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Abstract

The paper develops a dynamic Input Output - Stock Flow consistent model based on the Supermultiplier approach. This framework integrates the dimension of output determination with the system of relative prices. Through this model, we define the determinants of the public debt-to-GDP ratio and the conditions for its stability. The main results of the research show that:

i) Given the interest rate, the saving rate, the tax rate, the industrial profit rate and the coefficients of production there exist a *steady-state* value of the public debt-to-GDP ratio ingrained in the economic system. This result calls into question the idea of imposing budgetary rules with threshold levels independently from the very specific features of each economic system;

ii) Expansions in the level of public expenditure have a permanent effect on the public debt-to-GDP ratio only in the presence of the accelerator effect, that is, through an induced increase in the share of private indebtedness on GDP and aggregate debt. Because of the accelerator channel, the public debt-to-GDP ratio depends on the capital intensity of the aggregate production process and, thus, on the system of relative prices. With this respect, the capital intensity determines the elasticity of private indebtedness with respect to one-point change in public spending;

iii) Conversely to the neoclassical argument, the relationship between the interest rate and public debt-to-GDP ratio goes from the first to the second. In particular, changes in the interest rate modify the public debt-to-GDP ratio through both variations in the quantitative and value dimension. Such variations have a puzzling effect on the steady-state value of the public debt-to-GDP ratio. For instance, the reverse capital deepening implies that an increase in the interest rate produces a decrease in the public debt-to-GDP ratio.

Finally, we point out that, in contrast to the standard argument proposed by mainstream macroeconomics, the condition of fiscal balance jointly a positive differential between the growth rate of output and the interest rate has no relevance for the stability conditions of the public debt-to-GDP ratio. In this regard, we develop a taxonomy of the growth regimes depicted by the model deriving such conditions in each scenario. The necessary condition of stability is the absence of budgetary constraints, it becomes sufficient when one of the following is respected: the growth rate of primary public expenditure is higher than zero, the interest rate is higher than zero or the propensity to consume out of wealth is non-zero.

Keywords: Fiscal policy, Monetary policy, Public debt-to-GDP ratio, SFC models, Input-Output

JEL classification: E12, E17, E42, E43, E52, E62

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Introduction

In the last decades, a long-standing discussion about the relationship between public debt and GDP growth has characterized the economic and policy debate. The common argument across mainstream literature is that high levels of public debt would amper economic growth and austerity policies are pinpointed as the measures needed to guarantee the sustainability of public finances boosting contemporarily GDP growth.

In this regard, the debate has focused, among other things, upon the apparently unescapable recommendation to lower the debt-to-GDP ratio. Such indicator has been extensively used as a benchmark for public debt sustainability and, as in the famous parameters of the 1992 Maastricht Treaty and the 2012 Fiscal Compact, it has become a specific objective for economic policy agenda. In this sense, the Eurozone austerity agenda has been implemented and justified in order to respect the thresholds fixed in such treaties.

Within neoclassical literature, an "unprecise" high level of public debt is supposed to be a cause of low growth rates, both in the short- and long-run via several channels: high public debt can adversely affect capital accumulation and growth via higher long-term interest rates (Gale and Orzag, 2003; Baldacci and Kumar, 2010), higher future distortionary taxation (Barro, 1979; Dotsey, 1994), lower future public infrastructure spending (Aizenmann et al., 2007), higher inflation (Sargent and Wallace 1981; Barro 1995; Cochrane 2011), and greater uncertainty about prospects and policies. In more extreme cases of a debt crisis, by triggering a banking or currency crisis, these effects can be magnified (Burnside et al., 2001; Hemming et al., 2003).

Anyway, on the empirical ground of analysis, there is still no consensus among mainstream authors on the relationship between these two variables and models provides ambiguous results. "Growth in Time of Debt" represents the reference contribution for this line of thought. In these articles, Reinhart and Rogoff (2010a, 2010b) assert that "whereas the link between growth and debt seems relatively weak at normal debt levels, median growth rates for countries with public debt over roughly 90 per cent of GDP are about one per cent lower that otherwise" (RR 2010a p. 573). This work has provided significant support for the austerity agenda that has been ascendant in Europe and USA since 2010. According to the authors, the two papers formed the basis for testimony before the Senate Budget Committee and represent the only evidence cited in the "Paul Ryan Budget" on the consequence of high public debt for economic growth.

Starting from this seminal contribution a large strand of literature has investigated this relationship attempting to identify the possible non-linearities and the threshold beyond which public debt harms GDP growth. For instance, Cane et al. (2010) find a similar non-linear effect on growth above 77% of GDP. However, lower levels of public debt contribute to increase investment and get faster economic growth. In Cecchetti et al. (2011) the threshold is fixed around 85% of GDP. Abbas and Chrisitnes (2010) using panel data of low-income and emerging countries describe a positive contribution to economic growth when domestic debt presents moderate levels, but when it represents more than 35 % of bank deposits it has a negative impact. Egert (2015) shows that the threshold can be lower than 90%, even between 20% and 60% of GDP depending on data frequency, time and country dimension and other assumptions. Baum et al (2013), studying the short-term impact of debt in 12

Euro area economies, detect a positive effect below 67% and a negative one above 95%. Minea and Parent (2002) find the same kind of non-linearity but with different thresholds.

Others authors tried to include the possibility of a reverse causality running from growth to debt. But also around this issue, there is currently no consensus. Ferreira (2009), analyzing 20 OECD countries, founded a bidirectional causal relationship between public debt and growth. Pasunte-Avjovin and Sanso-Navarro (2015), using a panel bootstrap Granger causality test, show that government debt does not cause real GDP growth. Gomez and Sosvilla-Rivero (2015) confirm that, considering the whole sample period 1980-2013, there is no negative causation between public debt and GDP growth. They find an inverse Granger-causality only from 2007 to 2009 above a debt threshold that goes from 56% to 103%. Ramos-Herrera and Sosvilla- Rivero, using the Word Bank's classification for income, initially indicate that the countries which present the lowest public debts are characterized by the highest economic growth. Nevertheless, this result change when they analyse the countries by income level. In this regard, they conclude that debt overhang effects cannot be related to a specific debt threshold and that the relationship between public debt and growth is complex.

Finally, as admitted by Panizza and Presbiero (2013), there is not a robust result because small changes in data or the econometric methods yield different results concerning the causal relationship. In this regard, also the econometric work of Reinhart and Rogoff (2010a and 2010b) has been proved to be misleading. Hendon et al. (2013), replicating their exercise founded coding errors, selective exclusion of available data, and unconventional weighting of summary statistics. They demonstrated that once properly calculated, the average real GDP growth rate for countries carrying a public debt-to-GDP ratio of over 90% is actually 2.2 per cent, not -0.1 per cent. That is, the average GDP growth at public debt-to-GDP ratio over 90% is not different than when the debt-to-GDP ratio is lower.

On the theoretical side, it is tricky to find an explanation of the impact of the high level of public debt on GDP growth. In the Ramsey-Cass-Koopmans model (Ramsey 1928, Cass 1965, Koopmans 1965), Ricardian equivalence holds, government bonds do not represent net wealth for households and the public debt is neutral to long-run output[2]. In the Blanchard Overlapping Generation model (1985), instead, the public debt slightly crowds out physical capital and reduces long-run output. In this model the structure of the economy is the same as in the RCK model, the only difference is the finite horizon of households. Anyway, as estimated by Dombi and Dadak, in this model a percentage point change in debt-to-GDP ratio reduce the steady-state per-capita output only by 0.008-0.032 per cent (Dombi and Dedak, 2019). Thus, also in this model, fiscal policies result to be neutral. As demonstrated by Evans (1991) these insignificant results depend on the fact that the Ricardian equivalence is still a good approximation of the model.

Laxton and Symansky (1997) and Faruqee (2003) shows that once some relevant life-cycle aspects of household behaviour are included in the model the "burden" of public debt can be considerable in the Blanchard model as well[3]. However, the value of the output-loss with respect to an increase in the public debt-to-GDP ratio

[2] Barro (1974) demonstrated that in a neoclassical word, if intergenerational links prevail, government bonds do not represent net wealth for the households and the Ricardian equivalence holds.

[3] Laxton and Symansky (1997) introduce wages following a hump-shaped life-cycle pattern, while Faruque (2003) include death probability which increases with the age.

is very sensitive to the parameter calibration: a decrease in the intertemporal elasticity of substitution bring the impact of public debt on GDP growth to zero (Faruqee and Laxton, 2000).

In general, to have a robust and significant crowding-out effect in the neoclassical model, it is needed that the saving ratio is constant and exogenous, as in the Solow model (1956). This is particularly valid in the endogenous growth model (Mankiw, Romer and Weil, 1992), where the saving ratio affects also the long-run growth of technological progress. Dedak and Dombi (2018) show that in a human capital augmented Solow model a one percentage point increase in the debt-to-GDP ratio reduces the long- run output by 0.167 per cent[4].

Ultimately, in the neoclassical models, the possibility of having a harmful effect of public debt on GDP depends on whether the Ricardian equivalence holds or not. The absence of intergenerational linkages or an exogenous saving rate implies that the Ricardian equivalence does not hold and an increase in public expenditure financed by deficit or taxes causes a decrease in the average saving ratio.

The general idea is that given the full-employment homogenous output, the higher is the level of public expenditure, the higher will be the share of total output that will be consumed and not saved[5]. Because savings determine the variation in the capital stock in the next production period, an increase in public expenditure depresses the long-run capital accumulation. If the Ricardian equivalence holds (as in the RCK model with infinite horizon), the negative effect on aggregate savings caused by an increase in public expenditure is offset by an increase in the saving rate of households, thus the long-run accumulation process is "preserved". Conversely in a model where the intergenerational link is absent, the future tax burden of the present deficit financing will fall to some extent on new generations, thus households do not modify their consumption plan[6]. For the same reason, in models where households saving rate is exogenous, an increase in public debt affect negatively the aggregate saving ratio.

Anyway, it is important to outline that in these models the impact of the public debt on GDP growth can be assessed only in terms of variations or via exercises in comparative statistics. In this sense, there is no way to determine a certain threshold beyond which the public debt-to-GDP ratio is harmful to economic growth.

As also argued by Krugman (2021), there is no theoretical reason in neoclassical models for having the non-linearity depicted by some empirical-mainstream works: "[…]There isn't a growth cliff at a debt ratio of 90 per cent. Nothing in standard macro suggested either of these things should be true. All that supported them was bad statistical analysis […]". In this respect, the common assertion "*high-level of public debt*" result to be unjustified, precisely because the term "high" has only the traits of emptiness and no quantification methodology is present.

In this regard, it is important to clarify that in the above-mentioned models, it is not the level of public debt per se which is detrimental for capital accumulation, but it is public expenditure. Indeed the same mechanisms

[4] Anyway, this effect strongly depends on the value of the saving rate and the growth rate of population. The effect decrease with an increase in the saving rate.

[5] It is worth noting that, implicitly, it is assumed that the total output absorbed by public sector is totally consumed (no public investments). In Ausher (2000) debt is used to finance productive public capital, in this case public debt has a positive impact on the growth rate.

[6] In Blanchard model, the effect is negligible because the endogenous variation of the saving rate in response to public expenditure is still in process even if not with a total displacement.

would apply in the case public expenditure is completely financed by taxes (thus, without any changes in the accumulation of public debt). In this context, the public debt-to-GDP ratio is just a reflection of the share of public expenditure on total output. Along this line, according to the well-known crowding-out effect, an increase in public spending, reducing the amount of loanable funds, would cause an increase in interest rates and (therefore) a reduction in private investments.

Within neoclassical literature, there are other attempts to connect directly the level of debt through the interest rate variation (without passing through the learnable found theory). For instance, Alesina and Perotti (1997) and Alesina and Ardagna (2010) argue that a reduction in public spending would produce a reduction in interest rates also through a change in expectations. A credible decrease in public spending would lead to a decrease in the expected debt-to-GDP ratio and will thus lower the probability of the sovereign default. A reduced risk premium will be required and, in turn, lower long-term interest rates and corresponding higher asset prices will emerge (Ardagna, 2004). Consequently, both the wealth effect, generated by an increase in the asset price and the corresponding decrease in the interest rates should stimulate private investments (Alesina and Ardagna, 2010).

In conclusion, it seems to be absent any theoretical reasons for the neoclassical research branch to empirically looking for a threshold for debt-to-GDP ratio above which the public debt is detrimental for growth. Consistently with such a theoretical approach, the empirical analysis should be limited only to the impact of public expenditure on economic growth[7].

However, as argued, although there is neither empirical nor theoretical justification even within the neoclassical strand, the idea that high levels of debt-to-GDP ratios are detrimental for economic growth is widely held among institutions, economists and policy makers.

In this paper, we are not questioning if the reduction of debt-to-GDP ratio would *per se* be beneficial for the economy, neither if such an indicator has some empirical or theoretical meaning for the public debt sustainability. Within a framework reproducing the fondant feature of a monetary economy of production, we are going to highlight which are the determinants of the public debt-to-GDP ratio and the conditions for its stabilization, with particular reference to the monetary nature of savings and their ex-post determination with respect to public spending and investments.

With this aim, we develop a theoretical framework that integrates the dimension determining output and the level of employment with the dimension determining value. That is, adopting the SFC methodology, we integrate a demand-led growth model based on the Supermultiplier mechanism with the system of relative prices. Indeed, as it will be shown, the public debt-to-GDP ratio depends both on the determinants of the level of production and public deficit, that is the propensity to consume, the tax rate and the interest rate and on the capital intensity of the production process, thus on the factors affecting the system of relative prices that are the coefficients of production, the rate of profit and the interest rate[8]. With this respect, the interest rate represents

[7] Of course, neoclassical authors have developed many empirical work that goes on this direction.
[8] The system of relative prices affect the public debt-to-GDP ratio modifying the differential of sectorial indebtedness and the private debt-to-GDP ratio.

the *trade union* between the two dimensions: variations in the interest rate impact on the public debt-to-GDP affecting simultaneously the quantitative dimension and the system of relative prices. For the sake of clarity, the analysis of the impact of changes in the interest rate will be carried out considering the effect on one dimension at a time. The results will be supported by both the simulation of the IO-SFC model and the analytical solution based on the baseline version of the SM.

Along this line, our analysis wants to outline that when we move from a neoclassical world to a framework that takes rigorously into consideration the monetary nature of the production process, money cannot appear ex-post with respect to the determination of the equilibrium quantities and prices. At the same time, the real output cannot be intended as distributed in real terms. In this context, money demand cannot be derived on the basis of agents' necessity to exchange the predetermined equilibrium quantities, nor the stock of money can be determined from an external authority that fixes the supply[9]. On the contrary, money enters into circulation within the process of production through autonomous spending and investment decisions, while the residual stock of money will depend on the households savings and portfolio decisions.

In this sense, as is clear from the SFC framework, public expenditure and public deficit can only increase the amount of deposits and, more generally, the financial resources available to the private sector. In this regard, if one assumes - as mainstream authors do - a negative relationship between the amount of "loanable funds" and interest rates on loans, investments could only benefit from an increase in public debt. Consistently with the principle of endogenous money, the financial resources available - at a given moment – in the economic system are not given but are endogenous to the expenditure of the system as a whole[10]. Therefore, they do not respond to the logic of "scarcity" dictated by the laws of supply and demand[11].

For this reason, also the not purely neoclassical thesis according to which an increase in the public deficit would produce an increase in interest through an increase in the demand for money for transactional purposes appears unfounded. Indeed, since the stock of money and “lonable funds” are determined ex-post with respect to spending decisions it cannot be considered as exogenous to the economic system. In particular, the existence itself of such loanable funds would be in discussion without an ex-ante injection of purchasing power.

The rest of the paper is organized as follows: Section 2 presents the framework of the model. Section 3 and 4 analyses the determinants of the public debt-to-GDP ratio considering respectively the quantitative and the

[9] It should be noted that the nature of the difference in results does not concern the inclusion or non-inclusion of the demand for money for speculative purposes.

[10] Of course, in this approach, as demonstrated by the authors belonging to the endogenous money theory, the interest rate cannot be determined from the intersection of the supply of founds and the demand for savings. In this sense, the expansion of public expenditure can affect investments only through the accelerator.

[11] This issue is linked to the debate on the stock-flow inconsistency of the IS-LM model (Godley and Shaikh, 2002). In this sense, the analysis of the monetary and real sphere cannot be conducted by watertight compartments, firstly determining the real quantities and, only later, the quantity of money requested or necessary to carry out the relevant transactions.
To this extent, thinking in terms of the IS-LM model, each movement of the IS produces a simultaneous expansion of the LM without any coordinated intervention of the CB. In this sense, the stock of money cannot be considered as exogenous with respect to the trend of the expenditure components. On the contrary, the stock of money simply represents a residual of the purchasing power injections related to these components and it is determined by savings and portfolio decisions of households.

value dimension. Section 5 discusses the stability conditions of the public debt-to-GDP, both through analytical derivations and simulations. Section 6 presents the results of policy experiments. Section 7 concludes.

1. The theoretical background of the model

The *demand-led* growth model developed in this paper is based on the monetary circuit framework (MC) and presents the same feature of the SM (Serrano 2015, Cesaratto et al. 2003): the capacity adjustment principle, the exogenously given normal rate, an autonomous component of demand and adaptive expectations.

