Three Remarks On Asset Pricing

Olkhov, Victor

Independent

24 July 2021

Three Remarks On Asset Pricing

Victor Olkhov

Moscow, Russia

victor.olkhov@gmail.com

ORCID: 0000-0003-0944-5113

Abstract

We consider the time interval Δ during which the market trade time-series are averaged as the key factor of the consumption-based asset-pricing model that causes modification of the basic pricing equation. The duration of Δ determines Taylor series of investor's utility over current and future values of consumption. We present consumption at current and future moments as sums of their mean values and perturbations during Δ of the price at current moment t and perturbations of the payoff at day $t+1$. For linear and quadratic Taylor series approximations of the basic equation we obtain new relations on mean price, mean payoff, their volatilities, skewness and amount of asset ξ_{max} that delivers max to investor's utility. The stochasticity of market trade time-series defines random properties of the asset price time-series during Δ. We introduce new market-based price probability measure entirely determined by frequency-based probability measures of the market trade value and volume. The conventional frequency-based price probability is a very special case of the market-based price probability measure when all trade volumes during Δ equal unit. Prediction of the market-based price probability at horizon T equals forecast of the market trade value and volume probabilities at same horizon. The similar Taylor series and probability measures alike to market-based price probability can be used as approximations of different versions of asset pricing, financial and economic models that describe relations between economic and financial variables averaged during some time interval Δ.

Keywords : asset pricing, volatility, price probability, market trades

JEL: G12

This research received no support, specific grant or financial assistance from funding agencies in the public, commercial or nonprofit sectors. We welcome funding our studies.

1. Introduction

Predictions of asset prices define the main desires of investors, traders and all the participants of financial markets. Last decades give a great progress in asset price valuation and setting. Starting with Hall and Hitch (1939) many researchers investigate the price theory (Friedman 1990; Heaton and Lucas 2000) and the factors those impact markets (Fama 1965), equilibrium economy (Sharpe 1964), fluctuations (Mackey 1989) macroeconomics (Cochrane and Hansen 1992) and business cycles (Mills 1946; Campbell 1998). Muth (1961) initiated studies on the dependence of asset pricing on the expectations and numerous scholars developed his ideas further (Lucas 1972; Malkiel and Cragg 1980; Campbell and Shiller 1988; Greenwood and Shleifer 2014). Many researchers describe the price dynamics and references (Goldsmith and Lipsey 1963; Campbell 2000; Cochrane and Culp 2003; Borovička and Hansen 2012; Weyl 2019) give only a small part of them.

Asset pricing depends on price fluctuations and volatility. The mean price trends and the price volatility are the most important issues that impact investors' expectation. Description of volatility is inseparable from price modeling (Hall and Hitch 1939; Fama 1965; Stigler and Kindahl 1970; Tauchen and Pitts 1983; Schwert 1988; Mankiw, Romer and Shapiro 1991; Brock and LeBaron 1995; Bernanke and Gertler 1999; Andersen et.al. 2001; Poon and Granger 2003; Andersen et.al. 2005). The list of references can be continued as hundreds and hundreds of articles describe different faces of the price-volatility puzzle.

Simple and practical advises on the price modeling and forecasting among the most demanded by investors. Different price models were developed to satisfy and saturate investors' desires. We refer only some pricing models (Ferson et.al. 1999; Fama and French 2015) and studies on Capital Asset Pricing Model (CAPM) (Sharpe 1964; Merton 1973; Cochrane 2001; Perold 2004). Cochrane (2001) shows that CAPM includes different versions of asset pricing as ICAPM and consumption-based pricing model (Campbell 2002) are CAPM variations. Further we consider Cochrane (2001) as clear and consistent presentation of CAPM basis, problems and achievements. Resent study (Cochrane 2021) complements the rigorous asset price description with deep and justified general considerations of the nature, problems and possible directions for further research.

Despite the fact that asset pricing, risk, uncertainties and financial markets were studied with a great accuracy and solidity there are still "some" problems left. We assume that the core economic difficulties and the fundamental economic relations may still impede further significant development of the price theory. To explain the nature of the existing economic obstacles that may hamper price forecasting we consider three interrelated remarks that impact asset pricing.

We outline that any averaging of economic and financial variables presented by time-series is performed during some time interval *Δ*. The choice of *Δ* allows derive Taylor series of the basic pricing equation for variables averaged during *Δ*. Linear and quadratic approximations by price and payoff variations during *Δ* give simple equations on mean price and payoff, their volatilities, skewness and other factors that define CAPM. The core factor of any asset price model is the price probability measure. We present the reasons to replace the conventional price probability *P(p)* proportional to the frequency of trades at price *p* during an interval *Δ* by a different price probability measure entirely determined by the probability measures of the market trade value and volume time-series during *Δ*. Indeed, price $p(t_i)$ of any particular trade at moment t_i is a coefficient between the trade value $C(t_i)$ and the trade volume $U(t_i)$

$$C(t_i) = p(t_i)U(t_i) \tag{1.1}$$

Time-series of market trade value $C(t_i)$ and volume $U(t_i)$ during *Δ* determine the market value and volume probabilities proportional to the frequency of trades at particular value and volume. It impossible independently define probabilities of three variable - trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ – those match equation (1.1). For any given *Δ* and probabilities of the trade value and volume, price probability must be result of equation (1.1). We define a market-based price probability measure that match (1.1) as function of probability measures of the market trade value and volume time-series during *Δ*. The market-based probability measure reflects randomness of market trades and its prediction at horizon *T* equals forecasting market probability measures at same horizon *T*.

It is convenient consider asset pricing having the single reference that describes almost all extensions and model variations within the uniform frame. We propose that readers are familiar with Cochrane (2001) and refer this monograph for any notions and clarifications.

In Sec.2 we remind main CAPM notions. In Sec.3 we consider remarks on the time scales and introduce the averaging interval *Δ* of the market trade and price time-series. In Sec.4 – remarks on Taylor series generated by the averaging interval *Δ*. We expand the utility functions by Taylor series and in linear and quadratic approximations by the price and payoff variations to consider the idiosyncratic risk, the utility max conditions and the impact of price-volume correlations. In Sec.5 we introduce the new market-based price probability measure and briefly consider its implications on asset pricing. Sec.7 – Conclusion. In App.A we collect some calculations that define maximum of investor's utility. In App.B we present simple approximations of the price characteristic function.

Equation (4.5) means equation 5 in the Sec. 4 and (A.2) – notes equation 2 in Appendix A. We assume that readers are familiar with basic notions of probability, statistical moments, characteristic functions and etc.

2. Brief CAPM Assumptions

The general frame that determines all CAPM versions and extensions states: "All asset pricing comes down to one central idea: the value of an asset is equal to its expected discounted payoff" (Cochrane 2001; Cochrane and Culp 2003; Hördahl and Packer 2007; Cochrane 2021). Let's follow (Cochrane 2001) and briefly consider CAPM notions and assumptions. The basic consumption-based equation has form:

$$p = E[m\, x] \tag{2.1}$$

In (2.1) p denotes the asset price at moment t, $x=p_{t+1}+d_{t+1}$ – payoff,, p_{t+1} - price and d_{t+1} - dividends at moment $t+1$, m - the stochastic discount factor and E – mathematical expectation at moment $t+1$ made by the forecast under the information available at moment t. Cochrane (2001) considers relations (2.1) in various forms to show that almost all models of asset pricing united by the title CAPM can be described by the similar equations. We shall consider (2.1) and refer (Cochrane 2001) for all CAPM extensions. For convenience we briefly reproduce consumption-based derivation of (2.1). Cochrane "models investors by a utility function defined over current c_t and future c_{t+1} values of consumption. c_t and c_{t+1} denotes consumption at date t and $t+1$."

