Pay-as-you-go social security and educational subsidy in an overlapping generations model with endogenous fertility and endogenous retirement

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Abstract

This study analytically investigates the effects of pay-as-you-go social security and educational subsidies on the fertility rate, retirement age, and GDP per capita growth rate in an overlapping generations model, where parents invest resources toward their children's human capital. We find that an old agent retires fully when his or her labor productivity is low and retires later when the labor productivity is high. Under the unique balanced-growth-path (BGP) equilibrium, when an old agent is still engaged in work, tax rates are neutral to the fertility rate, higher tax rates encourage him or her to retire earlier, a higher social security tax rate depresses the GDP per capita growth rate, and a higher tax rate for educational subsidies can accelerate growth. However, when an old agent fully retires, higher tax rates increase the fertility rate, a higher social security tax rate lowers the GDP per capita growth rate, and a higher tax rate for educational subsidies boosts growth. Additionally, if an old agent's labor productivity increases, the fertility rate also increases. We also conduct numerical simulations and analyze how an old agent's labor productivity affects the retirement age, fertility rate, and GDP per capita growth rate under the BGP equilibrium.

Keywords: Pay-as-you-go social security; educational subsidy; fertility; endogenous retirement; GDP per capita growth rate

JEL Classification: H55, J26, J13, I25

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1 Introduction

Educational subsidies and social security systems are two common public policies in many developed countries. The important role of human capital accumulation in economic growth has been recognized since the seminal papers of Lucas (1988) and Becker et al. (1990). To increase the accumulation of human capital, many countries have been providing public education or educational subsidies in recent decades.[1] However, countries that have implemented social security systems are facing population aging due to lower fertility rates and increased longevity. These factors threaten the sustainability of pay-as-you-go social security systems.

Educational subsidies and social security are financed by tax revenues. While the provision of educational subsidies transfers resources from the middle-age generation to the young one, the implementation of a social security system transfers resources from the middle-age generation to the old one. As such, increases in the burden of the social security system affect the allocation of tax revenues, which may in turn affect economic growth. Kaganovich and Zilcha (1999) analyzes how the allocation of tax revenues between these two public expenditures affects economic growth. The current study differs by endogenizing agents' decisions on the retirement age and fertility rate.

Two methods to mitigate the increased burden of the social security system are increasing the tax revenue and reducing payments. Tax revenues from labor income tax depend on the number of workers, as well as workers' human capital. To increase the number of workers, raising the retirement age is one solution. Since labor income tax rates affect an old agent's retirement behavior, in this study, we allow an old agent to determine his or her retirement age and examine how changes in the labor income tax rates affect this agent's retirement decision and economic growth.[2] Furthermore, to accumulate more human capital, providing educational subsidies is one solution. Following Becker et al. (1990), we assume parents make joint decisions on fertility and educational investments for their children. Since parents decide on both the quality and number of children, the provision of an educational subsidy reduces the cost of education for children, which in turn affects parents' decision on the number of children or the number of workers.

We consider an overlapping generations (OLG) model for a small open economy, under which the fertility rate and retirement age are endogenously determined, and the government provides pay-as-you-go social security and educational subsidies. Focusing on a balanced-growth-path (BGP) equilibrium, we present comparative statics on how the government's policies affect the fertility rate, retirement age, and GDP per capita growth rate. Our findings can be summarized as follows.

An old agent's retirement age depends on his or her labor productivity, and when his or her labor productivity is low (high), under a unique BGP equilibrium, he or she fully retires (provides a positive amount of labor). Regarding the fertility rate, we find that it reacts differently depending on whether an old agent fully retires under the BGP equilibrium. Specifically, the fertility rate under the BGP equilibrium where an old agent fully retires is

[1] Bovenberg and Jacobs (2005) shows that optimal educational subsidies ensure the efficiency of human capital accumulation.

[2] Chen (2018) and Cipriani (2018) compare pension benefits under the assumption of an old agent's exogenous retirement age with those under the assumption of an old agent's endogenous retirement age, and discuss how the model setting affects the results.

increasing in both tax rates, while that under the BGP equilibrium where an old agent provides a positive amount of labor is neutral to the tax rates. This difference stems from the fact that, when the tax rate changes, an old agent adjusts his or her retirement age under the BGP equilibrium where he or she supplies a positive amount of labor. When an old agent fully retires, an increase in the tax rates increases pay-as-you-go social security benefits, which increases lifetime income and the fertility rate. However, when an old agent works, if pay-as-you-go social security benefits increase, he or she supplies a less positive amount of labor and lifetime income does not change, nor does the fertility rate. Additionally, we show that the fertility rate is higher under the BGP equilibrium where an old agent supplies a positive amount of labor than that under the BGP equilibrium where an old agent fully retires, because a child is a normal good in the model and a higher labor productivity of an old agent induces higher lifetime income. We also find that an increase in tax rates will discourage an old agent from working more, meaning he or she will retire earlier. Regarding the GDP per capita growth rate, an increase in the pay-as-you-go social security tax rate lowers the growth rate because of the well-known crowding-out effect under both types of BGP equilibria. However, an increase in the tax rate for educational subsidies will accelerate the growth rate, because parents invest more in their children's education, which is the growth engine in the model.

Additionally, we describe the relationship between the GDP per capita growth rate under the BGP equilibrium and an old agent's labor productivity. Although we cannot prove this analytically, we numerically show that the GDP per capita growth rate can increase or decrease as an old agent's labor productivity increases. A key variable is the time cost of rearing a child. Since an agent faces a quantity–quality trade-off regarding children, when the time cost is high (low), the number of children an agent has decreases (increases) and an agent invests more (less) in each child, which accelerates (depresses) growth.

Many studies have investigated the effects of pay-as-you-go social security and of educational subsidies on fertility, retirement, and economic growth. Most of them, except Kaganovich and Zilcha (1999), examine the effects of either of the two policies on some of the three economic variables. For instance, how the pay-as-you-go social security affects the fertility rate has been studied by Wigger (1999), Cigno and Werding (2007), Miyazaki (2013), and so on. Wigger (1999) and Miyazaki (2013) show that a medium size of the pay-as-you-go social security tax rate will maximize the fertility rate in an OLG model. Our results suggest that endogenizing an old agent's retirement age may affect the relationship between the pay-as-you-go social security tax rate and the fertility rate. Meanwhile, Hu (1979), Momota (2003), Fenge and Pestieau (2005), Michel and Pestieau (2013), Chen (2018), Miyazaki (2019), Cipriani and Fioroni (2021), and others analyze the relationship between pay-as-you-go social security and an old agent's labor supply or decision regarding retirement. Hu (1979) and Momota (2003) investigate how the length of the retirement period changes at equilibrium as the pension tax rate and pension benefits change, although they do not consider the case of no labor supply for old agents. Because our model is more tractable than theirs, we include the condition of an old agent's labor productivity under which he or she supplies no labor and above which he or she supplies a positive amount of labor. Additionally, we show that an increase in the pay-as-you-go social security tax rate always encourages an old agent to retire earlier, a result that has remained ambiguous in the literature. Michel and Pestieau (2013) and Miyazaki (2019) examine whether pay-as-you-go social security can make competitive equilibrium allocations socially optimal even in an OLG model with endogenous retirement. Michel and Pestieau (2013) claim that additional policies are necessary to achieve the first-best allocation at a competitive equilibrium, whereas Miyazaki (2019) show the condition on an old agent's labor productivity under which the first-best allocation is implementable. Chen (2018) examine the effects of fertility and retirement age on long-run pay-as-you-go pensions and show that the effects depend on whether retirement age is exogenously given. Recently, Cipriani and Fioroni (2021) examined how exogenous longevity affects fertility, retirement, and pensions in an OLG model with an exogenous growth setting. They show that when retirement age is determined endogenously, the fertility rate at equilibrium depends on the pension tax rate. In our model, when an old agent supplies positive labor, the fertility rate at equilibrium is neutral to the tax rate, which is different from the result shown in Cipriani and Fioroni (2021). This difference suggests that a choice between an exogenous growth setting and an endogenous growth setting with endogenous retirement can be important when policy effects are considered. Regarding the effects of educational subsidies on fertility and growth, Chen (2015) finds that increases in educational subsidies increase both the quantity of children and the quality of adults based on an OLG model in which the government provides educational subsidies in skills training for adults. Zhang (1997) and Zhang and Casagrande (1998) examine these effects on fertility and growth using a dynastic model. Zhang and Casagrande (1998) shows that education subsidies have no effect on fertility and that an increase in the education subsidy tax rate raises parental education and the GDP per capita growth rate. We derive similar results in an OLG model when an old agent supplies a positive amount of labor, but our results differ from theirs when the agent supplies no labor. This difference implies that endogenizing an old agent's labor supply matters in this case too.

As mentioned above, Kaganovich and Zilcha (1999) examines the effects of the allocation of tax revenues between education vouchers and pay-as-you-go social security on growth under a stationary equilibrium. One of the findings of this study is how much parents care about their children's human capital and the level of tax revenues for education vouchers, which are key parameters that determine the level of economic growth. Our model extends theirs by endogenizing the fertility rate and retirement age. Since tax revenue is affected by the number of workers, endogenizing the fertility rate and retirement age can derive different results from those from Kaganovich and Zilcha (1999). We also show that the sizes of the two tax rates play a role in determining whether an educational subsidy accelerates growth.

The remainder of this paper is organized as follows. Section 2 describes the model and Section 3 characterizes the BGP equilibrium. In Section 4, we investigate the relationship between an old agent's labor productivity and the fertility rate, retirement age, and GDP per capita growth rate under the BGP equilibrium, both analytically and numerically. Section 5 concludes the paper. All proofs can be found in the Appendix.

2 Model

Time is discrete and continues forever: $t = 1, 2, \ldots$. We consider an OLG model in a small open economy, in which capital freely moves across countries and labor is a country-specific factor.[3] The interest rate is equal to the world interest rate. For simplicity, we assume that the world interest rate is constant over time, denoted by $\bar{r} > 0$. Given a neoclassical production function, the capital per unit of effective labor, $\bar{k}$, and the wage, $\overline{w}$, are constant over time.

An agent lives for three periods: childhood, young age, and old age. During childhood, the agent makes no decisions. A young agent is endowed with one unit of time and decides on the amount of savings, s_t, how many children to have, n_t, and how much to spend on private education for his or her children. A young agent spends qn_t units of time to raise n_t children, where $q > 0$. After production occurs, a young agent receives labor income $\overline{w} > 0$. A young agent pays tax for transfer to retired old agents and for educational subsidies. The tax rate is denoted by $\tau_1, \tau_2 \in [0, 1)$, where τ_1 is the tax rate for the transfer to retired old agents and τ_2 is the tax rate for educational subsidies. If a young agent has n_t children and spends e_t on private education for each of his or her children, then the agent's budget constraint is:

$$s_t + (1 - \phi_t)e_t n_t = (1 - \tau_1 - \tau_2)(1 - qn_t)\overline{w}h_t, \tag{1}$$

where $\phi_t \in [0, 1)$ of the education cost is subsidized and h_t is the young agent's human capital in period t. If a young agent spends e_t for the human capital accumulation of his or her children, their human capital in period $t+1$ will be:

$$h_{t+1} = \theta e_t^{\eta} h_t^{1-\eta}, \tag{2}$$

where $\theta > 0$ and $\eta \in (0, 1)$.

A young agent's savings at date t, s_t, are used for production at $t+1$ as a capital good without any transformation cost from one unit of a consumption good to capital good. When a young agent's savings at date t are s_t and the agent becomes old at $t+1$, the old agent receives interest income $(1 + \bar{r})s_t$. Additionally, an old agent is endowed with one unit of time. If an agent works for l^o_{t+1} units of time, he or she receives labor income $l^o_{t+1}\chi h_t \overline{w}$, where $\chi \geq 0$ is an old agent's labor productivity relative to that of a young agent. Moreover, an agent has to pay tax and, thus, his or her disposable labor income is $(1 - \tau_1 - \tau_2)l^o_{t+1}\chi h_t \overline{w}$. Once an old agent retires, he or she is eligible to receive retirement benefits $(1 - l^o_{t+1})P_{t+1}$. An old agent's budget constraint in period $t+1$ is:

$$c^o_{t+1} = (1 + \bar{r})s_t + (1 - \tau_1 - \tau_2)l^o_{t+1}\chi h_t \overline{w} + (1 - l^o_{t+1})P_{t+1}. \tag{3}$$

[3]The same setting is considered in Chen and Miyazaki (2018).

