Original divine proportions of general competitive equilibrium

Malakhov, Sergey

28 October 2021

Sergey Malakhov
Ph.D., Applied Economics
Pierre Mendes France University (Grenoble-Alpes)

Original Divine Proportions of General Competitive Equilibrium.

Abstract

The proof of the invisible hand discovers many interesting peculiarities of the general competitive equilibrium at times when Adam Smith was working on the 'Wealth of Nations'. If his self-interested producer allocates his time between production and delivery to the 'the door' of the buyer with zero search costs and unintentionally maximizes customer's consumption-leisure utility, both the marginal rate of transformation of production into delivery and the marginal rate of substitution of leisure for consumption become equal to the golden ratio conjugate whereas the sales-costs of production ratio becomes equal to the golden ratio itself. While the golden ratio was called by Luca Pacioli, the founder of the modern accounting, as the divine proportion, this paper contributes to the deeper understanding of the Adam Smith's natural theology approach to the analysis of social processes.

JEL Classification: D11, D63, D83

Key words: golden ratio, invisible hand, divine proportion, general competitive equilibrium

---

This paper represents the abridged version of the article "Divine Proportion of Invisible Hand: a new look at Adam Smith's natural theology", written in Russian for the *Journal of Institutional Studies* (forthcoming).

Introduction

The proof of the invisible hand (Malakhov 2020, 2021a, 2021b) demonstrates the successful meeting of both uninformed buyer and seller under wage and price dispersion.[1] The seller allocates his time between production and delivery whereas the buyer allocates his time between labor, search, and leisure.

The consumer searches for the satisficing price. But when he buys the quantity demanded with respect to the satisficing price, he unintentionally maximizes his consumption-leisure utility, now with respect to the equilibrium price, which is equal to the minimal willingness to pay 'at doors' of consumers with zero search costs as well as to the willingness to accept of consumers with positive search costs.

Making the delivery, the producer 'sells' to his customers some leisure and unintentionally contributes to the optimal consumer allocation of time. The set of 'time in production – time in delivery' trade-offs results in the multiple local equilibria with respect to different customers' willingness to pay.

But one of these local equilibria represents the particular state of the market. When the producer sells at 'the door' for a consumer with zero search costs, his price is equal to the equilibrium price. By this way, the prototype of the general competitive equilibrium emerges. This prototype has very specific attributes. The analysis of these attributes gives an idea that originally the market was equilibrated much more than the economic theory could imagine.

This paper describes the appearance of the prototype of the general competitive equilibrium under wage and price dispersion and defines its ideal proportions. From this point of view the idea of the invisible hand in the

---

[1] The articles (Malakhov 2020,2021a,2021b) have been published off-line in Russian, but they have been synthesized on-line in English on the author' personal page at Berkeley Electronic Press site in the papers 'Work of the Invisible Hand: the gravitation between sellers and buyers on the consumption-leisure production possibility frontier' (https://works.bepress.com/sergey_malakhov/25/) and 'Invisible Hand Equilibrium in Family: the gravitation between men and women in marriage markets' (https://works.bepress.com/sergey_malakhov/27/) (S.M.)

environment of natural prices, wages, and profits, constructed by Adam Smith in the 'Wealth of Nations', looks more valuable than a simple literary metaphor.

The optimal matching under wage and price dispersion

We take again the farmer who allocates his time between production on the farm and delivery to some point between the farm and the downtown. While his working time is constant, his total costs *TC* are also constant. But he is indifferent where to sell because he gets advantage from the wage dispersion and his sales are always equal to his total costs:

$$(1) \quad TC = PQ = const$$

Choosing the place for sales, he determines the time for delivery T_d. It gives him as the residual the time for production T_f and the quantity Q to be produced under constant average AC_f and marginal costs MC_f on farming. Keeping in mind the constant $PQ{=}TC$ value, he also gets the price P for the quantity Q. This price gives him the total costs on delivery TC_d with respect to the total costs on farming TC_f:

$$(2) \quad TC_d = TC - TC_f$$

And the price becomes equal to the total of both average and marginal costs on production and delivery:

$$(3) \quad P = \frac{TC}{Q} = \frac{TC_f}{Q} + \frac{TC_d}{Q} = AC_f + AC_d = MC_f + MC_d$$