$$U(c_t;\ c_{t+1}) = u(c_t) + \beta E[u(c_{t+1})] \tag{2.2}$$

$$c_t = e_t - p\xi \quad ; \quad c_{t+1} = e_{t+1} + x\xi \tag{2.3}$$

$$x = p_{t+1} + d_{t+1} \tag{2.4}$$

Here (2.3) e_t and e_{t+1} "denotes original consumption level (if the investor bought none of the asset), and ξ denotes the amount of the asset he chooses to buy" (Cochrane 2001). A payoff x (2.4) is determined by a price p_{t+1} and a dividend d_{t+1} of asset at moment $t+1$. Cochrane calls β as "subjective discount factor that captures impatience of future consumption". *E[...]* in (2.2) denotes mathematical expectation of the random utility due to the random payoff x (2.4) made at moment $t+1$ by forecast on base of information available at moment t. The first-order maximum condition for (2.2) by amount of asset ξ is fulfilled by putting derivative of (2.2) by ξ equals zero (Cochrane 2001):

$$\max_{\xi} U(c_t;\ c_{t+1}) \leftrightarrow \frac{\partial}{\partial \xi} U(c_t;\ c_{t+1}) = 0 \tag{2.5}$$

From (2.2-2.5) it is obvious that:

$$p = \beta E\left[\frac{u'(c_{t+1})}{u'(c_t)}\, x\right] = E[mx] \quad ; \quad m = \beta \frac{u'(c_{t+1})}{u'(c_t)} \quad ; \quad \frac{d}{dc} u(c) = u'(c) \qquad (2.6)$$

and (2.6) reproduces (2.1) for m (2.6). This completes the brief derivation of the basic equation (2.1; 2.6) and we refer Cochrane (2001) for any further details.

3. Remarks on Time Scales

We start with simple remarks on averaging procedure and time scales. Any averaging of market trade time-series delivers the mean values during some time interval *Δ*. The averaging procedure can be different but any such procedure aggregates the time-series during an interval *Δ*. The choice of the averaging interval *Δ* defines the *internal* time scale of the problem under consideration. Prediction of asset price at time-horizon *T* at "the next day" *t+1* = *t+T* defines the *external* time scale of the problem. Relations between the *internal Δ* and *external T* scales determine evolution of the averaged variables, sustainability and accuracy of the model description. Financial variables – price, volatility, beta – averaged during the interval *Δ* can behave irregular or randomly on time scales *T* for *T*>> *Δ*. This effect mentioned, for example, by Cochrane (2021): "Another great puzzle is how little we know about betas. In continuous-time diffusion theory, 10 seconds of millisecond data should be enough to measure betas with nearly infinite precision. In fact, betas are hard to measure and unstable over time". It's clear, that averaging of time-series during the interval *Δ* smooth perturbations with scales less than *Δ*. If factors that disturb the market have a time scale *d* > *Δ* then variables averaged during *Δ* will demonstrate irregular or random properties during the term *T*. It is clear that the price, payoff and discount factor are under impact of the disturbing factors with different time scales. Eventually, the choice of the averaging interval *Δ* is important for asset pricing, but sadly it is not the main trouble.

As we note the averaging interval *Δ* defines the *internal* time scale of the problem. In simplest case averaging of the price time-series during the interval *Δ* that equals 1 min, 1 hour, 1 day, 1 week establish the least time divisions of "the Clocks" of the problem under consideration that equal 1 min, 1 hour, 1 day, 1 week and moments of time *t(k)* of the model after averaging during *Δ* take form

$$t(k) = t(0) + k\,\Delta \quad ; \quad k = 0, \pm 1, \pm 2, \ldots \qquad (3.1)$$

Here t_0 can be the moment "to-day". It is reasonable use the same time scale divisions "to-day" at moment *t* and the "next-day" at *t+1*. Indeed, time scale divisions can't be measured "to-day" in hours and "next-day" in weeks. The utility (2.2) "to-day" at moment *t* and the "next-day" at *t+1* should have the same time divisions. Averaging of any time-series at the "next-day" at $t+1$ during the interval Δ undoubtedly implies averaging "to-day" at moment t during same time interval Δ and vise-versa. Thus, as the utility (2.2) is averaged at t+1 during some interval Δ, then the utility (2.2) also should be averaged at moment t and take form:

$$U(c_t;\ c_{t+1}) = E_t[u(c_t)] + \beta E[u(c_{t+1})] \tag{3.2}$$

We denote $E_t[..]$ in (3.2) as mathematical expectation "to-day" at moment t during Δ. It does not matter how one considers the price time-series "to-day" – as random or as irregular. Mathematical expectation $E_t[..]$ performs smoothing of the random or irregular time-series via aggregating data during Δ under particular probability measure. Mathematical expectations $E_t[..]$ and $E[..]$ within identical averaging intervals Δ establish identical time division of the problem at moments t and $t+1$ in (3.2). Hence, relations similar to (2.5; 2.6) should derive modification of the basic pricing equation in the form:

$$E_t[p\, u'(c_t)] = \beta E\ [x\, u'(c_{t+1})] \tag{3.3}$$

Cochrane (2001) takes "subjective discount factor" β as non-random and we follow same assumption. Mathematical expectation in the left side $E_t[...]$ assesses mean price p at moment t during Δ. In the right side $E[xu'(c_{t+1})]$ on base of data available at moment t forecasts the average of $xu'(c_{t+1})$ at moment $t+1$ within the same averaging interval Δ.

Brief resume 1. The choice of the averaging interval Δ, transition to time divisions (3.1) and analysis of dependence of mean variables on duration of Δ establish the key problems of any economic or financial model that describe relations between averaged variables.

4. Remarks on Taylor series

Relation (2.5) presents first-order condition at point ξ_{max} that delivers maximum to investor's utility (2.2) or (3.2). Let us choose the averaging interval Δ and take the price p at moment t during the interval Δ and the payoff x at moment $t+1$ during the interval Δ as:

$$p\ =\ p_0 + \delta p\ ; \quad x = x_0 + \delta x \tag{4.1}$$

$$E_t[p] = p_0\ ;\ E[x] = x_0\ ;\ E_t[\delta p] = E[\delta x] = 0\ ;\ \sigma^2(p) = E_t[\delta^2 p]\ \ ;\ \ \sigma^2(x) = E[\delta^2 x] \tag{4.2}$$

The relations (4.1; 4.2) give the average price p_0 and its volatility $\sigma^2(p)$ at moment t and the average payoff x_0 its volatility $\sigma^2(x)$ at moment $t+1$. We underline, that we consider averaging during Δ as averaging of a random or as smoothing of an irregular variable. Thus $E_t[p]$ – at moment t smooth random or irregular price p during Δ and $E[x]$ – averages the random payoff x during Δ at moment $t+1$. We call both procedures as mathematical expectations. We present the derivatives of utility functions in (3.3) by Taylor series in a linear approximation by δp and δx during Δ:

$$u'(c_t) = u'(c_{t;0}) - \xi u''(c_{t;0})\delta p \quad ; \quad u'(c_{t+1}) = u'(c_{t+1;0}) + \xi u''(c_{t+1;0})\delta x \tag{4.3}$$

$$c_{t;0} = e_t - p_0\xi \;\; ; \quad c_{t+1;0} = e_{t+1} + x_0\xi$$

Now substitute (4.3) into (3.3) and obtain equation (4.4):

$$u'(c_{t;0})p_0 - \xi u''(c_{t;0})\sigma^2(p) = \beta u'(c_{t+1;0})x_0 + \beta\xi u''(c_{t+1;0})\sigma^2(x) \qquad (4.4)$$

Taylor series are simplest entry-level mathematical tool like as ordinary derivatives and we see no sense refer any studies those also use Taylor or ordinary derivatives in asset pricing. However, Cochrane (2001) uses Taylor expansions. We underline: Taylor series and (4.1-4.4) are determined by the duration of the averaging interval Δ. The change of Δ implies change of the mean price p_0, the mean payoff x_0 and their volatilities $\sigma^2(p)$, $\sigma^2(x)$ (4.2). Equation (4.4) is a linear approximation by the price and payoff fluctuations of the first-order max conditions (2.5) and assesses the root ξ_{max} that delivers maximum to the utility $U(c_t;c_{t+1})$ (3.2)

$$\xi_{max} = \frac{u'(c_{t;0})p_0 - \beta u'(c_{t+1;0})x_0}{u''(c_{t;0})\sigma^2(p) + \beta u''(c_{t+1;0})\,\sigma^2(x)} \qquad (4.5)$$

We note that (4.5) is not an "exact" equation on ξ_{max} as utilities u' and u'' depend also on ξ_{max} as it follows from (4.3). However, (4.5) gives an assessment of ξ_{max} in a linear approximation by Taylor series δp and δx averaged during Δ. Let underline that the ξ_{max} (4.5) depends on the price volatility $\sigma^2(p)$ at moment t and on the payoff volatility $\sigma^2(x)$ at moment $t+1$ (4.2).

