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ABSTRACT

This paper offers an approach to construct a family of extraction paths for nonrenewables that guarantee long-run sustainability of an imperfect economy. A path from this family leads to a monotonic growth of output with a decreasing rate of growth if a sustainability condition holds. Otherwise, the path leads either to a bounded decline or U-shaped path of output. In this sense, the paper extends neoclassical results and provides a bridge between neoclassical and degrowth theories because neoclassical tools are used to quantify degrowth scenarios. The offered path can be incentive-compatible for climate change problems because it reduces the extraction of polluting minerals consistently with the IPCC goals. That is, the climate-benefiting emission cuts by the parties of climate agreements may be guided by purely "egoistic" motives—to make own economies long-run sustainable.

1. Introduction

Ninety years ago Hotelling (1931) wrote: "Contemplation of the world's disappearing supplies of minerals, forests, and other exhaustible assets has led to demands for regulation of their exploitation." These demands receive now increasing support from awareness that sustainability goals should coordinate market activities with the needs of future generations and keep the economy within the Earth's limits.

Sustainability goals are becoming increasingly important for policymakers. Some governments strategic plans declare a need for intertemporal policy regulation (e.g. USDS, 1998; Gosduma, 2014; EC, 2021). These documents require coordination of current actions with scientific updates to achieve sustainable social and economic development.

However, the current state of research does not provide a technique for practical estimation of sustainable distribution of limited nonrenewable resources among generations (Section 2). Moreover, the 17 sustainability goals formulated by United Nations do not reflect this urgent problem.

This paper offers a closed-form expression for a path of resource extraction that guarantees long-run sustainability[1] of an imperfect economy. The path is specific for technology and initial conditions. The approach differs from previous work in that feasible extraction paths are determined by asymptotic sustainability and then the investment rule maximizes a welfare criterion. Similarly to climate change problem, a social planner can realize the path by tax/subsidy policies (e.g. Acemoglu et al., 2016), which may be combined with non-price interventions (e.g. Stiglitz, 2019).

The importance of asymptotic results follows from the Hotelling's question about the amount of resource that must be reserved for our remote descendants. This question can be formulated in terms of the properties of the tail of resource distribution among generations. As Hotelling (1931) put it, "Problems of exhaustible assets are peculiarly liable to become entangled with the infinite." The asymptotic properties are obviously connected with the short-run extraction because the thicker is the tail of resource distribution, the less should be current extraction.

Cairns (2008) states that "there is no observable indicator of whether an economy is being sustained." This paper offers such an indicator in Theorem 1. The indicator works if the elasticity of substitution between the resource and produced capital is not less than one and extraction follows the offered path. If this indicator shows current unsustainability, the extraction path leads to a bounded decline or U-shaped

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[1]In a sense of nondecreasing consumption.

path of output. In this sense, the paper extends neoclassical results for the cases when monotonic growth is not feasible and provides a bridge between neoclassical approach and degrowth theories because neoclassical tools are used to quantify degrowth scenarios.

The offered path can be incentive-compatible for climate change problems because it reduces the extraction of polluting minerals consistently with the IPCC goals. That is, the climate-benefiting emission cuts by the parties of climate agreements may be guided by purely "egoistic" motives—to make own economies long-run sustainable.

The paper structure is as follows: Section 2 provides a review of the relevant literature; Section 3 offers sustainability conditions, a long-run sustainable path of extraction, and the output scenarios depending on satisfaction of these conditions; Section 4 discusses the questions of optimality; Section 5 illustrates the sensitivity of optimal paths to uncertain parameters; Section 6 shows the connection of the results with the problems of climate change; Section 7 concludes.

2. Investment and sustainability: a review

> All happy families are alike; each unhappy family is unhappy in its own way.
>
> Leo Tolstoy

The first saving rule for a neoclassical economy with a nonrenewable resource, obtained by Solow (1974b) and Hartwick (1977), shows the importance of sustaining production base by substituting disappearing resource with produced capital. The balance equation in this economy is

$$c = q - \dot{k}, \tag{1}$$

which connects a proper aggregate of per capita consumption c,[2] production $q(k, r)$, where r is the resource flow, and investment $\dot{k}$ into properly measured produced capital k.

By (1), the path of $\dot{k}$ is an intuitive tool to control all issues related to c including sustainability, and the standard Hartwick rule also uses this tool. A brief and elegant derivation of this result, provided in Hartwick (2003)[3], deserves a special attention.

The rule leads to constant consumption ($\dot{c} = 0$) if two conditions hold: (i) investing resource rents ($\dot{k} = rq_r$) and (ii) a necessary condition of dynamic efficiency (Hotelling rule: $q_k = \dot{q}_r/q_r$). By (1) and (i), $\dot{k} = q(k, r) - c = rq_r$. Then time derivative of (1) with (i) and (ii) yields the result:

$$\dot{c} = q_k r q_r + q_r \dot{r} - \dot{r} q_r - r\dot{q}_r = (q_k q_r - \dot{q}_r)r = 0. \tag{2}$$

Conditions (i) and (ii) lead to a decreasing path of extraction[4] which can be realized by restructuring the production and consumption to reduce natural resource dependance and maintain consumption aggregate constant. *How does this rule relate to sustainability?*

Note that q here is *any*[5] function, which can create an impression that consumption can be always kept constant by investing resource rent into capital for any technology and extraction path if government interventions support the prices consistent with (ii). Indeed, there are policy-oriented studies (e.g. Ologunde et al., 2020) that refer to (i) as to a sufficient condition for sustainability even without (ii).

It is known,[6] however, that sustainability is impossible if, for example, q is a CES function and the *effective long-run* elasticity of substitution[7] between the resource and capital

[2] Consumption should include expenses on health, education, and other valuables that determine quality of life (e.g. UNDP, 2021).

[3] Hartwick (2003) provides also a review of some generalizations.

[4] See, e.g., equation (A.4) in Appendix A.

[5] q should be twice continuously differentiable.

[6] E.g., Dasgupta and Heal (1974); Solow (1974b).

[7] Effective elasticity of substitution (EES) is what empirical studies estimate. It depends on aggregation and institutions. A review on EES between capital and labor is in Knoblach and Stöckl (2020). Since EES may be nonmonotonic, the assumption of constant EES is an approximation.

(σ_{KR}) is less than one. Then where is a magic trick?

Recall that Hartwick (2003) stated that "This result is local in time" and the production function must be "with substitutability among inputs."[8] An example below illustrates general issues[9] reflected in these disclaimers. The example considers a non-optimal economy under (i) and (ii) with the data that are close to real economies. The point of interest is the effect of changes in σ_{KR} while other parameters, including the resource-labor and capital-labor substitutabilities, are fixed. Calculations are provided in Appendix A.

Example 1. An economy with a constant population $N = 100$ mln exploits an oil reserve $S_0 = 7,500$ mln tonnes (54,975 mln bbl) or $s_0 = 75$ tonnes/capita. At $t = 0$ (in years), the economy uses oil at rate $r_0 = 3$ tonnes / (capita × year) ($R_0 = 5.17$ mln bbl/day) and the services of produced capital $k_0 = 7 \cdot 10^4$ \$/capita. The output is $q = A_\rho f_\rho(k, r)$ \$/(capita×year), where $\rho = 1 - 1/\sigma_{KR}$ is a substitution parameter, A_ρ is TFP[10] and f_ρ is a CES function with the parameters $\alpha = 0.3, \beta = 0.2$ and $\sigma_{KR} = 1/(1-\rho)$. Consider (a) $\sigma_{KR} = 0.9$; (b) $\sigma_{KR} = 1$; and (c) $\sigma_{KR} = 1.1$.

As shown in Appendix A, economy (a) is inefficient, unsustainable and even unsurvivable.[11] Unsustainability is inevitable due to low σ_{KR} – the economy is doomed to collapse as $r \to 0$. Inefficiency (despite satisfaction of (ii) and exhaustion of the resource stock) follows from capital overaccumulation. Rule (i) requires to invest 67% of output in produced capital at $t = 0$ and this share increases to 71% by the time of collapse after 31 years. By that time, $k = 2.3 \cdot 10^5$, which is useless for $t > 31$. During these 31 years, consumption is, indeed, constant at 2,330 \$/(capita×year). Inefficiency can be illustrated by a simple decapitalization, when both s_0 and k_0 are exhausted in 59.8 years, and c increases from $c_0 = 2,330$ to $c(59.8) = 7,090.6$ and then collapses.

Economy (c) is sustainable but inefficient because of underextraction: r quickly (asymptotically) goes to zero and the total extracted amount is $42.3 < s_0$.[12] Investment share w asymptotically decreases from $w_0 = 0.21$ to zero and capital grows to $k_{max} = 7.47 \cdot 10^5$. The initial consumption $c_0 = 5,586$ is, indeed, maintained infinitely.

Unfortunately, $\sigma_{KR} > 1$ is not sufficient for sustainability.[13] Consider a scenario for $\sigma_{KR} = 1.1$ where, like in reality, r is not decreasing (Jackson and Smith, 2014). Let (i) still hold but $r \equiv r_0$ during 25 years satisfying efficiency condition $\int_0^{25} r dt = s_0$. Rule (ii) does not hold: the ratio $\dot{q}_r/(q_r q_k)$ is essentially less than one, which is close to empirics (Gaudet, 2007). In 25 years, w decreases from 0.21 to 0.20 and then drops to zero (no resource, no investment); consumption grows to 8,361 and then drops to 860. That is, the economy is inefficient and unsustainable but may be survivable depending on the subsistence definition.

Numerical estimates show that only for $\bar{\sigma}_{KR} = 1.0315$ is the economy in Example 1 "happy"—both efficient and sustainable ($c \equiv c_0$ for any $t \geq 0$). For $\sigma_{KR} < \bar{\sigma}_{KR}$, the economy "cannot afford" permanently constant consumption. It requires either short-run sacrifices or future consumption drop. For $\sigma_{KR} > \bar{\sigma}_{KR}$, the prescribed path (stag-

[8] Both neoclassical and ecological economists accept that *there is long-run* substitutability between many types of natural resources and produced capital (e.g. Cleveland and Ruth, 1997, Section 3.1).

[9] Mitra et al. (2013) provide general conditions of existence and efficiency for constant-consumption paths.

[10] TFP includes all other factors that influence output besides k and r. A_ρ are chosen to make the initial outputs Q_0 equal for different ρ in order to compare qualitative effects of ρ around $\rho = 0$, where CES function is discontinuous. A_0 makes q_0 close to the average per capita NNI of upper middle-income countries in 2018 (current \$US, World Bank). TFP dimensions depend on ρ and convert input units into the units of output. For $\rho = 0$, TFP unit is $\$^{1-\alpha} \times \text{tonne}^{-\beta}/(\text{capita}\times\text{year})^{1-\beta}$. Reviews on the nature of TFP are in Hulten (2001) and Lipsey and Carlaw (2004).

[11] Economy is unsurvivable if consumption drops below subsistence.

[12] Theoretically, efficiency can be recovered by a discontinuous shift in r_0 leading to $\int_0^{\infty} r dt = s_0$. However, discontinuity is infeasible here due to low empirical short-run price elasticities of oil demand and supply (Baumeister and Peersman, 2013). This infeasibility resulted, e.g., in negative oil prices in 2020 due to sharp drop in demand caused by COVID-19.

[13] This is a counterexample for the claim of Sesmero and Fulginiti (2016) that CES production function leaves an important question unanswered: is $\sigma_{KR} > 1$ sufficient for sustainability?

nation) is too restrictive because the economy can afford a monotonic growth that depends on technology ($\sigma_{KR}, \alpha, \beta$) and initial conditions.

In case (b), the production function is

$$q = Ak^{\alpha} r^{\beta}, \tag{3}$$

This case is unsustainable because the economy uses up all the resource in 39.1 years with $c \equiv c_0 = 5,663$ and then collapses with capital $k(39.1) = 1.25 \cdot 10^5$. Of course, similarly to case (a), decapitalization could alleviate the collapse. However, this choice is questionable because, similarly to case (c), there are resource policies that allow to avoid the collapse. Sections below examine case (b) in more detail.

Example 1 shows that a general saving rule may become either overinvestment or underinvestment. Sustainability of a real economy needs specificity because real economies are imperfect with respect to assumptions of general rules and every economy is imperfect in its own way.

In particular, there is a watershed: $\sigma_{KR} < 1$ call for policies such as decapitalization that cushion inevitable doom. Therefore, too pessimistic assumptions about the *long-run* substitutability ($\sigma_{KR} < 1$) are unacceptable because they may lead to actions that can cause collapse even if sustainability is possible.

On the contrary, $\sigma_{KR} \geq 1$ needs rules that help to avoid collapse and improve wellbeing by choosing the paths of extraction and (positive) investment. Moreover, even if it is known that $\sigma_{KR} > 1$, there may exist such $\bar{\sigma}_{KR}$ that a general rule leads to unsustainability for $1 < \sigma_{KR} < \bar{\sigma}_{KR}$ or inefficiency for $\sigma_{KR} > \bar{\sigma}_{KR}$.

Hamilton and Hartwick (2005) generalize condition (i) for the case of locally growing consumption using the notion of genuine saving offered by Hamilton and Clemens (1999), which includes depreciation of natural capital besides net investment. The proof of this generalization also uses (ii) and is similar to (2). The result is also local, depending on σ_{KR}. For $\dot{c} > 0$, the only "happy" economy (sustainable and efficient) requires higher $\bar{\sigma}_{KR}$ or, for the same $\bar{\sigma}_{KR}$ and $\sigma_{KR} < \bar{\sigma}_{KR}$, the collapse is faster than for $\dot{c} = 0$ since growth is more resource-consuming (e.g. Bazhanov, 2013).

