Price and quality competition in a mixed duopoly : Differential game approach

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30 August 2021

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Abstract

This paper investigates price and quality competition in a mixed duopoly market, where a state-owned welfare-maximizing public firm competes against a profit-maximizing private firm. We use a differential game approach with a Hotelling spatial competition framework. We extend Cellini et al. (2018) by incorporating a state-owned public firm and derive open- and closed-loop solutions. The steady-state quality levels are optimal in the open-loop solution. Numerical results show that the steady-state quality level of the public firm in the closed-loop solution does not necessarily lower than that in the open-loop solution. As a private firm's investment is large, the public firm's incentive for quality improvement increases since there exists intertemporal strategic substitutability between investment and quality. Competition and privatization policies are neutral under the open-loop solution but not under the closed-loop solution. Competition policy improves social welfare with an increase in quality and privatization policy improves it with an decrease in quality in the closed-loop solution.

JEL classification: H42 L13
Keywords : Mixed oligopoly, Privatization, Differential-game, Quality.

1 Introduction

Though many state-owned public firms have been privatized over the last decades, state-owned public firms still play essential roles in healthcare industries, for instance China, Japan, and Korea. In China and Korea, it is common for patients to choose treatments that are not covered by public insurance. In Japan, some advanced medical

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care is not covered by public insurance, for example, treatments of cancer and infertility. Providers are usually allowed to freely determine prices of treatments not covered by public insurance. Cellini et al. (2018) discuss price and quality competition between profit-maximizing firms in healthcare markets whose prices are not regulated in US and European countries by employing a differential game approach. The purpose of this paper is to discuss the price and quality competition in the mixed market by employing a differential game approach. In these countries, advances in medical technology have made it possible for people to receive higher medical care and has increased life expectancy. On the other hand, the provision of high-quality medical treatment and population aging entails an increase in health care costs. We also discuss reforms in the healthcare industries.

We extend Cellini et al. (2018), which employ a differential game approach with a Hotelling spatial competition framework to a mixed duopoly model. We use the Hotelling framework, whose line represents geographical distances between individuals and hospitals. We build the model as follows: a public firm maximizes the discounted stream of social welfare. In contrast, a private firm maximizes the discounted stream of profit. We use differential game approach, where quality provision requires investments. Treatment quality is a major concern in healthcare markets. Provision of high-quality medical care requires expensive medical equipment (e.g., MRI, CT) and the training of doctors. Recently, the increase in health care costs is a serious problem. The static model cannot analyze the effect of an increase in health care costs due to competition in quality among firms. We also investigate how competition and privatization policies affect social welfare.

We derive two kinds of solutions represented by different decision rules: the open-loop decision rule and the closed-loop decision rule. If the firms use open-loop decision rules, they can observe the initial state but cannot observe the evolution of state. They commit to optimal plans at the beginning of the game and stick to them over the time horizon. If firms use closed-loop decision rules, they can observe the evolution of state and their actions depend on the observed state. Therefore, there is a dynamic strategic interaction in the closed-loop solution. The firm's quality improvement affects its opponents' decisions. We compare the steady-state levels of qualities of the firms in two solution concepts.

When firms use open-loop decision rules, quality is optimally provided. The reason is that, in our model, there is no strategic interaction under the open-loop solution and the marginal profit of quality is equal to the marginal social welfare of quality as in standard spatial competition models. When firms use closed-loop decision rules, the numerical results indicate that the steady-state quality level of the public firm can be higher or lower than that in the open-loop solution. Two effects determine this relationship.

Firstly, the public firm's incentive for quality improvement decreases as its share decreases. The dynamic strategic interaction increases the private firm's incentive to increase quality in the closed-loop solution. The public firm's market share is lower as the private firm ' s quality level is higher. In this situation, the incentive of the public firm for quality improvement in the closed-loop solution is lower than in the open-loop solution.

Secondary, an increase of the public firm's quality level reduces the optimal quality investment of the private firm since there is intertemporal strategic substitutability between quality and investment in the model. Then, a higher investment level by the public firm has a trade-off between today's lower investment cost by the private firm, which is cost saving today, and future lower quality level by the private firm, which is future efficiency loss. Therefore, the steady-state quality of the public firm in the closed-loop solution is higher than that in the open-loop solution when the public firm's incentive to crowd out the private firm's investment is sufficiently large.

Ishibashi and Kaneko (2008) also consider quality and price competition in a static Hotelling model with partial privatization. They show that the first best is achieved in the simultaneous-move game. They also showed that the social-welfare maximizing public firm's quality level is always less than the first-best quality level in the sequential-move games, as in Matsumura and Matsushima (2004). However, we indicate that the quality of the social-welfare maximizing public firm in the closed-loop solution can be higher than that in the open-loop solution while quality is optimally provided in the open-loop solution. Cellini et al. (2018) show that steady-state quality is increasing in the degree of competition in the closed-loop solution. This result also holds in our model. Furthermore, numerical analysis indicates that the difference between steady-state qualities is increasing in the degree of competition and competition policy improves social welfare in the mixed duopoly. It is in contrast to the numerical result that privatization policy improves social welfare by decreasing quality in the closed-loop solution.

Our paper is related to the literature that deals with R&D activities in a mixed oligopoly. Laine and Ma (2017) and Kuo et al. (2019) also investigate price and quality competition in the mixed duopoly market within the Hotelling framework. Sanjo (2009) investigates quality competition under price regulation by employing the Hotelling framework. A vast amount of literature investigates cost-reducing activities in the mixed oligopoly. Nishimori and Ogawa (2002) show that the entry of private firms reduces the public firm's incentive to undertake cost-reducing investment. Matsumura and Matsushima (2004) obtain that private firm engages in excessive strategic cost-reducing activities than the public firm. Heywood (2009) shows that an R&D rivalry reduces the optimal degree of privatization. Buehler and Wey (2014) show that a state-owned firm crowds out investment by a private firm under several conditions. On the other hand, there are a few pieces of literature studying mixed markets employing a differential game approach.[1]

[1] Futagami et al. (2019) employ a differential game approach to investigate optimal privatization of a dynamic mixed oligopoly model.

Our paper is also related to the literature that deals with hospital quality competition in a Hotelling framework with a differential game approach. Brekke et al. (2010) and Cellini et al. (2018) set quality as a stock variable. Brekke et al. (2010) find that quality levels are equal under open- and closed-loop solutions when the marginal cost is constant under price regulation. On the other hand, Cellini et al. (2018) shows that quality level is lower under the closed-loop solution than under the open-loop solution in an unregulated market since there is a strategic incentive to reduce quality investments to dampen future price competition. Brekke et al. (2012) and Siciliani et al. (2013) set demands as stock variables. They assume that the demands move sluggishly over time. Siciliani et al. (2013) investigate quality competition by introducing a motivated provider that maximizes a weighted sum of total utility of the consumers the provider serve and its profit, and find that a decrease of the number of consumers the regulator serve reduces provider's incentive for quality improvement. By taking quality as a state variable, Bisceglia et al. (2019) also investigate quality competition where regional regulators consider surpluses of their residents. In this paper, we introduce a firm who considers not only its profit and utilities of consumers but also its competitor's profit. We find that sufficient large investment by the private firm increases the social-welfare maximizing public firm's incentive for quality improvement. If there exists dynamic strategic interaction between quality and quality investment, the incentive of the provider, who considers its competitor's profit, is affected by not only its share but also the trade-off between the competitor's investment cost and the competitor's future quality level.

The rest of the paper is organized as follows. In section 2, we present the model. In section 3 and section 4, we derive the open-loop solution and the closed-loop solution. Section 5 discuss steady-state qualities and the competition and privatization policies by using numerical analysis. Section 6 concludes the paper.

