To VaR, or Not to VaR, That is the Question

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ABSTRACT

We consider the core problems of the conventional value-at-risk (VaR) based on the price probability determined by frequencies of trades at a price p during an averaging time interval Δ. To protect investors from risks of market price change, VaR should use price probability determined by the market trade time-series. To match the market stochasticity we introduce the new market-based price probability measure entirely determined by probabilities of random market time-series of the trade value and volume. The distinctions between the market-based and frequency-based price probabilities result different assessments of VaR and thus can cause excess losses. Predictions of the market-based price probability at horizon T equals the forecasts of the market trade value and volume probability measures.

Keywords: value-at-risk, risk measure, price probability, market trades

JEL : C10, E37, G11, G32

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1. Introduction

The value-at-risk as the risk measure was proposed in the late 60s almost 50 years ago as a respond to the request of JP Morgan's Chairman Dennis Weatherstone. "It was of JP Morgan, at the time the Chairman of JP Morgan, who clearly stated the basic question that is the basis for VaR as we know it today – "how much can we lose on our trading portfolio by tomorrow's close?""(Allen, Boudoukh and Saunders, 2004). The response of JP Morgan's team on Weatherstone's question results in presenting the VaR models by RiskMetrics Group and further development by (Longerstaey and Spencer, 1996; CreditMetrics™, 1997; Duffie and Pan, 1997; Laubsch and Ulmer, 1999; Mina and Xiao, 2001; Holton, 2003; Allen, Boudoukh and Saunders, 2004; Mina, 2005; Choudhry, 2013; Auer, 2018).

Due to (Longerstaey and Spencer, 1996) "Value-at-Risk is a measure of the maximum potential change in value of a portfolio of financial instruments with a given probability over a pre-set horizon." Since then Value-at-Risk or VaR becomes a standard tool for the risk assessment and was studied in hundreds articles. As usual, the roots of any good concept like VaR can by found much early than it is noted by RiskMetrics "official mythology" and Holton (2002) takes the VaR back to 1922. We are cannot refer all those who contributed to VaR as one of most effective and useful risk measures and mention only few (Malkiel, 1981; Linsmeier and Pearson 1996; Marshall and Siegel, 1996; Simons, 1996; Duffie and Pan, 1997; Berkowitz and O'Brien, 2001; Manganelli and Engle, 2001; Kaplanski and Kroll, 2002; Holton, 2003; Jorion, 2006; Aramonte, Rodriguez and Wu 2011). Since RiskMetrics publications the VaR concept occupied the permanent position in the risk management monographs (Choudhry, 2013; Horcher, 2015). Various forms of the VaR were developed for the risk assessment of market portfolios, corporate and credit risk, financial risk management (Sanders and Manfredo, 1999; Jondeau, Poon and Rockinger, 2007; Adrian and Brunnermeier, 2011; Aramonte, Rodriguez and Wu, 2011; Andersen et.al., 2012; Auer, 2018). VaR concept plays the important role in bank and security risk regulations (FRS, 1998; Amato and Remolona, 2005; CESR, 2010). Wide usage of VaR as a risk measure is explained by its clear and general concept. Let's take price probability measure $f(p)$:

$$\int dp \;\; f(p) = 1 \tag{1.1}$$

and choose small number ε<<1. Then one can derive the price $p(\varepsilon)$:

$$\int_0^{p(\varepsilon)} dp \;\; f(p) = \varepsilon \tag{1.2}$$

Price $p(\varepsilon)$ determines the bottom line of possible losses with probability $1-\varepsilon$

$$p(\varepsilon) \le p \;\; with\, probability \;\;\; 1-\varepsilon \tag{1.3}$$

Simple relations (1.1-1.3) give firm and clear ground for VaR. Only some "easy" problems left: how to chose and forecast the price probability measure *f(p)* ?