It is clear that sequential iterations may give more accurate approximations of ξ_{max}. Nevertheless, our approach and (4.5) give a new look on the basic equation (2.6; 3.3). If one follows the standard derivation of (2.6) (Cochrane, 2001) and neglects the averaging at moment t in the left side (3.3), then (2.6; 4.5) give

$$\xi_{max} = \frac{u'(c_t)p - \beta u'(c_{t+1;0})x_0}{\beta u''(c_{t+1;0})\sigma^2(x)} \qquad (4.6)$$

Relations (4.6) show that even the standard form of the basic equation (2.6) hides dependence of ξ_{max} on the payoff volatility $\sigma^2(x)$ at moment $t+1$. If one has the independent assessment of ξ_{max} then can use it to present (4.6) in a way alike to the basic equation (2.6):

$$p = \frac{u'(c_{t+1;0})}{u'(c_t)}\beta x_0 + \xi_{max}\frac{u''(c_{t+1;0})}{u'(c_t)}\beta\sigma^2(x) \qquad (4.7)$$

One can transform (4.7) alike to (2.6):

$$p = m_0 x_0 + \xi_{max} m_1 \sigma^2(x) \qquad (4.8)$$

$$m_0 = \frac{u'(c_{t+1;0})}{u'(c_t)}\beta \;\; ; \;\; m_1 = \frac{u''(c_{t+1;0})}{u'(c_t)}\beta \qquad (4.9)$$

For the given ξ_{max} equation (4.8) in a linear approximation by Taylor series describes dependence of the price p at moment t (3.1) on the mean discount factors m_0 and m_1 (4.9), the mean payoff x_0 (4.1) and the payoff volatility $\sigma^2(x)$ during Δ. Let underline that while the mean discount factor $m_0>0$, the mean discount factor $m_1<0$ because the utility $u'(c_t)>0$ and $u''(c_t)<0$ for all t. Hence, for (4.8) valid:

$$p < m_0 x_0 \quad ; \quad \xi_{max} m_1 \sigma^2(x) < 0$$

We underline that (4.6-4.9) have sense for the given value of ξ_{max}. Equation (4.8) in a linear approximation by Taylor series δx during the interval Δ describes the modified CAPM statement: *the value of an asset is equal the mean payoff x_0 discounted by the mean factor m_0 minus payoff volatility $\sigma^2(x)$ discounted by factor $|m_1|$ and multiplied by the amount of asset ξ_{max} that delivers maximum to the investor's utility (2.2).* As the price p in (4.8) should be positive hence ξ_{max} should obey inequality (4.10):

$$0 < \xi_{max} < -\frac{u'(c_{t+1;0})}{u''(c_{t+1;0})} \frac{x_0}{\sigma^2(x)} \tag{4.10}$$

Taking into account (4.3) it is easy to show for (4.10) that for the conventional power utility (Cochrane 2001) (A.2):

$$u(c) = \frac{1}{1-\alpha} c^{1-\alpha} \quad ; \quad \frac{u'(c)}{u''(c)} = -\frac{c}{\alpha} \quad ; \quad 0 < \alpha \le 1$$

inequality (4.10) valid always if

$$\alpha\, \sigma^2(x) < x_0^2$$

For this approximation (4.10) limits the value of ξ_{max}. If one takes (4.5) then obtains equations similar to (4.8; 4.9):

$$m_0 = \frac{u'(c_{t+1;0})}{u'(c_{t;0})} \beta > 0 \,;\; m_1 = \frac{u''(c_{t+1;0})}{u'(c_{t;0})} \beta < 0 \,;\; m_2 = \frac{u''(c_{t;0})}{u'(c_{t;0})} < 0 \tag{4.11}$$

$$p_0 = m_0 x_0 + \xi_{max}[m_1 \sigma^2(x) + m_2 \sigma^2(p)] \tag{4.12}$$

We use the same notions m_0, m_1 to denote the discount factors taking into account replacement of $u'(c_t)$ in (4.9) by $u'(c_{t;0})$ in (4.11; 4.12). Modified basic equation (4.12) at moment t describes dependence of the price p_0 on the price volatility $\sigma^2(p)$ at moment t, the mean payoff x_0 and payoff volatility $\sigma^2(x)$ at moment $t+1$ averaged during same interval Δ. Equation (4.15) illustrate well-known practice that high volatility $\sigma^2(p)$ of the price at moment t and high forecast of payoff volatility $\sigma^2(x)$ at moment $t+1$ may cause decline of the mean price p_0 at moment t. We leave the detailed analysis of (4.5-4.12) for the future.

4.1 The Idiosyncratic Risk

Here we briefly consider the case of the idiosyncratic risk for which the payoff x in (2.6) is not correlated with the discount factor m at moment $t+1$ (Cochrane 2001):

$$cov(m, x) = 0 \tag{4.13}$$

In this case equation (2.6) takes form:

$$p = E[mx] = E[m]E[x] + cov(m,x) = E[m]x_0 = \frac{x_0}{R_f} \quad (4.14)$$

The risk-free rate R^f in (4.14) is known ahead (Cochrane, 2001). Taking into account (4.3) in a linear approximation by δx Taylor series for derivative of the utility $u'(c_{t+1})$:

$$u'(c_{t+1}) = u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x \quad (4.15)$$

Hence, the discount factor m (2.6) takes form:

$$m = \beta\frac{u'(c_{t+1})}{u'(c_t)} = \frac{\beta}{u'(c_t)}[u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x]$$

$$E[m] = \bar{m} = \beta\frac{u'(c_{t+1;0})}{u'(c_t)}$$

$$\beta E\left[\frac{u'(c_{t+1})}{u'(c_t)}\right]x_0 = \frac{x_0}{R_f} \quad ; \quad E[u'(c_{t+1})x] = 0$$

and

$$\delta m = m - \bar{m} = \frac{\beta}{u'(c_t)}u''(c_{t+1;0})\xi\delta x$$

Hence, (4.13) implies:

$$cov(m,x) = E[\delta m\delta x] = \beta\frac{u''(c_{t+1;0})}{u'(c_t)}\xi_{max}\sigma^2(x) = 0 \quad (4.16)$$

That causes zero payoff volatility $\sigma^2(x)=0$. Of course zero payoff volatility does not model market reality but (4.16) reflects restrictions of the linear approximation (4.15). To overcome this discrepancy let take into account Taylor series up to the second degree by $\delta^2 x$:

$$u'(c_{t+1}) = u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x + u'''(c_{t+1;0})\xi^2\delta^2 x \quad (4.17)$$

$$m = \beta\frac{u'(c_{t+1})}{u'(c_t)} = \frac{\beta}{u'(c_t)}[u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x + u'''(c_{t+1;0})\xi^2\delta^2 x] \quad (4.18)$$

For this case the mean discount factor $E[m]$ takes form:

$$E[m] = \bar{m} = \frac{\beta}{u'(c_t)}[u'(c_{t+1;0}) + u'''(c_{t+1;0})\xi^2\sigma^2(x)] \quad (4.19)$$

and variations of the discount factor δm:

$$\delta m = m - \bar{m} = \frac{\beta}{u'(c_t)}[u''(c_{t+1;0})\xi\delta x + u'''(c_{t+1;0})\xi^2\{\delta^2 x - \sigma^2(x)\}$$