From Equations (1) and (3), an agent's lifetime budget constraint is:

$$(1-\phi_t)e_t n_t + \frac{c^o_{t+1}}{1+\bar{r}} = (1-\tau_1-\tau_2)(1-qn_t)\overline{w}h_t + \frac{(1-\tau_1-\tau_2)l^o_{t+1}\chi h_t\overline{w} + (1-l^o_{t+1})P_{t+1}}{1+\bar{r}}. \tag{4}$$

An agent's lifetime utility is represented by the following function:

$$U(n_t, h_{t+1}, c^o_{t+1}, l^o_{t+1}) := \ln(c^o_{t+1}) + \sigma\ln(n_t h_{t+1}) + \gamma\ln(1-l^o_{t+1}),$$

where $\sigma > 0$ and $\gamma > 0$ are, respectively, the weights on the preference over children and their human capital and leisure relative to consumption when they are old. Hence, an agent's problem is:

$$\begin{aligned} \max_{n_t, e_t, c^o_{t+1}, l^o_{t+1}} \quad & U(n_t, h_{t+1}, c^o_{t+1}, l^o_{t+1}) \\ \text{s.t.} \quad & \text{Equations (2), (4),} \\ & 0 \leq n_t \leq \frac{1}{q} \text{ and } 0 \leq l^o_{t+1} \leq 1. \end{aligned} \tag{5}$$

The last constraint is the time constraint for both a young agent and an old agent. Note that since the utility from the number of children and leisure is represented by $\sigma\ln(n_t h_{t+1})$ and $\gamma\ln(l^o_{t+1})$, $n_t > 0$ and $l^o_{t+1} > 0$ must hold in the solution to the maximization problem. Therefore, Equation (5) can be replaced by $n_t \leq \frac{1}{q}$ and $l^o_{t+1} \leq 1$.

We assume that an initially old agent has saved $s_0 > 0$ at date 0, that the initial old population is normalized to 1, and that $n_0 > 0$ is given. Let N_t be the population of young agents in period t. Then, the population of young agents in period $t+1$ is:

$$N_{t+1} = n_t N_t.$$

In every period, the government balances the following budget constraints for all $t \geq 1$:

$$N_t P_{t+1}(1-l^o_{t+1}) = \tau_1\overline{w}[N_{t+1}(1-qn_{t+1})h_{t+1} + N_t\chi l^o_{t+1}h_t] \tag{6}$$

and

$$N_t\phi_t e_t n_t = \tau_2\overline{w}[N_t(1-qn_t)h_t + N_{t-1}\chi l^o_t h_{t-1}]. \tag{7}$$

Equation (6) implicitly ignores the case of $l^o_{t+1} = 1$. Since an old agent does not choose $l^o_{t+1} = 1$ at equilibrium, the government budget constraint is well defined.

The equilibrium concept is thus a standard *perfect foresight competitive equilibrium*. The formal definition is as follows.

Definition 2.1. *Given τ_1, τ_2, $\overline{w}$, $\bar{r}$, s_0, and n_0, an* **equilibrium** *consists of an allocation $(l^o_1, c^o_1, (n_t, e_t, c^o_{t+1}, l^o_{t+1})_{t=1}^{\infty})$, a sequence of retirement benefits $(P_{t+1})_{t=0}^{\infty}$, and a sequence of subsidization of the cost of education $(\phi_t)_{t=1}^{\infty}$, such* *that:*

1. *given* τ_1, τ_2, $\overline{w}$, $\overline{r}$, P_{t+1}, *and* ϕ_t, $(n_t, e_t, c^o_{t+1}, l^o_{t+1})$ *solves an agent's problem for all* t.

2. *given* τ_1, τ_2, $\overline{w}$, $\overline{r}$, *and* P_1, (l^o_1, c^o_1) *solves an initial old agent's problem.*

3. *The government's budget constraints in Equations (6) and (7) are satisfied for all* t.

An equilibrium is a **BGP** *equilibrium if all per capita variables grow at a constant rate.*

3 The BGP equilibrium analysis

The first-order conditions to an agent's problem are:

$$\begin{aligned} \frac{\sigma}{1+\overline{r}}c^o_{t+1} &= n_t[(1-\tau_1-\tau_2)\overline{w}h_t q + (1-\phi_t)e_t], && (8)\\ \frac{\eta\sigma}{1+\overline{r}}c^o_{t+1} &= (1-\phi_t)n_t e_t, && (9)\\ \gamma c^o_{t+1} &\geq (1-l^o_{t+1})[(1-\tau_1-\tau_2)\overline{w}h_t\chi - P_{t+1}] \text{ with equality holding if } l^o_{t+1}>0. && (10) \end{aligned}$$

3.1 The BGP equilibrium where an old agent supplies a positive amount of labor

First, we characterize an equilibrium under which an old agent supplies strictly positive amounts of labor, that is, Equation (10) holds with equality.

By inserting Equations (8), (9), and (10) into Equation (4), we obtain:

$$c^o_{t+1} = \widehat{c}^{o*}_{t+1} := \frac{(1+\overline{r})(1-\tau_1-\tau_2)\overline{w}h_t}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\overline{r}}\right). \tag{11}$$

From Equations (8) and (9), we have:

$$n_t(1-\tau_1-\tau_2)\overline{w}h_t q = \frac{(1-\eta)\sigma}{1+\overline{r}}c^o_{t+1}. \tag{12}$$

By plugging Equation (11) into Equation (12):

$$n_t = \widehat{n}^* := \frac{(1-\eta)\sigma}{q(1+\sigma+\gamma)}\left(1+\frac{\chi}{1+\overline{r}}\right). \tag{13}$$

Note that the fertility rate is constant over time and does not depend at all on the tax rates. For this fertility rate to satisfy constraint $n_t \leq \frac{1}{q}$, we impose the following assumption on parameters.

Assumption 3.1. *An old agent's labor productivity satisfies:*

$$\chi \leq \overline{\chi} := (1+\overline{r})\frac{1+\gamma+\eta\sigma}{(1-\eta)\sigma}. \tag{14}$$

Since $\bar{r} \geq 0$, $\overline{\chi}$ can be greater than 1, which implies that an old agent's labor productivity is greater than that of a young agent. It is natural to assume $\chi \leq 1$. However, since $\chi > 1$ does not matter in our study, we allow χ to take values higher than 1.

Proposition 3.1. *Under a BGP equilibrium where an old agent supplies a positive amount of labor, a change in the tax rates does not affect the fertility rate at all, that is,* $\frac{\partial \widehat{n}^*}{\partial \tau_i} = 0$ *for all* $i = 1, 2$.

The fertility rate under the BGP equilibrium where an old agent supplies a positive amount of labor is not affected by the changes in the tax rates, τ_1 and τ_2, at all. A change in τ_1 and τ_2 reduces the labor income in both young and old ages. However, a reduction in the old age labor income is compensated by social security benefits. Additionally, an increase in the tax rates reduces the opportunity cost of having children. These effects are cancelled out in the BGP equilibrium. Therefore, the fertility rate does not change at all, even if τ_1 and τ_2 change.

From Equations (9), (11), and (13), we obtain:

$$e_t = \frac{q\eta(1-\tau_1-\tau_2)\overline{w}}{(1-\eta)(1-\phi_t)} h_t. \tag{15}$$

Equations (2) and (15), the GDP per capita growth rate at equilibrium is:

$$g_{t+1} = \frac{h_{t+1}}{h_t} - 1 = \theta \left[\frac{q\eta(1-\tau_1-\tau_2)}{(1-\phi_t)(1-\eta)} \overline{w} \right]^{\eta} - 1. \tag{16}$$

From Equations (6), (10), and (11), labor supply of old agents at equilibrium is:

$$\begin{aligned}(1-\tau_2) l^o_{t+1} \chi \quad &= \quad \frac{1-\tau_1-\tau_2}{1+\sigma+\gamma} \left[\chi(1+\sigma) - \gamma(1+\bar{r}) \right] \\ &- \quad \frac{(1-\eta)\sigma}{(1+\sigma+\gamma)q} \left(1 + \frac{\chi}{1+\bar{r}} \right) \tau_1 \theta \left[\frac{q\eta(1-\tau_1-\tau_2)}{(1-\phi_t)(1-\eta)} \overline{w} \right]^{\eta} \left[1 - \frac{(1-\eta)\sigma}{1+\sigma+\gamma} \left(1 + \frac{\chi}{1+\bar{r}} \right) \right].\end{aligned} \tag{17}$$

At equilibrium, the parental education subsidy, ϕ_t, must satisfy the following equation, which results from rewriting Equation (7):

$$\frac{\phi_t q \eta}{(1-\phi_t)(1-\eta)} (1-\tau_1-\tau_2)\widehat{n}^* = \tau_2 \left[(1-q\widehat{n}^*) + \frac{\chi l^o_t}{\widehat{n}^*} \frac{h_{t-1}}{h_t} \right]. \tag{18}$$

By inserting $\widehat{n}^*$, l^o_t, and $\frac{h_{t-1}}{h_t}$ into Equation (18), we obtain:

$$\begin{aligned}\frac{\phi_t}{1-\phi_t} \frac{\eta\sigma}{1+\sigma+\gamma} \left(1 + \frac{\chi}{1+\bar{r}} \right) \quad &= \quad \frac{\tau_2}{1-\tau_2} \left[1 - \frac{(1-\eta)\sigma}{1+\sigma+\gamma} \left(1 + \frac{\chi}{1+\bar{r}} \right) \right] \\ &+ \quad \frac{q}{(1-\eta)\sigma} \frac{1+\bar{r}}{1+\bar{r}+\chi} \frac{\tau_2}{1-\tau_2} \left[\chi(1+\sigma) - \gamma(1+\bar{r}) \right] \frac{(1-\phi_{t-1})^{\eta}}{\theta \left[\frac{q\eta(1-\tau_1-\tau_2)}{1-\eta} \overline{w} \right]^{\eta}}.\end{aligned} \tag{19}$$

Note that, when $\tau_2 = 0$, $\phi_t = 0$. Assume that $\tau_2 > 0$. Since the left-hand side of Equation (19) is strictly increasing in ϕ_t and its right-hand side is strictly decreasing in ϕ_{t-1}, we can show there exists a unique $\widehat{\phi}^* \in (0,1)$ that satisfies:[4]

$$\begin{aligned}\frac{\widehat{\phi}^*}{1-\widehat{\phi}^*}\frac{\eta\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right) &= \frac{\tau_2}{1-\tau_2}\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right] \\ &+ \frac{q}{(1-\eta)\sigma}\frac{1+\bar{r}}{1+\bar{r}+\chi}\frac{\tau_2}{1-\tau_2}\left[\chi(1+\sigma)-\gamma(1+\bar{r})\right]\frac{(1-\widehat{\phi}^*)^\eta}{\theta\left[\frac{q\eta(1-\tau_1-\tau_2)}{1-\eta}\overline{w}\right]^\eta}. \end{aligned} \tag{20}$$

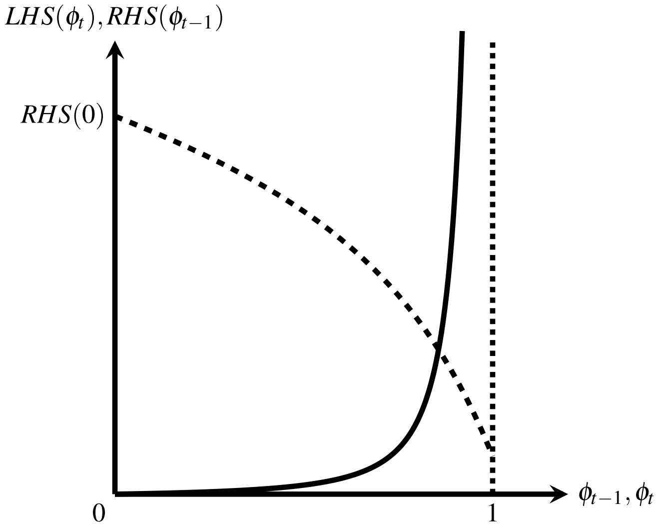

Figure 1: The line represents the left-hand side and the dashed line the right-hand side of Equation (19). A unique crossing point is $\widehat{\phi}^*$.