Really, there is a constant return on scale for any place of sales, i.e., for any quantity to be sold. While the marginal and average costs on farming stay constant for any output, the marginal and average costs on delivery vary from one point of sale to another. But they stay constant for the given quantity to be produced and delivered:

$$(4) \quad MC_d = AC_d = \frac{TC_d}{Q} = \frac{TC - TC_f}{Q} = P - MC_f$$

Any quantity Q to be supplied means a) the increase in delivery dQ_d with respect to the option where sales are made at the farm and when the time for delivery T_d is equal to zero; b) the decrease in production dQ_d also with respect to sales on the farm. And we can get the following total differential for farming and delivery:

$$(5)\quad dTC_{FD} = dQ_f MC_f + dQ_d MC_d = 0$$

where the value dQ_f is equal to the cut in production and the dQ_d value means the final supply Q.

From here we get the marginal transformation rate of farming into delivery:

$$(6)\quad RPT_{FD} = -\frac{dQ_f}{dQ_d} = \frac{MC_d}{MC_f} = \frac{AC_d}{AC_f} = \frac{TC_d}{TC_f} = \frac{T_d}{T_f}$$

Equation 6 rearranges Equation 3 into the following form:

$$(7)\quad P = MC_f\left(1 + \frac{MC_d}{MC_f}\right) = MC_f\left(1 + \frac{T_d}{T_f}\right)$$

The producer knows nothing about the consumer; neither his willingness to pay nor his allocation of time between labor L, search S, and leisure H. The producer wants only one thing – to sell the PQ value. Both P and Q values are given by his T_d/T_f allocation of time. In addition, his offer should be competitive. Here the value MC_f becomes crucial. If we come back to Chapter VIII of the 'Wealth of Nations', we can follow Adam Smith's reasoning on the wage of independent producer who is both the master and the workman and who gets two distinct revenues – the profits of stocks and the wages of labor (Smith 2000, p.75). Here we get almost the same case. The profit is equal to the total costs on delivery, which rewards farmer's commercial skills, and the wages are equal to total costs on farming. But to be competitive, the marginal costs on farming MC_f should be equal to wages of independent workmen, who are employed in neighbouring village. But these independent workmen should keep at the competitive market an option – either to work on the farm or to work on the factory. It means that at the general competitive equilibrium farmer's marginal costs on production MC_f are equal to the equilibrium wage rate w_e.

However, the competitive value of the marginal costs on farming MC_f don't impede to sell fruits and vegetables to low-wage rate customers. For them, the farmer chooses the point of sale not far from the farm, and they spend some time S on the search for the low price. As a result, the local equilibrium appears.

This equilibrium exists only in farmer's coordinate system. There, the values of the time of farming T_f, the quantity Q, and the price P become dependent on the point of sales, i.e., on the time for delivery T_d. The consumer literally enters into this space with his pre-determined marginal rate of substitution of leisure for consumption *MRS (H for Q)*.

It has been demonstrated with the help of l'Hopital rule that the value *MRS (H for Q)* appears not at the moment of purchase but at the moment of intention to buy, when $Q, L, S \to 0$ (Malakhov 2020; 2021a). The application of l'Hopital rule for the moment of intention to buy result in the unit elasticity of consumption with respect to the total costs on purchase, or $e_{Q/(L+S)}=1$. This conclusion confirms consumer's stable preferences, which are exhibited here by the marginal rate of substitution of leisure for consumption in natural terms:

$$(8) \quad MRS\ (H\ for\ Q) = \frac{Q}{L+S}$$

To present it in monetary terms, we should come back to the inner mechanism of satisficing purchase (Malakhov 2021a).

Under the traditional problem of search for the fixed quantity demanded Q (Stigler 1961), we get the intersection of $QP(S)$ curve and labour income $wL(S)$ curve with regard to the time of search S when the consumer chooses the first offer QP_P below his willingness to pay $WTP=wL_0$:

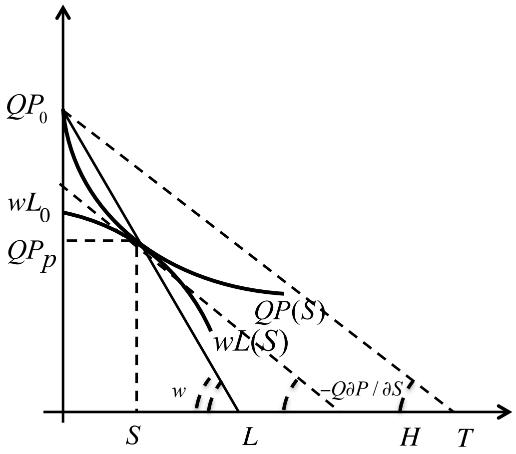

Fig.1. Behavioral satisficing choice

where S – the search; L – labor; H – leisure; T – time horizon until next purchase; Q – quantity demanded; w – wage rate; wL_0 – willingness to pay; P_P – purchase price.