In this case

$$cov(m,x) = E[\delta m\delta x] = [u''(c_{t+1;0})\xi\sigma^2(x) + u'''(c_{t+1;0})\xi^2\gamma^3(x)] = 0 \quad (4.20)$$

$$\gamma^3(x) = E[\delta^3 x] \quad ; \quad Sk(x) = \frac{\gamma^3(x)}{\sigma^3(x)} \quad (4.21)$$

$Sk(x)$ – denotes normalized payoff skewness at moment $t+1$ treated as the measure of asymmetry of the probability distribution during Δ. For approximation (4.18) from (4.20; 4.21) obtain relations on the skewness $Sk(x)$ and ξ_{max}:

$$\xi_{max}\, Sk(x)\sigma(x) = -\frac{u''(c_{t+1;0})}{u'''(c_{t+1;0})} \tag{4.22}$$

For the conventional power utility (A.2) and (4.3) relations (4.22) take form

$$\xi_{max} = \frac{e_{t+1}}{(1+\alpha)Sk(x)\sigma(x) - x_0} \tag{4.23}$$

It is assumed that second derivative of utility $u''(c_{t+1})<0$ always negative and third derivative $u'''(c_{t+1})>0$ is positive and hence the right side in (4.22) is positive. Hence to get positive ξ_{max} for (4.23) for the power utility (A.2) the payoff skewness $Sk(x)$ should obey inequality (4.24) that defines the lower limit of the payoff skewness $Sk(x)$:

$$Sk(x) > \frac{x_0}{(1+\alpha)\sigma(x)} \tag{4.24}$$

In (4.14) R_f denotes known risk-free rate. Hence, (4.19; 4.22; 4.24) define relations:

$$\frac{\beta}{u'(c_t)}\left[u'(c_{t+1;0}) + u'''(c_{t+1;0})\xi_{max}{}^2\sigma^2(x)\right] = \frac{1}{R_f}$$

$$\xi_{max}{}^2\sigma^2(x) = \frac{1}{\beta R_f}\frac{u'(c_t)}{u'''(c_{t+1;0})} - \frac{u'(c_{t+1;0})}{u'''(c_{t+1;0})}$$

$$Sk^2(x) = \frac{R_f}{1-m_0R_f}\frac{m_1^2}{m_3} > \frac{x_0^2}{(1+\alpha)^2\sigma^2(x)} \;\; ; \;\; m_0 < 1/R_f$$

$$\frac{\sigma^2(x)}{x_0^2} > \frac{m_3}{m_1^2}\frac{1-m_0R_f}{(1+\alpha)^2R_f} \tag{4.25}$$

Inequality (4.25) establishes the lower limit on the payoff volatility $\sigma^2(x)$ normalized by the mean payoff x_0^2. The lower limit in the right side of (4.25) is determined by the discount factors (4.26), the risk-free rate R_f and the conventional power utility' factor α (A.2).

$$m_0 = \beta\frac{u'(c_{t+1;0})}{u'(c_t)} \;\; ; \; m_1 = \beta\frac{u''(c_{t+1;0})}{u'(c_t)} \;\; ; \;\; m_3 = \beta\frac{u'''(c_{t+1;0})}{u'(c_t)} \tag{4.26}$$

The coefficients in (4.26) differ a little from (4.1) as (4.26) takes the denominator $u'(c_t)$ instead of $u'(c_{t;0})$ in (4.11) but we use the same letters to avoid extra notations. The similar calculations for (3.2; 3.3) describe both the price volatility $\sigma^2(p)$ and the skewness $Sk(p)$ at moment t and the payoff volatility $\sigma^2(x)$ and the skewness $Sk(x)$ at moment $t+1$. Further approximations by Taylor series of the utility derivative $u'(c_t)$ up to δ^3p and $u'(c_{t+1})$ up to δ^3x similar to (4.17) give assessments of kurtosis of the price probability at moment t and kurtosis of the payoff probability at moment $t+1$ estimated during interval Δ. We leave these exercises for future.

4.2 The Utility Maximum

The relations (2.5) define the first-order condition that determines the amount of asset ξ_{max} that delivers the max to the utility $U(c_t;\, c_{t+1})$ (2.2; 3.2). To confirm that function $U(c_t;\, c_{t+1})$ has max at ξ_{max} , the first order condition (2.5) must be supplemented by condition:

$$\frac{\partial^2}{\partial \xi^2} U(c_t;\ c_{t+1}) < 0 \tag{4.27}$$

Usage of (4.27) gives interesting consequences. From (2.2–2.4) and (4.27) obtain:

$$p^2 > -\frac{\beta}{u''(c_t)} E[x^2\, u''(c_{t+1})\,] \tag{4.28}$$

Take the linear Taylor series expansion of the second derivative of the utility $u''(c_{t+1})$ by δx

$$u''(c_{t+1}) = u''\left(c_{t+1;0}\right) + u'''\left(c_{t+1;0}\right)\xi\delta x$$

Then (4.28) takes form:

$$p^2 > -\beta \frac{u''(c_{t+1;0})}{u''(c_t)} [x_0{}^2 + \sigma^2(x)] - \beta \frac{u'''(c_{t+1;0})}{u''(c_t)} \xi_{max}\ [2x_0\sigma^2(x) + \gamma^3(x)] \tag{4.29}$$

For the power utility (A.2) simple calculations (see App.A) give relations on (4.27; 4.29). If the payoff volatility $\sigma^2(x)$ multiplied by factor $(1+2\alpha)$ is less then mean payoff $x_0{}^2$ (4.30; A.5)

$$(1 + 2\alpha)\sigma^2(x) < x_0{}^2 \quad ; \quad \frac{1}{3} \le \frac{1}{1+2\alpha} < 1 \tag{4.30}$$

Then (4.29) is always valid. If payoff volatility $\sigma^2(x)$ is high (A.6)

$$(1 + 2\alpha)\sigma^2(x) > \ x_0{}^2$$

Then (4.29) valid only for ξ_{max} (A.6):

$$\xi_{max} < \frac{e_{t+1}[x_0{}^2 + \sigma^2(x)]}{x_0\ [(1 + 2\alpha)\sigma^2(x) - x_0{}^2]}$$

However, this upper limit for ξ_{max} can be high enough. The same but more complex considerations can be presented for (3.2).

$$E_t[p^2 u''(c_t)] < -E[\beta x^2\, u''(c_{t+1})\,]$$

Brief resume 2. Almost all financial and economic models and asset pricing models in particular, describe relations between averaged or smoothed variables. Usage of Taylor series determined by the duration of averaging interval Δ may help derive successive approximations determined by expansions by linear, quadratic, etc., fluctuations of the variables under consideration averaged during Δ.

5. Remarks on the Price Probability Measure

As usual the problems that are the most common and "obvious" hide the most difficulties. The price probability measure is exactly the case of such hidden complexity.

All asset pricing models assume that it is possible forecast the probability of random price *p* and payoff *x* at horizon *T*. Let us consider the choice and forecasting of the price probability measure as most interesting and complex problem of finance.

The usual treatment of the price *p* probability "is based on the probabilistic approach and using A. N. Kolmogorov's axiomatic of probability theory, which is generally accepted now"

(Shiryaev 1999). The conventional definition of the price probability is based on the frequency of trades at a price *p* during the averaging interval *Δ*. The economic ground of such choice is simple: it is assumed that each trade of *N* trades during *Δ* has equal probability ~ *1/N*. If there are *m(p)* trades at the price *p* then probability *P(p)* of the price *p* during *Δ* equals *m(p)/N*. The frequency of the particular event is absolutely correct, general and conventional approach to the probability definition. The conventional frequency-based approach to the price probability uses different assumptions on form of the price probability measure and checks how almost all standard probability measures fit the market random price. As standard probabilities we refer (Walck 2007; Forbes et.al. 2011). Parameters that define standard probabilities permit calibrate each in a manner that increase the plausibility and consistency with the observed market price time-series. For different assets, options and markets different standard probabilities are tested and applied to fit and predict the random price dynamics as well as possible.

