{q} - R_f p$ be the vector of expected excess returns and consider the following mean-variance optimization problem

$$\max_{x\in\mathbb{R}^K} \mu_w(e,p,x) \quad \text{s.t.} \quad \sigma_w(x)\le\sigma, \tag{3}$$

where $\sigma \ge 0$. This is a linear maximization problem on a convex set first considered by Markowitz (1952). Its solution is well known and given in the following proposition.

Proposition 1
Let $\pi \neq 0$, V be positive definite and $\sigma \ge 0$. Then the solution to (3) is

$$x_{\text{eff}}(\sigma,\pi,V) := \frac{\sigma}{\langle \pi, V^{-1}\pi\rangle^{\frac{1}{2}}} V^{-1}\pi.$$

Let e, $\overline{q}$, π, and V be arbitrary but fixed with $\pi = \bar{q} - R_f p \neq 0$ and consider the curve

$$\sigma \mapsto \mu_w\big(e,p,x_{\text{eff}}(\sigma,\pi,V)\big).$$

Inserting $x_{\text{eff}}(\sigma, \pi, V)$, one gets a straight line

$$\mu = \mu_w\big(e, p, x_{\text{eff}}(\sigma, \pi, V)\big) = R_f e \, + \, \langle \pi, V^{-1}\pi\rangle^{\frac{1}{2}} \, \sigma \tag{4}$$

whose slope is the so-called market price of risk $\langle \pi, V^{-1}\pi\rangle^{\frac{1}{2}}$. Hence equation (4) is nothing but the *efficient frontier* expressed in terms of wealth rather than returns. Indeed, if $r = \frac{w(e,p,q,x_{\text{eff}}(\sigma,\pi,V))}{e} - 1$ denotes the return of $x_{\text{eff}}(\sigma, \pi, V)$, then (4) implies the well-known efficient frontier

$$\mu_r = r_f + \rho \, \sigma_r,$$

stating that the relationship between the expected return μ_r and the standard deviation σ_r of any efficient portfolio is linear. The efficient frontier (4) is now used to formulate the *two-fund separation theorem.*

Theorem 1
Let the hypotheses of Assumption 1 be satisfied. Then for any $e > 0$ and any $\pi \in \mathbb{R}^K$ with $\langle \pi, V^{-1}\pi\rangle^{\frac{1}{2}} < \rho_U$, the optimization problem (1) has a unique maximizer $x_\star \in \mathbb{R}^K$. The corresponding asset demand function Φ takes the form

$$x_\star = \Phi(e, \pi, V) := \begin{cases} \frac{\varphi(e, \langle \pi, V^{-1}\pi\rangle^{\frac{1}{2}})}{\langle \pi, V^{-1}\pi\rangle^{\frac{1}{2}}} \, V^{-1}\pi, & \text{if } \pi \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

where for each $\rho \in [0, \rho_U)$,

$$\varphi(e, \rho) := \operatorname*{argmax}_{\sigma \geq 0} U\big(R_f e + \rho \, \sigma, \sigma\big) \tag{6}$$

is bounded from above.

The formulation of Theorem 1 is, in essence, that of Böhm & Chiarella (2005, Lemma 2.3). A quite similar formulation has been applied by Rochet (1992) and others to describe the behavior of commercial banks, see also Freixas & Rochet (1997, Chap. 8) and references therein. The present formulation of Theorem 1, however, accounts for two special cases that the literature usually does not consider. Firstly, the case of a finite limiting slope ρ_U, in which the objective function $\sigma \mapsto U(R_f e + \rho\sigma, \sigma)$ in (6) becomes unbounded for each $\rho \geq \rho_U$ so that the asset demand function (5) is undefined whenever $\rho \geq \rho_U$. Nielsen (1987), as an exception, has noted that asset demand may become unbounded for $\rho \geq \rho_U$ but without stating the explicit form (5). Secondly, the case in which the investor will not participate in the asset market which occurs if either the expected excess return π is zero or her willingness to take risk $\varphi(e, \rho)$ is zero.

Proof of Theorem 1. The proof proceeds in 5 steps.