Hamilton (2016) makes important further steps by relaxing condition (ii) (economy can be inefficient) and decoupling the paths of extraction and investment. Hamilton offers a path with $\dot{r}/r = -\phi := -(r_0/s_0) < 0$ implying $r = \tilde{r}(t) = r_0 \exp(-\phi t)$, which guarantees $\int_0^{\infty} r dt = s_0$ and $r > 0$ for any $t \geq 0$. Along $\tilde{r}$, an economy with any σ_{KR} applies the rule $\dot{c} = q_k g - \dot{g}$, where $g = \dot{k} - \phi n r$ is genuine saving and n is the value of reserve in SEEA (2014) units. Hamilton claims that the economy is sustainable along $\tilde{r}$ if the offered saving rule applies at each point in time. However, as Bazhanov (2020) shows, the output q of economy (3) drops to zero along $\tilde{r}$ even if $\dot{k} \equiv q$. The problem is that the benchmark investment $\phi n r$ is not linked to economy's investment ability. That is, a prescribed $\dot{k}$ may exceed output.

The current paper continues this research line in a sense that economy is imperfect and the searches for extraction and investment paths are separated. This paper differs in that sustainability determines a family of long-run sustainable extraction paths, and then the parameter of this family and investment rule maximize a welfare criterion. Another difference is that this paper, to avoid the drawbacks of general rules, examines the most interesting case of $\sigma_{KR} = 1$, which reflects the research question, and then analyzes sensitivity of consumption to σ_{KR} and other uncertain parameters.

It is difficult to estimate empirically the long-run σ_{KR} due to uncertainty of technical change. However, as argued above, it is unsafe to assume that this parameter is inherently less than one because such an assumption can lead to economy-collapsing policies while sustainability may be possible. On the other hand, if the assumption is $\sigma_{KR} > 1$, the resource is not necessary for production ($q(k, 0) > 0$), which may lead to collapse due to resource-wasting policies if the assumption is too optimistic. Therefore, as Solow (1974b) put it, "only the Cobb-Douglas remains."

Another important uncertainty is the dynamics of TFP, which interpretations differ in theory (Lipsey and Carlaw, 2004) and in positive empirical studies leading to essentially different results (Schatzer et al., 2019).[14] While the efforts to increase TFP and switch from the use of limited resources to backstop technologies are important (Solow, 1974a; Easterly and Levine, 2001; Bretschger and Smulders, 2012; Sessmero and Fulginiti, 2016), there is evidence (Brander, 2010; Byrne et al., 2016) that the pace of innovation in the crucially important sectors is cause for concern. Therefore, consequentialism suggests that normative works such as sustainability studies should not assume too optimistic technical change or TFP to avoid overshoot and collapse.

In this paper, TFP compensates for population change, which must be bounded from above (e.g. Dasgupta and Dasgupta, 2021), and capital depreciation if $\dot{k}$ is not *net* investment. That is, A is constant and there is no capital decay. This assumption is a "worst-case" scenario for TFP that still allows for bounded and unbounded growth of output.

3. Extraction and sustainability

3.1. Asymptotic sustainability

The notion of sustainability requires to treat future generations in the same way as the current one. Therefore, a search for sustainable extraction paths can start from securing the needs of our descendants. It turns out, that such "altruism" can bring some unexpected analytical benefits expressed below in Proposition 2 and Theorem 1.

Assume for clearness, that $\dot{k} = wq$, where $w = const \in (0, 1)$,[15] q is determined by (3), and the balance equation (1) holds. If $w = const$, monotonicity of q implies monotonicity of c. This assumption is relaxed in subsection 5.3.

The needs of future generations are expressed below as asymptotic monotonicity: $\lim_{t\to\infty} \dot{q}(t) \geq 0$. Assume that r is strictly positive and twice continuously differentiable for all $t \geq 0$. Lemma below provides a simple rule for pre-selection of the paths r that *may* lead to global[16] sustainability. That is, the paths that do not satisfy the rule cause a decreasing output for economy (3) at least in the remote future.

Lemma 1. *Economy (3) is sustainable, that is* $\dot{q}(t) \geq 0$ *for all* $t \geq 0$, *only if* $\lim_{t\to\infty} \dot{r}/r^{1+\beta} = 0$.

This lemma disqualifies the paths r with too thin tails. The result immediately implies the following necessary sustainability condition that is less strict but simpler to verify because it does not require estimation of β.

Proposition 1. *Economy (3) with any* $w \in (0, 1)$ *is sustainable, that is* $\dot{q}(t) \geq 0$ *for all* $t \geq 0$, *only if* $\lim_{t\to\infty} \dot{r}/r = 0$.

It is surprising, that unsustainable extraction paths, which can be easily weeded out by Proposition 1, may be prescribed by a normative approach. For example, an important result of Dasgupta and Heal (1974) proves that utilitarian criterion with any positive rate of discount leads to $\lim_{t\to\infty} \dot{r}/r < 0$ for a wide class of production functions including (3). Moreover, this criterion prescribes optimal paths of consumption and output that, in accord with Proposition 1, decrease to zero for model (3). Hamilton (2016) offers another example of a path with $\dot{r}/r < 0$, which is discussed in Section 2.

One more source of unsustainable paths is business as usual—reliance on predictions of possible future extraction

[14] Empirical studies usually consider the models without resource, e.g. $Q = AK^{\alpha}L^{1-\alpha}$, implying that A includes all factors except for capital and labor. For studies of resource effects, TFP should be recalculated with $Q = AK^{\alpha}R^{\beta}L^{1-\alpha-\beta}$ as a base production function.

[15] By World Bank data, w oscillates in narrow ranges well inside (0,1).

[16] Global, here, means global in time, not geographically.

rates. For example, a well-known "oil peak" theory uses historical data to calibrate bell-shaped Hubbert curves. A general form of such a curve is $r_H(t) = 2r_{\max}/\{1 + \cosh[a(t - t_{\max})]\}$, where $r_{\max} > 0$ and $t_{\max} > 0$ are the maximum rate of extraction and the correspondent year respectively, and $a > 0$ is a parameter. For this curve, $\lim_{t\to\infty} \dot{r}_H/r_H = -a < 0$, that is, r_H is unsustainable. Therefore, even if the resource market is able to follow this path, a social planner that cares about sustainability should apply available tools to avoid this path by redistributing the extraction in favor of the future. Note that the form of the Hubbert curve was selected as the best fit for historical market-driven patterns of extraction (Laherrere, 2000). This fact is another evidence of market failure in extractive industry, which is a standard reason for policy interventions.

These examples show that Lemma 1 and Proposition 1 sensibly restrict the set of patterns of extraction that *may* be sustainable. However, these statements do not guarantee yet that a path of extraction from this set leads to sustainability even in the remote future. The following proposition is formulated for the paths that are pre-selected by Lemma 1. The proposition specifies such r that *guarantee* an asymptotically non-decreasing output.

Proposition 2. *Assume* $\lim_{t\to\infty} \dot{r}/r^{1+\beta} = 0$. *Economy (3) is asymptotically sustainable iff (if and only if)* $\lim_{t\to\infty} \ddot{r}r/\dot{r}^2 = 1 + \beta/\alpha + \varepsilon$, *where* $\varepsilon \geq 0$ *is an asymptotic growth parameter.*

As the proof in Appendix B shows, the growth of q is faster the larger is ε. In order to define a proper place of asymptotically sustainable path r in the family of convex functions, recall, for self-sufficiency of the paper, the following facts.

Proposition 3. 1. *$r(t)$ is convex iff $\ddot{r}(t) \geq 0$.*
2. *$r(t)$ is log-convex iff $\ln[r(t)]$ is convex or $\ddot{r}r/\dot{r}^2 \geq 1$.*
3. *$r(t)$ is strictly log-convex if the inequality above is strict.*
4. *$r(t)$ is strongly convex with parameter $m > 0$ iff $\ddot{r}(t) \geq m$.*
5. *$r(t)$ is strongly log-convex with parameter $m > 0$ iff $\ln[r(t)]$ is strongly convex with parameter $m > 0$ or $\ddot{r}r/\dot{r}^2 \geq 1 + m(r/\dot{r})^2$.*

A direct comparison of the condition in Proposition 2 and facts 2 and 3 in Proposition 3 imply a more tight necessary sustainability condition than the one in Proposition 1:

Proposition 4. *Economy (3) is sustainable for all $t \geq 0$ only if r is asymptotically strictly log-convex.*

A technical benefit of Proposition 4 is the same as the one of Proposition 1: there is no need to estimate α and β. The cost is that the condition is still only necessary. It spots the paths that *may* be sustainable but does not guarantee sustainability even in the remote future. It is intuitive, that a path that *guarantees* sustainability must be technology-specific.

As Proposition 2 shows, the more is the resource share (β) and the faster is a desirable growth (ε) the stronger must be the tail convexity. The latter means that the tail of r must be thicker, leaving more of the initial stock to the future.

Since the asymptotic strict log-convexity is not enough to guarantee sustainability, the tails of sustainable paths must belong to a more restricted subset of convex functions. Does it mean that sustainable paths are strongly log-convex?

Fact 5 in Proposition 3 implies that strong log-convexity is too strong compared to the prescription of Proposition 2. This is because $\lim_{t\to\infty} \ddot{r}r/\dot{r}^2 = \infty$ for a strongly log-convex path that is a "candidate-sustainable" by Proposition 1.

Hence, the tail of a sustainable path must be between strictly log-convex and strongly log-convex:

Corollary 1. *Assume* $\lim_{t\to\infty} \dot{r}/r^{1+\beta} = 0$. *Economy (3) is asymptotically sustainable iff $r(t)$ is asymptotically strongly log-convex with a variable parameter $v(t) = (\beta/\alpha)(\dot{r}/r)^2$.*

Asymptotic sustainability conditions sort out unsuitable paths for the future. However, these conditions are only necessary for global sustainability. One of the problems is the same as with any attempt to construct a sustainability indicator that does not reflect the requirement of distributing a given finite stock s_0 over the infinite period of time. That is, an indicator must be connected to a necessary sustainability (and efficiency) condition $\int_0^\infty r dt = s_0$.

The results of this section raise the questions: What are the paths that satisfy Proposition 2 for any $t \geq 0$? What are the conditions that guarantee global sustainability for these paths, and what happens if these conditions do not hold? The following subsection provides the answers.

3.2. Global sustainability conditions

After securing sustainability for the remote future, the condition of Proposition 2 can be applied to all $t \geq 0$ leading to a family of extraction paths. Then the requirement of global sustainability yields a sustainability condition. It is interesting that this condition must hold only at the initial moment. Future sustainability follows from abiding a specific extraction path. This result is formulated below with a detailed proof provided in Appendix C.

Theorem 1. *Economy (3) is globally sustainable along*

$$r(t) = r_0(1 + r_1 t)^{-\alpha/(\beta+\alpha\varepsilon)}, \tag{4}$$

where $r_1 = r_0(\beta + \alpha\varepsilon)/\left\{s_0[\alpha(1-\varepsilon) - \beta]\right\} > 0$, *iff*

$$p_0^s \dot{s}_0 + \dot{k}_0 \geq 0, \tag{5}$$

where $p_0^s = \beta k_0/\{s_0[\alpha(1-\varepsilon) - \beta]\}$ *is a sustainability accounting price of natural capital in the units of produced capital for economy (3) along (4) at* $t = 0$, *and* $\varepsilon \in [0, (\alpha - \beta)/\alpha)$ *is an asymptotic growth parameter.*

This result requires neither optimality nor efficiency, and the conditions of global sustainability are specified for an "imperfect" initial state. Moreover, even if (5) does not hold, a path from family (4) with $\varepsilon > 0$ provides, as shown below in Corollary 3, a transition to global sustainability. A special case of (4) with $\varepsilon = 0$ can be derived from the proof of Proposition 5b in Stiglitz (1974) by solving for two constants of integration and assuming $\dot{q} \equiv 0$.[17]

Inequality (5) with $\varepsilon = 0$ coincides with the condition of potential sustainability offered in Bazhanov (2011). Theorem 1 is more general because it offers the conditions that guarantee not only *potential* sustainability, like condition (5) itself, but *global* sustainability along a path from family (4). Moreover, the proposition guarantees an asymptotic growth for $\varepsilon > 0$. The bound $\varepsilon < (\alpha - \beta)/\alpha$ or $\alpha(1-\varepsilon) > \beta$ is a generalized Solow (1974b)-Stiglitz (1974) convergence condition for $\varepsilon \geq 0$. When (5) holds, it implies a tighter bound:

$$\varepsilon \leq \bar{\varepsilon} = (\alpha - \beta)/\alpha - \beta k_0 r_0/(\alpha \dot{k}_0 s_0). \tag{6}$$

The LHS of (5) has a familiar form of net investment, although (5) is not a saving rule and p_0^s, in general, is neither a competitive price, nor marginal productivity q_r (Hartwick, 2003), nor the average unit value of the asset (SEEA, 2014). In terms of Dasgupta and Mäler (2000), the difference is determined by the differences in goals and, consequently, in the allocation mechanisms. p_0^s *shows by how much* k_0 *must be increased to compensate for a unit of the extracted resource in a sense that the economy is still able to maintain infinitely constant consumption along (4) with* $\varepsilon = 0$.[18] Path (4), unlike the paths in previous studies, guarantees the *long-run* sustainability regardless of possible current overextraction. Corollary 2 below provides the relation between p^s and q_r.

Condition (5) can be reformulated in terms of the change in s_0, namely $\dot{s}_0 + p_0^k \dot{k}_0 \geq 0$, where $p_0^k = 1/p_0^s$ shows how much of the extracted resource can be compensated by a unit increase in k_0 given that the economy is still able to maintain infinitely constant consumption along (4) with $\varepsilon = 0$.

