Mechanical analyses and derivations of money velocity

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Abstract

The equation of exchange is derived from a standpoint encompassing the physics and economics thereof, whereby the maximisation of a money value function, increasing in real output and decreasing in the real money supply, while accounting for time and space, subjected to a money constraint, at the macroeconomic level, gives rise to an optimal level of real output thereby, expressing the liquidity demand coefficient as the inverse quotient of space over time. The fusion of such a liquidity demand coefficient expression with the money constraint, which is the equilibrium Cambridge equation, in turn gives rise to an equation for space, being the position of money, whose differentiation is precisely instantaneous money velocity and thence the exchange equation as presented by Fisher. The present analysis also derives money position on account of non-constant instantaneous money velocity as instantiated by Fisher, advancing a framework for the macroeconomy's general money value function and money constraint in the process. It likewise advances simulations of non-constant average and instantaneous money velocity, with a particular application to a stylised closed macroeconomy. It finally proceeds to remodel instantaneous money velocity through the use of ordinary differential equations (ODEs) for the money equations of motion, both generally, by letting the sum of the three equal a corrected exponential random walk with drift, and through a money force model, of free accumulation with financial assets resistance. This work thus remarks in sum that money velocity as customarily calculated, taught and understood is not univocal.

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Contents

1. Introduction

1.1 Exchange equation. 110 years ago Irving Fisher [1] formalised the eponymous exchange equation, which John Stuart Mill [8] and David Hume [3] had materialised before him, and thereby introduced the concept of money velocity, in a closed economy framework. Specifically, for positive nominal money supply M_S, prices p, money velocity v and real output y there arises the exchange equation together with one for

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money velocity $v: \ vM_S = py$, where $M_S, \ p, \ v, \ y \in \mathbb{R}_{++}$, whence $v = M_S^{-1} py = M_S^{-1} Y$. Moreover, real money supply $m_S = p^{-1} M_S$, whence money velocity $v = m_S^{-1} y$.

1.2 Cambridge equation. Such a $vM_S = py$ equation was thus born intrinsically classical, that is, at a time when Keynesianism had not yet arisen (or returned, if to have truly reprised corporativism and mercantilism). Keynesianism did not grow to contest it, however, but expanded on it from the standpoint of money demand, which essentially models demand for liquidity, in the broader framework of the Cambridge equation, due to Alfred Marshall [4], Arthur Cecil Pigou [10] and John Maynard Keynes [6]. More clearly, money demand m_D is envisaged as a positive fraction of real output $y: \ m_D = \kappa y$, where $\kappa \in (0, \ 1)$ and $m_D \in \mathbb{R}_{++}$; κ is specifically termed the money demand coefficient. Now, at equilibrium real money supply $m_S = m_D = \kappa y \longrightarrow y = \kappa^{-1} m_S$ such that money velocity $v = m_S^{-1} y = m_S^{-1} \kappa^{-1} m_S = \kappa^{-1}$.

Indeed, because money demand m_D increases in non-negative real interest rate r with deceleration, money being ever sought, money demand coefficient κ can be envisaged as an increasing and concave function thereof, dictating proportionality between inverse money velocity v^{-1} and real interest rate $r: \ \ \kappa : \mathbb{R}_+ \to \mathbb{R}_{++}$ such that $\kappa = \kappa(\overset{+-}{r})$, whence $v^{-1} \propto r$. Real interest rate r in turn decreases in real money supply m_S (debatably) at convexity and increases in money demand m_D at concavity: $r = r(\overset{-+}{m}_S, \ \overset{+-}{m}_D)$.

Strictly speaking, in fact, money demand m_D is envisaged to increase in real interest rate r as a partial inverse thereof, together with its coefficient κ, which also decreases at convexity in real output $y: \ m_D = m_D(\overset{++}{r}, \ \overset{+-}{m}_S)$, whence $\kappa = \kappa(\overset{+=}{m}_D, \ \overset{-+}{y}) = \kappa(\overset{++}{r}, \ \overset{+-}{m}_S, \ \overset{-+}{y})$, that is, $\kappa = y^{-1} m_D, \ \kappa_{m_D} = y^{-1}, \ \kappa_{m_D m_D} = 0, \ \kappa_y = -y^{-2} m_D$ and $\kappa_{yy} = 2y^{-3} m_D$. For completeness, real money supply $m_S = m_S(\overset{-+}{r}, \ \overset{+-}{m}_D)$, whence money demand coefficient $\kappa = \kappa(\overset{+=}{m}_S, \ \overset{-+}{y}) = \kappa(\overset{-+}{r}, \ \overset{+-}{m}_D, \ \overset{-+}{y})$, since at equilibrium money demand coefficient $\kappa = y^{-1} m_S$, all else equal.

By way of example, given parameter $\alpha \in (0, \ 1)$, money demand coefficient $\kappa = r^\alpha = y^{-1} m_D = y^{-1} m_S$ at equilibrium, whence real interest rate $r = \kappa^{\frac{1}{\alpha}} = (y^{-1} m_D)^{\frac{1}{\alpha}} = (y^{-1} m_S)^{\frac{1}{\alpha}}$; to be sure, $\kappa_r = \alpha r^{\alpha - 1}$ and $\kappa_{rr} = (\alpha^2 - \alpha) r^{\alpha - 2}$. Greater specificity accordingly suggests real interest rate $r = m_S^{-\alpha} m_D^{1-\alpha}$, given parameter $\alpha \in (0, \ 1)$, whence money demand $m_D = (r m_S^\alpha)^{\frac{1}{1-\alpha}} = \kappa y \longrightarrow \kappa = y^{-1} (r m_S^\alpha)^{\frac{1}{1-\alpha}}$ and real money supply $m_S = (r^{-1} m_D^{1-\alpha})^{\frac{1}{\alpha}} = \kappa y \longrightarrow \kappa = y^{-1} (r^{-1} m_D^{1-\alpha})^{\frac{1}{\alpha}}$ at equilibrium; to be sure, $r_{m_S} = -\alpha m_S^{-\alpha-1} m_D^{1-\alpha}, \ r_{m_S m_S} = (\alpha^2 + \alpha) m_S^{-\alpha-2} m_D^{1-\alpha}, \ r_{m_D} = (1 - \alpha) m_S^{-\alpha} m_D^{-\alpha}$ and $r_{m_D m_D} = (\alpha^2 - \alpha) m_S^{-\alpha} m_D^{-\alpha-1}$.

1.3 Money quantity theory. Nominal exchange equation $vM_S = Y$ became subsequently functional in classical economics' comeback against Keynesianism, led by Milton Friedman's [2] monetarism and the resurgence of the quantity theory of money.

The quantity theory of money specifically posits that increments in nominal money supply M_S do not affect real output y and real money supply m_S, forasmuch as correspondent to ones in prices p, which are therefore flexible in the short run as well; consequently, in nominal exchange equation $vM_S = Y$, all else equal, that which increases in nominal output Y are only prices $p: \ y = y(\overset{+}{m}_S), \ m_S = m_S(\overset{+}{M}_S, \ \overset{-}{p}), \ y_{m_S} m_{S_{M_S}} = 0$ and $m_{S_{M_S}} = 0$ because $\uparrow M_S = \uparrow p \bar{m}_S$, whereby $M_{S_t} = p_t$; thus, *ceteris paribus,* $\bar{v} \uparrow M_S = \bar{v} \uparrow p \bar{m}_S = \uparrow p \bar{y} = \uparrow Y$. In terms of equilibrium Cambridge equation $m_S = \kappa y$ money demand coefficient κ cannot vary either: $\bar{m}_S = \bar{\kappa} \bar{y}$.

Keynesian price rigidity by contrast holds that in the short run prices p are rigid and that increments in nominal money supply M_S positively affect real money supply m_S and real output y therethrough, which increments, all else equal, can in turn affect money velocity $v: \ y = y(\overset{+}{m}_S), \ m_S = m_S(\overset{+}{M}_S, \ \overset{-}{p})$ and $\uparrow M_S = \bar{p} \uparrow m_S$, whereby $M_{S_t} = m_{S_t}$, whence $y_{m_S} m_{S_{M_S}} > 0$ and $m_{S_{M_S}} > 0$; thus, *ceteris paribus,* (i) $\bar{v} \uparrow M_S = \bar{v} \bar{p} \uparrow m_S = \bar{p} \uparrow y = \uparrow Y$, whence $\bar{v} = \uparrow m_S^{-1} \uparrow y$, whereby $y_{M_S} = m_{S_{M_S}}$, (ii) $\uparrow v \uparrow M_S = \uparrow v \bar{p} \uparrow m_S = \bar{p} \uparrow y = \uparrow Y$, whence $\uparrow v = \uparrow m_S^{-1} \uparrow y$, whereby $y_{M_S} > m_{S_{M_S}}$, (iii) $\downarrow v \uparrow M_S = \downarrow v \bar{p} \uparrow m_S = \bar{p} \uparrow y = \uparrow Y$, whence $\downarrow v = \uparrow m_S^{-1} \uparrow y$, whereby $y_{M_S} < m_{S_{M_S}}$. In terms of equilibrium Cambridge equation $m_S = \kappa y$ money demand coefficient κ varies *qua* inverse of money velocity v, namely, in accordance with the relation between the effects of increments in nominal money supply M_S on real output y and real money supply $m_S: \ \kappa = v^{-1} = y^{-1} m_S$.

In fact, real output y is an increasing function of real money supply m_S and money demand m_D; alternatively, it is a decreasing function of prices p and an increasing function of nominal money supply M_S and money demand m_D. The latter in turn increases in positive autonomous money demand am_D and decreases in positive autonomous prices ap. Prices p similarly increase in autonomous prices ap, changed nominal money supply $\dot{M}_S$ and changed autonomous money demand $a\dot{m}_D$ and decrease in changed autonomous prices $\dot{ap}$, changed variables being all positive. Formally: $y = y(\overset{+}{m}_S,\ \overset{+}{m}_D)$, $m_D = m_D(a\overset{+}{m}_D,\ \overset{-}{ap})$ and $p = p(\overset{+}{\dot{ap}},\ \overset{+}{\dot{M}}_S,\ a\overset{+}{\dot{m}}_D,\ \overset{-}{\dot{ap}})$, where $am_D,\ ap,\ \dot{M}_S,\ a\dot{m}_D,\ \dot{ap} \in \mathbb{R}_{++}$. Autonomous money demand am_D decreases in positive demand taxation t_D, thereby increasing in demand subsidies, and increases in positive domestic demand d_y, discretionally proxied by confidence: $am_D = am_D(\overset{-}{t}_D,\ \overset{+}{d}_y)$, where $t_D,\ d_y \in \mathbb{R}_{++}$. Autonomous prices ap increase in supply taxation t_S, thereby decreasing in supply subsidies, capital return rk and real wages w and decrease in technology tc, exogenous variables being all positive: $ap = ap(\overset{+}{t}_S,\ \overset{+}{rk},\ \overset{+}{w},\ \overset{-}{tc})$, where $t_S,\ rk,\ w,\ tc \in \mathbb{R}_{++}$. It follows that exogenous variables $t_D,\ d_y,\ t_S,\ rk,\ w,\ tc,\ M_S$ are ultimately increasing functions of non-negative time $t:\ \ f = f(\overset{+}{t})$, for $f = t_D,\ d_y,\ t_S,\ rk,\ w,\ tc,\ M_S$, where $t \in \mathbb{R}_+$.

On such an account, Karl Marx [7] could be said to have held that increments in real output y ultimately due to autonomous money demand am_D, all else equal, correspond to ones in money velocity v, at constant prices $p:\ y_{m_D} m_{D_{am_D}}$ is such that, *ceteris paribus,* $\uparrow v\bar{M}_S =\uparrow v\bar{p}\bar{m}_S = \bar{p}\uparrow y =\uparrow Y$, whence $\uparrow v = \bar{m}_S^{-1}\uparrow y$. He should likewise have held that decrements in real output y and real money supply m_S ultimately due to autonomous prices ap, all else equal, correspond to varying money velocity v, at risen prices $p:\ y_p p_{ap}$ and $m_{S_p} p_{ap}$ are such that, *ceteris paribus,* (i) $\bar{v}\uparrow p\downarrow m_S =\uparrow p\downarrow y$, whence $\bar{v} =\downarrow m_S^{-1}\downarrow y$, whereby $y_p p_{ap} = m_{S_p} p_{ap}$, (ii) $\downarrow v\uparrow p\downarrow m_S =\uparrow p\downarrow y$, whence $\downarrow v =\downarrow m_S^{-1}\downarrow y$, whereby $y_p p_{ap} > m_{S_p} p_{ap}$, (iii) $\uparrow v\uparrow p\downarrow m_S =\uparrow p\downarrow y$, whence $\uparrow v =\downarrow m_S^{-1}\downarrow y$, whereby $y_p p_{ap} < m_{S_p} p_{ap}$, noting that $p_{ap} \gtreqless -y_p p_{ap}$ and $p_{ap} \gtreqless -M_{S_p} p_{ap}$ in order for Y and M_S to respectively rise, remain unvaried or fall.

Nominal exchange equation $vM_S = Y$ is therefore today an undisputed tenet of macroeconomics at large and money velocity v a regularly elaborated statistic of national accounting.

1.4 Literary derivation of the exchange equation. All non-axiomatic identities can be derived, that is, all identities can be derived which are not first principles. Now, nominal exchange equation $vM_S = Y$ is not deemed an axiomatic identity, but one of accountancy, yet precisely because regarded as an accounting identity does it lack a formal derivation. In other words, however axiomatic may its explanations be, micro-founded or not, they remain material, because of nominal exchange equation $vM_S = Y$'s accountancy status.

To be sure, Kenrick Hunte [5] did attempt to micro-found it, to derive it from an optimisation problem of representative agents, that is to say, but the only derivation from first principles was thereby that of real output y over real money supply m_S, in terms of wealth and parameters, not money velocity v, whose equality to real output y over real money supply m_S was taken as given.

For completeness, Marco Vinicio Monge Mora [9] micro-founded Cambridge equation $m_D = \kappa y$ more recently, perhaps though falling prey to potential extensions of the "Anything goes" theorem, whereby the excess demands of rational, utilitarian agents do not perforce originate the canonical excess demand in the aggregate.

1.5 Mechanical acceptation of the exchange equation. Be that as it may, the adoption of physical notions in economics is to this day unusual, being a metaphysical science, not only concerned with phenomena but also with noumena. The origins of Fisher's formal use of velocity in economics, building indeed upon Mill, Marx and Hume, are probably best therefore sought in the decisive years of his cultural formation, in the history of the economist.

Indeed, one of Fisher's doctoral advisors was the physicist Josiah Willard Gibbs; it thus appears appropriate to primitively envisage money velocity v in terms of money position and time t, that is, the time t employed by real money supply m_S to be exchanged for real output y, money being properly understood as potential output.

This article's main contribution is therefore that of a mechanical analysis and derivation of exchange equation $vm_S = y$ and money velocity v. Secondary contributions involve: the expression of money position as a function of money demand coefficient κ; constancy of money velocity v and thus of money demand coefficient κ and of real interest rate r, however unlifelike; the expression of money demand coefficient κ and of real interest rate r in terms of money position and time t.

The present analysis shall also derive money position on account of non-constant instantaneous money velocity as presented by Fisher, advancing a framework for the macroeconomy's general money value function and money constraint in the process. It shall likewise advance simulations of non-constant average and instantaneous money velocity, with a particular application to a stylised closed macroeconomy. It shall finally proceed to remodel instantaneous money velocity through the use of ODEs for the money equations of motion, both generally, by letting the sum of the three equal a corrected exponential random walk with drift, and through a money force model, of free accumulation with financial assets resistance.

2. Mechanical analysis of the exchange equation

2.1 Fisher formulation of the exchange equation. Fisher had initially formulated nominal exchange equation $vpm_S = py$ in terms of quantity q, measuring economic transactions; quantity q was specifically utilised *en lieu* of real output y, which would have subsumed quantity q with the advent of national accounting (itself undoubtedly exhibiting limitations analogous to those of quantity q's measurement). Consequently, nominal exchange equation $vpm_S = pq$ was not unidimensional, one of direct aggregates, that is to say, but featured a row vector of prices $\mathbf{p}^\top$ in order to match a column vector of quantity $\mathbf{q}: \ vM_S = \mathbf{p}^\top\mathbf{q} = Q$, where $\mathbf{p}^\top \in \mathbb{R}^{1\times n}_{++}$ and $\mathbf{q} \in \mathbb{R}^{n\times 1}_{++}$.

The reason for which nominal money supply M_S, alongside money velocity v, was instead unidimensional is its characterisation, in terms of price row vector $\mathbf{p}^\top$ and real money supply column vector $\mathbf{m}_S$, which records those commodities particularly employed as a medium of exchange: $M_S = \mathbf{p}^\top\mathbf{m}_S$, where $\mathbf{m}_S \in \mathbb{R}^{n\times 1}_{++}$.

The multidimensionality of Fisher's nominal exchange equation $v\mathbf{p}^\top\mathbf{m}_S = \mathbf{p}^\top\mathbf{q}$ therefore dictated money velocity v's computation in nominal terms, for real terms involve a price p inversion, which the price $\mathbf{p}^\top$ row vector excludes by definition: $v = M_S^{-1}Q$. The multidimensional computation of money velocity v would have been moreover impeded by real money supply column vector $\mathbf{m}_S$, which admits of no inversion either.

The advent of national accounting not only replaced quantity q with real output y, as seen, but allowed for the unidimensional treatment of nominal exchange equation $vpm_S = py$.

2.2 Retrospection. Once again, money velocity $v = m_S^{-1}y$. An attempted derivation could employ Monge Mora's micro-foundation of Cambridge equation $m_D = \kappa y$ and the subsequent equation of money velocity v with inverse money demand coefficient κ^{-1} at equilibrium, as viewed hereinbefore: $m_S = m_D = \kappa y \longrightarrow \kappa^{-1} = m_S^{-1}y$, through micro-foundations, and $v = \kappa^{-1} = m_S^{-1}y$; money velocity v would not have although originated from first principles. An alternative derivation of nominal exchange equation $vpm_S = py$ which may allow money velocity v to originate axiomatically could trace the inter-connexion between nominal money supply $M_S = pm_S$ and nominal equilibrium Cambridge equation $pm_S = \kappa py$.

Specifically, the latter derivation premultiplies nominal money supply M_S by non-negative parameter v, treated as arbitrary: $M_S = pm_S \longrightarrow vM_S = vpm_S$, where $v \in \mathbb{R}_+$. Nominal equilibrium Cambridge equation $M_S = \kappa Y$ is then rearranged to express nominal output py in terms of money demand coefficient κ and nominal money supply $pm_S: \ M_S = pm_S = \kappa Y = \kappa py \longrightarrow py = \kappa^{-1}pm_S$. Weighted nominal money supply vpm_S therefore equals nominal output py if and only if parameter v equals inverse money demand coefficient κ^{-1}, being thereby restricted to the positive real line: $vpm_S = py = \kappa^{-1}pm_S$ if and only if $v = \kappa^{-1}$, whence $v \in \mathbb{R}_{++}$.

Micro-foundations apart, the difference between the latter derivation and the former is that in the latter parameter v is not even envisaged as money velocity, but is left arbitrary, rendering it stronger. In sum, howbeit, neither one originates money velocity v from first principles.

2.3 Classical mechanics. A physical vector features magnitude and direction, whereas a physical scalar only features magnitude; a vectorial variation in space or position is called displacement and a scalar one distance. Correspondingly, velocity is a vectorial variation in space given one in time t and speed is a scalar variation in space given one in time t. The treatment of money velocity v therefore dictates the use of displacement and velocity hereby. Constant velocity is moreover constant speed in a constant direction, that is, in a straight line.