The SM extends to the long-run principle of effective demand and combines the role of non-capacity creating autonomous components of demand with the accelerator mechanism. Firms try to adjust productive capacity to match the expected demand in correspondence with the normal or desired degree of capacity utilization. The long-run income is the result of the interaction between the multiplier and accelerator mechanisms and savings adjust to investments by variations in the level of production and the corresponding production capacity. In this approach, the distribution is determined exogenously starting from historical, institutional and social factors, inherent in the bargaining power of the classes and social norms regarding the fairness of remuneration (see, for example, Stirati, 1994 and Levrero, 2013). The main result of the model is that, in the long-run, the output growth rate converges towards that of the autonomous component while the degree of capacity utilization converges to the normal one.

The present model combines these features with a different economic structure regarding the "origins" of the aggregate demand taking as reference the monetary circuit theory[12]. To this extent, wages and inputs are paid in advance and firms need to estimate both current and future demand[13]. Conversely, because of the economic structure built on the income/expenditure scheme, the SM implicitly assumes an on-spot economy: all firms and sectors know in advance current demand[14] and wages and profits are paid ex-post with respect to sales realization (production is constantly equal to demand)[15]. At last, as in the Supermultiplier investments are completely induced and firms try to adapt productive capacity in order to maintain a normal degree of capacity utilization.

We choose to develop this model because it is immune to the critique addressed to the traditional version of the SM (Nikiforos 2018; Skott 2019), while it preserves its main features. Indeed, in this framework, there is no need to assume an exogenous growth rate of the autonomous component in the long run. In the SM the autonomous component represents the initial injection of money into the system and the original source of

[12] See Di Domenico (2020) for the reference model and a detailed discussion about.

[13] The future demand is the expected demand for the period in which the capital good would be available.

[14] They have to estimate only future demand in fixing investment decisions.

[15] The injection of purchasing power carried out in correspondence with the autonomous component of a given period triggers a multiplicative sequence in the periods to come which is intertwined with the multiplicative sequences triggered by the autonomous expenditure realized in the previous periods and to be realized in future periods. For instance, an autonomous demand in the period t equal to 50, generates a production of equal value and a downstream payment of incomes equal to 50. In the following period, a portion of this income will constitute the induced consumption demand generating, together with the value of the autonomous component of this period, a production and income of equal value. The income distributed at the end of that period will form the consumption base for the next period, and so on.

demand, without this the system results to be undetermined and the economic system could not be represented. Thus, such kind of assumption is unavoidable.

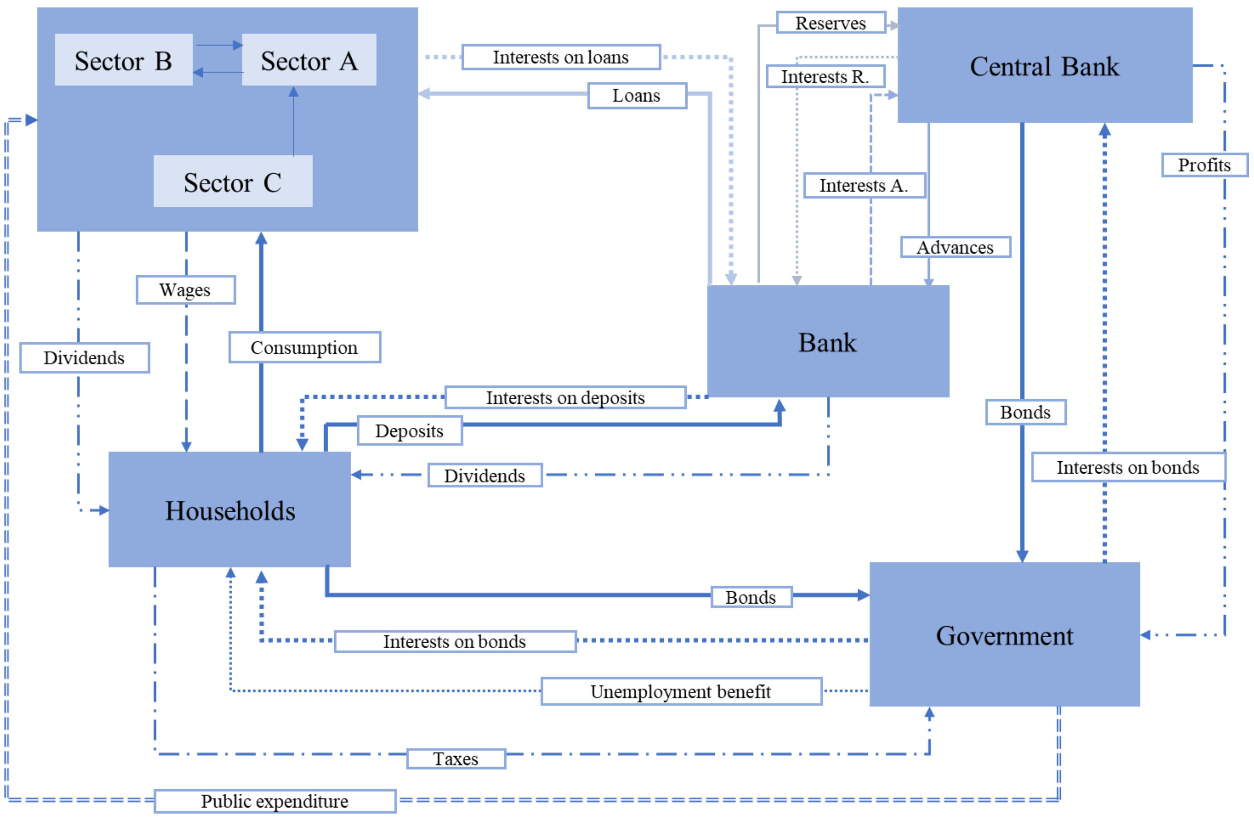
On the contrary, in the MC, aggregate demand is endogenously generated by firms' anticipations in financing production costs (wages and input), thus adaptive expectations on current demand take the place of the Supermultiplier exogenous injection of purchasing power. This makes it possible that all the components of demand, including the autonomous one, can be considered endogenous in the long run. For this reason, the system can be determined independently to the assumption on the long-run growth rate of the autonomous component. This differential approach is particularly relevant for the assumption of an exogenous long-term trend of autonomous consumption[16].

[16] To this respect, the consumption function can be formalized in various way, i.e. path dependent consumption function or traditional post-keynesian function where consumption depend both on income and wealth. The central issue is that these autonomous component of demand do not constitute the initial injection of money into the system but are originally financed by firm anticipations, thus, in monetary terms, by firms indebtedness themselves.

2.1 The Input Output - Stock Flow Consistent model

The model represents a closed economy with five sectors: Government, households (workers and capitalists), production sector, commercial bank and Central Bank (CB).

The economic system is described by the flow diagram of the following figure:



The productive sector consists of two basic sectors producing investment goods and one consumption sector. Sector C produces the consumer good by means of labour and commodity A, sector A produces by means of labour and commodity B and sector B produces by means of labour and commodity A.

Firms operating in Sector C define the level of production for each period starting from the expectations on future demand and the desired level of inventories. They invest in order to match the expected demand in correspondence with a normal degree of capacity utilization. Firms C pay in advance wages and capital, while firms A e B produce on-spot based on orders received. Thus, the indebtedness of sector C is devoted to both the advancement of wages and capital, while sector A and sector B ask for loans only to acquire fixed capital. They pay current costs from revenues[17].

The price of commodities is set according to the logic of production costs and the markup (the industrial rate of profit) and interest rate are set exogenously.

[17] Because of we are assuming that sector C pays the capital good at the time of the order, this also applies if the period of the capital good is longer than one

The banking sector makes loans by passively accommodating the credit demand of firms and collects deposits from households. These hold their savings in the form of deposits and government bonds. The CB acts as a lender of last resort, acquiring the residual part of public bonds not demanded by the private sector. The only autonomous component considered in the model is direct primary expenditure.
The circular flow of material that characterize an economy with two or more basic commodities would require the Inverse Leontief matrix to solve the simultaneity in the determination of inputs production. For the sake or realism, because Sector A and Sector B does not solve a simultaneous system to determine the quantity to produce and they cannot take as given the quantity desired of the other sector, they use expectations on demand to fix the desired production.
In Appendix 2, the transaction matrices and balance sheets describing the economy are reported.

2.1.1 Consumer Sector

Sector C uses as inputs labor and fixed capital:

$$l_c \oplus k_c \rightarrow C$$

Firms C fix current production ($y^d_{t,i}$) based on expected demand ($q^e_{t,i}$) and it is determined through adaptive expectations. In addition, firms consider a store of inventories to address the discrepancies between expected demand and realized one.

$$q^e_{t,i} = q^e_{t-1,i} + \alpha(q^r_{t-1,i} - q^e_{t-1,i})$$

$$y^d_{t,i} = \max\{0, q^e_{t,i}(1+\sigma^T) - inv_{t-1,i}\}$$

Where σ^T is the desired ratio of inventory on sales and $inv_{t-1,i}$ is the amount of inventories from the previous period. The degree of capacity utilization in correspondence of the planned production is:

$$\omega^e_{t,i} = \min\left\{1, \frac{y^d_{t,i} v^*_{c_{t,i}}}{k_{t,i}}\right\}$$

Where $v^*_{c_{t,i}}$ is the capital/output ratio in correspondence with the full utilization of the productive capacity. Given the amount of capital needed to produce $y^d_{t,i}$ and the capital/labor ratio ($\alpha_{c_{t,i}}$) it is possible to determine the labor demand:

$$L^d_{t,i} = \frac{\omega^e_{t,i} k_{t,i}}{\alpha_{t,i}\, v_{t,i}} = y^d_{t,i} l_c$$

$k_{i,t}$ is the amount of fixed capital and l_c is the amount of required working hours per unit of output.
The feasible production is[18]:

[18] The production function is characterized by fixed coefficient of production (Leontieff technology).

$$y_{t,i} = \min\left(\frac{L_{t,i}h^m}{l_{t,i}};\frac{k_{t,i}}{v_{t,i}^*}\right)$$ [19]

Investment function:

Firms C adjust productive capacity in order to satisfy expected demand in correspondence of a normal (desired) degree of capacity utilization (in the period in which the capital will be available, that is in $t + dk$[20]):

$$I_{t,i} = \max\{0; q_{t+dk,i}^e(1+\sigma^T)v_{t,i}^n - k_{t+dk,i}\}$$

where v_c^n is the ratio between capital and normal output, $k_{t+dk,i}$ is the residual capital in the period in which the ordered capital would be installed if the investments were not made[21], $g_{t+1,i}^e$ is the expected growth rate of demand.

$$q_{t+dk,i}^e = q_{t,i}^e(1+g_{t,i}^e)$$

$$g_{t,i}^e = g_{t-1,i}^e + \alpha\left(g_{t-1,i}^r - g_{t-1,i}^e\right)$$

The stock of capital in period t is composed by the residuals of capital goods installed in the previous z + 1 periods (vintage capital goods), with z representing the life span of the capital good:

$$k_{t,i} = \sum_{j=t-z+1}^{t} k_{j,i}^{ins}\left(\frac{j+z-t}{z}\right)$$

where $k_{j,i}^{ins} = I_{t-dk,i}$ is the amount of capital installed in period j and corresponds to the gross investment carried out dk previous periods. The total deterioration in each period consists of the sum of the deterioration of capital goods installed in the previous z periods (including the current one):

$$deterioration_{t,i} = \sum_{j=t-z+1}^{t} \frac{k_{j,i}^{ins}}{z}$$

Amortization is needed to compute unit costs and profits. The amortization for computing unit cost includes both the cost of capital and the cost of debt service considering the realized leverage[22]:

$$amortization_{t,i} = \frac{1}{a\,z}\sum_{j=t-z+1}^{t} p_{i,indexK}k_j^{ins}(1+r_jbl_j)(j+z-t)$$

where r_j e l_j represent, respectively, the interest rate in the period in which the debt was contracted and the *leverage* realized in purchasing the capital good.

[19] Where $l_{i,t} = \frac{v}{\alpha}$, that is the ratio between capital/output ratio and capital/labor ratio.

[20] dk is the number of periods needed to produce one unit of the capital good..

[21] The capital good ordered in period t is installed in period $t + dk$.

[22] See Appendix 1 in Di Domenico (2020) for an explanation on computation of amortization and unit cost.

$K_{j,i}^{ins}$ is the installed capital in period j from sector i and $p_{i,indexK}$ is its price (because z is the useful life of the capital, it goes back up to a maximum of z periods in the depreciation calculation). $a = \sum_{i=1}^{z} \frac{i}{z}$ e $b = \frac{1}{a\,z} \sum_{i=1}^{z} \frac{i^2+i}{2}$ are the multiplying factors for the computation, respectively, of the interest accrued on loan granted in a given period and of the (potential) cumulated production (in correspondence of the degree of normal use) over the useful life of capital good. Because the capital good has a finite useful life, a constant absolute depreciation of installed capital is adopted and, therefore, the depreciation rate is increasing.

2.1.2 Capital sector A and B

$$l_a + b_a \rightarrow A$$

$$l_b + a_b \rightarrow B$$

l_a and l_b are the amount of working hours per unit of output, a_b and b_a represent the amortizations of fixed capital. Given the number of periods required to produce the K good (dk)[23], the quantities that sector i wishes to produce in each period is:

$$yd_{t,i} = \sum_{j=t-dk}^{t} \frac{orders_{j,i}}{dk}$$

Where the summation corresponds to the number of capital goods ordered to sector i, from previous dk periods to period t[24]. Investments, labour demand, capital deterioration and amortization are computed in the same way as for Sector C.

2.1.3 System of price equations

Prices are determined according to the normal-cost pricing (Andrews, 1949; Andrews and Brunner, 1975). The unit cost (which takes into account the different ages of the capital goods) is defined in correspondence with the normal degree of capacity utilization and amortization is computed adopting the full cost methodology[25].

The unit cost in each sector is:

$$c_{t,i} = \frac{\overline{w}_{t,i} * L_{n_{t,i}}}{y_{t,i}^{n}} (1 + ra_s z_s) + \frac{am_{t,i}}{y_{t,i}^{n}} =$$

$$\left(\overline{w}_{t,i} \frac{v_{t,i}^{*}}{\alpha_{t,i}} (1 + ra_s z_s) + \frac{\frac{1}{a\,z} \sum_{j=t-z+1}^{t} p_{k_j} K_j^{ins} (1 + r_j bl_j)(j + z - t)}{\frac{k_t \omega^n}{v_{t,i}^{*}}} \right) =$$

[23] dk is a key parameter in determing the magnitude of the interaction between multiplier and accelerator.

[24] In each period a portion of $1/dk$ of each order is produced.

[25] According with the principle of opportunity cost, the interest rate is applied to all the inputs of productions or all the anticipations (See Pivetti, 1985, 1991). That is, a leverage equal to one is adopted to compute the unit cost.

$$\left(\bar{w}_{t,i} l_c (1 + r a_s z_s) + \frac{\frac{1}{a\, z} \sum_{j=t-z+1}^{t} p_{k_j} K_j^{ins} (1 + r_j b l_j)(j + z - t)}{\frac{\omega^n}{v_{t,i}^*} \sum_{j=t-z+1}^{t} K_j^{ins} \left(\frac{j + z - t}{z} \right)} \right)$$

where $L_{n_{t,i}}$ is the amount of working hours corresponding to the normal degree of capacity utilization, $y_{n_{t,i}}$ is the normal production, $\bar{w}_{t,i}$ is the nominal wage, $am_{t,i}$ is the amortization, p_{k_j} is the price of the capital acquired in period j, a_s is the multiplicative factor to compute the total debt service and z_s is the payback time of short loans (aimed at advancing wages)[26]. If the profit rate, the interest rate and the coefficient of productions are constant over time and $l_j = 1$ (full-cost pricing), the equation of unit cost is reduced to:

$$c_{t,i} = w l_c (1 + r a_s z_s) + \frac{v_c^* p_k}{a \omega^n} (1 + rb)$$

Thus the system of prices can be expressed as follows:

$$\boldsymbol{pA}(1 + m) + a_n w = \boldsymbol{p}$$

Where:

$$A = \begin{bmatrix} 0 & \alpha & 0 \\ \beta & 0 & 0 \\ \gamma & 0 & 0 \end{bmatrix}$$

$$\vartheta = \frac{v_a^*}{a \omega^n} (1 + rb)$$

$$\beta = \frac{v_b^*}{a \omega^n} (1 + rb)$$

$$\gamma = \frac{v_c^*}{a \omega^n} (1 + rb)$$

Solving the system we get the expression of prices:

$$p_a = w(1 + m) \frac{l_a + \vartheta l_b (1 + m)}{1 - \vartheta \beta (1 + m)^2}$$

$$p_b = w(1 + m) \frac{l_b + \beta l_a (1 + m)}{1 - \vartheta \beta (1 + m)^2}$$

$$p_c = w[l_c (1 + r a_s z_s) + \gamma \frac{(1 + m)[l_a + \vartheta l_b (1 + m)]}{1 - \vartheta \beta (1 + m)^2}](1 + m)$$

[26] In the case of Sector A and B this value is zero.

