Semi-Endogenous or Fully Endogenous Growth? A Unified Theory.

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Abstract

Is growth ultimately fully endogenous or semi-endogenous? Three decades of theoretical and empirical growth economics have kept both possibilities open. Here, R & D-driven growth is a general combination of both semi-endogenous and fully endogenous mechanisms.

I demonstrate that if the semi-endogenous growth component is indispensable to the actual growth mechanism, the long-run growth rate follows the semi-endogenous growth predictions. Conversely, if the semi-endogenous growth is non-essential and the world population experiences slow growth, the fully endogenous growth mechanism could dictate the long run, even if it is not essential.

If no other (third) growth mechanism exists, a criterion sufficient to ascertain the essentiality of semi-endogenous growth is that reduced research consistently leads to fewer innovations.

If an unknown third growth engine exists, the steady state remains semi-endogenous, provided the essentiality criterion is met. Regardless of how this third factor impacts short-term growth, semi-endogenous growth will prevail in the long run.

JEL classification: O30, O40.
Keywords: Strong scale effect; Semi-endogenous growth; Fully endogenous growth.

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1 Introduction

The worldwide population keeps growing. How shall we design an R&D-driven growth model in the presence of non-zero population growth? Ignoring population growth could be an option for some specific microeconomic theoretical questions, but it would make a growth model unable to match macroeconomic data. Consequently, the design of any data-consistent growth model imposes an excruciating choice between two approaches with diverging long-run policy implications: shall growth be eventually endogenous or semi-endogenous? This question has haunted the modern growth theory at least since Jones' (1995b) path-breaking article. After more than a quarter-century of debate and one Nobel Prize, both growth approaches pervade macroeconomics.

Consequently, it is safe to say that the actual aggregate growth process is either semi-endogenous or fully endogenous, or a combination of the two. No other growth mechanism exists, according to current R&D-driven growth theory.

The semi-endogenous growth theory keeps the door open to the very non-rivalry of ideas in the innovation process: any individual researcher's stock of cumulated ideas can facilitate new ideas. Nevertheless, in Romer (1990), as in Grossman and Helpman (1991) and Aghion and Howitt's (1992) "creative destruction" version, this implied a strong scale effect[1] (Jones, 2005) that conflicted with data, as Jones (1995a and b) has highlighted. This finding motivated Jones (1995b), Kortum (1997), and Segerstrom (1998) semi-endogenous growth variant: the knowledge of the existing stock of existing ideas facilitates innovation but with decreasing returns. An important implication of such a solution is that per-capita GDP could only grow with a growing population. Key to this solution is the idea that R&D's (TFP impact-adjusted) productivity declines the more knowledge accumulates, as recently empirically confirmed by Bloom, Jones, Van Reenen, and Webb (2020).[2]

A different group of scholars, such as Smulders and Van de Klundert (1995), Dinopoulous and Thompson (1998), Peretto (1998), Young (1998), and Howitt (1999), have proposed another class of endogenous growth models, immune to the "strong scale effect", despite dynamic returns to ideas to be constant. In these models, it is not the single researcher that matters, but research as a fraction of the population (Jones, 1999 and 2005). Several microfoundations exist for this solution to the scale effect problem. Still, the basic structure is common to all: by diluting the individual research contribution on a more significant labour force, a larger population reduces the effect of their research effort in proportion. This idea stresses the industrial dynamics and cross-sector dilution of innovation, with population partitioning and specializing innovative efforts.[3]

Both theories are well-motivated, illuminating essential aspects of the innovative process. The importance of cross-sectoral non-rivalry - which characterized semi-endogenous growth - is undeniable, as is the concept of an increase in specialization accompanying population growth - which marks the endogenous growth dilution approach. In this paper, I study the consequence of assuming that both aspects are essential to the innovation process: the TFP

[1] Which counterfactually predicted that the steady-state per-capita GDP growth rate would increase with the economy's population size.

[2] Also see Venturini (2012).

[3] For recent evidence favouring scale-free endogenous growth theory, see Minniti and Venturini (2017). Also, see Madsen (2008).

growth rate would be zero if one were absent. I will show that endogenous growth essentiality does not dictate the long-run growth endogeneity property. Instead, the semi-endogenous growth essentiality is the key to a long-term prediction.