Tridimensional space $\mathbf{x}(t)$ is a function of unidimensional time t, whereby time t can be null: $\mathbf{x} \in \mathbb{R}^3_{++}$ and $t \in \mathbb{R}_+$ such that $\mathbf{x} : \mathbb{R}_+ \to \mathbb{R}^3_+$. For any object across space $\mathbf{x}(t)$ and time t, average velocity $\hat{\mathbf{v}}(t)$ is thus defined as the change in space $\Delta\mathbf{x}(t)$ given a change in time Δt, that is, a variation in position $\Delta\mathbf{x}(t)$ given a discrete one in time $\Delta t :\ \hat{\mathbf{v}}(t) := \frac{\Delta\mathbf{x}(t)}{\Delta t}$ such that $\hat{\mathbf{v}} : \mathbb{R}^4_+ \to \mathbb{R}^4_+$. Instantaneous velocity $\mathbf{v}(t)$ is therefrom defined as the change in space $d\mathbf{x}(t)$ given an infinitesimal change in time dt, that is, a variation in position $d\mathbf{x}(t)$ given a continuous one in time dt, being also understood as the object's continuing velocity if it stopped accelerating at a given instant: $\mathbf{v}(t) := \lim_{\Delta t \to 0} \frac{\Delta\mathbf{x}(t)}{\Delta t} = \frac{d\mathbf{x}(t)}{dt}$ such that $\mathbf{v} : \mathbb{R}^4_+ \to \mathbb{R}^4_+$. In both the discrete and the continuous case velocity is a function of tridimensional space $\mathbf{x}(t)$ and unidimensional time t and thus ultimately of the latter, whereby the change in space $\mathbf{x}(t)$ can be null, but not that in time t. Moreover, space $\mathbf{x}(t)$ curvature is instantaneous acceleration $\mathbf{a}(t)$, introduced hereinafter, and that of instantaneous velocity $\mathbf{v}(t)$ is instantaneous jerk $\mathbf{j}(t)$, whose signs depend on the space $\mathbf{x}(t)$ function. The Galilean definition of speed $S(t)$ as the quotient of distance $D(t)$ and time t is finally discrete and therefore equal to average speed $\hat{S}(t) :\ S(t) := \frac{D(t)}{t} \longleftrightarrow \hat{S}(t) = \frac{\Delta x(t)}{\Delta t}$.

Space $\mathbf{x}(t)$ is consequently the sum of changes in space $d\mathbf{x}(t)$ across indefinite continuous time t net of arbitrary constant C, namely, it is the sum of instantaneous velocities $\mathbf{v}(t)$ given infinitesimal changes in time t net of arbitrary constant $C :\ \forall C \in \mathbb{R}^3_+,\ \mathbf{x}(t) = \int d\mathbf{x}(t) - C = \int \mathbf{v}(t)dt - C = \int \mathbf{x}'(t)dt - C = \mathbf{x}(t) + C - C$. The sum of changes in space $d\mathbf{x}(t)$ across (i) definite continuous time t_1 and (ii) discrete time $t \equiv t_j$ is in turn (i) space $\mathbf{x}_j(t_1)$ and (ii) discrete displacement $\Delta\mathbf{x}(t) \equiv \Delta\mathbf{x}(t_j)$, respectively: $\mathbf{x}_1(t_1) = \int_0^1 d\mathbf{x}(t) = \int_0^1 \mathbf{v}(t)dt,\ \mathbf{x}_2(t_1) = \int_1^2 d\mathbf{x}(t) = \int_1^2 \mathbf{v}(t)dt$ etc. such that $\mathbf{x}(t) = \sum_{j=1}^{\infty} \mathbf{x}_j(t_1) = \sum_{j=1}^{\infty} \int_{j-1}^{j} d\mathbf{x}_j(t_1) = \sum_{j=1}^{\infty} \int_{j-1}^{j} \mathbf{v}(t)dt$ and, $\forall 0,\ j \in \mathbb{Z}_+,\ \Delta\mathbf{x}(t) \equiv \Delta\mathbf{x}(t_j) = \int_0^j d\mathbf{x}(t) = \int_0^j \mathbf{v}(t)dt = \int_0^j \mathbf{x}'(t)dt = \mathbf{x}(t)|_0^j = \mathbf{x}(j) - \mathbf{x}(0) = \mathbf{x}_j - \mathbf{x}_0$. Summing changes in space $d\mathbf{x}(t)$ across discrete time $t \equiv t_j$ is indubitably counter-intuitive, because of the association of instantaneous velocities $\mathbf{v}(t)$ given infinitesimal changes in time t with discrete changes in time t, but they are only nominal, short of a contradiction in terms: in reality such velocities are average, precisely because of the discrete changes in time t. Average velocity $\hat{\mathbf{v}}(t)$ is consequently discrete displacement $\Delta\mathbf{x}(t) \equiv \Delta\mathbf{x}(t_j)$ weighted at the inverse change in discrete time $t \equiv t_j :\ \forall 0,\ j \in \mathbb{Z}_+,\ \hat{\mathbf{v}}(t) := \frac{\Delta\mathbf{x}(t)}{\Delta t} \equiv \frac{\Delta\mathbf{x}(t_j)}{\Delta t_j} = (j-0)^{-1} \int_0^j \mathbf{v}(t)dt$.

Derivatively, average acceleration $\hat{\mathbf{a}}(t)$ is defined as the change in average velocity $\Delta\hat{\mathbf{v}}(t)$ given a change in time Δt and instantaneous acceleration $\mathbf{a}(t)$ is defined as the change in instantaneous velocity $d\mathbf{v}(t)$ given an infinitesimal change in time $dt :\ \hat{\mathbf{a}}(t) := \frac{\Delta\hat{\mathbf{v}}(t)}{\Delta t} = \frac{\Delta^2\mathbf{x}(t)}{\Delta t^2}$ and $\mathbf{a}(t) := \lim_{\Delta t \to 0} \frac{\Delta\hat{\mathbf{v}}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta^2\mathbf{x}(t)}{\Delta t^2} = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{x}(t)}{dt^2}$. An object's momentum $\mathbf{M}(t)$ is therefrom the product of its mass m and instantaneous velocity $\mathbf{v}(t)$ and the net force $\mathbf{F}(t)$ acting upon it is a product of its mass m and instantaneous acceleration $\mathbf{a}(t)$, being Isaac Newton's Second law of motion: $\mathbf{M}(t) := m\mathbf{v}(t) = m\mathbf{x}'(t)$ and $\mathbf{F}(t) := m\mathbf{a}(t) = \mathbf{M}'(t) = m\mathbf{v}'(t) = m\mathbf{x}''(t)$. Newton's First law of motion is that of inertia, whereby velocity $\mathbf{v}(t)$ is constant, dictating that the net force $\mathbf{F}(t)$ acting upon an object with positive mass m is null if and only if its acceleration $\mathbf{a}(t)$ is null: $\forall m \in \mathbb{R}_{++},\ \mathbf{F}(t) = m\mathbf{a}(t) = \mathbf{0}$ if and only if $\mathbf{a}(t) = \mathbf{0}$, whereby $\mathbf{v}(t) = \mathbf{k},\ \forall \mathbf{k} \in \mathbb{R}^4_{++}$. Newton's third law of motion is finally known as "Action reaction", whereby the force object A exerts on object B is neutralised by one reciprocal to it: $\mathbf{F}_{A/B}(t) = -\mathbf{F}_{B/A}(t) \longrightarrow \mathbf{F}_{A/B}(t) + \mathbf{F}_{B/A}(t) = 0$.

2.4 Instantaneous money velocity and instantaneous money acceleration. As seen, national accounting allows for the unidimensional treatment of nominal exchange equation $vpm_S = py$. Money velocity $v = m_S^{-1}y$ and its derivations are therefrom envisaged as instantaneous, considering infinitesimal changes in time t to the end of furthering specificity. For simplicity, bold vectorial notation is relinquished throughout.

If the considered object is real money supply m_S, that is, money, then money position or space $x(t)$ is the indefinite integral of instantaneous money velocity $v(t)$ with respect to time t net of arbitrary constant C, namely, the sum of changes in money position or space $dx(t)$ across indefinite continuous time $t :\ x(t) = \int dx(t) - C = \int v(t)dt - C = \int_k^t v(t)dt - C = \int x'(t)dt - C = \int_k^t x'(t)dt - C = \int_k^t m^{-1}ydt - C = x(t)|_k^t - C = [x(t) - x(k)] - C = x(t) + C - C = m_S^{-1}yt|_k^t - C = (m_S^{-1}yt - m_S^{-1}yk) - C = m_S^{-1}yt + C - C,\ \forall k \in \mathbb{R}_+$, since $x : \mathbb{R}_+ \to \mathbb{R}^3_+$. Specifically, money position $x(t)$ is real output y over real money supply m_S weighted at time t, namely, instantaneous money velocity $v(t)$ weighted at time $t :\ x(t) = m_S^{-1}yt = v(t)t$.

Instantaneous money velocity $v(t)$ is therefrom the quotient of money position $x(t)$ and time t, in turn equivalent to that of real output y and real money supply $m_S :\ v(t) = t^{-1}x(t) = t^{-1}m_S^{-1}yt = m_S^{-1}y$.

Indeed, instantaneous money velocity $v(t)$ is a function of (i) unidimensional positive real output y, real money supply m_S and time t or, reflectively, of (ii) unidimensional positive time t and tridimensional positive space $x(t)$, inclusive of time $t: \ v: \mathbb{R}^3_{++} \to \mathbb{R}_{++}$, since $y, \ m_S, \ t \in \mathbb{R}_{++}$ in domain $\mathbb{R}^3_{++}$ and suitably in codomain $\mathbb{R}_{++}$. Time t is particularly positive forasmuch as figuring inversely, whence follows instantaneous money velocity $v(t)$'s positivity.

Instantaneous money acceleration $a(t)$, the instantaneous money velocity $v(t)$ derivative with respect to time t, that is to say, is moreover null, alongside money force $F(t): \ a(t) = v'(t) = 0$, whereby $F(t) = m \cdot 0 = 0$. The implicit assumption, which was that of Fisher, is that real output y over real money supply m_S is a constant function of time $t: \ m_S^{-1}(t)y(t) = m_S^{-1}y$; such is however false, as discussed hereinafter. Such in fact implies that instantaneous money velocity $v(t)$ and money direction are constant, money travelling along a straight path: $\int a(t)dt - C = \int v'(t)dt - C = v(t) + C - C = \int 0dt - C = K - C, \ \forall K, \ C \in \mathbb{R}_+$, whence $v(t) = K - C = k$. The path money travels is straight not because it does not circulate, but because all economic transactions are intuited as being situated along a rectilinear trajectory, whereby money stops travelling in each transaction as it is exchanged for consumable output, to then recommence.

The constancy of instantaneous money velocity $v(t)$ in turn implies that of money position $x(t)$ and time t and thence of real output y and real money supply m_S or of offsetting changes therein; the constancy or offsetting change of money position $x(t)$ therefrom confirms that of the other variables or, respectively, of suitable changes therein: $v(t) = k = t^{-1}x(t) = m_S^{-1}y$, whereby (i) $\bar{v}(t) = \bar{t}^{-1}\bar{x}(t) = \bar{m}_S^{-1}\bar{y}$ and $\bar{x}(t) = \bar{m}_S^{-1}\bar{y}\bar{t}$, (ii) $\bar{v}(t) =\uparrow t^{-1} \uparrow x(t) =\uparrow m_S^{-1} \uparrow y$ and $\uparrow x(t) =\uparrow m_S^{-1} \uparrow y \uparrow t$ or (iii) $\bar{v}(t) =\downarrow t^{-1} \downarrow x(t) =\downarrow m_S^{-1} \downarrow y$ and $\downarrow x(t) =\downarrow m_S^{-1} \downarrow y \downarrow t$.

Cases (ii) and (iii) are however specious, for changes in real output y and real money supply m_S ultimately signify that real output y over real money supply m_S is a non-constant function of time t and the definitions of instantaneous money velocity $v(t)$ and money position $x(t)$ are perforce altered as a result: $m_S^{-1}(t)y(t) \neq m_S^{-1}y$, whereby $v(t) = m_S^{-1}(t)y(t)$ and $x(t) = \int v(t)dt - C = \int m_S^{-1}(t)y(t)dt - C$.

2.5 Money position and money demand coefficient. Instantaneous money velocity $v(t) = \kappa^{-1}$, on account of equilibrium Cambridge equation $m_S = \kappa y$, as viewed hereinbefore, dictating money position $x(t)$ as inverse money demand coefficient κ^{-1} weighted at time $t: \ v(t) = t^{-1}x(t) = t^{-1}m_S^{-1}yt = t^{-1}\kappa^{-1}t = \kappa^{-1}$, whereby $v: \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ in terms of domain elements κ and t, and $x(t) = v(t)t = m_S^{-1}yt = \kappa^{-1}t$.

Consequently, just as constant instantaneous money velocity $v(t)$ decreases in money demand coefficient κ at an increasing rate so does money position $x(t)$, increasing at inflexion in time $t: \ v = v(\overset{-+}{\kappa}, \ \overset{=}{t})$ and $x = x(\overset{-+}{\kappa}, \ \overset{+=}{t})$, since $v_\kappa = -\kappa^{-2}, \ v_{\kappa\kappa} = 2\kappa^{-3}, \ x_\kappa = v_\kappa t, \ x_{\kappa\kappa} = v_{\kappa\kappa}t, \ x_t = \kappa^{-1}$ and $x_{tt} = v_t = 0$.

Instantaneous money velocity $v(t) = t^{-1}x(t) = \kappa^{-1}$ also implies inverse instantaneous money velocity $v^{-1}(t) = x^{-1}(t)t = \kappa$, as does money position $x(t) = v(t)t = \kappa^{-1}t$. Wherefore, money demand coefficient κ is ultimately a constant function of time $t: \ \kappa = x^{-1}(t)t = v^{-1}(t) = y^{-1}m_S = y^{-1}(t)m_S(t)$, whence $\kappa(t) = y^{-1}m_S$ and $\kappa_t = 0$.

2.6 Real interest rate. Now, since money demand coefficient $\kappa(t)$ is an increasing and concave function of real interest rate r it follows that inverse instantaneous money velocity $v^{-1}(t)$ behaves in the same way: $\kappa(\overset{+-}{r}) = \kappa(t) = v^{-1}(t)$. Consequently, real interest rate r is a κ^{-1} function of money demand coefficient $\kappa(t)$ if and only if function $\kappa(r)$ is bijective; real interest rate r is thereby also a κ^{-1} function of (i) inverse instantaneous money velocity $v^{-1}(t)$, of (ii) time t over money position $x(t)$ and of (iii) real money supply m_S over real output $y: \ r = \kappa^{-1}[\kappa(t)] = \kappa^{-1}[v^{-1}(t)] = \kappa^{-1}[x^{-1}(t)t] = \kappa^{-1}(y^{-1}m_S)$ if and only if $\kappa: \mathbb{R}_+ \to \mathbb{R}_{++}$, in terms of domain element $\kappa(t)$, is bijective.

Forasmuch as money demand coefficient $\kappa(t)$ is an ultimate function of time t so is real interest rate $r: \ r = \kappa^{-1}[\kappa(t)] = \kappa^{-1}[v^{-1}(t)] = \kappa^{-1}[x^{-1}(t)t] = \kappa^{-1}(y^{-1}m_S) = \kappa^{-1}[y^{-1}(t)m_S(t)]$, whence $r(t) = \kappa^{-1}[\kappa(t)]$. Yea, real interest rate $r(t) = \kappa^{-1}(y^{-1}m_S) = \kappa^{-1}(y^{-1}m_D)$, on account of equilibrium Cambridge equation $m_S = m_D = \kappa y$. For time t variant real output y, real money supply m_S and money demand m_D, real interest rate $r(t)$ therefore decreases in real output $y(t)$ insofar as money demand coefficient $\kappa(t)$ may decrease therein too, money demand $m_D(t)$ remaining constant, and *vice versa:* provided $y(t), \ m_S(t), \ m_D(t) \neq y, \ m_S, \ m_D, \downarrow r(t) = \kappa^{-1}[\downarrow \kappa(t)] = \kappa^{-1}[\uparrow y^{-1}(t)\bar{m}_D(t)]$, since $\bar{m}_D(t) =\downarrow \kappa(t) \uparrow y(t)$, and $\uparrow r(t) = \kappa^{-1}[\uparrow \kappa(t)] = \kappa^{-1}[\bar{y}^{-1}(t) \uparrow m_D(t)]$, since $\uparrow m_D(t) =\uparrow \kappa(t)\bar{y}(t)$.

Constant $k = v(t) = \kappa^{-1}(t)$ finally dictates inverse constant $k^{-1} = v^{-1}(t) = \kappa(t) = \kappa[r(t)]$ and, for a bijective function $\kappa[r(t)]$, real interest rate $r(t) = \kappa^{-1}(k^{-1})$, namely, constant real interest rate $r(t)$. The implications of a non-accelerating real interest rate $r(t)$ would be methodological, if only; specifically, on accepting instantaneous money velocity $v(t)$ as presented by Fisher the time t derivative of real interest rate $r(t)$ must be zero: $r(t) = \kappa^{-1}[\kappa(t)] = \kappa^{-1}[v^{-1}(t)] = \kappa^{-1}(k^{-1})$, whereby $r_t = 0$. Otherwise seen: instantaneous money velocity $v(t)$ as presented by Fisher is constant; now, instantaneous money velocity $v(t)$ is also inverse money demand coefficient $\kappa^{-1}(t)$ and such is a decreasing and concave κ^{-1} function of real interest rate $r(t)$; therefore, whenever such a $\kappa^{-1}[r(t)]$ function may be invertible real interest rate $r(t)$ is constant. Formally: $v(t) = k = \kappa^{-1}(t) = \kappa^{-1}[\overline{\overline{r(t)}}]$, whence $r(t) = \kappa(k)$, for bijective $\kappa^{-1} : \mathbb{R}_{+} \rightarrow \mathbb{R}_{++}$.

As seen, however, instantaneous money velocity $v(t)$ can theoretically feature a non-zero time t derivative, owing to both real output y and real money supply m_S as non-constant functions of time t, allowing real interest rate $r(t)$ to vary too, as expected: $v_y y_t$ or $v_{m_S} m_{S_t} \neq 0$ and $r_y y_t$ or $r_{m_S} m_{S_t} \neq 0$ are normatively possible, provided $v(t) = m_S^{-1}(t)y(t) \neq m_S^{-1}y$ and $r(t) = \kappa[\kappa^{-1}(t)] = \kappa[v(t)] = \kappa[m_S^{-1}(t)y(t)] \neq \kappa(m_S^{-1}y)$.

2.7 Applications. By way of example, observing that average velocity $\hat{\mathbf{v}}(t) := \lim_{dt\to\infty} \frac{d\mathbf{x}(t)}{dt} = \frac{\Delta\mathbf{x}(t)}{\Delta t}$, average money velocity $\hat{v}(t)$ and its derivations can be examined: $\hat{v}(t) := \lim_{dt\to\infty} \frac{dx(t)}{dt} = \frac{\Delta x(t)}{\Delta t} = \lim_{dt\to\infty} m_S^{-1}y = m_{St}^{-1}y_t \longrightarrow \Delta x(t) = \hat{v}(t)\Delta t = m_{St}^{-1}y_t\Delta t$, whence $v_{1t} = \hat{v}(t_1) = \Delta t_1^{-1}\Delta x(t_1) = [t-(t-1)]^{-1}[x(t) - x(t-1)] = x_t - x_{t-1} = m_{St}^{-1}y_t$ etc. The gist of such a discretisation is that real output y and real money supply m_S are assumed to be white noises, whereby white noises under continuousness must be indexed by time t under discreteness, varying thereat. The white noise realisation normally envisaged under discrete time t is in fact *una tantum,* being null at all other temporal periods, varying no longer, thereby mimicking its fixation once and for all under the continuous equivalent.