For a general q with n types of resources r_i, and produced capital as a numeraire, the definition of sustainability accounting price follows from the equality $\dot{q} = q_k \dot{k} +$

[17] Stiglitz (1974) came to a second order differential equation for r by differentiating $\dot{q}/q = \alpha\dot{k}/k + \beta\dot{r}/r$ and substituting $q_k = \dot{q}_r/q_r$ as an efficiency condition, whereas here the second order equation comes from the requirement of asymptotic sustainability.

[18] Definition of p_0^s follows from equality in (5) with $\dot{s}_0 = -r_0 = -1$.

$\sum_{i=1}^{n} q_{r_i}\dot{r}_i$, which implies $\dot{k}+\sum_{i=1}^{n} p^{s_i}\dot{s}_i = \dot{q}/q_k$, where

$$p^{s_i} = (q_{r_i}/q_k)\cdot(-\dot{r}_i)/r_i. \tag{7}$$

For economy (3) along the path (4) formula (7) is

$$p^s = [\beta k r_0/r]/\left\{s_0[\alpha(1-\varepsilon)-\beta](1+r_1 t)\right\},$$

which becomes p_0^s provided in Theorem 1 for $t=0$.

The formulas for p_0^s and p_0^k are intuitive. They show, for example, that the maximum *sustainable* extraction $r_0^{\max} = -\dot{s}_0^{\max} = p_k\dot{k}_0$ is lower if β is higher (the economy depends more on the resource); α is lower (the impact of capital on output is lower); s_0/k_0 is lower (due to the concavity of q in both r and k); ε is higher (an asymptotic growth is faster).

It is not surprising that resource value for sustainability p^s may exceed a current market price or marginal productivity q_r, which are local measures. For example, p_0^s of oil in Example 1 (b) with $\varepsilon=0$ is $p_0^s = 254.7$ \$/bbl, whereas $q_r|_{t=0} = 64.4$ \$/bbl. For this economy, condition (5) does not hold: $\dot{k}_0 - p_0^s r_0 = -\$4184$, which can be interpreted either as underinvestment or overextraction valued at p_0^s.

Feasibility of investment. The relation between p^s and q_r is linked with the feasibility of a smallest *sustainable* investment $\dot{k}^{\min}$ (leads to constant consumption) in a sense that $\dot{k}^{\min}$ should be less than output q. By Theorem 1, $\dot{k}_0^{\min} = p_0^s r_0$ guarantees sustainability along (4) but it is not obvious that $p_0^s r_0 < q_0$. Recall that this kind of "benchmark investment" for the Hartwick rule is $\dot{k}^{\min} = q_r r$, which is always feasible for economy (3): $q_r r = \beta q < q$. For imperfect economies, prescribed investments are not feasible automatically. The corollary below provides a feasibility condition that follows immediately from (3) and the formula for p_0^s.

Corollary 2. *For economy (3), $p_0^s r_0 < q_0$ iff*

$$A > \beta k_0^{1-\alpha} r_0^{1-\beta}/\left\{s_0[\alpha(1-\varepsilon)-\beta]\right\}, \tag{8}$$

and $p_0^s \le q_r|_{t=0}$ iff

$$A \ge k_0^{1-\alpha} r_0^{1-\beta}/\left\{s_0[\alpha(1-\varepsilon)-\beta]\right\}. \tag{9}$$

According to (3), a low TFP A (e.g., due to inefficient institutions) can lead to a low output q despite high k and intensity of extraction r/s. Condition (8) shows that if A is too low, the whole output is not enough to compensate for the extracted resource in order to maintain at least constant consumption. That is, inefficiencies of real economies may cause infeasibility of prescribed sustainability investments (Bazhanov, 2015). But even if (8) holds, $\dot{k}_0^{\min}$ can be close to q_0, prescribing consumption $c_0 = q_0 - \dot{k}_0^{\min}$ below a subsistence level. In this sense, condition (9) guarantees a reasonable level of $\dot{k}_0^{\min}$ by requiring a higher A.

Paths of output. If (8) does not hold, any feasible investment violates (5) and the economy is (locally) unsustainable. This situation raises an important question for real economies: What are the scenarios of output along path (4) depending on ε and initial conditions? The corollary below (proof is in Appendix D) provides the answer.

Corollary 3. *The path of output along (4) is*

$$\begin{aligned} q = & q_0\left[(1-k_2)(1+r_1 t)^{-\beta(1-\alpha)/(\beta+\alpha\varepsilon)}\right.\\ & \left.+k_2(1+r_1 t)^{\alpha\varepsilon/(\beta+\alpha\varepsilon)}\right]^{\alpha/(1-\alpha)}, \end{aligned} \tag{10}$$

where $k_2 = wAs_0(1-\alpha)[\alpha(1-\varepsilon)-\beta]yk_0^{\alpha-1}r_0^{\beta-1}$ and $y = 1/[\beta(1-\alpha)+\alpha\varepsilon]$. This path is

1. $\varepsilon = 0$, *(a) stagnation $q \equiv q_0$ iff $\dot{k}_0 = p_0^s r_0$ or $k_2 = 1$;*

(b) bounded monotonic growth $q = q_0[k_2-(k_2-1)(1+r_1 t)^{-(1-\alpha)}]^{\alpha/(1-\alpha)}$ with $q \to q^\infty := q_0(k_2|_{\varepsilon=0})^{\alpha/(1-\alpha)}$ as $t \to \infty$ iff $\dot{k}_0 > p_0^s r_0$ or $k_2 > 1$;

(c) bounded monotonic decline $q = q_0[k_2+(1-k_2)(1+r_1 t)^{-(1-\alpha)}]^{\alpha/(1-\alpha)}$ with $q \to q^\infty$ as $t \to \infty$ iff $\dot{k}_0 < p_0^s r_0$ or $k_2 < 1$;

2. $\varepsilon > 0$, *(a) unbounded monotonic growth with $\dot{q}_0 \ge 0$ iff $\dot{k}_0 \ge p_0^s r_0$ or $k_2 \ge \beta(1-\alpha)y$;*

(b) U-shaped path with $\dot{q}_0 < 0$ iff $\dot{k}_0 < p_0^s r_0$ or $k_2 < \beta(1-\alpha)y$ or $w < \hat{w} := \beta k_0^{1-\alpha} r_0^{1-\beta}/\{As_0[\alpha(1-\varepsilon)-\beta]\}$. Moreover, q attains a unique minimum

$$t^{\min} = \left\{\left[\alpha\varepsilon k_2/[\beta(1-\alpha)(1-k_2)]\right]^{-(\beta+\alpha\varepsilon)y} - 1\right\}/r_1, \tag{11}$$

$$q^{\min} = q(t^{\min}) = q_0 \left\{k_2/[\beta(1-\alpha)]\right\}^{\alpha\beta y} \times \times \left[\alpha\varepsilon/(1-k_2)\right]^{-\alpha^2\varepsilon y/(1-\alpha)} y^{-\alpha/(1-\alpha)}, \quad (12)$$

where $q^{\min}|_{\varepsilon\to+0} = q^\infty$, $q^{\min}|_{\varepsilon\to(\alpha-\beta)/\alpha} = 0$, *and follows an unbounded monotonic growth for* $t > t^{min}$. *The initial value* q_0 *is recovered at*

$$t^{rec} = \left[\bar{z}^{(\beta+\alpha\varepsilon)/(\alpha\varepsilon)} - 1\right]/r_1, \text{ where } \bar{z} > 1 \quad (13)$$

is a unique solution to $(1-k_2)z^{-\beta(1-\alpha)/(\alpha\varepsilon)} = 1 - k_2 z$.

This corollary shows that path (4) helps to avoid an *explicit collapse*—an output that goes to zero. It is important that there is no explicit collapse even if economy overextracts at the current moment, that is, (5) does not hold.[19] In this case, the output follows a monotonic decline to a sustainable level q^∞ if $\varepsilon = 0$ (scenario 1(c)) or a U-shaped path if $\varepsilon > 0$ (scenario 2(b)). Obviously, current overextraction hurts future generations, and the values of $p_0^s r_0$ that are essentially higher than $\dot{k}_0$ may lead to an *implicit collapse*, which can be defined by consumption ($(1-w)q^\infty$ in case 1(c) or $(1-w)q^{\min}$ in case 2(b)) that is below a critical level c_s (e.g. subsistence). In this case, economy may collapse due to riots and wars rather than from starvation as in the explicit collapse. That is, an implicit collapse may happen even if the resource is not necessary for production ($q(k,0) > 0$).[20]

The implicit collapse in case 2(b) may also result from ε that is close to $(\alpha-\beta)/\alpha$. Using the term of Chichilnisky (1996), this choice of ε is the dictatorship of the future: The requirement of a fast future growth redistributes the resource into the future and may lead to a short- or middle-run collapse even if the discrepancy between $\dot{k}_0$ and $p_0^s r_0$ is low and transition to sustainability is possible.

Corollary 3 provides a range of paths of output depending on ε, w, and initial conditions (satisfaction of (5)). The initial state cannot be changed at $t = 0$, whereas ε pinpoints the path of extraction from family (4) and w specifies the paths of output and consumption.[21] The following section discusses peculiarities of selection of optimal paths.

4. Sustainability and optimality

The practical value of consumption (utility) paths crucially depends on criterion. The problems of choosing a criterion for ranking infinite streams are reviewed, for example, in Asheim et al. (2010). This section concentrates on practical questions of this ranking. It is known[22] that sustainability is a constraint in normative problems. Following this paradigm, Corollary 3 offers a test range of feasible paths of output that are either globally sustainable if (5) holds, or allow to avoid explicit collapse if (5) does not hold. This test range can be used for consequentialist analyses of normative approaches. The following subsections show that (5) is a watershed that separates qualitatively different normative questions, which illustrates "the dependence of justice evaluation on the context" (Konow, 2003).

4.1. How to minimize the risk of implicit collapse?

Implicit collapse may result from the lack of basic goods rather than from a low level of output directly. Therefore, this subsection considers the aggregate consumption $c = (1-w)q$ rather than q as a welfare indicator.

If (5) does not hold, c decreases along (4) at least in the short-run. Assume that implicit collapse eventually leads to zero consumption for all succeeding generations. Then the solution that minimizes the risk of collapse provides a constraint for the optimal growth after the period of decline.

To illustrate numerically the properties of the paths of c and their sensitivities to ε and w, the choice of the best values of ε and w is split below into a two-step process using the data of Example 1(b). First step shows the effects of ε under

[19] $q^{\min}$ can be close to zero in case 2(b) but it is always positive since the feasibility of ε requires $\varepsilon < (\alpha-\beta)/\alpha$.

[20] See a discussion in Dasgupta and Heal (1979), p. 197, which develops the definition of *essential* resource given in Dasgupta and Heal (1974).

[21] Recall that family (4) does not depend on w.

[22] E.g., Pezzey (1997).

ε	c_{10}	$\Delta c_{10}(\%)$	$t^{\min}$ (years)	$c^{\min}$	t^{rec} (years)
0.333	1396	13.07	782	834	$2.6{\cdot}10^9$
0.275	3862	3.76	274	2847	$3.3{\cdot}10^5$
0.10	4786	1.67	661	3463	$1.9{\cdot}10^7$
0.01	4983	1.27	14337	3243	$1.0{\cdot}10^{43}$

Table 1
$c(t)$ in Example 1(b) along (4) with $w = 0.2$ and $c_0 = 5663$.

the standard Hartwick rule ($w = \beta$, that is, $c = 0.8q$). The second step considers the reaction of some paths, selected on the first step, to the choice of w.

Choice of ε. The future costs of current overextraction are as follows: for $\varepsilon = 0$, c monotonically decreases from \$5663 to c^∞ =\$3142; for $\varepsilon > 0$, c follows U-shaped paths described in Table 1 depending on ε, where c_{10} is consumption after 10 years and $\Delta c_{10} = 1 - (c_{10}/c_0)^{0.1}$ is the average percent decline in consumption during 10 years. Recall that ε is upper bounded by $(\alpha - \beta)/\alpha$, which is 1/3 in this example.

The path of c is asymmetric with respect to $t^{\min}$: fast decline for $t \le t^{\min}$ and long recovery for $t > t^{\min}$. Dictatorship of the future is represented by $\varepsilon = 0.333$—a fast and deep drop in c which is a cost of a relatively fast growth after 782 years. The opposite case, $\varepsilon = 0$, is the dictatorship of the present constrained by (4) because the short-run values of c are the highest among feasible paths. The costs are ever decreasing consumption and a possible implicit collapse.

Two values of ε in Table 1 are optimal with respect to different criteria. $\varepsilon = 0.1$ is maximin-optimal ($\max_\varepsilon[c^{\min}(\varepsilon)]$). Maximin is practical in this situation despite its known drawback – lack of sensitivity to other values of $c > c^{\min}$. Possible collapse turns this drawback into advantage—a full concentration is needed to avoid extinction.

By Kahneman and Tversky (1979), "the carriers of value are changes in wealth or welfare, rather than final states." That is, a fast drop in c (low $\dot{c}$) may lead to an implicit collapse. Then the problem is to find $\max_\varepsilon W\{u[c(t,\varepsilon), \dot{c}(t,\varepsilon)]\}$ where W is a criterion (e.g. $\min_t$) and $u(c,\dot{c})$ with $u_c > 0, u_{\dot{c}} > 0$ accounts for both c and $\dot{c}$.

The exact form of criterion is context-specific and depends, besides the observable data, on intangibles such as the levels of people's awareness and sympathy to the future, disutility from long recession, credibility of government(s), strength of international agreements, etc.

If the risk of collapse is low, ε can reduce transition time. In this example, $\varepsilon = 0.275$ provides the fastest transition to sustainability[23]—"just" in 274 years. Moreover, c_0 recovers by millions of years earlier than for $\varepsilon = 0.1$. The cost is a faster and deeper transition decline of c compared to $\varepsilon = 0.1$.