2 Model

Consider the line segment $[0, 1]$, where firm 0 is located at 0 and firm 1 at 1. The locations of firms are fixed. Let firm 0 be a fully state-owned public firm and firm 1 be a private firm. The product of firm i has quality q_i ($i = 0, 1$). A unit mass of consumers is distributed uniformly on the line segment. Each consumer demands one unit of product or service. The utility of a consumer located at $x \in [0, 1]$ who will buy from firm i, located at $z_i \in \{0, 1\}$, is given by

$$U(x, z_i) = v + kq_i - \tau|x - z_i| - p_i, \tag{1}$$

where p_i is the price of firm i, $v > 0$ is the gross valuation of consumption, $k > 0$ is the marginal willingness to pay for quality, and $\tau > 0$ is the marginal transportation cost.

The consumer who is indifferent between buying from firm 0 and from firm 1 is located at

$$x_0^D = \max\left[0, \min\left[\frac{1}{2} + \frac{k(q_0 - q_1)}{2\tau} - \frac{p_0 - p_1}{2\tau}, 1\right]\right]. \tag{2}$$

Therefore, the demand for firm 0 is x_0^D and that for firm 1 is $1 - x_0^D$.

We assume that firms can not adjust quality instantaneously. Quality improvement requires investment. Firms invest in improving their quality at each point in time $t \in [0, \infty)$. Assuming time is continuous, the law of motion of quality is given by

$$\dot{q}_i(t) = I_i(t) - \delta q_i(t), \tag{3}$$

where $I_i(t)$ is the investment in quality at time t and quality depreciates at a constant rate, $\delta > 0$.

We assume that firms have identical production technologies. The cost function of firm i is given by

$$C(x_i^D(t), I_i(t), q_i(t)) = cx_i^D(t) + \frac{1}{2}(\gamma I_i(t)^2 + \beta q_i(t)^2), \tag{4}$$

where $c > 0$ is a constant marginal cost of production and γ and β are positive constants. We assume that the cost function is increasing and strictly convex in quality and investment. Expensive materials and skilled workers are required to achieve a higher quality level. Besides, a higher quality level requires higher maintenance costs. Profit of firm i is

$$\begin{aligned} &\pi_i(p_0(t), p_1(t), I_i(t), q_0(t), q_1(t)) = \\ &\qquad (p_i(t) - c)x_i(p_0(t), p_1(t), q_0(t), q_1(t)) - \frac{\gamma}{2}I_i(t)^2 - \frac{\beta}{2}q_i(t)^2. \end{aligned} \tag{5}$$

The consumer surplus at each point in time is

$$\begin{aligned} CS &\quad (p_0(t), p_1(t), q_0(t), q_1(t)) \\ &= \int_0^{x_0^D(t)} (v + kq_0(t) - p_0(t) - \tau z)dz + \int_{x_0^D(t)}^1 (v + kq_1(t) - p_1(t) - \tau(1 - z))dz, \\ &= v + (kq_0(t) - p_0(t))x_0^D(t) + (kq_1(t) - p_1(t))(1 - x_0^D(t)) \\ &\qquad - \frac{\tau}{2}\{x_0^D(t)^2 + (1 - x_0^D(t))^2\}. \end{aligned} \tag{6}$$

The producer surplus at each point in time is

$$\begin{aligned} &PS(p_0(t), p_1(t), I_0(t), I_1(t), q_0(t), q_1(t),) \\ &\quad = \pi_0(p_0(t), p_1(t), I_0(t), q_0(t), q_1(t)) + \pi_1(p_0(t), p_1(t), I_1(t), q_0(t), q_1(t)). \end{aligned} \tag{7}$$

Therefore, the social welfare at each point in time is

$$\begin{aligned}
&SW(I_0(t), I_1(t), q_0(t), q_1(t)) \\
&\quad = CS(p_0(t), p_1(t), q_0(t), q_1(t)) + PS(p_0(t), p_1(t), I_0(t), I_1(t), q_0(t), q_1(t)), \\
&\quad = v - c + kq_0(t)x_0^D(t) + kq_1(t)(1 - x_0^D(t)) \\
&\qquad -\frac{\tau}{2}\left\{x_0^D(t)^2 + (1 - x_0^D(t))^2\right\} - \frac{\gamma}{2}(I_0(t)^2 + I_1(t)^2) - \frac{\beta}{2}(q_0(t)^2 + q_1(t)^2).
\end{aligned} \tag{8}$$

The objective function of firm 0 over the infinite time horizon is

$$\int_0^\infty SW(I_0(t), I_1(t), q_0(t), q_1(t)) \ \mathrm{e}^{-\rho t} dt, \tag{9}$$

where ρ is the constant discount rate, The objective function of firm 1 over the infinite time horizon is

$$\int_0^\infty \pi_1(p_0(t), p_1(t), I_t(t), q_0(t), q_1(t)) \ \mathrm{e}^{-\rho t} dt. \tag{10}$$

From equations (9) and (10), we can see that the objective function of firm 0 depends on both I_0 and I_1, while that of firm 1 depends only on I_1.

We derive the open-loop solution and the closed-loop solution, which have different information sets available to firms. If firms use open-loop decision rules, firms can observe the initial state but cannot observe the evolution of state. Firms have to decide the optimal plan at the beginning of the game and do not revise it over the time horizon. Hence, the open-loop solution is weakly time consistent.[2] If firms use closed-loop decision rules, firms observe the evolution of state over time. Firms choose price and quality depend on the current state. Hence, the closed-loop solution is time consistent.

3 Open-loop solution

The maximization problem for firm 0 is

$$\max_{I_0(t), p_0(t)} \int_0^\infty SW(t) \ e^{-\rho t} dt,$$

$$\begin{aligned}
\text{subject to} \quad & \dot{q_0}(t) = I_0(t) - \delta q_0(t), \\
& \dot{q_1}(t) = I_1(t) - \delta q_1(t), \\
& q_0(0) = q_{00} > 0, \\
& q_1(0) = q_{10} > 0.
\end{aligned}$$

[2]The open-loop solution is weakly time consistent if optimal rules do not change when firms reconsider their rules on the equilibrium trajectory. That is time consistent if it holds not only on the equilibrium trajectory but also off the trajectory.

The maximization problem for firm 1 is

$$\max_{I_1(t),p_1(t)} \int_0^\infty \pi_1(t)\ e^{-\rho t} dt,$$

$$\begin{aligned}\text{subject to}\quad \dot{q_0}(t) &= I_0(t) - \delta q_0(t),\\ \dot{q_1}(t) &= I_1(t) - \delta q_1(t),\\ q_0(0) &= q_{00} > 0,\\ q_1(0) &= q_{10} > 0.\end{aligned}$$

Solving maximization problems, we have the following results. In the rest of this paper, we omit time argument t for notational simplicity. Firms set equal equilibrium prices as follows:

$$p_0 = c + \tau - k(q_0 - q_1), \tag{11}$$

$$p_1 = c + \tau + k(q_1 - q_0). \tag{12}$$

From (11) and (12), we can easily obtain that $(p_0 - kq_0) - (p_1 - kq_1) > 0$ iff $q_1 > q_0$. Hence, firm 0 is less aggressive if its quality is lower than that of firm 1.