More clearly: $y,\ m_S = \varepsilon_1,\ \varepsilon_2 \sim \mathcal{N}(0,\ \sigma^2)$, thus, $\Delta x(t_1) = \int_{t-1}^{t} x'(t)dt = x(t)|_{t-1}^{t} = x(t) - x(t-1) = x_t - x_{t-1} = \int_{t-1}^{t} m_S^{-1}ydt = m_S^{-1}yt|_{t-1}^{t} = m_S^{-1}y(t-t+1) = m_S^{-1}y = m_{St}^{-1}y_t = v_{1t}(t-t+1) = v_{1t} = \hat{v}(t_1)(t-t+1) = \hat{v}(t_1) \longrightarrow x_t = x_{t-1} + m_{St}^{-1}y_t = x_{t-1} + v_{1t}$, where $y_t,\ m_{St} = \varepsilon_{1t},\ \varepsilon_{2t} \sim \mathcal{N}(0,\ \sigma^2)$; recapitulating, $v_{1t} = (t-t+1)^{-1}(x_t - x_{t-1}) = x_t - x_{t-1} = m_{St}^{-1}y_t = \varepsilon_{2t}^{-1}\varepsilon_{1t}$. On the other hand, in order for the discretisation of other time t variant variables to occur their analytical form must be cognised and their integrals accordingly evaluated; if real output y and real money supply m_S are not hereby assumed to be white noises, but time t variant all the same, then the discretisation of their attendant quotient $m_S^{-1}(t)y(t)$ invokes the evaluation of its integral, as worked hereinafter: if $v(t) = m_S^{-1}(t)y(t) \neq m_S^{-1}y = \varepsilon_2^{-1}\varepsilon_1$ then $v_{1t} = \int_{t-1}^{t} m_S^{-1}(t)y(t)dt$.

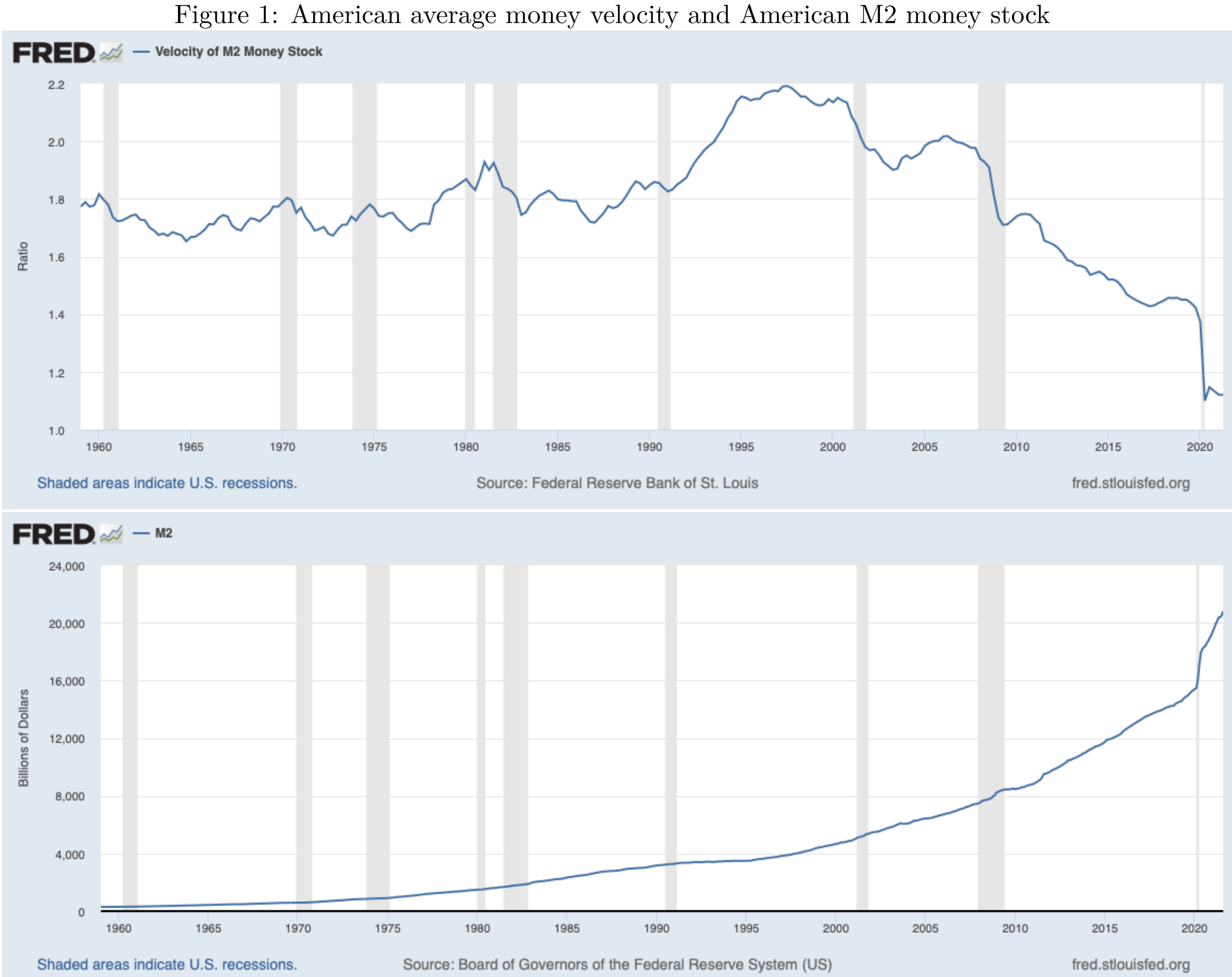
That established, if real output y amounts to 10 commodities, real money supply m_S to 2 thereof, time variation Δt_3 amounts to 3 seconds s, money displacement $\Delta x(t_3)$ is measured in metres m and direction is contemplated then average money velocity $v_{3t} = \hat{v}(t_3)$ equals 5 metres per second $s^{-1}m$ and money displacement $\Delta x(t_3)$ is 15 metres m : $v_{3t} = \hat{v}(t_3) = m_{St}^{-1}y_t = 2^{-1}10 = 5s^{-1}m$ and $\Delta x(t_3) = v_{3t}\Delta t_3 = \hat{v}(t_3)\Delta t_3 = 5s^{-1}m(3s) = 15m$. Such signifies that in 3 elapsed seconds s money travelled 15 metres m at an average velocity $v_{3t} = \hat{v}(t_3)$ of 5 metres per second $s^{-1}m$, that is, the 2 real money supply m_S commodities out of the 10 available of real output y were displaced by 15 metres m in the range of 3 seconds s at an average velocity $v_{3t} = \hat{v}(t_3)$ of 5 metres per second $s^{-1}m$.

A more concrete example is that of American average money velocity v_{1t} in the first graph below, featuring time t variations; it spans the 1959-2021 period and is recorded according to a quarterly annual basis. American average money velocity v_{1t} per quarter is specifically measured as the seasonally adjusted ratio of the quarterly American nominal gross domestic product (GDP) to the quarterly average of the American M2 money stock: $v_{1t} = M_{St}^{-1}Y_t = (p_t m_{St})^{-1}p_t y_t$, where Y_t is American nominal GDP and M_{St} the American average M2 money stock at a quarterly frequency, adjusted seasonally. In the second quarter (Q2) of 2021 American average money velocity v_{1t} amounted to 1.12; in other words, in Q2 2021 American nominal output Y_t exceeded American nominal money supply M_{St} by 1.12 times.

The FRED describes average money velocity v_{1t} in America as the number of times one American Dollar is spent to purchase (priced) commodities per unit of time t, specifying that if average money velocity v_{1t} increases then so have economic transactions, together with money displacement $\Delta x(t)$. Now, average money velocity $\hat{v}(t)$'s unit of measure is metres per second $s^{-1}m$, therefore, forasmuch as one quarter contains 7,884,000 seconds American money displacement $\Delta x(t)$ per quarter must be multiplied thereby. Consequently, in Q2 2021 American money displacement $\Delta x(t)$ amounted to metres $1.12s^{-1}m(7,884,000s) = 8,830,080m$, that is, kilometres $8,830km$ *circa,* where $7,884,000 = (60)(60)(24)(365)(4)^{-1}$ are indeed seconds s in terms of minutes, hours, days, years and quarters, respectively. By adopting the FRED's unit of measure of average money velocity $\hat{v}(t)$, expenses per second $s^{-1}e$, that is to say, American money displacement $\Delta x(t)$ in Q2 2021 was instead kilo-expenses $8,830ke$.

For a given time variation Δt higher average money velocity v_{1t} ultimately suggests a rise in real output y_t or a fall in real money supply m_{St} and an attendant rise in money displacement $\Delta x(t)$, namely, money travels a longer distance because of the fallen equilibrium demand therefor: (i) $\uparrow v_{1t} =\downarrow m_{St}^{-1}\bar{y}_t =\downarrow m_{Dt}^{-1}\bar{y}_t =\downarrow \kappa_{1t}^{-1}$, (ii) $\uparrow v_{1t} = \bar{m}_{St}^{-1} \uparrow y_t = \bar{m}_{Dt}^{-1} \uparrow y_t =\downarrow \kappa_{1t}^{-1}$ or (iii) $\uparrow v_{1t} =\downarrow m_{St}^{-1} \uparrow y_t =\downarrow m_{Dt}^{-1} \uparrow y_t =\downarrow \kappa_{1t}^{-1}$, since at equilibrium $m_S = m_D$ and $\int_{t-1}^{t} \kappa(t)dt = \int_{t-1}^{t} \chi'(t)dt = \int_{t-1}^{t} y^{-1}m_S dt = \chi(t)|_{t-1}^{t} = \chi(t) - \chi(t-1) = \chi_t - \chi_{t-1} = y^{-1}m_S t|_{t-1}^{t} = y^{-1}m_S(t - t + 1) = y^{-1}m_S = y_t^{-1}m_{St} = \kappa_{1t}(t - t + 1) = \kappa_{1t}$, *ceteris paribus.* Lower average money velocity v_{1t} specularly suggests a fall in real output y_t or a rise in real money supply m_{St} and an attendant fall in money displacement $\Delta x(t)$, money travelling a shorter distance because of the risen equilibrium demand therefor: (i) $\downarrow v_{1t} =\uparrow m_{St}^{-1}\bar{y}_t =\uparrow m_{Dt}^{-1}\bar{y}_t =\uparrow \kappa_{1t}^{-1}$, (ii) $\downarrow v_{1t} = \bar{m}_{St}^{-1} \downarrow y_t = \bar{m}_{Dt}^{-1} \downarrow y_t =\uparrow \kappa_{1t}^{-1}$ or (iii) $\downarrow v_{1t} =\uparrow m_{St}^{-1} \downarrow y_t =\uparrow m_{Dt}^{-1} \downarrow y_t =\uparrow \kappa_{1t}^{-1}$, *ceteris paribus.* American average money velocity v_{1t}, together with American displacement $\Delta x(t)$, was highest during the 1990s, throughout the internet bubble, and began to drastically decrease, beyond the 1960-1990 period, as of the Great recession.

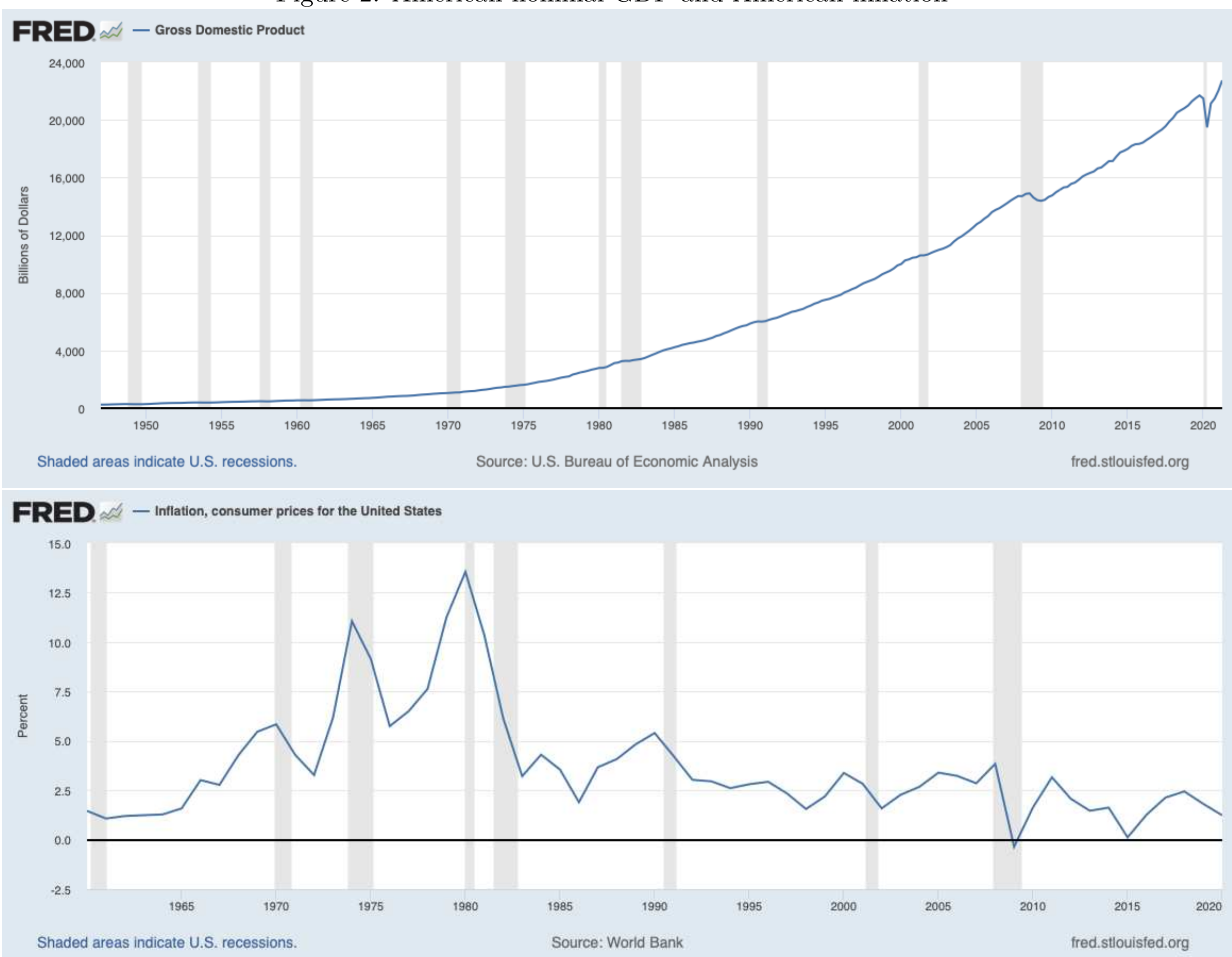
Figure 1: American average money velocity and American M2 money stock



Note. These graphs are taken from the Federal reserve economic data (FRED) and plot American average money velocity v_{1t} and American M2 money stock across the 1959-2021 period at a quarterly and monthly frequency, respectively. The former is computed as the ratio between the American nominal GDP and the American M2 money stock, which normally encompasses the monetary base and deposits. Grey areas represent recessions (i.e. GDP decline over two consecutive quarters).

Throughout the 1990s the Federal reserve's monetary policy had become one of price stability, under chairs Paul Volcker and Alan Greenspan, leading to the Great moderation. The 1970s had marked the resurgence of the quantity theory of money and the advent of monetarism, as viewed hereinbefore, and the late 1970s the start of the attendant policies of inflation targeting on the part of the Federal reserve and other world central banks. Such policies alongside the technological progress of internet widespread diffusion suggest sustained growth in American average money velocity v_{1t} during the 1990s on account of American nominal GDP growth, in the main, that is, of falling prices p_t and rising real output y_t; instantaneously: $\uparrow v(t) \uparrow M_S(t) =\uparrow Y(t)$, in real terms, $\uparrow v(t) \downarrow p(t) \uparrow m_S(t) =\downarrow p(t) \uparrow y(t)$, whereby $y_p p_{-ap(t)} > -p_{-ap(t)}$ and $m_{S_p} p_{-ap(t)} > -p_{-ap(t)}$, whence $\uparrow v(t) =\uparrow m_S^{-1}(t) \uparrow y(t)$, whereby $y_p p_{-ap(t)} > m_{S_p} p_{-ap(t)}$.

Figure 2: American nominal GDP and American inflation



Note. These two graphs are taken from the FRED and plot American nominal GDP and American inflation across the 1947-2021 and 1960-2020 periods at a quarterly and annual frequency, respectively. Grey areas represent recessions.

Throughout the 2010s the Federal reserve's monetary policy turned expansionary, in order to front the Great recession begun in 2008. The monetary expansions had in fact begun shortly after the outbreak of the Great recession, through chair Benjamin Bernanke's quantitative easing, thereby suggesting a drastic fall in American average money velocity v_{1t} during the 2008-2010 period on account of American nominal GDP decline and American M2 money stock growth, that is, of falling real output y_t and rising real money supply m_{St}; instantaneously: $\downarrow v(t) \uparrow M_S(t) =\downarrow Y(t)$, in real terms, $\downarrow v(t)\bar{p}(t) \uparrow m_S(t) = \bar{p}(t) \downarrow y(t)$, whence $\downarrow v(t) =\uparrow m_S^{-1}(t) \downarrow y(t)$, whereby $-y_{-m_D(t)} > y_{M_S(t)}$, noting that prices $p(t)$ are instantaneously fixed forasmuch as not yet negatively affected by the fall in $am_D(t)$.

As American nominal GDP began to grow again throughout the 2010s, in view of the monetary expansions, American average money velocity v_{1t} should have begun to grow again too, as it temporarily did, but the persistent rise in the American M2 money stock ultimately caused it to fall even more; instantaneously: $\downarrow v(t) \uparrow M_S(t) =\uparrow Y(t)$, in real terms, $\downarrow v(t)\bar{p}(t) \uparrow m_S(t) = \bar{p}(t) \uparrow y(t)$, whence $\downarrow v(t) =\uparrow m_S^{-1}(t) \uparrow y(t)$, whereby $m_{S_{M_S(t)}} > y_{M_S(t)}$, noting that prices $p(t)$ can be averaged throughout (rising and thence falling afresh) and thus taken as instantaneously fixed.

The "COVID-19" worldwide emergency finally catalysed an unprecedented decline in American nominal GDP, almost to parity with its M2 money stock, whose growth rate was by then decelerating, under chairs Janet Yellen and Jerome Powell especially, disrupting almost all economic transactions and thereby causing a corresponding reduction in American average money velocity v_{1t}; instantaneously: $\downarrow v(t) \uparrow M_S(t) =\downarrow Y(t)$, in real terms, $\downarrow v(t)\bar{p}(t) \uparrow m_S(t) = \bar{p}(t) \downarrow y(t)$, whence $\downarrow v(t) =\uparrow m_S^{-1}(t) \downarrow y(t)$, noting that prices $p(t)$ are instantaneously fixed forasmuch as not yet negatively affected by the fall in $am_D(t)$ once again.

Noteworthy are also the two increments in American average money velocity v_{1t} during the recessions of the 1970s, whereby American nominal GDP decline was perforce outshone by an even greater decline in the American M2 money stock, however graphically imperceptible; instantaneously: $\uparrow v(t) \downarrow M_S(t) =\downarrow Y(t)$, in real terms, $\uparrow v(t) \uparrow p(t) \downarrow m_S(t) =\uparrow p(t) \downarrow y(t)$, whereby $p_{ap(t)} < -y_p p_{ap(t)}$ and $p_{ap(t)} < -m_{S_p} p_{ap(t)}$, whence $\uparrow v(t) =\downarrow m_S^{-1}(t) \downarrow y(t)$, whereby $m_{S_{p(t)}} > y_{p(t)}$.

Out of recessions, such as in 1960s, 70s and 80s, the decrements in American average money velocity v_{1t} were by contrast ultimately due to increments in the American M2 money stock; instantaneously: $\downarrow v(t) \uparrow M_S(t) =\uparrow Y(t)$, in real terms, $\downarrow v(t)\bar{p}(t) \uparrow m_S(t) = \bar{p}(t) \uparrow y(t)$, whence $\downarrow v(t) =\uparrow m_S^{-1}(t) \uparrow y(t)$, whereby $m_{S_{M_S(t)}} > y_{M_S(t)}$, noting that prices $p(t)$ are instantaneously fixed forasmuch as not yet positively affected by the rise in $M_S(t)$.