Choice of w. Investment, as reviewed in Section 2, is a conventional tool for adjusting consumption. This tool can work simultaneously with the choice of extraction path. This paper, however, finds first a family of extraction paths that guarantee the long-run sustainability of output. Then the best (in terms of a welfare measure and criterion) path from this family can be selected using a saving rule.

In particular, the standard Hartwick rule prescribes $w = \beta = const$ for the base case ($\sigma_{KR} = 1$) of this paper. The following corollary (proof is in Appendix E) shows that a constant w that maximizes $c^{\min}$ in scenario 2(b) always exceeds β. To guarantee that the economy remains in scenario 2(b) for any $w \in (0,1)$ and $\varepsilon \in (0, (\alpha-\beta)/\alpha)$, the corollary considers only the cases where the benchmark investment is infeasible ($p_0^s r_0 \ge q_0$).[24]

Corollary 4. *For $p_0^s r_0 \ge q_0$, a unique w that maximizes $c^{\min}$*

[23] Some studies (e.g. Cairns and Martinet, 2021) define sustainability in terms of maximin value implying that economy may be considered sustainable if this value increases despite current consumption decline. This approach, as argued above, requires a very accurate definition of utility to avoid a short-run implicit collapse (despite the growth of sustainability indicator) due to a fast drop in current consumption.

[24] Recall that scenario 2(b) does not exist for any combinations of ε and $\bar{w} > \hat{w}(\varepsilon)$ if $\hat{w}(\varepsilon) < 1$. That is the economy can switch to a sustainable path 1(a), 1(b), or 2(a) (depending on ε and $\bar{w}$) at $t = 0$ just by reducing ε and increasing w to $\bar{w}$ if $\bar{w}$ is "politically" feasible (not too close to one). Corollary 4 considers more complicated cases when this switch is impossible.

ε	w	c_{10}	$\Delta c_{10}(\%)$	$t^{\min}$ (years)	$c^{\min}$	t^{rec} (years)
0.154	0.672	2088	9.50	50	2000	1387
0.084	0.257	4543	2.18	530	3505	$1.09{\cdot}10^7$

Table 2
$c(t)$ in Example 1(b) along (4) with optimized w.

in scenario 2(b) of Corollary 3 is $w^* =$

$$\alpha/2+\left[k_0^{1-\alpha}r_0^{1-\beta}(\beta+\alpha\varepsilon)-d\right]/(2As_0[\alpha(1-\varepsilon)-\beta]), \quad (14)$$

where the expression for d is provided in the proof. Moreover, $\beta < w^* < \alpha, w^*|_{\varepsilon\to 0} = \alpha$, *and* $w^*|_{\varepsilon\to(\alpha-\beta)/\alpha} = \beta$.

It is important that w^* is feasible and specific for technological parameters and initial conditions, which allows to avoid the drawbacks of general rules discussed in Section 2. For any parameters, the maximin investment for scenario 2(b) exceeds the resource rent ($w^* > \beta$). Note also that $w^* < \alpha$ holds not only for steady, efficient growth as shown in (Stiglitz, 1974, condition (15)).

Substitution of (14) into $c^{\min}$ leads to $c^{\min}(\varepsilon) = [1 - w^*(\varepsilon)]q^{\min}(\varepsilon)$, where $q^{\min}(\varepsilon)$ is given by (12), and maximization in ε yields the maximin solution. The second line of Table 2 illustrates how $w^*(\varepsilon)$ and a search in ε improve the solution with $w = \beta$ and $\varepsilon = 0.1$ provided in Table 1. The improvements include, besides higher $c^{\min}$, faster transition and recovery. The costs are paid by the generations with $t < t^{\min}$, including a discontinuous drop in c_0 from 5663 to 5259 (7%). This drop may be feasible if government reduces some spending (e.g., military expenses or fossil fuel subsidies) rather than cutting private consumption.

An increase in w can essentially reduce transition time. By (11), $t^{\min}$ monotonically decreases in w, therefore, if the goal is to minimize $t^{\min}$, w should be constrained by a lower bound for consumption:

$$\min_{w,\varepsilon}[t^{\min}(w,\varepsilon)] \text{ s.t. } c^{\min}(w,\varepsilon) \geq \bar{c}.$$

For example, the first line of Table 2 shows that the boundary solution of this problem with $\bar{c} = 2000$ reduces $t^{\min}$ from 274 (Table 1) to 50 years. The cost, however, is a very sensitive drop in c_0 from 5663 to 2322 (59%), which, using the argument above, can cause implicit collapse at $t = 0$. The problem can be approached by a variable $w = w(t)$ with $w(0) = w_0$, which is considered in subsection 5.3 below.

4.2. Optimal growth: bounded or unbounded?

When condition (5) holds ($\dot{k}_0 \geq p_0^s r_0$), Corollary 3 provides a family of sustainable paths, which include stagnation, bounded and unbounded growth depending on ε.

However, not all sustainable paths can be considered as desirable and intergenerationally just. The case $\dot{k}_0 = p_0^s r_0$ with $\varepsilon = 0$ leads to stagnation, and economy is optimal for the standard Hartwick rule, which connects this study with previous results.[25] Stagnation is a trivial and the least attractive form of intergenerational justice because it does not provide any gains, which, by Kahneman and Tversky (1979), are important carriers of value.

The author conducted a survey of his students during a number of years about the choice among combinations of current consumption sacrifices and the rates of growth.[26] In these combinations, a larger sacrifice led to a faster concave growth but later time of overtaking the level of stagnation. The respondents' choice was always in the range from 1.8% to 3.5% growth at $t = 0$ with the corresponding overtaking time from 5 to 10 years. A small loss today was always better than "loss of a dream:" nobody selected stagnation when future growth was possible even at a cost of sacrifice.[27]

Perpetual stagnation is avoidable by increasing either w (discontinuous drop in c at $t = 0$) or ε. After increase in

[25] Indeed, formula (4) for r coincides with (A.5), where k_1 for $w = \beta$ becomes $k_1 = \dot{k}_0/k_0 = \beta r_0/[s_0(\alpha-\beta)] = r_1$, and with the one in Bazhanov (2013, p. 344) using $s_0 = \int_0^\infty r dt$, which gives $r_0 = [s_0 k_0^{\alpha-1}(\alpha-\beta)]^{1/(1-\beta)}$.

[26] The survey was in the form of open discussion of the patterns of sustainable growth provided in Fig. 1 of Bazhanov (2013).

[27] The rate of time preference varies interpersonally and over time but saving rates are always separated from zero (e.g. World Bank data).

ε, condition (5) will not hold at $t = 0$, which is considered above in Section 4.1. An increase in w with $\varepsilon = 0$ leads to scenario 1(b)—a bounded growth.

Assume that $\dot{k}_0 > p_0^s r_0$ for any $\varepsilon \in [0, \hat{\varepsilon}]$, where $\hat{\varepsilon} < \bar{\varepsilon}$ and $\bar{\varepsilon}$ is defined by (6). If utility is $u(c, \dot{c})$ with $u_c > 0$ and $u_{\dot{c}} > 0$, intergenerational justice[28] can be expressed as

$$u(c, \dot{c}) = \bar{u} = const\ \forall t \geq 0 \text{ with } u(0, \dot{c}) = u(c, 0) = 0, \quad (15)$$

where the last condition formalizes the aversions to collapse and stagnation. For example, Bazhanov (2013) provides intergenerationally just paths satisfying (15) for utility $u = \dot{c}^\gamma c^{1-\gamma} = (\dot{c}/c)^\gamma c$, where $\gamma \geq 0$. A family of maximin-optimal growth paths for this u and $q = k^\alpha r^\beta$ is $c = c_0(1 + \varphi t)^\gamma$, where $\varphi = \dot{c}_0/(c_0\gamma)$ and γ is bounded by nonrenewable resource constraint: $\gamma < (\alpha - \beta)/(1 - \alpha)$, which for conventional values of α and β is less than one. That is, optimal paths of c are unbounded but concave (decelerating), and intergenerational justice means that *a slower future growth is compensated by a higher level of consumption*.

By Kahneman and Tversky (1979), the smaller the gain is, the more valuable a unit of the gain is. That is, for concave $c(t)$, a unit increase in c becomes more valuable in time, which may lead to projection bias (Loewenstein et al., 2003). The bias can be reduced by a proper factor in utility, specifying the condition of intergenerational justice (15) as follows:

$$u(c, \dot{c}) = (\dot{c}h)^\gamma c = \bar{u} = const\ \forall t \geq 0, \quad (16)$$

where $\gamma \geq 0$ and $\dot{h} \geq 0$. Due to a variance in individual time preferences, γ and h can be specified empirically only as ranges. Then (16) raises the questions: what are the ranges for γ and h, and if some of the paths in Corollary 3 satisfy these ranges? While the first question can be addressed by a separate empirical study, an answer to the second one is given in the corollary below (proof is in Appendix F), which, for simplicity, assumes $k_2 > 1$ and $\alpha < 0.5$.

[28] On the concepts of justice between generations see, e.g. Dasgupta (2005). This paper follows Solow (1974b) but applies maximin to $u(c, \dot{c})$ rather than to the level of consumption c.

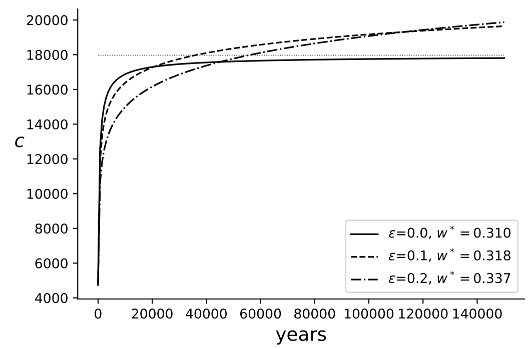

Figure 1: Which path of c is the fairest?

Corollary 5. *If there are $\varepsilon \in [0, \hat{\varepsilon}]$, such that $u(c, \dot{c}) = (\dot{c}h)^\gamma c$ represents intertemporal utility with $\gamma = \alpha/(1 - 2\alpha)$ and $h = h(t) = (1 + r_1 t)^{\beta/(\beta+\alpha\varepsilon)}/$*

$$/\left\{\alpha\varepsilon k_2/[(k_2 - 1)\beta(1 - \alpha)] + (1 + r_1 t)^{-[\beta-\alpha(\beta-\varepsilon)]/(\beta+\alpha\varepsilon)}\right\}$$

with $\dot{h} > 0$, then the paths $c = (1 - w^*)q$ correspondent to these ε and q determined by Corollary 3 can be considered as candidates for intergenerationally just paths in the sense of (16) with constant utility $\bar{u}^* = c_0(w^*)\left\{c_0(w^*)[k_2(w^*) - 1]\alpha\beta r_1/(\beta + \alpha\varepsilon)\right\}^{\alpha/(1-2\alpha)}$ maximized by

$$w^*(\varepsilon) = \alpha + k_0^{1-\alpha} r_0^{1-\beta}[\beta - \alpha(\beta - \varepsilon)]/\{A s_0[\alpha(1 - \varepsilon) - \beta]\},$$

where $w^*(\varepsilon)$ increases in ε and $w^*(\varepsilon) > \alpha\ \forall \varepsilon \in [0, \hat{\varepsilon}]$.

Figure 1 illustrates the difficulty of selecting the best path of growing c. The figure uses the result of Corollary 5 and the data of Example 1(b) with a larger initial reserve ($s_0 = 4000$) to satisfy condition (5) for any $\varepsilon \in [0, 0.2]$.

A finitely living agent should choose bounded growth ($\varepsilon = 0$) because it is life-cycle undominated. However, the choice of a planner who treats generations equally may be different. Behind the Rawls (1971) "veil of ignorance"—the respondents do not know to which generation they belong—a stagnation-averse agent may select a path with $\varepsilon > 0$ because the path with $\varepsilon = 0$ quickly approaches a constant

$c^{\max} = 17972$, and consumption gains are impalpable during life cycle starting from some $t = T$.

It is questionable if it makes sense to determine in practice the fairest sustainable path with high accuracy as opposed to a plausible path from a range of "fair-candidates." Besides personal projection biases, the parameters that determine the value of $\bar{u}^*$ and the path of c are uncertain and change over time. The following section illustrates the sensitivity of the consumption path to some of these parameters.

5. Sensitivity

The analysis above uses a most pessimistic model that still allows for sustainable consumption growth. Assume that a planner enables the extraction to follow path (4) constructed for model (3). However, actual technological parameters and the stock estimate s_0 are more optimistic than the data used for calibration. As Solow (1974a) put it, "a correct theory of optimal social policy will have to take account of technological uncertainty (and perhaps also uncertainty about the true size of mineral reserves)." How do these misspecifications affect the path of consumption? The question can be reformulated: How do the efforts in reducing natural resource dependence affect consumption compared to the effects of ε and w considered in Section 4? The following subsections illustrate the answer.

5.1. Changes in technology

$\sigma_{KR} > 1$. Let an economy follow a path of extraction (4) estimated for $\sigma_{KR} = 1$, whereas in reality $\sigma_{KR} > 1$. For example, the economy overuses the resource at $t = 0$ and the planner chooses a maximin-optimal U-shaped path of c described in the second line of Table 2. Table 3 illustrates the reaction of this path to increases in σ_{KR} given other parameters fixed. Suboptimality of path (4) for $\sigma_{KR} = 1.01$ (second line of Table 3) redistributes consumption from the short and middle run to the future. The resulting path is in the middle between maximin-optimal and fastest-transition paths from Table 2. Consumption overtakes the path for $\sigma_{KR} = 1$ only after 3809 years. Of course, using flexibility of path (4), the resource can be redistributed from the future to present by reducing ε. Indeed, numerical search in ε and w, using (3) for r and equation $\dot{k} = wA_\rho(\alpha k^\rho + \beta r^\rho)^{1/\rho}$ for k with $\rho = 1 - 1/1.01$ yields a better solution: $c^{\min} = 3056$ at $t^{\min} = 123$ for $\varepsilon = 0$ and $w = 0.44$, which is still a U-shaped path whereas for $\sigma_{KR} = 1$ with $\varepsilon = 0$, c monotonically decreases to an asymptote c^∞ by scenario 1(c) of Corollary 3.