However, one may ask a simple question: does the conventional frequency-based approach to the price probability fit the random market pricing? The asset price is a result of the market trade and it seems reasonable that the market trade randomness should govern the price stochasticity. We propose the new market-based price probability measure that is different from the conventional frequency based probability and is entirely determined by the probability measures of the market trades values and volumes.

Let note that almost 30 years ago the volume weighted average price (VWAP) was introduced and is widely used now (Berkowitz et.al. 1988; Buryak and Guo 2014; Busseti and Boyd 2015; Duffie and Dworczak 2018; CME Group 2020). The definition of the VWAP during the interval *Δ* is follows. Let take that during *Δ* (5.3) there are *N* market trades at moments $t_{i,}$ $i=1,...N$. Then the VWAP *p(1)* (5.1) at moment *t* equals

$$p(1) = \frac{1}{U(1)} \sum_{i=1}^{N} p(t_i)U(t_i) = \frac{C(1)}{U(1)} \quad ; \quad C(t_i) = p(t_i)U(t_i) \tag{5.1}$$

$$C(1) = \sum_{i=1}^{N} C(t_i) = \sum_{i=1}^{N} p(t_i)\, U(t_i) \quad ; \quad U(1) = \sum_{i=1}^{N} U(t_i) \tag{5.2}$$

$$\Delta = \left[t - \frac{\Delta}{2}\, , t + \frac{\Delta}{2}\right] \quad ; \quad t_i \in \Delta\, , \; i = 1, \ldots N \tag{5.3}$$

We consider time-series of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ as random variables during averaging interval *Δ* (5.3). Relations (1.1; 5.1) at moment t_i define the price $p(t_i)$ of trade value $C(t_i)$ and volume $U(t_i)$. The sum *C(1)* of values $C(t_i)$ (5.2) and sum *U(1)* of volumes $U(t_i)$ (5.2) of *N* trades during *Δ* (5.3) define the VWAP *p(1)* (5.1).

We hope that our readers can distinguish the difference between notations of consumption c_t (2.2; 2.3) and utility U (2.2) in Sections 2-4 and trade value $C(t_i)$ and volume $U(t_i)$ (5.1) in current Section.

It is obvious, that VWAP (5.1) can be equally determined (5.4) by the mean value $C_m(1)$ (5.5) and the mean volume $U_m(1)$ (5.6) of N trades during Δ:

$$C_m(1) = p(1)U_m(1) \tag{5.4}$$

Mean trade value $C_m(1)$ and mean trade volume $U_m(1)$ are determined by conventional frequency-based probabilities $\nu(C_k)$ of the trade value and $\mu(U_k)$ of the trade volume:

$$\nu(C_k) = \frac{1}{N}\, m(C_k) \;\; ; \; C_m(1) = E[C(t_i)] = \sum C_k\, \nu(C_k) = \frac{1}{N}\sum_{i=1}^{N} C(t_i) \tag{5.5}$$

$$\mu(U_k) = \frac{1}{N}\, m(U_k) \;\; ; \; U_m(1) = E[U(t_i)] = \sum U_k\, \mu(U_k) = \frac{1}{N}\sum_{i=1}^{N} U(t_i) \tag{5.6}$$

The mean VWAP price $p(1)$ (5.4) is a coefficient between the mean value $C_m(1)$ (5.5) and the mean volume $U_m(1)$ (5.6).

However, it is obvious that probabilities of trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ time-series that match equations (1.1; 5.1) cannot be determined independently. Given probabilities of trade value $C(t_i)$ and volume $U(t_i)$ time-series during Δ that match (1.1; 5.1) should determine the price probability measure. Asset pricing should follow the market trade probability distributions of value $C(t_i)$ and volume $U(t_i)$ time-series. Let note that relations (1.1; 5.1) for each particular trade at moment t_i give:

$$C^n(t_i) = p^n(t_i)U^n(t_i) \;\; ; \;\; n = 1, 2, \ldots \tag{5.7}$$

We use (5.7) to define *nth* statistical moment $p(n)$ of the price similar to (5.4; 5.7) as coefficient between *n-th* statistical moment $C_m(n)$ of the value and $U_m(n)$ of the volume

$$C_m(n) = p(n)U_m(n) \tag{5.8}$$

We define *n-th* statistical moments of the value $C_m(n)$ and volume $U_m(n)$ by the conventional frequency-based probabilities $\nu(C_k)$ and $\mu(U_k)$ (5.5; 5.6):

$$C_m(n) = \sum C_k^n\, \nu(C_k) = \frac{1}{N}\sum_{i=1}^{N} C^n(t_i) \;\; ; \;\; U_m(n) = \sum U_k^n\, \mu(U_k) = \frac{1}{N}\sum_{i=1}^{N(t)} U^n(t_i) \tag{5.9}$$

Averaging of (5.7) and definition (5.8) imply no correlations between *n-th* power of the volume $U^n(t_i)$ and *n-th* power of the price $p^n(t_i)$:

$$C_m(n) = E[C^n(t_i)] = E[p^n(t_i)U^n(t_i)] = E[p^n(t_i)]E[U^n(t_i)] = p(n)U_m(n) \tag{5.10}$$

However, relations (5.10) does not cause statistical independence between random volume $U(t_i)$ and random price $p(t_i)$. Not at all, from (5.7-5.9) one can obtain that *n-th* power of the random volume $U^n(t_i)$ correlates with *m-th* power of the random price $p^m(t_i)$ if $n \neq m$.

It is obvious that price statistical moments $p(n)$ (5.8) differ from statistical moments $\pi(n)$ generated by frequency-based price probability $P(p) \sim m(p)/N$:

$$\pi(n) = E[p^n(t_i)] = \frac{1}{N}\sum_{i=1}^{N} p^n(t_i)$$

$$\pi(n) = \frac{1}{N}\sum_{i=1}^{N} p^n(t_i) = \frac{1}{N}\sum_{i=1}^{N} \frac{C^n(t_i)}{U^n(t_i)} \neq \frac{\sum_{i=1}^{N} C^n(t_i)}{\sum_{i=1}^{N} U^n(t_i)} = \frac{C_m(n)}{U_m(n)} = p(n)$$

The difference between the frequency-based *π(n)* and market-based *p(n)* price statistical moments determine the distinctions between two approaches to definition of the price probability. Only if during *Δ* (5.3) all trade volumes equal unit $U(t_i)=1$ then price statistical moments *p(n)* equal statistical moments of market value $C_m(n)$ and for this special case the market trade price stochasticity is described by usual frequency-based probability measure *P(p) ~ m(p)/N.*

$$if\ U(t_i) = 1\ for\ all\ \ i = 1,..N, \qquad then \quad \pi(n) = C_m(n) = \ p(n)$$

The set of price statistical moments (5.8) determines Taylor series of the price characteristic function *F(x)* (Shephard 1991; Shiryaev 1999; Klyatskin 2005; 2015):

$$F(x) = 1 + \sum_{i=1}^{\infty} \frac{i^n}{n!} p(n)\, x^n \tag{5.11}$$

Fourier transform of the price characteristic function *F(x)* determines price probability measure *η(p)* and vise versa as:

$$\eta(p) = \int dx\, F(x) \exp(-ixp) \quad ; \quad F(x) = \int dp\, \eta(p) \exp(ixp) \tag{5.12}$$

$$p(n) = \frac{d^n}{(i)^n dx^n} F(x)|_{x=0} = \int dp\, \eta(p) p^n \tag{5.13}$$

For brevity in (5.12) we omit normalizing factors proportional to *(2π)*. However, Taylor series (5.11) of the price characteristic function *F(x)* do not allow directly define price probability measure *η(p)* via Fourier transform (5.12). Nevertheless, Taylor series (5.11) permit derive successive approximations of the characteristic function $F_k(x)$ and their Fourier transforms define successive approximations $\eta_k(p)$ of the price probability measure. In App.B we consider simple successive approximations of the price characteristic function that take into account finite number members of the Taylor series (5.11).