The following equation and state equations (3) describe the system.

$$\dot{I}_i = (\rho + \delta)I_i + \frac{\beta}{\gamma}q_i - \frac{k}{2\tau\gamma}\{\tau + k(q_i - q_j)\}, \quad i = 1, 2,\ i \neq j. \tag{13}$$

This is symmetry across the firms in spite of the difference in their purposes. The steady-state quality level in the open-loop solution is

$$q^{OL} = \frac{k}{2(\beta + \gamma\delta(\delta + \rho))}. \tag{14}$$

We here assume $\beta\tau - k^2 > 0$ to ensure the existence of the open-loop solution. Details of derivations are described in Appendix A. From (14) and Appendix B, we can see that the steady-state quality level in the open-loop solution coincides with the first-best steady state quality level.

Proposition 1 *If the firms use open-loop decision rules, the steady-state quality levels are optimal.*

The reason is that the private firm's marginal profit of quality is equal to the marginal social welfare of quality in the model, that is, $\frac{\partial SW(q_0,q_1)}{\partial q_1} = \frac{\partial \pi_1(q_0,q_1)}{\partial q_1} = kx_1$. An increase in q_1 increases aggregate utilities of consumers who choose firm 1 by kx_1. It also increases aggregate utility of consumers who choose firm 1 by inducing new demand. The second effect is cancelled out by a decrease in aggregate utility of consumers who shift from firm 0 to firm 1 due to increase in q_1 since the total demand is fixed. Hence, $\frac{\partial SW(q_0,q_1)}{\partial q_1} = kx_1$ holds.

An increase in q_1 directly increases firm 1's demand and indirectly affects it through prices. The second effects vanish and $\frac{\partial \pi_1(q_0,q_1)}{\partial q_1} = kx_1$ holds since the demand function is linear and symmetric in prices. Cellini et al. (2018) assume two symmetric profit maximising firms and states that quality is optimally provided when they use open-loop decision rules. It is obvious that the same result holds for the mixed duopoly.

4 Closed-loop solution

In this section, we derive the closed-loop solution. Firms' price and investment decisions depend on qualities at each point in time, when they use closed-loop decision rules. We use the value function approach to derive the closed-loop solution. From (9) and (10), the Hamilton-Jacobi-Bellman (HJB) equation of firm 0 is given by

$$\rho V^0(q_0,q_1) = \max\left\{ SW + V^0_{q_0}(q_0,q_1)(I_0 - \delta q_0) + V^0_{q_1}(q_0,q_1)(I_1 - \delta q_1) \right\}, \tag{15}$$

and that of firm 1 is given by

$$\rho V^1(q_0,q_1) = \max\left\{ \pi_1 + V^1_{q_1}(q_0,q_1)(I_1 - \delta q_1) + V^1_{q_0}(q_0,q_1)(I_0 - \delta q_0) \right\}, \tag{16}$$

where $V^i(q_0,q_1)$ is firm i's value function and π_i and SW are given by (5) and (8), respectively. Since the game can be characterized as a linear quadratic game,[3] we guess the value functions as

$$V^0(q_0,q_1) = \alpha_0 + \alpha_1 q_0 + \alpha_2 q_1 + \frac{\alpha_3}{2}q_0^2 + \frac{\alpha_4}{2}q_1^2 + \alpha_5 q_0 q_1, \tag{17}$$

$$V^1(q_0,q_1) = \varepsilon_0 + \varepsilon_1 q_1 + \varepsilon_2 q_0 + \frac{\varepsilon_3}{2}q_1^2 + \frac{\varepsilon_4}{2}q_0^2 + \varepsilon_5 q_0 q_1, \tag{18}$$

where α_i and ε_i $(i = 0,1,2,3,4,5)$ are unknown constant parameters.

Maximization of the right-hand sides of (15) and (16), respectively, yields the optimal pricing rules as

$$p_0 = c + \tau - k(q_0 - q_1), \tag{19}$$

$$p_1 = c + \tau + k(q_1 - q_0). \tag{20}$$

Equation (20) indicates that, for all else equal, firm 1 reacts to an increase in q_0 by reducing its price to compensate for its demand loss. Equation (19) indicates that, for all else equal, firm 0 reacts to an increase in q_1 by raising its price. The reason is that

[3]See Dockner et al. (2000, ch.7).

it is desirable for the public firm, which is concerned with the social welfare, to increase the market share with a higher quality level. As a result, firm 0 is less aggressive if its quality is lower than that of firm 1, as mentioned in Section 3.

Maximization of the right-hand sides of (15) and (16) gives quality investment rules as

$$
\begin{aligned}
I_0 = \phi^0(q_0, q_1) &= \frac{\alpha_1 + \alpha_3 q_0 + \alpha_5 q_1}{\gamma}, && (21)\\
I_1 = \phi^1(q_0, q_1) &= \frac{\varepsilon_1 + \varepsilon_3 q_1 + \varepsilon_5 q_0}{\gamma}. && (22)
\end{aligned}
$$

If α_5 is positive (negative), there exists an intertemporal strategic complementarity (substitutability) between q_1 and I_0 (See Jun and Vives (2004)). The higher q_1 is, the higher (lower) the optimal quality investment level of firm 0 is. The same is true for ε_5.

Substituting (17), (18), (19), (20), (21), and (22) into (15) and (16) and collecting terms with equal powers of q_i result in simultaneous equations shown in Appendix C. Solutions of the simultaneous equations satisfying the following conditions constitute a globally asymptotically stable equilibrium.[4]

$$
\begin{aligned}
(\alpha_3 - \gamma\delta) + (\varepsilon_3 - \gamma\delta) &< 0, && (23)\\
(\alpha_3 - \gamma\delta)(\varepsilon_3 - \gamma\delta) - \alpha_5\varepsilon_5 &> 0, && (24)\\
(\alpha_3 - \varepsilon_3)^2 + 4\alpha_5\varepsilon_5 &> 0. && (25)
\end{aligned}
$$

Derivation of the above conditions are given in Appendix D. There is a unique globally asymptotically stable closed-loop solution, as shown in Appendix E.

5 Numerical Analysis

In this section, we rely on numerical analysis to derive the closed-loop solution. The parameter set is as follows: $\rho = 0.04$, $\delta = 0.05$, $c = 0$, $k = 1$, $v = 10$, $\beta = 5$. We consider two cases: $\gamma = 8$ and $\gamma = 24$. We derive the steady-state values in qualities, investments, prices, profits, the demand for firm 0, the consumer surplus, the producer surplus, and social welfare for $\tau = 3$ and $\tau = 7$. The numerical results are shown in Table 3, Table 1, Table 2, and Table 4. The steady-state levels of quality in the open-loop solution on Table 1 are obtained by substituting the parameters into equation (14). Each parameter set in this section satisfies the restriction, $\beta\tau - k^2 > 0$, derived in Section 3. The steady-state levels of quality in the pure duopoly in the closed-loop solution on Table 4 are obtained by substituting the parameters into outcomes derived by Cellini et al. (2018).

[4]We assume $-2\alpha_5\varepsilon_5 + (1-\theta)\varepsilon_5^2 + (\gamma s)^2 + \gamma(\beta - 2fg) > 0$ and $-2\alpha_5\varepsilon_5 + (\gamma s)^2 + \gamma(\beta - 2hg) > 0$ to ensure the solutions are real values.

5.1 Steady-state quality

The numerical results show that the public firm's steady-state quality level in the closed-loop solution can be larger than in the open-loop solution depending on parameters. Table 1 and Table 2 show that q_0^{CL} is smaller than q^{OL}. However, in the case of $\gamma = 8$ and $\tau = 3$, q_0^{CL} is smaller than q^{OL}. In this case, the tables show that $q_0^{CL} = 0.09932$ and $q^{OL} = 0.09929$. There is a question why the public firm chooses overinvestment in the case of $\gamma = 8$ and $\tau = 3$ while it chooses underinvestment in other cases when firms use closed-loop decision rules. It can be explained by dynamic strategic interaction between firms.