Given that $\widehat{\phi}^*$, $\widehat{l}^{o*}$ is:

$$\begin{aligned}\widehat{l}^{o*} &= \frac{1}{1+\sigma+\gamma}\frac{1-\tau_1-\tau_2}{1-\tau_2}\left[1+\sigma-\frac{\gamma(1+\bar{r})}{\chi}\right] \\ &- \frac{(1-\eta)\sigma\theta}{(1+\sigma+\gamma)q}\left(\frac{1}{\chi}+\frac{1}{1+\bar{r}}\right)\frac{\tau_1}{1-\tau_2}\left[\frac{q\eta(1-\tau_1-\tau_2)}{(1-\widehat{\phi}^*)(1-\eta)}\overline{w}\right]^\eta\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right]. \end{aligned} \tag{21}$$

Since $\widehat{l}^{o*} > 0$, in this case, the parameters must satisfy:

$$\begin{aligned}\chi &> \frac{\gamma(1+\bar{r})}{1+\sigma} \\ &+ \frac{\tau_1}{1-\tau_1-\tau_2}\frac{(1-\eta)\sigma}{q(1+\sigma)}\left(1+\frac{\chi}{1+\bar{r}}\right)\theta\left[\frac{q\eta(1-\tau_1-\tau_2)}{(1-\widehat{\phi}^*)(1-\eta)}\overline{w}\right]^\eta\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right]. \end{aligned} \tag{22}$$

[4]See Figure 1. For the formal proof, see the proof of the next proposition in the Appendix.

Otherwise, an old agent supplies no labor; this case is analyzed in the next section. Plugging $\widehat{\phi}^*$ into Equations (15) and (16), we have:

$$\widehat{e}_t^* = \frac{q\eta(1-\tau_1-\tau_2)\overline{w}}{(1-\eta)(1-\widehat{\phi}^*)} h_t$$

and

$$\widehat{g}^* = \theta \left[\frac{q\eta(1-\tau_1-\tau_2)}{(1-\widehat{\phi}^*)(1-\eta)} \overline{w} \right]^{\eta} - 1.$$

Additionally, retirement benefits at equilibrium are:

$$\widehat{P}_{t+1}^* = \frac{\widehat{n}^* \tau_1 \overline{w}(1+\widehat{g}^*)(1-q\widehat{n}^*) + \overline{w}\widehat{l}^{o*}\chi\tau_1}{1-\widehat{l}^{o*}} h_t.$$

The next proposition is a statement of the existence of a unique BGP equilibrium where an old agent supplies a positive amount of labor.

Proposition 3.2. *Assume the parameters satisfy Equation (22). Then, there is a unique BGP equilibrium characterized by* $\widehat{c}_{t+1}^{o*}, \widehat{n}^*, \widehat{\phi}^*, \widehat{l}^{o*}, \widehat{e}_t^*, \widehat{g}^*$, *and* $\widehat{P}_{t+1}^*$. *Additionally,* $\frac{\widehat{c}_{t+1}^*}{\widehat{c}_t^*} = \frac{\widehat{e}_{t+1}^*}{\widehat{e}_t^*} = \frac{\widehat{P}_{t+1}^*}{\widehat{P}_t^*} = 1+\widehat{g}^*$ *and* $\frac{\widehat{n}_{t+1}^*}{\widehat{n}_t^*} = \frac{\widehat{\phi}_{t+1}^*}{\widehat{\phi}_t^*} = \frac{\widehat{l}_{t+1}^{o*}}{\widehat{l}_t^{o*}} = 1$.

Now, we examine how $\widehat{l}^{o*}$ changes with the tax rate. An increase in tax rates decreases the opportunity cost of leisure in old age, which encourages an old agent to retire earlier. An increase in tax rates can also decrease the disposable lifetime income, which encourages an old agent to work more. Because of these opposite effects, how tax rates affect an old agent's decision on his or her labor supply may depend on other parameters and functional forms.

Proposition 3.3. *Assume the parameters satisfy Equation (22). Then, a slight increase in the pension tax rate* τ_1 *decreases the labor supply of the old agent under the BGP equilibrium. If* τ_1 *and* τ_2 *satisfy* $0 < \tau_1 < \min\{(1-\tau_2)^2, (1-\eta)(1-\tau_2)\}$, *a slight increase in* τ_2 *also decreases the labor supply of the old agent under the BGP equilibrium. If* $\tau_1 = 0$, *a slight increase in* τ_2 *has no effect on* $\widehat{l}^{o*}$.

This proposition implies that an increase in both tax rates discourages an old agent from working. In other words, higher tax rates incentivize an old agent to retire earlier. Therefore, the model suggests that, since the substitution effect dominates the income effect of an increase in tax rates, an increase in the pension tax rate and the tax for parental education subsidies drives agents to retire earlier. Note that the value of τ_1 is important when the effect of τ_2 on the labor supply of old agents is considered. When $\tau_1 = 0$, there exist no pension benefits. Since the utility function in the model is a natural logarithm, the substitution and income effects are cancelled out. Therefore, when there are no pension benefits, an old agent's labor supply does not change with τ_2.

Finally, we examine the effects of tax rates on the growth rate, $\widehat{g}^*$, under the BGP equilibrium. From Equation

(16):

$$\widehat{g}^* = \frac{h_{t+1}}{h_t} - 1 = \theta \left[\frac{q\eta(1-\tau_1-\tau_2)}{(1-\widehat{\phi}^*)(1-\eta)} \overline{w} \right]^{\eta} - 1. \tag{23}$$

An increase in both tax rates decreases the disposable labor income, which in turn decreases parental education, with a negative effect on the growth rate. An increase in the pension tax rate, τ_1, discourages an old agent from working, as in Proposition 3.3, which decreases the tax revenue for the parental education subsidies. Because of these negative effects on growth, an increase in τ_1 decreases the growth rate under the BGP equilibrium.

For an increase in τ_2, as the first effect, the decrease in the disposable labor income is the same as for τ_1. Additionally, from Proposition 3.3, an increase in τ_2 makes an old agent work for a shorter period under certain conditions. However, different from the case of τ_1, an increase in τ_2 can increase the tax revenue for the parental education subsidies, which can increase $\widehat{\phi}^*$. If $\widehat{\phi}^*$ increases, from Equation (23), $\widehat{g}^*$ can also increase when τ_2 increases.

Proposition 3.4. *Assume the parameters satisfy Equation (22). Then, a slight increase in τ_1 reduces the growth rate under the BGP equilibrium. If a slight increase in τ_2 increases the growth rate under the BGP equilibrium, $\tau_1 < (1-\tau_2)^2$ must be satisfied.*

It is certain that an increase in the pension tax depresses growth under the BGP equilibrium, whereas it is ambiguous whether an increase in τ_2 can accelerate growth under the BGP equilibrium. Note that, in Proposition 3.4, $\tau_1 < (1-\tau_2)^2$ is a *necessary* condition that an increase in τ_2 boosts economic growth under the BGP equilibrium. However, if tax rates satisfy a certain condition, then the economy under the BGP equilibrium may grow more rapidly as the tax rate for the parental education subsidies increase.

3.2 The BGP equilibrium where an old agent supplies no labor

Assume the parameters do not satisfy Equation (22), that is:

$$\begin{aligned} \chi &\leq \frac{\gamma(1+\bar{r})}{1+\sigma} \\ &+ \frac{\tau_1}{1-\tau_1-\tau_2} \frac{(1-\eta)\sigma}{q(1+\sigma)} \left(1+\frac{\chi}{1+\bar{r}}\right) \theta \left[\frac{q\eta(1-\tau_1-\tau_2)}{(1-\widehat{\phi}^*)(1-\eta)} \overline{w} \right]^{\eta} \left[1 - \frac{(1-\eta)\sigma}{1+\sigma+\gamma} \left(1+\frac{\chi}{1+\bar{r}}\right) \right]. \end{aligned} \tag{24}$$

In this case, an old agent supplies no labor. Therefore, inserting $l^o_{t+1} = 0$, Equations (8), and (9) into Equation (4), we obtain:

$$c^o_{t+1} = \frac{1+\bar{r}}{1+\sigma} \left[(1-\tau_1-\tau_2) h_t \overline{w} + \frac{P_{t+1}}{1+\bar{r}} \right]. \tag{25}$$

From Equations (8) and (9), we have:

$$n_t = \frac{\sigma(1-\eta)}{(1-\tau_1-\tau_2)q\overline{w}h_t}\frac{1}{1+\sigma}\left[(1-\tau_1-\tau_2)\overline{w}h_t + \frac{P_{t+1}}{1+\overline{r}}\right]. \tag{26}$$

Combining Equations (9), (25), and (26), we have:

$$e_t = \frac{\eta(1-\tau_1-\tau_2)q\overline{w}h_t}{(1-\phi_t)(1-\eta)}. \tag{27}$$

From the above equation, the GDP per capita growth rate at equilibrium is:

$$g_{t+1} = \frac{h_{t+1}}{h_t} - 1 = \theta\left[\frac{\eta(1-\tau_1-\tau_2)q\overline{w}}{(1-\phi_t)(1-\eta)}\right]^\eta - 1. \tag{28}$$

Under this equilibrium, Equations (7) and (27) lead to:

$$\phi_t = \frac{(1-\eta)\tau_2(1-qn_t)}{\eta(1-\tau_1-\tau_2)qn_t + (1-\eta)\tau_2(1-qn_t)}. \tag{29}$$

Therefore, Equation (28) can be rewritten as:

$$g_{t+1} = \theta\left[\frac{\eta(1-\tau_1-\tau_2)qn_t + (1-\eta)\tau_2(1-qn_t)}{n_t}\frac{\overline{w}}{1-\eta}\right]^\eta - 1. \tag{30}$$

Furthermore, from Equations (6) and (26), we have:

$$n_t = \frac{\sigma(1-\eta)}{q(1+\sigma)} + \frac{\sigma(1-\eta)\tau_1 n_t(1-qn_{t+1})}{(1-\tau_1-\tau_2)q(1+\sigma)(1+\overline{r})}\frac{h_{t+1}}{h_t}. \tag{31}$$

Lemma 3.1. *Under the BGP equilibrium where an old agent fully retires, the fertility rate must be constant, that is, for some $\widetilde{n}^* > 0$, $n_{t+1} = n_t = \widetilde{n}^*$ for all t.*

From this lemma, let $\widetilde{n}^*$ be the fertility rate under the BGP equilibrium. Then, $\widetilde{n}^*$ satisfies:

$$\widetilde{n}^* = \frac{\sigma(1-\eta)}{q(1+\sigma)} + \frac{\sigma(1-\eta)}{q(1+\sigma)}\frac{\tau_1\theta\widetilde{n}^*(1-q\widetilde{n}^*)}{(1+\overline{r})(1-\tau_1-\tau_2)}\left[\frac{\eta(1-\tau_1-\tau_2)q\widetilde{n}^* + (1-\eta)\tau_2(1-q\widetilde{n}^*)}{\widetilde{n}^*}\frac{\overline{w}}{1-\eta}\right]^\eta, \tag{32}$$

where Equation (30) is plugged into Equation (31).