In the late 60s RiskMetrics developed the first approximations of the VaR. The standard treatment of Value-at-Risk (Longerstaey and Spencer, 1996) is based on the price probability *f(p)* determined by number (frequency) of trades at price *p*. To define the price probability *f(p)* one should chose certain time averaging interval *Δ*, collect all *N* trades with asset *A* during interval *Δ* and count the number *m(p)* of trades at price *p*. Investor may choose the time interval *Δ* to be equal an hour, a day, a week or whatever. The duration of *Δ* impacts the properties of the price probability measure *f(p)*. The frequency-based price probability *f(p)* and a mean price *E[p]* during the interval *Δ* equals

$$f(p) = \frac{1}{N} m(p) \;;\; E[p] = \frac{1}{N} \sum_k p_k m(p_k) = \frac{1}{N} \sum_{i=1}^{N} p(t_i) \tag{1.4}$$

We note as *E[…]* to define mathematical expectation. If one choose *ε*=5% then with probability 95% (1.2; 1.3) all trade prices *p* during interval *Δ* will be higher than *p(5%)*. Hence *M* shares of asset *A* with probability 95% will have value more or equal than *p(5%)M*. Investor may choose the benchmark 1%, 3% or whatever and obtain the lower estimate of asset *A* value or possible losses – with probability 99%, 97% etc.

As the first approximation RiskMetrics Group (Longerstaey and Spencer, 1996) assumed that the frequency-based price probability measure (1.1; 1.4) *f(p)* of trades at price *p* takes form of standard Normal distribution. "A standard property of the Normal distribution is that outcomes less than or equal to 1,65 standard deviations below the mean occur only 5 percent of the time" (Longerstaey and Spencer, 1996). Investors use this result for years as risk assessment of portfolio losses. Further researchers investigate the way to forecast the frequency-based price probability *f(p)* (1.4), estimate the deviation of price probability *f(p)* (1.4) from normal distribution, explain the "fat tails" of the observed price probability and etc. These problems are difficult and till now are far from final solution.

We discus the core problems of the conventional VaR concept: the price probability and its prediction. We show that the frequency-based definition of the price probability (1.4) is definitely not the only one and most likely not the correct one. We consider random time-series of market trade value and volume as origin of price stochasticity and introduce price probability as consequence of probability measures of the trade value and volume. VaR should protect investors from risks of random change of market price and hence market trade probability should determine market price probability. Below we derive market-based price probability entirely determined by probabilities of market trade value and volume. The distinctions between the market-based price probability and the conventional frequency-based price probability (1.4) result differences in VaR assessments of *p(ε)* (1.2; 1.3) and hence can cause excess losses.

We propose that readers are familiar with methods of stochastic systems, statistical moments, characteristic functions and etc.

2. Price probability

Each market trade at moment t_i is described by its value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$:

$$C(t_i) = p(t_i)U(t_i) \tag{2.1}$$

One can consider market trade time-series as irregular or random. To study market laws, the trade time-series are averaged or smoothed during certain averaging time interval *Δ*. We consider random time-series of market trade value $C(t_i)$ and volume $U(t_i)$ during the averaging interval *Δ* as origin of price time-series $p(t_i)$ (2.1). Duration of the averaging interval *Δ* defines the number of members of the time-series of the value $C(t_i)$ and volume $U(t_i)$ and thus impacts the properties of their probability distributions. For convenience we take that moments t_i belong to the averaging interval *Δ* near moment *t* if:

$$t - \frac{\Delta}{2} \le t_i \le t + \frac{\Delta}{2} \; ; \quad i = 1, \dots N(t) \tag{2.2}$$

We consider the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ (2.1) during the interval *Δ* (2.2) as random variables. It is impossible independently define probabilities of random value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ during *Δ* those match equation (2.1). Given probabilities of value $C(t_i)$ and volume $U(t_i)$ for (2.1) should define probability of price $p(t_i)$. Below we derive the probability of price $p(t_i)$ determined by probabilities of random trade value and volume during *Δ* that match (2.1).