Step 1. The monotonicity properties of U imply $U(R_f e, 0) > U(R_f e, \sigma)$ for all $\sigma > 0$. Thus $x_\star = 0$ for $\pi = 0$. For the remainder of the proof, assume therefore that $\pi \neq 0$. Then $\mu_w(e, p, x) = R_f e + \langle \pi, x\rangle$ is a linear function of x and

$$\sigma_w(x) = \langle x, Vx\rangle^{\frac{1}{2}}$$

is a convex function of x, as it defines an norm on $\mathbb{R}^K$. Since U is increasing in μ, decreasing in σ, and strictly concave, it follows that the objective function $x \mapsto U\big(\mu_w(e,p,x), \sigma_w(x)\big)$ is strictly concave in x. Hence, if a bounded solution exists, it is uniquely determined.

Step 2. The first order conditions for $x_\star \neq 0$ are

$$\begin{aligned} 0 &\overset{!}{=} \mathrm{D}_x U\big(\mu_w(e,p,x_\star), \sigma_w(x_\star)\big) \\ &= \underbrace{\frac{\partial U}{\partial \mu}\big(\mu_w(e,p,x_\star), \sigma_w(x_\star)\big)}_{>0}\, \pi + \underbrace{\frac{\partial U}{\partial \sigma}\big(\mu_w(e,p,x_\star), \sigma_w(x_\star)\big)}_{<0}\, \frac{1}{\langle x_\star, V x_\star\rangle^{\frac{1}{2}}} V x_\star. \end{aligned} \tag{7}$$

Rearranging and using (2), one has

$$x_\star = \frac{\sigma_w(x_\star)}{S\big(\mu_w(e,p,x_\star), \sigma_w(x_\star)\big)}\, V^{-1}\pi. \tag{8}$$

Hence a non-zero solution to (1) must be an efficient portfolio. Inserting (8) into $\sigma_w(x_\star) := \langle x_\star, V x_\star\rangle^{\frac{1}{2}}$, we see that

$$\sigma_w(x_\star) = \frac{\sigma_w(x_\star)}{S\big(\mu_w(e,p,x_\star), \sigma_w(x_\star)\big)}\, \langle \pi, V^{-1}\pi\rangle^{\frac{1}{2}}.$$

Therefore, the first order conditions (7) take the form

$$\begin{cases} x_\star & \overset{!}{=} \frac{\sigma_w(x_\star)}{\langle \pi, V^{-1}\pi\rangle^{\frac{1}{2}}} V^{-1}\pi, \\ \langle \pi, V^{-1}\pi\rangle^{\frac{1}{2}} & \overset{!}{=} S\big(\mu_w(e,p,x_\star), \sigma_w(x_\star)\big). \end{cases} \tag{9}$$

Step 3. Let $\sigma_\star \geq 0$ be arbitrary and set

$$\tilde{x} := \frac{\sigma_\star}{\langle \pi, V^{-1}\pi\rangle^{\frac{1}{2}}}\, V^{-1}\pi \quad \text{and} \quad \rho = \langle \pi, V^{-1}\pi\rangle^{\frac{1}{2}}. \tag{10}$$

Then

$$\begin{aligned} \sigma_w(\tilde{x}) &= \langle \tilde{x}, V\tilde{x}\rangle^{\frac{1}{2}} = \sigma_\star, \\ \mu_w(\tilde{x}) &= R_f e + \langle \pi, \tilde{x}\rangle = R_f e + \rho\, \sigma_\star. \end{aligned}$$

Thus $\tilde{x}$ as defined in (10) satisfies the first order conditions (9) if and only if $\sigma_\star$ satisfies

$$S\big(R_f e + \rho\, \sigma_\star, \sigma_\star\big) \overset{!}{=} \rho. \tag{11}$$

Step 4. Observe that the first order condition of the maximization problem

$$\max_{\sigma \geq 0} U(R_f e + \rho\, \sigma, \sigma) \tag{12}$$

is exactly equation (11). The strict concavity of U implies that for each fixed $\rho \geq 0$, the objective function $\sigma \mapsto U(R_f e + \rho\,\sigma, \sigma)$ is strictly concave. We have $0 \leq S(R_f e, 0)$ and for any $0 \leq \rho < \rho_U$,

$$S\big(R_f e + \rho\,\sigma, \sigma\big) > \rho \quad \text{for all sufficiently large } \sigma. \tag{13}$$

Thus for each $0 \leq \rho < \rho_U$, the function $\sigma \mapsto U(R_f e + \rho\,\sigma, \sigma)$ is decreasing for all sufficiently large σ. This shows that problem (12) admits a unique maximizer

$$\sigma_\star := \varphi(e, \rho) < \infty, \tag{14}$$

given by (6). Moreover, any strictly positive maximizer (14) solves (11).