Lemma 3.2. *There exists a unique $\widetilde{n}^* \in \left(\frac{\sigma(1-\eta)}{q(1+\sigma)}, \frac{1}{q}\right)$ that satisfies Equation (32).*

Substituting $\widetilde{n}^*$ into Equations (29), (30) and (6), under the BGP equilibrium:

$$\widetilde{\phi}^* = \frac{(1-\eta)\tau_2(1-q\widetilde{n}^*)}{\eta(1-\tau_1-\tau_2)q\widetilde{n}^* + (1-\eta)\tau_2(1-q\widetilde{n}^*)},$$

$$\widetilde{g}^* = \theta \left[\frac{\eta(1-\tau_1-\tau_2)q\widetilde{n}^* + (1-\eta)\tau_2(1-q\widetilde{n}^*)}{\widetilde{n}^*} \frac{\overline{w}}{1-\eta} \right]^{\eta} - 1, \tag{33}$$

and

$$\widetilde{P}^*_{t+1} = \tau_1 \overline{w}\widetilde{n}^*(1-q\widetilde{n}^*)h_{t+1}.$$

From Equation (25), under the BGP equilibrium, an old agent's consumption is:

$$\widetilde{c}^{o*}_{t+1} = \frac{1+\overline{r}}{1+\sigma}\overline{w}\left[(1-\tau_1-\tau_2) + \frac{\tau_1\widetilde{n}^*(1-q\widetilde{n}^*)}{1+\overline{r}}(1+\widetilde{g}^*)\right]h_{t+1},$$

and the parental investment for a child's human capital accumulation is:

$$\widetilde{e}^*_t = \frac{\eta(1-\tau_1-\tau_2)q\overline{w}h_t}{(1-\widetilde{\phi}^*)(1-\eta)}.$$

Proposition 3.5. *Assume the parameters satisfy Equation (24). Then, there exists a unique BGP equilibrium characterized by* $\widetilde{c}^{o*}_{t+1}, \widetilde{e}^*_t, \widetilde{n}^*, \widetilde{l}^{o*}(=0), \widetilde{P}^*_{t+1}$, $\widetilde{\phi}^*$, *and* $\widetilde{g}^*$. *Additionally,* $\frac{\widetilde{c}^*_{t+1}}{\widetilde{c}^*_t} = \frac{\widetilde{e}^*_{t+1}}{\widetilde{e}^*_t} = \frac{\widetilde{P}^*_{t+1}}{\widetilde{P}^*_t} = 1+\widetilde{g}^*$, *and* $\frac{\widetilde{n}^*_{t+1}}{\widetilde{n}^*_t} = \frac{\widetilde{\phi}^*_{t+1}}{\widetilde{\phi}^*_t} = 1$.

How does $\widetilde{n}^*$ change as tax rates change? Without general equilibrium effects, Equation (26) implies that an increase in tax rates increases the fertility rate. This is because the opportunity cost of having children decreases as tax rates increase. However, for general equilibrium effects, this may not be necessarily true, because of the effect of the pay-as-you-go social security benefits, P_{t+1}. These benefits depend on children's human capital. Since an increase in the tax rates decreases the disposable labor income tax, it can discourage a young agent from investing in his or her children, which decreases the pay-as-you-go social security benefits, thus reducing the number of children because a child is a normal good. Therefore, the effects of a change in the tax rate are not straightforward. However, when τ_1 is sufficiently low, we obtain the following results.

Proposition 3.6. *Assume the parameters satisfy Equation (24). Then, for a* τ_1 *sufficiently close to 0 but greater than 0, an increase in* τ_i *increases the fertility rate,* $\widetilde{n}^*$, *under the BGP equilibrium for all* $i = 1,2$. *Moreover, if* $\tau_1 = 0$, *an increase in* τ_1 *increases the fertility rate, whereas a change in* τ_2 *has no effect on the fertility rate.*

The detailed condition can be found in the proof of this proposition in the Appendix. One implication of this proposition is that the effect of τ_2 on the fertility rate depends on τ_1. Specifically, when $\tau_1 = 0$, a change in τ_2 has no effect on the fertility rate. Since $\tau_1 = 0$ implies no pension benefits, an agent has no incentive to have more children. Therefore, a change in τ_2 does not affect the fertility rate under the BGP equilibrium where an old agent fully retires. However, when $\tau_1 > 0$ but is sufficiently small, the opportunity cost of having more children decreases as τ_2 increases. Consequently, an increase in τ_2, which can be considered unrelated to the fertility rate, can actually increase this rate.

Next, we consider the effects of a small change in τ_1 and τ_2 on $\widetilde{g}^*$:

$$\widetilde{g}^* = \theta\left(\frac{\overline{w}}{1-\eta}\right)^{\eta}\left[\eta(1-\tau_1-\tau_2)q + \frac{(1-\eta)\tau_2(1-q\widetilde{n}^*)}{\widetilde{n}^*}\right]^{\eta} - 1.$$

First, an increase in τ_i can decrease the disposable labor income in young age, which reduces the amount of parental education subsidies. Second, a change in τ_2 can increase the parental education subsidies. Depending on which effect is stronger, we determine how the tax rates affect the growth rate under the BGP equilibrium.

Proposition 3.7. *Assume the parameters satisfy Equation (24). Then, an increase in τ_1 depresses the growth rate. Contrary to the case of τ_1, if τ_1 is sufficiently small, an increase in τ_2 boosts the growth rate.*

4 Discussion

4.1 Policy implications

Our analysis indicates that whether an old agent retires fully or partially depends on his or her labor productivity and tax rates. In order to change an agent's retirement decision, what a policy maker can do is set tax rates. Considering the fact that many developed countries face shrinking working-age populations, one idea to increase the number of workers is to reduce tax rates, which encourages old workers to work longer.

Regarding the effects of tax rates on fertility rate, how fertility rates change as tax rates change depends on whether an old agent retires fully or partially. When an old agent fully retires because his or her labor productivity is too low or tax rates are too high, the fertility rate will not change at all as tax rates change. Specifically, even if more educational subsidies are provided by raising tax rates, such a policy may not increase the fertility rate. In contrast, when an old agent works, an increase in tax rates can increase the fertility rate. To increase the fertility rate in this case, a policy maker should increase the tax rates, although such a policy discourages old workers from working longer. A policy maker should pay attention to this trade-off.

Regarding the effects of tax rates on growth rates, whether an old agent retires fully or partially does not matter. In either case, an increase in the pension tax rate lowers the growth rate whereas an increase in tax rates for educational subsidies increases the growth rate. Hence, if a policy maker wants to boost the growth rate, he or she should raise the tax rate for educational subsidies, provide more educational subsidies to young agents, and reduce the pension tax rate.

4.2 Effects of an old agent's labor productivity

In this section, we analyze the effects of an old agent's labor productivity on the fertility rate, retirement age, and growth rate of GDP per capita. To proceed with our analysis, we need to discuss the condition for χ above which an old agent works and below which he or she fully retires under the BGP equilibrium. For this purpose, we refer to Equation (22):

$$
\begin{aligned}
\chi &> \frac{\gamma(1+\bar{r})}{1+\sigma} \\
&+ \frac{\tau_1}{1-\tau_1-\tau_2}\frac{(1-\eta)\sigma}{q(1+\sigma)}\left(1+\frac{\chi}{1+\bar{r}}\right)\theta\left[\frac{q\eta(1-\tau_1-\tau_2)}{(1-\widehat{\phi}^*)(1-\eta)}\overline{w}\right]^{\eta}\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right]. \\
&= \frac{\gamma(1+\bar{r})}{1+\sigma}+\frac{\tau_1}{1-\tau_1-\tau_2}\frac{1+\sigma}{1+\sigma+\gamma}\widehat{n}^*(1-q\widehat{n}^*)(1+\widehat{g}^*).
\end{aligned} \tag{34}
$$

Since the second term in the right-hand side of Equation (34) is greater than or equal to 0 for all τ_1 and τ_2, if $\chi \leq \frac{\gamma(1+\bar{r})}{1+\sigma}$, an old agent retires fully under the BGP equilibrium. Therefore, when an old agent's labor productivity relative to a young agent's productivity is not sufficiently high, all results in Section 3.2 hold.

Let χ^* be a cutoff value of χ, above which an old agent works under the BGP equilibrium. That is, χ^* satisfies Equation (34) with equality. Since $\widehat{\phi}^*$ is an implicit function of χ, it is difficult to show the uniqueness of χ^*. However, under Assumption 3.1, we can prove the existence of χ^* in $(0,\overline{\chi})$.

Lemma 4.1. *There exists $\chi^* \in (0,\overline{\chi})$ so that, for all $\chi > \chi^*$, an old agent works under the BGP equilibrium.*

4.2.1 Fertility rate

First, we examine the relationship between the fertility rate and an old agent's labor productivity. We make the following statement:

Proposition 4.1.

1. *$\widehat{n}^*$ is increasing in χ.*
2. *$\widehat{n}^* > \widetilde{n}^*$ for all $\chi > \chi^*$.*

Since a child is a normal good in our model, an increase in the lifetime income due to an increase in χ incentivizes an agent to have more children.

One policy implication of this result is that letting an old agent choose his or her retirement timing might be a good idea to boost the fertility rate in developed countries, because old people in developed countries are healthier and more productive than before.

4.2.2 Retirement age

Next, we investigate how an old agent's retirement age changes with labor productivity. Taking the derivative of $\widehat{l^{o*}}$ with respect to χ, we obtain:

$$
\begin{aligned}
\frac{\partial \widehat{l^{o*}}}{\partial \chi} &= \frac{1}{1+\sigma+\gamma}\frac{1-\tau_1-\tau_2}{1-\tau_2}\frac{\gamma(1+\bar{r})}{\chi^2} \\
&+ \frac{(1-\eta)\sigma\theta}{(1+\sigma+\gamma)q}\frac{1}{\chi^2}\frac{\tau_1}{1-\tau_2}\left[\frac{q\eta(1-\tau_1-\tau_2)}{(1-\widehat{\phi}^*)(1-\eta)}\overline{w}\right]^\eta\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right], \\
&- \frac{(1-\eta)\sigma\theta}{(1+\sigma+\gamma)q}\left(\frac{1}{\chi}+\frac{1}{1+\bar{r}}\right)\frac{\tau_1}{1-\tau_2}\left[\frac{q\eta(1-\tau_1-\tau_2)}{(1-\widehat{\phi}^*)(1-\eta)}\overline{w}\right]^\eta\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right] \\
&\quad \times\frac{\eta}{1-\widehat{\phi}^*}\frac{\partial\widehat{\phi}^*}{\partial\chi}, \\
&+ \frac{(1-\eta)\sigma\theta}{(1+\sigma+\gamma)q}\left(\frac{1}{\chi}+\frac{1}{1+\bar{r}}\right)\frac{\tau_1}{1-\tau_2}\left[\frac{q\eta(1-\tau_1-\tau_2)}{(1-\widehat{\phi}^*)(1-\eta)}\overline{w}\right]^\eta\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\frac{1}{1+\bar{r}}. \qquad (35)
\end{aligned}
$$

From Equation (20):

$$
\frac{\partial\widehat{\phi}^*}{\partial\chi} = -\frac{\frac{\widehat{\phi}^*}{1-\widehat{\phi}^*}\frac{\eta\sigma}{1+\sigma+\gamma}\frac{1}{1+\bar{r}}+\frac{\tau_2}{1-\tau_2}\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\frac{1}{1+\bar{r}}-\frac{q}{(1-\eta)\sigma}\frac{(1+\bar{r})\tau_2}{1-\tau_2}\frac{(1+\bar{r})(1+\sigma+\gamma)}{(1+\bar{r}+\chi)^2}\frac{(1-\widehat{\phi}^*)^\eta}{\theta\left[\frac{q\eta(1-\tau_1-\tau_2)}{1-\eta}\overline{w}\right]^\eta}}{\frac{1}{(1-\widehat{\phi}^*)^2}\frac{\eta\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)+\frac{q}{(1-\eta)\sigma}\frac{1+\bar{r}}{1+\bar{r}+\chi}\frac{\tau_2}{1-\tau_2}[\chi(1+\sigma)-\gamma(1+\bar{r})]\frac{\eta(1-\widehat{\phi}^*)^{\eta-1}}{\theta\left[\frac{q\eta(1-\tau_1-\tau_2)}{1-\eta}\overline{w}\right]^\eta}}.
$$

To know the sign of Equation (35), we need to know the sign of $\frac{\partial\widehat{\phi}^*}{\partial\chi}$. However, it is difficult to analytically know the sign of $\frac{\partial\widehat{l^{o*}}}{\partial\chi}$. Therefore, we use a numerical simulation.