Let us mention well-known tools that describe any random variable. One can describe price as a random variable *p* by price probability measure *η(p)* or by price characteristic function *F(x)* (Shiryaev, 1999; Klyatskin, 2005; Gardiner, 2009; Klyatskin, 2015). Statistical properties of a random variable can depend on moment *t* that defines the averaging interval *Δ* (2.2) but for simplicity we do not consider it here. Relations between probability measure *η(p)* and characteristic function *F(x)* of random variable *p* are well known. Fourier transform of price characteristic function *F(x)* defines price probability measure *η(p)* and vice versa (for brevity we omit factors proportional to *(2π)*):

$$\eta(p) = \int dx \;\; F(x)\, exp\, {-ipx} \quad ; \quad F(x) = \int dp \;\; \eta(p)\, exp\, ipx \tag{2.3}$$

Price probability measure *η(p)* and characteristic function *F(x)* define price statistical moments *p(n)*:

$$p(n) = \int dp\ p^n\ \eta(p) = i^{-n} \frac{d^n}{dx^n}\ F(x)|_{x=0} \quad (2.4)$$

Price statistical moments *p(n)* define Taylor series of price characteristic function *F(x)*:

$$F(x) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!}\ p(n)\ x^n \quad (2.5)$$

Let us assume that frequency-based probabilities of the trade value $C(t_i)$ and volume $U(t_i)$ random variables during the averaging interval Δ (2.2) are known. The trade value probability *ν(C)* defines *n-th* statistical moments $C_m(n)$ of the value

$$\nu(C_k) = \frac{1}{N}\ m(C_k)\ \ ;\ C_m(n) = E[C^n(t_i)] = \sum C_k^n\ \nu(C_k) = \frac{1}{N} \sum_{i=1}^{N} C^n(t_i) \quad (2.6)$$

The trade volume probability *μ(U)* defines *n-th* statistical moments $U_m(n)$ of the volume:

$$\mu(U_k) = \frac{1}{N}\ m(U_k)\ \ ;\ U_m(n) = E[U^n(t_i)] = \sum U_k^n\ \mu(U_k) = \frac{1}{N} \sum_{i=1}^{N} U^n(t_i) \quad (2.7)$$

In (2.6; 2.7) $m(C_k)$ and $m(U_k)$ define number of trades at value C_k and number of trades at volume U_k respectively. Hence one can use trade value and trade volume statistical moments (2.6; 2.7) and equation (2.1) to define statistical moments of random price $p(t_i)$ time-series.

Let remind that almost 30 years ago (Berkowitz et.al 1988) introduced the volume weighted average price (VWAP) and it is widely used now (Buryak and Guo, 2014; Guéant and Royer, 2014; Busseti and Boyd, 2015; Padungsaksawasdi and Daigler, 2018; CME Group, 2020). The VWAP *p(1)* or 1-st statistical moment determined by the trade value $C(t_i)$ and volume $U(t_i)$ time-series during the interval Δ (2.2) can be determined as:

$$C(1) = \sum_{i=1}^{N} C(t_i)\ \ ;\ \ U(1) = \sum_{i=1}^{N} U(t_i)\ \ ;\ C(1) = p(1)U(1) \quad (2.8)$$

Using (2.6; 2.7) relations (2.8) for *p(1)* can be presented in an equal form as:

$$C_m(1) = p(1)U_m(1) \quad (2.9)$$

$C_m(1)$ (2.6) and $U_m(1)$ (2.7) denote the mean value and the mean volume of *N* trades during Δ (2.2). The mean price *p(1)* or 1-st statistical moment of price is determined by the mean value $C_m(1)$ and mean volume $U_m(1)$ (2.9). We outline that mean trade value $C_m(1)$ and mean trade volume $U_m(1)$ are determined by frequency-based probabilities (2.6; 2.7). To derive the set of price statistical moments *p(n)* that defines price characteristic function *F(x)* as (2.5) we take *n-th* power of each term in (2.1) and for all *n=1,2,…* obtain:

$$C^n(t_i) = p^n(t_i)\ U^n(t_i) \quad (2.10)$$

It is obvious that (2.10) is a direct consequence of (2.1) and we use (2.10) to define all price statistical moments *p(n)*. Let us mention that VWAP *p(1)* (2.9) is derived using implicit assumption that trade volume $U(t_i)$ and price $p(t_i)$ time-series are not correlated during the interval Δ (2.2). Indeed, (2.9) implies that

$$C_m(1) = E[C(t_i)] = E[p(t_i)U(t_i)] = E[p(t_i)]E[U(t_i)] = p(1)U_m(1) \quad (2.11)$$