2.1.3 Households

Consumption demand is a function of the income and wealth stock. The basic idea is that there is a target wealth-to-income ratio that a consumer would like to attain over time (Godley and Lavoie, 2007):

$$\text{Workers: } c_{i,t}^{D,w} = \min\left(YD_{t,i}c_{1,w} + V_{t-1,i}c_{2,w}\right)$$

where:

$$YD_{t,i} = \begin{cases} \left(w_{t,i}h_{t,i}^{work} + M_{t-1}r_{t-1}^{m}\right)\left(1-\tau^{work}\right) \text{ if employed} \\ \left(w_{gov} + M_{t-1}r_{t-1}^{m}\right)\left(1-\tau^{work}\right) \text{ otherwise} \end{cases}$$

where $h_{t,i}^{work}$ are the monthly worked hours, w_{gov} is the unemployment benefit and τ^{work} is the tax rate on workers income. Workers held all their wealth in the form of deposits: $V_{t,i} = M_{t,i}$.

The capitalist consumption function is:

$$c_{i,t}^{D,\pi} = \min\left(YD_{t-1,i}c_{1,\pi} + V_{t-1,i}c_{2,\pi}, YD_{t-1,i} + M_{t-1,i}\right)$$

Capitalists income is made up of dividends distributed by firms and banks, the interest accrued on deposits and public bonds:

$$YD_{t-1,i} = \left(Div_{t-1,i} + M_{t-1,i}r_{t-1}^{m} + B_{t,i}^{h}r_{t-1}^{b}\right)\left(1-\tau^{\pi}\right)$$

$Div_{t-1,i}$ are dividends and τ^{π} is the tax rate on capitalists income.

The stock of capitalist wealth is made up of deposits and government bonds:

$$V_{t,i} = M_{t,i} + B_{t,i}^{h}$$

$$V_{t,i} = V_{t-1,i} + YD_{t-1,i} - C_{t,i}$$

The demand for government bonds is a function of the stock of wealth, disposable income and the interest rate (Tobin, 1982):

$$\frac{B_{t,i}^{d}}{V_{t,i}} = \lambda_0 + \lambda_1 r_t^{b} + \lambda_2\left(\frac{YD_{t,i}}{V_{t,i}}\right)$$

2.1.6 Commercial Bank

The banking sector consists of one single commercial bank. This plays a passive role, supporting the credit demand of firms and collecting household deposits. In determining the interest rate on loans, the bank sets a markup on the rate set by the central bank. The interest rate on loans is higher than that on deposits. The difference between loans and deposits is held as reserves at the CB (reserves accrue at an interest rate equal to that of deposits).

2.1.7 Government

The public sector has an exogenous component of expenditure which is expressed in terms of consumer goods (non-capacity creating expenditure) and an endogenous component that is the debt service. Government accounting is:

$S_g = G_t - \theta Y_t + r_b B_{-1} - F_t^{cb}$ where F_t^{cb} are distributed profits by CB, θ is the tax rate, B_t is the stock of public debt. The supply of public bonds is:

$B_t = B_{t-1} - S_g$

$B_t = B_{h,t} + B_{cb,t}$

2.1.8 Central Bank

Central bank profits depend on interest earned on public bonds (B_{t-1}^{cb}), advances *(A_{t-1})* and from interests paid on reserves (H_{t-1}^{cb}).

$$\pi_t^{cb} = B_{t-1}^{cb} r_{t-1}^{b} - H_{t-1}^{cb} r_{t-1}^{riserve} + A_{t-1} r_{t-1}^{a}$$

CB acts as the lender of last resort in the public bonds market:

$$B_t^{cb} = B_t - \sum_{i}^{ncap} B_{t,i}^{d}$$

Where $\sum_i^{ncap} B_{t,i}^{d}$ is the amount of bonds held by capitalists.

Since households hold their savings in the form of deposits or public bonds, the amount of bonds purchased by the CB is equal to the amount of households deposits.

See Appendix 2 for all the equations regarding the stock-flow consistency, financing of production decisions and accountancy.

Redundant equation (check for consistency)

$M_t = L_t + B_{cb,t}$

The redundant equation expresses the indirect relationship between bonds purchased by the CB and the share of savings held in the form of deposits. The difference between aggregate deposits and bank loans corresponds to public bonds held by the CB, which in turn are equal to the commercial bank's reserves held at the CB:

$M_t - L_t = B_{cb,t} = R_{cb,t}$

Regarding the last equations, in the literature, SFC models are normally closed by requiring that the cash held by households is equal to the net money creation of the CB (that corresponds to the amount of public bond held by the CB). In this model, since households do not hold cash, the part of the savings corresponding to the share of government debt held by the CB is held by households in the form of deposits. Therefore, the total amount of deposits is equal to the sum of loans created by the commercial bank and government bonds purchased by the CB. The difference between total deposits and loans is held by the commercial bank in the form of reserves; as a consequence, the reserves are equal to the amount of public bonds purchased by the CB[27].

2. Determinants of the public debt-to-GDP ratio: the quantitative dimension

3.1 Interest rate, saving rate and output growth

This paragraph analyses the implications for the traditional conclusions of the SM since the supermultiplier mechanism is dropped within an SFC framework and, therefore, the effect of an exogenous trend of primary public expenditure on the debt stock and related effects on income determination are included.

In general terms, in the traditional SM model, the long-run growth rate converges to the exogenously given growth rate of the autonomous component. The inclusion of endogenous debt service implies that a part of the output growth rate is endogenously determined by household saving decisions that, in turn, determine the growth rate of public debt service. Indeed, the public debt service represents another autonomous component whose pattern results to be endogenously determined and, exactly like all autonomous components, contributes to the determination of the growth rate of output

The endogenously determined share of output growth can be positive only if in the steady-state the growth rate of savings, and symmetrically the growth rate of public debt, are positive. In particular, if the exogenously given growth rate of primary public expenditure is zero, the growth rate of the economy becomes only a function of the interest rates accrued on public bonds and correspond to the growth rate of public debt[28]. In this sense, the interest accrued on the public debt becomes part of the income and consumption of households, thus public debt service, adding new purchasing power into the system, becomes an autonomous component.

Since the current deficit is a function of the multiplier effect, the higher the savings rate, the higher is the deficit and the accumulation of public debt and the growth rate of interest expenditure. In this sense, a higher saving rate producing a higher growth rate of public debt service can affect positively the growth rate of output[29]. Of course, this relationship does not mean that a higher saving ratio is preferable to foster economic growth. Conversely, in correspondence with lower levels of the saving rate, the same growth rate of the total public expenditure can be reached by an equal expansion in the growth rate of primary public expenditure. In this case, the output growth rate would be the same but, due to a higher value of the multiplier, the value of the public debt-to-GDP ratio would be lower.

Let's analyse these mechanisms more in detail.

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[27] See Appendix 4 for further explanation.

[28] In dominant macroeconomic models, the dual nature of public debt is often overlooked: a liability for the government, but at the same time an asset for the private sector. Accrued interests on bonds do not represent an exit to a blind alley for the economy as a whole; on the contrary, they stimulate private consumption levels and tax revenues from it. Thus, following the same effects as an expansion of direct public expenditure, if the accelerator channel is operational, the income stream generated from accrued interest on bonds may lead to reductions in the debt ratio. These mechanisms are consistently traced within SFC models such as those proposed by Godley and Lavoie (2007) and Ryoo and Skott (2013).

[29] This is analytically valid as long as the interest accrued on savings becomes part of the current income of households and then re-enters (at least in part) into circulation through consumer spending.

Figure 3.1 reports the results of model simulations, according to the different combinations of c_1 and c_2. In order to isolate the effect of the interest rate, the growth rate of direct public expenditure is set to zero. The endogenously determined growth rate of the economy is negatively correlated with the propensities to consume ($c_1$and c_2) and it is positively correlated with the interest rate. The graph presents the values of c_1 and c_2, respectively on the x-axis and y-axis. The z-axis shows the output growth rate. Each surface represent the relationship between the propensity to consume and the GDP growth rate in correspondence of different levels of the interest rate.

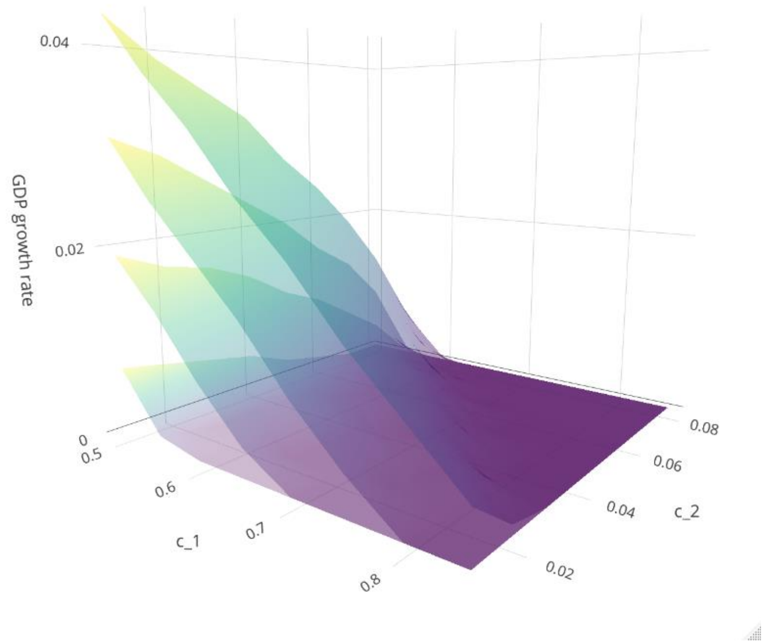

Figure 3.1: Equilibrium growth rate and different combinations of consumption propensity on income (x-axis) and lagged wealth (y-axis). Each surface represent a different level of the interest rate..

For the combinations of $c1$ and c_2 in the purple zone (i.e. for high values of the propensity to consume), the economy reaches a steady-state where households achieve the desired wealth-to-income ratio: public deficit and accumulation of savings (and public debt) are zero. The range of change of the interest rate does not modify the steady-state growth rate of output (they affect exclusively the equilibrium income level).

In this scenario, considering the traditional SM system expressed in a sequential tune[30], we can derive the equilibrium values of income and wealth analytically (for the sake of simplicity let us assume that all wealth is held in the form of government bonds):

[30] See Appendix 1 for the general system. See Di Domenico 2020 about the advantages of the sequential methodology with respect to the traditional formulation of the SM expressed in terms of sequence of simultaneous systems.

$$\begin{cases} Y_t = (Y_{t-1}(1-\theta))c_1 + V_{t-1}c_2 + +r_b B_{-1} + G_t + I_t - \delta K_t{}^{31} & (3.1) \\ G_t - (G_t + Y_{t-1}(1-\theta)c_1 + V_{-1}c_2 + rV_{-1})\theta + r_b B_{-1} = 0 & (3.2) \end{cases}$$

$$Y = \frac{V_{-1}(c_2 + r_b) + G_t}{1-(1-\theta)c_1}{}^{32} \quad (3.3)$$

After some manipulation, we can get the expression of income and public debt in equilibrium:

$$V^* = \frac{G_t(1-\theta)(1-c_1)}{r_b(1-\theta)[1-c_1(1-2\theta)] + c_2\theta}{}^{33} \quad (3.4)$$

$$Y^* = \frac{\bar{G}[c_2 + 2(1-\theta)r_b]}{r_b(1-\theta)[1-c_1(1-2\theta)] + c_2\theta} \quad (3.5)$$

Deriving the income equation with respect to c_2, we get:

$$\frac{\partial Y^*}{\partial c_2} = \frac{Gr_b(1-\theta)(1-2\theta)(1-c_1)}{(r_b(1-\theta)[1-c_1(1-2\theta)] + c_2\theta)^2}$$

For $0 < \theta < 1$ *e* $0 < c_1 < 1$, the numerator and denominator (square of a binomial) are always positive. Therefore, if the growth effect given by the savings essay is absent, a decrease of c_3 (that is an increase of c_2) produces an increase in the equilibrium level of income.

In the same way, deriving by c_1 we obtain:

$$\frac{\partial Y^*}{\partial c_1} = \frac{Gr_b(1-\theta)(1-2\theta)[c_2 + 2(1-\theta)r_b]}{(r_b(1-\theta)[1-c_1(1-2\theta)] + c_2\theta)^2}$$

The expression is always positive, as the propensity to consume increases, the equilibrium income increases.

The steady-state with $\Delta S > 0$ (therefore, with positive public deficits) is generated only if the increase in household savings has such an expansive effect on income that it prevents the achievement of the desired wealth-to-income ratio[34]. This mechanism implies a positive accumulation rate of private savings in the long term and, therefore, of public debt service. This "allows" the expansive effect to be produced from the above-mentioned accumulation of public debt.

Thus, the endogenously determined growth rate of the economy is positively correlated with the desired wealth-to-income ratio and the interest rate on government bonds. In the transition phase, the households' attempt to increase the share of savings is not effective since it leads to an increase in accrued interests. In the steady-state, the growth rate of savings equals that of output and public debt.

Similar results can be found in You and Dutt (1996). In their contribution, the authors develop a post-Keynesian model in which they trace the short- and long-term effects of public expenditure and the accumulation of public debt on the growth rate of the economy and the distribution of income. The results of their empirical

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[31] In steady state $Y_{t-1}(1-\theta)c_1 + V_{t-1}c_2 = Y_{t-1}(1-\theta)$ e $Y_{t-1} = Y_t$ and investments are equal to the replacement of impaired capital $I_t = \delta k_t$

[32] $V_{-1} = B_{-1}$

[33] See Appendix 1.1 for the procedure.

[34] This dynamic characterizes, by construction, the scenario in which $c_2 = 0$. In this case, if the interest rate is higher than zero, the accumulation of savings never runs out and, with that, the accumulation of public debt. In this case, the output growth rate endogenously determined is always positive.

analysis show how an increase in public debt stimulates the growth rate through an increase in income and consumption of public bonds holders.

To this extent, the results of traditional SM could only be confirmed if households held all their savings in the form of cash or if the interest rate on public bonds and deposits is zero. When a share of savings is held in the form of deposits or government bonds, the growth rate of the semi-autonomous component depends on households' savings decisions.

These results can be extended to the case where workers and banks purchase government bonds. In this context, the effect on the growth rate, with the same interest rate and savings rate, is higher than in the baseline scenario. Indeed, the amount of interest accrued on public debt would be higher and, with it, all the effects induced by the growth rate of public interest expenditure. A further consequence of the introduction of the purchase of bonds by the commercial bank is the lowering in the threshold at which the combination of r and s triggers the growth effect.

3.2 Interest rate, saving rate and public debt-to-GDP ratio

This section analyses the determinants of the public debt-to-GDP ratio (d^*) concerning the quantitative dimension. To isolate only the effects that operate through this channel, we keep constant the system of relative prices, hence the coefficient of production, the rate of profit and the interest rate which enters in prices equations are fixed. With this respect, it is important to clarify the assumption which necessarily descends from this type of analysis: when the interest rate on public bonds changes the distribution of income does not change. This implies that different interest rate does not comove. Such assumption will be removed in a second step.

The central issue is the following: given the interest rate, the saving rate and the tax rate, the resulting steady-state deficit-to-GDP and public debt-to-GDP ratios are intrinsic to the economic system and are uniquely determined. For the sake of reasoning, we will define such value as the *natural public debt-to-GDP ratio*.

Figure 4.1reports the variation of d^* when c_1 and c_2 change. For each combination of c_1 and c_2, the effect of a change in the interest rate on public bonds is reported. On the x-axis and y-axis, there are respectively the values of c_1 and c_2, the z-axis represents the steady-state values of the public debt-to-GDP ratio.

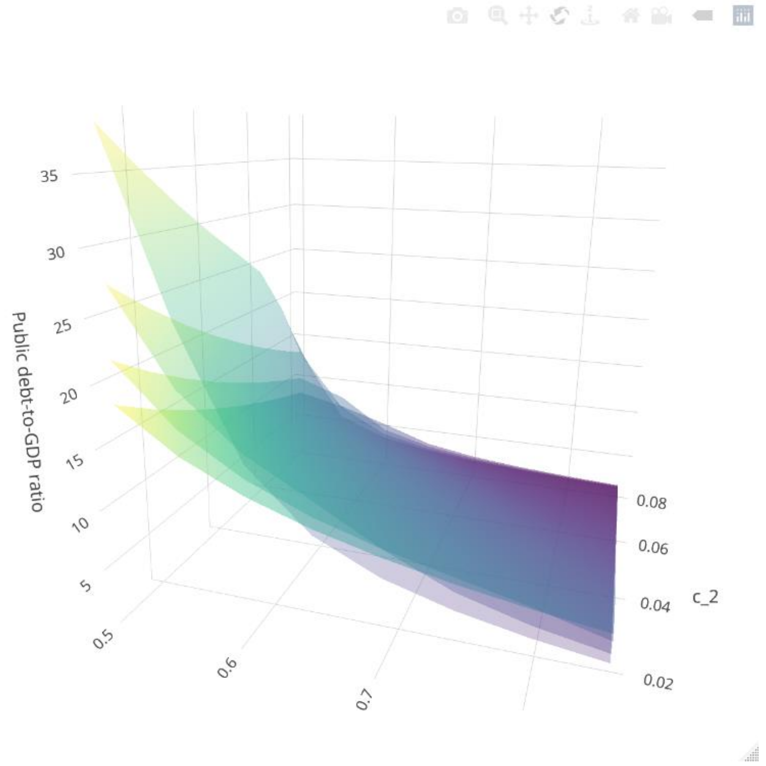


Figure 4.1: Equilibrium values of the Debt-to-GDP ratio as the propensity to consume on current income (x-axis) and wealth (y-axis) change. Each surface represents one level of the interest rate.