Step 5. If $\sigma_\star$ as given by (14) solves the first order conditions (11), then $x_\star = \tilde{x}$ as defined in (10) solves (9) and hence is the interior solution to the maximization problem (1). If, on the contrary, (13) holds for all $\sigma \geq 0$, then

$$U(R_f e, 0) > U\big(R_f e + \langle \pi, V^{-1}\pi\rangle^{\frac{1}{2}}\sigma, \sigma\big) \quad \text{for all } \sigma > 0.$$

Since $\sigma_w(x) > 0$ for any efficient portfolio $x \neq 0$, then $x_\star = 0$. □

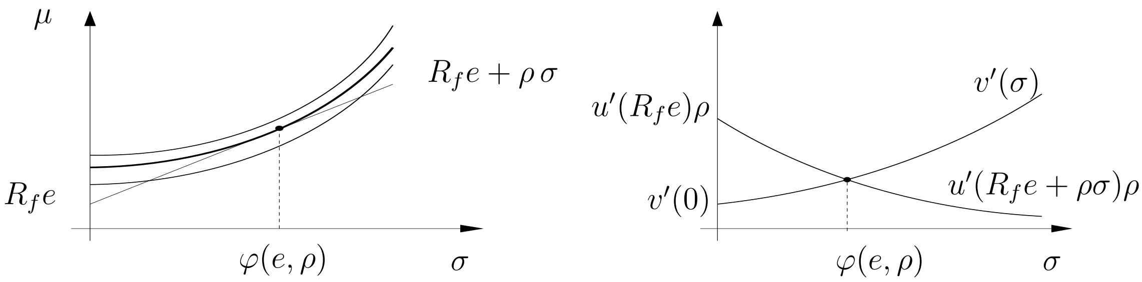
The separation theorem is illustrated in Figure 1 (a). The proof was obtained straightforwardly from comparing the first-order conditions corresponding to the two maximization problems (1) and (3): in an interior optimum, the marginal rate of substitution between risk and return has to be equal to the price of risk ρ.[4] The boundary case in which the investor does not partake in the market for risky assets was treated separately.

The interpretation is standard. The separation theorem states that given π and V, the optimal portfolio $x_\star$ is collinear to $x_{\text{eff}}(1)$ and hence efficient. Thus investors with the same beliefs $(\overline{q}, V)$ will invest in the same two funds, the risk-free asset on the one hand and a 'mutual fund' with the same mix of risky assets. Since $\sigma_w(x_\star) = \varphi(e, \rho)$, the function φ describes the investor's willingness to take risk. Unlike the portfolio mix, this willingness and hence the amount of the endowment invested into risky assets depends on the investor's preferences.

4 Risk-taking behavior

The separation theorem may be used to describe the risk-taking behavior of investor and to address existence and uniqueness issues in the CAPM. As recognized by Lajeri & Nielsen (2000), the willingness to take risk function φ can be characterized by properties of the risk aversion $S(\mu, \sigma)$. The following result is a slight generalization of Dana (1999, Prop. 3.1 and 3.4) and Hens, Laitenberger & Löffler (2002, Lemma 1). It includes a condition formulated in Assumption (ii) that guarantees the surjectivity of φ and which seems to have been overlooked by Dana and Hens et al. It also includes the boundary case in which $\frac{\partial U}{\partial \sigma}(\mu, 0)$ may be positive.

[4] The important assumption here is that the investor may short-sell risky assets as wells as borrow and lend at the risk-free rate r_f. Such constraints are investigated in Lintner (1965) and Black (1972).



(a) Separation principle (b) Optimality with separable utility

Figure 1: Portfolio selection

Proposition 2
Let Assumption 1 be satisfied and assume, in addition, that U is twice continuously differentiable. Let $R_f e > 0$ be arbitrary but fixed. Then $\varphi(e,\cdot)$ is differentiable on $(0,\rho_U)$ except for $\rho = S(R_f e, 0)$. Moreover:

(i) $\varphi(e,\rho) = 0$ *for all* $0 \le \rho \le S(R_f e, 0)$.