(γ,τ)	q_0	q_1	I_0	I_1	p_0	p_1	π_0	π_1	x_0	CS	PS	SW
(8,3)	0.09929	0.09929	0.00496	0.00496	3	3	1.47526	1.47526	0.5	6.34929	2.95052	9.29980
(8,7)	0.09929	0.09929	0.00496	0.00496	7	7	3.47526	3.47526	0.5	1.34929	6.95052	8.29980
(24,3)	0.09789	0.09789	0.00489	0.00489	3	3	1.47576	1.47576	0.5	6.34789	2.95152	9.29940
(24,7)	0.09789	0.09789	0.00489	0.00489	7	7	3.47576	3.47576	0.5	1.34789	6.95152	8.29940

Table 1: Steady state values in the open-loop solution

(γ,τ)	q_0	q_1	I_0	I_1	p_0	p_1	π_0	π_1	x_0	CS	PS	SW
(8,3)	0.09932	0.20925	0.00497	0.01046	3.10993	3.10993	1.47323	1.50204	0.48168	6.29536	2.97527	9.27063
(8,7)	0.09927	0.20292	0.00496	0.01015	7.10365	7.10365	3.47450	3.50107	0.49260	1.29783	6.97556	8.27339
(24,3)	0.09785	0.20607	0.00489	0.01030	3.10823	3.10823	1.47383	1.50274	0.48196	6.29471	2.97657	9.27127
(24,7)	0.09784	0.19997	0.00489	0.01000	7.10213	7.10213	3.47504	3.50170	0.49271	1.29715	6.97674	8.27389

Table 2: Steady state in the closed-loop solution (Mixed duopoly)

Table 1 and Table 2 show that the private firm's steady-state quality is higher in the closed-loop solution than in the open-loop solution.[5] When firms use closed-loop decision rules, firms react to changes in their competitors' quality stock. The public firm reacts to an increase in q_1 by raising its price, p_0 as shown in Equation (19). This means that the higher quality investments by the private firm today will lead to weaker

[5] Cellini et al. (2018) consider the case where firm 0 in this model is a profit-maximizing firm. Firm 0, the profit-maximizing firm, reacts to an increase in q_1 by raising its price, p_0. Hence, the private firm's steady-state quality is lower in the closed-loop solution than in the open-loop solution. Brekke et al. (2010) analyzes the case where prices are regulated in Cellini et al. (2018) and shows that the open-loop and closed-loop solutions coincide. In these papers, the results obtained by dynamic models are similar to the ones obtained by static models, contrasting with our results.

price competition in the future. Since weak price competition increases the private firm's profit, this dynamic strategic interaction increases the private firm's incentive for quality improvement compared to that in the open-loop solution.

The private firm's overinvestment in quality reduces the public firm's incentive for quality improvement. It is well known that the incentive of the public firm for quality improvement is increasing in its share.[6] As the number of consumers who enjoy the benefit from the quality improvement of firm 0 decreases, the marginal social benefit by increasing q_0 is smaller. The public firm's market share is lower as the private firm's quality level is higher, as shown in Ishibashi and Kaneko (2008).

(γ, τ)	I_0	I_1
(8, 3)	0.62635 -5.68352 q_0 -0.10578 q_1	1.27120 -5.57196 q_1 -0.21734 q_0
(8,7)	0.61908 -5.74415 q_0 -0.04515 q_1	1.24646 -5.69799 q_1 -0.09130 q_0
(24,3)	1.05709 -9.22074 q_0 -0.18179 q_1	2.14450 -9.02919 q_1 -0.37334 q_0
(24,7)	1.04530 -9.32493 q_0 -0.07760 q_1	2.10416 -9.24563 q_1 -0.15689 q_0

Table 3: Optimal investment rules in the closed-loop solution

The public firm's incentive for quality improvement may be increased by dynamic strategic interaction between quality and investment. The numerical results show that there is intertemporal strategic substitutability. The coefficient parameters, α_5 and ε_5, are negative from Table 3. That is, an increase of the social welfare-maximizing public firm's quality level provokes a reduction of the private firm's quality investment. By this intertemporal strategic substitutability, the public firm faces a trade-off between reducing the private firm's investment cost and encouraging the private firm's quality improvement. Suppose that the private firm reacts to an increase in q_0 by reducing its quality investment. This reduction results in the lower investment cost of the private firm today and the lower quality level of the private firm in the future. The lower investment cost is instantaneous gain, and the lower future's quality level is future loss from the viewpoint of social welfare maximization. If the investment cost of the private firm is sufficiently large, the former effect dominates the later effect. Therefore, the social welfare-maximizing public firm can have an incentive for quality investment to crowd out the private firm's quality investment.

The public firm's steady-state quality level in the closed-loop solution can be larger than in the open-loop solution since there exists intertemporal strategic substitutability. All else equal, the private firm has a stronger incentive for quality improvement as γ and τ is small.[7] The higher private firm's quality level reduces the social welfare-maximizing

[6] See Nishimori and Ogawa (2002), Matsumura and Matsushima (2004), and Ishibashi and Kaneko (2008).

[7] When γ is close to zero, the constraint conditions do not hold.

public firm's share. At the same time, the higher private firm's investment increases the social welfare-maximizing public firm's incentive for crowding out the private firm's investment. If the latter effect dominates the former effect, the public firm's steady-state quality level in the closed-loop solution is larger than in the open-loop solution. Ishibashi and Kaneko (2008) show that the first best is achieved in the one-shot game simultaneous moves. This result is related to the outcome of the open-loop solution in this paper. However, our outcome in the closed-loop solution contrasts to the results in Ishibashi and Kaneko (2008) who indicate that the social welfare-maximizing public firm's quality level is lower than the optimal level in the one-shot game with sequential moves.

5.2 Policies

We investigate how competition and privatization policies affect social welfare in the closed-loop solution. Table 2 show that steady-state quality levels are increasing in the degree of competition. This result is consistent with Cellini et al. (2018), who show that steady-state quality is increasing in the degree of competition in the pure duopoly. Furthermore, we indicate that the difference between steady-state qualities is increasing in the degree of competition in the mixed duopoly. Table 2 shows that $q_1 - q_0$ is 0.10365 in the case of $\tau = 7$ and 0.10992 in the case of $\tau = 3$ when $\gamma = 8$. It also shows that x_0 is 0.10213 in the case of $\tau = 7$ and 0.10822 in the case of $\tau = 3$ when $\gamma = 24$. Regardless of these results, competitive policy increases social welfare since the market share of public firm decreases.

(γ, τ)	q_i	I_i	p_i	π_i	x_0	CS	PS	SW
(8,3)	0.06643	0.00332	3	1.48892	0.5	6.31643	2.97785	9.29428
(8,7)	0.06629	0.00331	7	3.48897	0.5	1.31629	6.97794	8.29423
(24,3)	0.06549	0.00327	3	1.48915	0.5	6.31549	2.97830	9.29379
(24,7)	0.06535	0.00327	7	3.48919	0.5	1.31535	6.97839	8.29374

Table 4: Steady state in the closed-loop solution (Pure duopoly)

We also find that privatization of the public firm improves steady-state social welfare. Table 2 show that social welfare in the mixed duopoly is 9.27063 and Table 4 show that that in the pure duopoly which is 9.29428 in the case of $\gamma = 8$ and $\tau = 3$. Hence, privatization improves social welfare. The same results hold for other parameter sets. Steady-state quality levels in the closed-loop solution is less than that in the open-loop solution as shown in Table 1. This implies that privatization reduced costs. Qualities is optimally provided and these policies do not affect social welfare in the open-loop solution. However, competition policy improves social welfare by increasing quality and privatization policy improves it by decreasing quality in the closed-loop solution.