The above presentation of American average money velocity on the part of the FRED could be alternatively read as American instantaneous money velocity $v(t)$ fitted for discrete time $t: \ \forall t \in \mathbb{Z}_+, \ v(t) = M_S^{-1}(t)Y(t) = [p(t)m_S(t)]^{-1}p(t)y(t) = v_t = M_{St}^{-1}Y_t = (p_t m_{St})^{-1}p_t y_t = \hat{v}(t)$. Its time t variation would obviously cause it to depart from a strictly Fisherian presentation thereof: $v(t) = M_S^{-1}(t)Y(t) \neq M_S^{-1}Y$. Such a translation of instantaneous money velocity $v(t)$ to the data would in fact be preferable to its erstwhile discretisation, for in the absence of quotient $m_S^{-1}(t)y(t)$'s analytical form real output y and real money supply m_S could be only imposed *qua* white noises, behoving them to be time t variant on account of the data, as viewed hereinbefore, conflicting with their mutual causality. Analogously, average money velocity $v_3 = \hat{v}(3) = m_{S3}^{-1}y_3 = v(3) = m_S^{-1}(3)y(3) = 5s^{-1}m$ and money displacement $\Delta x(3) = v_3\Delta t = \hat{v}(3)\Delta t = v(3)3 = 5s^{-1}m(3s) = 15m$ hereinbefore.

3. Mechanical derivation of the exchange equation

3.1 Money value function and money constraint. Money position $x(t) = m_S^{-1}yt \longrightarrow yt = m_S x(t)$, moreover, expressing an equivalence between real output y weighted by the time t taken to produce it and real money supply m_S weighted by the distance $x(t)$ it travelled, whereby greater time t suggests greater production of real output y, however time t invariant, and greater distance $x(t)$ suggests more transactions or employment of real money supply m_S, respectively.

An increase in money position $x(t)$ could for instance be matched by an increase in real output y or time t or by a decrease in real money supply m_S : (i) $\bar{m}_S \uparrow x(t) =\uparrow y\bar{t}$; (ii) $\bar{m}_S \uparrow x(t) = \bar{y} \uparrow t$; (iii) $\downarrow m_S \uparrow x(t) = \bar{y}\bar{t}$; (iv) $\bar{m}_S \uparrow x(t) =\uparrow y \uparrow t$; (v) $\downarrow m_S \uparrow x(t) = \bar{y} \uparrow t$; (vi) $\downarrow m_S \uparrow x(t) =\uparrow y \uparrow t$. Cases (i) and (iii) are however afresh specious, for changes in real output y and money distance $x(t)$ require changes in time t; the latter require them by definition, while the former on account of reason, whereby real output y cannot vary irrespective of time t.

A money value function of the macroeconomy can therefore be envisaged, whereby it may be twice continuously differentiable and specifically increasing and concave in real output y, decreasing and convex in real money supply m_S, increasing in time t, decreasing in money position $x(t)$, at a constant rate in both: $V[\overset{+-}{y}, \ \overset{+=}{t}, \ \overset{-+}{m_S}, \ \overset{-=}{x(t)}] \in C^2$ such that, $\forall \alpha, \ \beta \in (0, \ 1], \ V = \frac{y^\alpha t}{\alpha} - \frac{m_S^\beta x(t)}{\beta}$; to be sure, $V_y = y^{\alpha-1}t, \ V_{yy} = (\alpha-1)y^{\alpha-2}t, \ V_t = \frac{y^\alpha}{\alpha}, \ V_{x(t)} = -\frac{m_S^\beta}{\beta}, \ V_{tt} = V_{x(t)x(t)} = 0, \ V_{m_S} = -m_S^{\beta-1}x(t)$ and $V_{m_S m_S} = (1-\beta)m_S^{\beta-2}x(t)$.

Money value function V is not of a representative agent, but of the macroeconomy, as stated, modelling money welfare and intercepting all objections stemming from potential extensions of the "Anything goes" theorem. Time t weighted real output yt is increasing because it produces ophelimity, real output y being consumable, while distance weighted real money supply $m_S x(t)$ does not, real money supply m_S not being consumable and being thereby decreasing.

Such a value function V is then necessarily constrained by equilibrium Cambridge equation $\kappa y = m_S$, in that real money supply m_S is perforce a fraction of real output $y : \ \forall \kappa \in (0,\ 1),\ m_S = \kappa y$. Equilibrium Cambridge equation $\kappa y = m_S$ is therefore the macroeconomy's money constraint, wherein demanded real money supply m_S is supplied by a κ share of real output y.

3.2 Constrained maximisation. Optimal solutions for real output y and real money supply m_S are given by the first order conditions (FOCs) of the attendant Hamiltonian equation, wherein λ is the Lagrangian multiplier, namely, they are its null partial derivatives, signalling the elusion of suboptimal changes: $\mathcal{H} = V[y,\ t,\ m_S,\ x(t)] + \lambda(\kappa y - m_S) = \frac{y^\alpha t}{\alpha} - \frac{m_S^\beta x(t)}{\beta} + \lambda(\kappa y - m_S)$. A necessary condition for optimal solutions is the non-singularity of real output y and real money supply m_S, which the constrained maximisation problem meets by virtue of their own definition. A sufficient condition for optimal solutions is money value function V's concavity, since $max\ V = -min\ (-V)$, which the constrained maximisation problem does not meet, forasmuch as partial derivatives $V_{yy} < 0,\ V_{m_S m_S} > 0$ and $V_{yy}V_{m_S m_S} - V_{y m_S} = V_{yy}V_{m_S m_S} < 0$, meeting thresholds $V_{yy} \leq 0$ and $V_{yy}V_{m_S m_S} \leq V_{y m_S}$, but failing threshold $V_{m_S m_S} \leq 0$.

Absent loss of generality, time t and money position $x(t)$ can be thus regarded as parameters: $V[y,\ m_S;\ t,\ x(t)]$. Accordingly, real money supply $m_S + \xi = 1 \longrightarrow m_S = 1 - \xi$ and $\xi = 1 - m_S$, all else equal: *ceteris paribus,* $m_S + \xi = 1$, whence $\xi \in \mathbb{R}$. Consequently, money value function $V[\overset{+-}{y},\ \overset{+-}{\xi - 1};\ t,\ x(t)] = \frac{y^\alpha t}{\alpha} + \frac{(\xi-1)^\beta x(t)}{\beta} = \frac{y^\alpha t}{\alpha} - \frac{m_S^\beta x(t)}{\beta}$, since partial derivatives $V_{\xi-1} = (\xi - 1)^{\beta-1}x(t)$ and $V_{(\xi-1)(\xi-1)} = (\beta - 1)(\xi - 1)^{\beta-2}x(t)$. As a result, the constrained maximisation problem is necessary and sufficient for optimal solutions: $\underset{\{y,\ m_S\}}{max}\ V = \frac{y^\alpha t}{\alpha} - \frac{m_S^\beta x(t)}{\beta}$ subject to $m_S = \kappa y$.

The FOCs under scrutiny are $\mathcal{H}_y = y^{\alpha-1}t + \lambda\kappa = 0 \longrightarrow t = -y^{1-\alpha}\lambda\kappa$ and $\mathcal{H}_{m_S} = -m_S^{\beta-1}x(t) - \lambda = 0 \longrightarrow -m_S^{\beta-1}x(t) = \lambda$. The second FOC can be substituted into the first to give rise to time $t = -y^{1-\alpha}[-m_S^{\beta-1}x(t)]\kappa \longrightarrow \kappa = y^{\alpha-1}m_S^{1-\beta}x^{-1}(t)t$, which is an equation expressing money demand coefficient κ in terms of real output y, real money supply m_S, money position $x(t)$ and time t. The FOC of Lagrangian multiplier λ is then $\mathcal{H}_\lambda = \kappa y - m_S = 0 \longrightarrow \kappa y = m_S$, being the equilibrium Cambridge equation anew.

The substitution of such a money demand coefficient $\kappa = y^{\alpha-1}m_S^{1-\beta}x^{-1}(t)t$ into real money supply $m_S = \kappa y$, stemming from the optimal path of Lagrangian multiplier λ, therefrom gives rise to equilibrium Cambridge equation $\kappa y = y^{\alpha-1}m_S^{1-\beta}x^{-1}(t)ty = y^\alpha m_S^{1-\beta}x^{-1}(t)t = m_S \longrightarrow y^\alpha m_S^{1-\beta}t = m_S x(t)$ or money position $x(t) = m_S^{-1}y^\alpha m_S^{1-\beta}t = m_S^{-\beta}y^\alpha t$, whence instantaneous money velocity $v(t) = x'(t) = m_S^{-\beta}y^\alpha$ and instantaneous money acceleration $a(t) = v'(t) = x''(t) = 0$. Exchange equation $v(t)m_S^\beta = y^\alpha \longrightarrow v(t)(pm_S)^\beta = (py)^\alpha = v(t)M_S^\beta = Y^\alpha$ ensues.

Finally, real money supply $m_S = y^\alpha m_S^{1-\beta}x^{-1}(t)t \longrightarrow m_S = [y^\alpha x^{-1}(t)t]^{\frac{1}{\beta}}$ above, whence money value function $V = \frac{y^\alpha t}{\alpha} - \frac{m_S^\beta x(t)}{\beta} = \frac{y^\alpha t}{\alpha} - \frac{\{[y^\alpha x^{-1}(t)t]^{\frac{1}{\beta}}\}^\beta x(t)}{\beta} = \frac{y^\alpha t}{\alpha} - \frac{y^\alpha x^{-1}(t)tx(t)}{\beta} = \frac{y^\alpha t}{\alpha} - \frac{y^\alpha t}{\beta} = \frac{(\beta-\alpha)y^\alpha t}{\alpha\beta} \equiv \varsigma y^\alpha t$. On admitting parametrisation $\alpha = \beta = 1$, money value function $V = 0$ and instantaneous money velocity $v(t) = m_S^{-1}y$, as desired, that is, precisely as instantiated by Fisher.

3.3 Unconstrained maximisation. The substitution of real money supply $m_S = \kappa y$, *qua* money constraint, into money value function V gives rise to an unconstrained maximisation problem: $V = \frac{y^\alpha t}{\alpha} - \frac{m_S^\beta x(t)}{\beta} = \frac{y^\alpha t}{\alpha} - \frac{(\kappa y)^\beta x(t)}{\beta}$ such that $\underset{y}{max}\ V = \frac{y^\alpha t}{\alpha} - \frac{(\kappa y)^\beta x(t)}{\beta}$. The FOC of real output y is thus $V_y = y^{\alpha-1}t - \kappa(\kappa y)^{\beta-1}x(t) = 0 \longrightarrow \kappa = [y^{1-\beta}x^{-1}(t)y^{\alpha-1}t]^{\frac{1}{\beta}}$, which expresses money demand coefficient κ in terms of real output y, money position $x(t)$ and time t.

Such an equation $\kappa = [y^{1-\beta}x^{-1}(t)y^{\alpha-1}t]^{\frac{1}{\beta}}$ can be therefrom substituted into real money supply $m_S = \kappa y$, *qua* money constraint, in order to give rise to equilibrium Cambridge equation $\kappa y = [y^{1-\beta}x^{-1}(t)y^{\alpha-1}t]^{\frac{1}{\beta}}y = m_S \longrightarrow x(t) = [m_S^{-1}y(y^{1-\beta}y^{\alpha-1}t)^{\frac{1}{\beta}}]^\beta = (m_S^{-1}y)^\beta(y^{1-\beta}y^{\alpha-1}t) = m_S^{-\beta}y^\alpha t$, being money position, whence instantaneous money velocity $v(t) = x'(t) = m_S^{-\beta}y^\alpha$, instantaneous money acceleration $a(t) = v'(t) =$

$x''(t) = 0$ and exchange equation $v(t)m_S^\beta = y^\alpha \longrightarrow v(t)(pm_S)^\beta = (py)^\alpha = v(t)M_S^\beta = Y^\alpha$ anew.

The substitution of the above money demand coefficient $\kappa = [y^{1-\beta}x^{-1}(t)y^{\alpha-1}t]^{\frac{1}{\beta}}$ into unconstrained money value function V finally yields the same result: $V = \frac{y^\alpha t}{\alpha} - \frac{(\kappa y)^\beta x(t)}{\beta} = \frac{y^\alpha t}{\alpha} - \frac{\{[y^{1-\beta}x^{-1}(t)y^{\alpha-1}t]^{\frac{1}{\beta}}y\}^\beta x(t)}{\beta} = \frac{y^\alpha t}{\alpha} - \frac{y^{1-\beta}x^{-1}(t)y^{\alpha-1}ty^\beta x(t)}{\beta} = \frac{y^\alpha t}{\alpha} - \frac{y^\alpha t}{\beta} = \frac{(\beta-\alpha)y^\alpha t}{\alpha\beta} \equiv \varsigma y^\alpha t$.

3.4 Average money velocity. The expression of real output y and real money supply m_S as non-constant functions of time t in the said optimisation problem gives rise to time t variant exchange equation $v(t)m_S^\beta(t) = y^\alpha(t) \longrightarrow v(t)[p(t)m_S(t)]^\beta = [p(t)y(t)]^\alpha = v(t)M_S^\beta(t) = Y^\alpha(t)$. Instantaneous money velocity $v(t)$ can therefrom be fitted for discrete time t, corresponding to average money velocity $\hat{v}(t)$ therethrough, as the case hereinbefore: $\forall t \in \mathbb{Z}_+,\ v(t) = M_S^{-\beta}(t)Y^\alpha(t) = v_t = M_{St}^{-\beta}Y_t^\alpha = \hat{v}(t)$.

The imposition of natural logarithms consequently consents the identification of instantaneous money velocity $v(t)$'s natural logarithm, inversely weighted by parameter α, with the attendant linear time series regression's residual $\hat{u}_t :\ lnv(t) = \alpha lnY(t) - \beta lnM_S(t) \longrightarrow lnY(t) = \frac{\beta}{\alpha}lnM_S(t) + \alpha^{-1}lnv(t) = lnY_t = \gamma lnM_{St} + u_t \longrightarrow lnY_t = \hat{\gamma}lnM_{St} + \hat{u}_t$, where $\hat{\gamma}$ and is $\gamma \equiv \frac{\beta}{\alpha}$ estimated through generalised least squares, whence ensues $\hat{u}_t$. Residual $\hat{u}_t$ is thus a statistic for average money velocity $\hat{v}(t) :\ \hat{u}_t = \alpha^{-1}lnv(t) = lnY(t) - \hat{\gamma}lnM_S(t) = \alpha^{-1}lnv_t = lnY_t - \hat{\gamma}lnM_{St}$.

Indeed, Fisherian instantaneous money velocity $v(t) = M_S^{-1}(t)Y(t)$ is only a declension of instantaneous money velocity $v(t) = M_S^{-\beta}(t)Y^\alpha(t)$, whereby parameter $\alpha = \beta = 1$ and natural logarithm $lnv(t) = lnY(t) - lnM_S(t)$ therethrough. Consequently, average money velocity $v_t = M_{St}^{-1}Y_t$, other than departing from strictly Fisherian average money velocity $v_t = M_S^{-1}Y$, need not represent average money velocity $v_t = M_{St}^{-\beta}Y_t^\alpha$ (e.g. American application above). In the absence of knowledge with regard to parameters α and β the elaboration of residual $\hat{u}_t = \alpha^{-1}lnv(t)$ would nevertheless be preferable to that of natural logarithm $lnv(t) = lnY(t) - lnM_S(t)$ as an approximation of average money velocity $\hat{v}(t)$, forasmuch as parametrisation $\alpha = \beta = 1$, *per se* restrictive, would not have occurred.

4. Time variant money velocity

4.1 Non-constant average money velocity. As viewed hereinbefore, a non-zero change in average velocity $\hat{v}(t)$ signifies the presence of average acceleration $\hat{a}(t)$, which average money velocity $\hat{v}(t)$ features as both theoretically possible and practically verified (e.g. American application above): $\hat{\mathbf{a}}(t) := \lim_{dt\to\infty}\frac{d^2\mathbf{x}(t)}{dt^2} = \lim_{dt\to\infty}\frac{d\mathbf{v}(t)}{dt} = \frac{\Delta^2\mathbf{x}(t)}{\Delta t^2} = \frac{\Delta\hat{\mathbf{v}}(t)}{\Delta t}$ and $\hat{a}(t) = \lim_{dt\to\infty}\frac{dv(t)}{dt} = \frac{\Delta\hat{v}(t)}{\Delta t} \neq 0$, being normatively possible and positively verified in the case of money, whence $a_{1t} = \hat{a}(t_1) = \Delta t_1^{-1}\Delta\hat{v}(t_1) = [t-(t-1)]^{-1}[\hat{v}(t_1)-\hat{v}(t_1-1)] = v_{1t} - v_{1t-1} = (x_t - x_{t-1}) - (x_{t-1} - x_{t-2}) = x_t - 2x_{t-1} + x_{t-2} \neq 0$ etc. thereby; it suitably follows that $\hat{a}(t) = \frac{\Delta\hat{v}(t)}{\Delta t} = \Delta t^{-1}\Delta m_{St}^{-1}y_t \neq 0$ and $a_{1t} = v_{1t} - v_{1t-1} = m_{St}^{-1}y_t - m_{St-1}^{-1}y_{t-1} \neq 0$, *ceteris paribus.*

A more generic specification of average money velocity $\hat{v}(t)$, reinforcing its normative possibility, could for instance evolve according to a random walk with drift, wherein the white noise is normally distributed and the standard deviation and the mnemonic drift are positive: $\forall\varepsilon \sim \mathcal{N}(0,\ \sigma^2),\ \sigma,\ \mu \in \mathbb{R}_{++},\ v(t) = x'(t) = m_S^{-1}(t)y(t) = (\mu+\sigma\varepsilon)t \longrightarrow a(t) = v'(t) = \mu+\sigma\varepsilon$, thus, $\hat{a}(t) = \lim_{dt\to\infty}\frac{dv(t)}{dt} = \frac{\Delta\hat{v}(t)}{\Delta t} = \Delta t^{-1}\Delta\hat{m}_S^{-1}(t)\hat{y}(t) = \lim_{dt\to\infty}(\mu+\sigma\varepsilon) = \mu + \sigma\varepsilon_t$, whence $a_{1t} = v_{1t} - v_{1t-1} = \mu + \sigma\varepsilon_t \longrightarrow v_{1t} = \mu + v_{1t-1} + \sigma\varepsilon_t$. In fact: $a_{1t} = \int_{t-1}^t a(t)dt = \int_{t-1}^t v'(t)dt = v(t)|_{t-1}^t = v(t) - v(t-1) = v_{1t} - v_{1t-1} = m_S^{-1}(t)y(t)|_{t-1}^t = m_S^{-1}(t)y(t) - m_S^{-1}(t-1)y(t-1) = m_{St}^{-1}y_t - m_{St-1}^{-1}y_{t-1} = \int_{t-1}^t \mu + \sigma\varepsilon dt = (\mu+\sigma\varepsilon)t|_{t-1}^t = (\mu+\sigma\varepsilon)(t-t+1) = \mu + \sigma\varepsilon_t$; recapitulating, $a_{1t} = v_{1t} - v_{1t-1} = m_{St}^{-1}y_t - m_{St-1}^{-1}y_{t-1} = \mu + \sigma\varepsilon_t$. As seen, time t variant white noises under continuousness, such as white noise ε, must be indexed by time t under discreteness; inversely, time t invariant variables, namely, parameters, such as drift μ, must not.

The difference between such a specification of average money velocity $\hat{v}(t)$ and that following Fisher's instantiation of instantaneous money velocity $v(t)$ above is the following: (i) $v(t) = m_S^{-1}(t)y(t) = (\mu+\sigma\varepsilon)t$, whence $v_{1t} = m_{St}^{-1}y_t = \mu + m_{St}^{-1}y_{t-1} + \sigma\varepsilon_t = \mu + v_{1t-1} + \sigma\varepsilon_t$; (ii) $v(t) = m_S^{-1}y = \varepsilon_2^{-1}\varepsilon_1$, whence $v_{1t} = m_{St}^{-1}y_t = \varepsilon_{2t}^{-1}\varepsilon_{1t}$. In the first specification, in fact, instantaneous money velocity $a(t)$ can be fitted for discrete time $t :\ \forall t \in \mathbb{Z}_+,\ a(t) = (\mu+\sigma\varepsilon) = a_t = \mu + \sigma\varepsilon_t = \hat{a}(t)$.