(ii) *If* $\lim_{\mu\to\infty} \dfrac{S(R_f e + \mu, \sigma)}{\mu} = 0$ *for all* $\sigma > 0$, *then for each risk* $\sigma \in \mathbb{R}_+$, *there exists a market price of risk* $\rho_\sigma \in [0,\rho_U)$ *such that* $\varphi(e,\rho_\sigma) = \sigma$.

(iii) $\varphi(e,\rho)$ *is increasing for all* $\rho \in [S(R_f e,0),\rho_U)$, *if and only if*

$$\frac{\partial S}{\partial \mu}\big(R_f e + \rho\,\varphi(e,\rho), \varphi(e,\rho)\big)\,\varphi(e,\rho) < 1 \quad \textit{for all} \quad \rho \in [S(R_f e,0),\rho_U).$$

(iv) *For each* $\rho \in [S(R_f e,0),\rho_U)$, $\varphi(e,\rho)$ *is decreasing in* $e \ge 0$, *if and only if*

$$\frac{\partial S}{\partial \mu}\big(R_f e + \rho\,\varphi(e,\rho), \varphi(e,\rho)\big) > 0 \quad \textit{for} \quad e \ge 0.$$

Proof. The demand function $\varphi(e,\rho)$ exists by Theorem 1. Its differentiability follows from the implicit function theorem. The other properties are derived as follows.

(i) Given $\rho \ge 0$, the first order conditions are

$$S(R_f e + \rho\,\sigma, \sigma) \stackrel{!}{=} \rho. \tag{15}$$

Hence, the solution $\sigma_\star = \varphi(e,\rho)$ is a boundary solution $\varphi(e,\rho) = 0$ for all $\rho \le S(R_f e, 0)$. It is an interior one with $\varphi(e,\rho) > 0$ for all $\rho > S(R_f e, 0)$.

(ii) Substituting $\mu = \rho\sigma$ into (15), the first order conditions take the form

$$\frac{\mu}{\sigma} = S(R_f e + \mu, \sigma). \tag{16}$$

Since $\lim\limits_{\mu\to\infty} S(R_f e+\mu,\sigma)/\mu = 0$ for each $\sigma > 0$, there exists a unique $\mu_\sigma > 0$ that solves (16). Substituting back, we see that for each $\sigma > 0$, there exists $\rho_\sigma = \frac{\sigma}{\mu_\sigma}$ such that $\varphi(e, \rho_\sigma) = \sigma$.

(iii) Analogously to Hens, Laitenberger & Löffler (2002, Lemma 1), Assertion (iii) follows from differentiating (15) with respect to ρ.

(iv) Assertion (iv) follows from differentiating (15) with respect to e.

□

The innovative part of Proposition 2 is the limiting condition in Assertion (ii) which ensures that the willingness to take risk φ is surjective on $\mathbb{R}_+$, implying that an investor is prepared to take on any amount of risk, provided that the market price of risk is high enough. This condition is automatically satisfied if the limiting slope ρ_U is finite. However, the condition is essential when ρ_U is infinite and is missing in Dana (1999) and Hens, Laitenberger & Löffler (2002). Böhm (2002) has realized that surjectivity of φ may fail to hold but without being able to properly identify its structural cause.

Assertion (iii) states conditions under which φ is increasing in the price of risk. For sake of completeness we have included a result by Lajeri & Nielsen (2000) in Assertion (iv) which states that risk in an inferior good if risk aversion is decreasing in means.

For additive separable utility functions of the form $U(\mu,\sigma) = u(\mu) - v(\sigma)$, the risk aversion (2) becomes

$$S(\mu,\sigma) = \frac{v'(\sigma)}{u'(\mu)},$$

implying

$$\frac{\partial S}{\partial \mu}(\mu,\sigma) = \frac{-v'(\sigma)u''(\mu)}{u'(\mu)^2}.$$

In this case it is straightforward to verify that condition (iii) is reduced to the elasticity condition of Dana (1999, Prop. 3.4) which takes the following form:

(iii') $\varphi(e,\rho)$ is strictly increasing for all $\frac{v'(0)}{u'(R_f e)} < \rho < \rho_U$, if and only if the elasticity of $u'(R_f e+\mu)$ satisfies $\frac{u''(R_f e+\mu)\mu}{u'(R_f e+\mu)} > -1$ for all $\mu > 0$.

For the separable case, the proof of Proposition 2 is illustrated in Figure 1 (b). Proposition 2 will be further illustrated in Section 6 below.