In (2.11) we denote mathematical expectation during the interval *Δ as E[…]*. Let use the same assumption for *n-th* degrees of the trade volume $U^n(t_i)$ and price $p^n(t_i)$ and take that *n-th* power of trade volume $U^n(t_i)$ and price $p^n(t_i)$ time-series are not correlated during the interval *Δ* (2.2). Then averaging of (2.10) during the interval *Δ* (2.2) gives:

$$C_m(n) = E[C^n(t_i)] = E[p^n(t_i)U^n(t_i)] = E[p^n(t_i)]E[U^n(t_i)] = p(n)U_m(n) \qquad (2.12)$$

Relations (2.12) define *n-th* statistical moment of price *p(n)* via *n-th* statistical moments of the trade value $C_m(n)$ (2.6) and trade volume $U_m(n)$ (2.7). Let underline that no correlations between *n-th* degrees of volume $U^n(t_i)$ and price $p^n(t_i)$ time-series (2.12) do not imply that the volume and price are statistically independent. It is easy to show that for $n \neq m$ time-series $p^n(t_i)$ correlate with time-series $U^m(t_i)$ and

$$E[p^n(t_i)U^m(t_i)] \neq E[p^n(t_i)]E[U^m(t_i)]$$

We repeat that *nth* statistical moment of the trade value $C_m(n)$ and volume $U_m(n)$ are determined by the frequency-based probabilities (2.6; 2.7). Relations (2.12) can take form alike to VWAP (2.8; 2.9):

$$p(n) = \frac{1}{U(n)} \sum_{i=1}^{N} p^n(t_i)\, U^n(t_i) = \frac{C(n)}{U(n)} = \frac{C_m(n)}{U_m(n)} \qquad (2.13)$$

$$C(n) = NC_m(n) = \sum_{i=1}^{N} C^n(t_i) \quad ; \quad U(n) = NU_m(n) = \sum_{i=1}^{N} U^n(t_i) \qquad (2.14)$$

Functions *C(n)* and *U(n)* (2.14) define sums of *n-th* degrees of the trade value $C^n(t_i)$ and volume $U^n(t_i)$ during the averaging interval *Δ* (2.2). Relations (2.12-2.14) define the set of price *n-th* statistical moments *p(n)* for all *n=1,2,…* and hence define Taylor series of the price characteristic function *F(x)* (2.5). Relations (2.6; 2.7) define statistical moments $C_m(n)$ of the value and statistical moments $U_m(n)$ of the volume via their probability measures *ν(C)* and *μ(U)*. Thus one can consider (2.12-2.14) as derivation of the price characteristic function *F(x)* (2.5) via frequency-based probability measures *ν(C)* and *μ(U)* (2.6; 2.7) of the trade value and the trade volume.

It is obvious that price statistical moments *p(n)* (2.12- 2.14) differ from statistical moments *π(n)* (2.15) generated by frequency-based price probability (1.4) during *Δ* (2.2).

$$\pi(n) = \frac{1}{N}\sum_{i=1}^{N} p^n(t_i) = \frac{1}{N}\sum_{i=1}^{N} \frac{C^n(t_i)}{U^n(t_i)} \neq \frac{\sum_{i=1}^{N} C^n(t_i)}{\sum_{i=1}^{N} U^n(t_i)} = \frac{C_m(n)}{U_m(n)} = p(n) \qquad (2.15)$$

Only if during the averaging interval *Δ* (2.2) all trade volumes $U(t_i)$ equal unit:

$$if\ U(t_i) = 1\ for\ i = 1, \dots N\ \ then\ \ \pi(n) = \frac{1}{N}\sum_{i=1}^{N} \frac{C^n(t_i)}{U^n(t_i)} = C_m(n) = p(n) \qquad (2.16)$$

The difference between the market-based price probability (2.3-2.5; 2.12-2.14) and frequency-based price probability (1.4; 2.15) that is illustrated by (2.15; 2.16) impacts the VaR assessment of *p(ε)* (1.2-1.3) and thus results the origin of unexpected losses.