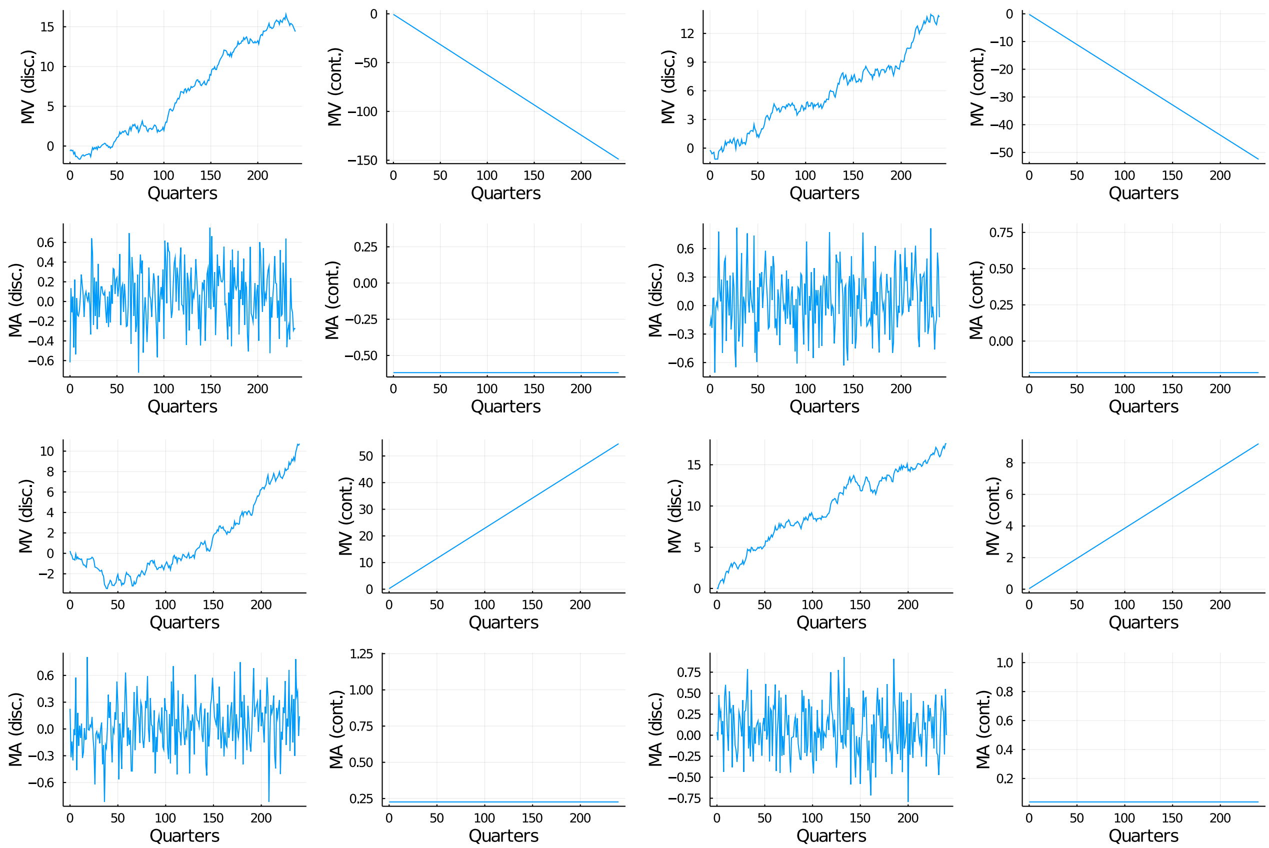
4.2 Simulations. Simulations of discrete money velocity $\hat{v}(t)$ and continuous money velocity $v(t)$ and of discrete money acceleration $\hat{a}(t)$ and continuous money acceleration $a(t)$ at a 240 quarter horizon (i.e. 60

years) are thus graphed hereinafter. Discrete money velocity $\hat{v}(t)$ and discrete money acceleration $\hat{a}(t)$ are respectively simulated as average money velocity $v_{1t} = \mu + v_{1t-1} + \sigma\varepsilon_t$ and average money acceleration acceleration $a_{1t} = \mu + \sigma\varepsilon_t$, whereas their continuous counterparts $v(t)$ and $a(t)$ are respectively simulated as instantaneous money velocity $v(t) = (\mu + \sigma\varepsilon)t$ and instantaneous money acceleration $a(t) = \mu + \sigma\varepsilon$, as viewed hereinbefore.

For simplicity, average money velocity $v_{1t} \equiv v_t$ and average money acceleration $a_{1t} \equiv a_t$ and are computed thus: $\forall t \in [1,\ 240] \subset \mathbb{Z}_{++},\ v_1 = \mu + \sigma\varepsilon_1,\ v_2 = \mu + v_1 + \sigma\varepsilon_2$ etc. and $a_1 = \mu + \sigma\varepsilon_1,\ a_2 = \mu + \sigma\varepsilon_2$ etc. Instantaneous money velocity $v(t)$ and acceleration $a(t)$ are computed thus: $\forall t \in [1,\ 240] \subset \mathbb{Z}_{++},\ v(1) = \mu + \sigma\varepsilon,\ v(2) = (\mu + \sigma\varepsilon)2$ etc. and $a(1) = (\mu + \sigma\varepsilon),\ a(2) = (\mu + \sigma\varepsilon)$ etc. It is immediately discerned that instantaneous money velocity $v(t)$ and instantaneous money acceleration $a(t)$ are respectively linear and constant throughout.

Standard deviation $\sigma = 0.3$ and mnemonic drift $\mu = 0.05$ all through; the former simulates a 30% deviation from the white noise's mean and the latter mimics a 2% annual growth rate.

Figure 3: Money velocity and money acceleration simulations



Note. The above graphs plot four simulations, at a 240 quarter horizon, of average money velocity $v_{1t} = \mu + v_{1t-1} + \sigma\varepsilon_t$ and average money acceleration $a_{1t} = \mu + \sigma\varepsilon_t$ and instantaneous money velocity $v(t) = (\mu + \sigma\varepsilon)t$ and instantaneous money acceleration $a(t) = \mu + \sigma\varepsilon$. "MV (disc.)" and "MA (disc.)" stand for discrete money velocity and discrete money acceleration, average money velocity v_{1t} and average money acceleration a_{1t}, that is to say. "MV (cont.)" and "MA (cont.)" stand for continuous money velocity and continuous money acceleration, that is, instantaneous money velocity $v(t)$ and instantaneous money acceleration $a(t)$. Standard deviation $\sigma = 0.3$ and mnemonic drift $\mu = 0.05$.

4.3 Non-constant instantaneous money velocity. Instantaneous money velocity $v(t)$ as plotted above is indeed not as that presented by Fisher, which excludes the presence of instantaneous money acceleration $a(t):\ v(t) = m_S^{-1}y$ such that $a(t) = v'(t) = 0$. As seen, however, instantaneous money acceleration $a(t)$ is theoretically possible, whereby real output y and real money supply m_S are non-constant functions of time $t:\ a(t) = v'(t) \neq 0$ is normatively possible, given $v(t) = m_S^{-1}(t)y(t) \neq m_S^{-1}y$.

Instantaneous money velocity $v(t) = m_S^{-1}(t)y(t) = (\mu + \sigma\varepsilon)t$ hereinbefore for instance implies money position $x(t) = \int v(t)dt - C = \int m_S^{-1}(t)y(t)dt - C = \int(\mu + \sigma\varepsilon)tdt - C = \frac{(\mu+\sigma\varepsilon)t^2}{2} + C - C$, whence also inverse instantaneous money velocity $v^{-1}(t) = [(\mu + \sigma\varepsilon)t]^{-1} = \kappa(t) = \kappa[r(t)]$.

More broadly, instantaneous money velocity $v(t) = m_S^{-1}(t)y(t)$ gives rise to money position $x(t) = \int v(t)dt - C = \int m_S^{-1}(t)y(t)dt - C = [y(t)\int m_S^{-1}(t)dt - \int \left(\int m_S^{-1}(t)dt\right) y'(t)dt] - C$: in $\int m_S^{-1}(t)y(t)dt$ one discerns $f(t) = y(t)$ and $g'(t) = m_S^{-1}(t)$, whence $f'(t) = y'(t)$ and $g(t) = \int m_S^{-1}(t)dt$, for parts integration $\int f(t)dg(t) = f(t)g(t) - \int g(t)df(t)$, where $df(t) = f'(t)dt$ and $dg(t) = g'(t)dt$. In particular, function $g(t) = \int m_S^{-1}(t)dt \longrightarrow g'(t) = m_S^{-1}(t) \longrightarrow h(t) \equiv g'^{-1}(t) = m_S(t)$ if and only if function $h : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is bijective. Similarly, instantaneous money acceleration $a(t) = v'(t) = \frac{y'(t)m_S(t)-y(t)m'_S(t)}{m_S^2(t)}$: in $m_S^{-1}(t)y(t)$ one discerns $f(t) = y(t)$ and $h(t) = m_S(t)$ for quotient rule $\frac{f'(t)h(t)-f(t)h'(t)}{h^2(t)}$.

4.4 General maximisation. For simplicity, symbolic logic notation outside of colons is henceforth expanded. The general money value function and money constraint of the macroeconomy are therefore $V = V[y(t),\ h(t),\ x(t);\ \Theta] \in C^2$ and $h(t) = h[y(t);\ \Phi]$, respectively, where parameter sets $\Theta,\ \Phi \subset \mathbb{R}_+$; the unconstrained general money value function thence is $V = V\{y(t),\ h[y(t);\ \Phi],\ x(t);\ \Theta\} = V[y(t),\ x(t);\ \Theta,\ \Phi]$. The general unconstrained maximisation problem is therefrom $\underset{y(t)}{max}\ V[y(t),\ x(t);\ \Theta,\ \Phi]$, whereby the doubly continuous differentiability of general money value function V and the non-singularity of real output $y(t)$ ultimately render it necessary and sufficient for an optimal real output $y(t)$ solution. FOC $V_y = 0$ is generalised as $y'(t) = y'[y(t),\ x(t);\ \Theta,\ \Phi] = 0$, whence follow parameter $\phi_i = \phi_i[y(t),\ x(t);\ \Theta]$, for parameters $\phi_{i\in\mathbb{N}_+} \in \Phi$, and general real money supply or money constraint $h(t) = h[y(t),\ x(t);\ \Theta]$, finally yielding general money position $x(t) = y(t)g(t) - \int g(t)y'(t)dt$.

Fisher's instantiation of instantaneous money velocity $v(t)$, studied hereinbefore, specifies the general unconstrained maximisation problem as follows: $V[y(t),\ h(t),\ x(t);\ \Theta] = V[y(t),\ x(t);\ \Theta,\ \Phi] = V[y,\ m_S,\ x(t);\ t] = V[y,\ x(t);\ t,\ \kappa] = yt - m_Sx(t) = yt - \kappa yx(t)$, $h[y(t);\ \Phi] = h(y;\ \kappa) = m_S = \kappa y$, $y'[y(t),\ x(t);\ \Theta,\ \Phi] = y'[y,\ x(t);\ t,\ \kappa] = t - \kappa x(t) = 0$, $\phi_i[y(t),\ x(t);\ \Theta] = \phi_i[y,\ x(t);\ t] = \kappa = x^{-1}(t)t$, $h[y(t),\ x(t);\ \Theta] = h[y,\ x(t);\ t] = m_S = x^{-1}(t)ty$ and $x(t) = m_S^{-1}yt$.

In such a formulation time t effectively emerges as parameter $\theta \in \Theta$ precisely because of the invariance real output y and real money supply m_S ultimately feature in its regards, as unrealistic as it be. In fact, their ultimate time t invariance is even worse than the quantity theory of money, whereby exchange equation $v(t)p(t)m_S(t) = p(t)y(t)$ features variations only in prices $p(t)$, for in such a formulation real output y and real money supply m_S can be absurdly envisaged as mere constants.

4.5 Macroeconomy. A time t variant instantiation of instantaneous money velocity $v(t) = m_S^{-1}(t)y(t)$, in accordance with the macroeconomic laws presented in Section 1, could unfold thus:

$y = y(\overset{+-}{m_S},\ \overset{+-}{m_D}) = m_S^{\alpha_1}m_D^{1-\alpha_1}$,

$m_S = m_S(\overset{+-}{M_S},\ \overset{-+}{p}) = M_S^{\alpha_2}p^{\alpha_2-1}$ such that $p \in \mathbb{R}_{++}$,

$m_D = m_D(\overset{+-}{am_D},\ \overset{-+}{ap}) = am_D^{\alpha_3}ap^{\alpha_3-1}$ such that $ap \in \mathbb{R}_{++}$,

$p = p(\overset{+-}{ap},\ \overset{+-}{\dot{M}_S},\ \overset{+-}{a\dot{m}_D},\ \overset{-+}{a\dot{p}}) = ap^{\alpha_4}\dot{M}_S^{\beta_4}a\dot{m}_D^{\gamma_4}a\dot{p}^{-\delta_4}$ such that $a\dot{p} \in \mathbb{R}_{++}$,

$M_S,\ ap,\ am_D = f(t) \in C^2$ such that $M_S,\ am_D = e^{\sigma_i\varepsilon_i t}$, $\forall i = M_S,\ am_D$, and $ap = e^{(\mu+\sigma_{ap}\varepsilon_{ap})t}$,

$\forall\alpha_{i\in[1,\ 4]}$, β_4, $\gamma_4 \in (0,\ 1)$, $\delta_4 = 1 - (\alpha_4 + \beta_4 + \gamma_4)$, generic white noise $\varepsilon \sim \mathcal{N}(0,\ \sigma^2)$, $t \in \mathbb{R}_+$ and generic standard deviation and generic mnemonic drift $\sigma,\ \mu \in \mathbb{R}_{++}$, whence $M_S,\ ap,\ am_D,\ a\dot{p},\ (\dot{M}_S a\dot{m}_D),\ p,\ m_D,\ m_S,\ y \in \mathbb{R}_{++}$.

Consequently, instantaneous money velocity $v(t) = m_S^{-1}(t)y(t) = [e^{\alpha_2\sigma_{M_S}\varepsilon_{M_S}t}p^{\alpha_2-1}(t)]^{-1}m_S^{\alpha_1}(t)m_D^{1-\alpha_1}(t) = e^{-\alpha_2\sigma_{M_S}\varepsilon_{M_S}t}p^{1-\alpha_2}(t)m_S^{\alpha_1}(t)m_D^{1-\alpha_1}(t)$, whence instantaneous money acceleration $a(t) = -\alpha_2\sigma_{M_S}\varepsilon_{M_S}e^{-\alpha_2\sigma_{M_S}\varepsilon_{M_S}t}p^{1-\alpha_2}(t)m_S^{\alpha_1}(t)m_D^{1-\alpha_1}(t) = -\alpha_2\sigma_{M_S}\varepsilon_{M_S}m_S^{-1}(t)y(t) = \theta m_S^{-1}(t)y(t)$ and money position $x(t) = (-\alpha_2\sigma_{M_S}\varepsilon_{M_S})^{-1}e^{-\alpha_2\sigma_{M_S}\varepsilon_{M_S}t}p^{1-\alpha_2}(t)m_S^{\alpha_1}(t)m_D^{1-\alpha_1}(t) = (-\alpha_2\sigma_{M_S}\varepsilon_{M_S})^{-1}m_S^{-1}(t)y(t) = [\theta m_S(t)]^{-1}y(t)$. The greater the path travelled by money the higher the velocity and the lower the real interest rate, after all.

General real money supply or money constraint $h(t) = h[y(t),\ x(t);\ \Theta]$ is therefrom specified as $m_S(t) = [\theta x(t)]^{-1}y(t)$, preceded by parameter $\phi_i = \phi_i[y(t),\ x(t);\ \Theta]$ declined as $\phi = [\theta x(t)]^{-1}$ and by general FOC $y'(t) = y'[y(t),\ x(t);\ \Theta,\ \Phi] = 0$ declined as $\theta^{-1} - \phi x(t) = 0$. General money value function

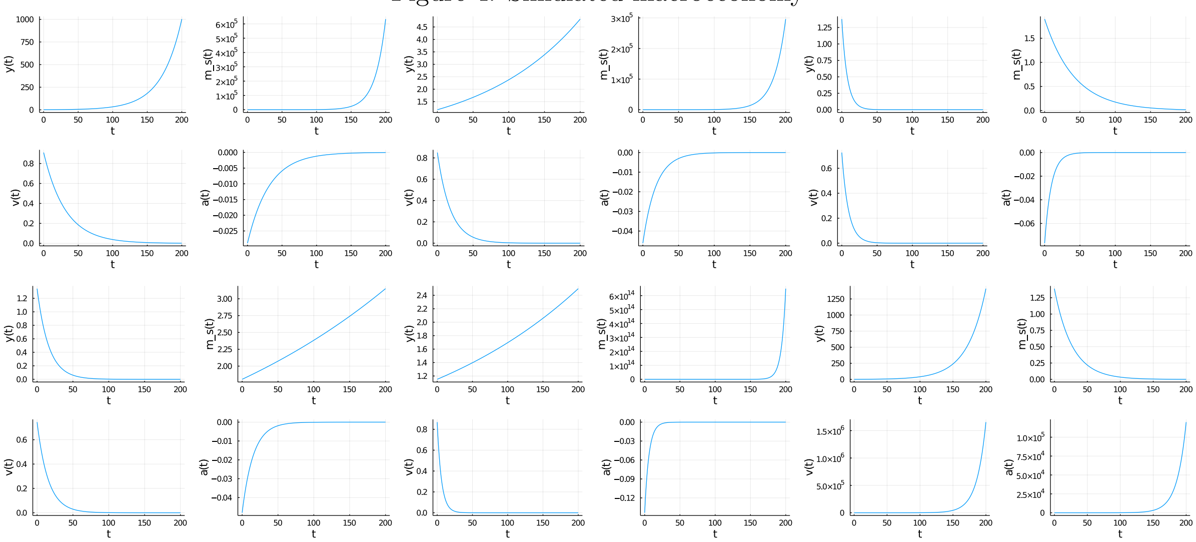
$V = V[y(t),\ h(t),\ x(t);\ \Theta]$ is therefore finally specified as $V = \theta^{-1}y(t) - m_S(t)x(t)$, alongside general money constraint $h(t) = h[y(t);\ \Phi]$ declined as $m_S(t) = \phi y(t)$, wherein parameter ϕ is the money demand coefficient. Money value function $V = \theta^{-1}y(t) - m_S(t)x(t)$ is afresh a decreasing function of real money supply $m_S(t)$ and money position $x(t)$ and an increasing one of real output $y(t)$, weighted by parameter θ^{-1}.

Parameter $\theta = -\alpha_2\sigma_{M_S}\varepsilon_{M_S}$ and it weights real output $y(t)$ in an inverse fashion; specifically, the greater the product of nominal money supply $M_S(t)$'s standard deviation and white noise $\sigma_{M_S}\varepsilon_{M_S}$, itself weighted by nominal money supply $M_S(t)$'s negative coefficient $-\alpha_2$ in real money supply $m_S(t)$, *qua* divisor within the money velocity $v(t)$ equation, the lower the money value V generated by augmented real output $\theta^{-1}y(t)$, namely, the more dampened will real output $y(t)$ in money value V be.

The substitution of money demand coefficient $\phi = [\theta x(t)]^{-1}$ into money constraint $m_S(t) = \phi y(t)$ and thence money value function $V = \theta^{-1}y(t) - m_S(t)x(t)$ lastly once again yields money value $V = \theta^{-1}y(t) - m_S(t)x(t) = \theta^{-1}y(t) - \phi y(t)x(t) = \theta^{-1}y(t) - [\theta x(t)]^{-1}y(t)x(t) = 0$.

The macroeconomy's parametrisation is then the following: $\alpha_{i\in[1,\ 3]} = 0.5,\ \alpha_4,\ \beta_4,\ \gamma_4,\ \delta_4 = 0.25,\ \mu = 0.05$ and, $\forall i = M_S,\ ap,\ am_D,\ \sigma_i = 0.3$. As conveyed, instantaneous money velocity $v(t)$ and instantaneous money acceleration $a(t)$ are both ultimately modelled as exponential functions of time t, in terms of a standard deviation augmented white noise, behaving accordingly.