5 Existence and uniqueness of equilibrium

An abundant amount of the literature has addressed the existence and uniqueness of asset-market equilibria in the traditional CAPM and its extensions, e.g., see Nielsen (1988, 1990a,b), Allingham (1991), Dana (1993, 1999), or Hens, Laitenberger & Löffler (2002). For the case under consideration we follow a basic line of reasoning in Dana

(1999, Sec. 3) which has been adapted in Böhm & Chiarella (2005) and Böhm (2002), respectively.

Consider an asset market with $i = 1, \ldots, I$ investors who are all characterized by Assumption 1. Suppose that utility functions and endowments are heterogeneous but that their expectations regarding future gross returns are identical and given by $(\overline{q}, V)$. Let $e^{(i)}$ denote investor i's endowment and $\varphi^{(i)}$ be her willingness to take risk derived from some $U^{(i)}$ satisfying Assumption 1. For fixed endowments $e^{(i)}$, define *aggregate willingness to take risk* by

$$\phi(\rho) := \sum_{i=1}^{I} \varphi^{(i)}(e^{(i)}, \rho), \qquad \rho \in [0, \overline{\rho}), \tag{17}$$

where $\overline{\rho} := \min\{\rho_U^{(i)} : i = 1, \ldots, I\}$ is the minimum of all limiting slopes $\rho_U^{(i)}$ of $U^{(i)}$. In light of the examples which will be presented below, $\overline{\rho}$ may be either finite or infinite. Note that the domain of definition of ϕ may be smaller than $\mathbb{R}_+$.[5]

Set $\sigma_{\max} := \sup\{\phi(\rho) : \rho \in [0, \overline{\rho})\}$ for the upper bound of risk, the investors are willing to accept. The following result is a refinement of Böhm & Chiarella (2005, Lemma 2.5) that includes the case in which the limiting slope $\overline{\rho}$ is finite.

Theorem 2
Let $(\overline{q}, V)$ and $e^{(1)}, \ldots, e^{(I)} > 0$ be given. Assume that aggregate willingness to take risk $\phi : [0, \overline{\rho}) \to \mathbb{R}_+$ is a continuous map with respect to ρ. Then the following holds:

(i) For each $0 \neq x_m \in \mathbb{R}_+^K$ with $\langle x_m, V x_m \rangle^{\frac{1}{2}} < \sigma_{\max}$, there exists an asset-market equilibrium with market-clearing prices

$$p_\star = \tfrac{1}{R_f}\left[\overline{q} - \frac{\rho_\star}{\langle x_m, V x_m \rangle^{\frac{1}{2}}} V x_m\right], \tag{18}$$

where $\rho_\star \in (0, \overline{\rho})$ is a solution to (20).

(ii) If, in addition, ϕ is strictly increasing with respect to all ρ for which $\phi(\rho) > 0$, then the asset-market equilibrium (18) is uniquely determined.

Proof. Using Theorem 1, the asset-market equilibrium condition takes the form

$$\frac{\phi(\langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}})}{\langle \pi, V^{-1}\pi \rangle^{\frac{1}{2}}} V^{-1}\pi = x_m, \tag{19}$$

where $\pi = \overline{q} - R_f p \neq 0$ is the vector of excess returns. Computing the standard deviation of the wealth associated with x_m and with the portfolio on the l.h.s of (19), $\pi_\star$ is a solution to (19), if $\rho_\star := \langle \pi_\star, V^{-1}\pi_\star \rangle^{\frac{1}{2}} \in (0, \overline{\rho})$ solves

$$\phi(\rho) = \langle x_m, V x_m \rangle^{\frac{1}{2}}. \tag{20}$$

[5] This fact has been overlooked in Böhm & Chiarella (2005).

Vice versa, if some $\rho_\star \in (0, \overline{\rho})$ solves (20), then $\pi_\star := \frac{\rho_\star}{\langle x_m, V x_m \rangle^{\frac{1}{2}}} V x_m$ is a solution to (19). Hence in equilibrium, aggregate willingness to take risk must be equal to the aggregate risk of the market $\langle x_m, V x_m \rangle^{\frac{1}{2}}$. The existence of $\rho_\star$ now follows from the intermediate-value theorem, observing that $\phi(0) = 0$.

□

The price function (18) as such is quite standard in finance and has been used in the form of a temporary equilibrium map in Böhm & Chiarella (2005).