3. Price probability approximations

Taylor series of price characteristic function *F(x)* (2.5) do not permit directly derive price probability measure *η(p)* via inverse Fourier transform (2.3). However, Taylor series (2.5) opens the way for successive approximations $F_k(x)$, *k=1,2..* of the price characteristic function *F(x)* that allows Fourier transforms (2.3) and result probability approximations $\eta_k(p)$. The most essential way to approximate price characteristic function *F(x)* (2.5) is to take finite number of Taylor series' terms and define approximate price characteristic function $F_K(x)$ that for $n \leq k$ determines price statistical moments $p_k(n)$ that are equal p(n) (2.11; 2.13). Let define such approximation $F_k(x)$ of characteristic function *F(x)* (2.5) as:

$$F_k(x) = \exp\left\{\sum_{m=1}^{k} \frac{i^n}{m!}\, a_m\, x^m\right\} \tag{3.1}$$

Let define approximate price statistical moments $p_k(n)$ similar to (2.4):

$$p_k(n) = \int dp\;\, p^n\, \eta_k(p) = i^{-n} \frac{d^n}{dx^n}\, F_k(x)|_{x=0} \tag{3.2}$$

For $n \leq k$ let require that $p_k(n)$ (3.2) be equal *p(n)* (2.12; 2.13):

$$p_k(n) = p(n) = \frac{C_m(n)}{U_m(n)} \quad ; \quad n \leq k \tag{3.3}$$

For *k=1* the approximation $F_1(x)$ is trivial (see 2.4):

$$F_1(x) = \exp\{i\, a_1 x\} \;\; ; \; p_1(1) = -i \frac{d}{dx} A_1(x)|_{x=0} = a_1 = p(1)$$

Approximate characteristic function $F_1(x)$ defines trivial approximation of price probability measure $\eta_1(p)$:

$$\eta_1(p) = \int dx\;\, A_1(x)\, exp\, {-ipx} = \delta\big(p - \;p(1)\big)$$

For *K=2* approximation $F_2(x)$ describes the Gaussian probability measure $\eta_2(p)$:

$$F_2(x) = \exp\left\{i\, p(1)x - \frac{a_2}{2} x^2\right\} \tag{3.4}$$

It is easy to show that due to (3.2; 3.3)

$$p_2(2) = -\frac{d^2}{dx^2} A_2(x)|_{x=0} = a_2 + p^2(1) = p(2)$$

Thus a_2 has meaning of price volatility $\sigma^2(p)$

$$a_2 = \sigma^2(p) = E\left[(p - p(1))^2\right] = p(2) - p^2(1) \tag{3.5}$$

and Fourier transform (2.3) for $F_2(x)$ gives Gaussian price probability measure $\eta_2(p)$:

$$\eta_2(p) = \frac{1}{(2\pi)^{\frac{1}{2}} \sigma(p)} \exp\left\{-\frac{(p-p(1))^2}{2\sigma^2(p)}\right\} \tag{3.6}$$

For *K=3* approximation $F_3(x)$:

$$F_3(x) = \exp\left\{i\, p(1)x - \frac{\sigma_p^2(t)}{2}x^2 - i\, a_3 x^3\right\} \qquad (3.7)$$

Using (3.2; 3.3) one can obtain:

$$p_3(3) = a_3 + 3p(1)\sigma^2(p) + \; p^3(1) = p(3)$$

$$a_3 = p(3) - 3p(1)\sigma^2(p) - \; p^3(1)$$

Coefficient a_3 in (3.7) defines price skewness $Sk(p)$ as:

$$Sk(p) = E\left[\left(p - p(1)\right)^3\right] = a_3 + 3p^3(1)$$

Approximation of price probability measure $\eta_3(p)$ determined by characteristic function $F_3(x)$ and further approximations of (2.5) requires separate consideration.

4. Discussion

The VaR as risk measure is successfully used for almost half a century and we hope it may serve further. However the problems with effective usage of the VaR are really tough. The VaR concept is perfect and simple. However economic reality is too complex to be described by easy conventional frequency-based price probability (1.4) that up now is the ground for the risk assessments for hundreds of billions of dollars worth of assets. Actually, the different price probabilities define different assessments of $p(\varepsilon)$ (1.2-1.3) that are the essence of the VaR. Any inaccuracy of the $p(\varepsilon)$ cost many millions dollars of excess losses.