We provide numerical examples by calibrating parameter values to match the U.S. economy. The parameters are calibrated under the assumption that one period (generation) corresponds to 30 years. The production function is assumed to display a Cobb–Douglas form: $F(K,L) = K^\alpha L^{1-\alpha}$. The capital share of output, α, is set at $\frac{1}{3}$. Reflecting recent low interest rates, the interest rate is assigned to 1% per year. Since one period is 30 years, $1+\bar{r} = (1.01)^{30} = 1.348$. Given the interest rate, the capital per effective unit of labor (k) equals 0.123. This implies that the wage ratio equals 0.332. In Kitao (2014), the weight on leisure relative to consumption for old agents is $\gamma = 0.5123$. Following de la Croix and Doepke (2003), we assume that the weight on fertility relative to consumption for the young (σ) is 0.906.[5] The cost of raising one child is (q) set at 0.075. The elasticity of human capital with respect to education, η, is set to $\eta = 0.635$. In Kitao (2014), social security tax rate is set to 10.6%. In that case, τ_1 is set to 10.6%. Analogously, since the labor income tax rate is set to 22.1%, we set $\tau_2 = 11.5\%$. The productivity θ in the production function of human capital is set to 9.9 to match the annual growth rate 1%.[6]

Figure 2 (a) expresses how $\widehat{\phi}^*$ changes as χ changes. In this calibration, $\widehat{\phi}^*$ is constant when χ is low, and $\widehat{\phi}^*$ strictly decreases when χ is high. This implies $\frac{\partial\widehat{\phi}^*}{\partial\chi} < 0$. From Equation (35), $\frac{\partial\widehat{l^{o*}}}{\partial\chi} > 0$, as is shown in Figure 2 (b). Since one period is 30 years, an old agent will work at most for $0.05 \times 30 = 1.5$ years in this calibrated model.

[5] In de la Croix and Doepke (2003), the lifetime utility function is $\ln(c_t) + \beta\ln(d_{t+1}) + \sigma\ln(n_t h_{t+1})$. They set $\beta = (0.99)^{120} = 0.299$ and

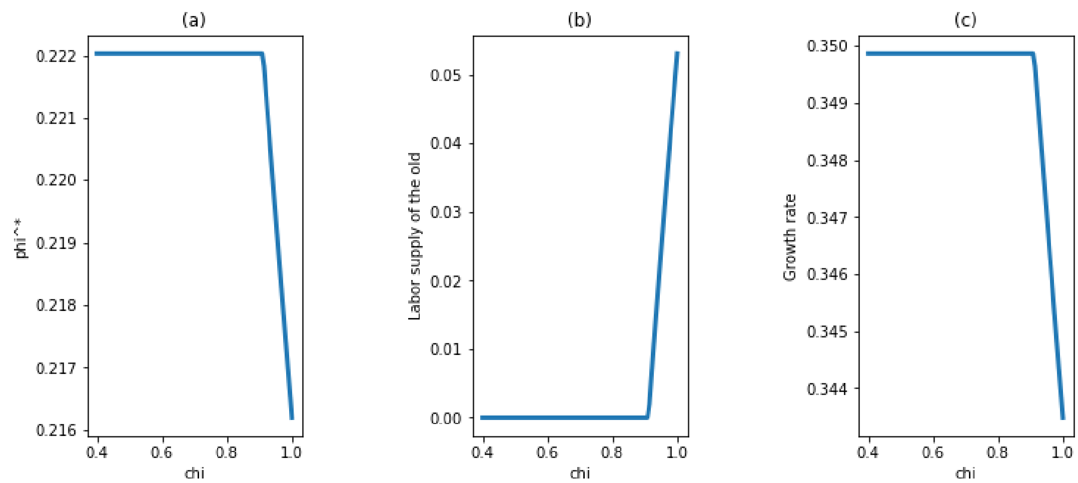

Figure 2: Simulation results

4.2.3 Growth rate of GDP per capita

Finally, we investigate the relationship between the GDP per capita growth rate and an old agent's labor productivity. Recall that the GDP per capita growth rate under the BGP equilibrium where an old agent fully retires is:

$$\widetilde{g}^* = \theta\left(\frac{\overline{w}}{1-\eta}\right)^{\eta}\left[\eta(1-\tau_1-\tau_2)q + \frac{(1-\eta)\tau_2(1-q\widetilde{n}^*)}{\widetilde{n}^*}\right]^{\eta} - 1.$$

Since an old agent fully retires, $\widetilde{g}^*$ does not depend on χ at all. Recall, also that the GDP per capita growth rate under the BGP equilibrium where an old agent works is:

$$\widehat{g}^* = \frac{h_{t+1}}{h_t} - 1 = \theta\left[\frac{q\eta(1-\tau_1-\tau_2)}{(1-\widehat{\phi}^*)(1-\eta)}\overline{w}\right]^{\eta} - 1.$$

As in the previous section, it is difficult to know analytically how $\widehat{\phi}^*$ changes with χ. In fact, $\widehat{g}^*$ can increase and decrease as χ increases. To prove this, we consider the following numerical examples.

Since $\frac{\partial\widehat{\phi}^*}{\partial\chi} < 0$, $\widehat{g}^*$ will decrease as χ increases, as is shown in Figure 2 (c). In this model, an agent faces a quantity-quality trade-off for children. Since higher χ incentivizes an agent to have more children, the education that parents give their children will be low. Therefore, as an old agent's labor productivity increases, the growth rate of GDP per capita will be lower.

We summarize the effects of an old agent's labor productivity on fertility, retirement age, and economic growth rate in Table 1.

$\sigma = 0.271$. Since $\beta = 1$ in our model, σ in our model is $\frac{0.271}{0.299} = 0.906$.

[6] In 30 years, the growth rate is $(1+0.01)^{30} = 1.348$, that is, 34.8%.

	Low χ	High χ
Retirement	Fully	Partially
Fertility rate	$\frac{\partial n^*}{\partial \tau_i} = 0$	$\frac{\partial \tilde{n}^*}{\partial \tau_i} > 0$
Growth rate	$\frac{\partial g^*}{\partial \tau_1} < 0$ $\frac{\partial g^*}{\partial \tau_2} > 0$	$\frac{\partial \tilde{g}^*}{\partial \tau_1} < 0$ $\frac{\partial \tilde{g}^*}{\partial \tau_2} > 0$

Table 1: Summary of equilibrium for different types of labor productivity of old agents

5 Concluding remarks

In this study, we consider the effects of a pay-as-you-go social security and parental education subsidies on the fertility rate, labor supply of old agents, and GDP per capita growth rate under an OLG model. We show the following: under a BGP equilibrium where an old agent supplies labor, the fertility rate is neutral to the tax rates and the labor supply of old agents decreases as the tax rates increase. However, when the pay-as-you-go social security tax rate is 0, the labor supply of old agents is neutral to the parental education subsidies. Further, the pay-as-you-go social security tax rate depresses the GDP per capita growth rate, while the tax rate of parental education subsidies accelerates it. Under a BGP equilibrium where an old agent fully retires, the tax rates can increase the fertility rate. The way in which the tax rates affect the GDP per capita growth rate is the same as under a BGP equilibrium where an old agent works. Additionally, we analytically and numerically show that an old agent's labor productivity changes the fertility rate, labor supply of old agents, and GDP per capita growth rate.

One of the implications of this study is that researchers should carefully decide whether an old agent endogenously determines his or her labor supply in an OLG model when the effects of government policies on economic variables are examined. This is because how economic variables change as government policies change depends on whether an old agent works at equilibrium. Another implication is that the effect of the tax rate for educational subsidies on the fertility rate, retirement age, and growth rate depends on the pay-as-you-go social security tax rate. Since many countries use these two policies, if the pay-as-you-go social security tax rate is not taken into account, then the effect of the educational subsidy can be incorrectly measured.

The arguments in this paper are based on a small open economy and, thus, we can ignore agents' savings or the interest rate, which helps us derive analytical results. However, it might be interesting to investigate how our results change in the case of a closed economy. Additionally, agents are homogeneous in this paper. However, in the real world, agents' labor productivity in old age differs among individuals. If one old agent's labor productivity is different from another's, then the tax neutrality result, Proposition 3.2, can be violated. We will leave this analysis for future research.

A Appendix

A.1 Proof of Proposition 3.2

Proof. First, we show there exists a unique $\widehat{\phi}^* \in (0,1)$ that satisfies Equation (19) with $\widehat{\phi}^* = \phi_{t-1} = \phi_t$.

Let $\widehat{LHS}(\phi_t)$ and $\widehat{RHS}(\phi_{t-1})$ be the left- and right-hand sides of Equation (19), respectively. Then, $\widehat{LHS}(\phi_t)$ is strictly increasing in ϕ_t, $\widehat{LHS}(0) = 0$, and $\lim_{\phi_t \to 1} \widehat{LHS}(\phi_t) = +\infty$. $\widehat{RHS}(\phi_{t-1})$ is strictly decreasing in ϕ_{t-1}, because $\chi > \frac{\gamma(1+\bar{r})}{1+\sigma}$ from Equation (22). Additionally, $\widehat{RHS}(1) = \frac{\tau_2}{1-\tau_2}\left[1 - \frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1 + \frac{\chi}{1+\bar{r}}\right)\right] \geq 0$ and is finite. This implies that $\widehat{RHS}(0) > 0$. These facts show there is a unique $\widehat{\phi}^* \in (0,1)$ so that $\widehat{LHS}(\phi^*) = \widehat{RHS}(\phi^*)$.

Given that $\widehat{\phi}^*$, $\widehat{l^{o*}}$ is uniquely determined by Equation (21). Therefore, there exists a unique BGP equilibrium where an old agent supplies strictly positive amount of labor. *Q.E.D.*

A.2 Proof of Proposition 3.3

Proof. Taking the derivative of $\widehat{l^{o*}}$ with respect to τ_1, we have:

$$
\begin{aligned}
\frac{\partial \widehat{l^{o*}}}{\partial \tau_1} &= -\frac{1}{1+\sigma+\gamma}\frac{1}{1-\tau_2}\left[1+\sigma-\frac{\gamma(1+\bar{r})}{\chi}\right] \\
&- \frac{(1-\eta)\sigma\theta}{(1+\sigma+\gamma)q}\left(\frac{1}{\chi}+\frac{1}{1+\bar{r}}\right)\frac{1}{1-\tau_2}\left[\frac{q\eta(1-\tau_1-\tau_2)}{(1-\widehat{\phi}^*)(1-\eta)}\overline{w}\right]^{\eta}\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right] \\
&+ \frac{(1-\eta)\sigma\theta}{(1+\sigma+\gamma)q}\left(\frac{1}{\chi}+\frac{1}{1+\bar{r}}\right)\frac{\tau_1}{1-\tau_2}\left(\frac{q\eta\overline{w}}{1-\eta}\right)^{\eta}\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right] \\
&\quad \times \eta\left(\frac{1-\tau_1-\tau_2}{1-\widehat{\phi}^*}\right)^{\eta-1}\left[-\frac{1}{1-\widehat{\phi}^*}+\frac{1-\tau_1-\tau_2}{(1-\widehat{\phi}^*)^2}\frac{\partial\widehat{\phi}^*}{\partial\tau_1}\right].
\end{aligned} \tag{36}
$$

To know the sign of $\frac{\partial \widehat{l^{o*}}}{\partial \tau_1}$, we need to know $\frac{\partial \widehat{\phi}^*}{\partial \tau_1}$.