Figure 4: Simulated macroeconomy



Note. These graphs simulate real output $y(t) = m_S^{\alpha_1}(t)m_D^{1-\alpha_1}(t)$, real money supply $m_S(t) = M_S^{\alpha_2}(t)p^{\alpha_2-1}(t)$, instantaneous money velocity $v(t) = m_S^{-1}(t)y(t)$ and instantaneous money acceleration $a(t) = \theta m_S^{-1}(t)y(t)$ according to the above specification of the macroeconomy, at a time t horizon of 200 quarters (i.e. 50 years). Parameterisation spells $\alpha_{i\in[1,\ 3]} = 0.5,\ \alpha_4,\ \beta_4,\ \gamma_4,\ \delta_4 = 0.25,\ \mu = 0.05$ and, $\forall i = M_S,\ ap,\ am_D,\ \sigma_i = 0.3$.

5. General money velocity

5.1 Money equations of motion. A broader approach to the formulation of instantaneous money velocity $v(t)$ suggests the conception of the money equations of motion through an ODE endowed with initial conditions. The money equations of motion are those of instantaneous money acceleration $a(t)$, instantaneous money velocity $v(t)$ and money position $x(t)$.

Now, since instantaneous money acceleration $a(t)$, instantaneous money velocity $v(t)$ and money position $x(t)$ are functions of the macroeconomy and can theoretically all be non-zero their sum equals a generic, non-negative real function of real output y, real money supply m_S, money demand m_D, prices p, nominal money supply M_S, aggregate money demand am_D and aggregate prices ap; at null time t money position $x(t)$ and instantaneous money velocity $v(t)$ are moreover void: $x''(t) + x'(t) + x(t) = f(y,\ m_S,\ m_D,\ p,\ M_S,\ am_D,\ ap) \equiv f$, where $f : \mathbb{R}^7_+ \to \mathbb{R}_+$, with initial conditions $x(0) = x'(0) = 0$.

In contrast with Subsection 4.5, the domain variables of function f are admitted as null forasmuch as their equations are not hereby specified; even though the sum of the money equations of motion be negative, the negativity of function f's codomain is instead excluded. The solution of such an ODE is consequently elaborated as follows:

$x''(t)+x'(t)+x(t)=f\longrightarrow(e^{zt})''+(e^{zt})'+e^{zt}=0\longrightarrow z^2+z+1=0\longrightarrow z_{1,\,2}=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm i\sqrt{3}}{2}$, suggesting $x(t)=x_c(t)+x_p=C_1e^{\frac{-t}{2}}sin\left(\frac{\sqrt{3}t}{2}\right)+C_2e^{\frac{-t}{2}}cos\left(\frac{\sqrt{3}t}{2}\right)+x_p=\left[C_1e^{\frac{-t}{2}}sin\left(\frac{\sqrt{3}t}{2}\right)+C_2cos\left(\frac{\sqrt{3}t}{2}\right)\right]e^{\frac{-t}{2}}+x_p$;

$x_p=k,\ \forall k\in\mathbb{R}_+$, such that $f=k''+k'+k=k=x_p$, whence $x(t)=x_c(t)+x_p=\left[C_1e^{\frac{-t}{2}}sin\left(\frac{\sqrt{3}t}{2}\right)+C_2cos\left(\frac{\sqrt{3}t}{2}\right)\right]e^{\frac{-t}{2}}+f$;

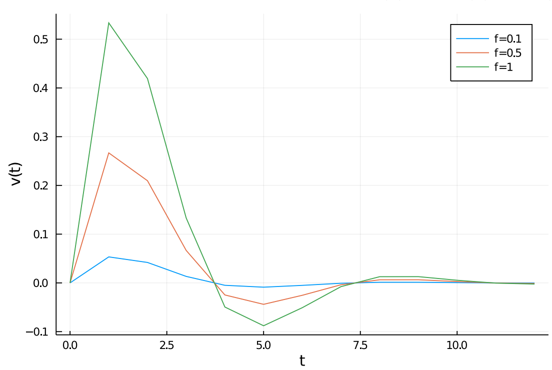
$x(0)=C_2+f=0\longrightarrow C_2=-f$ and $x'(0)=\frac{-C_2}{2}+\frac{C_1\sqrt{3}}{2}=0\longrightarrow C_1=\frac{C_2}{\sqrt{3}}=\frac{-f}{\sqrt{3}}$, since $x'(t)=C_1\left[-\frac{1}{2}e^{\frac{-t}{2}}sin\left(\frac{\sqrt{3}t}{2}\right)+\frac{\sqrt{3}}{2}e^{\frac{-t}{2}}cos\left(\frac{\sqrt{3}t}{2}]\right)\right]+C_2\left[-\frac{1}{2}e^{\frac{-t}{2}}cos\left(\frac{\sqrt{3}t}{2}\right)-\frac{\sqrt{3}}{2}e^{\frac{-t}{2}}sin\left(\frac{\sqrt{3}t}{2}\right)\right]=\left\{C_1\left[\frac{\sqrt{3}}{2}cos\left(\frac{\sqrt{3}t}{2}]\right)-\frac{1}{2}sin\left(\frac{\sqrt{3}t}{2}\right)\right]-C_2\left[\frac{1}{2}cos\left(\frac{\sqrt{3}t}{2}\right)+\frac{\sqrt{3}}{2}sin\left(\frac{\sqrt{3}t}{2}\right)\right]\right\}e^{\frac{-t}{2}}$;

therefore, money position $x(t)=\left[-fcos\left(\frac{\sqrt{3}t}{2}\right)-\frac{f}{\sqrt{3}}sin\left(\frac{\sqrt{3}t}{2}\right)\right]e^{\frac{-t}{2}}+f=\left\{1-\left[cos\left(\frac{\sqrt{3}t}{2}\right)+\frac{1}{\sqrt{3}}sin\left(\frac{\sqrt{3}t}{2}\right)\right]e^{\frac{-t}{2}}\right\}f$.

Therefrom, instantaneous money velocity $v(t)=x'(t)=\left\{f\left[\frac{1}{2}cos\left(\frac{\sqrt{3}t}{2}\right)+\frac{\sqrt{3}}{2}sin\left(\frac{\sqrt{3}t}{2}\right)\right]-\frac{f}{\sqrt{3}}\left[\frac{\sqrt{3}}{2}cos\left(\frac{\sqrt{3}t}{2}]\right)-\frac{1}{2}sin\left(\frac{\sqrt{3}t}{2}\right)\right]\right\}e^{\frac{-t}{2}}=\frac{2f}{\sqrt{3}}e^{\frac{-t}{2}}sin\left(\frac{\sqrt{3}t}{2}\right)$ and instantaneous money acceleration $a(t)=v'(t)$ accordingly.

Absent loss of generality, function f's codomain can be restricted to a semi-open interval between 0 and 1, assessing instantaneous money velocity $v(t)$ over time t therethrough: $f:\mathbb{R}^7_+\rightarrow(0,\ 1]$ in $v(t)=\frac{2f}{\sqrt{3}}e^{\frac{-t}{2}}sin\left(\frac{\sqrt{3}t}{2}\right)$ over $t\in\mathbb{R}_+$. Test function f values 0.1, 0.5 and 1 can be specifically selected to the end of a plot. As of the mid seventh quarter instantaneous money velocity $v(t)$ is stabilised around 0, irrespective of the macroeconomic constant's magnitude, and the smaller function f's codomain the stabler instantaneous money velocity $v(t)$ initially.

Figure 5: Simulations of instantaneous money velocity in $x''(t)+x'(t)+x(t)=f$ with $x(0)=x'(0)=0$



Note. This graph plots simulations of instantaneous money velocity $v(t)=\frac{2f}{\sqrt{3}}e^{\frac{-t}{2}}sin\left(\frac{\sqrt{3}t}{2}\right)$ for function $f=0.1,\ 0.5,\ 1$ at 12 quarter horizon (i.e. 3 years). It is the time t derivative of money position $x(t)$, *qua* solution to ODE $x''(t)+x'(t)+x(t)=f$ with initial conditions $x(0)=x'(0)=0$.

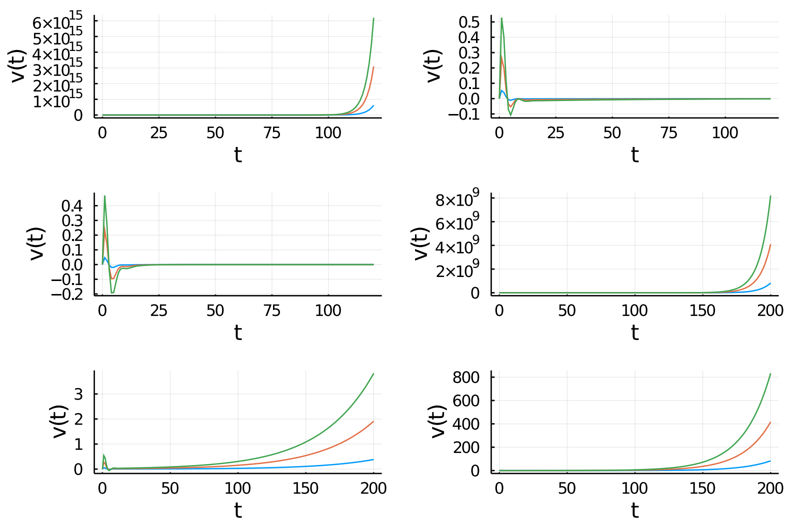
5.2 Time variation. Macroeconomic function f is an ultimate function of t as well, however, oscillating about a trend, at least. Specifically, the business cycle dynamics of a closed economy are summarily assumed to mimic a sine oscillation about a trend, along the lines of function $f=\alpha(sint+\beta t)$, for parameters $\alpha,\ \beta\in\mathbb{R}$. Function f can thus be said to model the dynamics of real output $y(t)$ within a closed economy. The intuition behind the associated ODE $x''(t)+x'(t)+x(t)=\alpha(sint+\beta t)$ is that the money equations of motion depend upon the prototypical business cycle of a closed economy, following a balanced growth or decline path.

A prototypical business cycle is albeit not that systematic, whereby real output $y(t)$ oscillations about a trend are instead symmetric and deterministic, evoking the more befitting function $f = \alpha e^{(\mu+\sigma\varepsilon)t}$, for parameter $\alpha \in \mathbb{R}$, white noise $\varepsilon \sim \mathcal{N}(0,\ \sigma^2)$ and standard deviation and mnemonic drift $\sigma,\ \mu \in \mathbb{R}_{++}$. Thence the associated ODE $x''(t) + x'(t) + x(t) = \alpha e^{(\mu+\sigma\varepsilon)t} = \alpha e^{\beta t}$, through which the money equations of motion depend upon business cycle dynamics like to those of a random walk with drift, that is, they amount to a correction of real output $y(t)$, whose movements over time t around the balanced growth path are ultimately random. An instance of real output $y(t)$ could be real output $y(t) = m_S^{\alpha_1}(t) m_D^{1-\alpha_1}(t)$, as studied hereinbefore.

ODE $x''(t) + x'(t) + x(t) = \alpha e^{\beta t}$ admits the hypothesis of particular solution $x_p(t) = ae^{\beta t} : (ae^{\beta t})'' + (ae^{\beta t})' + (ae^{\beta t}) = \alpha e^{\beta t} \longrightarrow a(\beta^2 + \beta + 1) = \alpha \longrightarrow a = \frac{\alpha}{\beta^2+\beta+1}$, whence $x(t) = x_c(t) + x_p(t) = \left[C_1 sin\left(\frac{\sqrt{3}t}{2}\right) + C_2 cos\left(\frac{\sqrt{3}t}{2}\right)\right] e^{\frac{-t}{2}} + \frac{\alpha e^{\beta t}}{\beta^2+\beta+1}$. On such an account instantaneous money velocity $v(t) = \left\{C_1\left[\frac{\sqrt{3}}{2} cos\left(\frac{\sqrt{3}t}{2}]\right) - \frac{1}{2} sin\left(\frac{\sqrt{3}t}{2}\right)\right] - C_2\left[\frac{1}{2} cos\left(\frac{\sqrt{3}t}{2}\right) + \frac{\sqrt{3}}{2} sin\left(\frac{\sqrt{3}t}{2}\right)\right]\right\} e^{\frac{-t}{2}} + \frac{\alpha\beta e^{\beta t}}{\beta^2+\beta+1}$. Initial conditions $x'(0) = x(0) = 0$ finally generate money position $x(t) = \frac{-\alpha\left[2\sqrt{3}\beta sin\left(\frac{\sqrt{3}t}{2}\right) - 3e^{(\beta+0.5)t} + 2\sqrt{3} sin\left(\frac{\sqrt{3}t}{2} + \frac{\pi}{3}\right)\right] e^{\frac{-t}{2}}}{3(\beta^2+\beta+1)}$ and instantaneous money velocity $v(t) = x'(t) = \frac{\alpha\left[3\beta e^{(\beta+0.5)t} - 2\sqrt{3}\beta cos\left(\frac{\sqrt{3}t}{2} + \frac{\pi}{6}\right) + 2\sqrt{3} sin\left(\frac{\sqrt{3}t}{2}\right)\right] e^{\frac{-t}{2}}}{3(\beta^2+\beta+1)}$.

For parameters $\alpha = 0.1,\ 0.5,\ 1,\ \sigma = 0.3$ and $\mu = 0.05$ and triple random values of white noise ε per parameter α at a 120 and 200 quarter horizon (i.e. 30 and 50 years) instantaneous money velocity $v(t)$ is displayed below.

Figure 6: Simulations of instantaneous money velocity in $x''(t) + x'(t) + x(t) = \alpha e^{\beta t}$ with $x(0) = x'(0) = 0$



Note. The above graphs plot simulations of instantaneous money velocity $v(t) = \frac{\alpha\left[3\beta e^{(\beta+0.5)t} - 2\sqrt{3}\beta cos\left(\frac{\sqrt{3}t}{2} + \frac{\pi}{6}\right) + 2\sqrt{3} sin\left(\frac{\sqrt{3}t}{2}\right)\right] e^{\frac{-t}{2}}}{3(\beta^2+\beta+1)}$ for parameters $\alpha = 0.1,\ 0.5,\ 1$ (blue, orange and green), $\sigma = 0.3$ and $\mu = 0.05$ at a 120 and 200 quarter horizon, respectively, thrice each. It is the time t derivative of money position $x(t)$, *qua* solution to ODE $x''(t) + x'(t) + x(t) = \alpha e^{(\mu+\sigma\varepsilon)t} = \alpha e^{\beta t}$ with initial conditions $x(0) = x'(0) = 0$.

Despite ODE $x''(t)+x'(t)+x(t) = \alpha e^{\beta t}$'s closed form solution it is to be solved numerically too. Foremost, if the ODE is not stiff then explicit methods are warranted. Stiffness is thus assessed first. Second order linear ODE $x''(t) + x'(t) + x(t) = \alpha e^{\beta t}$ is primarily rewritten as a first order ODE system: $y_1(t) = x(t),\ y_2(t) = y_1'(t) = x'(t) = v(t)$ and $y_2'(t) = y_1''(t) = x''(t) = a(t)$ such that $y_2'(t) = -y_2(t) - y_1(t) + \alpha e^{\beta t}$ and $Y'(t) = [y_1'(t)\ y_2'(t)]^\top = [y_2(t)\ -y_2(t) - y_1(t)]^\top + [0\ \alpha e^{\beta t}]^\top = Y(t) + G(t)$. The eigenvalues of such a system's Jacobian matrix are subsequently elaborated: $J = [(\frac{\partial Y_1}{\partial y_1}\ \frac{\partial Y_1}{\partial y_2})\ (\frac{\partial Y_2}{\partial y_1}\ \frac{\partial Y_2}{\partial y_2})]^\top = [(0\ 1)\ (-1\ \ -1)]^\top$ such that $det[J(\lambda)] = det[(-\lambda\ 1)\ (-1\ \ -1-\lambda)]^\top = 0 \longrightarrow \lambda + \lambda^2 + 1 = 0 \longrightarrow \lambda_{1,\ 2} = z_{1,\ 2}$, *ceteris paribus.* A first order ODE system's stiffness ratio is then that between the greatest and the smallest real eigenvalue in modulus, wherethrough a system is stiff if such a ratio is most greater than 1: $\forall \underline{\lambda}_{J(\lambda)},\ \ldots,\ \overline{\lambda}_{J(\lambda)} \in \mathbb{R},\ SR = \frac{|\overline{\lambda}_{J(\lambda)}|}{|\underline{\lambda}_{J(\lambda)}|} \gg 1$ for a stiff system; thus, $Re(z_{1,\,2}) = \frac{-1}{2}$ and $SR = \frac{0.5}{0.5} = 1 \not\gg 1$ hereby.

The pronounced non-stiffness of such a system thence warrants the adoption of the explicit Euler method or the like to the end of a numerical approximation. Explicit Euler method $Y_{j+1} = Y_j + hF(Y_j,\ t_j) = Y_j + hF_j$, where factor h is a temporal step size such that time $t_j = jh,\ \forall j \in [0,\ J] \subset \mathbb{Z}_+$ and summand $Y_j \approx Y(t_j)$, namely, it approximates function $Y(t)$'s discretisation $Y(t_j)$. Function $F(Y,\ t) = Y'(t) = Y(t) + G(t)$, whence factor $F_j = Y(t_j) + G(t_j) \approx Y_j + G(t_j)$, initial value $Y_0 = Y(0) = [y_2(0)\ \ -y_2(0) - y_1(0)]^\top = [v(0)\ \ -v(0) - x(0)]^\top = [0\ 0]^\top$ and temporal step size $h = 0.1$ are hereby exploited to elaborate functional discretisation $Y(200)$, that is, the numerical solution after 200 time quarters (i.e. 50 years). Maximum steps $J = h^{-1}t_J$, implying $2000 = 0.1^{-1}200$, whence steps $j \in [0,\ 2000]$ hereby. The step solutions are elaborated as follows:

$Y_1 = Y_0 + hF_0 = [0\ 0]^\top + 0.1[Y_0 + G(0)] = 0.1[G(0)] = 0.1[0\ \alpha e^{\beta 0}]^\top = [0\ 0.1\alpha]^\top$,

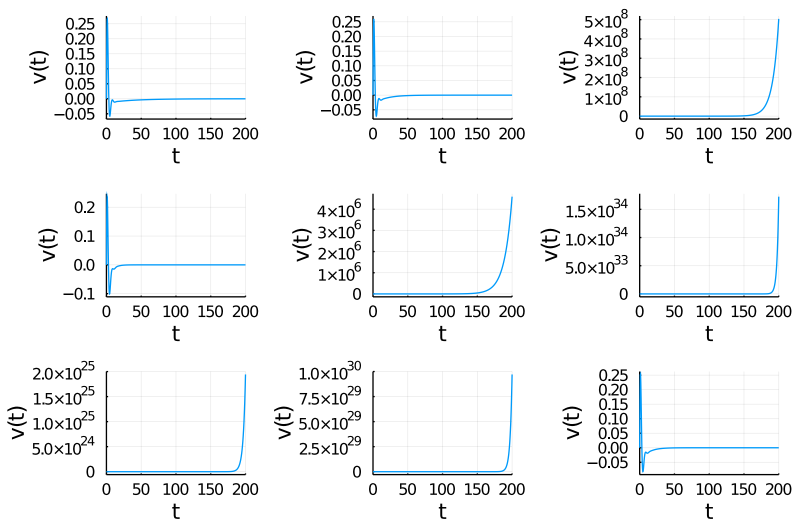
$Y_2 = Y_1 + hF_1 = [0\ 0.1\alpha]^\top + 0.1[Y_1 + G(1)] = [0\ 0.1\alpha]^\top + 0.1([0\ 0.1\alpha]^\top + [0\ \alpha e^{\beta 1}]^\top) = [0\ 0.1\alpha]^\top + ([0\ 0.1^2\alpha]^\top + [0\ 0.1\alpha e^{\beta 1}]^\top) = [0\ 0.1\alpha(1 + 0.1 + e^{\beta 1})]^\top$,

$\vdots$

$Y_{2000} = Y_{1999} + hF_{1999} = \ldots = [0\ 1.105 \times 10^8]^\top \approx [v(200)\ \ -v(200) - x(200)]^\top = [x'(200)\ a(200) - \alpha e^{\beta 200}]^\top$,

having parameterised $\alpha = 0.1,\ 0.5,\ 1,\ \sigma = 0.3$ and $\mu = 0.05$, for triple random values of white noise ε per parameter α. Results, obtained by means of JULIA's automatic method, are displayed in the nine graphs below.