6 Conclusion

This study investigates price and quality competition in the mixed duopoly in which firms need investment to improve their qualities. We extend Cellini et al. (2018) to a mixed duopoly model by considering that one of the firms is the public firm concerns with social welfare. If the firms use open-loop decision rules, quality provision is socially optimal. However, if the firms use closed-loop decision rules, the steady-state quality level of the social welfare-maximizing public firm may be higher than the first-best steady-state quality level. If there is intertemporal strategic substitutability, the social welfare-maximizing public firm's steady-state quality level in the closed-loop solution can be higher than that in the open-loop solution. Ishibashi and Kaneko (2008) showed that the social-welfare maximizing public firm's quality level is always less than the first-best quality level in sequential-move games by using the static Hotelling model. Finally, the competition and privatization policies improve social welfare. Our model does not consider partial privatization developed by Matsumura (1998). Solving differential games with the privatization rate of the public firm as a control variable remains as a matter to be discussed further.

Appendices

A Derivation of the open-loop solution

A.1. Derivation of the steady-state quality

The current-value Hamiltonian for firm 0 is

$$\begin{aligned} H^0 = CS + (p_0 - c)\left\{\frac{1}{2} + \frac{k(q_0 - q_1)}{2\tau} - \frac{p_0 - p_1}{2\tau}\right\} - \frac{\gamma}{2}I_0^2 - \frac{\beta}{2}q_0^2 \\ + (p_1 - c)\left\{\frac{1}{2} + \frac{k(q_1 - q_0)}{2\tau} - \frac{p_1 - p_0}{2\tau}\right\} - \frac{\gamma}{2}I_1^2 - \frac{\beta}{2}q_1^2 + \mu_0^0(I_0 - \delta q_0) + \mu_0^1(I_1 - \delta q_1), \end{aligned} \tag{A-1}$$

where μ_0^0 and μ_0^1 are the current value co-state variables associated with the two state equations and CS is given by (6). With the help of $\frac{\partial CS}{\partial p_0} = -x_0^D$, $\frac{\partial CS}{\partial q_0} = kx_0^D$, and

$\frac{\partial CS}{\partial q_1} = k(1 - x_0^D)$, we can easily obtain the first-order conditions as follows:

$$\mu_0^0 = \gamma I_0, \tag{A-2}$$

$$p_1 - p_0 = 0, \tag{A-3}$$

$$\dot{\mu}_0^0 = (\rho + \delta)\mu_0^0 + \beta q_0 - k\left\{\frac{1}{2} + \frac{k}{2\tau}(q_0 - q_1)\right\}, \tag{A-4}$$

$$\dot{\mu}_0^1 = (\rho + \delta)\mu_0^1 + \beta q_1 - k\left\{\frac{1}{2} + \frac{k}{2\tau}(q_1 - q_0)\right\}. \tag{A-5}$$

and the transversality conditions are $\lim_{t\to\infty} e^{-\rho t}\mu_0^i(t)q_i(t) = 0, i = 0, 1$.

The current-value Hamiltonian for firm 1 is

$$H^1 = (p_1 - c)\left\{\frac{1}{2} + \frac{k(q_1 - q_0)}{2\tau} - \frac{p_1 - p_0}{2\tau}\right\} - \frac{\gamma}{2}I_1^2 - \frac{\beta}{2}q_1^2 + \mu_1^1(I_1 - \delta q_1) + \mu_1^0(I_0 - \delta q_0), \tag{A-6}$$

where μ_1^1 and μ_1^0 are the current value co-state variables associated with the two state equations. The first-order conditions are

$$\mu_1^1 = \gamma I_1, \tag{A-7}$$

$$\frac{1}{2} + \frac{k(q_1 - q_0)}{2\tau} - \frac{p_1 - p_0}{2\tau} - \frac{p_1 - c}{2\tau} = 0, \tag{A-8}$$

$$\dot{\mu}_1^1 = (\rho + \delta)\mu_1^1 + \beta q_1 - \frac{k}{2\tau}(p_1 - c), \tag{A-9}$$

$$\dot{\mu}_1^0 = (\rho + \delta)\mu_1^0 + \frac{k}{2\tau}(p_1 - c), \tag{A-10}$$

and the transversality conditions are $\lim_{t\to\infty} e^{-\rho t}\mu_1^i(t)q_i(t) = 0, i = 0, 1$.

Equations (A-3) and (A-8) yield the equilibrium prices as

$$p_0 = c + \tau - k(q_0 - q_1), \tag{A-11}$$

$$p_1 = c + \tau + k(q_1 - q_0). \tag{A-12}$$

Combining (A-2), (A-4), (A-7), and (A-9), we obtain

$$\dot{I}_i = (\rho + \delta)I_i + \frac{\beta}{\gamma}q_i - \frac{k}{2\tau\gamma}\left\{\tau + \frac{1}{2\theta + 1}k(q_i - q_j)\right\}, \quad i = 1, 2,\ i \neq j. \tag{A-13}$$

Imposing $\dot{q}_i = \dot{I}_i = 0$, we get the steady-state quality in the open-loop solution as

$$q^{OL} = \frac{k}{2(\beta + \gamma\delta(\delta + \rho))}. \tag{A-14}$$

A.2. Concavity of Hamiltonians

Differentiating (A-1) yields the following:

$$H^0_{I_0 I_0} = -\gamma < 0, \tag{A-15}$$

$$H^0_{p_0 p_0} = -\frac{1}{2\tau} < 0, \tag{A-16}$$

$$H^0_{q_0 q_0} = -\beta + \frac{k^2}{2\tau} < 0, \tag{A-17}$$

$$H^0_{I_0 I_0} H^0_{p_0 p_0} - (H^0_{I_0 p_0})^2 = \frac{\gamma}{2\tau} > 0, \tag{A-18}$$

$$H^0_{p_0 p_0} H^0_{q_0 q_0} - (H^0_{p_0 q_0})^2 = \frac{1}{4\tau^2}[2\beta\tau - k^2], \tag{A-19}$$

$$H^0_{q_0 q_0} H^0_{I_0 I_0} - (H^0_{q_0 I_0})^2 = \gamma[\beta - \frac{k^2}{2\tau}]. \tag{A-20}$$

The Hessian of Hamiltonian, H^0, is negative semidefinite if equations (A-19) and (A-20) are positive. Therefore, Hamiltonian, H^0 is concave in I_0, p_0, and q_0 if $2\beta\tau - k^2 > 0$. With a similar calculation, we find that Hamiltonian, H^1, is concave in I_1, p_1, and q_1 if $4\beta\tau - k^2 > 0$.

A.3. Uniqueness and Stability for the steady state

The open-loop system is

$$\begin{pmatrix} \dot{I}_0(t) \\ \dot{q}_0(t) \\ \dot{I}_1(t) \\ \dot{q}_1(t) \end{pmatrix} = \begin{pmatrix} \rho+\delta & \frac{\beta}{\gamma}-\phi & 0 & \phi \\ 1 & -\delta & 0 & 0 \\ 0 & \phi & \rho+\delta & \frac{\beta}{\gamma}-\phi \\ 0 & 0 & 1 & -\delta \end{pmatrix} \begin{pmatrix} I_0(t) \\ q_0(t) \\ I_1(t) \\ q_1(t) \end{pmatrix} + \begin{pmatrix} -\frac{k}{2\gamma} \\ 0 \\ -\frac{k}{2\gamma} \\ 0 \end{pmatrix}, \tag{A-21}$$

where $\phi \equiv \frac{k^2}{2\tau\gamma}$. Letting λ be eigenvalues of the coefficient matrix, the characteristic equation is

$$\left\{(\rho+\delta-\lambda)(\delta+\lambda) + \frac{\beta}{\gamma}\right\}\left\{(\rho+\delta-\lambda)(\delta+\lambda) + \frac{\beta}{\gamma} - 2\phi\right\} = 0, \tag{A-22}$$

and its roots are

$$\lambda = \frac{\rho}{2} \pm \sqrt{\frac{\rho^2}{4} + (\rho+\delta)\delta + \frac{\beta}{\gamma}}, \quad \frac{\rho}{2} \pm \sqrt{\frac{\rho^2}{4} + (\rho+\delta)\delta + \left(\frac{\beta}{\gamma} - 2\phi\right)}. \tag{A-23}$$

Hence, the matrix has two positive and two negative real roots only if $\frac{\beta}{\gamma} - 2\phi > 0$. This implies that there exists a unique saddle path converging to the steady-state point if $\beta\tau - k^2 > 0$.