Prediction of the price probability measure *η(p)* (5.12) at horizon *T* equals forecasts of all statistical moments *p(n)* (5.8) and thus forecasts of all trade statistical moments $C_m(n)$ and $U_m(n)$ (5.9). That equals prediction of the market probability measures *ν(C)* of value and *μ(U)* volume (5.5; 5.6). One may consider the definition of the market-based price probability measure *η(p)* through the market trade probability measures *ν(C)* and *μ(U)* as formal mathematical expression of the famous phrase: “You can’t beat the market”.

As simple case let us consider the price volatility $\sigma^2(p)$ determined by price statistical moments *p(1)* and *p(2)* (5.8):

$$\sigma^2(p) = E_t\left[\left(p - p(1)\right)^2\right] = p(2) - p^2(1) = \frac{C_m(2)}{U_m(2)} - \frac{C_m^2(1)}{U_m^2(1)} \quad (5.14)$$

Description of price volatility $\sigma^2(p)$ (5.14) equals modeling statistical moments $C_m(1), C_m(2)$ of the value and $U_m(1), U_m(2)$ of the volume (5.9), or equally – modeling the $C(n)$ and $U(n)$, $n=1,2$ (5.15) of the trades during Δ:

$$C(n) = \sum_{i=1}^{N} C^n(t_i) = \sum_{i=1}^{N} p^n(t_i)U^n(t_i) \ ; \quad U(n) = \sum_{i=1}^{N} U^n(t_i) \quad (5.15)$$

$$C(n) = N\, C_m(n) \quad ; \quad U(n) = N\, U_m(n)$$

The price statistical moments $p(n)$ (5.8) and the price volatility $\sigma^2(p)$ (5.14) can be presented equally via $C(n)$ and $U(n)$ (5.15):

$$C(n) = p(n)U(n) \quad (5.16)$$

$$\sigma^2(p) = \frac{C(2)}{U(2)} - \frac{C^2(1)}{U^2(1)} \quad (5.17)$$

To forecast the price volatility $\sigma^2(p)$ (5.14; 5.17) one should predict the mean price $p(1)$ and the mean square price $p(2)$ and that implies forecasting of $C(1), C(2), U(1), U(2)$ (5.15-5.17). Forecasting of the mean price $p(1)$ averaged during Δ equals prediction of $C(1)$ and $U(1)$ (5.15) – sums of values $C(t_i)$ and volumes $U(t_i)$ of trades during Δ. Sums of the trade value $C(1)$ and volume $U(1)$ are composed by fist-degree variables and their evolution can be described by current economic theory that models macroeconomic variables determined as sums of the first-degree variables. Indeed, almost all macro variables are composed as sums of agents' variables of the first degree. Macro investment, credits, consumption and etc., are composed as sums (without duplication) of the first-degree investment, credits, consumption of all agents in the economy. Basically to some extend these variables are described by current macroeconomic theory.

However, price volatility $\sigma^2(p)$ (5.17) is a sample of the second-degree variable. Indeed, price volatility $\sigma^2(p)$ depends on $C(2)$ and $U(2)$ (5.15) determined as sums of squares of trade values and volumes during Δ. The similar second-degree macroeconomic variables can be determined as sums (without duplication) of squares of investment, credits, consumption and etc. of all agents in the economy. These second-degree macroeconomic variables can describe volatilities of macro investment, credits, consumption and etc. Description of the second-degree macro variables as well as description of $C(2)$ and $U(2)$ requires development of the second-order economic theory (Olkhov, 2021a). It is so just because no second-degree variables are considered in the current macroeconomic models at all. Moreover, description of the price skewness $Sk(p)$ requires *3-d* statistical moments $p(3)$ of price:

$$C(3) = p(3)U(3)$$

Hence, to predict the price skewness *Sk(p)* one should forecast *C(3)* and *U(3)* and should develop the third-order economic theory that models sums of the *3-d* power of the market trade values *C(3)* and volumes *U(3).* Forecasts of price kurtosis require development of the forth-order economic theory and so on.

However, above considerations do not determine correct or incorrect price probability measure. Economics is a social science and investors are free in their trade decisions based on their personal expectations, habits, beliefs, financial and social "myths & legends". Investors are free to choose any definition of the price probability measure they prefer.

Brief resume 3. Price probability measure should be determined by market probabilities of the trade value and volume. Relations similar to (5.1; 5.7) describe common link between variables in economic and finance. Returns, interest and growth rates, tax and consumption rates and etc. are determined as coefficients between two other variables at the same or at different moments. Definition of statistical moments $x(n)$ like (5.8) of economic or financial variable x via statistical moments $A_m(n)$ of variable A and $B_m(n)$ of variable B that are described similar to (5.1; 5.7):

$$A = x\,B$$

allows describe random properties of returns, interest and growth rates, economic and financial variables x using assumptions on probabilities of variables A and B.

6. Conclusion

1. We derive modification of the basic pricing equation (3.3) that takes into account averaging of investor's utility during Δ at moment t and $t+1$. The choice of Δ and description of the dependence of the mean price, payoff, volatilities and etc., on duration of Δ are important for any asset pricing model.

2. Taylor series expansions of common (2.6) and modified (3.3) basic pricing equations permit derive and new approximate relations on mean price, payoff, their volatilities, skewness and etc. As example of linear Taylor series expansion of (3.3) we mention (4.12)

$$p_0 = m_0 x_0 + \xi_{max}[m_1 \sigma^2(x) + m_2 \sigma^2(p)]$$

that describes mean price p_0 at t as function of mean payoff x_0 and payoff volatility $\sigma^2(x)$ at $t+1$, price volatility $\sigma^2(p)$ at t and amount of assets ξ_{max} that delivers max to investor's utility and equals the root of equation (3.3). More new results are presented in Sec.4. Taylor series expansions generated by the averaging interval Δ can de considered for any asset pricing models and for economic and financial models that consider relations between averaged variables. Taylor series expansions can be applied to any financial model that considers mean price, payoff or other averaged variables. Linear, quadratic or higher expansions of Taylor series can help describe relations on mean variables, their volatilities and etc.

3. We introduce new market-based price probability measure through definition of all price *n-th* statistical moments *p(n)* (5.8). Price statistical moments *p(n)* are determined by *n-th* statistical moments of the trade value $C_m(n)$ and volume $U_m(n)$ (5.9). Our approach to definition of the market-based price probability measure valid and should be applied to constructing probability measures of other non-additive macroeconomic and financial variables like interest rates, returns, growth rates and etc. Macroeconomic additive variables as investment and loans, trade value and volume, profits and consumption are determined as sum of corresponding variables of economic agents. Ratios of additive variables define non-additive variables as price, growth rates, interest rates and etc. For example price is the ratio of trade value to trade volume. GDP growth rate is the ratio of GDP at date *t* to GDP at *t-1*. Ratios of non-additive also can define "subordinate" non-additive variables. For example, ratios of prices at moment *t* and moment *t-1* define "subordinate" non-additive variables – returns. Economic modeling of all additive variables allows describe all non-additive variables. Frequency-based probabilities define random properties of additive economic variables like trade value, volume of loans, investment, profits, value-added and etc. Probability measures of additive variables determine probability measures of non-additive variables. Respectively, the probabilities of non-additive variables define probability measures of "subordinate" non-additive variable. For example, probabilities of price at moment *t* and *t-1* define probability measure of returns. Probability measures of returns are determined by set of statistical moments of the trade value and volume at moments *t* and *t-1*.

Current economic theories describe evolution of variables composed as sums of *1-st* degree variables of economic agents. Description of sums of *2-d* degree agent's variables requires development of *2-d* order economic theory (Olkhov, 2021a). Such models can help describe price volatility as function of *2-d* statistical moments of market trades. Dependence of the price probability on set of *n-th* statistical moments of the market trade value and volume creates certain difficulties for efficient price probability predictions.