In particular, given the level of taxation, the steady-state debt-to-GDP ratio is positively correlated with the desired wealth-to-income ratio of households, so it is a negative function of the propensity to consume out of income and the propensity to consume out of wealth. Interest rate variations have a puzzling effect.

The relationship between saving rate and d_t is explained by two mechanisms that simultaneously affect both the nominator and denominator of d_t. On the one side, given the level of public expenditure, a higher propensity to consume implies a higher multiplier and, thus, a higher level of GDP. On the other side, the higher is the propensity to consume implies, the lower is the public deficit: specularly, the higher is the propensity to consume, the lower is the stock of savings held by households and the lower is the symmetric stock of public debt. For these reasons, the debt-to-GDP ratio is a positive function of the aggregate saving ratio: an increase in the propensity to consume reduce the public debt-to-GDP ratio decreasing the nominator and increasing the denominator.

The net effect of changes in the interest rate depends on the combination of the value of these parameters and the value of the tax rate. As the level of tax rate decreases, the weight that changes in interest rates have on disposable income increases, via a higher multiplier effect.

An increase in the interest rate has two effects that work in opposite directions: on the one hand, it has a positive effect through the expansion of public expenditure and hence GDP and tax revenues (income effect), and on the other hand, it increases the cost of debt service (cost effect). The resultant of these two effects determines the net impact on the debt-to-GDP ratio. Figure 4.1 shows the puzzle which emerges and the stationary values of the debt-to-GDP ratio as the components that determine the aggregate savings rate vary (c_1 and c_2).

The puzzling effect produced by interest rate variation depends on the saving ratio: when the latest is high, the so-called income effect is dominant, vice versa when the saving ratio is lower the multiplier effect dominates on the income effect. In the first case, the expansive effect of an increase in the interest rate on the GDP growth rate is such that it more than offsets the increase in the debt stock (the yellow dots represent a higher interest rate).

The following plot shows the effects of interest rate changes in the case of a propensity to consume (c_1) equal to 0.4. As we can see, changes in interest rates have a twofold effect on the growth rate of disposable income: the first effect is due to higher accrued interests and the second is due to the change in the composition of the household portfolio (the proportion of savings held in the form of bonds increases).

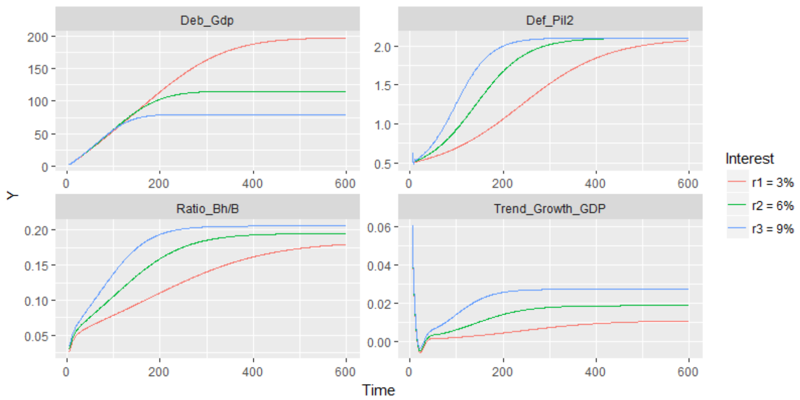


Figure 4.2: Equilibrium values of Debt-to-GDP ratio, deficit-to-GDP ratio, % households bond on public debt and growth rate of GDP as interest rate changes.

Both these two dynamics contribute to the expansionary effect on GDP of interest rate changes via an increase in public spending. In this scenario, an increase in the interest rate generates a decrease in the debt-to-GDP ratio. Figure 4.3, instead, shows the impact of interest rate changes at lower values of the wealth/income ratio ($c_3 = \frac{1-c_1}{c_2}$), respectively with $c_2 = 0.02$ and 0.06 (that is for a lower saving rate).

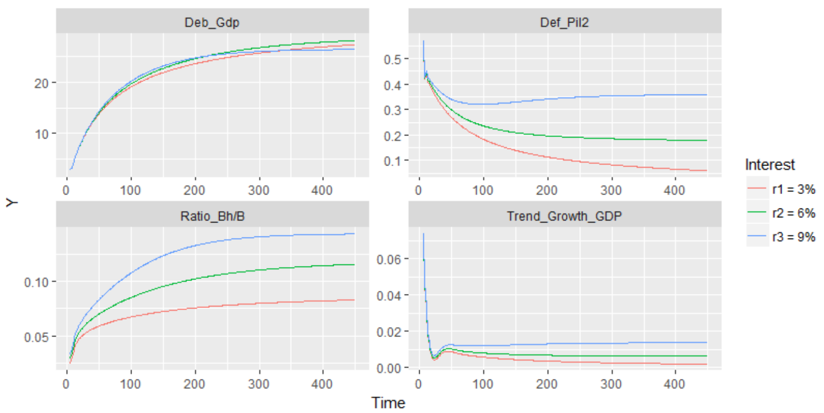


Figure 4.3: Equilibrium values of Debt-to-GDP ratio, deficit-to-GDP ratio, % households bond on public debt and growth rate of GDP as interest rate changes ($c_2 = 0.02$).

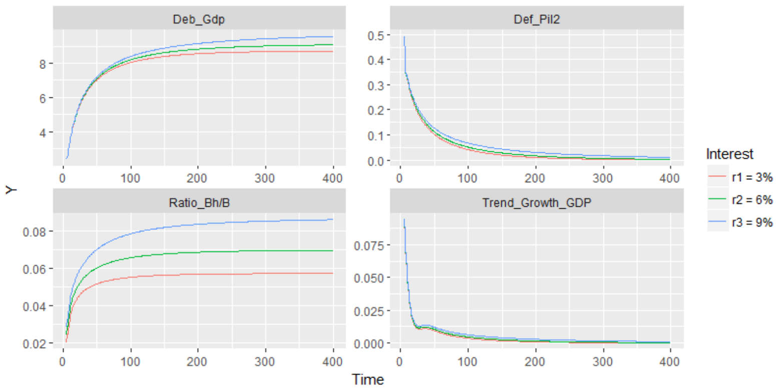


Figure 4.4: Equilibrium values of Debt-to-GDP ratio, deficit-to-GDP ratio, % households bond on public debt and growth rate of GDP as interest rate changes ($c_2 = 0.06$).

As the savings rate falls, the expansive effect of positive interest rate changes is not sufficient to offset the increase in the debt stock, so that the public debt-to-GDP ratio increases (Case 2).

When the propensity to consume is high, households achieve the desired wealth-to-income ratio. In the steady-state, there is no accumulation of savings, and consequently, the growth rate is zero (the growth rate of public debt service is zero). In this scenario (Case 3 – Figure 4.4), as the expansive effect of interest rates disappears, any positive change in the interest rate on government bonds implies a higher public debt-to-GDP ratio.

3. Stability of the public debt-to-GDP ratio

The contributions in mainstream macroeconomics commonly conclude that an interest rate lower than the growth rate of the economy jointly to the imposition of a primary balanced budget is the necessary condition to guarantee the reduction or the stability of the public debt-to-GDP ratio (Diamond, 1965; Giavazzi and Pagano, 1990; Barrett, 2018; Mehrotra and Sergeyes, 2018, Blanchard 2019).

This result comes from the arithmetic of the public debt-to-GDP ratio:

$$d_t = \frac{1 + r_t}{1 + g_t} d_{t-1} + x_t \quad (5.1)$$

Where x_t is the primary deficit-to-GDP ratio, d_t is the public debt-to-GDP ratio, r_t is the interest rate on public bonds and g_t is the nominal GDP growth rate. Following this arithmetic, by imposing d_t equal to d_{t-1}, the condition that stabilises the debt-to-GDP ratio can be derived:

$$x = \left(\frac{g - r}{1 + g}\right) d \approx (g - r) d \quad (5.2)$$

If $x_t = 0$, if the growth rate of public debt is equal to the interest rate[35], the debt-to-GDP ratio has an explosive trend if $r > g$. To this respect, the mainstream policy recommendation to stabilize or reduce d^* is that imposing a primary balance budget constraint, the growth rate of the economy has to be equal or higher than the interest rate.

In general, this interpretation of the arithmetic of the public debt-to-GDP ratio derives from the (implicit) hypothesis of an absence of interdependence between interest expenditure, primary deficit and output and, in general terms, that public expenditure does not affect the output. This would be justifiable only within a theoretical framework (the one behind the neoclassical theory) in which the level and the distribution of output are pre-determined with respect to the production process, the final output is exchanged in real terms across the participants and the homogenous good can be either a consumer good or a capital good. Under these conditions, it could exist market automatisms that, via changes in prices and wages, can bring the economic system towards the full use of resources. In this case expansions of public expenditure cannot modify the output and, least of all, the private wealth. Supply factors determine the amount of output and private wealth.

In contrast to these positions, in this section, we will show how the interpretation of the arithmetic of public debt-to-GAP ratio changes when we move from a neoclassical word to a monetary economy of production where monetary savings can exit only ex-post with respect to public spending and investment decisions, symmetrically to the accumulation of public and private debt. In this context, as the principle of effective demand determines output, the stability conditions cannot be determined without considering the co-movement between interest rate and primary deficit, through the variation of output and fiscal revenues.

In this sense, we will show that, if public budget constraints are absent, the condition $x = \left(\frac{g-r}{1+g}\right) d$ is fulfilled by an endogenous adjustment of the primary deficit and the output growth rate.

This implies that the condition on $r - g$ jointly with a primary balance budget does not have *per se* any relevance for the determination of the public debt-to-GDP ratio in the long term (see Aspromourgos et al. 2009 for a review of the literature on the subject). Conversely, the imposition of the public budget constraint that is not consistent with the natural public debt-to-GDP ratio of the economy[36] produces an explosive trajectory of the public debt-to-GDP ratio.

Three different levels of analysis will be used to demonstrate such results:

1. Analytical solutions of a system reproducing only the multiplier mechanism;
2. Analytical solutions of a system reproducing the interaction between the multiplier and the accelerator in a two-sector model (Sector C and K);
3. Simulations of the IO-SFC model.

[35] However, even following Blanchard's reasoning, the growth rate of public debt could be equal to r_t only if the entire public debt was in private hands. Otherwise, the interest rate actually paid by the Government would be: : $r\alpha(1-\tau)$. $1-\alpha$ is the portion of debt held by the CB on which no interest rate is paid, under to the condition that the counterpart of such debt is held by households in the form of cash. Conversely, if such counterpart is held in the form of deposits, the interest rate actually paid by the government on this share of the debt is equal to the interest rate on reserves.

[36] Determined by the tax rate, saving rate, interest rate and capital/output ratio.

4.1 Public debt stability in the simple income/expenditure model

Let's analyse the stability condition within the dynamic version of the Income-Expenditure model including endogenous public debt service and propensity to consume out-of-wealth higher than zero:

$$\begin{cases} B_t = B_{t-1} + \bar{G} - \theta Y_t + S_{t-1} r^{37} & (5.3) \\ Y_t = Y_{t-1}\, c_1\, (1-\theta) + S_{t-1}(c_2 + r) + \bar{G} & (5.4) \\ S_t = S_{t-1}(1-c_2) + Y_{t-1}(1-\theta)(1-c_1) & (5.5) \end{cases}$$

The redundant equation is:

$$S_t = B_t - Y_t(1-\theta) \ (5.6)$$

B_t is the public debt, S_t is the stock of household savings, r is the interest rate on public debt, θ is the average tax rate, $\bar{G}$ is primary public expenditure and it is constant. For the sake of simplicity, we assume that households hold their savings in the form of public bonds[38] (we would have the same situation assuming that the interest rate on deposits is equal to the interest rate on public bonds)

Re writing 5.4:

$$S_{t-1} = \frac{Y_t - Y_{t-1}\, c_1\, (1-\theta) - \bar{G}}{c_2 + r} \quad (5.7)$$

Substituting 5.7 in the 5.5, after some manipulations, we get the second-order difference equation which describes the intertemporal dynamic of income:

$$Y_{t+1} - Y_t\, [c_1\, (1-\theta) + 1 - c_2] - Y_{t-1}(1-\theta)[c_2 + r(1-c_1) - c_1] = \bar{G} c_2$$

The solution is:

$$Y_t = \frac{2^{-1-t}\bar{G}[\mathrm{c}_2\mathrm{a}(-2^{1+t} + (b-a)^t + (b+a)^t) - d((b-a)^t - (b+a)^t)]}{(-\theta c_2 + (-1+c_1)(-1+\theta)r)a} \quad (5.8)$$

Where:

$$a = \sqrt{(-1 + c_1(\theta-1) + c_2)^2 + 4(\theta-1)(c_1 - c_2 + (-1+c_1)r)}$$

$$b = 1 + c_1(1-\theta) - c_2$$

$$d = c_2(1 + c_1(-1+\theta) - 2\theta + c_2) + 2(-1+c_1)(-1+\theta)r$$

Given that $g_G = 0$, depending on the value of c_2 and r, two different regimes emerge: *interest-led* or stationary state with zero-growth. Adopting the limit of 5.8 for t that goes to infinity we can study the long-run dynamic:

---

[37]Even though households holds all their wealth in the form of public bonds, during the transition phase, the amount of savings held by households is necessarily lower than the public debt (the remaining part is held by CB). The Government pays interests only on the share of public debt held by households while the share held by CB does not accrues interest since CB distribute profits to the Government.

[38]Or that, equivalently, the interest rate on deposits and reserves is equal to that on public bonds.

$$\lim_{t\to\infty} \frac{2^{-1-t}\bar{G}[\mathrm{c}_2\,\mathrm{a}\,(-2^{1+t}+(b-a)^t+(b+a)^t)-d((b-a)^t-(b+a)^t)]}{(-\theta c_2+(-1+c_1)(-1+\theta)r)a}$$

$$\begin{cases} 0<\log(b+a)>\log(2) \\ \log(b+a)<\log(b-a) \\ 0<\log(b-a)>\log(2) \end{cases} \rightarrow \begin{cases} b+a>0 & (1) \\ b+a>1 & (2) \\ b+a<2 & (3) \\ b+a<C & (4) \\ b-a<2 & (5) \\ b-a>0 & (6) \\ b-a>1 & (7) \end{cases}$$

Because $0 \le c_1, c_2, r < 1$ condition 1, 2, 4, 5, 6, 7 are always verified, thus condition 4 determines the two regimes:

$$c_1(1-\theta)-c_2+\sqrt{(-1+c_1(\theta-1)+c_2)^2+4(\theta-1)(c_1-c_2+(-1+c_1)r)}<1 \quad (5.9)$$

If the previous condition is satisfied, the long-run growth rate is zero and income, savings and public debt converges to a stationary value:

$$g_Y \approx 0$$

$$Y^* = \lim_{t\to\infty} Y_t = \frac{\mathrm{c}_2\bar{G}}{\theta(c_2+r-c_1r)-r(1-c_1)} \quad (5.10)$$

$$S^* = \frac{(1-c_1)(1-\theta)\bar{G}}{c_2\theta-r(1-\theta)(1-c_1)} \quad (5.11)$$

The stationary values of the private wealth-to-GDP and public debt-to-GDP ratios are:

$$\frac{S^*}{Y^*} = \frac{(1-c_1)(1-\theta)}{c_2}$$ [39]

$$\frac{B^*}{Y^*} = (1-\theta)\frac{1+c_2-c_1}{c_2}$$ [40]

Conversely, if $c_1(1-\theta)-c_2+\sqrt{(-1+c_1(\theta-1)+c_2)^2+4(\theta-1)(c_1-c_2+(-1+c_1)r)} \ge 1$, the interest-led regime emerges. That is, for high values of the propensity to save and interest rate, the steady-state growth rate is positive and converges to the following value:

$$g_Y^* = \lim_{t\to\infty}\frac{Y_t}{Y_{t-1}}-1 = \frac{2[\mathrm{c}_2\mathrm{a}(-2^{1+t}+(b-a)^t+(b+a)^t)-d((b-a)^t-(b+a)^t)]}{\mathrm{c}_2\mathrm{a}(-2^t+(b-a)^{t-1}+(b+a)^{t-1})-d((b-a)^{t-1}-(b+a)^{t-1})}-1$$
$$=\frac{a+b}{2}-1 \quad (5.12)$$

[39] In stationary state equation 5.4 became $S^* = \frac{Y^*(1-\theta)(1-c_1)}{c_2}$

[40] Substituting $S^* = \frac{Y^*(1-\theta)(1-c_1)}{c_2}$ in 5.6.