B Derivation of the first-best steady-state quality

In this Appendix, we derive the first-best steady-state quality. The social planner maximizes the social welfare given by (8) by choosing the market share and each firm's quality investment. The maximization problem for the social planner is given by

$$\max_{I_0, I_1, x_0^D} \int_0^\infty SW(t) \quad e^{-\rho t} dt,$$

$$\begin{aligned} \text{subject to} \quad & \dot{q_0}(t) = I_0(t) - \delta q_0(t), \\ & \dot{q_1}(t) = I_1(t) - \delta q_1(t), \\ \\ & q_0(0) = q_{00} > 0, \\ & q_1(0) = q_{10} > 0. \end{aligned}$$

The current-value Hamiltonian is

$$H = v - c + kq_0x_0^D + kq_1(1 - x_0^D) - \frac{\tau}{2}\{x_0^{D2} + (1 - x_0^D)^2\} - \frac{\gamma}{2}(I_0^2 + I_1^2) - \frac{\beta}{2}(q_0^2 + q_1^2) + \mu_0(I_0 - \delta q_0) + \mu_1(I_1 - \delta q_1), \tag{B-1}$$

where μ_0 and μ_1 are the current value co-state variables associated with the state equations. The first-order conditions are

$$k(q_0 - q_1) - 2\tau x_0^D + \tau = 0, \tag{B-2}$$

$$\mu_i = \gamma I_i, \tag{B-3}$$

$$\dot{\mu_i} = (\rho + \delta)\mu_i + \beta q_i - kx_i, \tag{B-4}$$

and the transversality conditions are $\lim_{t\to\infty} e^{-\rho t}\mu_i(t)q_i(t) = 0$, $i = 0, 1$. Because of symmetry across firms, the first-best steady-state quality, q^*, is

$$q^* = \frac{k}{2(\beta + \gamma\delta(\delta + \rho))}. \tag{B-5}$$

C First order conditions of Hamiltonian-Jacobi-Bellan equation

Hamilton-Jacobi-Bellman (HJB) equations of firms are

$$\rho V^0(q_0, q_1) = \max\left\{SW + V^0_{q_0}(q_0, q_1)(I_0 - \delta q_0) + V^0_{q_1}(q_0, q_1)(I_1 - \delta q_1)\right\}, \tag{C-1}$$

$$\rho V^1(q_0, q_1) = \max\left\{\pi_1 + V^1_{q_1}(q_0, q_1)(I_1 - \delta q_1) + V^1_{q_0}(q_0, q_1)(I_0 - \delta q_0)\right\}. \tag{C-2}$$

Substituting (2), (5), and (8) into (C-1) and (C-2), we have

$$\rho V^0(q_0,q_1) = \max\left\{ CS + (p_0 - c)\left(\frac{1}{2} + \frac{k(q_0-q_1)}{2\tau} - \frac{p_0-p_1}{2\tau}\right) - \frac{\gamma}{2}I_0^2 - \frac{\beta}{2}q_0^2 \right.$$
$$+(p_1 - c)\left(\frac{1}{2} + \frac{k(q_1-q_0)}{2\tau} - \frac{p_1-p_0}{2\tau}\right) - \frac{\gamma}{2}I_1^2 - \frac{\beta}{2}q_1^2$$
$$\left. +V^0_{q_0}(q_0,q_1)(I_0 - \delta q_0) + V^0_{q_1}(q_0,q_1)(I_1 - \delta q_1) \right\}, \quad \text{(C-3)}$$

$$\rho V^1(q_0,q_1) = \max\left\{ (p_1 - c)\left(\frac{1}{2} + \frac{k(q_1-q_0)}{2\tau} - \frac{p_1-p_0}{2\tau}\right) - \frac{\gamma}{2}I_1^2 - \frac{\beta}{2}q_1^2 \right.$$
$$\left. + V^1_{q_1}(q_0,q_1)(I_1 - \delta q_1) + V^1_{q_0}(q_0,q_1)(I_0 - \delta q_0) \right\}. \quad \text{(C-4)}$$

where CS is given by (6). Maximization of the right-hand side of (C-3) with respect to p_0 yields

$$\frac{p_1 - p_0}{2\tau} = 0. \quad \text{(C-5)}$$

Maximization of the right-hand side of (C-4) with respect to p_1 yields

$$\frac{1}{2} + \frac{k(q_1-q_0)}{2\tau} - \frac{p_1-p_0}{2\tau} - \frac{p_1-c}{2\tau} = 0. \quad \text{(C-6)}$$

By (C-5) and (C-6), optimal pricing rules are

$$p_0 = c + \tau + \frac{2\theta-1}{2\theta+1}k(q_0-q_1), \quad \text{(C-7)}$$
$$p_1 = c + \tau + \frac{1}{2\theta+1}k(q_1-q_0). \quad \text{(C-8)}$$

Maximization of the right-hand side of (C-3) with respect to I_0 and that of (C-4) with respect to I_1 yield

$$I_0 = \frac{1}{\gamma}V^0_{q_0}(q_0,q_1), \quad \text{(C-9)}$$
$$I_1 = \frac{1}{\gamma}V^1_{q_1}(q_0,q_1). \quad \text{(C-10)}$$

We conjecture the quadratic value function of each firm as

$$V^0(q_0,q_1) = \alpha_0 + \alpha_1 q_0 + \alpha_2 q_1 + \frac{\alpha_3}{2}q_0^2 + \frac{\alpha_4}{2}q_1^2 + \alpha_5 q_0 q_1, \quad \text{(C-11)}$$
$$V^1(q_0,q_1) = \varepsilon_0 + \varepsilon_1 q_1 + \varepsilon_2 q_0 + \frac{\varepsilon_3}{2}q_1^2 + \frac{\varepsilon_4}{2}q_0^2 + \varepsilon_5 q_0 q_1. \quad \text{(C-12)}$$

Using (C-11) and (C-12), we can rewrite (C-9) and (C-10) in the form of linear investment strategies as

$$I_0 = \frac{\alpha_1 + \alpha_3 q_0 + \alpha_5 q_1}{\gamma}, \tag{C-13}$$

$$I_1 = \frac{\varepsilon_1 + \varepsilon_3 q_1 + \varepsilon_5 q_0}{\gamma}. \tag{C-14}$$