I will also propose a simple rule to determine which growth rate will eventually win in the steady state. If a reduction in the number of researchers always reduces the growth rate, semi-endogenous growth has to be essential and dictate the long run.

Both theories are highly credited in growth economics. Hence, the time has come to settle this divide and get used to the idea that both approaches coexist in a unique framework. Given the importance of this issue, this paper claims that long-term predictions should be robust against much more general aggregations. It will show that significant results do not rely on any specific functional form. Therefore, only natural aggregation properties will be assumed to get striking long-term results analytically. I here prove that all that matters for our long-run results is what happens at the origin: when one of the growth engines turns off. For example, what happens to the growth rates if the semi-endogenous growth mechanism tends to zero? If the resulting growth rate tends to zero, we say that the semi-endogenous part is "essential": a little bit of it is needed for a positive growth rate.

Scarce studies in the literature intertwined these growth theories, asserting that both are instrumental to innovation. Cozzi (2017a and b) and Chu and Wang (2020) are notable exceptions. However, these studies have only run parametric exercises, postulating that the real-world technology growth rate aggregates the semi-endogenous and the fully endogenous solution linearly (in Cozzi, 2017a), with a CES (in Cozzi, 2017b) or with a Cobb-Douglas function (in Chu and Wang, 2020). Yet, these functions are adopted from well-known constant returns to scale production functions linking material inputs to outputs. However, why should a general research and innovation process, with all its thought process complexities, follow any of these off-the-shelf functions inherited from manufacturing production modelling? By embracing the coexistence of both approaches, this paper seeks to bridge the divide and show that long-term predictions can be derived from natural aggregation properties, sidestepping specific functional forms. We ground our findings in the fundamental question: What transpires when one of the growth engines is deactivated?

Our general growth aggregation always implies that innovative ideas are getting harder to find, consistent with Bloom et al. (2020) evidence. However, while the observation of decreasing research productivity is often associated only with the semi-endogenous mechanism, this paper qualifies this argument. Declining research productivity implies the presence of semi-endogenous growth. However, while research becomes increasingly demanding, long-term productivity growth could eventually be fully endogenous. Hence, the general theoretical results of this paper help avoid misleading interpretations of the empirical results of declining R&D productivity.

Last but not least, what would happen if there was a third source of growth, neither semi-endogenous nor fully endogenous? What if this still unknown growth source affected the general growth process of the economy with the other two in a general way? Would our main result still hold? The last part of this paper develops such a case. Indeed, this paper's Proposition 3 shows that if a steady state exists and the semi-endogenous growth component is essential, it will prevail in the long run. This result is of the highest importance because it renders our results immune to future theory.

Section 2 sets up the general model. Section 3 will show that if the semi-endogenous growth engine is essential, the steady state will only follow the semi-endogenous growth part. This result holds regardless of whether fully endogenous growth is essential. This case is helpful because, for example, no CES aggregator allows the essentiality of only one input.

Section 4 proves that if semi-endogenous growth is not essential, growth will eventually be fully endogenous if the population growth rate is low enough. Section 5 extends the result to the existence of a third growth paradigm. Section 6 concludes.

The Appendix nests our general steady-state result into a standard textbook Romer (1990) model for illustrative purposes.

2 Growth Mechanics

Let us assume the following aggregate production function:

$$Y_t = A_t L_{Yt}, \tag{1}$$

where Y_t is output at time t, A_t is total factor productivity, and L_{Yt} is labor employed in manufacturing. By definition, L_{Yt} is a fraction $0 < s_{Yt} < 1$ of the total labour force L_t, which in turn grows at the - possibly negative - constant net rate n. The complementary fraction $s_{At} = 1 - s_{Yt}$ of the labour force denotes the R&D labour share. Total R&D employment is then $L_{At} = s_{At} L_t$.

I assume that the manufacturing total factor productivity, A_t, grows according to the following general function:

$$\frac{A_t - A_{t-1}}{A_{t-1}} \equiv g_{At} = F(g_{\text{sem}t}, g_{\text{end}t}), \tag{2}$$

where

$$g_{\text{sem}t} = (A_{t-1})^{\varphi-1} (s_{At} L_{t-1})^{\lambda_1^A}$$

is the semi-endogenous growth rate (Jones, 1995b), while

$$g_{\text{end}t} = s_{At}^{\lambda_2^A}$$

is the fully endogenous growth rate without scale effects (Smulders and Van de Klundert, 1995).