Further description of the above relations, statistical moments, characteristic functions and probability measures for particular non-additive economic and financial variables will be the subject to future research.

Appendix A

Max of Utility

$$p^2 > -\beta \frac{u''(c_{t+1;0})}{u''(c_t)} [x_0{}^2 + \sigma^2(x)] - \beta \frac{u'''(c_{t+1;0})}{u''(c_t)} \xi_{max} [2x_0\sigma^2(x) + \gamma^3(x)] \qquad \text{(A.1)}$$

If the right side is negative then it is valid always. If the right side is positive – then there exist a lower limit on the price *p*. For simplicity, neglect term $\gamma^3(x)$ to compare with $2x_0\sigma^2(x)$ and take the conventional power utility *u(c)* (Cochrane 2001) as:

$$u(c) = \frac{1}{1-\alpha} c^{1-\alpha} \qquad \text{(A.2)}$$

Let us consider the case with negative right side for (A.1). Simple but long calculations give:

$$-\beta \frac{u''(c_{t+1;0})}{u''(c_t)} [x_0{}^2 + \sigma^2(x)] < \beta \frac{u'''(c_{t+1;0})}{u''(c_t)} \xi_{max}\, 2x_0\sigma^2(x)$$

$$\xi_{max}\, 2x_0\sigma^2(x) < -\frac{u'''(c_{t+1;0})}{u''(c_{t+1;0})} [x_0{}^2 + \sigma^2(x)] \qquad \text{(A.3)}$$

Let us take into account (A.2) and for (A.3) obtain:

$$\frac{u''(c)}{u'''(c)} = \frac{-\alpha c^{-\alpha-1}}{\alpha(1+\alpha)c^{-\alpha-2}} = -\frac{c}{1+\alpha} \quad ; \quad \xi_{max}\, 2x_0\sigma^2(x) < \frac{e_{t+1} + x_0\xi_{max}}{1+\alpha} [x_0{}^2 + \sigma^2(x)]$$

$$\xi_{max} x_0 \left[(1+2\alpha)\sigma^2(x) - x_0{}^2\right] < e_{t+1}[x_0{}^2 + \sigma^2(x)] \qquad \text{(A.4)}$$

Inequality (A.4) determines that the right side (A.1) is negative in two cases.

1. The left side in (A.4) is negative and

$$(1+2\alpha)\sigma^2(x) < x_0{}^2 \quad ; \quad \frac{1}{3} \le \frac{1}{1+2\alpha} < 1 \qquad \text{(A.5)}$$

Inequality (A.5) describes small payoff volatility $\sigma^2(x)$. In this case the right side of (A.1) is negative for all ξ_{max} and all price *p* and hence (4.27) that defines max of utility (2.5) is valid.

2. The left side in (A.4) is positive and

$$(1+2\alpha)\sigma^2(x) > x_0{}^2 \quad ; \quad \xi_{max} < \frac{e_{t+1}[x_0{}^2 + \sigma^2(x)]}{x_0\, [(1+2\alpha)\sigma^2(x) - x_0{}^2]} \qquad \text{(A.6)}$$

This case describes high payoff volatility and the upper limit on ξ_{max} to utility (2.5). Take the positive right side in (A.1). Then (A.4) is replaced by the opposite inequality

$$\xi_{max} x_0 \left[(1+2\alpha)\sigma^2(x) - x_0{}^2\right] > e_{t+1}[x_0{}^2 + \sigma^2(x)] \qquad \text{(A.7)}$$

It is valid for (A.6) only. (A.7) determines a lower limit on ξ_{max} to utility (2.5):

$$\xi_{max} > \frac{e_{t+1}[x_0{}^2 + \sigma^2(x)]}{x_0\, [(1+2\alpha)\sigma^2(x) - x_0{}^2]}$$

Appendix B

Approximations of the price characteristic function and probability measure

Presentation of price characteristic function by Taylor series expansion helps develop successive approximations of characteristic function. Derivation of approximation is a self-standing research and here we present few simple examples of such approximations only.

We consider the approximations of price characteristic function $F_k(x)$ and price probability measure $\eta_k(p)$ those fit simple condition. As such we require that approximation of the price characteristic function $F_k(x)$ and hence probability measure $\eta_k(p)$ determined by Fourier transform (5.12) define first k price statistical moments $p_k(n)$ as:

$$p_k(n) = p(n) = \frac{C_m(n)}{U_m(n)} \;\; ; \;\; n \leq k \qquad \text{(B.1)}$$

As $p_k(n)$ we denote price *n-th* statistical moments determined by approximation of price characteristic function $F_k(x)$ or probability measure $\eta_k(p)$:

$$p_k(n) = \frac{d^n}{(i)^n dx^n} F_k(x)|_{x=0} = \int dp\, \eta_k(p) p^n \qquad \text{(B.2)}$$

Price Statistical moments $p_k(n)$ for $n>k$ will be different from statistical moments $p(n)$ (5.10) but first k moments $p_k(n)$, $n \leq k$ will be equal to $p(n)$ (5.10).

Let us approximate price characteristic $F_k(x)$ function as

$$F_k(x) = \exp\left\{\sum_{m=1}^{k} \frac{i^n}{m!}\, a_m\, x^m\right\} \;\; ; \;\; k = 1,2,.. \qquad \text{(B.3)}$$

and require (B.1; B.2). Then obtain simple approximations.

For $k=1$ the approximate price characteristic function $F_1(x)$ and measure $\eta_1(p)$ are trivial:

$$F_1(x) = \exp\{i\, a_1 x\} \;\; ; \; p(1) = -i \frac{d}{dx} F_1(x)|_{x=0} = a_1 \qquad \text{(B.4)}$$

$$\eta_1(p) = \int dx \;\; A_1(x)\, exp -ipx = \delta\big(p - \; p(1)\big) \qquad \text{(B.5)}$$

For $k=2$ approximation $F_2(x)$ describes the Gaussian probability measure $\eta_2(p)$:

$$F_2(x) = \exp\left\{i\, p(1)x - \frac{a_2}{2} x^2\right\} \qquad \text{(B.6)}$$

It is easy to show (B.1; B.2) that

$$p_2(2) = -\frac{d^2}{dx^2} F_2(x)|_{x=0} = a_2 + p^2(1) = p(2) \qquad \text{(B.7)}$$

Hence:

$$a_2 = p(2) - p^2(1) = \sigma^2(p) \qquad \text{(B.8)}$$

Coefficient a_2 equals price volatility $\sigma^2(p)$ (5.14) and Fourier transform (2.3) for $F_2(x)$ gives Gaussian price probability measure $\eta_2(p)$:

$$\eta_2(p) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma(p)} \exp\left\{-\frac{(p-p(1))^2}{2\sigma^2(p)}\right\} \quad \text{(B.9)}$$

For *k=3* approximation $F_3(x)$ has form:

$$F_3(x) = \exp\left\{i\, p(1)x - \frac{\sigma_p^2(t)}{2}x^2 - i\, a_3 x^3\right\} \quad \text{(B.10)}$$

One can obtain (B.1; B.2) for a_3:

$$p_3(3) = i\frac{d^3}{dx^3}F_3(x)|_{x=0} = a_3 + 3p(1)\sigma^2(p) + p^3(1) = p(3)$$

$$a_3 = p(3) - 3p(1)\sigma^2(p) - p^3(1) \quad \text{(B.11)}$$

Coefficient a_3 defines price skewness *Sk(p)* as:

$$Sk(p) = E\left[\left(p - p(1)\right)^3\right] = a_3 + 3p^3(1) \quad \text{(B.12)}$$

Derivation of price probability measure $\eta_3(p)$ determined by characteristic function $F_3(x)$ (B.10; B.11) and further approximations $F_k(x)$ for $k>3$ require separate consideration.