We can verify that, also in this case, in absence of public budgetary constrain the growth rate of the stock of savings and public debt converge to the growth rate of GDP. Substituting $Y_{t-1} = \frac{S_t - S_{t-1}(1-c_2)}{(1-\theta)(1-c_1)}$ in equation 2.2b we get the second-order difference equation which describes the intertemporal dynamic of the stock of debt:

$$S_{t+1} - S_t[(1-c_2) + c_1\,(1-\theta)] + S_{t-1}(1-\theta)[c_1 - c_2 - r(1-c_1)] = \bar{G}(1-\theta)(1-c_1)$$

The solution is:

$$S_t = \frac{2^{-1-t}((b-\mathrm{a})^t((2(1+c-d)+b(-2+d))+\mathrm{a}d) - (b+\mathrm{a})^t((2(1+c-d)+b(-2+d))-\mathrm{a}d) - 2^{1+t}\mathrm{a}d)\bar{G}}{\mathrm{a}(-1+b-c)}$$

Where:

$$a = \sqrt{(1+c_1(1-\theta)-c_2)^2 - 4(1-\theta)[c_1-c_2-r(1-c_1)]}$$

$$b = 1 + c_1(1-\theta) - c_2$$

$$c = (1-\theta)[c_1 - c_2 - r(1-c_1)]$$

$$d = (1-\theta)(1-c_1)$$

In the long run – when $b + a > 2$, the growth rate of saving is:

$$g_S^* =$$
$$= \lim_{t\to\infty} \frac{(b-\mathrm{a})^t[(2(1+c-d)+b(-2+d))+\mathrm{a}d] - (b+\mathrm{a})^t[(2(1+c-d)+b(-2+d))-\mathrm{a}d] - 2^{1+t}\mathrm{a}d}{2\{(b-\mathrm{a})^{t-1}[(2(1+c-d)+b(-2+d))+\mathrm{a}d] - (b+\mathrm{a})^{t-1}[(2(1+c-d)+b(-2+d))-\mathrm{a}d] - 2^t\mathrm{a}d\}}$$
$$-1 = \frac{a+b}{2} - 1 \quad (5.13)$$

The growth rate of savings converges to the growth rate of income and the wealth-to-GDP ratio and public debt-to-GDP ratio stabilize in correspondence of the following values:

$$g_S^* = g_Y^* = g$$

$$\frac{S^*}{Y^*} = \frac{(1-c_1)(1-\theta)}{c_2 + g^*}$$

$$\frac{B^*}{Y^*} = (1-\theta)\frac{1 + c_2 - c_1}{c_2 + g^*}$$

Given the value of the interest rate, the interest-led regime is determined in correspondence with low values of the propensity to consume and the propensity to consume out-of-wealth.

The following table resumes all the steady-state solutions of the baseline model regarding different combinations of $r\ and\ c_2$.

	$c_2 = 0, r > 0$[41]	$c_2 > 0, r = 0$	$c_2 > 0, r > 0$		$c_2 = 0, r = 0$
g	$\frac{1}{2}(a+b-2)$[42]	0	0	$\frac{1}{2}(a+b-2)$	0
Y^*	*Growth*	$\frac{\bar{G}}{\theta}$	$\frac{c_2\bar{G}}{\theta(c_2+r-c_1 r)-r(1-c_1)}$	*Growth*	$\frac{\bar{G}}{1-c_1(1-\theta)}$
$\frac{S^*}{Y^*}$	$\frac{(1-c_1)(1-\theta)}{g}$	$\frac{(1-c_1)(1-\theta)}{c_2}$	$\frac{(1-c_1)(1-\theta)}{c_2}$	$\frac{(1-c_1)(1-\theta)}{c_2+g}$	*Explosive*
$\frac{B^*}{Y^*}$	$\frac{(1-\theta)(1-c_1+g)}{g}$	$\frac{(1-\theta)(1+c_2-c_1)}{c_2}$	$\frac{(1-\theta)(1+c_2-c_1)}{c_2}$	$\frac{(1-\theta)(1+c_2-c_1+g)}{c_2+g}$	*Explosive*

When public budget constraints are absent, the sufficient condition for the stabilization of the public debt-to-GDP ratio is that one among the propensity to consume out-of-wealth, the interest rate and the growth rate of primary public expenditure is higher than zero. Thus, the only case in which the public debt-to-GDP ratio does not stabilize is when both the interest rate and the propensity to consume out-of-wealth are zero. In this regard, it should be noted that the expression 5.8 is a generalized form of the traditional formulation of equilibrium income the simple income/expenditure model. When $r = 0$, $c_2 = 0$ and $g_G = 0$, the system converge to a stationary income ($g_Y = 0$). Indeed, form 5.8, after some manipulation we get [43]:

$$Y_t = \frac{\bar{G}\,\{[2c_1(1-\theta)]^t - 2^t\}}{2^t[c_1(1-\theta)-1]} = \frac{\bar{G}\,\{2^t[c_1^t(1-\theta)^t - 1]\}}{2^t[c_1(1-\theta)-1]} \quad (5.20)$$

When t goes to infinite:

$$g_Y = \lim_{t\to\infty} \frac{c_1^t(1-\theta)^t - 1}{c_1^{t-1}(1-\theta)^{t-1} - 1} - 1 = 0$$

$$\lim_{t\to\infty} \frac{\bar{G}\,[c^t(1-\theta)^t - 1]}{c(1-\theta)-1} \approx \frac{\bar{G}}{1-c_1(1-\theta)} \qquad Hp\text{: } 0 < c_1, \theta < 1 \quad (5.21)$$

In this case ($c_2 = 0, r = 0, g_G = 0$), the debt-to-GDP ratio explodes as income reaches a stationary level and the accumulation of savings continues:

$$\frac{B}{Y} \approx \infty$$

[41] See Appendix 2 for the analytical calculations.

[42] $a = \sqrt{(1+c_1(1-\theta)-c_2)^2 - 4(1-\theta)[c_1-c_2-r(1-c_1)]}$; $b = 1+c_1(1-\theta)-c_2$

[43] The procedure:

$$Y_t = \frac{\bar{G}2^{-t}\left\{\left[c(1-\theta)+1+\sqrt{[c(1-\theta)+1]^2-4(1-\theta)c}\right]^t - \left[c(1-\theta)+1-\sqrt{[c(1-\theta)+1]^2-4(1-\theta)c}\right]^t\right\}}{\sqrt{[c(1-\theta)+1]^2-4(1-\theta)c}}$$

$$Y_t = \frac{\bar{G}2^{-t}\left\{\left[c(1-\theta)+1+\sqrt{[c(1-\theta)-1]^2}\right]^t - \left[c(1-\theta)+1-\sqrt{[c(1-\theta)-1]^2}\right]^t\right\}}{\sqrt{[c(1-\theta)-1]^2}}$$

If the growth rate of autonomous component (secondary and primary public expenditure is equal to zero) and the desired wealth-to-income ratio is equal to infinity ($c_2 = 0$), in the stationary state the public debt necessarily grows indefinitely.

4.2 Public debt stability in the two-sector Supermultiplier model

We are going to derive analytically the steady-state values of the public debt-to-GDP ratio using a simplified form of the I-O model considering the capital sector as an aggregate sector (a two-sector Supermultiplier model). The system with a rigid accelerator is as follows:

$$\begin{cases} Y_t = C_t + G_t + I_t + B_{t-1}r - K_t\delta \\ C_t = Y_{t-1}c_1(1-\theta) + S_{t-1}c_2 \\ I_t = Y_t^e v - K_t(1-\delta) \\ Y_t^e = Y_{t-1} \\ B_t = B_{t-1}(1+r) + G_t - \theta Y_t \\ G_t = \bar{G} \end{cases}$$

This system can be expressed as a system of second-order difference equations:

$$\begin{cases} Y_t = Y_{t-1}(c(1-\theta)+v) + S_{t-1}(c_2+r) + \bar{G} - Y_{t-2}v & (5.2.1) \\ B_t = B_{t-1} + \bar{G} - \theta Y_t + S_{t-1}r & (5.2.2) \\ S_t = S_{t-1}(1-c_2) + Y_{t-1}(1-\theta)(1-c_1) & (5.2.3) \end{cases}$$

The redundant equation is:

$$S_t = B_t + Y_{t-2}v(1-\delta) - (Y_t + K_t)(1-\theta)$$

Given the value of the interest rate, for high values of the propensity to consume[44], the long-run income converge to a stationary value and the levels of the private wealth-to-income ratio and public debt-to-GDP ratios are as follows[45]:

$$\frac{S^*}{Y^*} = \frac{(1-c_1)(1-\theta)}{c_2} \quad (5.2.4)$$

$$\frac{B^*}{Y^*} = \frac{(1-\theta)(1-c1) - c_2[v(1-\delta(2-\theta)) - 1 + \theta]}{c_2} \quad (5.2.5)$$

As we can see from equation 5.2.5, the public debt-to-GDP ratio depends also on v that is the capital/output ratio. Because the capital/output can be measured only in value terms, d_t is a function of the system of relative prices, hence it will depend on the production coefficients, the rate of industrial profit and the interest rate. In this sense, the system of relative prices determines the elasticity of private indebtedness with respect to nominal public expenditure, which, in turn, depends on the capital intensity of the production process.

[44] In this scenario, there is no growth effect produced by the interest rate.
[45] See Appendix 2 for all computations.

4.3 Public debt stability in the IO-SFC model

Let's move on to the discussion on the more general conditions on the debt-to-GDP ratio by presenting the results of the IO-SFC model as the differential between the growth rate of the economy and the interest rate changes[46]. In particular, we study both growth regimes described above: i) the output growth rate is driven by the growth rate of primary public expenditure (*exogenous demand-led regime:* $g_Y = g_{G_{primary}}$); ii) the output growth rate is co-determined by the growth rate of primary spending and debt services: ($g_Y = f(g_{G_{primary}}, g_{G_{interests}}) = f\left(g_{G_{primary}}, h(r, c_1, c_2)\right)$)[47]. Let us consider the four scenarios resulting from the combination of these two regimes:

	$\boldsymbol{g = f(r_b, c_1, c_2) = 0}$	$\boldsymbol{g = f(r_b, c_1, c_2) > 0}$
$\boldsymbol{g_{G_{primary}} = 0}$	**A**	**B**
$\boldsymbol{g_{G_{primary}} > 0}$	**C**	**D**

In scenarios A and C, the combination of the aggregate saving rate and the interest rate is such that, if the growth rate of the semi-autonomous component is zero, there would be no output growth

In scenarios B and D, the combination of these values is such that an interest-led growth regime emerges: the growth rate of public debt service is positive. In scenario D, the resulting growth rate is co-determined by the endogenously determined growth rate and the growth rate of direct public expenditure.

In the following table, the values of public debt-to-GDP, deficit-to-GDP ratio and equilibrium growth rate are reported for each scenario:

[46] In this analysis changes in the interest rate on public bonds do not modify the interest rate on bank loans, so that the relative prices remains constant (the interest rate on loans enters into the price equations). Conversely, since a co-movement between the two rates is assumed, an increase in the interest rate reinforces the interest-led growth regime by favouring an higher accumulation rate of savings.

[47] In this case, with $g_G = 0$,, the growth rate of the economy is always lower than the interest rate on public bonds.

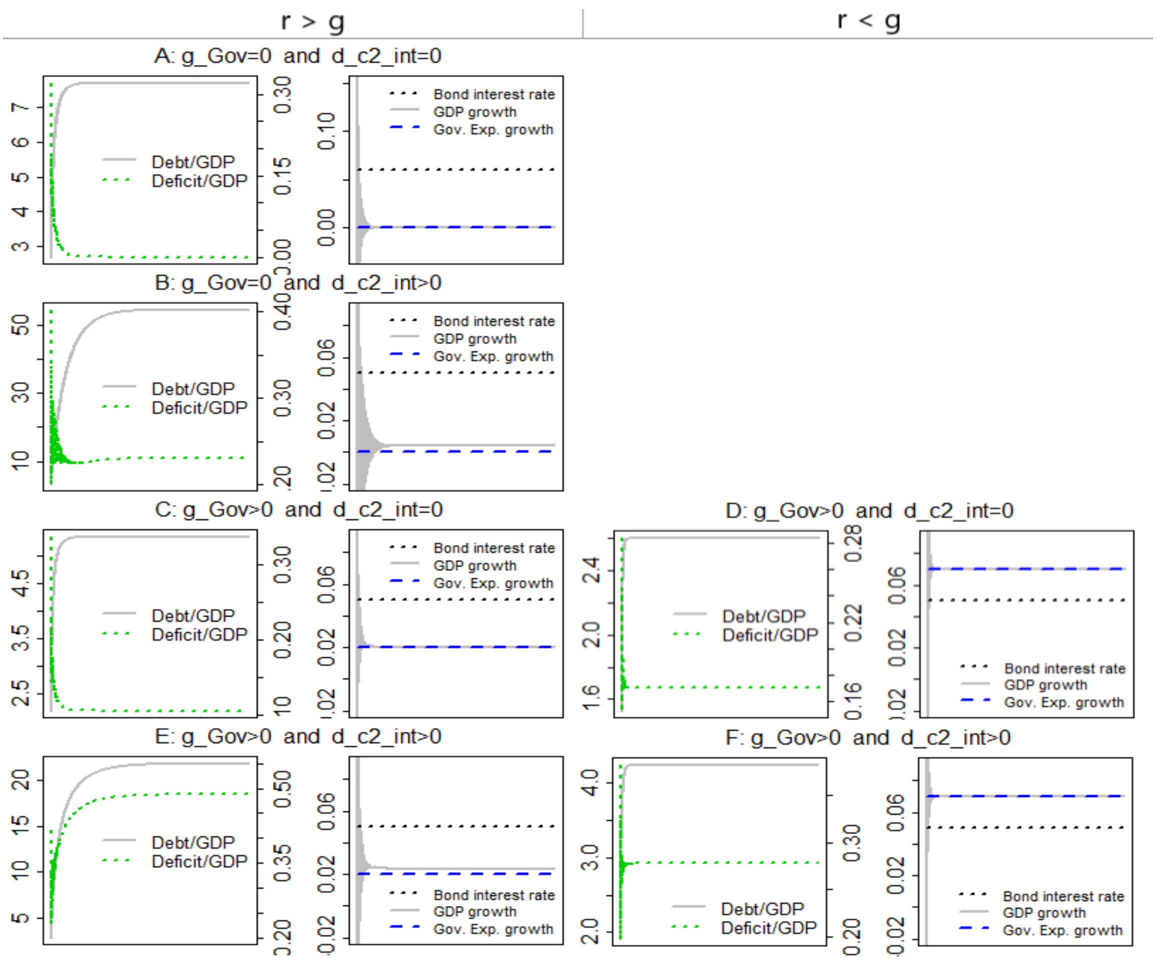


Figure 5.3.1: Equilibrium values of the Debt-to-GDP ratio according to different growth regimes and under two different scenarios: $r > g$ and $r < g$ [48]

As can be seen from these results, in all scenarios, although as the difference between r and g increases the equilibrium value of the debt-to-GDP ratio, such difference does not affect the stability of these indicators: all ratios always reach a steady-state value. This result is due to the expansionary effect on GDP and tax revenues produced by debt service expenditure in each period. Therefore, although the total deficit can assume positive values, the public expenditure for interests generate, via a multiplier and accelerator effect, a variation in income and tax revenues such that, during the course of the periods, even if $r > g$, the debt can never grow at a rate equal to $r\alpha(1-\tau)$[49].

[48]In the case A, the growth rate of direct public expenditure and the growth rate of the economy are zero: in the steady state the variation in the stock of savings is zero (households reach the desired wealth-to-income ratio) and the level of income is constant. In the case B, the growth rate of direct public expenditure is zero and the combination ℵ of the parameters is such as to generate an interest-led regime. In case C, a positive rate of the direct public expenditure component is included in the case A. Case E presents an interest-led regime together with a positive growth rate of the primary public expenditure. Finally, cases D and F have a growth rate of direct public expenditure (and therefore of total output) higher than the interest rate in the respective baseline cases C and E.

[49] For simplicity we use this expression, which is valid in the case of interest rates on deposits and on reserves equal to zero. τ is the average tax rate. α means the portion of bonds held by households, that is the share of bonds on which the Government actually pays interest rates. Indeed, the amount of interests paid to the CB by the Government, goes back to the latest through the distribution of CB profits. In any case, unless the households hold the amount of savings

In this framework, since the growth rate of public debt service has a positive effect on the income and wealth of the private sector, increasing consumption, investments and tax revenues, the growth rate of public expenditure (endogenous or exogenous) determines a primary budget surplus that grows, in steady-state, at a rate equal to that of the output. As a result, the growth rate of public debt is as follows:

$$Deb_t = Deb_{t-1}(1 + \alpha r) - Surplus_{primario} \quad (5.3)$$

$$g_D = \omega r_b - \frac{surplus\ primario_{t-1}}{Debt_{t-1}}(1 + g_Y) \quad (5.4)$$

α indicates the share of the debt on which the government pays interests (if the interest rates on deposits and reserves are zero, the interest rates paid on bonds held by the CB are zero since the profits of the CB are redistributed to the government). The graph below shows the case of the interest-led regime[50] with a constant level of primary government expenditure (case B):

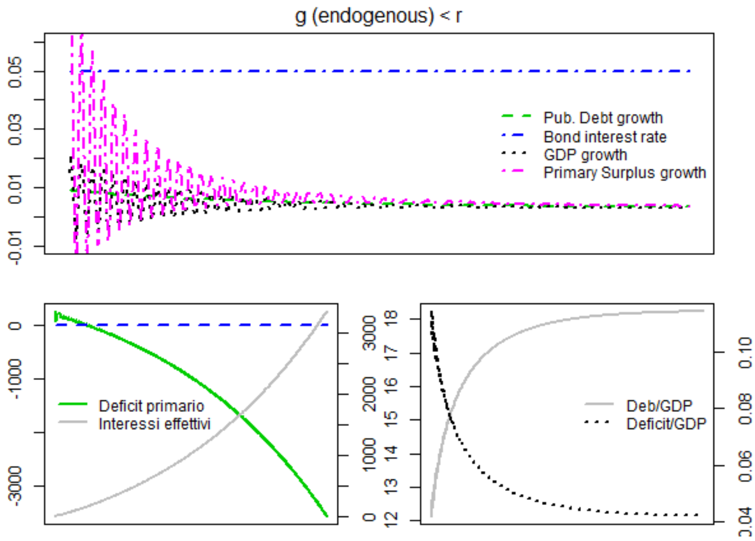


Figure 5.3.2: Growth rates of public finance indicators (case B)

Changes in the interest rate on public bonds produce a change in the growth rate of the economy and generate, via multiplier and accelerator effects, a growth rate of the primary surplus equal to the growth rate of the output. Due to the multiplier effect on income and tax revenues produced, the total deficit is always lower than the amount of interest accrued on government securities and, in particular, the public debt grows at the same rate of income: the growth rate of the stock of debt converges with that of the output.

corresponding to the public bonds held by CB in terms of cash, CB profits are not exactly equals to the amount of interests paid by the Government. Indirectly, the government pays an interest equal to the interest on deposits on the share of bonds held by the Central Bank. Indeed the amount of deposit created by the bonds held by CB are hold by commercial bank as reserves at the CB. Because of the amount of reserves - and the corresponding deposits generated by the same money injected by the CB at the time of purchase of the securities - on which the central bank pays an interest rate to the commercial bank.