Following Jones (1995b), in the steady state:

$$\begin{aligned} g_{\text{sem}} &= (1+n)^{\frac{\lambda}{1-\phi}} - 1 \simeq \frac{\lambda_1^A n}{1-\varphi}, \text{ if } n \geq 0 \text{ and} \\ g_{\text{sem}} &= 0, \text{ if } n < 0 \ . \end{aligned}$$

3 Semi-Endogenous Essentiality Matters

In this section, I will show that if the semi-endogenous R&D-driven growth mechanism must be at work for the economy to grow, then the steady-state growth rate will be semi-endogenous.

Let us make the following two assumptions:

Assumption 1. *Function F is non-negative, continuous, and strictly increasing in both its arguments for all possible* $(g_{\text{sem}t}, g_{\text{end}t}) \in R_{++}^{2}$[4]

This assumption is very general and natural. Given the R&D fraction of the labour force, it posits that growth will be higher in a more populated economy. Assumption 1 captures the focus on population size well that semi-endogenous growth theory has inherited from Romer's fully endogenous growth framework. Humans produce ideas, and these are non-rival inputs in the production of new ideas. Hence, the economy's scale matters in levels, even though the "strong scale effect" (Jones, 2005) in growth disappears.

At the same time, Assumption 1 shows that given the scale of the economy, the higher the fraction of it engaged in R&D, the higher the resulting growth rate. This way, it captures the fully endogenous growth element inherited from Romer (1990) but purged from the strong scale effect.

Assumption 2. *Function F is zero if* $g_{\text{sem}t}$ *is zero, that is:*

$$\begin{aligned} F(g_{\text{sem}t}, g_{\text{end}t}) &> 0, \ \forall g_{\text{end}t} \in \mathbb{R}_{+}, \text{if } g_{\text{sem}t} > 0, \text{ and} \\ F(g_{\text{sem}t}, g_{\text{end}t}) &= 0, \ \forall g_{\text{end}t} \in \mathbb{R}_{+}, \text{if } g_{\text{sem}t} = 0. \end{aligned}$$

Notice that Assumption 2 posits that the semi-endogenous growth mechanism is essential: no growth will occur if it is missing. The economy's scale cannot reach zero while still creating a significant growth rate of ideas. It leaves the door open to the fully endogenous growth mechanism to be or not to be essential. That is, a large economy with a negligible fraction of people or GDP involved in R&D may or may not generate significant growth.

As Jones (1995b), I am not describing a one-country economy exchanging ideas with the rest of the world because the imported innovation will spur growth. Instead, I am considering the growth rate of the whole world. Viewed this way, a negligible number in the function's variable would reflect quite a complex world, undoubtedly unable to display any meaningful growth performance. Hence, Assumption 2 is quite natural too.

The two general assumptions above are enough for us to prove our main result:

Proposition 1. *The steady-state growth rate,* g_A*, if it exists, is always semi-endogenous.*

Proof. In a steady state, the growth rate and R&D fraction of the labour force is constant, that is, $s_{At} = s_A$ and $g_{At} = g_A$. Hence we can rewrite (2) as:

$$g_A = F\left[(A_{t-1})^{\varphi-1}(s_A L_{t-1})^{\lambda_1^A}, s_A^{\lambda_2^A}\right] \tag{3}$$

If the population is non-increasing, $n \leq 0$ and the semi-endogenous growth part will tend to zero. In fact, $(A_{t-1})^{\varphi-1}(s_A L_{t-1})^{\lambda_1^A}$ tends to zero as $L_{t-1} \to 0$ or if $A_{t-1} \to \infty$. If the population is increasing, that is $n > 0$, the growth rate function $F\left[(A_{t-1})^{\varphi-1}(s_A L_{t-1})^{\lambda_1^A}, s_A^{\lambda_2^A}\right]$ is a positive constant if and only if both of its arguments are positive constants. Since $s_A^{\lambda_2^A}$ is

[4]With R_{++}^2, I mean the set of ordered pairs of strictly positive real numbers. Instead, R_{+}^2 means the set of ordered pairs of non-negative real numbers.

by construction a positive, only $(A_{t-1})^{\varphi-1}(s_A L_{t-1})^{\lambda_1^A}$ has to be a positive constant. But this happens if and only if the growth rate of $(A_{t-1})^{\varphi-1}$ is equal to the growth rate of $(L_{t-1})^{\lambda_1^A}$, that is, if and only if $g_{\text{sem}} = \frac{\lambda_1^A n}{1-\varphi}$. **QED**

Proposition 1 showed that if the semi-endogenous growth mechanism is essential to the growth process, the only steady state possible is the semi-endogenous steady state.