σ_{KR}	c_{10}	$\Delta c_{10}(\%)$	$t^{\min}$ (years)	$c^{\min}$	t^{rec} (years)
1	4543	2.18	530	3505	$1.09 \cdot 10^7$
1.01	4018	3.37	315	2482	10572
1.0315	4298	2.72	64	3671	496

Table 3
The effects of σ_{KR} on $c(t)$ in Example 1 along path (4).

No model is perfect, therefore a search for a "best" path, which may require high sacrifices (24% drop in c only at $t = 0$ and 46% total drop by $t^{\min}$ in the example above) is a questionable policy. The next example shows that a better way to improve the path of c is a further increase in resource-capital substitutability.

Section 2 shows that $\sigma_{KR} = 1.0315$ is "perfect" for economy in Example 1 to maintain constant c forever while using up all the resource. If this economy follows path (4) estimated for $\sigma_{KR} = 1$ (third line in Table 3), it has a 64-year period of consumption sacrifice from $c_0 = 5663$ to 3671 followed by unbounded growth with the fastest across examples above recovery of c_0—496 years. By the time of recovery for $\sigma_{KR} = 1.01$ (10572 years), per capita consumption of economy with $\sigma_{KR} = 1.0315$ is 20 times higher—\$112,528. For this economy, the same rule works as for $\sigma_{KR} = 1.01$: an increase in σ_{KR} should be combined with decrease in ε. Namely, for $\varepsilon = 0$ and $w = 0.32$, consumption starts grow-

α	β	c_{10}	$\Delta c_{10}(\%)$	$t^{\min}$ (years)	$c^{\min}$	t^{rec} (years)
0.3	0.2	4543	2.18	530	3505	$1.09 \cdot 10^7$
0.31	0.19	4603	2.05	231	3817	52139
0.29	0.21	4484	2.31	3070	3109	$5.4 \cdot 10^{19}$

Table 4
The effects of α and β on $c(t)$ in Example 1 along path (4).

ing after $t^{\min} = 40.5$ with $c^{\min} = 4060$ (28% total drop).

The examples in Table 3 confirm the effect known in medicine: combining some drugs may lead to an opposite result. The antidote is intuitive—increases in resource-capital substitutability should be accompanied with gradual redistribution of the resource from the future to present compared to path (4) for model (3) (not to the initial overextracting path!). The examples also show that if economy follows a path from family (4), some inevitable in practice deviations from planned consumption look better than explicit collapse in 39 years as shown in Example 1(b).

Changes in α and β. Assume that estimated α and β differ from actual α_+ and β_+. The assumption of other parameters fixed (including q_0) implies that the estimation of TFP A also differs from actual A_+. For the example above (first line of Table 3), assume that $\alpha_+ + \beta_+ = 0.5$ still holds, that is the production function is still homogeneous of degree one (Appendix A). Then the path of extraction r determined by (4) with $\alpha = 0.3$ and $\beta = 0.2$ leads to different paths of capital and consumption for α_+, β_+, and A_+. Capital is determined by the equation $\dot{k}_+ = wA_+k_+^{\alpha_+}r^{\beta_+}$, which yields $k_+ = k_0\left\{1+k_{2+}\left[(1+r_1t)^{[\beta-\alpha(\beta_+-\varepsilon)]/(\beta+\alpha\varepsilon)}\right]\right\}^{1/(1-\alpha_+)}$, where $k_{2+} = wA_+r_0^{\beta_+}(\beta+\alpha\varepsilon)(1-\alpha_+)/\left\{k_0^{1-\alpha_+}r_1[\beta-\alpha(\beta_+-\varepsilon)]\right\}$, leading to $c_+ = (1-w)A_+k_+^{\alpha_+}r^{\beta_+}$.

Table 4 does not reveal any paradoxes similar to the one resulted from increases in σ_{KR}. All generations are better off along path (4) if economy is less resource dependent, that is if $\alpha - \beta$ increases, and worse off if this difference decreases.

5.2. Underestimated stock as investment

Paths from family (4) depend on stock s_0, which is highly uncertain due to geological factors (e.g. McGlade, 2012). Moreover, future prices may or may not make low grade deposits economically valuable. Sustainability requires to use the lowest estimate of s_0 to avoid future collapse due to overestimation. Of course, underestimation of production possibilities leads to inefficiency. Bazhanov (2015) considers this inefficiency as an insurance against possible collapse. However, underestimation of stock is more than just an insurance.

Figure 2 compares the consumption paths for the example above (first line of Table 4) under different assumptions about s_0. Assume that the total additional stock that will be economically recoverable in the future equals current proven reserve s_0, and the current additional stock $\Delta s(t)$ is concave in time, for example, $\Delta s(t) = s_0(1 - e^{-0.001t})$. Figure 2 illustrates three scenarios: (a) precommitment for $s_0 = 75$ (dashes); (b) precommitment for $s_0 = 150$ (dots)—the planner takes a risk by adding $\Delta s(t)|_{t\to\infty}$ to s_0 at $t = 0$, which may end up in collapse; (c) consumption c is being estimated quarterly (solid line in Figure 2) while the path of extraction (4) is being reconstructed annually given the stock $s(t)$ with updates[29] as shown in Appendix G.

It is intuitive that, for some period of time, consumption in case (c) is between the one in cases (a) and (b) because the estimated resource stock in case (c) is always between 75 and 150. But why does case (c) consumption eventually exceed the one in case (b)? The answer is clear if marginal resource productivity q_r increases over time, which holds for model (3). That is, a unit of extracted resource is more productive in the future than in the present. Therefore, a cautious underestimation of resource stock works not only as an insurance

[29]For analytical goals, it may be convenient to express extraction via current stocks s and k using the criterion (e.g. Cairns and Martinet, 2021). In practice, however, it can be less robust than recalculation because preferences are changing in time (Loewenstein et al., 2003) and coordination of many activities requires the *time* path of extraction.

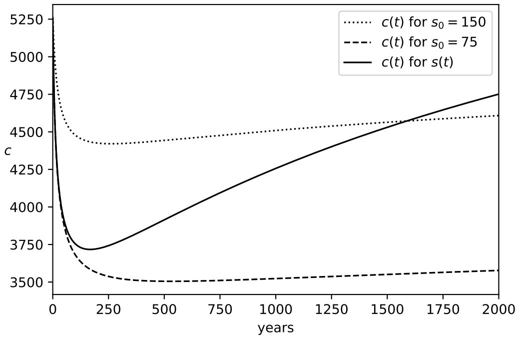

Figure 2: Dynamically reconstructed path of c (solid) compared to precommitment paths with $s_0 = 75$ and $s_0 = 150$.

against collapse but also as an investment.

5.3. Variable investment rate

Under the setup of this paper, a general problem of finding $w(t)$ and $\varepsilon(t)$ is to maximize $W\{u[c(t,w,\varepsilon),\dot{c}(t,w,\varepsilon)]\}$ subject to $\dot{k} = wq, \dot{s} = -r$, the constraints $0 < w_{\min} \le w \le w_{\max} < 1, 0 \le \varepsilon \le \varepsilon_{\max} < (\alpha - \beta)/\alpha$, and the initial conditions $k(0) = k_0, s(0) = s_0, w(0) = w_0, \varepsilon(0) = \varepsilon_0$, where q and r are determined by (3) and (4) respectively, and, as discussed in Section 4, W is context-specific.

However, the efforts to increase W should include also technological parameters (Section 5.1), which depend on the path of restructuring. This path is not easy to predict or control, calling for robust practical solutions for $r(\varepsilon)$ and w.

This subsection illustrates the sensitivity of consumption to changes in w under uncertainty of capital-resource substitutability given other parameters fixed. It is known[30] that an increase in current investment redistributes consumption from the present to future, and decrease leads to a reverse redistribution. Therefore, a hump-shaped $w(t)$ can improve the maximin value of a U-shaped consumption path constructed with a constant w^*.

Table 5 shows that $c^{\min}$ can be increased for all values of σ_{KR} considered in Section 5.1 (Table 3) for $W = \min_t[c(t)]$

[30]See, e.g. Solow (1974b).

σ_{KR}	c_{10}	$\Delta c_{10}(\%)$	$t^{\min}$ (years)	$c^{\min}$	t^{rec} (years)
1	4301	2.71	765	3566	$5.05{\cdot}10^7$
1.01	3816	3.87	215	2652	12012
1.0315	4085	3.21	46	3699	407

Table 5
The effect of variable w on $c(t)$ in Example 1 (cf. Table 3).

by the same simple piecewise linear w of the form $w(t) =$

$$\begin{cases} (w^{\max} - w_0)t/t^{\max} + w_0 & \mid t \in [0, t^{\max}] \\ (w^* - w^{\max})(t - t^{\max})/(t^* - t^{\max}) + w^{\max} & \mid t \in (t^{\max}, t^*] \\ w^* & \mid t > t^*, \end{cases}$$

where $w_0 = w(0), w^{\max} < 1$ is a maximal feasible w at a minimal feasible $t^{\max} > 0$, and $t^* > t^{\max}$ (parameter) is a time of switching to a sustainable growth investment policy. For Table 5, $w_0 = 0.2, t^{\max} = 5$ years, $w^{\max} = 0.3, t^* = 300, w^* = 0.257$. That is, there exists a $\sigma_{KR}-$robust $w(t)$ that improves the maximin value of a U-shaped path, and the best pattern of this $w(t)$ is context-specific.

6. Sustainable extraction and climate change

The results above show that the long-run sustainability of production may require a fast decrease in current extraction compared to business as usual. These results assume that resource always benefits production and utility, which does not hold for many minerals such as coal or petroleum. The theory (e.g. Stollery, 1998; Bazhanov, 2012) suggests that negative side effects from resource use call for further redistribution of extraction from the present to future.

Meanwhile, numerical estimates of the Intergovernmental Panel on Climate Change (IPCC) conclude that to limit global warming to 1.5°C requires halving GHG emissions by 2030 and meeting near-zero emissions by 2050 (Rogelj et al., 2018).[31] *How do these numbers relate to the neces-*

[31]Confirmed by IPCC (2021).

sity to reduce the use of polluting minerals just to achieve sustainable production while ignoring negative effects?

The approach of this paper provides a rough estimate. For example, the maximin-optimal path of oil extraction in a hypothetical economy described in the first lines of Tables 3 and 4 reduces extraction by 65% in 10 years and by 87% in 30 years. By EIA (2021), the share of fuels in petroleum products is 71% (May 2021). That is, if the reduction during the first 10 years happens due to switch to zero-emission energy, which can be realized by tax/subsidy policies (e.g. Acemoglu et al., 2016),[32] the use of petroleum fuels and the correspondent GHG emissions will be cut by 92%. The next 20 years will require further restructuring of consumption and production such as phasing out plastic cups, bags, etc.

The example illustrates that the goal of production sustainability may work as an *incentive-compatibility mechanism* for resource-extracting countries: this goal requires the same actions as the global goal of reducing GHG emissions. *If the parties of climate agreements are concerned about sustainability of their own economies, they should cut the extraction of polluting nonrenewables by the amounts consistent with the goals of agreements and regardless of the actions of other parties.*

Moreover, the output-sustaining cuts of extraction for nonrenewables are not limited by climate goals. Even under zero-emission use, planners should always keep updating the long-term programs of restructuring technologies and consumption to reduce per capita extraction of these resources. Practical realization of these programs may be achieved by the same tax/subsidy policies that lead to transition from dirty to clean technologies. Some studies use Integrated Assessment Models (IAM) to estimate these policies (e.g. Acemoglu et al., 2016; Golosov et al., 2014).

As is known, the prescriptions of IAM, the main tool of climate-economy modeling, highly depend on uncertain key parameters such as discount rate or damage functions. As Stern and Stiglitz (2021) put it, “In the presence of these extreme uncertainties …a full analysis is impossible.” IPCC conclusions induce an additional approach for rough estimates of mitigating policies. The conclusions imply that the *perceived* quality of life[33] for majority of Earth’s population, including the least advantageous, will decline with further growth of global temperature. Since the approach to justice evaluation depends on context (Konow, 2003), the climate context calls for the maximin to mitigate the worst damages while eliminating the most controversial parameter—discount rate—and, again, bringing closer the problem of climate change to the analysis of the current paper.

For a decreasing welfare indicator, the maximin, as illustrated in Sections 4.1 and 5.3, requires to start with the most aggressive “politically feasible” actions, which is consistent with climate mitigating investments that are less than 2% GDP annually (Stern and Stiglitz, 2021). This claim does not require full optimization since it follows from a simple logical exercise similar to the one in Solow (1974b).

Under uncertainties, the IAM prescriptions and the paths in the form of (4) can be used as first order approximations with further corrections after updates in knowledge (Section 5.2) possibly using *feedback control:* if emissions exceed the limits advised by IPCC or the extraction of a mineral exceeds an estimate given by (4), the correspondent taxes/subsidies should be incrementally increased.

7. Conclusions

Sustainability requires coordination of market activities with the ability of economy to satisfy the current and fu-

[32] Acemoglu et al. (2016) uses Hotelling rule for price dynamics of a polluting resource although the dynamic efficiency condition in this case is more complicated (Bazhanov, 2015, Lemma 1) and the real price increase is slow (Gaudet, 2007), which may require additional tax/subsidy efforts.

[33] On measurements see, e.g., Helliwell et al. (2020) or UNDP (2021).

ture consumption needs using limited stocks of nonrenewables. This ability depends on the consistency of intertemporal distribution of a stock with the possibility to gradually replace the resource with other factors. This paper assumes the weakest form of this possibility (unitary long-run elasticity of resource-capital substitution) that still gives a chance for sustainability. More pessimistic assumptions can lead to collapse-inducing policies such as complete decapitalization in finite time while sustainability may be possible.