[50] In the interest-led regime with zero growth rate of direct public expenditure, the endogenously determined growth rate is necessarily lower than the interest rate on public bonds.

In particular, the former tends to decrease after reaching its maximum value just after the first period (maximum deficit) and the latter progressively increase until the matching with the growth rate of debt. As periods go ahead, the cumulated effect of the multiplier and accelerator triggered by the growing public debt service is realized. At the same time, two effects increase the primary surplus of each period. On the one hand, the multiplier effects of the expenditures made in the previous periods contribute to increasing fiscal revenues not directly dependent on current expenditure (both through induced consumption on current income and consumption out-of-wealth, as well as on the multiplier effects of the investments realized in each period). On the other hand, the rate of growth of savings decreases as the achieved wealth-to-income ratio increases. These results confirm those presented in Godley and Lavoie (2007) and Ryoo and Skoot (2013). In particular, Godley and Lavoie (2007) argue that the stability of the debt-to-GDP ratio can be maintained even if the interest rate is higher than the GDP growth rate, provided that there are no budgetary constraints in public spending.

Unlike the model proposed by Godley and Lavoie, in which, although public debt service has an expansive effect on consumption, the growth rate is taken as exogenous and independent of the interest rate, in this model the two variables interact. In particular, in this model the expansive effect on consumption interacts with the accelerator mechanism, modifying the levels of the private debt-to-GDP ratio.

Finally, confirming the results of the previous models, the necessary condition to ensure the stability of d_t is the absence of public budget constraints. Latter become sufficient if one of the following conditions is verified:

$$c_2 > 0 \; or \; g_{G_{primary}} \; or \; r_b > 0$$

That is, it is sufficient that one of the following conditions is respected: 1) the desired wealth-to-income ratio different from infinity; 2) the growth rate of primary public expenditure is greater than zero; 3) the interest rate on public bonds is positive. As a consequence, it follows that the only scenario in which the debt-to-GDP ratio does not stabilise is when $c_2 = 0$ e $r_b = 0$ and $g_{G_{primary}} = 0$. This case is reported in Figure 5.3.4.

- $If: g_{G_{primary}}, c_2, r_b = 0 \; \rightarrow \alpha_3^T = \infty, \; : \Delta S > 0 \; \rightarrow Deficit > 0 \; \rightarrow g_Y = 0 \; e \; g_{\frac{B}{Y}} > 0$

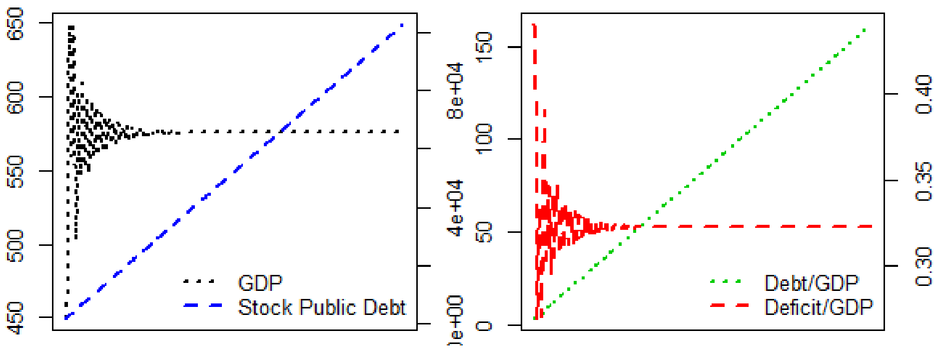


Figure 5.3.4: Evolution of the Debt-to-GDP ratio with $c_2 = 0 \wedge r_b = 0 \wedge \; g_{G_{direct}} = 0$

The only case in which the debt-to-GDP ratio presents an explosive trajectory is when the interest rate and the growth rate of the autonomous component are both equal to zero, and the desired wealth-to-income ratio is equal to infinity ($c_2 = 0$). This is the only case in which, in steady-state, households continue to accumulate savings (and therefore to generate a positive primary deficit for the public sector) without these having an expansive effect on income. It is worth noticing that in the only scenario in which the debt ratio does not stabilise, the difference between r and g is zero.

When the wealth-to-income ratio is finite ($c_2 > 0$), the debt-to-GDP ratio stabilises in all cases, even when the interest rate and the growth rate of direct public expenditure are zero. In this specific scenario, in the steady-state, the accumulation of savings and public deficit are zero:

$$\forall\ (c_1; c_2; r_b)\ |\ \alpha_3^T = \alpha_3^r\ e\ c_2 > 0 : \Delta S = 0\ \rightarrow Deficit = 0\ \rightarrow g_Y = 0\ e\ \frac{B^*}{Y^*} = k$$

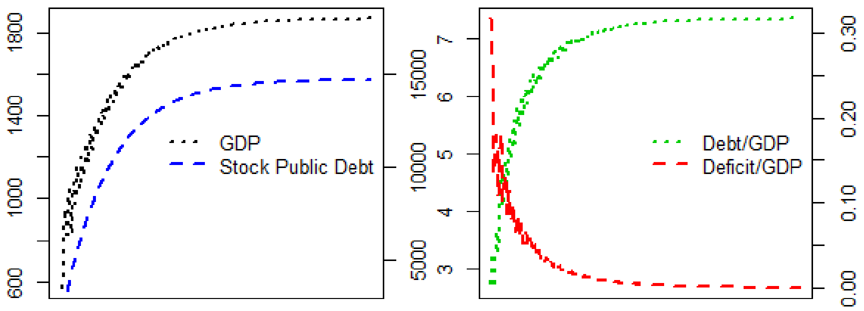


Figure 5.3.5: Evolution of the Debt-to-GDP ratio with $c_2 > 0\ \wedge g_{G_{diretta}} = 0\ \wedge g_{G_{interest}} = 0$

Finally, in all cases where the growth rate of the autonomous component (exogenously or endogenously determined) is positive, regardless of the value of propensity to consume, the growth rate of the primary surplus is such that the growth rate of the debt is equal to that of the output:

$$\forall\ (c_1; c_2; r_b)\ |\ \alpha_3^T \neq \alpha_3^r : \Delta S > 0\ \rightarrow Deficit > 0\ \rightarrow g_{G_{interests}} > 0 \rightarrow g_Y > 0\ e\ \frac{B^*}{Y^*} = k$$

$$o\ g_{G_{primary}} > 0$$

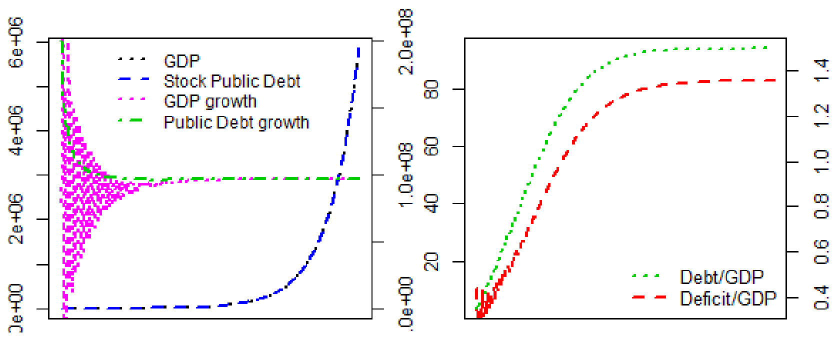


Figure 5.6: Evolution of the Debt-to-GDP ratio with $g_{G_{diretta}} > 0\ \vee\ g_{G_{interest}} > 0$

Trivially, in scenarios where the growth rate is partly (E) or fully (B) driven by the interest rate on bonds, the debt-to-GDP ratio is higher than in the case where growth is driven by the positive growth rate of direct government expenditure (case C). Indeed, the expansive effect of interest rates on output is realized only in correspondence with high savings rates[51].

In all scenarios, the positive growth rate of the direct public expenditure component, although it implies direct effects on public debt service, always reduces the difference between r and g e, therefore, the steady-state level of the Debt-to-GDP ratio (See the comparison between A and C or B and E). In particular, as can be seen by comparing the results between case C and D or E and F, a growth rate of direct public expenditure higher than the interest rate implies a lower Debt-to-GDP ratio[52].

Finally, since the growth rate endogenously determined from the expenditure for debt service is necessarily lower than the interest rate, the only way to decrease this differential, and therefore the debt-to-GDP ratio is represented by the change in the growth rate of primary public expenditure[53].

[51] As seen above, the debt-to-GDP ratio is positively correlated with the aggregate savings rate.

[52] As it will be shown in Section 5, this effect is due to the accelerator mechanism that triggers a higher share of private indebtedness with respect to the public one.

[53] The government cannot directly control the debt service expenditure, this depends on households' savings decisions (savings rate and portfolio decisions) and the interest rate set by the CB.

4. Public debt-to-GDP ratio, the accelerator and the system of relative prices

The level of public expenditure affects the level of public debt-to-GDP ratio only through the accelerator process. That is, an expansion of public expenditure can modify the steady-state value of d_t only if it indirectly modifies the stock of private debt. With this respect, higher is the change in private indebtedness caused by a one-point percentage variation in public expenditure, higher is the share of households savings covered by private debt and lower is the level of d^*. This mechanism is generally composed of both a level and composition effect. Namely, an expansion of public expenditure generates an increase in the total stock of capital, savings and the private debt-to-GDP ratio.

Conversely, as reported in Figure 6.1, if no accelerator effect is operational, the level of public expenditure does not affect the *steady-state* value of d^*. In the case the effect of public spending is limited to the multiplier, such expansion has only a transitory effect on d^*.

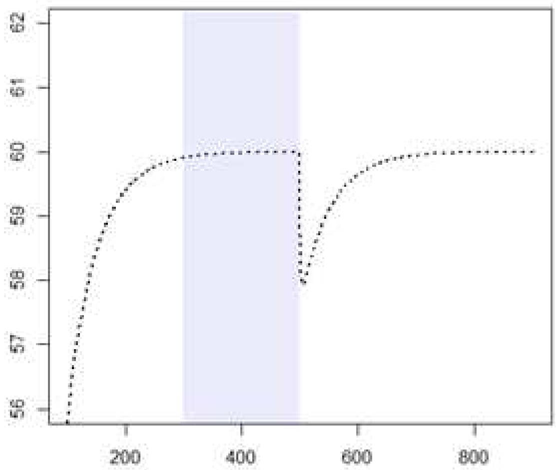


Figure 6.1: Change in the public debt-to-GDP ratio in response to a permanent increase in the level of public spending and the absence of the accelerator mechanism**.**

In this case, besides taxation, the public sector can affect the steady-state value of d_t only through variation in the growth rate of public spending, namely through a series of level effects that never run out. In general, given the set of parameters[54] the minimum achievable level is determined by the growth rate of labor force and productivity.

Conversely, when the accelerator effected is operational, the level of public spending can affect the steady-state value of the public debt-to-GDP ratio. As reported in Figure 6.2, a permanent increase in the level of public spending produces a positive change in the stationary level of income and a decrease in the public debt-to-GDP ratio together with an increase in the private debt-to-GDP ratio.

After the shock, the degree of capacity utilization increases and the accumulation rate becomes positive (transition phase) until reaching a new steady state in which the amount of capital installed is increased together with the level of replacement, firm indebtedness and households savings.

[54] Saving rate, tax rate, interest rate and coefficient of productions.

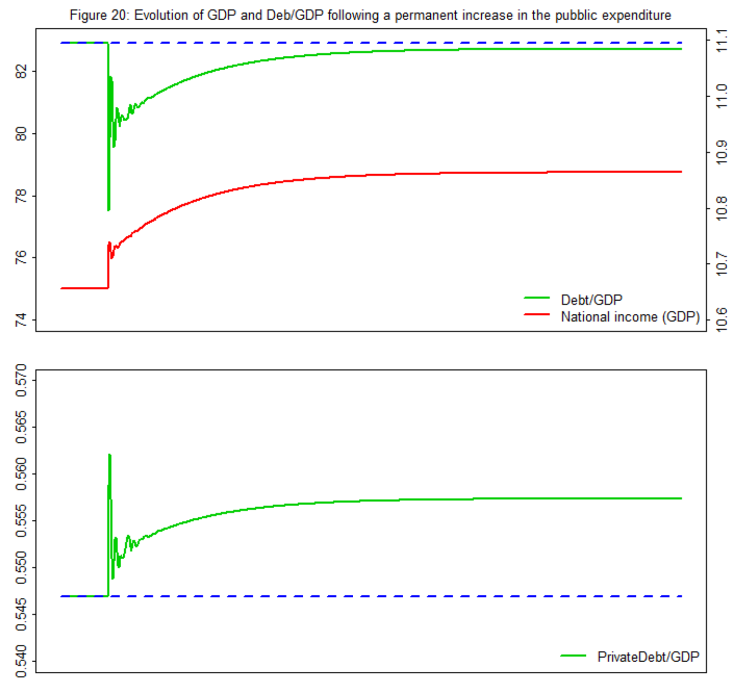


Figure 6.2: Evolution of GDP, public debt-to-GDP ratio and private debt-to-GDP ratio in response to a permanent increase in the level of public expenditure

Ultimately, in the presence of the accelerator effect, the public debt-to-GDP ratio decreases thanks to the fact that the household's desired wealth-to-income ratio is achieved through a higher increase in the share of the wealth covered by the production sector. Thus, both a level and composition effect are operational.

With the inclusion of investments, as shown in equation 5.2.5, d^* becomes a function of the capital intensity of the economic system. In this case, the capital intensity determines the elasticity of private indebtedness with respect to public expenditure[55] and the public expenditure can affect permanently the public debt-to-GDP ratio.

Because the capital/output can be measured only in value terms, d_t is a function of the system of relative prices, hence it will depend on the production coefficients, the rate of industrial profit and the interest rate.

Besides the dimension regarding the output determination, this is the second channel through which the interest rate affects the public debt-to-GDP ratio. As it is well known from the Capital controversy, a variation in the distribution of income has an unpredictable effect on the system of relative prices and the capital intensity of the aggregate production process.

If we consider a two-sector Supermultiplier model where the capital sector is intended as an integrated sector that uses only labor as an external factor of production, the relationship between the relative price of the capital good and d_t is linear. For instance, an increase in labour productivity in sector K produce a decrease in the relative price of K and, thus, an increase in d_t.

[55] The indebtedness of the production sector depends on the anticipations in fixed capital.

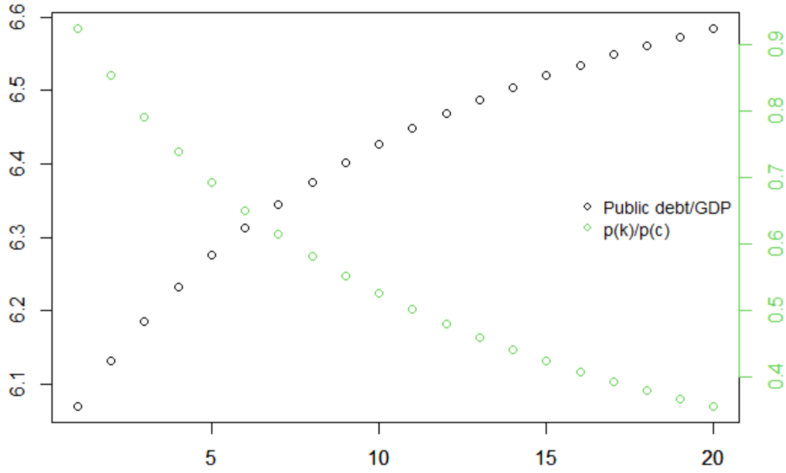


Figure 6.2: Public debt-to-GDP ratio and the relative price of the capital good in terms of consumer good

Anyway, when we consider the interdependency across the sectors producing the capital goods, such a relationship ceases to be so sharp and loses its linearity. Indeed, even though an increase in labor productivity in sector A would cause a decrease in the relative price of such commodity, it can cause an increase in the relative price of B in a way that the capital intensity ultimately decreases[56].

Let's analyze more in detail the effect of variations in the distribution through changes in the level of the interest rate. In this analysis, to isolate the effect of changes in relative prices on the public debt-to-GDP and keeping aside the quantitative channel, the interest rates on loans, deposits, advances and public bonds are equal to zero and the propensity to consume is equal across workers and capitalists.