Remark. No CES function F allows the essentiality of only one of its inputs. If the elasticity of substitutions is less than or equal to one, both inputs are essential, while if it is higher than one, no input is essential. Hence, the results of Proposition 1 would be impossible to obtain in the CES case analyzed by Cozzi (2017b).

3.1 A Microfoundation of Semi-endogenous Growth Essentiality

This section gives a simple, testable condition for semi-endogenous growth to be essential. In particular, I will make the following general assumption:

Assumption 3. *For each positive level of the stock of ideas, A_t, fewer researchers always imply fewer new ideas.*

We can now prove the following:

Lemma 1. *Under Assumptions 1 and 3, semi-endogenous growth is essential.*

Proof. Suppose that the semi-endogenous growth mechanism was not essential. Then $\frac{L_{At}}{L_t} > 0$ implies

$$g_{\text{endt}} = F\left(0, s_A^{\lambda_2^A}\right) = F\left[0, \left(\frac{L_{At}}{L_t}\right)^{\lambda_2^A}\right] > 0. \tag{4}$$

Let the population drop in half while maintaining s_A constant. Now, the number of researchers, following population, will be halved:

$$L_{At'} = 0.5 L_{At},\ t > t\prime.$$

Since

$$L_{t'} = 0.5 L_t,\ t > t\prime,$$

eq. (4) implies that

$$\begin{aligned} A_{t'} - A_{t'-1} &= A_{t'-1} g_{\text{endt}} = A_{t'-1} F\left[0, \left(\frac{L_{At'}}{L_{t'}}\right)^{\lambda_2^A}\right] = A_{t'-1} F\left[0, \left(\frac{0.5 L_{At}}{0.5 L_t}\right)^{\lambda_2^A}\right] \\ &= A_{t'-1} F\left[0, \left(\frac{L_{At}}{L_t}\right)^{\lambda_2^A}\right] = A_t - A_{t-1}, \end{aligned}$$

that is, the same flow of new ideas will be produced, which would violate our previous Assumption 3. **QED**

Lemma 1 provides a simple rule for the essentiality of semi-endogenous growth in a world where the only two possible growth regimes are semi-endogenous and fully endogenous. Given Proposition 1, if Assumption 3 is satisfied, the steady-state growth rate will be semi-endogenous.

4 What if Semi-Endogenous Growth were not Essential?

In this section, we will explore the case in which at least semi-endogenous growth is not essential. By Corollary 1, this means that our previous Assumption 3 is violated. Therefore, we will keep postulating Assumption 1 but drop both Assumptions 2 and 3. In particular, Assumption 2 will be replaced by the following:

Assumption 4. *Function F is positive if and only if the fully endogenous growth mechanism is positive, that is:*

$$\begin{aligned} F(g_{\text{sem}t}, g_{\text{end}t}) &> 0,\ \forall g_{\text{sem}t} \in \mathbb{R}_+, \text{if } g_{\text{end}t} > 0,\ \text{and} \\ F(g_{\text{sem}t}, g_{\text{end}t}) &= 0,\ \forall g_{\text{sem}t} \in \mathbb{R}_+, \text{if } g_{\text{end}t} = 0. \end{aligned}$$

Remark. Notice that the "if" part of Assumption 4 rules out that the semi-endogenous growth mechanism is essential. Otherwise, the positivity of the $g_{\text{end}t}$ would not suffice to make function F positive when $g_{\text{sem}t}$ is zero.