Since $\widehat{\phi}^*$ satisfies:

$$
\begin{aligned}
\frac{\widehat{\phi}^* q\eta}{(1-\widehat{\phi}^*)(1-\eta)}(1-\tau_1-\tau_2)\widehat{n}^* &= \tau_2(1-q\widehat{n}^*) + \frac{1}{\widehat{n}^*}\frac{\tau_2}{1-\tau_2}\frac{1-\tau_1-\tau_2}{1+\sigma+\gamma}\left[\chi(1+\sigma)-\gamma(1+\bar{r})\right] \\
&\quad \times \frac{(1-\widehat{\phi}^*)^{\eta}}{\theta\left[\frac{q\eta(1-\tau_1-\tau_2)}{1-\eta}\overline{w}\right]^{\eta}} \\
&- \frac{1}{\widehat{n}^*}\frac{\tau_2}{1-\tau_2}\frac{(1-\eta)\sigma}{(1+\sigma+\gamma)q}\left(1+\frac{\chi}{1+\bar{r}}\right)\tau_1\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right],
\end{aligned} \tag{37}
$$

$$
\frac{\partial\widehat{\phi}^*}{\partial\tau_1} = \frac{\left\{\begin{array}{c}\frac{\widehat{\phi}^* q\eta}{(1-\widehat{\phi}^*)(1-\eta)}\widehat{n}^* - \frac{1}{\widehat{n}^*}\frac{\tau_2}{1-\tau_2}\frac{(1-\eta)(1-\tau_1-\tau_2)^{-\eta}}{1+\sigma+\gamma}\left[\chi(1+\sigma)-\gamma(1+\bar{r})\right]\frac{(1-\widehat{\phi}^*)^{\eta}}{\theta\left(\frac{q\eta}{1-\eta}\overline{w}\right)^{\eta}} \\ -\frac{1}{\widehat{n}^*}\frac{\tau_2}{1-\tau_2}\frac{(1-\eta)\sigma}{(1+\sigma+\gamma)q}\left(1+\frac{\chi}{1+\bar{r}}\right)\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right]\end{array}\right\}}{\frac{q\eta}{1-\eta}(1-\tau_1-\tau_2)\frac{\widehat{n}^*}{(1-\widehat{\phi}^*)^2} + \frac{1}{\widehat{n}^*}\frac{\tau_2}{1-\tau_2}\frac{(1-\tau_1-\tau_2)^{1-\eta}}{1+\sigma+\gamma}\left[\chi(1+\sigma)-\gamma(1+\bar{r})\right]\frac{\eta(1-\widehat{\phi}^*)^{\eta-1}}{\theta\left(\frac{q\eta}{1-\eta}\overline{w}\right)^{\eta}}}.
$$

From this:

$$
\begin{aligned}
\frac{1-\tau_1-\tau_2}{(1-\widehat{\phi}^*)^2}\frac{\partial\widehat{\phi}^*}{\partial\tau_1} &= \frac{1}{1-\widehat{\phi}^*}\frac{\left\{\begin{array}{c}\frac{\widehat{\phi}^* q\eta}{(1-\widehat{\phi}^*)(1-\eta)}\widehat{n}^* - \frac{1}{\widehat{n}^*}\frac{\tau_2}{1-\tau_2}\frac{(1-\eta)(1-\tau_1-\tau_2)^{-\eta}}{1+\sigma+\gamma}\left[\chi(1+\sigma)-\gamma(1+\bar{r})\right]\frac{(1-\widehat{\phi}^*)^\eta}{\theta\left(\frac{q\eta}{1-\eta}\overline{w}\right)^\eta} \\ -\frac{1}{\widehat{n}^*}\frac{\tau_2}{1-\tau_2}\frac{(1-\eta)\sigma}{(1+\sigma+\gamma)q}\left(1+\frac{\chi}{1+\bar{r}}\right)\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right]\end{array}\right\}}{\frac{q\eta}{1-\eta}\frac{\widehat{n}^*}{1-\widehat{\phi}^*}+\frac{1}{\widehat{n}^*}\frac{\tau_2}{1-\tau_2}\frac{(1-\tau_1-\tau_2)^{-\eta}}{1+\sigma+\gamma}\left[\chi(1+\sigma)-\gamma(1+\bar{r})\right]\frac{\eta(1-\widehat{\phi}^*)^\eta}{\theta\left(\frac{q\eta}{1-\eta}\overline{w}\right)^\eta}} \\
&< \frac{1}{1-\widehat{\phi}^*},
\end{aligned} \tag{38}
$$

where we use:

$$
\begin{aligned}
&\left\{\begin{array}{c}\frac{\widehat{\phi}^* q\eta}{(1-\widehat{\phi}^*)(1-\eta)}\widehat{n}^* - \frac{1}{\widehat{n}^*}\frac{\tau_2}{1-\tau_2}\frac{(1-\eta)(1-\tau_1-\tau_2)^{-\eta}}{1+\sigma+\gamma}\left[\chi(1+\sigma)-\gamma(1+\bar{r})\right]\frac{(1-\widehat{\phi}^*)^\eta}{\theta\left(\frac{q\eta}{1-\eta}\overline{w}\right)^\eta} \\ -\frac{1}{\widehat{n}^*}\frac{\tau_2}{1-\tau_2}\frac{(1-\eta)\sigma}{(1+\sigma+\gamma)q}\left(1+\frac{\chi}{1+\bar{r}}\right)\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right]\end{array}\right\} \\
<\;& \frac{\widehat{\phi}^* q\eta}{(1-\widehat{\phi}^*)(1-\eta)}\widehat{n}^* \\
<\;& \frac{q\eta}{1-\eta}\frac{\widehat{n}^*}{1-\widehat{\phi}^*}+\frac{1}{\widehat{n}^*}\frac{\tau_2}{1-\tau_2}\frac{(1-\tau_1-\tau_2)^{-\eta}}{1+\sigma+\gamma}\left[\chi(1+\sigma)-\gamma(1+\bar{r})\right]\frac{\eta(1-\widehat{\phi}^*)^\eta}{\theta\left(\frac{q\eta}{1-\eta}\overline{w}\right)^\eta}.
\end{aligned}
$$

Therefore, the last term in Equation (36) is $-\frac{1}{1-\widehat{\phi}^*}+\frac{1-\tau_1-\tau_2}{(1-\widehat{\phi}^*)^2}\frac{\partial\widehat{\phi}^*}{\partial\tau_1}<0$, which implies $\frac{\partial\widehat{l^{o*}}}{\partial\tau_1}<0$.

Next, taking the derivative of $\widehat{l^{o*}}$ with respect to τ_2, we have:

$$
\begin{aligned}
\frac{\partial\widehat{l^{o*}}}{\partial\tau_2} &= -\frac{\tau_1}{(1-\tau_2)^2}\frac{1}{1+\sigma+\gamma}\frac{\chi(1+\sigma)-\gamma(1+\bar{r})}{\chi} \\
&- \frac{(1-\eta)\sigma\theta}{(1+\sigma+\gamma)q}\left(\frac{1}{\chi}+\frac{1}{1+\bar{r}}\right)\frac{\tau_1}{(1-\tau_2)^2}\left[\frac{q\eta(1-\tau_1-\tau_2)}{(1-\widehat{\phi}^*)(1-\eta)}\overline{w}\right]^\eta\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right] \\
&- \frac{(1-\eta)\sigma\theta}{(1+\sigma+\gamma)q}\left(\frac{1}{\chi}+\frac{1}{1+\bar{r}}\right)\frac{\tau_1}{1-\tau_2}\left(\frac{q\eta\overline{w}}{1-\eta}\right)^\eta\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right] \\
&\times \eta\left(\frac{1-\tau_1-\tau_2}{1-\widehat{\phi}^*}\right)^{\eta-1}\left[-\frac{1}{1-\widehat{\phi}^*}+\frac{1-\tau_1-\tau_2}{(1-\widehat{\phi}^*)^2}\frac{\partial\widehat{\phi}^*}{\partial\tau_2}\right].
\end{aligned} \tag{39}
$$

From Equation (37):

$$
\frac{\partial\widehat{\phi}^*}{\partial\tau_2} = \frac{\left\{\begin{array}{c}\frac{\widehat{\phi}^* q\eta}{(1-\widehat{\phi}^*)(1-\eta)}\widehat{n}^*+(1-q\widehat{n}^*)+\frac{1}{\widehat{n}^*}\frac{(1-\tau_1-\tau_2)^{-\eta}[1-\tau_1-\tau_2-\tau_2(1-\tau_2)(1-\eta)]}{(1-\tau_2)^2}\frac{[\chi(1+\sigma)-\gamma(1+\bar{r})]}{1+\sigma+\gamma}\frac{(1-\widehat{\phi}^*)^\eta}{\theta\left(\frac{q\eta}{1-\eta}\overline{w}\right)^\eta} \\ -\frac{1}{\widehat{n}^*}\frac{1}{(1-\tau_2)^2}\frac{(1-\eta)\sigma}{(1+\sigma+\gamma)q}\left(1+\frac{\chi}{1+\bar{r}}\right)\tau_1\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right]\end{array}\right\}}{\frac{q\eta}{1-\eta}(1-\tau_1-\tau_2)\frac{\widehat{n}^*}{(1-\widehat{\phi}^*)^2}+\frac{1}{\widehat{n}^*}\frac{\tau_2}{1-\tau_2}\frac{(1-\tau_1-\tau_2)^{1-\eta}}{1+\sigma+\gamma}\left[\chi(1+\sigma)-\gamma(1+\bar{r})\right]\frac{\eta(1-\widehat{\phi}^*)^{\eta-1}}{\theta\left(\frac{q\eta}{1-\eta}\overline{w}\right)^\eta}}. \tag{40}
$$

Note that the denominator of Equation (40) is strictly positive. Therefore, $\frac{\partial\widehat{\phi}^*}{\partial\tau_2}>0$ if and only if the numerator of Equation (40) is strictly positive. Including $\widehat{n}^*$ into $(1-q\widehat{n}^*)-\frac{1}{\widehat{n}^*}\frac{1}{(1-\tau_2)^2}\frac{(1-\eta)\sigma}{(1+\sigma+\gamma)q}\left(1+\frac{\chi}{1+\bar{r}}\right)\tau_1\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right]$, we have:

$$\left[1-\frac{\tau_1}{(1-\tau_2)^2}\right]\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right].$$

Since $1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)>0$, $(1-q\widehat{n}^*)-\frac{1}{\widehat{n}^*}\frac{1}{(1-\tau_2)^2}\frac{(1-\eta)\sigma}{(1+\sigma+\gamma)q}\left(1+\frac{\chi}{1+\bar{r}}\right)\tau_1\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right]>0$ if and only if $\tau_1<(1-\tau_2)^2$. If $\tau_1<(1-\tau_2)^2$, the numerator of Equation (40) is strictly positive, which implies $\frac{\partial\widehat{\phi}^*}{\partial\tau_2}>0$. Given $\frac{\partial\widehat{\phi}^*}{\partial\tau_2}>0$, consider the last two terms in Equation (39), that is:

$$\begin{aligned}
& - \frac{(1-\eta)\sigma\theta}{(1+\sigma+\gamma)q}\left(\frac{1}{\chi}+\frac{1}{1+\bar{r}}\right)\frac{\tau_1}{(1-\tau_2)^2}\left[\frac{q\eta(1-\tau_1-\tau_2)}{(1-\widehat{\phi}^*)(1-\eta)}\overline{w}\right]^{\eta}\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right] \\
& - \frac{(1-\eta)\sigma\theta}{(1+\sigma+\gamma)q}\left(\frac{1}{\chi}+\frac{1}{1+\bar{r}}\right)\frac{\tau_1}{1-\tau_2}\left(\frac{q\eta\overline{w}}{1-\eta}\right)^{\eta}\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right] \\
& \quad \times\eta\left(\frac{1-\tau_1-\tau_2}{1-\widehat{\phi}^*}\right)^{\eta-1}\left[-\frac{1}{1-\widehat{\phi}^*}+\frac{1-\tau_1-\tau_2}{(1-\widehat{\phi}^*)^2}\frac{\partial\widehat{\phi}^*}{\partial\tau_2}\right] \\
= & -\frac{(1-\eta)\sigma\theta}{(1+\sigma+\gamma)q}\left(\frac{1}{\chi}+\frac{1}{1+\bar{r}}\right)\frac{\tau_1}{1-\tau_2}\left[\frac{q\eta}{(1-\widehat{\phi}^*)(1-\eta)}\overline{w}\right]^{\eta}(1-\tau_1-\tau_2)^{\eta-1} \\
& \quad \times\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right]\left(\frac{1-\tau_1-\tau_2}{1-\tau_2}-\eta\right) \\
& \quad -\frac{(1-\eta)\sigma\theta}{(1+\sigma+\gamma)q}\left(\frac{1}{\chi}+\frac{1}{1+\bar{r}}\right)\frac{\tau_1}{1-\tau_2}\left(\frac{q\eta\overline{w}}{1-\eta}\right)^{\eta}\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right]\eta\left(\frac{1-\tau_1-\tau_2}{1-\widehat{\phi}^*}\right)^{\eta-1} \\
& \quad \times\frac{1-\tau_1-\tau_2}{(1-\widehat{\phi}^*)^2}\frac{\partial\widehat{\phi}^*}{\partial\tau_2}.
\end{aligned}$$

These are strictly negative if $\frac{1-\tau_1-\tau_2}{1-\tau_2}>\eta$ or $\tau_1<(1-\eta)(1-\tau_2)$, given $\frac{\partial\widehat{\phi}^*}{\partial\tau_2}>0$. Since the first term in Equation (39) is strictly negative, if $0<\tau_1<\min\{(1-\tau_2)^2,(1-\eta)(1-\tau_2)\}$, then $\frac{\partial\widehat{l}^{o*}}{\partial\tau_2}<0$. Inserting $\tau_1=0$ into Equation (39), $\frac{\partial\widehat{l}^{o*}}{\partial\tau_2}=0$. *Q.E.D.*

A.3 Proof of Proposition 3.4

Proof. To investigate the effects of τ_1 and τ_2 on $\widehat{g}^*$, we focus on:

$$\widehat{G}^*:=\frac{1-\tau_1-\tau_2}{1-\widehat{\phi}^*}.$$

Taking the derivative of $\widehat{G}^*$ with respect to τ_1, we have:

$$\frac{\partial\widehat{G}^*}{\partial\tau_1} = -\frac{1}{1-\widehat{\phi}^*}+\frac{1-\tau_1-\tau_2}{(1-\widehat{\phi}^*)^2}\frac{\partial\widehat{\phi}^*}{\partial\tau_1}<0,$$

which was shown for Equation (38).