Substituting (C-7), (C-8), and (C-11)-(C-14) into (C-3) and (C-4) results in

$$\begin{aligned}
&\rho\big(\alpha_0 + \alpha_1 q_0 + \alpha_2 q_1 + \frac{\alpha_3}{2} q_0^2 + \frac{\alpha_4}{2} q_1^2 + \alpha_5 q_0 q_1\big) \\
&\quad = v - c + k q_0 \left\{ \frac{1}{2} + \frac{1}{2\theta+1} \frac{k(q_0 - q_1)}{2\tau} \right\} + k q_1 \left\{ \frac{1}{2} + \frac{1}{2\theta+1} \frac{k(q_1 - q_0)}{2\tau} \right\} \\
&\qquad - \frac{\tau}{2} \left\{ \frac{1}{2} + \frac{1}{2\theta+1} \frac{k(q_0 - q_1)}{2\tau} \right\}^2 - \frac{\tau}{2} \left\{ \frac{1}{2} + \frac{1}{2\theta+1} \frac{k(q_1 - q_0)}{2\tau} \right\}^2 \\
&\qquad - \frac{1}{2\gamma}(\alpha_1 + \alpha_3 q_0 + \alpha_5 q_1)^2 - \frac{\beta}{2} q_0^2 - \frac{1}{2\gamma}(\varepsilon_1 + \varepsilon_3 q_1 + \varepsilon_5 q_0)^2 - \frac{\beta}{2} q_1^2 \\
&\qquad + \frac{1}{\gamma}(\alpha_1 + \alpha_3 q_0 + \alpha_5 q_1)^2 - \delta(\alpha_1 + \alpha_3 q_0 + \alpha_5 q_1) q_0 \\
&\qquad + \frac{1}{\gamma}(\alpha_2 + \alpha_4 q_1 + \alpha_5 q_0)(\varepsilon_1 + \varepsilon_3 q_1 + \varepsilon_5 q_0) - \delta(\alpha_2 + \alpha_4 q_1 + \alpha_5 q_0) q_1,
\end{aligned} \tag{C-15}$$

$$\begin{aligned}
&\rho\big(\varepsilon_0 + \varepsilon_1 q_1 + \varepsilon_2 q_0 + \frac{\varepsilon_3}{2} q_1^2 + \frac{\varepsilon_4}{2} q_0^2 + \varepsilon_5 q_0 q_1\big) \\
&\quad = \left\{ \tau + \frac{1}{2\theta+1} k(q_1 - q_0) \right\} \left\{ \frac{1}{2} + \frac{1}{2\theta+1} \frac{k(q_1 - q_0)}{2\tau} \right\} \\
&\qquad - \frac{1}{2\gamma}(\varepsilon_1 + \varepsilon_3 q_1 + \varepsilon_5 q_0)^2 - \frac{\beta}{2} q_1^2 \\
&\qquad + \frac{1}{\gamma}(\varepsilon_1 + \varepsilon_3 q_1 + \varepsilon_5 q_0)^2 - \delta(\varepsilon_1 + \varepsilon_3 q_1 + \varepsilon_5 q_0) q_1 \\
&\qquad + \frac{1}{\gamma}(\varepsilon_2 + \varepsilon_4 q_0 + \varepsilon_5 q_1)(\alpha_1 + \alpha_3 q_0 + \alpha_5 q_1) - \delta(\varepsilon_2 + \varepsilon_4 q_0 + \varepsilon_5 q_1) q_0.
\end{aligned} \tag{C-16}$$

Collecting terms with equal powers of q_i results in

$$\rho\alpha_0 - \frac{\alpha_1^2}{2\gamma} - \frac{\alpha_2\varepsilon_1}{\gamma} - \frac{\tau}{2} - (v - c - \frac{3}{4}\tau - \frac{\varepsilon_1^2}{2\gamma}) = 0, \tag{C-19}$$

$$(\rho+\delta)\alpha_1 - \frac{\alpha_1\alpha_3}{\gamma} - \frac{\alpha_2\varepsilon_5}{\gamma} - \frac{\alpha_5\varepsilon_1}{\gamma} + \frac{\varepsilon_1\varepsilon_5}{\gamma} - \frac{k}{2} = 0, \tag{C-20}$$

$$(\rho+\delta)\alpha_2 - \frac{\alpha_1\alpha_5}{\gamma} - \frac{\alpha_2\varepsilon_3}{\gamma} - \frac{\alpha_4\varepsilon_1}{\gamma} + \frac{\varepsilon_1\varepsilon_3}{\gamma} - \frac{k}{2} = 0, \tag{C-21}$$

$$(\frac{\rho}{2}+\delta)\alpha_3 - \frac{\alpha_3^2}{2\gamma} - \frac{\alpha_5\varepsilon_5}{\gamma} + \frac{\varepsilon_5^2}{2\gamma} + \frac{\beta}{2} - \frac{k^2}{4\tau} = 0, \tag{C-22}$$

$$(\frac{\rho}{2}+\delta)\alpha_4 - \frac{\alpha_4\varepsilon_3}{\gamma} - \frac{\alpha_5^2}{2\gamma} + \frac{\varepsilon_3^2}{2\gamma} + \frac{\beta}{2} - \frac{k^2}{4\tau} = 0, \tag{C-23}$$

$$(\rho+2\delta)\alpha_5 - \frac{\alpha_3\alpha_5}{\gamma} - \frac{\alpha_4\varepsilon_5}{\gamma} - \frac{\alpha_5\varepsilon_3}{\gamma} + \frac{\varepsilon_3\varepsilon_5}{\gamma} + \frac{k^2}{2\tau} = 0, \tag{C-24}$$

$$\rho\varepsilon_0 - \frac{\varepsilon_1^2}{2\gamma} - \frac{\varepsilon_2\alpha_1}{\gamma} - \frac{\tau}{2} = 0, \tag{C-25}$$

$$(\rho+\delta)\varepsilon_1 - \frac{\varepsilon_1\varepsilon_3}{\gamma} - \frac{\varepsilon_2\alpha_5}{\gamma} - \frac{\varepsilon_5\alpha_1}{\gamma} - k = 0, \tag{C-26}$$

$$(\rho+\delta)\varepsilon_2 - \frac{\varepsilon_1\varepsilon_5}{\gamma} - \frac{\varepsilon_2\alpha_3}{\gamma} - \frac{\varepsilon_4\alpha_1}{\gamma} + k = 0, \tag{C-27}$$

$$(\frac{\rho}{2}+\delta)\varepsilon_3 - \frac{\varepsilon_3^2}{2\gamma} - \frac{\varepsilon_5\alpha_5}{\gamma} + \frac{\beta}{2} - \frac{k^2}{2\tau} = 0, \tag{C-28}$$

$$(\frac{\rho}{2}+\delta)\varepsilon_4 - \frac{\varepsilon_4\alpha_3}{\gamma} - \frac{\varepsilon_5^2}{2\gamma} - \frac{k^2}{2\tau} = 0, \tag{C-29}$$

$$(\rho+2\delta)\varepsilon_5 - \frac{\varepsilon_3\varepsilon_5}{\gamma} - \frac{\varepsilon_4\alpha_5}{\gamma} - \frac{\varepsilon_5\alpha_3}{\gamma} + \frac{k^2}{\tau} = 0. \tag{C-30}$$

D Stability conditions of the closed-loop solution

Substituting (C-13) and (C-14) into (3) yields

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \end{bmatrix} = \begin{bmatrix} \frac{\alpha_3}{\gamma} - \delta & \frac{\alpha_5}{\gamma} \\ \frac{\varepsilon_5}{\gamma} & \frac{\varepsilon_3}{\gamma} - \delta \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} + \begin{bmatrix} \frac{\alpha_1}{\gamma} \\ \frac{\varepsilon_1}{\gamma} \end{bmatrix}. \tag{D-1}$$

Let the coefficient matrix be J. The trace and the determinant of J are

$$tr(J) = \left(\frac{\alpha_3}{\gamma} - \delta\right) + \left(\frac{\varepsilon_3}{\gamma} - \delta\right), \qquad det(J) = \left(\frac{\alpha_3}{\gamma} - \delta\right)\left(\frac{\varepsilon_3}{\gamma} - \delta\right) - \frac{\alpha_5\varepsilon_5}{\gamma^2}.$$