Figure 7: Numerical simulations of instantaneous money velocity in $x''(t) + x'(t) + x(t) = \alpha e^{\beta t}$ with $x(0) = x'(0) = 0$



Note. The above graphs plot numerical simulations of instantaneous money velocity $v(t)$ in ODE $x''(t) + x'(t) + x(t) = \alpha e^{(\mu+\sigma\varepsilon)t}$ with initial values $x(0) = x'(0) = 0$ according to JULIA's automatic method at a 200 quarter horizon. Parameters $\sigma = 0.3$ and $\mu = 0.05$ throughout. The three rows of graphs respectively present parameterisation $\alpha = 0.1,\ 0.5,\ 1$.

For consistent parameterisations, some of the solutions consequently trace the simulated macroeconomy above, wherein instantaneous money velocity $v(t) = m_S^{-1}(t)y(t) = e^{-\alpha_2\sigma_{M_S}\varepsilon_{M_S}t}p^{1-\alpha_2}(t)m_S^{\alpha_1}(t)m_D^{1-\alpha_1}(t)$ and is thus an exponential function of time t at core, dependent on white noises ε, their standard deviations σ, mnemonic drifts μ and suitable parameterisations. ODE $x''(t) + x'(t) + x(t) = \alpha e^{\beta t}$ therefore delivers perhaps the most general instantaneous money velocity $v(t)$ function conceivable.

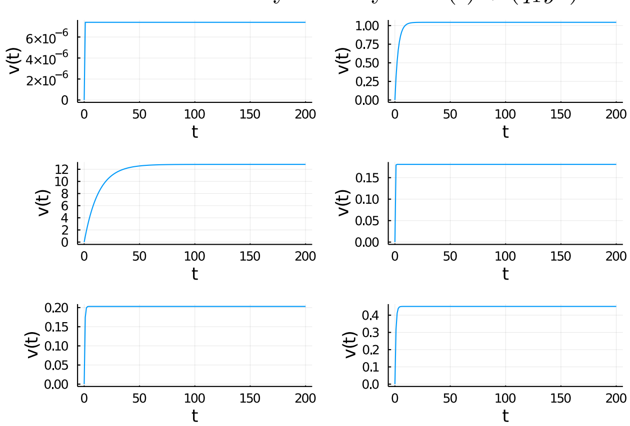
The economic significance is that it ultimately depends upon mere shocks, exogenous to the model and the economy, being undoubtedly poor in technical insight, phenomenally naive, in fact, but after all coherent with the "macro-founded" policy spectra provided by the time t variant version of Fisher's instantiation of instantaneous money velocity $v(t)$, nay, with the manner by which non-particular macroeconomic dynamics are to be conceived. Specifically, the time t variant version of Fisher's instantaneous money velocity $v(t)$ instantiation is expression $v(t) = m_S^{-1}(t)y(t)$, which could in turn equal anything axiomatic, as seen, and said axiomatic equalisation ultimately relies precisely upon parameters and exogenous variables.

5.3 Money force. An alternative ODE, to the end of bettering the specification of the money equations of motion, can be envisaged through the exploitation of money forces. Following the model of an object's free fall with air resistance, money force $F = ma(t)$ is equated to the difference between money gravitational force $F_g = mg$ and money resistance or friction force $F_r = bv(t)$, whereby money mass m, money gravity g and money drag coefficient b are positive: $\forall m,\ g,\ b \in \mathbb{R}_{++},\ F = ma(t) = F_g - F_r = mg - bv(t)$.

Money gravity g is an increasing function of capital return r_k and government bonds return r_b, forasmuch as money gravitates towards its exchange in virtue thereof, all else equal; the higher the interest rate on alternative assets, the higher the opportunity cost of money, the lower the demand for money and the higher the propensity to exchange it, all else unchanged: $g = g(\overset{+}{r_k},\ \overset{+}{r_b})$. Money drag coefficient b is likewise an increasing function of real interest rate r, forasmuch as increments therein increase money demand and offset the propensity to exchange it, all else equal; it also increases in money mass m and money surface area $A:\ b = b(\overset{+}{r},\ \overset{+}{m},\ \overset{+}{A})$.

An object's mass m is normally measured in kilograms kg, that is, in litres l and ultimately cube metres m^3, tangibly or at least historically; such an object's average velocity $\hat{v}(t)$ is similarly measured in metres per second $s^{-1}m$; now, because average money velocity $\hat{v}(t)$ is measured in terms of expenses per unit of time $t^{-1}e$ and because expenses e can themselves be recorded as real output y money mass m could be measured in terms of cube real output y^3. Otherwise put, an object's position $x(t)$ is measured in metres m and money position $x(t)$ is measured in terms of real output y; an object's mass m is measured in cube metres m^3; thus, money mass m is measured in terms of cube real output y^3, that is, positive quantity q_1 multiplied by cube real output $y^3:\ \forall q_1,\ y \in \mathbb{R}_{++},\ m = q_1 y^3$. Money surface area A is comparably measured in terms of square real output $y^2:\ \forall q_2,\ y \in \mathbb{R}_{++},\ A = q_2 y^2$.

Figure 8: Simulations of instantaneous money velocity in $v'(t) + (q_1\bar{y}^3)^{-1}\bar{b}v(t) = \bar{g}$ with $v(0) = 0$



Note. The above graphs plot simulations of instantaneous money velocity $v(t) = \frac{q_1\bar{y}^3\bar{g}}{\bar{b}}(1 - e^{\frac{-\bar{b}t}{q_1\bar{y}^3}})$ for parameters $q_1,\ \bar{g},\ \bar{b},\ \bar{y} \in [0,\ 1)$ at a 200 quarter horizon. It is the solution to ODE $v'(t) + (q_1\bar{y}^3)^{-1}\bar{b}v(t) = \bar{g}$ with initial condition $v(0) = 0$.

Money force $F = ma$ can thus be rewritten as follows: $F = ma(t) = F_g - F_r = mg - bv(t) = q_1 y^3 g - bv(t)$. As instantaneous money velocity $v(t)$ increases so does money resistance $F_r = bv(t)$, eventually equalling or even surpassing money gravitational force $F_g = mg$, being thereby itemised as non-initial money velocity $v_T(t)$ and subscripted by a terminal T for simplicity, therefrom derived: $0 \geq F_g - F_r = mg - bv_T(t) = q_1 y^3 g - bv_T(t) \longrightarrow v_T(t) \geq \frac{mg}{b} = \frac{q_1 y^3 g}{b}$. Focussing on the equality between the two forces, $v_T = b^{-1}mg$, that is to say, there emerges an ODE for instantaneous money velocity $v(t):\ F = ma(t) = mv'(t) = bv_T - bv(t) = b[v_T - v(t)] \longrightarrow v'(t) = m^{-1}b[v_T - v(t)] \longrightarrow v'(t) + m^{-1}bv(t) = m^{-1}bv_T = g$. It is solved thus:

$v'(t) + Bv(t) = G$, where $B \equiv m^{-1}b$ and $G \equiv g$, suggests $(e^{zt})' + Be^{zt} = 0 \longrightarrow z + B = 0 \longrightarrow z = -B$ such that $v(t) = C_1 e^{-Bt} + v_p$;

$v_p = \Gamma$ is hypothesised such that $\Gamma' + B\Gamma = G \longrightarrow \Gamma = \frac{G}{B}$, whence $v(t) = C_1 e^{-Bt} + \frac{G}{B}$;

the initial condition $v(0) = C_1 + \frac{G}{B} = 0 \longrightarrow C_1 = \frac{-G}{B}$, whence $v(t) = \frac{-G}{B} e^{-Bt} + \frac{G}{B} = \frac{G}{B}(1 - e^{-Bt}) = \frac{mg}{b}(1 - e^{\frac{-bt}{m}}) = v_T(1 - e^{\frac{-bt}{m}})$;

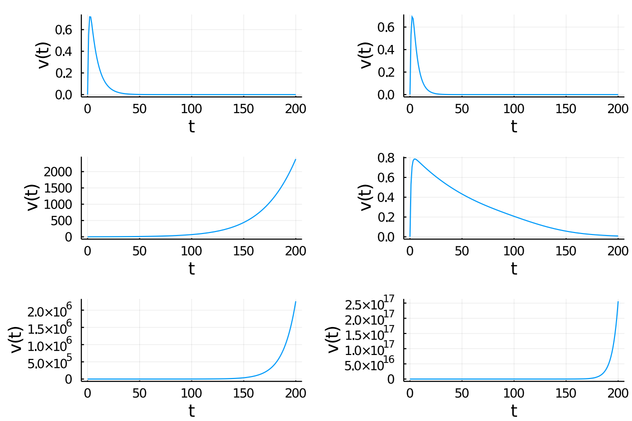
instantaneous money acceleration $a(t) = v'(t) = \frac{v_T b}{m} e^{\frac{-bt}{m}}$, whence $v'(t) + Bv(t) = \frac{v_T b}{m} e^{\frac{-bt}{m}} + \frac{bv_T}{m}(1 - e^{\frac{-bt}{m}}) = \frac{bv_T}{m}$;

money position $x(t) = \int v(t)dt - C = \int v_T(1 - e^{\frac{-bt}{m}})dt - C = v_T\left(t + \frac{me^{\frac{-bt}{m}}}{b}\right) + C - C$ and average money velocity $v_{1t} = \int_{t-1}^{t} v(t)dt = \int_{t-1}^{t} x'(t)dt = x(t) - x(t-1) = x_t - x_{t-1} = \int_{t-1}^{t} v_T(1 - e^{\frac{-bt}{m}})dt = v_T\left(t + \frac{me^{\frac{-bt}{m}}}{b}\right)|_{t-1}^{t} = v_T\left(t + \frac{me^{\frac{-bt}{m}}}{b}\right) - v_T\left[(t-1) + \frac{me^{\frac{-b(t-1)}{m}}}{b}\right] = v_T\left\{1 + \frac{m}{b}\left[e^{\frac{-bt}{m}} - e^{\frac{-b(t-1)}{m}}\right]\right\}$.

For fixed positive values of money gravity g, money drag coefficient b and real output y the equation for instantaneous money velocity $v(t)$ is rewritten thus: $\forall \bar{g},\ \bar{b},\ \bar{y} \in \mathbb{R}_{++},\ v(t) = \frac{q_1 \bar{y}^3 \bar{g}}{b}(1 - e^{\frac{-\bar{b}t}{q_1 \bar{y}^3}})$. Simulations thereof at a 200 quarter horizon (i.e. 50 years) are plotted above for parametrisation $q_1,\ \bar{g},\ \bar{b},\ \bar{y} \in [0,\ 1)$, thereby admitting contrapositions, though effectively rectified to $\bar{g} \in [0,\ 1)$ through the erstwhile $\bar{g},\ \bar{b},\ \bar{y} \in \mathbb{R}_{++}$.

A functional specification of money gravity g, money drag coefficient b and rates of interest $r_b,\ r_k$ and r can then be the following: $g = r_k^{\alpha_5} r_b^{1-\alpha_5}$, $b = r^{\alpha_6} m^{\beta_6} A^{\gamma_6}$, $r_b = r^{\alpha_7} r_k^{1-\alpha_7}$, $r_k = r^{\alpha_8}$ and $r = m_S^{-\alpha_9} m_D^{1-\alpha_9}$, $\forall \alpha_{i\in[5,\ 9]}$, $\beta_6 \in (0,\ 1)$ and $\gamma_6 = 1 - (\alpha_6 + \beta_6)$. Real output $y = m_S^{\alpha_1} m_D^{1-\alpha_1}$, as hereinbefore. Instantaneous money velocity $v(t)$ is therefrom expansible to the end of encompassing such functional specifications: $v(t) = \frac{mg}{b}\left(1 - e^{\frac{-bt}{m}}\right) = \frac{q_1 y^3 \left(r_k^{\alpha_5} r_b^{1-\alpha_5}\right)}{r^{\alpha_6} m^{\beta_6} A^{\gamma_6}}\left[1 - e^{\frac{-\left(r^{\alpha_6} m^{\beta_6} A^{\gamma_6}\right)t}{q_1 y^3}}\right] = \frac{\left(q_1 y^3\right)^{1-\beta_6}\left\{r^{\alpha_8 \alpha_5}\left[r^{\alpha_7} r^{\alpha_8(1-\alpha_7)}\right]^{1-\alpha_5}\right\}}{r^{\alpha_6}\left(q_2 y^2\right)^{\gamma_6}}\left\{1 - e^{-\left[r^{\alpha_6}\left(q_1 y^3\right)^{\beta_6 - 1}\left(q_2 y^2\right)^{\gamma_6}\right]t}\right\}$. Parametrisation lays down $\alpha_{i=5,\ 7,\ 9} = 0.5$, $\alpha_6,\ \beta_6,\ \gamma_6 = \frac{1}{3}$, $\alpha_8 = 0.75$ and $q_{i=1,\ 2} \in [0,\ 1)$, all else equal. Simulations of instantaneous money velocity $v(t)$ are thence displayed at a 200 quarter horizon (i.e. 50 years).

Figure 9: Simulations of instantaneous money velocity in $v'(t) + (q_1 y^3)^{-1} bv(t) = g$ with $v(0) = 0$



Note. The above graphs plot simulations of instantaneous money velocity $v(t) = \frac{q_1 y^3 g}{b}(1 - e^{\frac{-bt}{q_1 y^3}})$ at a 200 quarter horizon. Parametrisation spells $\alpha_{i=5,\ 7,\ 9} = 0.5$, $\alpha_6,\ \beta_6,\ \gamma_6 = \frac{1}{3}$, $\alpha_8 = 0.75$ and $q_{i=1,\ 2} \in [0,\ 1)$, with real output $y = m_S^{\alpha_1} m_D^{1-\alpha_1}$ as modelled above. It is the solution to ODE $v'(t) + (q_1 y^3)^{-1} bv(t) = g$ with initial condition $v(0) = 0$.

6. Conclusion

Equation of exchange $v(t)m_S = y$ has been therefore derived from a standpoint encompassing the physics and economics thereof, whereby the maximisation of money value function V, increasing in real output y and decreasing in real money supply m_S, while accounting for time t and space $x(t)$, subjected to money constraint $m_S = \kappa y$, at the macroeconomic level, gives rise to an optimal level of real output y thereby, expressing money demand coefficient κ as the quotient of time t over space $x(t)$. The fusion of such a money demand coefficient expression $\kappa = x^{-1}(t)t$ with money constraint $m_S = \kappa y$, which is the equilibrium Cambridge equation, in turn gives rise to an equation for space $x(t)$, being the position of money, whose differentiation is precisely instantaneous money velocity $v(t)$ and thence exchange equation $v(t)m_S = y$ as presented by Fisher.

Further insights of the present analysis have moreover been: the expression of money position $x(t)$ as function of money demand coefficient $\kappa(t)$, formally, $x(t) = \kappa^{-1}(t)t$; constancy of instantaneous money velocity $v(t)$ and thus of money demand coefficient $\kappa(t)$ and of the real interest rate $r(t)$, formally, $v(t) = k = \kappa^{-1}(t) = \kappa^{-1}[r(t)]$, thus, $r(t) = \kappa(k)$, however unlifelike; the expression of money demand coefficient $\kappa(t)$ and of the real interest rate $r(t)$ in terms of money position $x(t)$ and time t, formally, $\kappa(t) = v^{-1}(t) = x^{-1}(t)t$ and $r(t) = \kappa^{-1}(\kappa) = \kappa^{-1}[v^{-1}(t)] = \kappa^{-1}[x^{-1}(t)t]$.

The present analysis also derived money position $x(t) = [y(t) \int m_S^{-1}(t)dt - \int \left(\int m_S^{-1}(t)dt\right) y'(t)dt] - C$ on account of non-constant instantaneous money velocity $v(t) = m_S^{-1}(t)y(t)$, advancing a framework for the macroeconomy's general money value function and money constraint in the process. It likewise advanced simulations of non-constant average money velocity v_t and non-constant instantaneous money velocity $v(t)$, with a particular application to a stylised closed macroeconomy. It finally proceeded to remodel instantaneous money velocity $v(t)$ through the use of ODEs for the money equations of motion, both generally, by letting the sum of the three equal a corrected exponential random walk with drift, and through a money force model, of free accumulation with financial assets resistance.

In closing, extensions could especially contemplate the refinement (i) of an ODE endowed with initial conditions in order to better model the money equations of motion and therefrom better derive money position $x(t)$ explicitly or (ii) of macroeconomic money value function V in the orbit of an apposite optimisation problem, whereby both extensions and possibly more are to serve the aspiration of ameliorating the verisimilitude of an equation for instantaneous money velocity $v(t)$. Conclusively, this work thus remarks that money velocity as customarily calculated, taught and understood is not univocal.

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Appendix

Julia commands for money velocity and money acceleration simulations (wherein # must replace %).

```
using LinearAlgebra, Plots, Statistics

T=240+1; % Time period columns (240 quarters, 60 years, plus 1 base year)
s=0.3; % Standard deviation (output to money)
mu=0.05; % Drift (output to money)
w=randn(1, T); % White noise (output to money)
VDsim=zeros(1, T); % Discrete money velocity
VCsim=zeros(1, T); % Continuous money velocity
ADsim=zeros(1, T); % Discrete money acceleration
ACsim=zeros(1, T); % Continuous money acceleration

VDsim[:, 1].=mu.+s.*w[1, 1]; % Discrete money velocity at t=1
for j=2:T % t=2, ..., 241
VDsim[:, j].=mu.+VDsim[:, j-1].+s.*w[1, j]; % Discrete money velocity at t=2, ..., 241
end

for j=1:T % t=1, ..., 241
VCsim[:, j].=mu.*j.+s.*w[1, 1]*j; % Continuous money velocity at t=1, ..., 241
end

for j=1:T % t=1, ..., 241
ADsim[:, j].=mu.+s.*w[1, j]; % Discrete money acceleration at t=1, ..., 241
end

for j=1:T % t=1, ..., 241
ACsim[:, j].=mu.+s.*w[1, 1]; % Continuous money acceleration at t=1, ..., 241
end

t=0:T-1; % Time period domain (240 quarters, 60 years)
p1a=plot(t, VDsim[1, :], xlabel="Quarters", ylabel="MV (disc.)", label="")
p1b=plot(t, VCsim[1, :], xlabel="Quarters", ylabel="MV (cont.)", label="")
p2a=plot(t, ADsim[1, :], xlabel="Quarters", ylabel="MA (disc.)", label="")
p2b=plot(t, ACsim[1, :], xlabel="Quarters", ylabel="MA (cont.)", label="")
% plot(p1a, p1b, p2a, p2b, layout=(2, 2), title="")
% savefig("mvsJ.pdf")
```