The paper provides a closed-form expression for a family of extraction paths that guarantee long-run sustainability of an imperfect economy. A path from this family leads to a monotonic growth of output with a decreasing rate of growth if a sustainability condition holds. Otherwise, the path leads either to a bounded decline or U-shaped path of output depending on a parameter. That is, the offered approach allows to quantify degrowth scenarios.

The paper does not assume that a planner should commit to the offered path. It is known that inevitable variations in technologies and other uncertain parameters such as stock estimates and consumer preferences lead to dynamic inconsistency. A sensitivity analysis provides practical recommendations on the path corrections depending on these changes. For example, an increase in capital-resource substitutability may be accompanied by a slower decrease in the short run extraction. Another example shows that stock underestimation and dynamic reestimation of extraction path depending on stock updates works as an investment rather than just insurance against future collapse.

Theoretical results are illustrated with numerical examples for a hypothetical upper middle-income oil extracting economy. In particular, these estimates show that the long-run sustainability requires a fast short-run decrease of extraction consistently with the IPCC goals on cutting GHG emissions. That is, the offered approach may work as an incentive-compatibility mechanism for resource-extracting countries: domestic production sustainability requires the same actions as the global goal of mitigating climate change.

Quantification of sustainable extraction, for example, for oil extracting countries, requires estimation of technological parameters including TFP using (3) as a base model,[34] which is an important direction for further work, as well as the estimation of tax/subsidies that can make the extraction sustainable. Another direction is studying the alternatives for formula (4), which is not a unique expression for a family of asymptotically sustainable extraction paths satisfying Proposition 2. Separate work may pinpoint unique sets of paths, for example, for specific welfare criteria.

One more problem that needs constant updates is the path of effective long-run resource-capital substitutability. Unfortunately, as Cleveland and Ruth (1997) note, "Most of the work has focused on measuring substitution between energy, labor and manufactured capital". Although energy efficiency is important, it is intuitive that energy-capital complementarity and energy efficiency may be not relevant for sustainability if an economy uses only renewables. Empirical estimates of resource-capital substitutability, similar to the ones reviewed in Knoblach and Stöckl (2020) for capital-labor substitutability, could essentially advance sustainability studies by reducing the uncertainty of this key parameter.

A. Calculations for Example 1

By Theorem 1 in Uzawa (1962), a constant return to scale CES function with $\sigma_{KL} = \sigma_{RL} = 1$ and $\sigma_{KR} \neq 1$ is $Q = \bar{A}_\rho L^{1-\alpha-\beta}(\alpha K^\rho + \beta R^\rho)^{1/\rho}$, where $\rho = 1 - 1/\sigma_{KR}$ is a substitution parameter. Let $\bar{A}_\rho = AK_0^\alpha R_0^\beta/(\alpha K_0^\rho + \beta R_0^\rho)^{1/\rho}$,

[34] Recall that there are different variants of TFP estimates for a number of countries provided, e.g. in the Penn Tables (Feenstra et al., 2015). However, these estimates are based on a two-factor (capital-labor) model while the questions of sustainability with respect to nonrenewables require at least three-factor models (resulting TFP should not include the resource effect).

where $A = 200$ leads to $q_0 = 7,079$ \$/(capita×year) for any ρ. Denote $A_\rho = \bar{A}_\rho L^{1-\alpha-\beta}$. Then the per capita output is

$$q = A_\rho \left(\alpha k^\rho + \beta r^\rho\right)^{1/\rho} \tag{A.1}$$

and $q_r = A_\rho \beta r^{\rho-1} \left(\alpha k^\rho + \beta r^\rho\right)^{1/\rho-1}$. Rule (i) is $\dot{k} = r q_r = A_\rho \beta r^\rho \left(\alpha k^\rho + \beta r^\rho\right)^{1/\rho-1} \geq 0$ implying that the gross fixed capital formation as a share of q is $w = \dot{k}/q = \beta r^\rho/(\alpha k^\rho + \beta r^\rho) = \{(\alpha/\beta)(k/r)^\rho + 1\}^{-1}$ (when $r > 0$ and $\dot{r} < 0$, w increases for $\rho < 0$ and decreases for $\rho > 0$). Moreover, $w = 0$ if $r = 0$, implying discontinuity for $\rho < 0$. Then, by (2), $c(t) \equiv c_0 = q - \dot{k} = A_\rho \alpha k_0^\rho \big(\alpha k_0^\rho + \beta r_0^\rho\big)^{1/\rho-1}$ for $r > 0$. Equations $q = c_0/(1-w)$ and (A.1) lead to $c_0[1 + (\beta/\alpha)(r/k)^\rho] = A_\rho \alpha^{1/\rho} k[1 + (\beta/\alpha)(r/k)^\rho]^{1/\rho}$ which, denoting $\bar{k} = c_0/(A_\rho \alpha^{1/\rho})$, is $\bar{k} = k[1 + (\beta/\alpha)(r/k)^\rho]^{1/\rho-1}$ yielding $(r/k)^\rho = (\alpha/\beta) \cdot \big((\bar{k}/k)^{\rho/(1-\rho)} - 1\big)$ and then $r = k\{(\alpha/\beta)[(\bar{k}/k)^{\rho/(1-\rho)} - 1]\}^{1/\rho}$. Substitution into the saving rule provides the equations that govern this economy:

$$\dot{k}(t) = c_0 \left\{[\bar{k}/k(t)]^{\rho/(1-\rho)} - 1\right\}, \tag{A.2}$$

$$r(t) = k(t) \left((\alpha/\beta) \left\{[\bar{k}/k(t)]^{\rho/(1-\rho)} - 1\right\}\right)^{1/\rho}. \tag{A.3}$$

This substitution leads also to $w = 1 - (k/\bar{k})^{\rho/(1-\rho)}$. Derivative of (A.2) yields $\ddot{k} = -c_0 \rho (k/\bar{k})^{-1/(1-\rho)}/[\bar{k}(1-\rho)]$ implying the convexity of $k(t)$ for $\rho \leq 0$ and concavity for $\rho \geq 0$.

(a) $\sigma_{KR} = 0.9$. A numerical solution of (A.2) substituted into (A.3) yields a decreasing path of extraction.[35] Integration of this path shows that the initial reserve s_0 is extracted in 31 years resulting in the collapse of the economy. During this period, w increases from $w_0 = 0.67$ to $w_{\max} = 0.71$, k grows from $k_0 = 7 \cdot 10^4$ to $k_{\max} = 2.3 \cdot 10^5$, and consumption is, indeed, constant at $c(t) = (1 - w_0) q_0 = 2,330$.

Let (ii) hold and, at $t = 0$, rule (i) switches to a decapitalization: $\dot{k} = r_0 q_r(0) - 2bt$, where $b > 0$ is a parameter. Then the path of capital is $k(t) = k0 + r_0 q_r(0) t - bt^2$ and rule (ii), using (A.1), is $\alpha(1-\rho) k^\rho (\alpha k^\rho + \beta r^\rho)^{-1} (\dot{k}/k - \dot{r}/r) = A_\rho \alpha k^{\rho-1} (\alpha k^\rho + \beta r^\rho)^{1/\rho-1}$ yielding the equation for r: $\dot{r} = r\{\dot{k}/k - A_\rho[\alpha + \beta(r/k)^\rho]^{1/\rho-1}/(1-\rho)\}$ with the given $\dot{k}$ and k. An iterative numerical solution with arbitrary b, T and the conditions $\int_0^T r dt = s_0$ and $k(T) = 0$ yields $b = 99.0$ and $T = 59.8$. $c(t)$ increases as a concave function from $c_0 = 2,330$ to $c(T) = 7,090.6$ and then collapses.

(b) The case $\sigma_{KR} = 1$ is analyzed for any $w \in (0, 1)$. More generality than in cases (a) and (c) is used in Section 4. Rules (i) and (ii) provide the following differential equations in r and k with the initial conditions $r(0) = r_0$ and $k(0) = k_0$: $\dot{r} = -[\alpha A(1-w)/(1-\beta)] k^{\alpha-1} r^{1+\beta}$ and $\dot{k} = w A k^\alpha r^\beta$. Elimination of time leads to $r = c_r k^{-\alpha(1-w)/[w(1-\beta)]}$, where $c_r = r_0 k_0^{\alpha(1-w)/[w(1-\beta)]}$. Substitution of this r into the equation for $\dot{k}$ results in $k(t) = k_0(1 + k_1 t)^{w(1-\beta)/d}$, where $d = \alpha\beta + w(1-\alpha-\beta) > 0$ and $k_1 = A c_r^\beta d k_0^{-d/[w(1-\beta)]}/(1-\beta) = A r_0^\beta k_0^{\alpha-1} d/(1-\beta)$. Using this k in $r = c_r k^{-\alpha(1-w)/[w(1-\beta)]}$ and then substitutions of $k(t)$ and $r(t)$ into q yield

$$r(t) = r_0(1 + k_1 t)^{-\alpha(1-w)/d}, \tag{A.4}$$

$$k(t) = k_0(1 + k_1 t)^{w(1-\beta)/d}, \tag{A.5}$$

$$q(t) = q_0(1 + k_1 t)^{\alpha(w-\beta)/d}. \tag{A.6}$$

In accord with the "non-negative genuine investment rule" ($\dot{k} \geq r q_r$), output does not decline iff $w \geq \beta$. Moreover, by (A.4), the convergence of $\int_0^\infty r dt$ requires $w < \alpha$ since for $w \geq \alpha$, $d \geq \alpha(1-\alpha)$. That is, the rate of globally sustainable growth of q is decreasing and bounded from above. These two conditions imply $\beta \leq w < \alpha$, i.e., the Solow (1974b) convergence condition $\beta < \alpha$ holds.

When $w \geq \alpha$, the growth of q is unsustainable because it leads to exhaustion of the resource in finite time and $q = 0$. The same outcome results from a too high r_0. The maximum

[35] For analytical goals, $r(t)$ can be approximated, e.g., by $\tilde{r}(t) = r_0/(1 + r_1 t^{r_2})$, where, for the data of Example 1, $r_1 = 0.025$ and $r_2 = 0.85$.

$r_0 = r_0^{\max}(s_0)$ that allows to keep infinitely $r > 0$ along the path (A.4), can be found from $s_0 = \int_0^\infty r dt$. Namely,

$$r_0^{\max} = [s_0 A(\alpha - w)/k_0^{1-\alpha}]^{1/(1-\beta)}. \tag{A.7}$$

If $w < \alpha$ and $r_0 > r_0^{\max}$ then $s_0 = \int_0^T r dt$, where $T < \infty$:

$$T = \{[1 - (r_0^{\max}/r_0)^{1-\beta}]^{-d/[(\alpha-w)(1-\beta)]} - 1\}/k_1. \tag{A.8}$$

The economy collapses at $t = T$, i.e., $q = 0$ for all $t \geq T$.

(c) For $\sigma_{KR} = 1.1$, a numerical solution of (A.2) substituted into (A.3) yields a quickly decreasing path of extraction[36] and $k(t) \to k_{\max} = \bar{k} = c_0/(A_\rho \alpha^{1/\rho}) = 7.47 \cdot 10^5$. Consumption is constant for any $t \geq 0$ at $c_0 = 5,586$. For this technology, r_0 is too low for given s_0. Constant consumption under rules (i) and (ii) is an "artificial" restriction and can be maintained only if not all reserve is extracted: numerical integration yields $\int_0^\infty r dt = 42.3 < s_0$. An iterative search shows that only if $\bar{\sigma}_{KR} = 1.0315$, is the economy in Example 1 both sustainable ($c(t) \equiv c_0$ for any $t \geq 0$) and efficient under the rules (i) and (ii). For $\sigma_{KR} < \bar{\sigma}_{KR}$, the economy "cannot afford" permanent constant consumption. It requires either short-run sacrifices or ends up in collapse in the future. For $\sigma_{KR} > \bar{\sigma}_{KR}$, the economy can afford a sustainable (for any $t \geq 0$) growth, which is determined by economy's technology ($\sigma_{KR}, \alpha, \beta$) and the initial conditions.