Taking the derivative of $\widehat{G}^*$ with respect to τ_2, we have:

$$\begin{aligned}
\frac{\partial \widehat{G}^*}{\partial \tau_2} &= -\frac{1}{1-\widehat{\phi}^*} + \frac{1-\tau_1-\tau_2}{(1-\widehat{\phi}^*)^2}\frac{\partial \widehat{\phi}^*}{\partial \tau_2} \\
&= \frac{1}{1-\widehat{\phi}^*} \\
&\times \left[-1 + \frac{\left\{ \begin{array}{c} \frac{\widehat{\phi}^* q\eta}{(1-\widehat{\phi}^*)(1-\eta)}\widehat{n}^* + (1-q\widehat{n}^*) \\ +\frac{1}{\widehat{n}^*}\frac{(1-\tau_1-\tau_2)^{-\eta}[1-\tau_1-\tau_2-\tau_2(1-\tau_2)(1-\eta)]}{(1-\tau_2)^2}\frac{[\chi(1+\sigma)-\gamma(1+\bar{r})]}{1+\sigma+\gamma}\frac{(1-\widehat{\phi}^*)^\eta}{\theta\left(\frac{q\eta}{1-\eta}\overline{w}\right)^\eta} \\ -\frac{1}{\widehat{n}^*}\frac{1}{(1-\tau_2)^2}\frac{(1-\eta)\sigma}{(1+\sigma+\gamma)q}\left(1+\frac{\chi}{1+\bar{r}}\right)\tau_1\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right] \end{array} \right\}}{\frac{q\eta}{1-\eta}\frac{\widehat{n}^*}{1-\widehat{\phi}^*}+\frac{1}{\widehat{n}^*}\frac{\tau_2}{1-\tau_2}\frac{(1-\tau_1-\tau_2)^{-\eta}}{1+\sigma+\gamma}[\chi(1+\sigma)-\gamma(1+\bar{r})]\frac{\eta(1-\widehat{\phi}^*)^\eta}{\theta\left(\frac{q\eta}{1-\eta}\overline{w}\right)^\eta}} \right].
\end{aligned}$$

Recall that, if $\tau_1 \geq (1-\tau_2)^2$, $(1-q\widehat{n}^*) - \frac{1}{\widehat{n}^*}\frac{1}{(1-\tau_2)^2}\frac{(1-\eta)\sigma}{(1+\sigma+\gamma)q}\left(1+\frac{\chi}{1+\bar{r}}\right)\tau_1\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right] \leq 0$. Furthermore, if $\tau_1 \geq (1-\tau_2)^2$:

$$\begin{aligned}
& \frac{1}{\widehat{n}^*}\frac{(1-\tau_1-\tau_2)^{-\eta}[1-\tau_1-\tau_2-\tau_2(1-\tau_2)(1-\eta)]}{(1-\tau_2)^2}\frac{[\chi(1+\sigma)-\gamma(1+\bar{r})]}{1+\sigma+\gamma}\frac{(1-\widehat{\phi}^*)^\eta}{\theta\left(\frac{q\eta}{1-\eta}\overline{w}\right)^\eta} \\
\leq\ & \frac{1}{\widehat{n}^*}\frac{\tau_2}{1-\tau_2}\frac{(1-\tau_1-\tau_2)^{-\eta}}{1+\sigma+\gamma}[\chi(1+\sigma)-\gamma(1+\bar{r})]\frac{\eta(1-\widehat{\phi}^*)^\eta}{\theta\left(\frac{q\eta}{1-\eta}\overline{w}\right)^\eta}.
\end{aligned}$$

Therefore, since $\widehat{\phi}^* < 1$:

$$\frac{\left\{ \begin{array}{c} \frac{\widehat{\phi}^* q\eta}{(1-\widehat{\phi}^*)(1-\eta)}\widehat{n}^* + (1-q\widehat{n}^*) + \frac{1}{\widehat{n}^*}\frac{(1-\tau_1-\tau_2)^{-\eta}[1-\tau_1-\tau_2-\tau_2(1-\tau_2)(1-\eta)]}{(1-\tau_2)^2}\frac{[\chi(1+\sigma)-\gamma(1+\bar{r})]}{1+\sigma+\gamma}\frac{(1-\widehat{\phi}^*)^\eta}{\theta\left(\frac{q\eta}{1-\eta}\overline{w}\right)^\eta} \\ -\frac{1}{\widehat{n}^*}\frac{1}{(1-\tau_2)^2}\frac{(1-\eta)\sigma}{(1+\sigma+\gamma)q}\left(1+\frac{\chi}{1+\bar{r}}\right)\tau_1\left[1-\frac{(1-\eta)\sigma}{1+\sigma+\gamma}\left(1+\frac{\chi}{1+\bar{r}}\right)\right] \end{array} \right\}}{\frac{q\eta}{1-\eta}\frac{\widehat{n}^*}{1-\widehat{\phi}^*}+\frac{1}{\widehat{n}^*}\frac{\tau_2}{1-\tau_2}\frac{(1-\tau_1-\tau_2)^{-\eta}}{1+\sigma+\gamma}[\chi(1+\sigma)-\gamma(1+\bar{r})]\frac{\eta(1-\widehat{\phi}^*)^\eta}{\theta\left(\frac{q\eta}{1-\eta}\overline{w}\right)^\eta}} < 1,$$

which leads to $\frac{\partial \widehat{G}^*}{\partial \tau_2} < 0$. Therefore, if a slight increase in τ_2 accelerates the growth rate under the BGP equilibrium, then $\tau_1 < (1-\tau_2)^2$ must be satisfied. *Q.E.D.*

A.4 Proof of Lemma 3.1

Proof. From Equation (31):

$$\frac{n_{t+1}}{n_t} = \frac{(1-\tau_1-\tau_2)\overline{w} + \frac{\tau_1\overline{w}n_{t+1}(1-qn_{t+2})}{1+\bar{r}}\frac{h_{t+2}}{h_{t+1}}}{(1-\tau_1-\tau_2)\overline{w} + \frac{\tau_1\overline{w}n_t(1-qn_{t+1})}{1+\bar{r}}\frac{h_{t+1}}{h_t}}$$

holds. Let $\widetilde{k} = \frac{n_{t+1}}{n_t}$ and $\widetilde{g} = \frac{h_{t+1}}{h_t}$ under the BGP equilibrium. Then, from the above equation:

$$\begin{aligned}
&\frac{n_{t+1}}{n_t} = \frac{(1-\tau_1-\tau_2)\overline{w} + \frac{\tau_1 \overline{w}\widetilde{k}n_t(1-q\widetilde{k}^2 n_t)}{1+\overline{r}}\widetilde{g}}{(1-\tau_1-\tau_2)\overline{w} + \frac{\tau_1 \overline{w}n_t(1-q\widetilde{k}n_t)}{1+\overline{r}}\widetilde{g}} = \widetilde{k}, \\
\Rightarrow \quad &(1-\tau_1-\tau_2)\overline{w} + \frac{\tau_1 \overline{w}\widetilde{k}n_t(1-q\widetilde{k}^2 n_t)}{1+\overline{r}}\widetilde{g} = \widetilde{k}\left[(1-\tau_1-\tau_2)\overline{w} + \frac{\tau_1 \overline{w}n_t(1-q\widetilde{k}n_t)}{1+\overline{r}}\widetilde{g}\right], \\
\Rightarrow \quad &(1-\widetilde{k})(1-\tau_1-\tau_2)\overline{w} = \frac{\tau_1 \overline{w}\widetilde{k}n_t}{1+\overline{r}}\widetilde{g}qn_t\widetilde{k}(\widetilde{k}-1)
\end{aligned}$$

must hold. Since $1-\tau_1-\tau_2 > 0$ and all parameters are strictly positive, this equation holds if and only if $\widetilde{k} = 1$. This completes the proof. *Q.E.D.*

A.5 Proof of Lemma 3.2

Proof. Since at equilibrium, $n_t > 0$ for all t, dividing both sides of Equation (31) by n_t, we have:

$$1 - \frac{\sigma(1-\eta)}{q(1+\sigma)}\frac{1}{n_t} = \frac{\sigma(1-\eta)}{1+\sigma}\frac{\tau_1\theta(1-qn_{t+1})}{(1+\overline{r})(1-\tau_1-\tau_2)q}\left[\frac{\eta(1-\tau_1-\tau_2)qn_t + (1-\eta)\tau_2(1-qn_t)}{n_t}\frac{\overline{w}}{1-\eta}\right]^{\eta}.$$

Moreover, dividing both sides by the last term in the right-hand side of the above equation, we obtain:

$$\frac{1 - \frac{\sigma(1-\eta)}{q(1+\sigma)}\frac{1}{n_t}}{\left[\eta(1-\tau_1-\tau_2)q + \frac{(1-\eta)\tau_2(1-qn_t)}{n_t}\right]^{\eta}} = \frac{\sigma(1-\eta)}{1+\sigma}\frac{\tau_1\theta(1-qn_{t+1})}{(1+\overline{r})(1-\tau_1-\tau_2)q}\left(\frac{\overline{w}}{1-\eta}\right)^{\eta}.$$

Let

$$\widetilde{LHS}(n_t) := \frac{1 - \frac{\sigma(1-\eta)}{q(1+\sigma)}\frac{1}{n_t}}{\left[\eta(1-\tau_1-\tau_2)q + \frac{(1-\eta)\tau_2(1-qn_t)}{n_t}\right]^{\eta}}$$

and

$$\widetilde{RHS}(n_{t+1}) := \frac{\sigma(1-\eta)}{1+\sigma}\frac{\tau_1\theta(1-qn_{t+1})}{(1+\overline{r})(1-\tau_1-\tau_2)q}\left(\frac{\overline{w}}{1-\eta}\right)^{\eta}.$$

Note that $\widetilde{LHS}(\frac{\sigma(1-\eta)}{q(1+\sigma)}) = 0$ and that $\widetilde{LHS}(\frac{1}{q}) = \frac{\frac{1+\sigma\eta}{1+\sigma}}{[\eta(1-\tau_1-\tau_2)q]^{\eta}} > 0$. Since the numerator of $\widetilde{LHS}(n_t)$ is strictly increasing in n_t and the denominator is strictly decreasing in n_t, $\widetilde{LHS}(n_t)$ is strictly increasing in n_t on $(0, \frac{1}{q})$. However, $\widetilde{RHS}(n_{t+1})$ is strictly decreasing in n_{t+1} on $(0, \frac{1}{q})$, $\widetilde{RHS}(\frac{\sigma(1-\eta)}{q(1+\sigma)}) = \frac{\sigma(1-\eta)}{1+\sigma}\frac{\tau_1\theta\frac{1+\sigma\eta}{1+\sigma}}{(1+\overline{r})(1-\tau_1-\tau_2)q}\left(\frac{\overline{w}}{1-\eta}\right)^{\eta} > 0$, and $\widetilde{RHS}(\frac{1}{q}) = 0$. Thus, there is a unique $\widetilde{n}^* \in (\frac{\sigma(1-\eta)}{q(1+\sigma)}, \frac{1}{q})$ such that $\widetilde{LHS}(\widetilde{n}^*) = \widetilde{RHS}(\widetilde{n}^*)$. *Q.E.D.*