The matrix, J, has two negative eigenvalues if

$$tr(J) < 0, \quad det(J) > 0, \quad tr(J)^2 - 4det(J) > 0.$$

Hence, conditions for globally asymptotically stable equilibrium are given by

$$(\alpha_3 - \gamma\delta) + (\varepsilon_3 - \gamma\delta) < 0, \tag{D-2}$$

$$(\alpha_3 - \gamma\delta)(\varepsilon_3 - \gamma\delta) - \alpha_5\varepsilon_5 > 0, \tag{D-3}$$

$$(\alpha_3 - \varepsilon_3)^2 + 4\alpha_5\varepsilon_5 > 0. \tag{D-4}$$

E The derivation of a unique solution

Introducing $s = \frac{\rho}{2} + \delta$, $g = \frac{k^2}{\tau}$, we can rewrite (C-22)-(C-24), (C-28)-(C-30) as follows:

$$s\alpha_3 - \frac{\alpha_3^2}{2\gamma} - \frac{\alpha_5\varepsilon_5}{\gamma} + (1-\theta)\frac{\varepsilon_5^2}{2\gamma} + \frac{\beta}{2} - \frac{g}{4} = 0, \tag{C-22'}$$

$$s\alpha_4 - \frac{\alpha_4\varepsilon_3}{\gamma} - \frac{\alpha_5^2}{2\gamma} + (1-\theta)\frac{\varepsilon_3^2}{2\gamma} + (1-\theta)\frac{\beta}{2} - \frac{g}{4} = 0, \tag{C-23'}$$

$$2s\alpha_5 - \frac{\alpha_3\alpha_5}{\gamma} - \frac{\alpha_4\varepsilon_5}{\gamma} - \frac{\alpha_5\varepsilon_3}{\gamma} + (1-\theta)\frac{\varepsilon_3\varepsilon_5}{\gamma} + \frac{g}{2} = 0, \tag{C-24'}$$

$$s\varepsilon_3 - \frac{\varepsilon_3^2}{2\gamma} - \frac{\varepsilon_5\alpha_5}{\gamma} + \frac{\beta}{2} - \frac{g}{2} = 0, \tag{C-28'}$$

$$s\varepsilon_4 - \frac{\varepsilon_4\alpha_3}{\gamma} - \frac{\varepsilon_5^2}{2\gamma} - \frac{g}{2} = 0, \tag{C-29'}$$

$$2s\varepsilon_5 - \frac{\varepsilon_3\varepsilon_5}{\gamma} - \frac{\varepsilon_4\alpha_5}{\gamma} - \frac{\varepsilon_5\alpha_3}{\gamma} + g = 0. \tag{C-30'}$$

Solving (C-22') for α_3 gives

$$\alpha_3(\alpha_5, \varepsilon_5) = \gamma s \pm \sqrt{(\gamma s)^2 - 2\alpha_5\varepsilon_5 + \varepsilon_5^2 + \gamma(\beta - 2fg)}. \tag{E-1}$$

Also, solving (C-28') for ε_3 gives

$$\varepsilon_3(\alpha_5, \varepsilon_5) = \gamma s \pm \sqrt{(\gamma s)^2 - 2\alpha_5\varepsilon_5 + \gamma(\beta - 2hg)}. \tag{E-2}$$

We can distinguish four different cases as follows:

$$\text{Case1}: \alpha_3 = \gamma s + \sqrt{A} \quad \text{and} \quad \varepsilon_3 = \gamma s + \sqrt{B}, \tag{E-3}$$

$$\text{Case2}: \alpha_3 = \gamma s + \sqrt{A} \quad \text{and} \quad \varepsilon_3 = \gamma s - \sqrt{B}, \tag{E-4}$$

$$\text{Case3}: \alpha_3 = \gamma s - \sqrt{A} \quad \text{and} \quad \varepsilon_3 = \gamma s + \sqrt{B}, \tag{E-5}$$

$$\text{Case4}: \alpha_3 = \gamma s - \sqrt{A} \quad \text{and} \quad \varepsilon_3 = \gamma s - \sqrt{B}, \tag{E-6}$$

where $A \equiv -2\alpha_5\varepsilon_5 + \varepsilon_5^2 + (\gamma s)^2 + \gamma(\beta - 2fg)$ and $B \equiv -2\alpha_5\varepsilon_5 + (\gamma s)^2 + \gamma(\beta - 2hg)$. We impose $A \geq 0$ and $B \geq 0$ to ensure that α_3 and ε_3 are real numbers.

Eliminating α_4 from (C-23') and (C-24') gives

$$\left(\frac{\varepsilon_5}{\gamma}\right)\left\{\frac{\alpha_5^2}{2\gamma} + fg - \frac{\varepsilon_3(\alpha_5,\varepsilon_5)^2}{2\gamma} - \frac{\beta}{2}\right\}$$
$$= \left(s - \frac{\varepsilon_3(\alpha_5,\varepsilon_5)}{\gamma}\right)\left\{\left(s - \frac{\alpha_3(\alpha_5,\varepsilon_5)}{\gamma} + s - \frac{\varepsilon_3(\alpha_5,\varepsilon_5)}{\gamma}\right)\alpha_5 + \frac{\varepsilon_3(\alpha_5,\varepsilon_5)\varepsilon_5}{\gamma} + 2fg\right\}. \tag{E-7}$$

Similarly, eliminating ε_4 from (C-29') and (C-30') gives

$$\left(\frac{\alpha_5}{\gamma}\right)\left(\frac{\varepsilon_5^2}{2\gamma} + hg\right) = \left(s - \frac{\alpha_3(\alpha_5,\varepsilon_5)}{\gamma}\right)\left\{\left(s - \frac{\alpha_3(\alpha_5,\varepsilon_5)}{\gamma} + s - \frac{\varepsilon_3(\alpha_5,\varepsilon_5)}{\gamma}\right)\varepsilon_5 + 2hg\right\}. \tag{E-8}$$

Parameters, α_5 and ε_5, can be obtained by solving these equations.

Consider Case 4. We set parameters as $\rho = 0.04$, $\delta = 0.05$, $c = 0$, $k = 1$, $v = 10$, $\beta = 5$, $\tau = 3$, $\gamma = 8$. We assume that γ is higher than β to avoid multiple equilibria. The numerical analysis indicates that (E-7) and (E-8) crosses at E in the fourth quadrant, as shown in Figure 1. The point E is the only one point of intersection of (E-7) and (E-8) which satisfies the conditions of (D-2)-(D-4), $A \geq 0$, and $B \geq 0$. Moreover, all other parameters are uniquely determined by other equations. Case 1 and Case 2 can be ruled out because of (D-2). The solution to Case 3 violates the condition (D-3). Therefore, the numerical analysis indicates that the simultaneous equations (C-19)-(C-30) have a unique solution under restrictions.

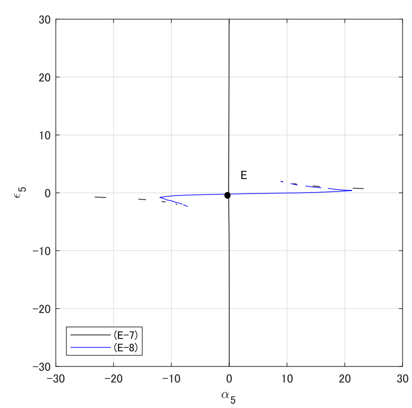

Figure 1: Configuration of α_5 and ε_5 of Case4

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