B. Proofs of Lemma 1 and Proposition 2

PROOF OF LEMMA 1. Using (3) and $\dot{k} = wq$, $\dot{q} = q_k \dot{k} + q_r \dot{r} = \alpha q^2 w/k + \beta q \dot{r}/r$. Then $\dot{q}$ is

$$\dot{q} = (\alpha q^2 w/k) \left[1 + (\beta/\alpha)(k\dot{r}/(rqw))\right].$$

Assume that $\dot{r} < 0$ and $r > 0$ for large enough t. Then $\dot{q} \geqslant 0$ for large enough t iff $(\beta/\alpha)(k\dot{r}/(rqw)) \geqslant -1$ or[37]

$$-\beta/\alpha \geqslant wAr^{1+\beta}/(k^{1-\alpha}\dot{r}) = rqw/(k\dot{r}). \tag{B.1}$$

The saving rule $\dot{k} = wq$ ($w \geq \hat{w} > 0$ and $q > 0$) implies $\lim_{t\to\infty} k^{1-\alpha} = \infty$.[38] Then $\lim_{t\to\infty} rqw/(k\dot{r})$ is upper bounded by $-\beta/\alpha$ (implying $\dot{q} \geqslant 0$ *may* hold for all $t \geqslant 0$) *only if* $\lim_{t\to\infty} \dot{r}/r^{1+\beta} = 0$, which is the lemma's claim. □

PROOF OF PROPOSITION 2. Assume $\lim_{t\to\infty} \dot{r}/r^{1+\beta} = 0$. By the proof of Lemma 1, a condition of asymptotic monotonicity of q follows from the expression for the limit of LHS in (B.1), that is $\lim_{t\to\infty}(wAr^{1+\beta}/\dot{r})/k^{1-\alpha} = \infty/\infty$. The L'Hôpital's rule yields $\lim_{t\to\infty}(wAr^{1+\beta}/\dot{r})/k^{1-\alpha} =$

$$= \frac{A}{1-\alpha} \lim_{t\to\infty} \left\{ w\left[(1+\beta)r^\beta \dot{r}^2 - r^{1+\beta}\ddot{r}\right]/\dot{r}^2 \right\} / k^{-\alpha}\dot{k}. \tag{B.2}$$

Substitution $\dot{k} = wq = wAk^\alpha r^\beta$ cancels out $k^{-\alpha}$ and wA. Then (B.2) is $\lim_{t\to\infty} \left[(1+\beta)r^\beta \dot{r}^2 - r^{1+\beta}\ddot{r}\right] / \left[\dot{r}^2 r^\beta (1-\alpha)\right]$, which is $\lim_{t\to\infty} \left[1 + \beta - r\ddot{r}/\dot{r}^2\right]/(1-\alpha)$ or $(1+\beta)/(1-\alpha) - \lim_{t\to\infty} r\ddot{r}/[\dot{r}^2(1-\alpha)]$. Then, by (B.1), inequality $\lim_{t\to\infty} \dot{q} \geqslant 0$ is equivalent to $-\lim_{t\to\infty} r\ddot{r}/[\dot{r}^2(1-\alpha)] \leqslant -(1+\beta)/(1-\alpha) - \beta/\alpha$. Multiplication by $-(1-\alpha)$ leads to $\lim_{t\to\infty} r\ddot{r}/\dot{r}^2 \geqslant 1 + \beta + \beta(1-\alpha)/\alpha = 1 + \beta/\alpha$, which implies the proposition statement: $\lim_{t\to\infty} \dot{q} \geqslant 0 \Leftrightarrow \lim_{t\to\infty} r\ddot{r}/\dot{r}^2 = 1 + \beta/\alpha + \varepsilon$, where $\varepsilon \geqslant 0$ is a parameter of asymptotic growth. □

[36] $r(t)$ can be approximated by $\tilde{r}(t) = r_0 \exp(-r_1 t^{r_2})$, where, for the data of Example 1, $r_1 = 0.121$ and $r_2 = 0.8294$.

[37] Recall that $rqw/(k\dot{r}) < 0$.

[38] If $\lim_{t\to\infty} k^{1-\alpha} = \hat{k} < \infty$, e.g. when $w(t)$ approaches zero faster than $1/t$, (B.1) becomes $\dot{r}/r^{1+\beta} \geq -\alpha w A/(\beta\hat{k})$ implying $\lim_{t\to\infty} \dot{r}/r^{1+\beta} = 0$.

C. Proof of Theorem 1

Path (4) comes from the requirement that the asymptotic sustainability condition of Proposition 2, $\lim_{t\to\infty} \ddot{r}r/\dot{r}^2 = 1 + \beta/\alpha + \varepsilon$, holds for any $t \ge 0$.

Recall that $\frac{d^2}{(dt)^2}(\ln r) = \left[\ddot{r}r/\dot{r}^2 - 1\right]/(r/\dot{r})^2$, which is $\left[\ddot{r}r/\dot{r}^2 - 1\right]\left[\frac{d}{dt}(\ln r)\right]^2$. Denote $x = \beta/\alpha + \varepsilon > 0$. Then the equation $\ddot{r}r/\dot{r}^2 = 1 + x$ becomes $\frac{d^2}{(dt)^2}(\ln r) = x\left[\frac{d}{dt}(\ln r)\right]^2$. Denote $f = \ln r$ or $r = \exp f$. Then r belongs to a general family $\ddot{f} = x\left(\dot{f}\right)^2$ with $r(0) = r_0$ and $\int_0^\infty r dt = s_0$.

Denote $g = \dot{f}$. Then $\ddot{f} = x\left(\dot{f}\right)^2$ is equivalent to $\dot{g} = xg^2$ or $dg/g^2 = xdt \Leftrightarrow -1/g = xt - 1/g_0$, where $g_0 = g(0)$ is a constant to be expressed via r_0 or s_0. The last equation yields $g(t) = g_0/(1 - g_0 xt)$ or $df = g_0 dt/(1 - g_0 xt)$, which integrates to $f(t) = -[\ln(1 - g_0 xt)]/x + f_0$, where $f_0 = \ln r_0$. Substitution $r = \exp f$ yields

$$r(t) = r_0(1 - g_0 xt)^{-1/x}, \tag{C.1}$$

where g_0 is to be found from $\int_0^\infty r dt = s_0$. By (C.1), $s_0 = r_0\int_0^\infty (1 - g_0 xt)^{-1/x} dt$, which converges only if $x < 1 \Leftrightarrow \varepsilon < 1 - \beta/\alpha = (\alpha - \beta)/\alpha$. Under this condition, $s_0/r_0 = \left[(1 - g_0 xt)^{1-1/x}\right]/[g_0(1 - x)]\Big|_0^\infty = -1/[g_0(1 - x)]$ implying $g_0 = -r_0/[s_0(1 - x)]$. Substitution of $1 - x = 1 - (\beta + \alpha\varepsilon)/\alpha$ leads to $g_0 = -r_0\alpha/\{s_0[\alpha(1 - \varepsilon) - \beta]\}$. After substitution of g_0, formula for $r(t)$ in (C.1) becomes $r(t) = r_0\left\{1 + r_0\alpha xt/\left[s_0(\alpha(1 - \varepsilon) - \beta)\right]\right\}^{-x}$.

Denote $r_1 = r_0\alpha x/\left\{s_0[\alpha(1 - \varepsilon) - \beta]\right\}$, which is positive since $\alpha(1 - \varepsilon) - \beta > 0$ by the convergence condition. Then substitution of x yields (4).

Given r, the saving rule $\dot{k} = wq$ allows to find k, which is needed to find global sustainability condition along (4). Indeed, $\dot{k} = wAk^\alpha r^\beta$, where $w = const \in (0, 1)$ or, by (4), $dkk^{-\alpha} = wAr_0^\beta(1 + r_1 t)^{-\alpha\beta/(\beta+\alpha\varepsilon)}dt$. Integration leads to $k^{1-\alpha}/(1-\alpha) = k_1(1+r_1 t)^{k_3} + C$, where $C = k_0^{1-\alpha}/(1-\alpha) - k_1$ using $k(0) = k_0$, $k_1 := wAr_0^\beta(\beta + \alpha\varepsilon)y/r_1 > 0$, $y := 1/[\beta(1 - \alpha) + \alpha\varepsilon)] > 0$, and $k_3 := 1/[y(\beta + \alpha\varepsilon)] > 0$. Substitution for r_1 leads to $k_1 = wAs_0[\alpha(1 - \varepsilon) - \beta]yr_0^{\beta-1}$. Denote $k_2 := k_1(1 - \alpha)k_0^{\alpha-1}$. Then

$$k(t) = k_0\left\{1 + k_2\left[(1 + r_1 t)^{k_3} - 1\right]\right\}^{1/(1-\alpha)}. \tag{C.2}$$

To find a condition that guarantees $\dot{q} \ge 0$ for all $t \ge 0$ (recall that path (4) guarantees only $\lim_{t\to\infty} \dot{q} \ge 0$), consider, similarly to the proof of Lemma 1, $\dot{q} = (A\alpha q^2 w/k) \times \left[1 + (\beta/\alpha)(k\dot{r}/(rqw))\right]$, which implies

$$\dot{q} \ge 0 \Leftrightarrow \dot{r}k^{1-\alpha}/r^{1+\beta} \ge -\alpha wA/\beta. \tag{C.3}$$

By (4), $\dot{r} = -\alpha r_0 r_1(1 + r_1 t)^{-\alpha/(\beta+\alpha\varepsilon)-1}/(\beta + \alpha\varepsilon)$ or $\dot{r} = \dot{r}_0(1 + r_1 t)^{-[\beta+\alpha(1+\varepsilon)]/(\beta+\alpha\varepsilon)}$, where, using the expression for r_1, $\dot{r}_0 = -\alpha r_0^2/\{s_0[\alpha(1 - \varepsilon) - \beta]\}$ (note that $\dot{r} < 0$ for all $t \ge 0$). Then LHS in (C.3) becomes $\dot{r}k^{1-\alpha}/r^{1+\beta} = \dot{r}_0 \times (1+r_1 t)^{-[\beta+\alpha(1+\varepsilon)]/(\beta+\alpha\varepsilon)} \times r_0^{-1-\beta}(1+r_1 t)^{[\alpha(1+\beta)]/(\beta+\alpha\varepsilon)} k_0^{1-\alpha} \times \left\{1 + k_2\left[(1 + r_1 t)^{k_3} - 1\right]\right\}$, which leads to $\dot{r}_0 r_0^{-1-\beta}k_0^{1-\alpha}(1 + r_1 t)^{-k_3}\left\{1 + k_2\left[(1 + r_1 t)^{k_3} - 1\right]\right\}$ and then $\dot{r}k^{1-\alpha}/r^{1+\beta} = \dot{r}_0 r_0^{-1-\beta}k_0^{1-\alpha}\left[k_2 + (1 - k_2)(1 + r_1 t)^{-k_3}\right]$.

Using the formulas for $\dot{r}_0$ and k_2, the second inequality in (C.3) can be written as follows:

$$k_2 \ge \beta(1 - \alpha)y\left[k_2 + (1 - k_2)(1 + r_1 t)^{-k_3}\right], \tag{C.4}$$

which, consistently with construction of path (4), holds for $t \to \infty$ because $\beta(1 - \alpha)y \le 1$ and $k_3 > 0$. Since RHS of (C.4) is monotonic in t, it remains to check (C.4) at $t = 0$. That is, (C.3) is equivalent to $\dot{q} \ge 0 \Leftrightarrow k_2 \ge \beta(1 - \alpha)y$. Using $\dot{k}_0 = wAk_0^\alpha r_0^\beta$ and $r_0 = -\dot{s}_0$, the last inequality can be written as $\dot{k}_0 s_0[\alpha(1 - \varepsilon) - \beta]/(\beta k_0) \ge -\dot{s}_0$ yielding the claim of the theorem: $\dot{q} \ge 0 \Leftrightarrow p_0^s\dot{s}_0 + \dot{k}_0 \ge 0$, where $p_0^s := \beta k_0/\left\{s_0[\alpha(1 - \varepsilon) - \beta]\right\}$. □

D. Proof of Corollary 3

By (3) and formulas (4) for r and (C.2) for k, $q =$

$$q_0(1+r_1t)^{-\alpha\beta/(\beta+\alpha\varepsilon)}\left\{1+k_2\left[(1+r_1t)^{k_3}-1\right]\right\}^{\alpha/(1-\alpha)},$$

where $k_2 = wAs_0(1-\alpha)[\alpha(1-\varepsilon)-\beta]yk_0^{\alpha-1}r_0^{\beta-1}$, $y = 1/[\beta(1-\alpha)+\alpha\varepsilon]$, and $k_3 = 1/[y(\beta+\alpha\varepsilon)]$ (see the proof of Theorem 1). Substitution for k_3 and rearrangements lead to expression (10) of the corollary.

The proof of Theorem 1 shows the equivalence of the following three inequalities: $\dot{q}_0 \geq 0$, $\dot{k}_0 \geq p_0^s r_0$ (condition (5) in Theorem 1), and $k_2 \geq \beta(1-\alpha)/[\beta(1-\alpha)+\alpha\varepsilon]$.

For $\varepsilon = 0$, the last inequality becomes $k_2 \geq 1$ and formula (10) is $q = q_0\left[k_2+(1-k_2)(1+r_1t)^{-(1-\alpha)}\right]^{\alpha/(1-\alpha)}$, yielding claims 1(a), 1(b), and 1(c) of the corollary.

For $\varepsilon > 0$, formula (10) implies that q unboundedly increases in the long run regardless of condition (5). If (5) does not hold, i.e., $\dot{k}_0 < p_0^s r_0$, or $\dot{q}_0 < 0$, then q attains a unique minimum $t^{\min} > 0$. Indeed, by (C.4), $\dot{q} = 0$ is equivalent to $k_2 = \beta(1-\alpha)y\left[k_2+(1-k_2)(1+r_1t)^{-k_3}\right]$, yielding $t^{\min} = \left(\left\{k_2[1/[\beta(1-\alpha)y]-1]/(1-k_2)\right\}^{-1/k_3}-1\right)/r_1$, which, after substitutions for k_3 and y, provides a unique positive $t^{\min}$ in the form of (11). Substitution of $1+r_1t^{\min} = \left\{\alpha\varepsilon k_2/[\beta(1-\alpha)(1-k_2)]\right\}^{-(\beta+\alpha\varepsilon)y}$ into (10) leads to

$$\begin{aligned}q^{\min} =& q_0\Big[(1-k_2)\left\{\alpha\varepsilon k_2/[\beta(1-\alpha)(1-k_2)]\right\}^{y\beta(1-\alpha)}\\ &+k_2\left\{\alpha\varepsilon k_2/[\beta(1-\alpha)(1-k_2)]\right\}^{-y\alpha\varepsilon}\Big]^{\alpha/(1-\alpha)}.\end{aligned}$$

Using $\beta(1-\alpha) = 1/y-\alpha\varepsilon$, it can be written as

$$\begin{aligned}q^{\min} =& q_0\Big[\alpha\varepsilon k_2/[\beta(1-\alpha)]\left\{\alpha\varepsilon k_2/[\beta(1-\alpha)(1-k_2)]\right\}^{-y\alpha\varepsilon}\\ &+k_2\left\{\alpha\varepsilon k_2/[\beta(1-\alpha)(1-k_2)]\right\}^{-y\alpha\varepsilon}\Big]^{\alpha/(1-\alpha)},\end{aligned}$$

or, after factoring out, as

$$\begin{aligned}q^{\min} =& q_0\left\{k_2/[\beta(1-\alpha)]\left\{\alpha\varepsilon k_2/[\beta(1-\alpha)(1-k_2)]\right\}^{-y\alpha\varepsilon}\times\right.\\ &\left.\times\,[\alpha\varepsilon+\beta(1-\alpha)]\right\}^{\alpha/(1-\alpha)},\end{aligned}$$

which, using the expression for y, yields (12). The boundary limits are: $q^{\min}|_{\varepsilon\to+0} = q_0\left\{k_2|_{\varepsilon=0}/[\beta(1-\alpha)]\right\}^{\alpha/(1-\alpha)}\times \lim_{\varepsilon\to+0}\left\{\alpha\varepsilon/(1-k_2)\right\}^{-\alpha^2\varepsilon y/(1-\alpha)}[\beta(1-\alpha)]^{\alpha/(1-\alpha)}$, where $\lim_{\varepsilon\to+0}\left\{\alpha\varepsilon/(1-k_2)\right\}^{-\alpha^2\varepsilon y/(1-\alpha)} = 1$. Then $q^{\min}|_{\varepsilon\to+0} = q_0\left\{k_2|_{\varepsilon=0}\right\}^{\alpha/(1-\alpha)}$. The limit $q^{\min}|_{\varepsilon\to(\alpha-\beta)/\alpha} = 0$ because $k_2|_{\varepsilon\to(\alpha-\beta)/\alpha} = 0$.