The straight line with the slope w, passing the intersection point, i.e., the purchase, gives us the QP_0 value on the $0Y$ axis and $(S+L)$ value on the $0X$ axis. The straight dotted line from the point QP_0 with the slope ($-Q\partial P/\partial S$), i.e., the tangent to the moment of purchase, gives us the value of the time horizon T on the $0X$ axis.

These considerations result in the following equation:

$$(9)\quad w(L+S) = -Q\frac{\partial P}{\partial S}T = QP_0$$

It is evident that Eq. 9 demonstrates consumer's willingness to accept WTA. But if he decides to re-sell the item, who can buy it? The answer also is evident. The consumer with positive search costs can re-sell the item to the consumer with zero search costs. Although the wage dispersion exists at the zero search level, there is no price dispersion. The P_0 value is the minimal price at the zero search level or the equilibrium price or $P_0=P_e$.

And we can take Eq. 9 as the budget constraint to some consumption-leisure utility function *U(Q,H)*, keeping in mind that for the given time horizon $T=L+S+H$ the value $\partial L/\partial H+\partial S/\partial H=-1$:

(10.1) $\mathcal{L} = U(Q,H) + \lambda(w(L+S) - QP_e)$

(10.2) $\frac{\partial \mathcal{L}}{\partial Q} = \frac{\partial U}{\partial Q} - \lambda P_e = 0$

(10.3) $\frac{\partial \mathcal{L}}{\partial H} = \frac{\partial U}{\partial H} + \lambda w\left(\frac{\partial L}{\partial H} + \frac{\partial S}{\partial H}\right) = \frac{\partial U}{\partial H} - \lambda w = 0$

(10.4) $\frac{\partial U/\partial H}{\partial U/\partial Q} = MRS(H\ for\ Q) = \frac{w}{P_e}$

It means that a satisficing choice with respect to the purchase price P has its implicit optimal replication with respect to the equilibrium price P_e.

So, the consumer enters into farmer's space with the following preferences:

(10.5) $MRS\ (H\ for\ Q) = \frac{Q}{L+S} = \frac{w}{P_e}$

However, if he pays the *PQ* value with satisfaction, we should confirm that his purchase is also optimal. But our means to prove this fact are very limited. The only thing we know that this *PQ* value represents a point in farmer's coordinate system, where it is described by $(T_d;T_f)$ allocation of his time.

The only mean, which can prove that the purchase is optimal, is the geometric normal to this point, taken from the origin of producer's coordinates, i.e., from the moment when he decides how to allocate his time between farming and delivery. But this normal tells us that the marginal rate of substitution of leisure for consumption is equal to the marginal rate of transformation of farming into delivery:

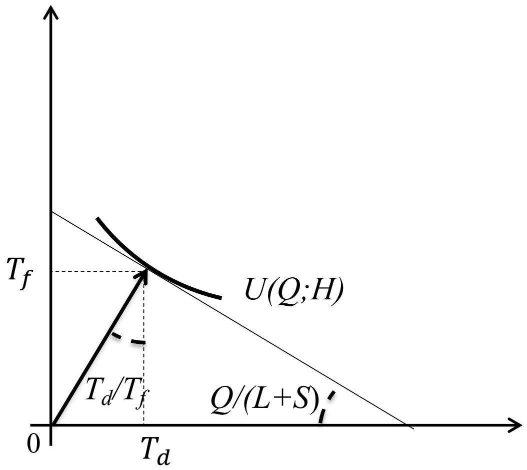

Fig.2.The optimal consumption-leisure choice
for the given allocation of producer's time

$$(11)\quad MRS\ (H\ for\ Q) = \frac{Q}{L+S} = \frac{w}{P_e} = \frac{T_d}{T_f} = \frac{MC_d}{MC_f} = RPT_{FD}$$

By this way we can construct the set of multiple equilibria under wage and price dispersion along the way from the farm to the downtown.