This result follows:

Proposition 2. *Under assumptions 1 and 4, the steady-state growth rate, g_A, if it exists, is fully endogenous if and only if*

$$n \leq \frac{F\left(0, s_A^{\lambda_2^A}\right)(1-\varphi)}{\lambda_1^A} \equiv \bar{n}. \tag{5}$$

Proof. In a steady state, the growth rate and R&D fraction of the labour force is constant, that is: $s_{At} = s_A$ and $g_{At} = g_A$. Hence we can rewrite (2) as:

$$g_A = F\left[(A_{t-1})^{\varphi-1}(s_A L_{t-1})^{\lambda_1^A}, s_A^{\lambda_2^A}\right] \tag{6}$$

If the population is non-increasing, $n \leq 0$ and the semi-endogenous growth part will tend to zero. In fact, $(A_{t-1})^{\varphi-1}(s_A L_{t-1})^{\lambda_1^A}$ tends to zero as $L_{t-1} \to 0$ or if $A_{t-1} \to \infty$. Consequently:

$$g_A \to F\left(0, s_A^{\lambda_2^A}\right) > 0. \tag{7}$$

Moreover, by Assumption 1, $F\left(0, s_A^{\lambda_2^A}\right)$ will increase in $s_A^{\lambda_2^A}$, which means that the long-run growth rate is fully endogenous.

If, instead, population is increasing, that is $n > 0$, two cases are possible:

(A) Condition (5) is satisfied, that is: $F\left(0, s_A^{\lambda_2^A}\right) \geq \frac{\lambda_1^A n}{1-\varphi} \equiv g_{\text{sem}}$;

(B) Condition (5) is not satisfied, that is: $F\left(0, s_A^{\lambda_2^A}\right) < \frac{\lambda_1^A n}{1-\varphi} \equiv g_{\text{sem}}$.

In case (A), by Assumption 1, it must be:

$$g_A = F\left[(A_{t-1})^{\varphi-1}(s_A L_{t-1})^{\lambda_1^A}, s_A^{\lambda_2^A}\right] \geq F\left(0, s_A^{\lambda_2^A}\right) > \frac{\lambda_1^A n}{1-\varphi}. \tag{8}$$

Consequently, $(A_{t-1})^{\varphi-1}(s_A L_{t-1})^{\lambda_1^A}$ will tend to zero and g_A will tend to $F\left(0, s_A^{\lambda_2^A}\right)$. This means that the steady-state growth rate is fully endogenous.

In case (B), $g_A = F\left(0, s_A^{\lambda_2^A}\right) < \frac{\lambda_1^A n}{1-\varphi}$ implies that $(A_{t-1})^{\varphi-1}(s_A L_{t-1})^{\lambda_1^A}$ will tend to infinity, which is inconsistent with a constant level of g_A. A constant level of g_A is achieved if and only if $(A_{t-1})^{\varphi-1}(s_A L_{t-1})^{\lambda_1^A}$ is a positive constant, that is if and only if $g_A = \frac{\lambda_1^A n}{1-\varphi}$. **QED**

Proposition 2 has shown that if the semi-endogenous growth mechanism is not essential to the growth process, the steady-state growth rate will be fully endogenous, depending on the population growth rate. However, notice that this does not constrain the essentiality of the fully endogenous growth part, which would be impossible with a CES aggregator like in Cozzi (2017a and b).

In light of Proposition 2, we can now generalize Cozzi's (2017a and b) result on the endogeneity of the threshold population growth rate, $\bar{n}$, below which the fully endogenous growth mechanism will dominate the steady state. In fact, $\bar{n}$ is a function of the R&D share of GDP, s_A, defined by this equation:

$$\bar{n}\left(s_A^{\lambda_2^A}\right) = \frac{F\left(0, s_A^{\lambda_2^A}\right)(1-\varphi)}{\lambda_1^A}. \tag{9}$$

Consequently, the higher s_A, the higher the population growth rate threshold $\bar{n}$, and the more likely a fully endogenous steady-state growth rate.

5 A Third Source of Growth

In principle, we cannot exclude that the current theory is limited and that other growth mechanisms - yet to be discovered - drive the growth process. Is our approach robust to this possibility? To check, let us extend the analysis to a general aggregator of semi-endogenous, fully endogenous, and any other third source of growth. Therefore, we shall assume the following:

Assumption 5. *Function F is non-negative, continuous and strictly increasing for all possible* $(g_{\text{sem}t}, g_{\text{end}t}, g_{\text{other}t}) \in R^3_{++}$

Notice the presence of $g_{\text{other}t}$, which denotes any source of growth that is neither semi-endogenous nor fully endogenous.