The first insight of the present formulation is that the equilibrium price of risk $\rho_\star$ is bounded from above by the lowest limiting slope $\overline{\rho}$, so that individual asset demands are always well defined in equilibrium. The second observation is that $\sigma_{\max}$ as defined before Theorem 2 is infinite if $\overline{\rho}$ is finite. This is an immediate consequence of Proposition 2 (ii), implying that an asset-market equilibrium exists for any market portfolio x_m if $\overline{\rho}$ is finite.

The third observation is that the aggregate demand for risk may be zero for all sufficiently low prices of risk. This is a slight generalization of Dana (1999, Prop. 3.4) which allows to include cases in which some of the investors will not invest into risky assets in equilibrium. The fourth observation is that the equilibrium price of risk responds to changes in second-moment beliefs V but not to changes in first-moment beliefs $\overline{q}$. As a consequence, any change in first-moment beliefs $\overline{q}$ changes the corresponding market-clearing asset prices in a linear fashion, irrespective of any nonlinearities in investors' utility functions.

The next corollary is immediate from Proposition 2 and Theorem 2 and corresponds to Dana (1999, Prop. 3.4) and Hens, Laitenberger & Löffler (2002, Thm. 1). It is a generalization of Böhm (2002, Thm. 3.2) for non-separable utility functions. The key observation is that aggregate willingness to take risk ϕ is invertible with respect to all $\rho \in (0, \overline{\rho})$, if all individual demand functions $\varphi^{(i)}$ are non-decreasing in ρ with at least one demand function being increasing for $\rho > 0$ and surjective on $\mathbb{R}_+$.

Corollary 1
Under the hypotheses of Theorem 2, suppose that the willingness to take risk of all investors is non-decreasing in ρ and that the preferences of at least one investor satisfies the conditions of Proposition 2 stated in (ii) and (iii). Then for any market portfolio $0 \neq x_m \in \mathbb{R}^K_+$, there exists a unique asset-market equilibrium.

6 An Illustration

The importance of the conditions in Proposition 2 for the existence of CAPM equilibria may be illustrated with four examples. Consider first the quasi-linear case

$$U(\mu, \sigma) = \mu - v(\sigma).$$

The assumptions on U imply $v' > 0$ and $v'' > 0$. Since $S(\mu, \sigma) = v'(\sigma)$, the limiting slope ρ_U is finite (infinite) if and only if $\lim_{\sigma \to \infty} v'(\sigma)$ is finite (infinite). As a consequence, surjectivity of φ is determined by v' alone. Moreover, $\varphi(e,\rho)$ is increasing in ρ for $\rho > \frac{v'(0)}{R_f e}$ and independent of e whenever $v'(0) = 0$.

Secondly, the fact that surjectivity may not obtain can also be seen from considering

$$U(\mu,\sigma) = \ln \mu - \sigma, \tag{21}$$

see also Böhm (2002). The limiting slope is $\rho_U = \infty$ and the willingness to take risk is given by

$$\varphi(e,\rho) = \max\left\{1 - \frac{R_f e}{\rho}, 0\right\}, \quad \rho \in \mathbb{R}_+.$$

In this case, the condition of Assertion (ii) is not satisfied, as

$$\lim_{\mu\to\infty} \frac{S(R_f e + \mu, \sigma)}{\mu} = \lim_{\mu\to\infty} \frac{R_f e + \mu}{\mu} = 1 > 0.$$

φ is increasing in ρ but the maximum risk that the investor is willing to accept is $\sigma_{\max} = 1$. The resulting offer curve is depicted in Figure 2 (a).

Thirdly, risk can be decreasing in ρ. This is seen from considering

$$U(\mu,\sigma) = -a \exp\left(-\frac{\mu}{a}\right) - \sigma, \quad a > 0, \tag{22}$$

where $u(\mu)$ is of the CARA form. The willingness to take risk is given by

$$\varphi(e,\rho) = \max\left\{\tfrac{1}{\rho}[a \ln \rho - R_f e], 0\right\}, \quad \rho \geq 0.$$

The limiting slope is again infinite, but in this case the two conditions stated in Assertions (ii) and (iii) of Proposition 2 are violated. For each $e > 0$, the map $\varphi(e,\rho)$ is unimodal with respect to ρ and attains its maximum at $\rho_{\max} = \exp\left(1 + \frac{R_f e}{a}\right)$. The maximum risk which the investor is willing to take is $\sigma_{\max} = a/\exp\left(1 + \frac{R_f e}{a}\right)$. Note that $\lim\limits_{\rho\to\infty} \varphi(e,\rho) = 0$. In the $\mu - \sigma$ plane, the 'offer curve' for risk is backward bending as in Figure 2 (b).