Under these conditions, variation in the interest rate can modify the public debt-to-GDP ratio through two channels:

1. Change in the capital intensity of the production process;
2. Change in the real value of profits.

Let's first consider the second mechanism. The endogenous nature of money ensures that variations in the interest rates or markups in the price of input commodities (in our case, A and B) do not affect the real value of profits. Indeed, the stock of money injected through the indebtedness of sectors that uses such commodities as inputs changes consistently with such variations[57]. But, because the stock of money is independent of the nominal price of the consumption good, an increase in the interest rate or markups causes an increase in the nominal price of the consumption good that is not compensated by an increase in the nominal stock of money.

[56] In general, if the change in production coefficients or profit rate produces both an increase in the price of the basic commodities that directly enter in the production of final output and an increase in the relative price of the basic commodities (m_i) that enter directly in the production of the basic commodities that have the largest vertical multiplier (commodity m_j), the public debt-to-GDP ratio will decrease.

[57] An increase in the nominal price of commodity j that enters in the production of commodity i causes proportional increase in the loan demand in sector j.

Such independence produces a reduction in the real value of nominal profit and the real consumption of capitalists as the interest rate increase. Indeed, the nominal amount of profit does not increase proportionally to the increase in the nominal price of consumer goods[58].

We can isolate this mechanism by adopting homogeneous capital/labor across sectors. Indeed, in this case, variations in the distribution do not affect the system of relative prices[59]. As it is shown in Figure 6.5, because of a decrease in the real value of profits, the public debt-to-GDP ratio rises when the interest rate increases.

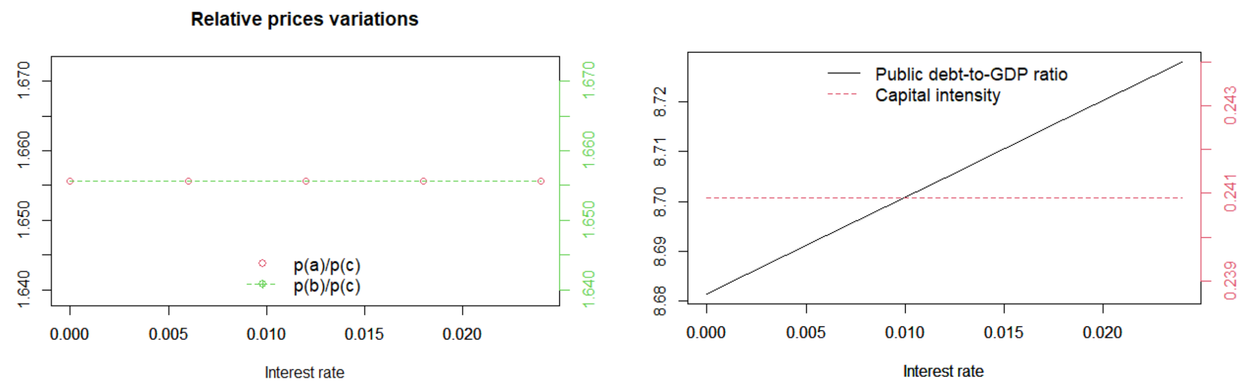


Figure 6.5

When we consider heterogeneous capital/labor across sectors, depending on the value of the capital/output in each sector, an increase in the interest rate can rise the public debt-to-GDP ratio through both a decrease in the capital intensity of the economy and the real value of profits (Figure 6.6)

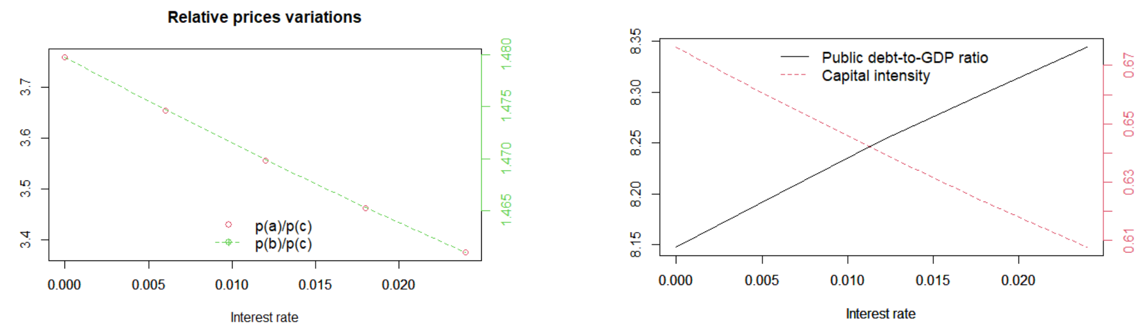
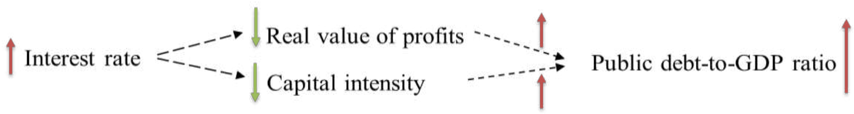


Figure 6.6



[58] Its increase is only due to the increase in input prices but does not respond to the increase in the price of the consumer good.

[59] This condition implies that the proportion of means of production with respect to the direct labor are equal across all sectors Indeed the ratio between the means of commodities and direct labor depends only on the capital/labor: $\frac{k_i}{l_I} = \frac{\frac{v_i^*}{a\omega^n}}{l_i} = \frac{\frac{v_i^*}{a\omega^n}}{\frac{v_i^*}{\alpha_i}} = \frac{\alpha_i}{a\omega^n}$. We are using $\frac{k_i}{l_I}$ because it is possible to say, with a certain degree of generality, that when the profit rate change the price of the commodity that have an higher $\frac{k_i}{l_I}$ with respect to the one of the numerarrie increase.

Conversely, for high values of the capital/output ratio in the input sectors with respect to the consumer sector, the phenomenon of capital reverse deepening can emerge. In this case, as shown in Figure 6.7[60], the increase in the interest rate causes a decrease in the public debt-to-GDP ratio.

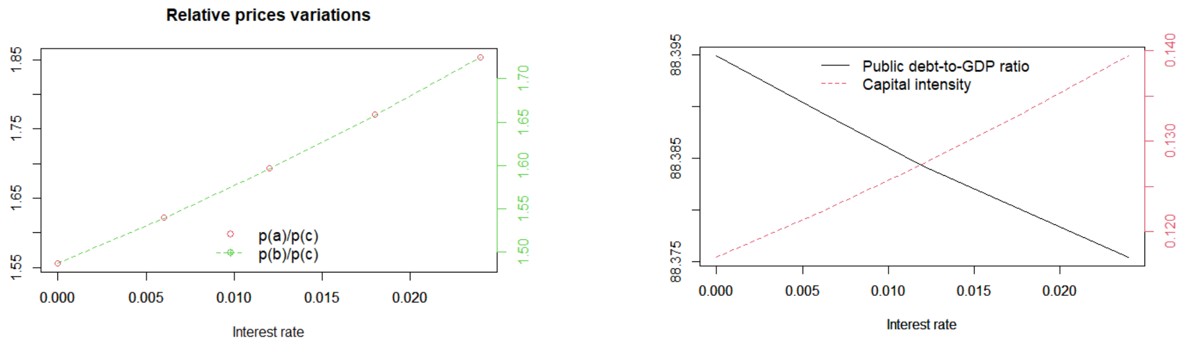
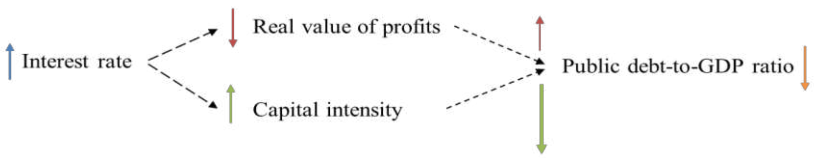


Figure 6.7

In this specific case, the increase in the capital intensity more than offset the effect generated by the decrease in the real value of profits.



Finally, let's analyze the effect of variation in the capital/output across sectors. These variations simultaneously modify the vertical multiplier and the relative price of commodities. We can derive the vertical multiplier using the Leontief inverse matrix:

$$A = \begin{bmatrix} 0 & \vartheta & 0 \\ \beta & 0 & 0 \\ \gamma & 0 & 0 \end{bmatrix}$$

$$(I - A)^{-1} = \begin{bmatrix} 1 & -\vartheta & 0 \\ -\beta & 1 & 0 \\ -\gamma & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{1}{1-\vartheta\beta} & \dfrac{\alpha}{1-\vartheta\beta} & 0 \\ \dfrac{\beta}{1-\vartheta\beta} & \dfrac{1}{1-\vartheta\beta} & 0 \\ \dfrac{\gamma}{1-\vartheta\beta} & \dfrac{\alpha\gamma}{1-\vartheta\beta} & 1 \end{bmatrix}$$

Thus, to produce one unit of consumer good it is required to produce a gross amount of A and B equal to:

$$m_a = \frac{1+\beta+\gamma}{1-\alpha\beta}$$

$$m_b = \frac{1+\alpha(1+\gamma)}{1-\alpha\beta}$$

[60] Parameters values: $v_a = 0.5, v_b = 0.4, v_c = 0.06$; $\alpha_a = 0.9$, $\alpha_b = 0.7, \alpha_c = 0.1$

If the ratio $\frac{m_a}{m_b} = \frac{1+\beta+\gamma}{1+\alpha(1+\gamma)}$ is higher than one, basic commodity A has the highest vertical multiplier, the opposite applies if the ratio is lower than one.

Now, we can study the effect of contemporaneous variation in the capital/output ratios within basic commodity sectors on the public debt-to-GDP ratio. Figure 6.8 reports the effect on public debt-to-GDP of changes in capital/output in sectors A and B.

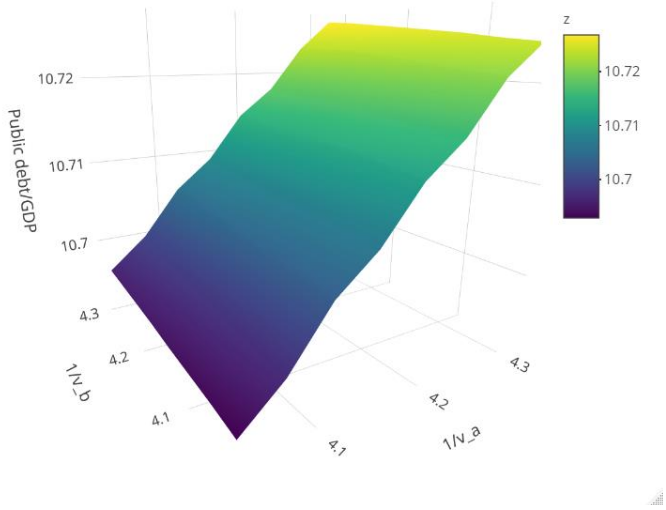


Figure 6.8: Public debt-to-GDP ratio (z-axis) as the reciprocal of capital/output in sector A and B change.

When v_b and v_a rise the public debt-to-GDP decreases. Indeed, d_t is negatively correlated with the capital intensity of the aggregate production process and this is positively correlated with the capital/output ratio of the various sectors (See Figure 6.4) Anyway, while such a relationship is strong for changes in v_a , it is quite debased for changes in v_b. Because A presents the highest vertical multiplier, the variation in v_a causes an higher change in capital intensity with respect to the same percentage variation in v_b[61]. So, the aggregate capital intensity is less responsive to variation in v_b[62].

[61] See Appendix 2.

[62] Of course, when the capital/output ratio changes also the vertical multuiplier changes, bu tin the range considered the multiplier of A rimanes higher than the multiplier of B.

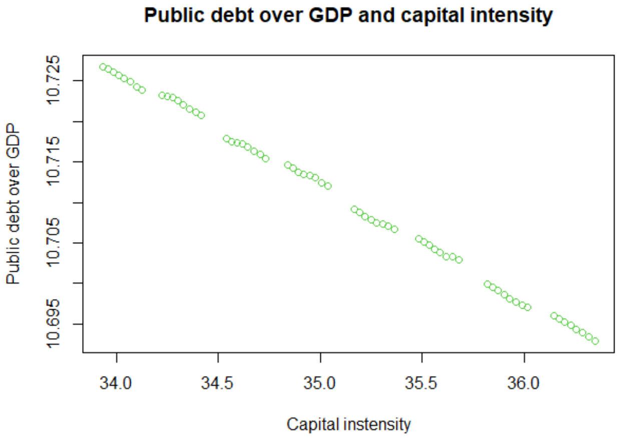
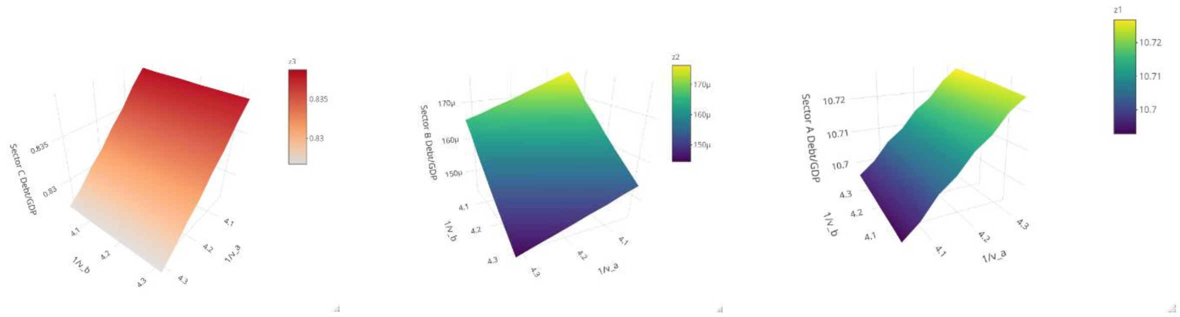


Figure 6.9: Public debt-to-GDP ratio (z-axis) and aggregate capital intensity

Such dynamic is reflected and explained by the variations of the productive sectors indebtedness over GDP: the Sector C and Sector B debt-to-GDP ratio rises as the capital/output in Sector A increase and, symmetrically, the public debt-to-GDP ratio decreases.



Conversely, if commodity B has the highest vertical multiplier, even though an increase in the capital/output in sector A causes an increase in the relative price, the steady-state value of the public debt-to-GDP ratio will increase. Indeed, such variations would produce a decrease in capital intensity.

5. Conclusions

In this paper, we have outlined that the "natural" level of the public debt-to-GDP ratio is determined by the saving rate, the tax rate, the interest rate, the industrial profit rate and the coefficients of production. The first two act through the determination of output and public deficit, the last two through the determination of the capital intensity of the aggregate production process. The level of the interest rate acts through both channels[63].

Given the economic system defined by such parameters, there exists a steady-state value of the public debt-to-GDP ratio which is ingrained in the system and that can be reached only in the absence of public budget constraints. In this sense, each attempt of the Government to modify public expenditure in order to respect a given public deficit-to-GDP ratio which is not coherent with the natural public debt-to-GDP ratio produce a growing or explosive pathway.

It has been pointed out that the government cannot modify the public debt-to-GDP ratio through permanent variation in the primary public expenditure. Indeed such ratio is determined always ex-post with respect to Government spending decisions and depends on the saving rate of household and the interest rate fixed by the CB. A permanent increase in public expenditure can modify the steady-state value of the public debt-to-GDP ratio only if the accelerator mechanism is at work, that is through an induced increase in the share of private debt over GDP. Indeed, the public debt-to-GDP ratio depends on the capital intensity of the aggregate production process which, in turn, determines the elasticity of private indebtedness with respect to one-percentage change in public expenditure. To this extend, Input-Output tables represent an important tool for fiscal and monetary policies which consider the level of the public debt-to-GDP ratio among their goals.

Finally, we have demonstrated that the necessary condition for the stabilization of the public debt-to-GDP ratio is the absence of public budgetary constrain. This condition becomes sufficient when one of the following is respected: the growth rate of (primary) public expenditure is greater than zero, the interest rate is higher than zero or the propensity to consume out of wealth is non-zero.

[63] It should be specified that also the profit rate, modifying the aggregate propensity to consume, affect the output dimension. But, differently from the interest rate, it has no impact on the growth rate of stocks and output.

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Appendix 1

The sequential version of the traditional Income/Expenditure model including public debt service is:

$$\begin{cases} Y_t = \bar{G}_t + C_t + rB_{t-1} \\ C_t = c(1-\tau)Y_{t-1} \\ B_t = \bar{G}_t - Y_t\theta + B_{t-1}(1+r) \\ S_t = Y_{t-1}(1-\tau)(1-c) \end{cases}$$

Which can be re-writed as a system of two first order difference equations with two variables:

$$\begin{cases} Y_t = \bar{G}_t + c(1-\tau)Y_{t-1} + rB_{t-1} \\ B_t = \bar{G}_t - Y_t\theta + B_{t-1}(1+r) \end{cases}$$

Please, note that if the interest rate on public bonds and deposits is equal to zero, the system is reduced to:

$$Y_{t+1} = Y_t c_1(1-\tau) + \bar{G}$$

The analytical solution is:

$$Y_t = \frac{\bar{G}[1 - c^t(1-\tau)^t]}{1 - c(1-\tau)}$$

Since $|1 - c_1(1-\tau)| < 1$, the system is asymptotically stable and for $t \to \infty, Y_t$ converges to $\frac{\bar{G}}{1-c_1(1-\tau)}$ that is the traditional solution of the I/E scheme.