Assumption 6. *Function F is zero if* $g_{\text{sem}t}$ *is zero, that is:*

$$\begin{aligned} F(g_{\text{sem}t}, g_{\text{end}t}, g_{\text{other}t}) &> 0,\ \forall g_{\text{end}t} \in \mathbb{R}_+, \text{if } g_{\text{sem}t} > 0, \text{ and} \\ F(g_{\text{sem}t}, g_{\text{end}t}, g_{\text{other}t}) &= 0,\ \forall g_{\text{end}t} \in \mathbb{R}_+, \text{if } g_{\text{sem}t} = 0. \end{aligned}$$

Assumption 6 implies that the semi-endogenous growth mechanism is essential: no growth will occur if it is zero.

Furthermore, now comes the new ingredient: $g_{\text{other}t}$. We will make no assumption on it, except remarking that it is constant in a steady state.

The two general assumptions above are enough for us to prove our main result:

Proposition 3. *The steady-state growth rate, g_A, if it exists, is always semi-endogenous.*

Proof. In a steady state, the growth rate and R&D fraction of the labour force is constant, that is, $s_{At} = s_A$ and $g_{At} = g_A$. Hence we can rewrite (2) as:

$$g_A = F\left[(A_{t-1})^{\varphi-1}(s_A L_{t-1})^{\lambda_1^A}, s_A^{\lambda_2^A}, g_{\text{other}}\right] \tag{10}$$

If the population is non-increasing, that is, $n \leq 0$, the semi-endogenous growth part will tend to zero. In fact, $(A_{t-1})^{\varphi-1}(s_A L_{t-1})^{\lambda_1^A}$ tends to zero as $L_{t-1} \to 0$ or if $A_{t-1} \to \infty$, which implies $g_A = 0$. If the population is increasing, that is $n > 0$, the growth rate function $F\left[(A_{t-1})^{\varphi-1}(s_A L_{t-1})^{\lambda_1^A}, s_A^{\lambda_2^A}, g_{\text{other}}\right]$ is a positive constant if and only if both of its arguments are positive constants. Since $s_A^{\lambda_2^A}$ is by construction a positive and g_{other} is constant in a steady-state, only $(A_{t-1})^{\varphi-1}(s_A L_{t-1})^{\lambda_1^A}$ has to be a positive constant. But this happens if and only if the growth rate of $(A_{t-1})^{\varphi-1}$ is equal to the growth rate of $(L_{t-1})^{\lambda_1^A}$, that is if and only if $g_{\text{sem}} = \frac{\lambda_1^A n}{1-\varphi}$. **QED**

Remark. Notice that Proposition 3 establishes that our main result is robust to the presence of a third unknown source of growth. Notice that the generality of the function implies that the third growth process could affect the actual growth process in any possible way. Therefore, whatever the empirical estimation for the transition, the steady state, if it exists, will be semi-endogenous.

6 Conclusions

This paper has proved that if the semi-endogenous growth mechanism captures an essential feature of the macroeconomic growth processes, the semi-endogenous growth will dictate the economy's steady-state growth rate. Moreover, in a world where only two growth processes exist, either semi-endogenous or fully endogenous, the steady-state will be semi-endogenous if the innovation stock systematically decreases as the aggregate R&D labour decreases.

Proposition 2 established that if the semi-endogenous growth part is not essential, the growth rate will be fully endogenous if the population growth rate is not too large. This result is independent of whether or not the fully endogenous growth mechanism is an essential part of the growth mechanism.

The essentiality of the semi-endogenous process has effects that go beyond the existence of only two growth processes: Proposition 3 has shown that even if there exists a third, yet unknown, growth engine, the steady-state will be semi-endogenous anyway. Therefore, the long run will be semi-endogenous, whatever happens in the transition due to additional growth dynamics.

Empirically, all combinations of growth frameworks studied in this paper predict that ideas are getting harder to find. However, in the cases of Section 4, the long-run consequences for growth are only sometimes semi-endogenous.