Finally, the limiting slope ρ_U may be finite. Consider

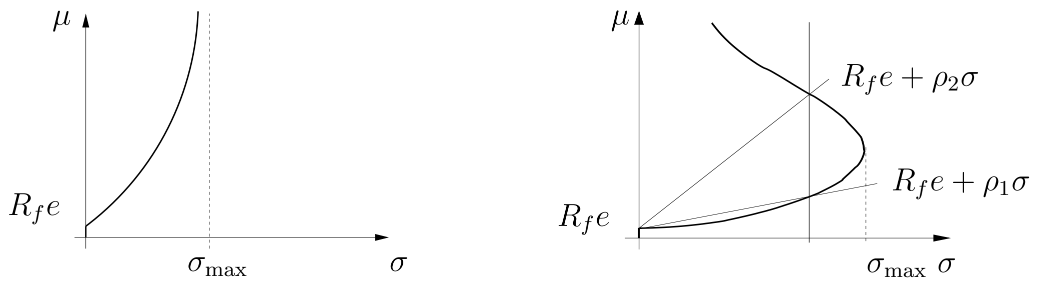
$$U(\mu,\sigma) = \sqrt{(\mu^2 + 2\mu)} - \sigma, \quad (\mu,\sigma) \in \mathbb{R}^2_+. \tag{23}$$

Then $\rho_U = 1$ and

$$\varphi(e,\rho) = \max\left\{\tfrac{1}{\rho}\left[\tfrac{1}{\sqrt{1-\rho^2}} - (R_f e + 1)\right], 0\right\}, \quad \rho \in [0,1).$$

An investor characterized by (23) is prepared to take any risk $\sigma \in \mathbb{R}_+$ as $\varphi(e,\cdot)$ is surjective on $\mathbb{R}_+$. However, in this case φ and hence the corresponding asset demand (5) is undefined for all $\rho \geq 1$.

More examples are found in Böhm (2002).



(a) Failure of surjectivity, $U(\mu, \sigma) = \ln \mu - \sigma$.

(b) Backward bending offer curve, $U(\mu, \sigma) = -\exp(-\mu) - \sigma$.

Figure 2: The willingness to take risk.

7 Conclusion

These notes provided an elementary and concise proof of the classical two-fund separation theorem and a generalized existence and uniqueness result for the CAPM with one risk-free asset, pointing out the importance of two limiting properties of the risk aversion associated with the investor's preferences. The formulation of the two-fund separation theorem is particularly amenable for disequilibrium models while the existence result allows for non-market participation in equilibrium.

A Appendix

Proof of Proposition 1. The Lagrangian function is

$$\mathcal{L}(x\,;\lambda) := Re + \langle \pi, x\rangle \;+\; \lambda[\bar{\sigma} - \langle x, Vx\rangle^{\frac{1}{2}}].$$

The first order conditions for a solution $(x_\star, \lambda_\star)$ are

$$\mathrm{D}_x\mathcal{L}(x_\star\,;\lambda_\star) = \pi + \lambda_\star \left[\frac{-2}{2\langle x_\star, Vx_\star\rangle^{\frac{1}{2}}} Vx_\star\right] \stackrel{!}{=} 0.$$

The complementary slackness conditions are

$$\lambda_\star\big[\sigma - \langle x_\star, Vx_\star\rangle^{\frac{1}{2}}\big] \stackrel{!}{=} 0.$$

Since $\mu_w(e, p, x)$ is increasing in x, $\lambda_\star > 0$ and the boundary condition must be binding. Hence

$$x_\star = \frac{\sigma}{\langle \pi, V^{-1}\pi\rangle^{\frac{1}{2}}}\, V^{-1}\pi.$$

□

References

ALLINGHAM, M. (1991): "Existence theorems in the CAPM", *Econometrica*, 59, 1169–1174.

BLACK, F. (1972): "Capital market equilibrium with restricted borrowing", *Journal of Business*, 45, 444–455.

BÖHM, V. (2002): "CAPM Basics", Discussion Paper 495, Department of Economics, Bielefeld University.