The time of recovery $t^{rec} > 0$ follows from the equation $q(t^{rec}) = q_0$, which, by (10), can be written as

$$(1-k_2)(1+r_1t)^{-\beta(1-\alpha)/(\beta+\alpha\varepsilon)} = 1-k_2(1+r_1t)^{\alpha\varepsilon/(\beta+\alpha\varepsilon)}.$$

Denoting $z := (1+r_1t)^{\alpha\varepsilon/(\beta+\alpha\varepsilon)}$, it becomes

$$(1-k_2)z^{-\beta(1-\alpha)/(\alpha\varepsilon)} = 1-k_2z.$$

Except for a trivial solution $z = 1$ ($t = 0$), this equation has a unique solution $\bar{z} > 1$ because (i) both LHS and RHS are monotonically decreasing in z, (ii) LHS asymptotically approaches zero while RHS becomes zero at finite z, (iii) the initial slope of LHS is steeper than the one of RHS. Property (iii) follows from $d/dz(\text{LHS})|_{z=1} < d/dz(\text{RHS})|_{z=1}$, which is $-\beta(1-\alpha)/(\alpha\varepsilon)(1-k_2) < -k_2$ or $k_2[\alpha\varepsilon+\beta(1-\alpha)]/(\alpha\varepsilon) < \beta(1-\alpha)/(\alpha\varepsilon)$. The last inequality holds because in case 2(b) ($\dot{q}_0 < 0$), $k_2 < \beta(1-\alpha)/[\alpha\varepsilon+\beta(1-\alpha)]$. Then, using the unique numerical solution $\bar{z} > 1$, the formula for z yields equation (13): $t^{rec} = \left[\bar{z}^{(\beta+\alpha\varepsilon)/(\alpha\varepsilon)}-1\right]/r_1$. □

E. Proof of Corollary 4

Denote $a_0 := k_2/w$. By (12), $c^{\min} = (1-w)q^{\min} =$

$$(1-w)w^{\alpha\beta y}\{a_0/[\beta(1-\alpha)]\}^{\alpha\beta y}\times$$
$$\times[(1-wa_0)/(\alpha\varepsilon)]^{\alpha^2\varepsilon y/(1-\alpha)}y^{-\alpha/(1-\alpha)} \text{ or}$$
$$c^{\min} = a_1(1-w)w^{\alpha\beta y}[(1-wa_0)/(\alpha\varepsilon)]^{\alpha^2\varepsilon y/(1-\alpha)},$$

where $a_1 = \{a_0/[\beta(1-\alpha)]\}^{\alpha\beta y}y^{-\alpha/(1-\alpha)}$. Then $\partial c^{\min}/\partial w =$

$$a_1 w^{\alpha\beta y}[(1-wa_0)/(\alpha\varepsilon)]^{\alpha^2\varepsilon y/(1-\alpha)}\times \tag{E.1}$$
$$\times\left\{-1+(1-w)\left[\alpha\beta y/w-\alpha^2\varepsilon y a_0/[(1-\alpha)(1-wa_0)]\right]\right\}$$

where $a_1 w^{\alpha\beta y}[(1-wa_0)/(\alpha\varepsilon)]^{\alpha^2\varepsilon y/(1-\alpha)} > 0$, implying that $\operatorname{sgn}(\partial c^{\min}/\partial w) = \operatorname{sgn}\{\cdot\}$. The range $w \in (0,1)$ contains at least one maximizer of $c^{\min}$ because $\{\cdot\}|_{w\to+0} = +\infty$ and $\{\cdot\}|_{w\to 1} = -1$. Since $0 < w < 1/a_0$, the equality $\{\cdot\} = 0$ can be written as

$$(1-w)\left[\alpha\beta y(1-\alpha)(1-wa_0)-w\alpha^2\varepsilon y a_0\right] = w(1-\alpha)(1-wa_0),$$

which, multiplied by $k_0^{1-\alpha}r_0^{1-\beta}[\beta(1-\alpha)+\alpha\varepsilon]/(1-\alpha)$, and using the formulas for a_0 and y, can be written as $b_2w^2 - b_1w + b_0 = 0$ with $b_2 = As_0[\alpha(1-\varepsilon)-\beta] > 0$ (since $\varepsilon < (\alpha-\beta)/\alpha$), $b_1 = k_0^{1-\alpha}r_0^{1-\beta}(\beta+\alpha\varepsilon) + \alpha As_0[\alpha(1-\varepsilon)-\beta] > 0$, and $b_0 = \alpha\beta k_0^{1-\alpha}r_0^{1-\beta} > 0$. The discriminant $D = b_1^2 - 4b_2b_0$ is positive because, by adding and subtracting $4\alpha As_0k_0^{1-\alpha}r_0^{1-\beta}(\beta+\alpha\varepsilon)[\alpha(1-\varepsilon)-\beta]$, it can be written as

$$D = \left\{k_0^{1-\alpha}r_0^{1-\beta}(\beta+\alpha\varepsilon)-\alpha As_0[\alpha(1-\varepsilon)-\beta]\right\}^2 +$$
$$+4As_0k_0^{1-\alpha}r_0^{1-\beta}\alpha^2\varepsilon[\alpha(1-\varepsilon)-\beta] > 0.$$

The two roots are real, distinct, and positive: $w^*_{1,2} =$

$$0.5\left\{\alpha+[k_0^{1-\alpha}r_0^{1-\beta}(\beta+\alpha\varepsilon)\mp d]/\left[As_0[\alpha(1-\varepsilon)-\beta]\right]\right\},$$

where $d := D^{1/2}$. The larger root w_2^* is irrelevant because $w_2^*|_{\varepsilon\to(\alpha-\beta)/\alpha} \to \infty$ and the infimum of w_2^* (b_2 monotonically decreases in ε) is $w_2^*|_{\varepsilon\to 0} = \hat{w}|_{\varepsilon\to 0}$, which is infeasible in scenario 2(b) of Corollary 3 ($w < \hat{w}$). Denote $w^* := w_1^*$.

Since $d > k_0^{1-\alpha}r_0^{1-\beta}(\beta+\alpha\varepsilon)-\alpha As_0[\alpha(1-\varepsilon)-\beta]$, $w^* < \alpha$. To show $w^* > \beta$, D can be rearranged as follows:

$$D = \left\{k_0^{1-\alpha}r_0^{1-\beta}(\beta+\alpha\varepsilon)-(2\beta-\alpha)As_0[\alpha(1-\varepsilon)-\beta]\right\}^2 -$$
$$-4As_0\beta[\alpha(1-\varepsilon)-\beta]^2\left[k_0^{1-\alpha}r_0^{1-\beta}-As_0(\alpha-\beta)\right],$$

where $k_0^{1-\alpha}r_0^{1-\beta} - As_0(\alpha-\beta) > 0$ since, by Corollary 2 with $\varepsilon \to 0$, $A \le \beta k_0^{1-\alpha}r_0^{1-\beta}/[s_0(\alpha-\beta)]$ for $p_0^s r_0 \ge q_0$ implying $k_0^{1-\alpha}r_0^{1-\beta} > \beta k_0^{1-\alpha}r_0^{1-\beta} \ge As_0(\alpha-\beta)$. Then $d < k_0^{1-\alpha}r_0^{1-\beta}(\beta+\alpha\varepsilon)-(2\beta-\alpha)As_0[\alpha(1-\varepsilon)-\beta]$ yielding $w^* > \beta$. The limits $w^*|_{\varepsilon\to 0} = \alpha$, and $w^*|_{\varepsilon\to(\alpha-\beta)/\alpha} = \beta$ follow directly from the expression for the smaller root w_1^*. □

F. Proof of Corollary 5

Formula (10) can be written as $q(t) = q_0 g(t)^{\alpha/(1-\alpha)}$ implying $c(t) = c_0 g(t)^{\alpha/(1-\alpha)}$, where $c_0 = (1-w)q_0$. For $w = const$, $\dot{c} = c_0 g^{-(1-2\alpha)/(1-\alpha)}\dot{g}\alpha/(1-\alpha)$, where, by (10), $\dot{g} = (1+r_1t)^{-\beta/(\beta+\alpha\varepsilon)}\left[\beta(1-\alpha)(k_2-1)(1+r_1t)^{-[\beta-\alpha(\beta-\varepsilon)]/(\beta+\alpha\varepsilon)} + \alpha\varepsilon k_2\right]r_1/(\beta+\alpha\varepsilon)$, which, using $k_2 > 1$, can be written as $\dot{g} = h^{-1}\beta(1-\alpha)(k_2-1)r_1/(\beta+\alpha\varepsilon)$, where h coincides with the expression given in the corollary. Then $\dot{g}h = \beta(1-\alpha)(k_2-1)r_1/(\beta+\alpha\varepsilon) = const$ and $[\dot{c}h]^{\alpha/(1-2\alpha)} = g^{-\alpha/(1-\alpha)}[c_0\alpha\beta(k_2-1)r_1/(\beta+\alpha\varepsilon)]^{\alpha/(1-2\alpha)}$ implying that $u(c,\dot{c}) = c[\dot{c}h]^{\alpha/(1-2\alpha)} = \bar{u} = c_0[c_0\alpha\beta(k_2-1)r_1/(\beta+\alpha\varepsilon)]^{\alpha/(1-2\alpha)}$ is constant overtime.

To maximize $\bar{u} = \bar{u}(w)$, consider $\tilde{u} = \left\{(1-w)[(1-w)(k_2(w)-1)]^{\alpha/(1-2\alpha)}\right\}^{(1-2\alpha)/\alpha} = (1-w)^{(1-\alpha)/\alpha}(k_2(w)-1)$, which is strictly concave in w and has the same maximizer as $\bar{u}$. Recall that $k_2(w) = wa_0$, where $a_0 = As_0(1-\alpha)[\alpha(1-\varepsilon)-\beta]k_0^{\alpha-1}r_0^{\beta-1}/[\beta(1-\alpha)+\alpha\varepsilon] > 0$. FOC for $\tilde{u}$ is $\partial\tilde{u}/\partial w =$

$-(1-w)^{(1-\alpha)/\alpha-1}(wa_0-1)(1-\alpha)/\alpha+(1-w)^{(1-\alpha)/\alpha}a_0 = 0$ or $(1-w)^{(1-\alpha)/\alpha-1}\{-wa_0(1-\alpha)/\alpha+(1-\alpha)/\alpha+a_0-wa_0\} = 0$, where $1-w > 0$ for feasible w. Then $\{\cdot\} = 0$, which is equivalent to $wa_0[(\alpha-1)/\alpha-1] = -(1-\alpha)/\alpha - a_0$ or $w = \alpha[1+(1-\alpha)/(\alpha a_0)]$, which after substitution for a_0 yields the expression for $w^*(\varepsilon)$. □

G. Updates in extraction path

A planner reconstructs the path (4) at $t_i, i = 1, 2, \dots$, as follows: $r^i(\tau) = r^i_0(1+r^i_0\delta\tau/s^i_0)^{-1/\delta-1}$, where $\tau \in [0, t_{i+1} - t_i]$, $\delta := (\beta+\alpha\varepsilon)/[\alpha(1-\varepsilon)-\beta]$, $r^i_0 = r^{i-1}(t_i - t_{i-1})$, $t_0 = 0$, $r^0(\tau)$ is given by (4), $r^0_0 = r_0$, $s^i_0 = s^{i-1}_0 - X^i + \Delta\bar{s}^i$, $s^0_0 = s_0$, X^i is the extracted amount in the period $[t_{i-1}, t_i]$, $\Delta\bar{s}^i$ is the additional stock in the same period. Then, similar to (C.2),

$$k^i(\tau) = k^i_0\left\{1 + k^i_2\left[(1+r^i_0\delta\tau/s^i_0)^{k_3} - 1\right]\right\}^{1/(1-\alpha)},$$

where $k^i_0 = k^{i-1}(t_i - t_{i-1})$, $k^0(\tau)$ is given by (C.2), $k^0_0 = k_0$, and $k^i_2 = wAs^i_0(k^i_0)^{\alpha-1}(r^i_0)^{\beta-1}(1-\alpha)y(\beta+\alpha\varepsilon)/\delta$, which yields $c^i(\tau) = (1-w)A(k^i(\tau))^\alpha(r^i(\tau))^\beta$.

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