A.6 Proof of Proposition 3.6

Proof. From Equation (32), we obtain:

$$\frac{\partial \widetilde{n}^*}{\partial \tau_1} = \frac{Z\frac{\widetilde{n}^*(1-q\widetilde{n}^*)}{1-\tau_1-\tau_2}\left[\eta(1-\tau_1-\tau_2)q+\frac{(1-\eta)\tau_2(1-q\widetilde{n}^*)}{\widetilde{n}^*}\right]^{\eta-1}\left[\begin{array}{c}\eta(1-\eta\tau_1-\tau_2)q\\ +\frac{(1-\eta)(1-\tau_2)\tau_2(1-q\widetilde{n}^*)}{(1-\tau_1-\tau_2)\widetilde{n}^*}\end{array}\right]}{1-Z\frac{\tau_1}{1-\tau_1-\tau_2}\left[\eta(1-\tau_1-\tau_2)q+\frac{(1-\eta)\tau_2(1-q\widetilde{n}^*)}{\widetilde{n}^*}\right]^{\eta-1}\left[\begin{array}{c}\eta(1-2q\widetilde{n}^*)(1-\tau_1-\tau_2)q\\ +\frac{(1-\eta)\tau_2(1-q\widetilde{n}^*)(1-\eta-2q\widetilde{n}^*)}{\widetilde{n}^*}\end{array}\right]}$$

and

$$\frac{\partial \widetilde{n}^*}{\partial \tau_2} = \frac{Z\frac{\tau_1(1-q\widetilde{n}^*)(1-\eta)}{1-\tau_1-\tau_2}\left[\eta(1-\tau_1-\tau_2)q+\frac{(1-\eta)\tau_2(1-q\widetilde{n}^*)}{\widetilde{n}^*}\right]^{\eta-1}\left[\eta+\frac{\tau_2(1-q\widetilde{n}^*)}{1-\tau_1-\tau_2}\right]}{1-Z\frac{\tau_1}{1-\tau_1-\tau_2}\left[\eta(1-\tau_1-\tau_2)q+\frac{(1-\eta)\tau_2(1-q\widetilde{n}^*)}{\widetilde{n}^*}\right]^{\eta-1}\left[\begin{array}{c}\eta(1-2q\widetilde{n}^*)(1-\tau_1-\tau_2)q\\ +\frac{(1-\eta)\tau_2(1-q\widetilde{n}^*)(1-\eta-2q\widetilde{n}^*)}{\widetilde{n}^*}\end{array}\right]},$$

where $Z := \frac{\sigma(1-\eta)}{1+\sigma}\frac{\theta}{(1+\bar{r})q}\left(\frac{\overline{w}}{1-\eta}\right)^{\eta}$. Since the numerators of both equations are positive, how the tax rates affect the fertility rate is determined by the signs of the denominators of both equations. That is, $\frac{\partial \widetilde{n}^*}{\partial \tau_i} > 0$ for $i = 1,2$ if and only if:

$$\begin{aligned} 1 \quad > \quad & Z\frac{\tau_1}{1-\tau_1-\tau_2}\left[\eta(1-\tau_1-\tau_2)q+\frac{(1-\eta)\tau_2(1-q\widetilde{n}^*)}{\widetilde{n}^*}\right]^{\eta} \\ & \times\frac{\eta(1-2q\widetilde{n}^*)(1-\tau_1-\tau_2)q+\frac{(1-\eta)\tau_2(1-q\widetilde{n}^*)(1-\eta-2q\widetilde{n}^*)}{\widetilde{n}^*}}{\eta(1-\tau_1-\tau_2)q+\frac{(1-\eta)\tau_2(1-q\widetilde{n}^*)}{\widetilde{n}^*}}. \end{aligned} \tag{41}$$

If τ_1 is sufficiently close to 0, the right-hand side of Equation (41) is sufficiently close to 0. Since it is continuous in τ_1, as long as τ_1 is sufficiently close to 0, Equation (41) holds, which implies that $\frac{\partial \widetilde{n}^*}{\partial \tau_i} > 0$ for all $i = 1,2$.

Specifically, when $\tau_1 = 0$, the numerator of $\frac{\partial \widetilde{n}^*}{\partial \tau_2} = 0$ for all τ_2, which implies that $\left.\frac{\partial \widetilde{n}^*}{\partial \tau_2}\right|_{\tau_1=0} = 0$. *Q.E.D.*

A.7 Proof of Proposition 3.7

Proof. Since $\widetilde{g}^*$ in Equation (33) is:

$$\widetilde{g}^* = \theta\left(\frac{\overline{w}}{1-\eta}\right)^{\eta}\left[\eta(1-\tau_1-\tau_2)q+\frac{(1-\eta)\tau_2(1-q\widetilde{n}^*)}{\widetilde{n}^*}\right]^{\eta}-1,$$

we focus on how

$$\widetilde{G}^* := \eta(1-\tau_1-\tau_2)q+\frac{(1-\eta)\tau_2(1-q\widetilde{n}^*)}{\widetilde{n}^*}$$

will change as tax rates change, instead of $\widetilde{g}^*$.

Taking the derivative of $\widetilde{G}^*$ with respect to τ_1, we have:

$$\frac{\partial \widetilde{G}^*}{\partial \tau_1} = -\eta q - (1-\eta)\tau_2 \frac{1}{(\widetilde{n}^*)^2} \frac{\partial \widetilde{n}^*}{\partial \tau_1}.$$

When the condition in Proposition 3.6 is satisfied, $\frac{\partial \widetilde{n}^*}{\partial \tau_1} > 0$. Therefore, $\frac{\partial \widetilde{G}^*}{\partial \tau_1} < 0$.

For τ_2, it is not as straightforward as the case of τ_1, because an increase in τ_2 can increase the parental education subsidies, which encourages an agent to invest in his or her children. Taking the derivative of $\widetilde{G}^*$ with respect to τ_2, we obtain:

$$\begin{aligned}\frac{\partial \widetilde{G}^*}{\partial \tau_2} &= -\eta q + (1-\eta)\frac{1-q\widetilde{n}^*}{\widetilde{n}^*} - (1-\eta)\tau_2 \frac{1}{(\widetilde{n}^*)^2}\frac{\partial \widetilde{n}^*}{\partial \tau_2} \\ &= -q + \frac{1-\eta}{\widetilde{n}^*}\left(1 - \frac{\tau_2}{\widetilde{n}^*}\frac{\partial \widetilde{n}^*}{\partial \tau_2}\right).\end{aligned}$$

We consider a special case here. Assume $\tau_1 = 0$ and $\chi \leq \frac{\gamma(1+\bar{r})}{1+\sigma}$. Then, $\widetilde{n}^* = \frac{\sigma(1-\eta)}{q(1+\sigma)}$ and $\frac{\partial \widetilde{n}^*}{\partial \tau_2} = 0$. Therefore, at $\tau_1 = 0$:

$$\frac{\partial \widetilde{G}^*}{\partial \tau_2} = -q + \frac{1-\eta}{\widetilde{n}^*} = \frac{1}{\widetilde{n}^*}\frac{1-\eta}{1+\sigma} > 0.$$

Since $\frac{\partial \widetilde{G}^*}{\partial \tau_2}$ is continuous in τ_1, when τ_1 is sufficiently close to 0, $\frac{\partial \widetilde{G}^*}{\partial \tau_2} > 0$. *Q.E.D.*

A.8 Proof of Lemma 4.1

Proof. Let

$$A(\chi) := \chi$$

and

$$B(\chi) := \frac{\gamma(1+\bar{r})}{1+\sigma} + \frac{\tau_1}{1-\tau_1-\tau_2}\frac{1+\sigma}{1+\sigma+\gamma}\widehat{n}^*(1-q\widehat{n}^*)(1+\widehat{g}^*).$$

First, since $\widehat{n}^*$ is continuous in χ,

$$\lim_{\chi\to\overline{\chi}} \widehat{n}^* = \frac{1}{q}.$$

Therefore, $\lim_{\chi\to\overline{\chi}}(1-q\widehat{n}^*) = 0$. Next, we consider $\lim_{\chi\to\overline{\chi}} \widehat{\phi}^*$. The first term on the right-hand side of Equation (20) goes to 0 as $\chi \to \overline{\chi}$. The left-hand side of Equation (20) and the second term on its right-hand side goes to some positive value as $\chi \to \overline{\chi}$. This implies that $\lim_{\chi\to\overline{\chi}} \widehat{\phi}^* < 1$. Hence, $\lim_{\chi\to\overline{\chi}}(1+\widehat{g}^*) < +\infty$. Therefore, $\lim_{\chi\to\overline{\chi}} B(\chi) = \frac{\gamma(1+\bar{r})}{1+\sigma}$. It is not difficult to verify $\overline{\chi}(= (1+\bar{r})\frac{1+\gamma+\eta\sigma}{(1-\eta)\sigma}) > \frac{\gamma(1+\bar{r})}{1+\sigma}$. Therefore, for a χ sufficiently close to $\overline{\chi}$, $A(\chi) > B(\chi)$. Since for $\chi < \frac{\gamma(1+\bar{r})}{1+\sigma}$, $A(\chi) < B(\chi)$, $A(\chi)$, and $B(\chi)$ are continuous in χ, there exists $\chi^* \in (0, \overline{\chi})$ so that $A(\chi^*) = B(\chi^*)$. From this argument, for all $\chi > \chi^*$, $A(\chi) > B(\chi)$ holds. *Q.E.D.*

A.9 Proof of Proposition 4.1

Proof. For 1, by differentiating $\widehat{n}^*$ with respect to χ, we have: $\frac{\partial \widehat{n}^*}{\partial \chi} = \frac{(1-\eta)\sigma}{q(1+\sigma+\gamma)} \frac{1}{1+\bar{r}} > 0$, which completes the proof.

For 2, when $\chi = \chi^*$, Equation (34) holds with equality. This means that, when $\chi = \chi^*$, under the BGP equilibrium, an old agent retires fully and the fertility rate and GDP per capita growth rate under the BGP equilibrium are $\widetilde{n}^*$ and $\widetilde{g}^*$, respectively. Therefore:

$$\chi^* = \frac{\gamma(1+\bar{r})}{1+\sigma} + \frac{\tau_1}{1-\tau_1-\tau_2} \frac{1+\sigma}{1+\sigma+\gamma} \widetilde{n}^*(1-q\widetilde{n}^*)(1+\widetilde{g}^*) \tag{42}$$

holds. Then:

$$\begin{aligned}
\widehat{n}^* &= \frac{(1-\eta)\sigma}{q(1+\sigma+\gamma)} \left(1+\frac{\chi}{1+\bar{r}}\right) > \frac{(1-\eta)\sigma}{q(1+\sigma+\gamma)} \left(1+\frac{\chi^*}{1+\bar{r}}\right) \\
&= \frac{(1-\eta)\sigma}{q(1+\sigma+\gamma)} \left[1+\frac{\gamma}{1+\sigma} + \frac{1}{1+\bar{r}} \frac{\tau_1}{1-\tau_1-\tau_2} \frac{1+\sigma+\gamma}{1+\sigma} \widetilde{n}^*(1-q\widetilde{n}^*)(1+\widetilde{g}^*)\right] \\
&= \frac{(1-\eta)\sigma}{q(1+\sigma)} \left[1+\frac{1}{1+\bar{r}} \frac{\tau_1}{1-\tau_1-\tau_2} \widetilde{n}^*(1-q\widetilde{n}^*)(1+\widetilde{g}^*)\right] = \widetilde{n}^*,
\end{aligned}$$

where the last equality is derived from Equations (32) and (33). *Q.E.D.*

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