The VaR is the assessment of the market price change risks and should deliver assurance that the price probability in the ground of the VaR can properly describe random properties of the market trade price. Economics and finance are social sciences and decades of public acceptance by investors and researchers of the frequency-based price probability (1.4) impact investment decisions much more than any our considerations that price probability should be determined by the probabilities of the market trade value and volume. Good or bad, but the times of simple solutions in economics and finance are over. Investors should adopt that the conventional and simple frequency-based approach to the market price probability (1.4) have almost nothing common with the random price generated by stochastic market trade time-series. As partial confirmation of that one can consider the VWAP that has no roots in the frequency-based price probability (1.4) and for almost 30 years is used as assessment of a mean price (2.11) in a line with a frequency-based mean price (1.4).

We underline that we do not neglect or call into question correctness of the frequency-based probability definition itself. Not for an instant. We state, that the frequency-based probabilities determine statistical moments of two additive random variables – the market trade value $C(t_i)$ and volume $U(t_i)$ time-series. Actually, it is impossible arbitrary determine probabilities of three mutually dependent time-series that are match equation (2.1). The choice of probabilities of the trade value $C(t_i)$ and volume $U(t_i)$ should uniquely determines the price probability $p(t_i)$ (2.1) during the interval Δ. That gives the self-consistent description of three random variables – the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ time-series.

As we show in (2.15; 2.16) the frequency-based price probability (1.4) coincides with price probability $\eta(p)$ (2.3) only if all trade volumes $U(t_i)=1$ during Δ. Actually in this case price probability measures (1.4) and (2.3) equal frequency-based probability measure $\nu(C)$ of trade values (2.6). And that case does not describe any real market at all. The differences between $p(\varepsilon)$ (1.2-1.3) determined by distinctions between frequency-based and market-based price statistical moments (2.15) become the origin of excess losses of many millions dollars worth.

To improve the risk assessment of the random market price change one should derive the market-based price probability at moment t and then forecast it at a horizon T that may equal a day, a week, a month or what ever. And that uncovers real difficulties that prohibit simple and easy assessment of $p(\varepsilon)$ (1.2-1.3). The origin of the difficulties is obvious: it is impossible predict the market price probability $\eta(p)$ at a horizon T without prediction of the market trade value $C(t)$ and volume $U(t)$ probabilities (2.6; 2.7) at the same horizon.

We outline only two issues. First, we repeat that the price probability measure is determined by the set of the price statistical moments $p(n)$ defined by statistical moments of the value $C_m(n)$ and the volume $U_m(n)$ (2.12). It is obvious that one can define the price statistical moments $p(n)$ using sums of *nth* degrees of the value $C(n)$ and *nth* degrees of the volume $U(n)$ during the interval Δ (2.14). Functions $C(n)$ and $U(n)$ describe the sums of the *nth* degree of value $C^n(t_i)$ and volume $U^n(t_i)$ of market trades during the interval Δ. Relations (2.14) allow present price statistical moments as (2.13). Thus, prediction of the price characteristic function $F(x)$ (2.5) depends on prediction of the sums $C(n)$ and $U(n)$ (2.14). Prediction of the mean price $p(1)$ depends on forecasting the sums $C(1)$ and $U(1)$ (2.8) of the market trade value and volume of the first degree. Forecasting the *1-st* degree sums (2.8) can be done by current economic models that describe evolution of macroeconomic variables determined as sum of agents' first-degree variables. For example, macroeconomic investment, credits and consumption are determined as sums (without duplication) of investment, credits and consumption of all agents in the economy during certain time interval Δ. Almost all macroeconomic variables are composed as sums of the *1-st* degree variables of all agents in the economy. Price volatility that impacts investment decisions, market trading and eventually the macroeconomic development is an example of the second-degree variable.