JULIA commands for the simulated macroeconomy (wherein # must replace %).

```
using LinearAlgebra, Plots, Statistics, SymPy

% Symbols
y, m_s, m_d, M_s, p, am_d, ap=Sym("y m_{S} m_{D} M_{S} p am_{D} ap"); % Endogenous variables
s_M_s, s_ap, s_am_d, w_M_s, w_ap, w_am_d, mu=Sym("s_{M_{S}} s_{ap} s_{am_{D}} w_{M_{S}} ...
    w_{ap} w_{am_{D}} mu"); % Exogenous variables and their parameters
a_1, a_2, a_3, a_4, b_4, g_4, d_4=Sym("a_{1} a_{2} a_{3} a_{4} b_{4} g_{4} d_{4}"); % ...
    Endogenous variables parameters
a, v, x, t=Sym("a v x t"); % Money mechanics variables

% Symbolic macroeconomy
M_s=exp(s_M_s*w_M_s*t); % Nominal money supply
ap=exp((s_ap*w_ap+mu)*t); % Aggregate prices
am_d=exp(s_am_d*w_am_d*t); % Aggregate money demand
dM_s=diff(M_s, t); % Changed nominal money supply
dap=diff(ap, t); % Changed aggregate prices
dam_d=diff(am_d, t); % Changed aggregate demand
y=(m_s^a_1)*(m_d^(1-a_1)); % Real output
m_s=(M_s^a_2)*(p^(a_2-1)); % Real money supply
m_d=(am_d^a_3)*(ap^(a_3-1)); % Money demand
p=(ap^a_4)*(dM_s^b_4)*(dam_d^g_4)*(dap^(-d_4)); % Prices
v=y/m_s; % Money velocity
a=diff(v, t); % Money acceleration
x=integrate(v, t); % Money position

% Numerical macroeconomy
rn_1=first(randn(1)); % White noise 1
```

```
rn_2=first(randn(1)); % White noise 2
rn_3=first(randn(1)); % White noise 3

M_sn=M_s(s_M_s=>0.3, w_M_s=>rn_1); % Numerical nominal money supply
apn=ap(s_ap=>0.3, w_ap=>rn_2, mu=>0.05); % Numerical aggregate prices
am_dn=am_d(s_am_d=>0.3, w_am_d=>rn_3); % Numerical aggregate money demand
dM_sn=diff(M_sn, t); % Numerical changed nominal money supply
dapn=diff(apn, t); % Numerical changed aggregate prices
dam_dn=diff(am_dn, t); % Numerical changed aggregate money demand
a_1n=a_1(a_1=>0.5); % Numerical parameter a_{1}
a_2n=a_2(a_2=>0.5); % Numerical parameter a_{2}
a_3n=a_3(a_3=>0.5); % Numerical parameter a_{3}
a_4n=a_4(a_4=>0.25); % Numerical parameter a_{4}
b_4n=b_4(b_4=>0.25); % Numerical parameter b_{4}
g_4n=g_4(g_4=>0.25); % Numerical parameter g_{4}
d_4n=d_4(d_4=>0.25); % Numerical parameter d_{4}
pn=(apn^a_4n)*(dM_sn^b_4n)*(dam_dn^g_4n)*(dapn^(-d_4n)); % Numerical prices
m_dn=(am_dn^a_3n)*(apn^(a_3n-1)); % Numerical money demand
m_sn=(M_sn^a_2n)*(pn^(a_2n-1)); % Numerical real money supply
yn=(m_sn^a_1n)*(m_dn^(1-a_1n)); % Numerical real output
vn=yn/m_sn; % Numerical money velocity
an=diff(vn, t); % Numerical money acceleration
% xn=integrate(vn, t);

t=0:200; % Time period domain (200 quarters, 50 years)
p1=plot(yn, t, ylabel="y(t)", xlabel="t", label="")
p2=plot(m_sn, t, ylabel="m_s(t)", xlabel="t", label="")
p3=plot(vn, t, ylabel="v(t)", xlabel="t", label="")
p4=plot(an, t, ylabel="a(t)", xlabel="t", label="")
% plot(p1, p2, p3, p4, layout=(2, 2), title="")
% savefig("mJ.pdf")
```

PYTHON commands for analytical solutions of ODEs $x''(t)+x'(t)+x(t)=f$ and $x''(t)+x'(t)+x(t)=\alpha e^{(\mu+\sigma\varepsilon)t}$ with initial conditions $x(0)=x'(0)=0$ (wherein # must replace %).

```
from sympy import *
import numpy as np

% Symbols
t, a, b, f=symbols('t a b f'); % Domain and parameters
x=Function('x'); % Codomain

% Ordinary differential equations
x_f=dsolve(diff(x(t), t, t)+diff(x(t), t)+x(t)-f, x(t)); % x''(t)+x'(t)+x(t)=f
x_s=dsolve(diff(x(t), t, t)+diff(x(t), t)+x(t)-(a*exp(b*t)), x(t)); % ...
    x''(t)+x'(t)+x(t)=alpha*exp(beta*t)
x_f, x_s

% Symbols
C1, C2=symbols('C1 C2'); % Solution coefficients

% First ODE solution
xo_f=f+(C1*sin(sqrt(3)*t/2)+C2*cos(sqrt(3)*t/2))*exp(-t/2); % x(t) (taken from x_f)
vo_f=diff(xo_f, t); % x'(t)
ao_f=diff(vo_f, t); % x''(t)
sc_f=simplify(ao_f+vo_f+xo_f-f); % Solution check

% Second ODE solution
xo_s=a*exp(b*t)/(b**2+b+1)+(C1*sin(sqrt(3)*t/2)+C2*cos(sqrt(3)*t/2))*exp(-t/2); % x(t) ...
    (taken from x_a)
vo_s=diff(xo_s, t); % x'(t)
ao_s=diff(vo_s, t); % x''(t)
sc_s=simplify(ao_s+vo_s+xo_s-a*exp(b*t)); % Solution check

sc_f, sc_s

```

```
% First ODE initial conditions
xo_f0=xo_f.subs(t, 0); % x(0)
vo_f0=vo_f.subs(t, 0); % x'(0)
C2_f0=solve(xo_f0, C2); % C2 from x(0)=0
C1_f0=solve(vo_f0, C1); % C1 from x''(0)=0

% Second ODE initial conditions
xo_s0=xo_s.subs(t, 0); % x(0)
vo_s0=vo_s.subs(t, 0); % x'(0)
C2_s0=solve(xo_s0, C2); % C2 from x(0)=0
C1_s0=solve(vo_s0, C1); % C1 from x''(0)=0

C2_f0, C1_f0, C2_s0, C1_s0

% First ODE solution with initial conditions
xo_fi=f+((sqrt(3)*(-f)/3)*sin(sqrt(3)*t/2)+(-f)*cos(sqrt(3)*t/2))*exp(-t/2); % x(t) ... (taken from xo_f0 and vo_f0)
vo_fi=diff(xo_fi, t); % x'(t)
ao_fi=diff(vo_fi, t); % x''(t)
sc_fi=simplify(ao_fi+vo_fi+xo_fi-f); % Solution check

% Second ODE solution with initial conditions
xo_si=a*exp(b*t)/(b**2+b+1)+((sqrt(3)*((-a/(b**2+b+1))*b**2+(-a/(b**2+b+1))*b+ \\
(-a/(b**2+b+1))-2*a*b)/(3*(b**2+b+1)))*sin(sqrt(3)*t/2)+ \\
(-a/(b**2+b+1))*cos(sqrt(3)*t/2))*exp(-t/2); % x(t) (taken from xo_s0 and vo_s0)
vo_si=diff(xo_si, t); % x'(t)
ao_si=diff(vo_si, t); % x''(t)
sc_si=simplify(ao_si+vo_si+xo_si-a*exp(b*t)); % Solution check

sc_fi, sc_si

% First ODE solution with initial conditions and parameters
vo_fis=simplify(vo_fi); % x'(t)
vo_fisn1=vo_fis.subs(f, 0.1); % x'(t) with f=0.1
vo_fisn2=vo_fis.subs(f, 0.5); % x'(t) with f=0.5
vo_fisn3=vo_fis.subs(f, 1); % x'(t) with f=1
plot(vo_fisn1, vo_fisn2, vo_fisn3, (t, 0, 12), ylabel='v(t)')

% Second ODE solution with initial conditions and parameters
vo_sis=simplify(vo_si); % x'(t)
w=np.random.randn(); % White noise
vo_sisn1=vo_sis.subs({a: 0.1, b: 0.05+0.3*w}).n(); % x'(t) with alpha=0.1, ... beta=mu+sigma*w=0.05+0.3*w
vo_sisn2=vo_sis.subs({a: 0.5, b: 0.05+0.3*w}).n(); % x'(t) with alpha=0.5, ... beta=mu+sigma*w=0.05+0.3*w
vo_sisn3=vo_sis.subs({a: 1, b: 0.05+0.3*w}).n(); % x'(t) with alpha=1, ... beta=mu+sigma*w=0.05+0.3*w
plot(vo_sisn1, vo_sisn2, vo_sisn3, (t, 0, 120), ylabel='v(t)') % (or (t, 0, 200))
```

JULIA commands for analytical solutions of ODEs $x''(t)+x'(t)+x(t)=f$ and $x''(t)+x'(t)+x(t)=\alpha e^{(\mu+\sigma\varepsilon)t}$ with initial conditions $x(0)=x'(0)=0$ (wherein # must replace %).

```
using LinearAlgebra, Plots, Statistics, SymPy

% Symbols
t, a, b, f=Sym("t a b f"); % Domain and parameters
x=SymFunction("x"); % Codomain

% First ODE solution with initial conditions
xo_fi=f+((sqrt(3)*(-f)/3)*sin(sqrt(3)*t/2)+(-f)*cos(sqrt(3)*t/2))*exp(-t/2); % x(t) ... (taken from xo_f0 and vo_f0 in Python)
vo_fi=diff(xo_fi, t); % x'(t)

% Second ODE solution with initial conditions
xo_si=a*exp(b*t)/(b^2+b+1)+((sqrt(3)*((-a/(b^2+b+1))*b^2+(-a/(b^2+b+1))*b+ \\
(-a/(b^2+b+1))-2*a*b)/(3*(b^2+b+1)))*sin(sqrt(3)*t/2)+ \\
```

```
(-a/(b^2+b+1))*cos(sqrt(3)*t/2))*exp(-t/2); % x(t) (taken from xo_s0 and vo_s0 in Python)
vo_si=diff(xo_si, t); % x'(t)

% First ODE solution with initial conditions and parameters
vo_fis=simplify(vo_fi); % x'(t)
vo_fisn1=vo_fis(f=>0.1); % x'(t) with f=0.1
vo_fisn2=vo_fis(f=>0.5); % x'(t) with f=0.5
vo_fisn3=vo_fis(f=>1); % x'(t) with f=1

t=0:12; % Time period domain (12 quarters, 3 years)
plot(vo_fisn1, t, ylabel="v(t)", xlabel="t", label="f=0.1")
plot!(vo_fisn2, t, ylabel="v(t)", xlabel="t", label="f=0.5")
plot!(vo_fisn3, t, ylabel="v(t)", xlabel="t", label="f=1")
savefig("memJ1.pdf")

% Second ODE solution with initial conditions and parameters
vo_sis=simplify(vo_si); % x'(t)
w=first(randn(1)); % White noise
vo_sisn1=vo_sis(a=>0.1, b=>0.05+0.3*w); % x'(t) with alpha=0.1, beta=mu+sigma*w=0.05+0.3*w
vo_sisn2=vo_sis(a=>0.5, b=>0.05+0.3*w); % x'(t) with alpha=0.5, beta=mu+sigma*w=0.05+0.3*w
vo_sisn3=vo_sis(a=>1, b=>0.05+0.3*w); % x'(t) with alpha=1, beta=mu+sigma*w=0.05+0.3*w

t=0:120; % Time period domain (120 quarters, 30 years, or t=0:200)
plot(vo_sisn1, t, ylabel="v(t)", xlabel="t", label="")
plot!(vo_sisn2, t, ylabel="v(t)", xlabel="t", label="")
p1=plot!(vo_sisn3, t, ylabel="v(t)", xlabel="t", label="")
% plot(p1, p2, p3, p4, p5, p6, layout=(3, 2), title="", label="")
% savefig("memJ2.pdf")
```

JULIA commands for numerical solutions of ODE $x''(t) + x'(t) + x(t) = \alpha e^{(\mu+\sigma\varepsilon)t}$ with initial values $x(0) = x'(0) = 0$ (wherein # must replace %).

```
using DifferentialEquations, Plots

function mv(dx, x, alpha, w, t) % X'(t)=X(t)+G(t)
 dx[1]=x[2]; % x'(t)=v(t)
 dx[2]=-x[2]-x[1]+alpha*exp((0.05+0.3*w)*t); % a(t)=-v(t)-x(t)+alpha*exp((mu+sigma*w)*t) ...
     (ODE)
end

alpha=0.5; % Parameter alpha (also 0.1 or 1)
w=first(randn(1)); % White noise
x0=[0; 0]; % X_0=X(0)=[v(0) -v(0)-x(0)]' (initial value)
tspan=(0, 200); % t in [0, 200] (time periods, 200 quarters, 50 years)
prob=ODEProblem(mv, x0, tspan);
sol=solve(prob);

p1=plot(sol, vars=(0, 2), xlabel="t", yaxis="v(t)", label="")
% plot(p1, p2, p3, p4, p5, p6, p7, p8, p9, layout=(3, 3), label="")
% savefig("memnJ2.pdf")
```

JULIA commands for analytical solutions of ODE $v'(t)+(q_1\bar{y}^3)^{-1}\bar{b}v(t) = \bar{g}$ with initial condition $v(0) = 0$ (wherein # must replace %).

```
using LinearAlgebra, Plots, Statistics, SymPy

% Symbols
t, b, m, v_T=Sym("t b m v_{T}"); % Domain and parameters
v=SymFunction("v"); % Codomain

% Ordinary differential equation
vs=dsolve(diff(v(t), t)+(b/m)*v(t)-(b/m)*v_T, v(t)) % v'(t)+(b/m)*v(t)=(b/m)*v_T

% Symbols
```

```
C1=Sym("C1"); % Solution coefficient

% ODE solution
vso=C1*exp(-(b*t)/m)+v_T; % v(t) (taken from vs)
sc=simplify(diff(vso, t)+(b/m)*vso-(b/m)*v_T) % Solution check

% ODE initial condition
vs0=vso(t=>0); % v(0)
C1_0=first(solve(vs0, C1)) % C1 from v(0)=0

% ODE with initial condition
vsoi=-v_T*exp(-(b*t)/m)+v_T % v(t) (taken from vs0)

% Symbols
b, m, v_T, t=Sym("b m v_{T} t"); % Domain and parameters
q_1, y, g=Sym("q_{1} y g"); % Parameters

% Expanded ODE with initial condition and parameters
m=q_1*(y^(3)); % Money mass
v_T=(m*g)/b; % Terminal money velocity
va=v_T*(1-exp(-(b*t)/m)); % Money velocity (taken from vsoi)

rn1=first(rand(1)); % Numerical value for money mass quantity q_{1}
rn2=first(rand(1)); % Numerical value for real output y
rn3=first(rand(1)); % Numerical value for money gravity g
rn4=first(rand(1)); % Numerical value for money drag coefficient b

mn=m(q_1=>rn1, y=>rn2); % Numerical money mass
v_Tn=v_T(m=>mn, g=>rn3, b=>rn4); % Numerical terminal money velocity
van=va(v_T=>v_Tn, b=>rn4, m=>mn); % Numerical money velocity

t=0:200; % Time period domain (200 quarters, 50 years)
p1=plot(van, t, ylabel="v(t)", xlabel="t", label="")
% plot(p1, p2, p3, p4, p5, p6, layout=(3, 2), title="")
% savefig("fmvJ.pdf")
```

JULIA commands for analytical solutions of ODE $v'(t)+(q_1y^3)^{-1}bv(t) = g$ with initial condition $v(0) = 0$ (wherein # must replace %).

```
using LinearAlgebra, Plots, Statistics, SymPy

% Symbols
y, m_s, m_d, M_s, p, am_d, ap=Sym("y m_{S} m_{D} M_{S} p am_{D} ap"); % Endogenous variables
s_M_s, s_ap, s_am_d, w_M_s, w_ap, w_am_d, mu=Sym("s_{M_{S}} s_{ap} s_{am_{D}} w_{M_{S}} ...
    w_{ap} w_{am_{D}} mu"); % Exogenous variables and their parameters
a_1, a_2, a_3, a_4, b_4, g_4, d_4=Sym("a_{1} a_{2} a_{3} a_{4} b_{4} g_{4} d_{4}"); % ...
    Endogenous variables parameters
v, t=Sym("v t"); % Money mechanics variables

m, A, r, r_k, r_b, g, b, v_T=Sym("m A r r_{k} r_{b} g b v_{T}"); % Money velocity domain
q_1, q_2, a_5, a_6, b_6, g_6, a_7, a_8, a_9=Sym("q_{1} q_{2} a_{5} a_{6} b_{6} g_{6} ...
    a_{7} a_{8} a_{9}"); % Money velocity parameters

% Symbolic macroeconomy
M_s=exp(s_M_s*w_M_s*t); % Nominal money supply
ap=exp((s_ap*w_ap+mu)*t); % Aggregate prices
am_d=exp(s_am_d*w_am_d*t); % Aggregate money demand
dM_s=diff(M_s, t); % Changed nominal money supply
dap=diff(ap, t); % Changed aggregate prices
dam_d=diff(am_d, t); % Changed aggregate demand
y=(m_s^a_1)*(m_d^(1-a_1)); % Real output
m_s=(M_s^a_2)*(p^(a_2-1)); % Real money supply
m_d=(am_d^a_3)*(ap^(a_3-1)); % Money demand
p=(ap^a_4)*(dM_s^b_4)*(dam_d^g_4)*(dap^(-d_4)); % Prices

m=q_1*(y^3); % Money mass
```

```
A=q_2*(y^2); % Money surface area
r=(m_s^(-a_9))*(m_d^(1-a_9)); % Real interest rate
r_k=r^a_8; % Capital return
r_b=(r^a_7)*(r_k^(1-a_7)); % Bond yield
g=(r_k^a_5)*(r_b^(1-a_5)); % Money gravity
b=(r^a_6)*(m^b_6)*(A^g_6); % Money drag coefficient
v_T=(m*g)/b; % Terminal money velocity
v=v_T*(1-exp(-(b*t)/m)); % Money velocity

% Numerical macroeconomy
rn_1=first(randn(1)); % White noise 1
rn_2=first(randn(1)); % White noise 2
rn_3=first(randn(1)); % White noise 3

M_sn=M_s(s_M_s=>0.3, w_M_s=>rn_1); % Numerical nominal money supply
apn=ap(s_ap=>0.3, w_ap=>rn_2, mu=>0.05); % Numerical aggregate prices
am_dn=am_d(s_am_d=>0.3, w_am_d=>rn_3); % Numerical aggregate money demand
dM_sn=diff(M_sn, t); % Numerical changed nominal money supply
dapn=diff(apn, t); % Numerical changed aggregate prices
dam_dn=diff(am_dn, t); % Numerical changed aggregate money demand
a_1n=a_1(a_1=>0.5); % Numerical parameter a_{1}
a_2n=a_2(a_2=>0.5); % Numerical parameter a_{2}
a_3n=a_3(a_3=>0.5); % Numerical parameter a_{3}
a_4n=a_4(a_4=>0.25); % Numerical parameter a_{4}
b_4n=b_4(b_4=>0.25); % Numerical parameter b_{4}
g_4n=g_4(g_4=>0.25); % Numerical parameter g_{4}
d_4n=d_4(d_4=>0.25); % Numerical parameter d_{4}
pn=(apn^a_4n)*(dM_sn^b_4n)*(dam_dn^g_4n)*(dapn^(-d_4n)); % Numerical prices
m_dn=(am_dn^a_3n)*(apn^(a_3n-1)); % Numerical money demand
m_sn=(M_sn^a_2n)*(pn^(a_2n-1)); % Numerical real money supply
yn=(m_sn^a_1n)*(m_dn^(1-a_1n)); % Numerical real output

q_1n=first(rand(1)); % Numerical money mass quantity q_{1}
q_2n=first(rand(1)); % Numerical money surface area quantity q_{2}

mn=q_1n*(yn^3); % Numerical money mass
An=q_2n*(yn^2); % Numerical money surface area
a_9n=a_9(a_9=>0.5); % Numerical parameter a_{9}
a_8n=a_8(a_8=>0.75); % Numerical parameter a_{8}
a_7n=a_7(a_7=>0.5); % Numerical parameter a_{7}
a_6n=a_6(a_6=>1/3); % Numerical parameter a_{6}
b_6n=b_6(b_6=>1/3); % Numerical parameter b_{6}
g_6n=g_6(g_6=>1/3); % Numerical parameter g_{6}
a_5n=a_5(a_5=>0.5); % Numerical parameter a_{5}
rn=(m_sn^(-a_9n))*(m_dn^(1-a_9n)); % Numerical real interest rate
r_kn=rn^a_8n; % Numerical capital return
r_bn=(rn^a_7n)*(r_kn^(1-a_7n)); % Numerical bond yield
gn=(r_kn^a_5n)*(r_bn^(1-a_5n)); % Numerical money gravity
bn=(rn^a_6n)*(mn^b_6n)*(An^g_6n); % Numerical money drag coefficient
v_Tn=(mn*gn)/bn; % Numerical terminal money velocity
vn=v_Tn*(1-exp(-(bn*t)/mn)); % Numerical money velocity

t=0:200; % Time period domain (200 quarters, 50 years)
p1=plot(vn, t, ylabel="v(t)", xlabel="t", label="")
% plot(p1, p2, p3, p4, p5, p6, layout=(3, 2), title="")
% savefig("fmvmJ.pdf")
```