Now we can rewrite Eq.7 in the following form:

$$(12)\quad P = MC_f\left(1+\frac{T_d}{T_f}\right) = MC_f\left(1+\frac{Q}{L+S}\right) = MC_f\left(1+\frac{w}{P_e}\right)$$

The divine proportions of the general competitive equilibrium

The general competitive equilibrium also represents the local equilibrium, this time 'at the doors' of the consumer with zero search costs and with the equilibrium wage rate w_e who pays the equilibrium price P_e:

This consideration transforms Eq. 12:

$$(13)\quad P_e = MC_f\left(1+\frac{T_d}{T_f}\right) = MC_f\left(1+\frac{w_e}{P_e}\right)$$

Eq.13 also can be transformed, now with the help of the assumption that the competitive marginal costs of farming are equal to the equilibrium wage rate or $MC_f=w_e$:

$$(14)\quad P_e = MC_f\left(1+\frac{w_e}{P_e}\right) = w_e\left(1+\frac{w_e}{P_e}\right)$$

We're making two more steps and get the following result:

$$(15.1)\quad \frac{P_e}{1+\frac{w_e}{P_e}} = w_e; \frac{P_e}{P_e+w_e} = \frac{w_e}{P_e}$$

$$(15.2)\quad \frac{P_e}{P_e+w_e} = \frac{w_e}{P_e} = \frac{a}{a+b} = \frac{b}{a} = 0{,}61803398\ldots = \frac{1}{\varphi} = \Phi$$

It means that ***at the general competitive equilibrium both the marginal rate of transformation of production into services and the marginal rate of substitution of leisure for consumption are equal to the golden ratio conjugate Φ***:

$$(15)\quad \frac{a}{a+b} = \frac{b}{a} = \frac{1}{\varphi} = \Phi = RPT_{FD} = \frac{MC_d}{MC_f} = \frac{w_e}{P_e} = -\frac{\partial Q}{\partial H} = MRS(H\ for\ Q)$$

But the equilibration doesn't stop here. It states the fact that the equilibrium price is the 'just price' because it is formed by the equilibrium trade mark-up m_e:

$$(16.1)\quad m_e = \frac{P_e - AC_f}{AC_f} = \frac{P_e - MC_f}{MC_f}$$

$$(16.2)\quad P_e = MC_f\left(1+\frac{w_e}{P_e}\right) = MC_f(1+m_e) = 1{,}61803398\ldots MC_f = \varphi MC_f$$

or ***the equilibrium mark-up is equal to the golden ratio conjugate while the equilibrium sales-costs of production ratio is equal to the golden ratio φ itself***.

Conclusion

We know that many academicians esteem the most famous Adam Smith's metaphor as the manifestation of his Calvinist background.[2] This paper brings us back to more ancient times, where we can find a friendship between two beautiful minds. Luca Pacioli, the founding father of modern accounting, was aware of the sacred nature of the golden ratio; he studied it and finally called La Proporzione

[2] One of the best reviews of theological grounds of Adam Smith's works is presented in Oslington (2012) (S.M.)

Divina. And he shared that idea with Leonardo da Vinci, who perpetuated it in his Vitruvian Man. Now we can understand that Adam Smith could really have good theological reasons for his best guess.

References

Malakhov, S. (2020) 'Proof of the Invisible hand: why the satisficing purchase becomes optimal?', *Terra Economicus*, 18 (2),70-84.

Malakhov, S. (2021a) ‘Force of Invisible Hand, Labor, and Just Price: basic principles of sorting and matching under wage and price dispersion.’ *Journal of Institutional Studies*, 13 (1),37-59.

Malakhov, S (2021b) ‘Equilibrium of Invisible Hand in the Family: the gravitation between men and women in marriage markets’. *Russian Journal of Economic Theory*, 18(3), 357-373.

Smith, A. (2000) ‘The Wealth of Nations.’ Random House, 1200 p.

Stigler, G. (1961) ‘The Economics of Information.’ *Journal of Political Economy*, 69(3), 213-225.

Oslington, P. (2011) ‘God and the Market: Adam Smith’s Invisible Hand.’ *Journal of Business Ethics* 108, 4, 429-438.