The sequential version of the Supermultiplier including public debt and public debt service is:

$$\begin{cases} Y_t = C_t + G_t + I_t + B_{t-1}r \\ C_t = Y_{t-1}c(1-\theta) \\ I_t = Y_t^e v - K_t(1-\delta) \\ Y_t^e = Y_{t-1} \\ B_t = B_{t-1}(1+r) + G_t - \theta Y_t \\ G_t = \bar{G} \end{cases}$$

From this system it is possible to get system of two second order difference equation whit two variables are GDPand public debt. Subsituing the expression of capital: $K_t = K_{t-1}(1-\delta) + I_{t-1} = Y_{t-2}v$ in the first equation, after some manipulations we have:

$$\begin{cases} Y_t - Y_{t-1}[c(1-\theta) + v] - Y_{t-2}v(1-\delta) + B_{t-1}r = \bar{G} \\ B_t = B_{t-1}(1+r) + \bar{G} - \theta Y_t \end{cases}$$

Appendix 1.1

$$G_t - \left(G_t + \frac{V_{-1}(c_2 - r_b) + G_t}{1-(1-\theta)c_1}(1-\theta)c_1 + V_{-1}c_2 + V_{-1}r_b\right)\theta + r_b B_{-1} = 0$$

$$G_t(1-\theta)\left(\frac{1-c_1}{1-(1-\theta)c_1}\right) - \frac{V_{-1}(c_2 + r_b)(1-\theta)c_1\theta}{1-(1-\theta)c_1} - V_{-1}c_2\theta + V_{-1}r_b(1-\theta) = 0$$

$$G_t(1-\theta)\left(1 - \frac{c_1\theta}{1-(1-\theta)c_1}\right) = V_{-1}(\frac{(c_2 + r_b)(1-\theta)c_1\theta + c_2\theta(1-(1-\theta)c_1) + r_b(1-\theta)(1-(1-\theta)c_1)}{1-(1-\theta)c_1}$$

$$Y^* = \frac{\frac{G_t(1-\theta)(1-c_1)}{r_b(1-\theta)[1-c_1(1-2\theta)] + c_2\theta}(c_2 + r_b) + G_t}{1-(1-\theta)c_1}$$

$$Y^* = \frac{\frac{G_t(1-\theta)(1-c_1)(c_2+r_b) + G_t r_b(1-\theta)[1-c_1(1-2\theta)] + G_t c_2\theta}{r_b(1-\theta)[1-c_1(1-2\theta)] + c_2\theta}}{1-(1-\theta)c_1}$$

$$Y^* = \frac{\frac{G_t(1-\theta)(1-c_1)c_2 + G_t(1-\theta)(1-c_1)r_b + G_t r_b(1-\theta)[1-c_1(1-2\theta)] + G_t c_2\theta}{r_b(1-\theta)[1-c_1(1-2\theta)] + c_2\theta}}{1-(1-\theta)c_1}$$

$$Y^* = \frac{\frac{G_t r_b(1-\theta)\{(1-c_1) + [1-c_1(1-2\theta)]\} + G_t c_2(\theta + (1-\theta)(1-c_1))}{r_b(1-\theta)[1-c_1(1-2\theta)] + c_2\theta}}{1-(1-\theta)c_1}$$

$$Y^* = \frac{\frac{G[1-(1-\theta)c_1][c_2 + 2(1-\theta)r_b]}{r_b(1-\theta)[1-c_1(1-2\theta)] + c_2\theta}}{1-(1-\theta)c_1}$$

Appendix 2

Income/Expenditure model with endogenous public debt service and $c_2 = 0$.

$$\begin{cases} B_t = B_{t-1} + \bar{G} - \theta Y_t + S_{t-1} r & (2.1) \\ Y_t = Y_{t-1}\, c_1\,(1-\theta) + \bar{G} + r\, S_{t-1} & (2.2) \\ S_t = S_{t-1} + Y_{t-1}(1-\theta)(1-c_1) & (2.3) \end{cases}$$

Redundant equation:

$$S_t = B_t - Y_t(1-\theta) \quad (5.6)$$

From 5.4 we get the saving expression as a function of the income $S_{t-1} = \frac{Y_t - Y_{t-1}\, c_1\,(1-\theta) - G_t}{r}$, substituting this one into equation 5.5 we obtain the second-order non-homogeneous difference equation of income:

$$Y_{t+1} - Y_t\,[1 + c_1\,(1-\theta)] + Y_{t-1}\,(1-\theta)[c_1 - (1-c_1)r] = 0 \quad (5.9)$$

From this, after a few passages, it is possible to derive the general expression that describes the dynamics of income over time:

$$Y_t = \frac{\bar{G}\, 2^{-t}\left\{\left[c_1(1-\theta) + 1 + \sqrt{[c_1(1-\theta)+1]^2 - 4(1-\theta)[c_1 - (1-c_1)r]}\right]^t - \left[c_1(1-\theta) + 1 - \sqrt{[c_1(1-\theta)+1]^2 - 4(1-\theta)[c_1 - (1-c_1)r]}\right]^t\right\}}{\sqrt{[c_1(1-\theta)+1]^2 - 4(1-\theta)[c_1 - (1-c_1)_1 r]}}$$

(5.7)

For the sake of simplicity, we can rewrite in the following form:

$$Y_t = \frac{\bar{G}\, 2^{-t}[(a+b)^t - (a-b)^t]}{b} \quad (5.7\ bis)$$

Where:

$$a = 1 + c_!(1-\theta) \qquad (5.8)$$

$$b = \sqrt{[c_1(1-\theta)+1]^2 - 4(1-\theta)[c_1 - (1-c_1)r]} \quad (5.9)$$

From 1.151 bis, we can study the convergence of the growth rate in the long term. For $r > 0$:

$$\lim_{t\to\infty} \frac{Y_t}{Y_{t-1}} - 1 = \frac{(a+b)^t - (a-b)^t}{2[\ (a+b)^{t-1} - (a-b)^{t-1}]} - 1 \approx \frac{1}{2}(a+b-2) \qquad con\ a > 1; 0 < b < 1$$[64]

Thus, the output growth rate converges to:

$$g_y = \frac{1}{2}(a+b-2) = \frac{1}{2}\left(1 + c_1(1-\theta) + \sqrt{[c_1(1-\theta)+1]^2 - 4(1-\theta)[c\_1\ - (1-c)r]} - 2\right) \quad (5.10)$$

Now, let us verify the stability of the government debt-to-GDP ratio, looking at the conditions for convergence of the two growth rates (public debt and output). In particular, for analytical simplicity, we consider the ratio of private savings to GDP (which is symmetrical to public debt).

From 5.5 we can derive the expression of income as a function of savings.

$$Y_{t-1} = \frac{S_t - S_{t-1}}{(1-\theta)(1-c_1)}$$

Replacing this in 5.1, we can derive, after some manipulation, the non-homogeneous second-degree difference equation of savings:

$$S_{t+1} - S_t[1 +\ (1-\theta)c_1] + S_{t-1}(1-\theta)[c_1 - r\,(1-c_1)] = \bar{G}(1-c_1)(1-\theta) \ (5.11)$$

For the sake of simplicity, we can rewrite the latter as:

$$S_{t+1} - bS_t + cS_{t-1} = \bar{G}d \quad (5.12)$$

Where:

$$a = 1 +\ (1-\theta)c_1 \quad (5.13)$$

$$c = (1-\theta)[c_1 - r\,(1-c_1)] \quad (5.14)$$

$$d = (1-c_1)(1-\theta) \quad (5.15)$$

Slving 5.12 it is possible to fond the value of the stock of savings infunction of time:

$$S_t = \frac{1}{2^{t+1}b(a-c-1)}[(a+b)^t(2-a+b) - (a-b)^t(2-a-b) - 2^{t+1}b]\, d\,\bar{G}$$

Where:

[64] These are verified because: $0 < c_1 < 1; 0 < \theta < 1; 0 < r < 1$.

$$b = \sqrt{a^2 - 4c} = \sqrt{[1 + \ (1-\theta)c_1]^2 - 4(1-\theta)[c_1 - r\,(1-c_1)]} \quad (5.17)$$

In the long run, the growth rate of saving is:

$$\lim_{t\to\infty} \frac{S_t}{S_{t-1}} - 1 = \frac{(a+b)^t(2-a+b) - (a-b)^t(2-a-b) - 2^{t+1}b}{2[(a+b)^{t-1}(2-a+b) - (a-b)^{t-1}(2-a-b) - 2^t b]} - 1$$

Under the condition $a + E > 2$ (which is verified because of $0 < c_1 < 1; 0 < \theta < 1; 0 < r < 1$) the growth rate of savings and public debt converge to the following value:

$$g_S \approx \frac{a+b}{2} - 1 = \frac{(1-\theta)c_1 - 1 + \sqrt{(1 + \ (1-\theta)c_1)^2 - 4(1-\theta)[c_1 - r\,(1-c_1)]}}{2} \quad (5.18)$$

As we can see from 5.10 this is equal to the growth rate of output:

$$g_S = g_Y = g_B$$

In particular, the equilibrium value of wealth-to-GDP ratio is:

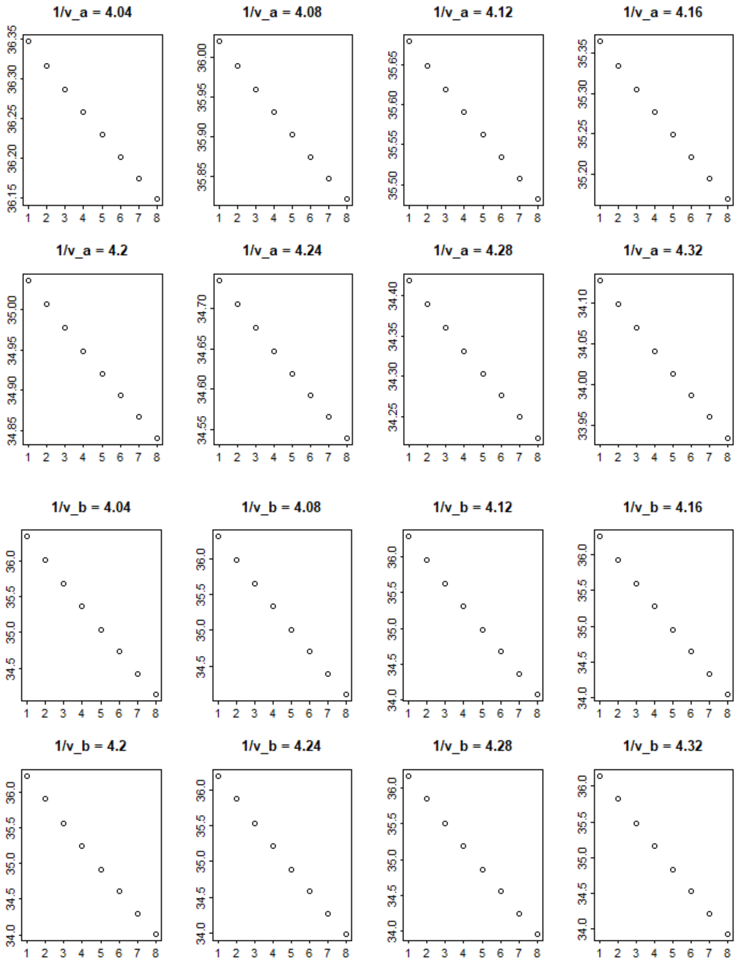
$$\frac{S^*}{Y^*} = \frac{(1-c_1)(1-\theta)}{g}{}^{65} = \frac{2(1-c_1)(1-\theta)}{(1-\theta)c_1 - 1 + \sqrt{(1 + \ (1-\theta)c_1)^2 - 4(1-\theta)[c_1 - r\,(1-c_1)]}} \quad (5.19)$$

Vice versa, when $a + b \leq 2$, the growth rate converges to zero. Indeed, in the case we are outside of the *interest-led regime*: in order to have $a + b \leq 2$ the interest rate have to be equal to zero.

[65] From 5.5 we get: $S_t = \frac{Y_t(1-\theta)(1-c_1)}{g}$.

Appendix 3

Figure 2.1 reports the variations in capital intensity when v_a and v_b changes.



Appendix 4

For instance, in the income/expenditure model without investments, the public expenditure (or any other autonomous component) triggers production and income generation while the deficit is the counterpart to the change in private savings. The initial injection of money, through the indebtedness of the government with the Central Bank, subdue to the original demand from which the process of production, income distribution and induced creation of further demand starts, triggering the multiplier effect along the next periods.

In the SFC approach, adopting the end-of-period accounting, the public deficit is calculated net of tax revenues generated by public expenditure, while the amount of bonds held by the CB depends on household portfolio decisions:

$$V_h = M + B_h = B_{pub}\text{[66]}$$

For example, if households decide to hold all their savings in the form of government bonds, the government debt to the CB is zero. Conversely, if households choose to keep all their savings in the form of deposits, the entire Government debt would be held by the CB, and the interest rate paid by the government would be equal to the interest rate on deposits (assuming that the interest rate on reserves is equal to the interest rate on deposits). The inclusion of induced investment brings the firm's indebtedness[67] into the model and, therefore, part of household savings will be generated symmetrically from the concession of bank loans. Since the demand for equities by households has not been included, firms' indebtedness only has as a counterpart the generation of deposits:

$$V_h = M + B_h = L + B_{pub}$$

Thus, in the absence of demand for cash, the counterpart to government bonds held by the CB ($B_{CB} = B_{pub} - B_h$), corresponds to a share of deposits held by households ($M = L + B_{CB}$).

[66] V_h is the wealth stock of households; M is the stock of deposit; B_h is the stock of public bonds held by households.
[67] In the SM, current costs are paid ex-post, once revenues have been realized. Current demand is known and production exactly equald demand. Therefore, in the income/expenditure model without investments in fixed capital, there is no indebtedness of the production sector (current costs are paid thorough the revenues; these revenues are generated thanks to the autonomous component and the multiplier effect triggered by such components).

Balance Sheet and Transaction Matrix

Table 1: Balance sheet matrix

Assets	Workers	Capitalists	Sector A	Sector B	Firm C	Bank	Government	BC	$\sum$
Check deposits[68]	$+M1_w$	$+M1_{cap}$	$+M1_a$	$+M1_c$	$+M1_c$	$-M1$			0
Time deposits	$+M1_w$	$+M1_{cap}$				$-M2$			0
HPM						$+H_b$		$-H$	0
Advances						$-A$		$-A$	0
Loans			$-L_a$	$-L_b$	$-L_c$	$+L$			0
Fixed Capital			$+K_{f_a}$	$+K_{f_b}$	$+K_{f_c}$				$+K_f$
Public bonds		$+B_{h,cap}$					$-B$	$+B_{cb}$	0
Net wealth	$-V_{h,w}$	$-V_{h,cap}$	$-V_a$	$-V_b$	$-V_c$	0	$+GD$	0	$-K_f$
$\sum$	0	0	0	0	0	0	0	0	0

[68] The distinction between "check" and "time" deposits is necessary for the sequential approach: income distributed at the end of the period (from which the consumption demand in the following period is generated) appears in the form of "check" deposits, while "time" deposits are the share of income saved and held in the form of deposits. Unlike check deposits, time deposits accrue interests in each period.

Tabella 1. Transaction Matrix

	Workers	Capitalists	Sector A	Sector B	Sector C	Government	Bank		Central Bank		Σ
							Current	Capital	Current	Capital	
Consumption	$-C_w$	$-C_{cap}$			$+C$						0
Investments in A			$+I$	$-I_b$	$-I_c$						0
Investments in B			$-I_a$	$+I$							0
Public expenditure					$+G$	$-G$					0
Wages	$+W$		$-W_a$	$-W_b$	$-W_c$						0
Taxes	$-T_w$	$-T_{cap}$				$+T$					0
Profits		$+Div_F$	$-Div_a$	$-Div_b$	$-Div_c$						0
Profits B		$+Div_B$					$-Div_B$				0
Profits BC						$+F_{cb}$			$-F_{cb}$		0
Int. Deposit	$+r_{m,t-1}M2_w$	$+r_{m,t-1}M2_{c,t-1}$					$-r_{m,t-1}M2$				0
Int. Loans			$-r_{l-1}L_{a,t-1}$	$-r_{l-1}L_{b,t-1}$	$-r_{l-1}L_{c,t-1}$		$+r_{l-1}L_{t-1}$				0
Int. Bond		$+i_{t-1}B_{h,t-1}$				$-i_{t-1}B_{t-1}$			$+i_rB_{bc,t-1}$		0
Int. Reserves							$+r_{r-1}H_{t-1}$		$-r_{r-1}H_{t-1}$		0
Int. Advances							$-r_{a,t-1}A_{t-1}$		$+r_{a,t-1}A_{t-}$		0
ΔDeposit time	$-\Delta M2_w$	$-\Delta M2_{cap}$						$+\Delta M2$			0
Δ Deposit check	$-\Delta M1_w$	$-\Delta M1_w$						$+\Delta M1$			0
Δ Loans			$+\Delta L_a$	$+\Delta L_b$	$+\Delta L_c$			$-\Delta L$			0
Δ Bonds		$-\Delta B_h$				$+\Delta B$			$-\Delta B_{bc}$		0
Δ Reserves								$-\Delta H$	$+\Delta H$		0
Σ	0	0	0	0	0	0	0	0	0	0	0