BÖHM, V. & C. CHIARELLA (2005): "Mean Variance Preferences, Expectations Formations, and the Dynamics of Random Asset Prices", *Mathematical Finance*, 15(1), 61–97.

BÖHM, V., N. DEUTSCHER & J. WENZELBURGER (2000): "Endogenous Random Asset Prices in Overlapping Generations Economies", *Mathematical Finance*, 10(1), 23–38.

CHIARELLA, C., R. DIECI & X. HE (2008): "On the dynamic behaviour of asset prices in disequilibrium: a survey", in *Dynamics and Evolution of Financial Markets*, ed. by T. Hens & K. R. Schenk-Hoppé, Handbook of Finance. Elsevier, in preparation.

COPELAND, T. E., J. F. WESTON & K. SHASTRI (2005): *Financial Theory and Corporate Policy.* Pearson Addison Wesley, Boston, 4th edn.

CUTHBERTSON, K. (1996): *Quantitative Financial Economics.* John Wiley & Sons, New York.

DANA, R. A. (1993): "Existence and uniqueness of equilibria when preferences are additively separable", *Econometrica*, 61(4), 953–957.

——— (1999): "Existence, Uniqueness and Determinacy of Equilibrium in CAPM with a Riskless Asset", *Journal of Mathematical Economics*, 32(2), 167–175.

FAMA, E. F. & M. H. MILLER (1972): *The Theory of Finance.* Holt, Rinehart and Winston, New York.

FREIXAS, X. & J.-C. ROCHET (1997): *Microeconomics of Banking.* MIT Press, Cambridge (Mass.), 5. edn.

HENS, T., J. LAITENBERGER & A. LÖFFLER (2002): "Two Remarks on the Uniqueness of Equilibria in the CAPM", *Journal of Mathematical Economics*, 37, 123–132.

HENS, T. & K. R. SCHENK-HOPPÉ (2008): *Dynamics and Evolution of Financial Markets.* Elsevier, in preparation.

LAJERI, F. & L. T. NIELSEN (2000): "Parametric Characterizations of Risk Aversion and Prudence", *Economic Theory*, 15(2), 469–476.

LeRoy, S. & J. Werner (2001): *Principles of Financial Economics.* Cambridge University Press, Cambridge, UK.

Lintner, J. (1965): "The Valuation of Risky Assets and Selection of Risky Investment in Stock Portfolios and Capital Budgets", *Review of Economics and Statistics*, 47, 13–37.

Markowitz, H. (1952): "Portfolio Selection", *Journal of Finance*, 7, 77–91.

Merton, R. C. (1972): "An analytic derivation of the efficient portfolio frontier", *Journal of Financial and Quantitative analysis*, 7(4), 1851–1872.

Mossin, J. (1966): "Equilibrium in a Capital Asset Market", *Econometrica*, 34, 268–290.

Nielsen, L. T. (1987): "Portfolio Selection in the Mean-Variance Model: A Note", *Journal of Finance*, 42(5), 1371–1376.

——— (1988): "Uniqueness of Equilibrium in the Classical Capital Asset Pricing Model", *Journal of Financial and Quantitative Analysis*, 23, 329–336.

——— (1990a): "Existence of Equilibrium in CAPM", *Journal of Economic Theory*, 52, 223–231.

——— (1990b): "Existence of Equilibrium in CAPM without a riskless asset", *Review of Economic Studies*, 57, 315–324.

Rochet, J.-C. (1992): "Capital requirements and the behaviour of commercial banks", *European Economic Review*, 36(5), 1137–1170.

Rockafellar, T. R. (1970): *Convex Analysis.* Princton University Press, Princton (NJ).

Sharpe, W. F. (1964): "Capital Asset Pricing: A Theory of Market Equilibrium Under Conditions of Risk", *Journal of Finance*, 19, 425–442.

Tobin, J. (1958): "Liquidity Preference as Behavior Towards Risk", *Review of Economic Studies*, 25, 65–86.

Wenzelburger, J. (2002): "Sequentielle Portefeuille-Entscheidungen auf Finanzmärkten", Mimeo, Bielefeld University.

——— (2008): "Perfect Forecasting, Behavioral Heterogeneities, and Asset Prices", in *Dynamics and Evolution of Financial Markets*, ed. by T. Hens & K. R. Schenk-Hoppé, Handbook of Finance. Elsevier, in preparation.