Appendix

A Romer (1990) Example

A large class of growth models satisfy our assumed technology represented by function $F(\cdot,\cdot)$. Our theory can apply to any R&D-driven growth model, like the prominent models byAcemoglu and Cao (2015), Akcigit and Kerr (2018), Garcia-Macia, Hsieh and Klenow (2019), Buera and Oberfeld (2020), Peters (2020), and Jones (2022).

While we used a constant labour share assumption to ease exposition, we could adapt our result to an entirely microfounded case. For example, let us assume a general Romer (1990) economy in which the introduction of new varieties (horizontal innovation) drives growth.

Each household optimizes its per capita consumption, c_t, according to

$$\max \int_0^\infty e^{-(\rho-n)t} \frac{c_t^{1-\varepsilon}-1}{1-\varepsilon} dt, \text{ with } \varepsilon > 0,$$

where $\rho > n$ is the subjective rate of time preferences, and ε is the inverse of the intertemporal elasticity of substitution.

Letting r_t denote the real interest rate, the Euler equation is

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\varepsilon},$$

which in equilibrium implies:[5]

$$r_t = \rho + \varepsilon g_{At}.$$

In a steady-state $r = \rho + \varepsilon g_A$.

The final good is produced in a perfectly competitive industry, according to

$$Y_t = L_{Yt}^{1-\alpha} \int_0^{A_t} x_{it}^{\alpha} di,$$

where A_t is the mass of intermediate product varieties: each intermediate product $i \in [0, A_t]$ is used in production in amount x_{it}.

Under perfect competition, real wage equals the marginal product of labour:

$$w_t = \frac{\partial Y_t}{\partial L} = (1-\alpha)\frac{Y_t}{L},$$

and the real price of each intermediate good equals its marginal product

$$p_t = \frac{\partial Y_t}{\partial x_{it}} = \alpha L^{1-\alpha} x_{it}^{\alpha-1},$$

which can be interpreted as an inverse demand function for intermediate good $i \in [0, A_t]$.

This demand function is taken as given by the intermediate monopolist producer, i.e., by its blueprint's patent holder.

[5] In a stochastic environment, ε would also measure the degree of relative risk aversion implicit in the instantaneous CRRA utility function.

In face, each intermediate good is produced by a monopoly, which maximizes profits

$$\Pi_{it} = p_{it} x_{it} - x_{it},$$

where $p_{it} = \frac{\partial Y_t}{\partial x_{it}} = \alpha L_{1t}^{1-\alpha} x_{it}^{\alpha-1}$. Equilibrium production is

$$x_{it} = L_{Yt} \alpha^{\frac{2}{1-\alpha}} = x_t,$$

which is symmetric, as are the maximized profits:

$$\Pi_{it} = \frac{1-\alpha}{\alpha} L_{Yt} \alpha^{\frac{2}{1-\alpha}} \equiv \Pi_t.$$

In a balanced growth path, they grow at rate n, and the firm's present discounted value, V_t, becomes

$$V_t = \frac{(1-\alpha)\,\alpha^{\frac{1+\alpha}{1-\alpha}}}{(r-n)} (1-s_A) L_t,$$

which equals the variety patent value.

In symmetric equilibrium,

$$Y_t = A_t L_{Yt}^{1-\alpha} x^{\alpha},$$

and the real wage is

$$w_t = \frac{\partial Y_t}{\partial L_{Yt}} = (1-\alpha)\, A_t L_{Yt}^{-\alpha} x_t^{\alpha} = (1-\alpha)\, \alpha^{\frac{2\alpha}{1-\alpha}} A_t. \tag{11}$$

We will assume that varieties evolve according to:

$$\dot{A}_t = A_t F(g_{\text{sem}t}, g_{\text{end}t}), \tag{12}$$

where

$$\begin{aligned} g_{\text{sem}t} &= (A_t)^{\varphi-1} (s_{At} L_t)^{\lambda_1^A}, \text{ and} \\ g_{\text{end}t} &= s_{At}^{\lambda_2^A}, \end{aligned}$$

and $F(g_{\text{sem}t}, g_{\text{end}t})$ has the previously stated properties. We will here add constant returns to scale to facilitate equilibrium computation.