Nevertheless Gaussian approximation $F_2(x)$, $\eta_2(p)$ is trivial, (B.8; B.9) uncovers direct dependence of Gaussian price volatility $\sigma^2(p)$ (B.8; 5.14) on *2d* statistical moments of the trade value $C_m(2)$ and volume $U_m(2)$. Thus prediction of price volatility $\sigma^2(p)$ for Gaussian measure $\eta_2(p)$ (B.9) should follow non-trivial forecasting of the market trade value $C_m(2)$ and volume $U_m(2)$

References

Andersen, T., Bollerslev, T., Diebold, F.X, Ebens, H. (2001). The Distribution of Realized Stock Return Volatility, Journal of Financial Economics, 61, 43-76

Andersen, T.G, Bollerslev, T., Christoffersen, P.F., Diebold, F.X. (2005). Volatility Forecasting, CFS WP 2005/08, 1-116

Berkowitz, S.A., Dennis, E., Logue, D.E., Noser, E.A. Jr. (1988). The Total Cost of Transactions on the NYSE, *The Journal of Finance*, 43, (1), 97-112

Bernanke, B., Gertler, M. (1999). Monetary Policy and Asset Price Volatility. FRB of Kansas City, Economic Review, 4Q, 1-36

Borovička, J., Hansen, L.P. (2012). Examining Macroeconomic Models through the Lens of Asset Pricing. FRB Chicago

Brock, W.A., LeBaron, B.D. (1995). A Dynamic structural model for stock return volatility and trading volume. NBER, WP 4988, 1-46

Buryak, A., Guo, I. (2014). Effective And Simple VWAP Options Pricing Model, Intern. J. Theor. Applied Finance, 17, (6), 1450036, https://doi.org/10.1142/S0219024914500356

Busseti, E., Boyd, S. (2015). Volume Weighted Average Price Optimal Execution, 1-34, arXiv:1509.08503v1

Campbell, J.Y., Shiller, R.J. (1988). Stock Prices, Earnings And Expected Dividends. NBER, WP 2511, 1-42

Campbell, J.Y. (1998). Asset Prices, Consumption, and the Business Cycle. NBER, WP6485

Campbell, J.Y. (2000). Asset Pricing at the Millennium. Jour. of Finance, 55(4), 1515-1567

Campbell, J.Y. (2002). Consumption-Based Asset Pricing. Harvard Univ., Cambridge, Discussion Paper # 1974, 1-116

CME Group (2020) https://www.cmegroup.com/search.html?q=VWAP

Cochrane, J.H., Hansen, L.P.: Asset Pricing Explorations for Macroeconomics. Ed., Blanchard, O.J., Fischer, S. (1992). NBER Macroeconomics Annual 1992, v. 7, 115 – 182

Cochrane, J.H. (2001). Asset Pricing. Princeton Univ. Press, Princeton, US

Cochrane, J.H., Culp, C.L. (2003). Equilibrium Asset Pricing and Discount Factors: Overview and Implications for Derivatives Valuation and Risk Management. In: Modern Risk Management. A History, Ed. S.Jenkins, 57-92

Cochrane, J.H. (2021). Portfolios For Long-Term Investors. NBER, WP28513, 1-54

Duffie, D., Dworczak, P. (2018). Robust Benchmark Design. NBER, WP 20540, 1-56

Fama, E.F. (1965). The Behavior of Stock-Market Prices. J. Business, 38 (1), 34-105

Fama, E.F., French, K.R. (2015). A five-factor asset pricing model. J. Financial Economics, 116, 1-22

Ferson, W.E., Sarkissian, S., Simin, T. (1999). The alpha factor asset pricing model: A parable. J. Financial Markets, 2, 49-68

Forbes, C.., Evans, M., Hastings, N., Peacock, B. (2011). Statistical Distributions. Wiley

Friedman, D.D. (1990). Price Theory: An Intermediate Text. South-Western Pub. Co., US

Goldsmith, R.W., Lipsey, R.E. (1963). Asset Prices and the General Price Level, NBER, 166 – 189, in Studies in the National Balance Sheet of the United States, Ed. Goldsmith, R.W. and R. E. Lipsey

Greenwood, R., Shleifer, A. (2014). Expectations of Returns and Expected Returns. The Review of Financial Studies, 27 (3), 714–746

Hall, R.L, Hitch, C.J. (1939). Price Theory and Business Behaviour. Oxford Economic Papers, 2 Reprinted in T. Wilson and P. W. S. Andrews (eds.), Oxford Studies in the Price Mechanism (Oxford, 1951)

Heaton, J., Lucas, D. (2000). Stock Prices and Fundamentals. Ed. Ben S. Bernanke, B.S and J. J. Rotemberg, NBER Macroeconomics Annual 1999, v. 14., 213 – 264

Hördahl, P., Packer, F. (2007). Understanding asset prices: an overview. Bank for International Settlements, WP 34, 1-38

Klyatskin, V.I. (2005). Stochastic Equations through the Eye of the Physicist, Elsevier B.V.

Klyatskin, V.I. (2015). Stochastic Equations: Theory and Applications in Acoustics, Hydrodynamics, Magnetohydrodynamics, and Radiophysics, v.1, 2, Springer, Switzerland

Lucas, R.E. (1972). Expectations and the Neutrality of Money. J. Econ. Theory, 4, 103-124

Mankiw, N.G, Romer, D., Shapiro, M.D. (1991). Stock Market Forecastability and Volatility: A Statistical Appraisal, Rev.Economic Studies, 58,455-477

Mackey, M.C. (1989). Commodity Price Fluctuations: Price Dependent Delays and Nonlinearities as Explanatory Factors. J. Economic Theory, 48, 497-509

Malkiel, B., Cragg, J.G. (1980). Expectations and valuations of shares. NBER, WP 471

Merton, R.C. (1973). An Intertemporal Capital Asset Pricing Model, Econometrica, 41, (5), 867-887

Mills, F.C. (1946). Price-Quantity Interactions in Business Cycles. NBER, Prins.Univ., NY

Muth, J.F. (1961). Rational Expectations and the Theory of Price Movements, Econometrica, 29, (3) 315-335

Olkhov, V. (2020). Volatility Depend on Market Trades and Macro Theory. MPRA, WP102434

Olkhov, V. (2021a). Price, Volatility and the Second-Order Economic Theory. ACRN Jour. Finance & Risk Perspectives, Sp. Issue, 18th FRAP Conference, 10, 139-165

Olkhov, V. (2021b). To VaR, or Not to VaR, That is the Question. SSRN, WP 3770615

Perold, A.F. (2004). The Capital Asset Pricing Model. J. Economic Perspectives, 18(3), 3–24

Poon, S-H., Granger, C.W.J. (2003). Forecasting Volatility in Financial Markets: A Review, J. of Economic Literature, 41, 478–539

Schwert, G. (1988). Why Does Stock Market Volatility Change Over Time? NBER WP2798

Sharpe, W.F. (1964). Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. The Journal of Finance, 19 (3), 425-442

Shephard, N.G. (1991). From Characteristic Function to Distribution Function: A Simple Framework for the Theory. *Econometric Theory*, 7 (4), 519-529

Shiryaev, A.N. (1999). Essentials Of Stochastic Finance: Facts, Models, Theory. World Sc. Pub., Singapore. 1-852

Stigler, G.J, Kindahl, J.K. (1970). The Dispersion of Price Movements, NBER, 88 - 94 in Ed. Stigler, G.J, Kindahl, J.K. The Behavior of Industrial Prices

Tauchen, G.E., Pitts, M. (1983). The Price Variability-Volume Relationship On Speculative Markets, Econometrica, 51, (2), 485-505

Walck, C. (2011). Hand-book on statistical distributions. Univ.Stockholm, SUF–PFY/96–01

Weyl, E.G. (2019). Price Theory, AEA J. of Economic Literature, 57(2), 329–384

Xu, J. (2007). Price Convexity and Skewness. Jour Of Finance, 67 (5), 2521-2552