Indeed, the price volatility $\sigma^2(p)$ (3.5) during the averaging interval Δ is expressed by *1-st* $p(1)$ and *2-d* $p(2)$ (2.12) price statistical moments:

$$\sigma^2(p) = p(2) - p^2(1)$$

Thus prediction of the price volatility $\sigma^2(p)$ at a horizon T requires forecasting the sums of squares of the trade value $C(2)$ and volume $U(2)$ (2.14) at same horizon T. Predictions of the price volatility $\sigma^2(p)$ establish the core problems of options and derivatives markets and volatility trading (Black and Scholes, 1973; Whaley, 1993; Hull, 2009; Sinclair, 2013; Bennett, 2014). Volatility modeling and forecasting are among the most important subjects of financial theory (Poon and Granger, 2003; Andersen et.al., 2005; Brownlees, Engle and Kelly, 2011). We refer only a few of hundreds studies of volatility related issues. In Olkhov (2020b) we show that the market price probability that match (2.1) leads to the 2-dimensional Black-Scholes-Merton-like equation with two constant volatilities (Black and Scholes, 1973; Merton, 1973), impacts Heston (1993) stochastic volatility model, influences the non-linear option pricing and etc.

The essence of the VaR is the forecasting of $p(\varepsilon)$ (1.2-1.3) for the price probability measure $\eta(p)$ (2.3) at the horizon T. Prediction of the price probability measure is equivalent to prediction of all price statistical moments and thus the price volatility becomes the first obstacle on that long way. Indeed, the price volatility $\sigma^2(p)$ (3.1) depends on *2-d* price statistical moment $p(2)$ determined by sums of squares of trades values $C(2;t)$ and squares of trade volumes $U(2)$ (2.14) during Δ. As we mentioned, current economic theories consider variables determined by sums of the *1-st* degree variables only and do not describe any *2-d* degree macro variables at all. Description of the price volatility $\sigma^2(p)$ (3.1) requires modelling sums of squares of trade values $C(2)$ and volumes $U(2)$. One can consider sums of squares of investment, credits and consumption of all agents in the economy as a tool to describe volatilities of macro investment, credits and consumption. Predictions of the sums of squares of trade values $C(2)$ and volumes $U(2)$ and macroeconomic variables of the second-degree require development of a new second-order economic theory (Olkhov, 2021a; 2021b). Forecasting of the *n-th* price statistical moments $p(n)$ implies prediction of the sums of *nth*-degree of the value $C(n)$ and the volume $U(n)$ and hence development of the *nth*-order economic theory. In simple words – to predict VaR at the horizon T one should predict price probability at same horizon. To do that one should forecast market probabilities of trade value and volume or their statistical moments $C_m(n)$ and $U_m(n)$ (2.12) for all n at horizon T. To avoid here excess complexity we refer (Olkhov, 2021a; 2021b) for details.

5. Conclusion

To assess *p(ε)* (1.2-1.3) at horizon *T* one should forecast of the price probability at same horizon. That equals predictions at horizon *T* of all statistical moments of the market trade value $C_m(n)$ and volume $U_m(n)$ (2.6; 2.7) or equally the sums of *nth*-degrees of the value *C(n)* and the volume *U(n)* (2.14). Explicit prediction of the price probability measure *η(p)* (2.3) on base of exact forecasting of trade statistical moments $C_m(n)$ and $U_m(n)$ (2.6; 2.7) for all *n* seems to be almost impossible. However, relations (3.1-3.3) open the way for developing successive approximations of the price characteristic functions $F_k(x)$ and these approximations can help approximate *p(ε)* (1.2-1.3) at horizon *T*.

The choice of the averaging interval *Δ* plays crucial role for determining statistical moments of the value, volume and price. The duration of *Δ* defines the internal scale of smoothness for economic fluctuations and trade disturbances. Relations between the interval *Δ* and horizon *T* determine internal and external scales of macroeconomic modeling and different macroeconomic approximations.

We outline that the ground elements of the VaR concept – the choice and the forecasts of the price probability, are in the heart of the advanced economic and financial studies. After usage of VaR for 50 years, the main problems in the base of the VaR concept are still open. One who succeeds in forecasting of the market trade price probability could manage the world markets alone. This is not the worst incentive to solve the VaR problem.

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