Let us assume an R&D subsidy rate $\sigma \in [0, 1[$ financed with lump-sum taxes. Free entry (zero profit) into R&D implies:

$$\dot{A}_t V_t = w_t L_{At} (1-\sigma),$$

which in the steady state becomes:

$$A_t F(g_{\text{sem}t}, g_{\text{end}t}) \frac{(1-\alpha)\,\alpha^{\frac{1+\alpha}{1-\alpha}}}{r-n} (1-s_A) L_t = w_t s_A L_t (1-\sigma). \tag{13}$$

Let us focus on a steady state. Using (11), eq. (13) simplifies to

$$F(g_{\text{semt}}, g_{\text{end}}) = \frac{s_A(r-n)(1-\sigma)}{\alpha(1-s_A)}. \tag{14}$$

Condition (14) holds only if g_{semt} is constant, that is if $\frac{dg_{\text{sem}\,t}}{dt} = 0$, which only happens if $g_A = g_{\text{sem}} = \frac{\lambda_1^A n}{1-\varphi}$.

The Euler equation implies $r = \rho + \varepsilon g_{\text{sem}}$. Remembering that

$$g_{\text{sem}} = g_A = F(g_{\text{sem}}, g_{\text{end}}),$$

equation (14) allows us to solve for the steady-state R&D share of the labour force:

$$s_A = \frac{\alpha g_{\text{sem}}}{(\rho-n)(1-\sigma) + (\varepsilon+\alpha)\, g_{\text{sem}}}.$$

Notice that all macroeconomic variables are obtained independently of the characteristics of function $F(\cdot,\cdot)$. Comparative statics implies that s_A decreases with the impatience parameter, ρ, and increases with the R&D subsidy rate, σ.

We could here apply our theory to the simple model solved in Section 3.2, obtaining the same results if $n \geq \bar{n}$. If instead $n < \bar{n}$, equation (14) in steady-state will become

$$F(0, s_A^{\lambda_2^A}) = \frac{s_A(r-n)}{\alpha(1-s_A)}, \tag{15}$$

which we can implicitly solve for the steady-state R&D share of the labour force:

$$\frac{\alpha}{\frac{(\rho-n)(1-\sigma)}{F\left(0, s_A^{\lambda_2^A}\right)} + (\varepsilon+\alpha)} - s_A = 0.$$

Unlike in Section 3.2, here s_A is not independent of the characteristics of function $F(\cdot,\cdot)$. We can still analyse the comparative statics. For example, an increase in the subjective rate of interest, ρ, will imply

$$\frac{\partial s_A^{\lambda_2^A}}{\partial \rho} = \frac{\frac{\alpha}{\left(\frac{(\rho-n)(1-\sigma)}{F\left(0, s_A^{\lambda_2^A}\right)} + (\varepsilon+\alpha)\right)^2} \frac{1-\sigma}{\frac{(\rho-n)(1-\sigma)}{F\left(0, s_A^{\lambda_2^A}\right)}}}{\frac{\alpha}{\left(\frac{(\rho-n)(1-\sigma)}{F\left(0, s_A^{\lambda_2^A}\right)} + (\varepsilon+\alpha)\right)^2} F_2\left(0, s_A^{\lambda_2^A}\right) - 1} < 0$$

which, if $F_2\left(0, s_A^{\lambda_2^A}\right) < 0$, implies that s_A declines with impatience. Under the same condition, we can prove that

$$\frac{\partial s_A^{\lambda_2^A}}{\partial \sigma} = -\frac{\frac{\alpha}{\left(\frac{(\rho-n)(1-\sigma)}{F\left(0, s_A^{\lambda_2^A}\right)} + (\varepsilon+\alpha)\right)^2} \frac{\rho-n}{\frac{(\rho-n)(1-\sigma)}{F\left(0, s_A^{\lambda_2^A}\right)}}}{\frac{\alpha}{\left(\frac{(\rho-n)(1-\sigma)}{F\left(0, s_A^{\lambda_2^A}\right)} + (\varepsilon+\alpha)\right)^2} F_2\left(0, s_A^{\lambda_2^A}\right) - 1} > 0$$

meaning that that s_A increases with the R&D subsidy rate. While the results are qualitatively similar to those of Section 3.2, it is useful to notice that quantitatively speaking, the specific function form of $F(\cdot,\cdot)$ will matter for s_A.

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